J-K-Γ Model on Triangular Lattice

swang

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The model Hamiltonian:

$$H_{JK\Gamma} = J\sum_{<ij>}\mathbf{S}_i\cdot\mathbf{S}_j + K\sum_{<ij>}S_i^{\gamma(i,j)}S_j^{\gamma(i,j)} + \Gamma\sum_{<ij>}\left(S_i^{\alpha(i,j)}S_j^{\beta(i,j)} + S_i^{\beta(i,j)}S_j^{\alpha(i,j)}\right)$$

The Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^+ = \sigma^x + i\sigma^y = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \sigma^x - i\sigma^y = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

Schwinger Fermion representation:

$$S^x = \frac{1}{2}\left[c_\uparrow^\dagger, c_\downarrow^\dagger\right]\sigma^x\begin{bmatrix} c_\uparrow \\ c_\downarrow \end{bmatrix} \qquad S^y = \frac{1}{2}\left[c_\uparrow^\dagger, c_\downarrow^\dagger\right]\sigma^y\begin{bmatrix} c_\uparrow \\ c_\downarrow \end{bmatrix} \qquad S^z = \frac{1}{2}\left[c_\uparrow^\dagger, c_\downarrow^\dagger\right]\sigma^z\begin{bmatrix} c_\uparrow \\ c_\downarrow \end{bmatrix}$$

$$S^x = \frac{1}{2}\left(c_\uparrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\uparrow\right) \qquad S^y = \frac{i}{2}\left(-c_\uparrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\uparrow\right) \qquad S^z = \frac{1}{2}\left(c_\uparrow^\dagger c_\uparrow + c_\downarrow^\dagger c_\downarrow\right)$$

with the constraint: $c_\uparrow^\dagger c_\uparrow + c_\downarrow^\dagger c_\downarrow = 1$

Spin-singlet and -triplet pairing operators:

$$s_{ij}^\dagger = \frac{1}{\sqrt{2}}\left[c_{i\uparrow}^\dagger, c_{i\downarrow}^\dagger\right]PM_0\begin{bmatrix} c_{j\uparrow}^\dagger \\ \\ c_{j\downarrow}^\dagger \end{bmatrix}$$

where $PM_0 = i\sigma^y\sigma^0$ and σ^0 is 2×2 identity matrix.

$$t_{ij}^{\alpha\dagger} = \frac{1}{\sqrt{2}}\left[c_{i\uparrow}^\dagger, c_{i\downarrow}^\dagger\right]PM_\alpha\begin{bmatrix} c_{j\uparrow}^\dagger \\ \\ c_{j\downarrow}^\dagger \end{bmatrix}$$

where $PM_\alpha = i\sigma^y\sigma^\alpha$ and σ^α is one of the three Pauli matrices.

$$\sqrt{2}s_{ij}^\dagger = c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger - c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger \qquad \sqrt{2}t_{ij}^{z\dagger} = -c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger - c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger$$
$$\sqrt{2}t_{ij}^{x\dagger} = c_{i\uparrow}^\dagger c_{j\uparrow}^\dagger - c_{i\downarrow}^\dagger c_{j\downarrow}^\dagger \qquad \sqrt{2}t_{ij}^{y\dagger} = i\left(c_{i\uparrow}^\dagger c_{j\uparrow}^\dagger + c_{i\downarrow}^\dagger c_{j\downarrow}^\dagger\right)$$

$$\sqrt{2}c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger = s_{ij}^\dagger - t_{ij}^{z\dagger} \qquad \sqrt{2}c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger = -s_{ij}^\dagger - t_{ij}^{z\dagger}$$
$$\sqrt{2}c_{i\uparrow}^\dagger c_{j\uparrow}^\dagger = t_{ij}^{x\dagger} - it_{ij}^{y\dagger} \qquad \sqrt{2}c_{i\downarrow}^\dagger c_{j\downarrow}^\dagger = -t_{ij}^{x\dagger} - it_{ij}^{y\dagger}$$

Rewrite these spin interaction term using these pairing operators:

$$\begin{aligned}
4S_i^xS_j^x &= \left(c_{i\uparrow}^\dagger c_{i\downarrow} + c_{i\downarrow}^\dagger c_{i\uparrow}\right)\left(c_{j\uparrow}^\dagger c_{j\downarrow} + c_{j\downarrow}^\dagger c_{j\uparrow}\right)\\
&= c_{i\uparrow}^\dagger c_{i\downarrow}c_{j\uparrow}^\dagger c_{j\downarrow} + c_{i\uparrow}^\dagger c_{i\downarrow}c_{j\downarrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{i\uparrow}c_{j\uparrow}^\dagger c_{j\downarrow} + c_{i\downarrow}^\dagger c_{i\uparrow}c_{j\downarrow}^\dagger c_{j\uparrow}
\end{aligned}$$

$$\begin{aligned}
4S_i^yS_j^y &= -\left(-c_{i\uparrow}^\dagger c_{i\downarrow} + c_{i\downarrow}^\dagger c_{i\uparrow}\right)\left(-c_{j\uparrow}^\dagger c_{j\downarrow} + c_{j\downarrow}^\dagger c_{j\uparrow}\right)\\
&= -c_{i\uparrow}^\dagger c_{i\downarrow}c_{j\uparrow}^\dagger c_{j\downarrow} + c_{i\uparrow}^\dagger c_{i\downarrow}c_{j\downarrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{i\uparrow}c_{j\uparrow}^\dagger c_{j\downarrow} - c_{i\downarrow}^\dagger c_{i\uparrow}c_{j\downarrow}^\dagger c_{j\uparrow}
\end{aligned}$$

$$\begin{aligned}
4S_i^zS_j^z &= \left(c_{i\uparrow}^\dagger c_{i\uparrow} - c_{i\downarrow}^\dagger c_{i\downarrow}\right)\left(c_{j\uparrow}^\dagger c_{j\uparrow} - c_{j\downarrow}^\dagger c_{j\downarrow}\right)\\
&= c_{i\uparrow}^\dagger c_{i\uparrow}c_{j\uparrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{i\downarrow}c_{j\downarrow}^\dagger c_{j\downarrow} - c_{i\uparrow}^\dagger c_{i\uparrow}c_{j\downarrow}^\dagger c_{j\downarrow} - c_{i\downarrow}^\dagger c_{i\downarrow}c_{j\uparrow}^\dagger c_{j\uparrow}
\end{aligned}$$

$$\begin{aligned}
4\left(S_i^xS_j^x + S_i^yS_j^y\right) &= 2\left(c_{i\uparrow}^\dagger c_{i\downarrow}c_{j\downarrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{i\uparrow}c_{j\uparrow}^\dagger c_{j\downarrow}\right)\\
4S_i^zS_j^z - n_in_j &= -2\left(c_{i\uparrow}^\dagger c_{i\uparrow}c_{j\downarrow}^\dagger c_{j\downarrow} + c_{i\downarrow}^\dagger c_{i\downarrow}c_{j\uparrow}^\dagger c_{j\uparrow}\right)\\
4\mathbf{S}_i\cdot\mathbf{S}_j - n_in_j &= -s_{ij}^\dagger s_{ij}
\end{aligned}$$

$$\begin{aligned}
c_{i\uparrow}^\dagger c_{i\downarrow}c_{j\uparrow}^\dagger c_{j\downarrow} + c_{i\downarrow}^\dagger c_{i\uparrow}c_{j\downarrow}^\dagger c_{j\uparrow} &= c_{i\uparrow}^\dagger c_{j\uparrow}^\dagger c_{j\downarrow}c_{i\downarrow} + c_{i\downarrow}^\dagger c_{j\downarrow}^\dagger c_{j\uparrow}c_{i\uparrow}\\
&= \frac{1}{2}\left(t_{ij}^{x\dagger} - it_{ij}^{y\dagger}\right)\left(-t_{ij}^x + it_{ij}^y\right) + \frac{1}{2}\left(-t_{ij}^{x\dagger} - it_{ij}^{y\dagger}\right)\left(t_{ij}^x + it_{ij}^y\right)\\
&= -t_{ij}^{x\dagger}t_{ij}^x + t_{ij}^{y\dagger}t_{ij}^y
\end{aligned}$$

$$\begin{aligned}
c_{i\uparrow}^\dagger c_{i\downarrow}c_{j\downarrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{i\uparrow}c_{j\uparrow}^\dagger c_{j\downarrow} &= c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger c_{j\uparrow}c_{i\downarrow} + c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger c_{j\downarrow}c_{i\uparrow}\\
&= \frac{1}{2}\left(s_{ij}^\dagger - t_{ij}^{z\dagger}\right)\left(-s_{ij} - t_{ij}^z\right) + \frac{1}{2}\left(-s_{ij}^\dagger - t_{ij}^{z\dagger}\right)\left(s_{ij} - t_{ij}^z\right)\\
&= -s_{ij}^\dagger s_{ij} + t_{ij}^{z\dagger}t_{ij}^z
\end{aligned}$$

$$
\begin{aligned}
c^\dagger_{i\uparrow}c_{i\uparrow}c^\dagger_{j\uparrow}c_{j\uparrow}+c^\dagger_{i\downarrow}c_{i\downarrow}c^\dagger_{j\downarrow}c_{j\downarrow} &= c^\dagger_{i\uparrow}c^\dagger_{j\uparrow}c_{j\uparrow}c_{i\uparrow}+c^\dagger_{i\downarrow}c^\dagger_{j\downarrow}c_{j\downarrow}c_{i\downarrow} \\
&= \frac{1}{2}\left(t^{x\dagger}_{ij}-it^{y\dagger}_{ij}\right)\left(t^x_{ij}+it^y_{ij}\right)+\frac{1}{2}\left(-t^{x\dagger}_{ij}-it^{y\dagger}_{ij}\right)\left(-t^x_{ij}+it^y_{ij}\right) \\
&= t^{x\dagger}_{ij}t^x_{ij}+t^{y\dagger}_{ij}t^y_{ij}
\end{aligned}
$$

$$
\begin{aligned}
c^\dagger_{i\uparrow}c_{i\uparrow}c^\dagger_{j\downarrow}c_{j\downarrow}+c^\dagger_{i\downarrow}c_{i\downarrow}c^\dagger_{j\uparrow}c_{j\uparrow} &= c^\dagger_{i\uparrow}c^\dagger_{j\downarrow}c_{j\downarrow}c_{i\uparrow}+c^\dagger_{i\downarrow}c^\dagger_{j\uparrow}c_{j\uparrow}c_{i\downarrow} \\
&= \frac{1}{2}\left(s^\dagger_{ij}-t^{z\dagger}_{ij}\right)\left(s_{ij}-t^z_{ij}\right)+\frac{1}{2}\left(-s^\dagger_{ij}-t^{z\dagger}_{ij}\right)\left(-s_{ij}-t^z_{ij}\right) \\
&= s^\dagger_{ij}s_{ij}+t^{z\dagger}_{ij}t^z_{ij}
\end{aligned}
$$

$$
\begin{aligned}
4S^x_iS^x_j &= -s^\dagger_{ij}s_{ij}-t^{x\dagger}_{ij}t^x_{ij}+t^{y\dagger}_{ij}t^y_{ij}+t^{z\dagger}_{ij}t^z_{ij} \\
4S^y_iS^y_j &= -s^\dagger_{ij}s_{ij}+t^{x\dagger}_{ij}t^x_{ij}-t^{y\dagger}_{ij}t^y_{ij}+t^{z\dagger}_{ij}t^z_{ij} \\
4S^z_iS^z_j &= -s^\dagger_{ij}s_{ij}+t^{x\dagger}_{ij}t^x_{ij}+t^{y\dagger}_{ij}t^y_{ij}-t^{z\dagger}_{ij}t^z_{ij}
\end{aligned}
$$

$$
\begin{aligned}
S^x_iS^y_j+S^y_iS^x_j &= \frac{i}{4}\left(c^\dagger_{i\uparrow}c_{i\downarrow}+c^\dagger_{i\downarrow}c_{i\uparrow}\right)\left(-c^\dagger_{j\uparrow}c_{j\downarrow}+c^\dagger_{j\downarrow}c_{j\uparrow}\right)+\frac{i}{4}\left(-c^\dagger_{i\uparrow}c_{i\downarrow}+c^\dagger_{i\downarrow}c_{i\uparrow}\right)\left(c^\dagger_{j\uparrow}c_{j\downarrow}+c^\dagger_{j\downarrow}c_{j\uparrow}\right) \\
&= \frac{i}{2}\left(-c^\dagger_{i\uparrow}c_{i\downarrow}c^\dagger_{j\uparrow}c_{j\downarrow}+\frac{i}{2}c^\dagger_{i\downarrow}c_{i\uparrow}c^\dagger_{j\downarrow}c_{j\uparrow}\right) \\
&= -\frac{i}{4}\left(t^{x\dagger}_{ij}-it^{y\dagger}_{ij}\right)\left(-t^x_{ij}+it^y_{ij}\right)+\frac{i}{4}\left(-t^{x\dagger}_{ij}-it^{y\dagger}_{ij}\right)\left(t^x_{ij}+it^y_{ij}\right) \\
&= \frac{1}{2}\left(t^{x\dagger}_{ij}t^y_{ij}+t^{y\dagger}_{ij}t^x_{ij}\right)
\end{aligned}
$$

$$
S^\alpha_iS^\beta_j+S^\beta_iS^\alpha_j = \frac{1}{2}\left(t^{\alpha\dagger}_{ij}t^\beta_{ij}+t^{\beta\dagger}_{ij}t^\alpha_{ij}\right)
$$

where $\alpha,\beta=x,y,z$