GIANT: Globally Improved Approximate Newton Method for Distributed Optimization

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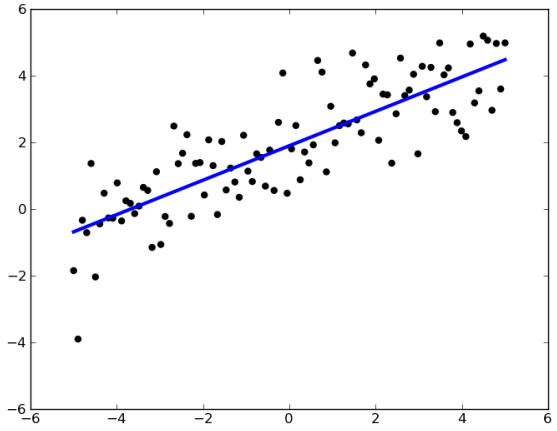
Background & Motivation

Optimization

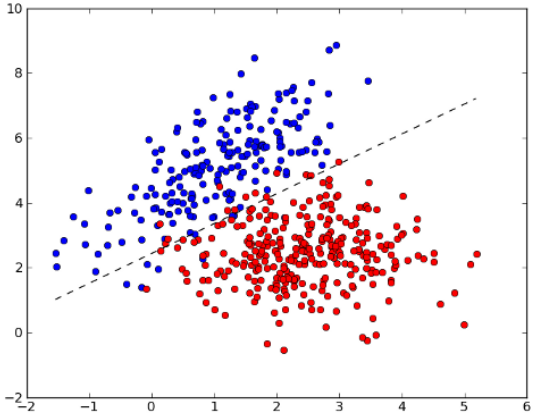
- We consider the *empirical risk minimization* problem:

$$\min_{\mathbf{w}\in\mathbb{R}^d} \left\{ f(\mathbf{w}) \triangleq \frac{1}{n}\sum_{j=1}^{n} l\left(\mathbf{w}\,;\,\mathbf{x}_j, y_j\right) + r(\mathbf{w}) \right\}$$

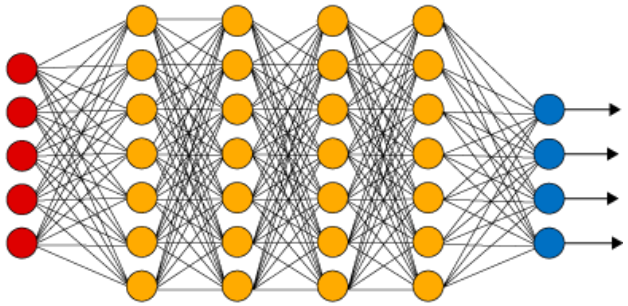
- Examples:



Linear Regression



Linear Classification



Neural Networks

Optimization

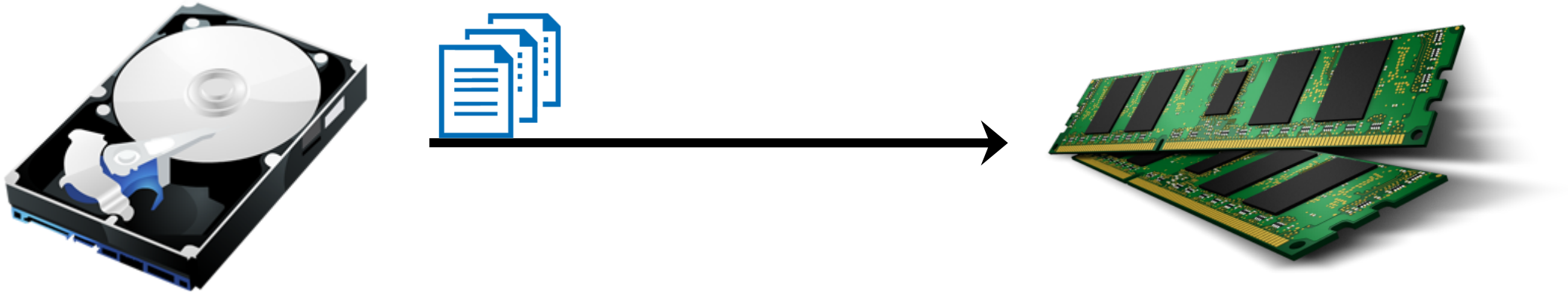
- How to solve the optimization problem $\min_{\mathbf{w}} f(\mathbf{w})$?

  1. Write some code / find a package.

Optimization

- How to solve the optimization problem $\min_{\mathbf{w}} f(\mathbf{w})$?

1. Write some code / find a package.
2. Load data to memory.



Disk

Random-access memory

Optimization

- How to solve the optimization problem $\min_{\mathbf{w}} f(\mathbf{w})$?

1. Write some code / find a package.
2. Load data to memory.
3. Run the code.

Optimization

- How to solve the optimization problem $\min_{\mathbf{w}} f(\mathbf{w})$?

  1. Write some code / find a package.
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- What if the data do not fit in memory?

Optimization

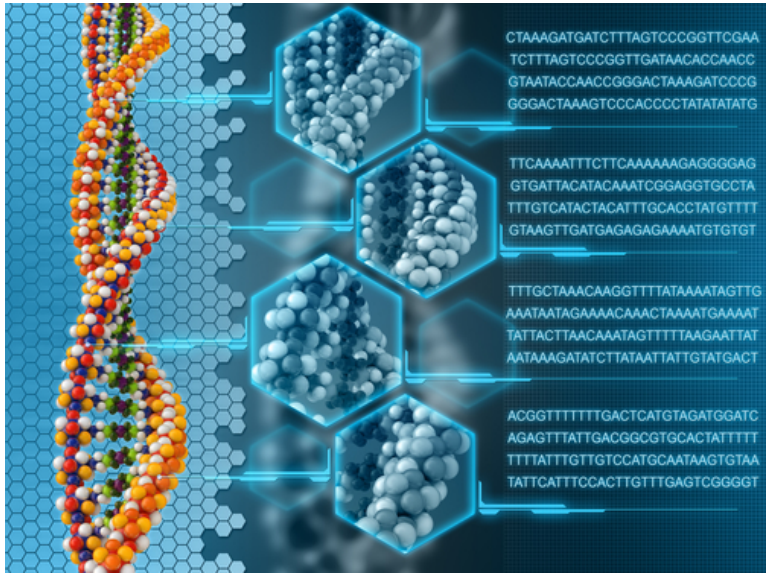
- How to solve the optimization problem $\min_{\mathbf{w}} f(\mathbf{w})$?

  1. Write some code / find a package.
  2. Load data to memory.
  3. Run the code.

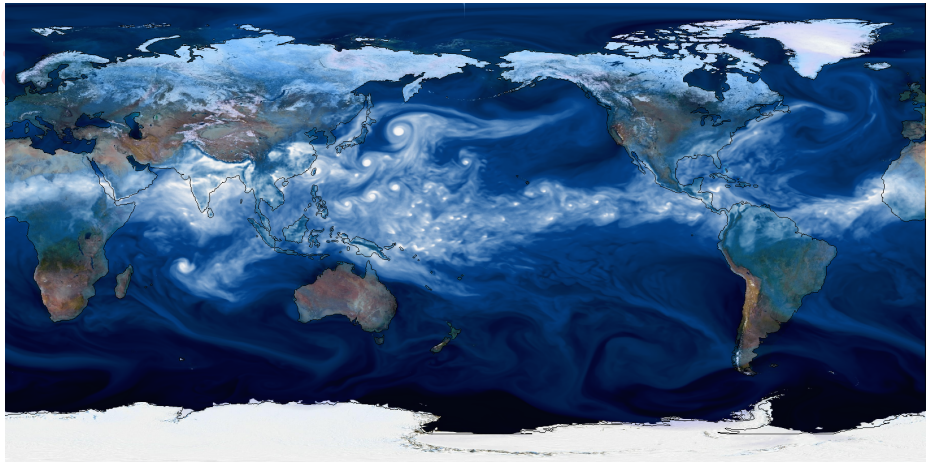
- What if the data do not fit in memory?
- What if the computation is too expensive for a single machine?

Optimization

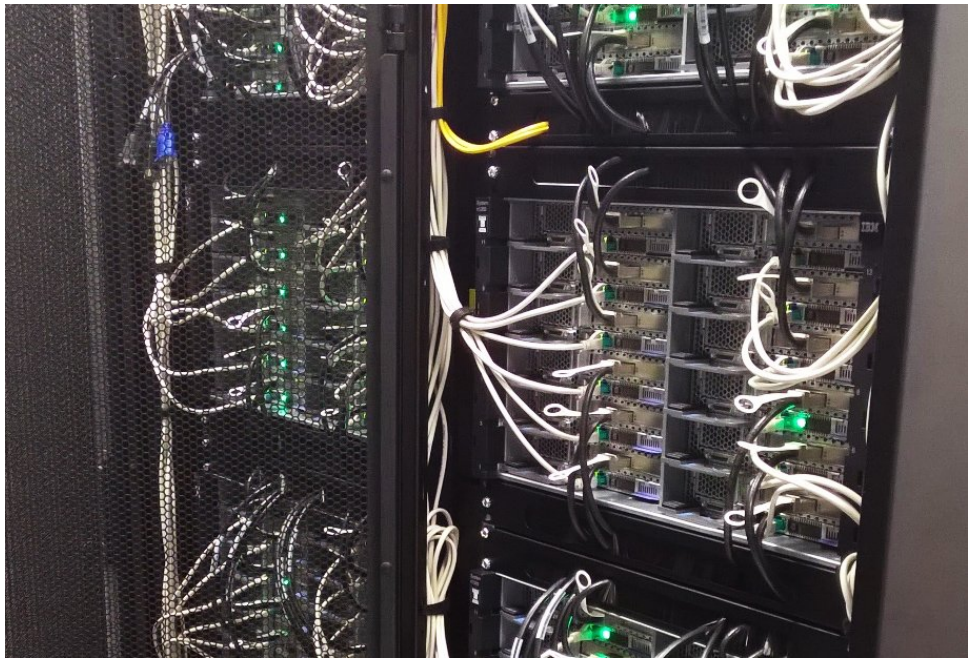
- How to solve the optimization problem $\min f(x)$ w
  1. Use some code / find a package
  2. Load data to memory
  3. Run the code.



- What if the data do not fit in memory?
  1. Store it out



Distributed Optimization

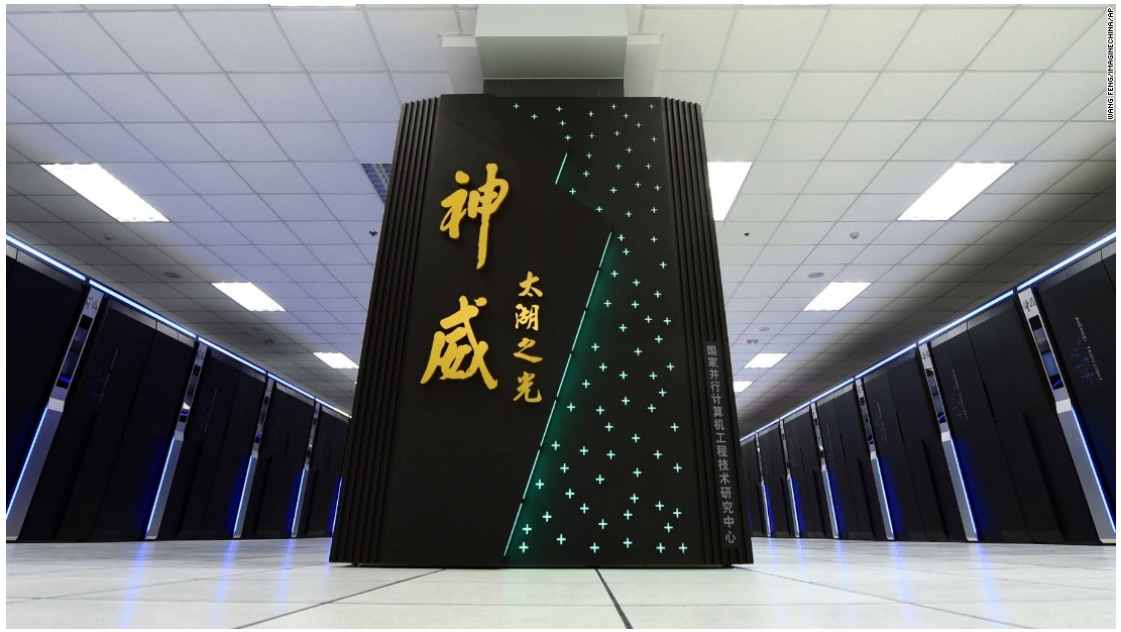


Computer clusters

Distributed Optimization

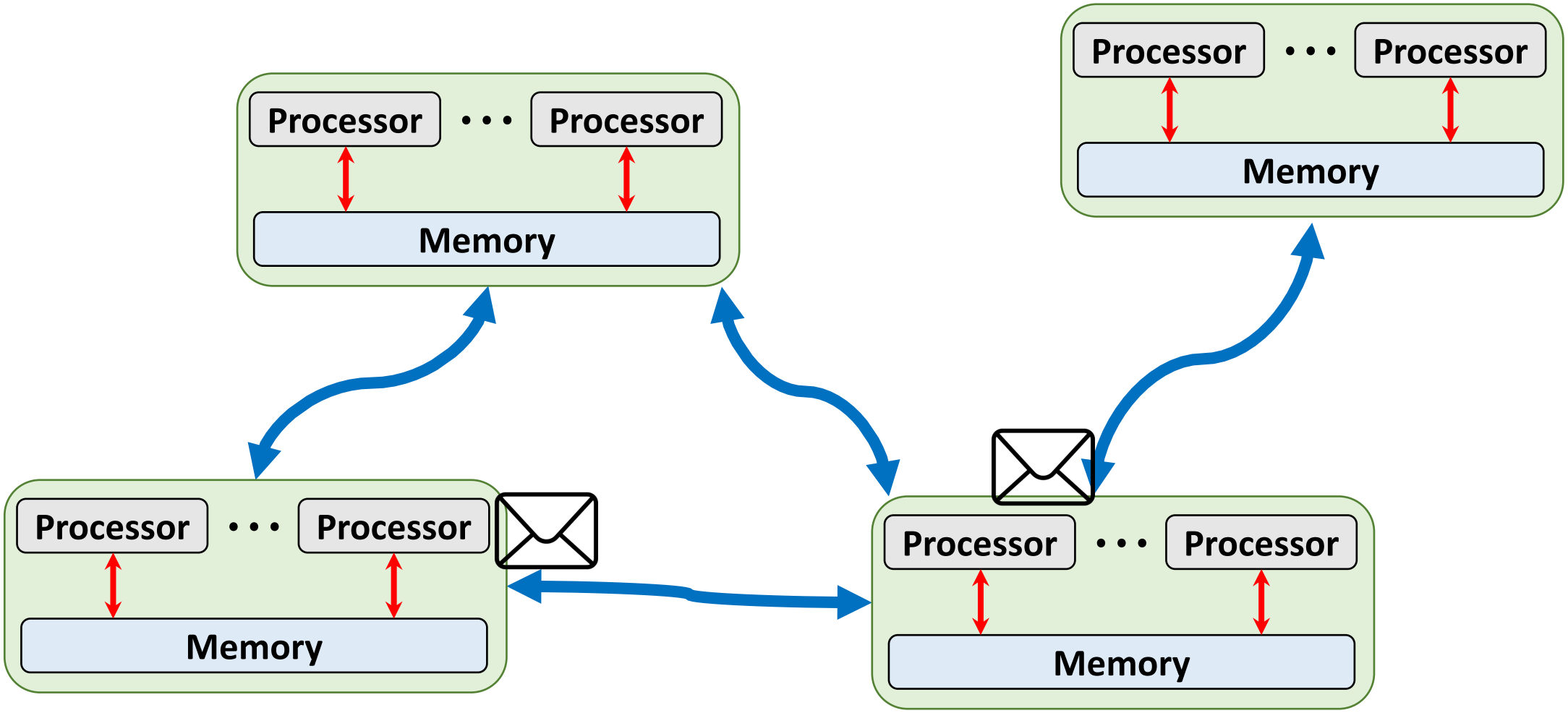
AMAZON EC2

amazon web services

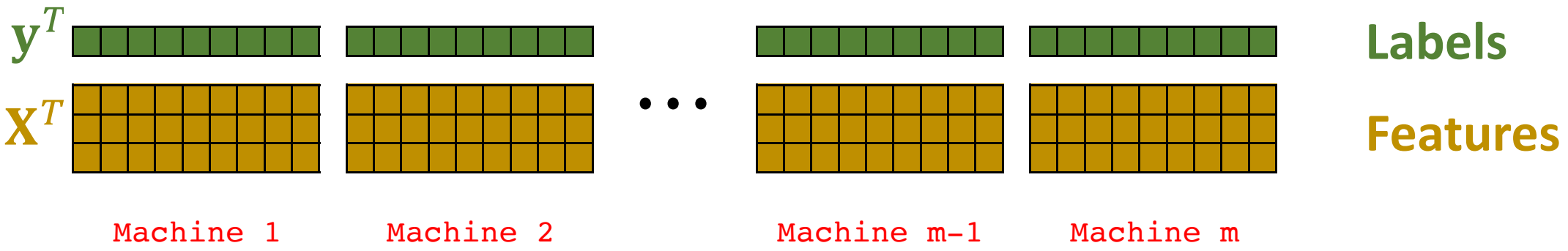


Supercomputer

Distributed Optimization



Distributed Optimization



- $(\mathbf{x}_1, y_1), \cdots, (\mathbf{x}_n, y_n)$ are split among m machines.

Distributed Optimization

Ideally,

- $\frac{1}{m}$ of the data fit in the memory of one machine;
- each machine does $\frac{1}{m}$ of the computation ➡ mx Speedup .

Distributed Optimization

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- $\frac{1}{m}$ of the data fit in the memory of one machine;
- each machine does $\frac{1}{m}$ of the computation ➡ ~~mx Speedup~~ .

Do not overlook the communication!

Distributed Optimization: Example

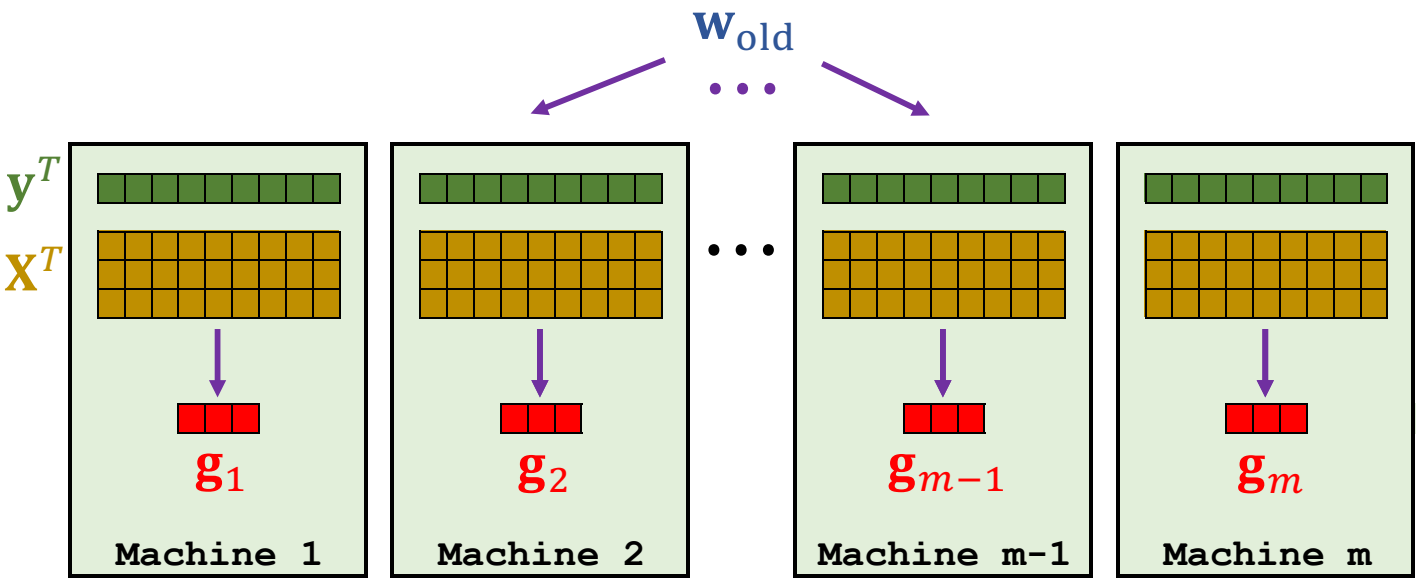
Solve the problem:

$$\min_{\mathbf{w}\in\mathbb{R}^d} \left\{ f(\mathbf{w}) \triangleq \frac{1}{n}\sum_{j=1}^{n} l\left(\mathbf{w}\,;\,\mathbf{x}_j, y_j\right) + r(\mathbf{w}) \right\}$$

Accelerated Gradient Descent (AGD) repeats:

1. Compute gradient: $g = \nabla f(\mathbf{w}_{\text{old}})$;
2. Update momentum: $\mathbf{z}_{\text{new}} = \beta \mathbf{z}_{\text{old}} + g, \quad 0 \le \beta < 1$;
3. Update model: $\mathbf{w}_{\text{new}} = \mathbf{w}_{\text{old}} - \alpha \mathbf{z}_{\text{new}}$.

Warm-up: Distributed AGD



Full gradient of $f(\cdot)$ at $\mathbf{w}_{\text{old}}$.

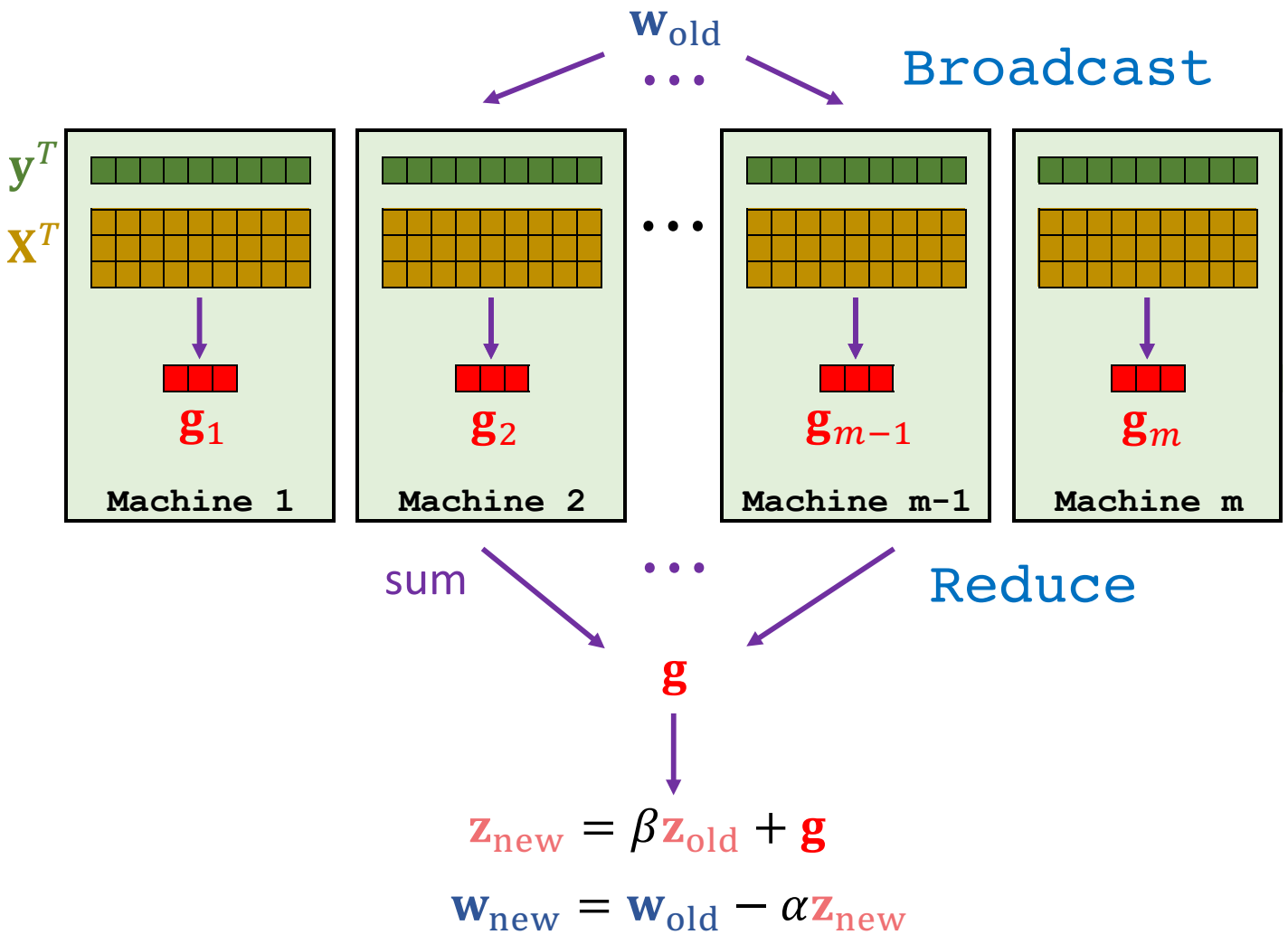
$$\mathbf{z}_{\text{new}} = \beta \mathbf{z}_{\text{old}} + \mathbf{g}$$

Momentum

$$\mathbf{w}_{\text{new}} = \mathbf{w}_{\text{old}} - \alpha \mathbf{z}_{\text{new}}$$

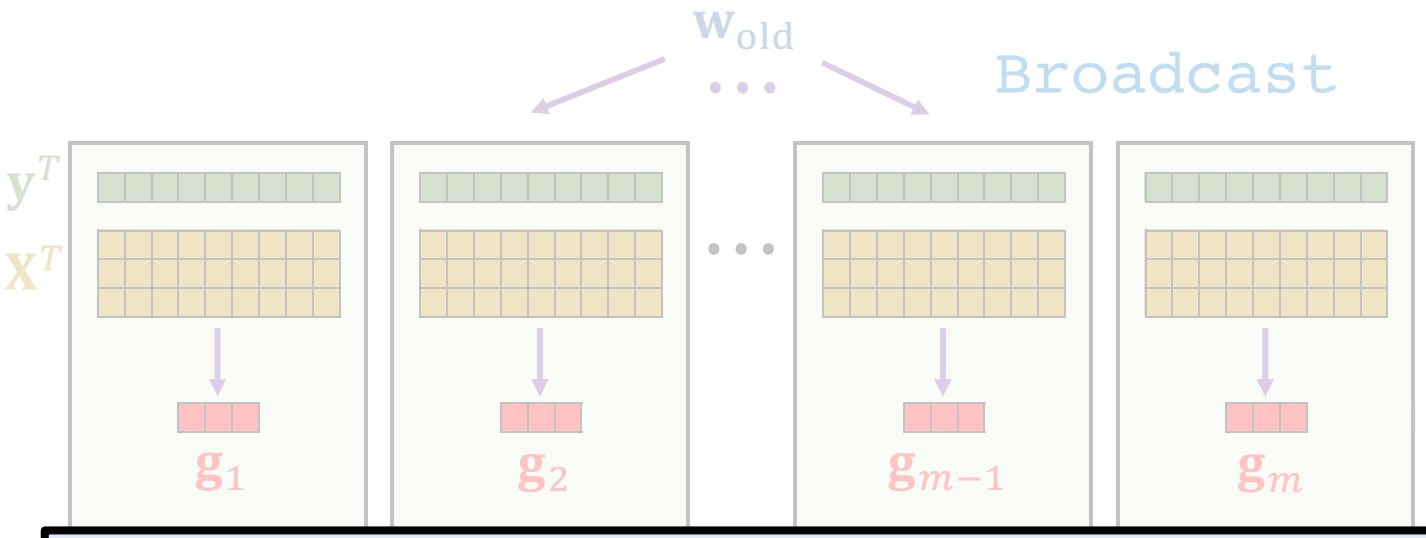
Update

Warm-up: Distributed AGD



- Time complexity: $O\left(\frac{nd}{m}\right)$ FLOPs per iteration.
- One `Broadcast` and one `Reduce` per iteration.
- Lots of iterations to converge ➔ lots of communications.

Warm-up: Distributed AGD



Cost = Computation + Communication

$$\mathbf{g}$$

$$\mathbf{z}_{\text{new}} = \beta \mathbf{z}_{\text{old}} + \mathbf{g}$$

$$\mathbf{w}_{\text{new}} = \mathbf{w}_{\text{old}} - \alpha \mathbf{z}_{\text{new}}$$

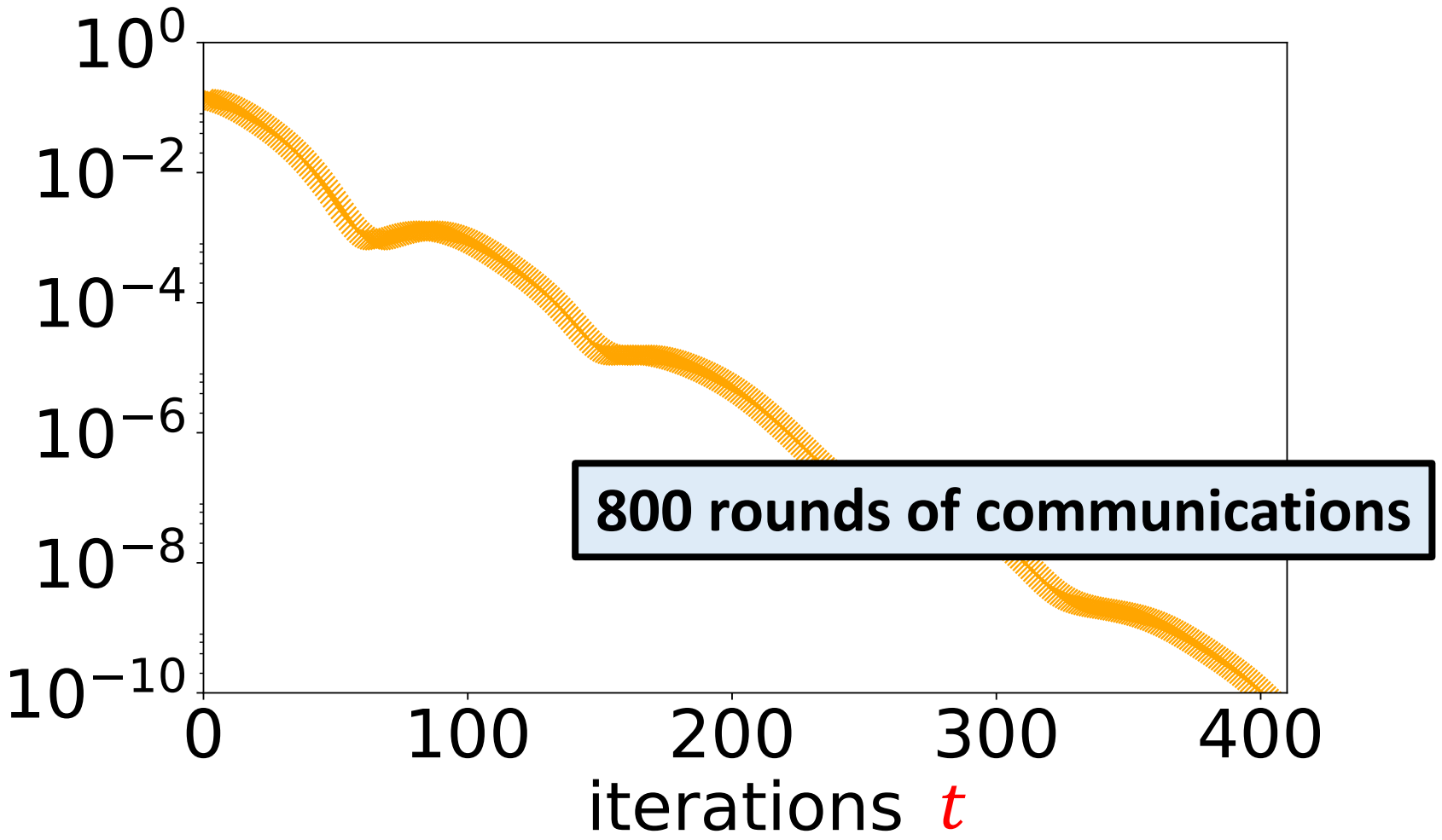
- Time complexity: $O\left(\frac{nd}{m}\right)$ FLOPs per iteration.
- One `Broadcast` and one `Reduce`
- Lots of iterations to converge ➔ lots of communications.

Example: Accelerated GD

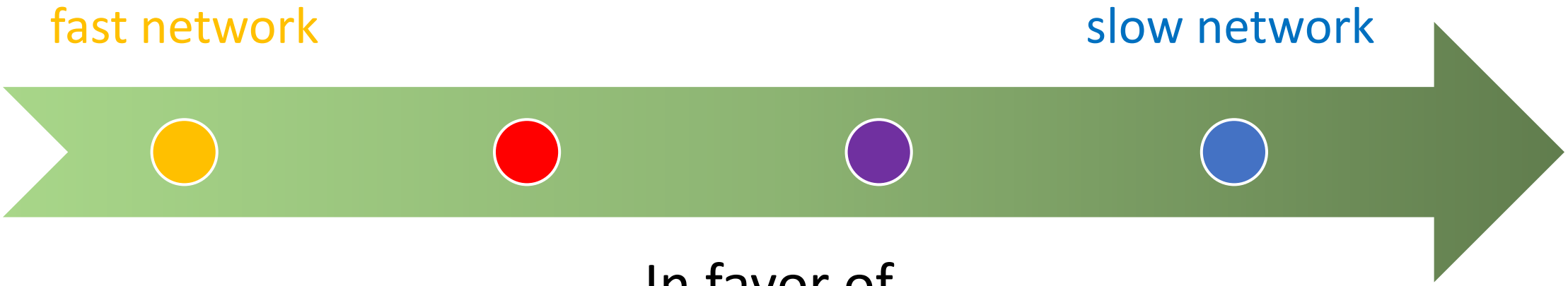
$f(\mathbf{w}_t) - f(\mathbf{w}^\star)$

iteration

optimal solution



FLOPs versus Communications



In favor of

- SGD
- SCD
- Mini-Batch SGD
- Block SCD
- Full GD
- L-BFGS

Single Machine	Cluster	Federated Learning

FLOPs versus Communications

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• SGD	• Mini-Batch SGD	• Full GD
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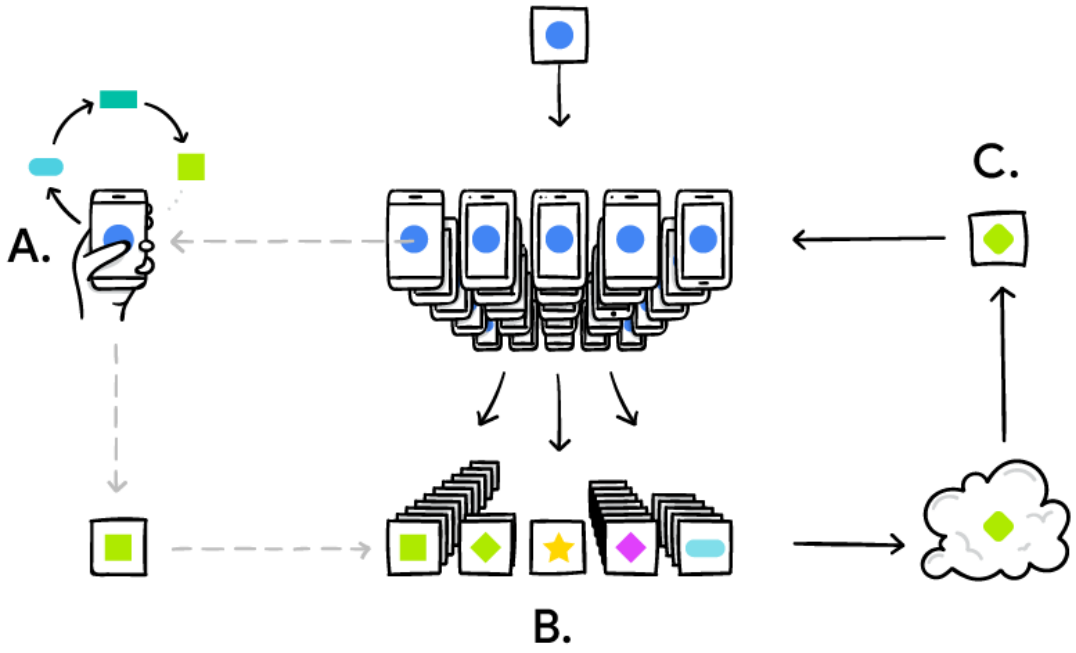
Single Machine	Cluster	Federated Learning

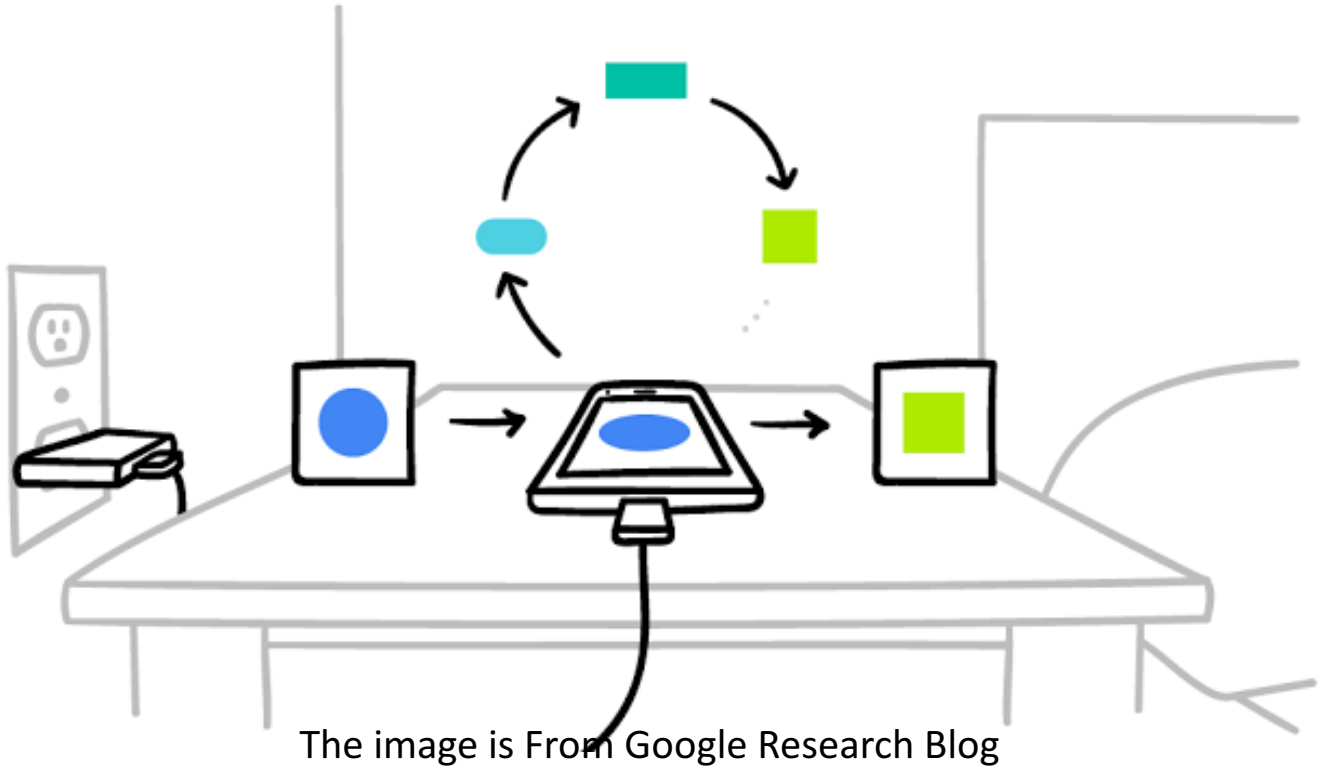
FLOPs versus Communications



The image is From Google Research Blog

Single Machine | Cluster | Federated Learning

FLOPs versus Communications



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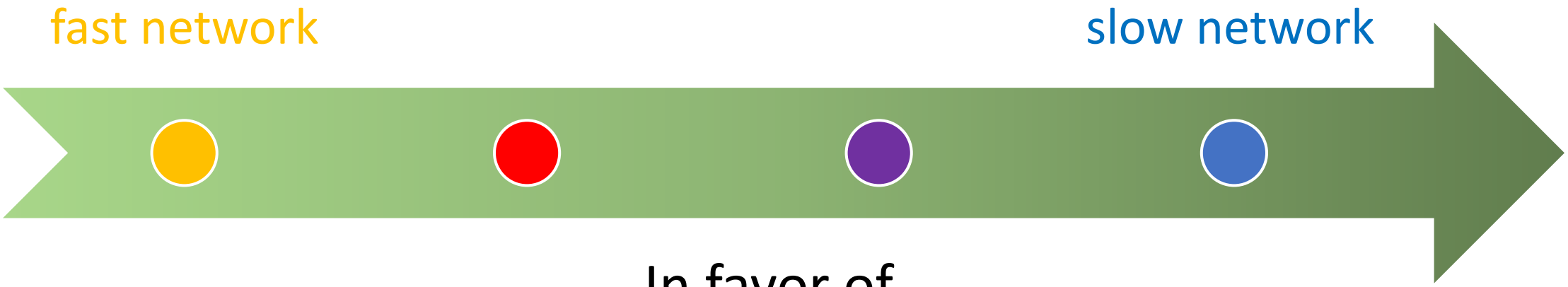
Single Machine | Cluster | Federated Learning

FLOPs versus Communications



In favor of

• SGD • SCD	• Mini-Batch SGD • Block SCD	• Full GD • L-BFGS	?	?

Single Machine	Cluster	Federated Learning

Distributed Optimization

Summary

1. For big-data problems, distributed optimization is very useful.
2. If the network is slow, then communication is the bottleneck.
   - Recall: `Cost = Computation + Communication`

GIANT: Overview

Globally Improved Approximate Newton (GIANT)

- GIANT is a distributed 2nd-order method.
- Each iteration has 4 rounds of communications.
  - `Broadcast` or `Reduce` of one vector.
- Much faster convergence than AGD in terms of communication.
  - Assume the objective function is strongly convex and smooth.

Globally Improved Approximate Newton (GIANT)

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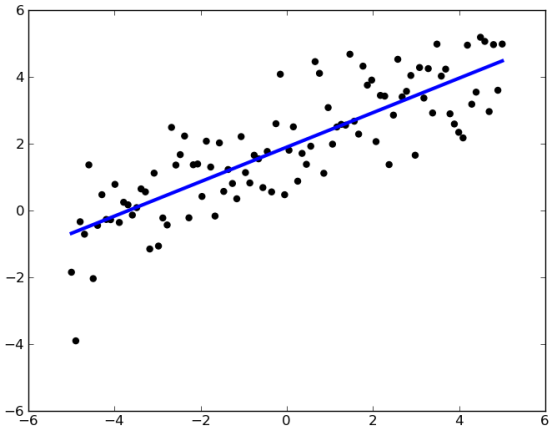
Globally Improved Approximate Newton (GIANT)

- Assume the objective function is strongly convex and smooth.
- **Examples**

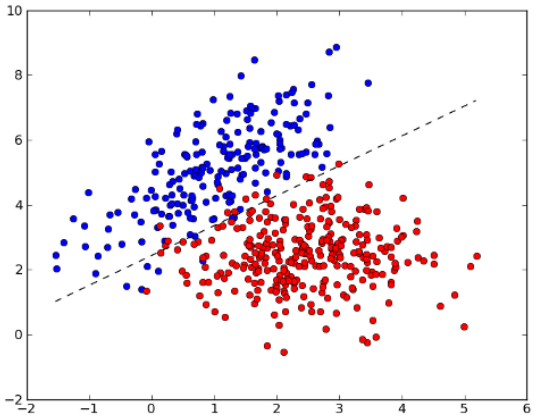
Linear regression

$$f(\mathbf{w}) = \frac{1}{2n}\sum_{j=1}^{n}\left(\mathbf{w}^T\mathbf{x}_j - y_j\right)^2 + \gamma\|\mathbf{w}\|_2^2$$



Logistic regression

$$f(\mathbf{w}) = \frac{1}{2n}\sum_{j=1}^{n}\log\left(1+e^{-y_j\mathbf{w}^T\mathbf{x}_j}\right) + \gamma\|\mathbf{w}\|_2^2$$

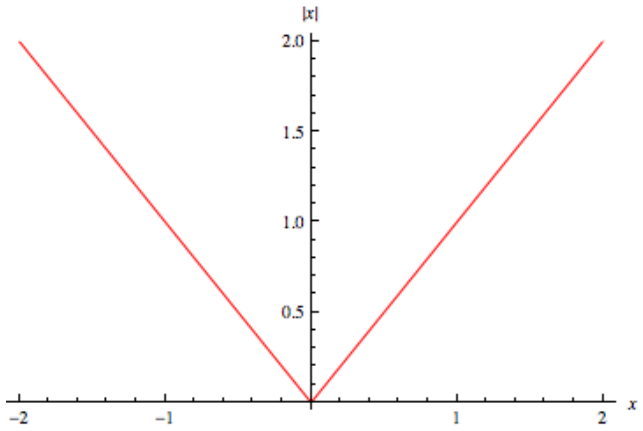


Globally Improved Approximate Newton (GIANT)

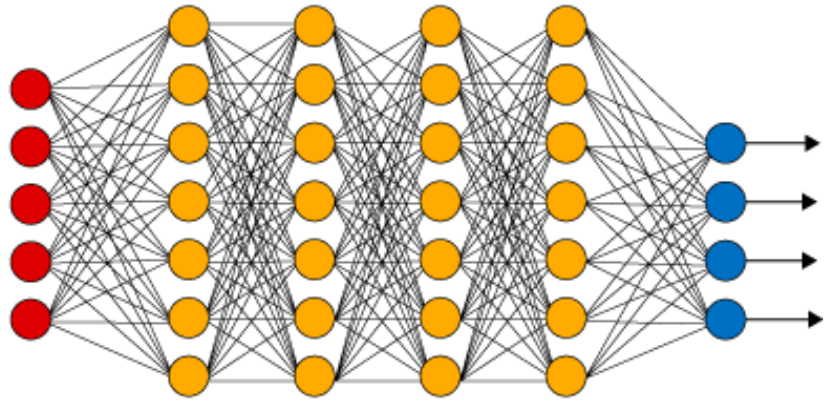
- Assume the objective function is strongly convex and smooth.
- **Counter-examples**

LASSO

$$f(\mathbf{w}) = \frac{1}{2n}\sum_{j=1}^{n}\left(\mathbf{w}^T\mathbf{x}_j - y_j\right)^2 + \gamma\|\mathbf{w}\|_1$$



Neural Networks



Globally Improved Approximate Newton (GIANT)

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The regularization is non-smooth!

Neural Networks

The objective is non-convex!

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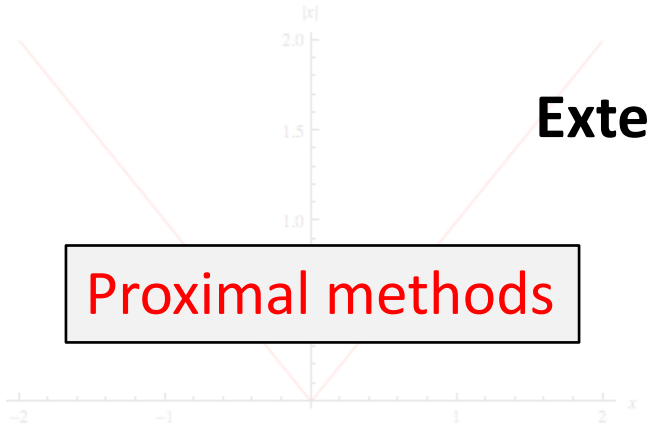
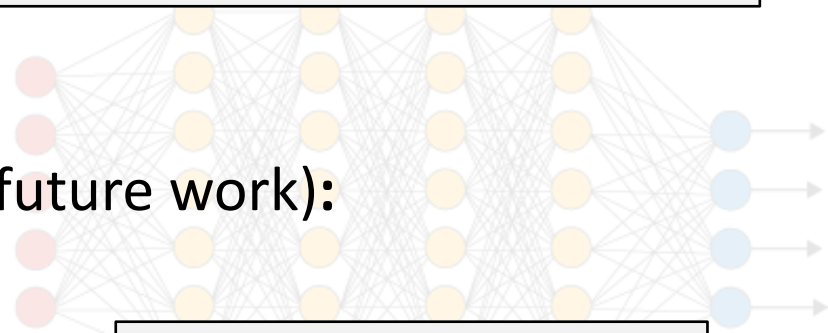


Extensions of GIANT (our future work)**:**

Proximal methods



Trust-region method

Globally Improved Approximate Newton (GIANT)

- Assume the objective function is strongly convex and smooth.

GIANT: Algorithm Description

Warm-up: Newton-CG

- Repeat until convergence
  1. Compute gradient $\mathbf{g}$ and Hessian $\mathbf{H}$;
  2. Solve $\mathbf{Hp} = \mathbf{g}$ by running tens/hundreds of CG steps;
  3. Update $\mathbf{w} \leftarrow \mathbf{w} - \alpha\mathbf{p}$ (find α by line search).

GIANT: Algorithm Derivation

Recall: Newton's direction is $\mathbf{p} = \mathbf{H}^{-1}\mathbf{g}$.

In parallel, form the approximations:

$\widetilde{\mathbf{H}}_1 \approx \mathbf{H}$ $\widetilde{\mathbf{H}}_2 \approx \mathbf{H}$ $\cdots$ $\widetilde{\mathbf{H}}_{m-1} \approx \mathbf{H}$ $\widetilde{\mathbf{H}}_m \approx \mathbf{H}$

In parallel, compute

$\widetilde{\mathbf{p}}_1 = \widetilde{\mathbf{H}}_1^{-1}\mathbf{g}$ $\widetilde{\mathbf{p}}_2 = \widetilde{\mathbf{H}}_2^{-1}\mathbf{g}$ $\cdots$ $\widetilde{\mathbf{p}}_{m-1} = \widetilde{\mathbf{H}}_{m-1}^{-1}\mathbf{g}$ $\widetilde{\mathbf{p}}_m = \widetilde{\mathbf{H}}_m^{-1}\mathbf{g}$

$\widetilde{\mathbf{p}} = \frac{1}{m}\sum_i \widetilde{\mathbf{p}}_i = \left(\frac{1}{m}\sum_i \widetilde{\mathbf{H}}_i^{-1}\right)\mathbf{g}$ approximates $\mathbf{p} = \mathbf{H}^{-1}\mathbf{g}$

GIANT: Algorithm Derivation

Recall: Newton's direction is $\mathbf{p} = \mathbf{H}^{-1}\mathbf{g}$.

$\tilde{\mathbf{p}} = \frac{1}{m}\sum_i \tilde{\mathbf{p}}_i = \left(\frac{1}{m}\sum_i \tilde{\mathbf{H}}_i^{-1}\right)\mathbf{g}$ approximates $\mathbf{p} = \mathbf{H}^{-1}\mathbf{g}$

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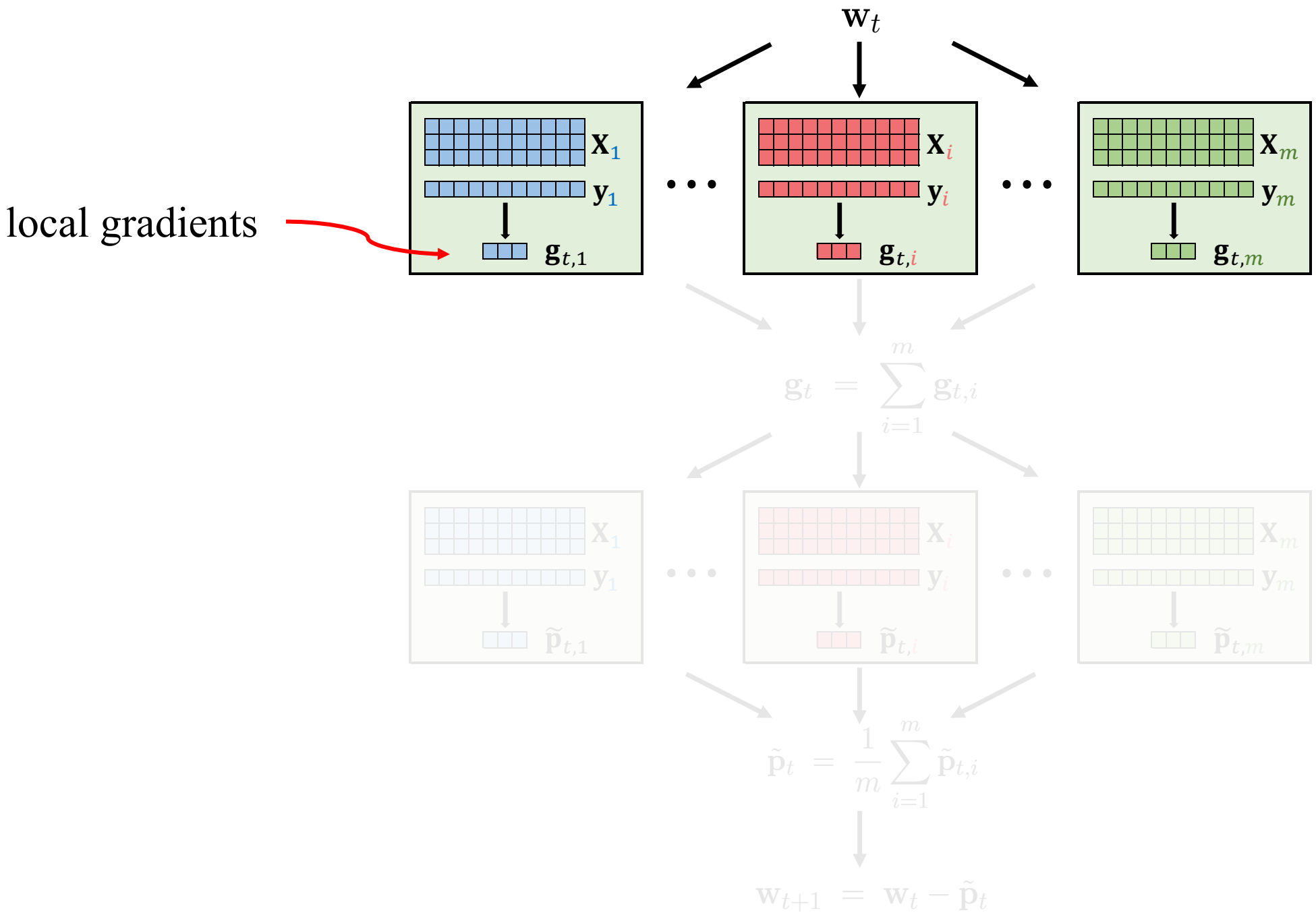
- GIANT uses the exact gradient $\mathbf{g}$.
- GIANT approximates the Hessian matrix $\mathbf{H}$ by $\left(\frac{1}{m}\sum_i \tilde{\mathbf{H}}_i^{-1}\right)^{-1}$.


local gradients
$\mathbf{w}_t$
$\mathbf{x}_1$
$\mathbf{y}_1$
$\mathbf{g}_{t,1}$
$\mathbf{x}_i$
$\mathbf{y}_i$
$\mathbf{g}_{t,i}$
$\mathbf{x}_m$
$\mathbf{y}_m$
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$\mathbf{g}_t = \sum_{i=1}^{m} \mathbf{g}_{t,i}$
$\tilde{\mathbf{p}}_{t,1}$
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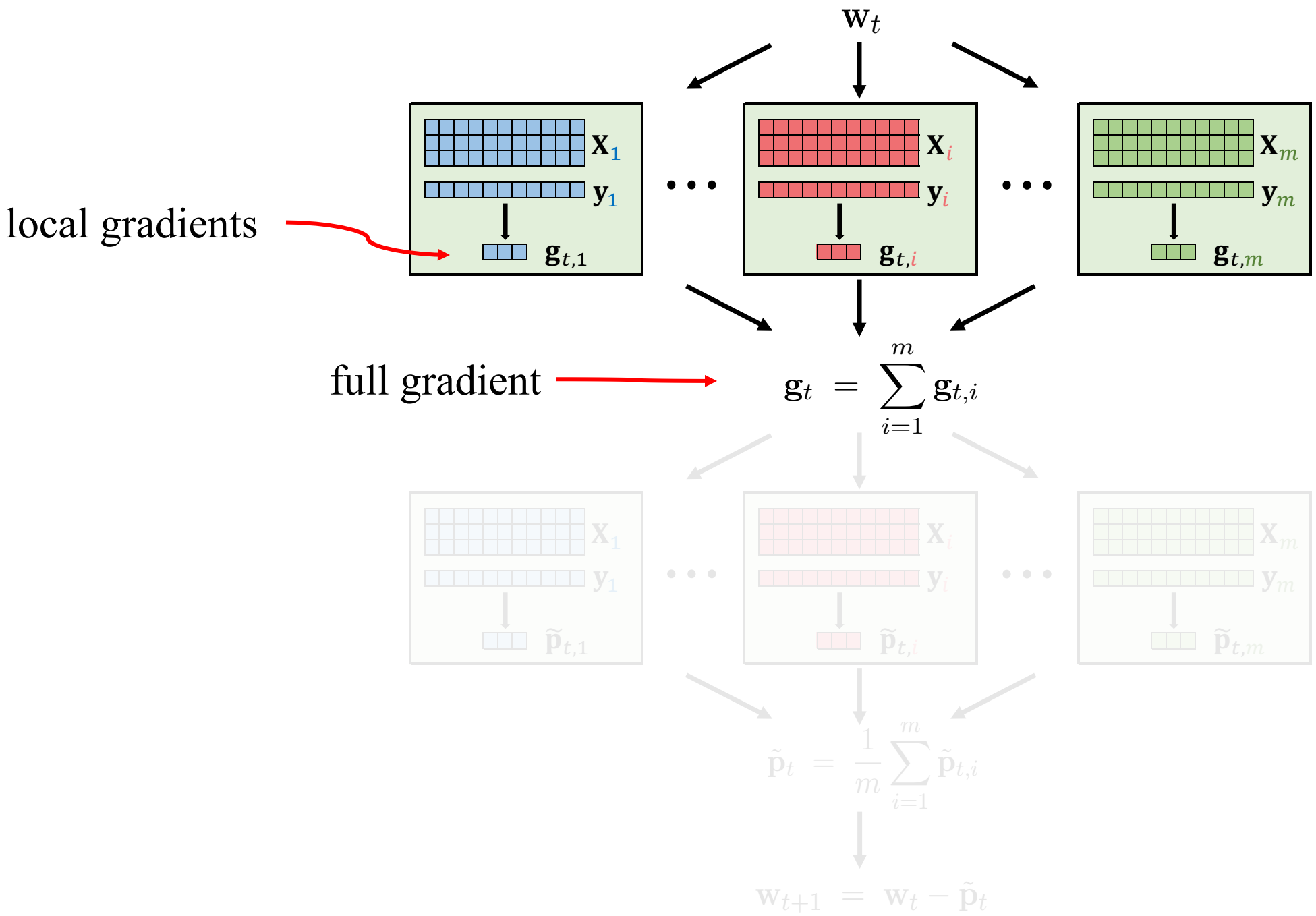
$\mathbf{w}_{t+1} = \mathbf{w}_t - \tilde{\mathbf{p}}_t$


$\mathbf{w}_t$
$\mathbf{X}_1$
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$\mathbf{X}_i$
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local gradients
full gradient
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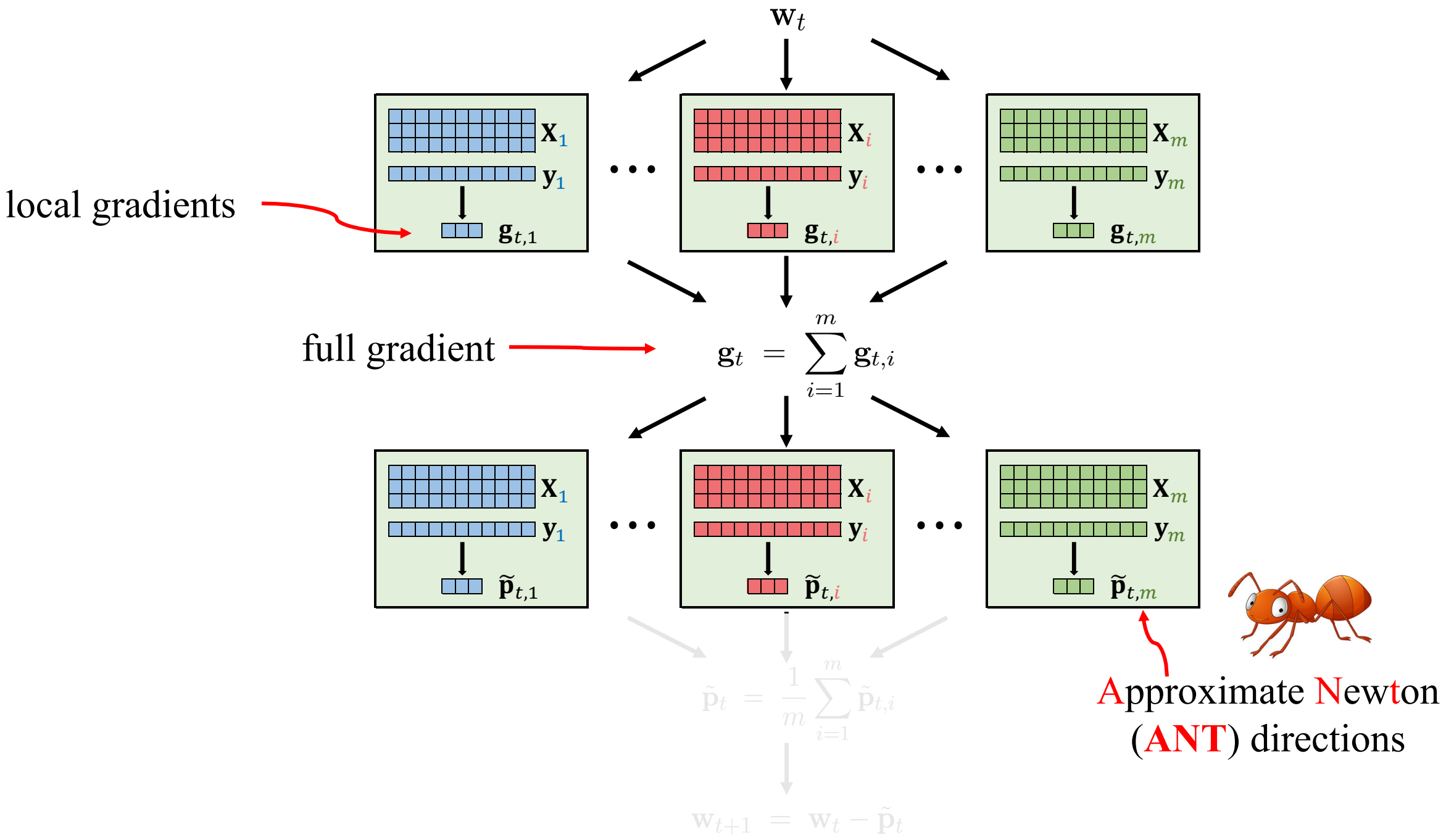
$\mathbf{w}_t$
local gradients
$\mathbf{X}_1$
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$\mathbf{g}_{t,1}$
$\mathbf{X}_i$
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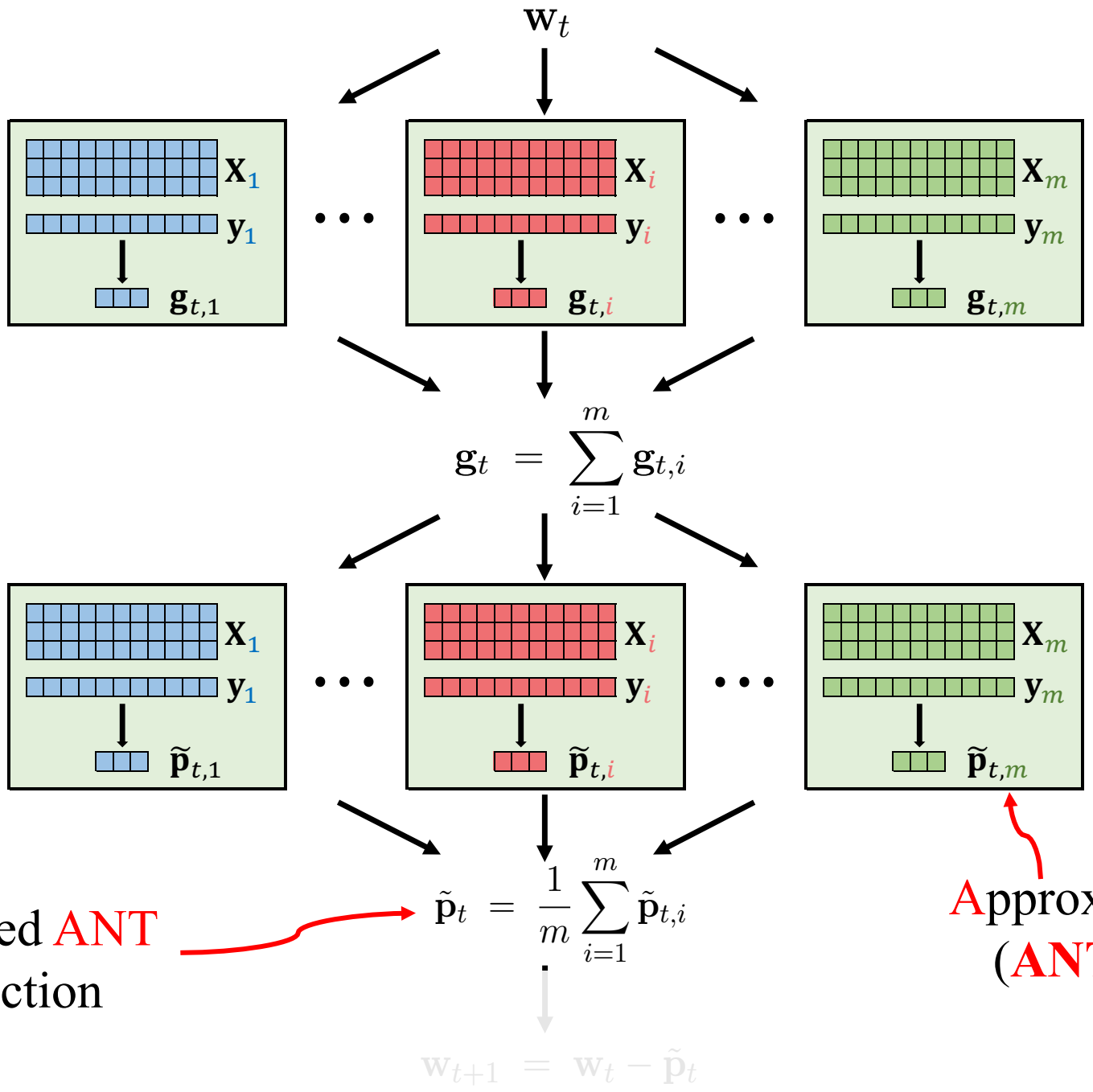
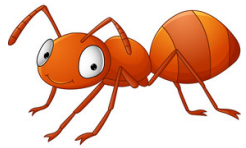
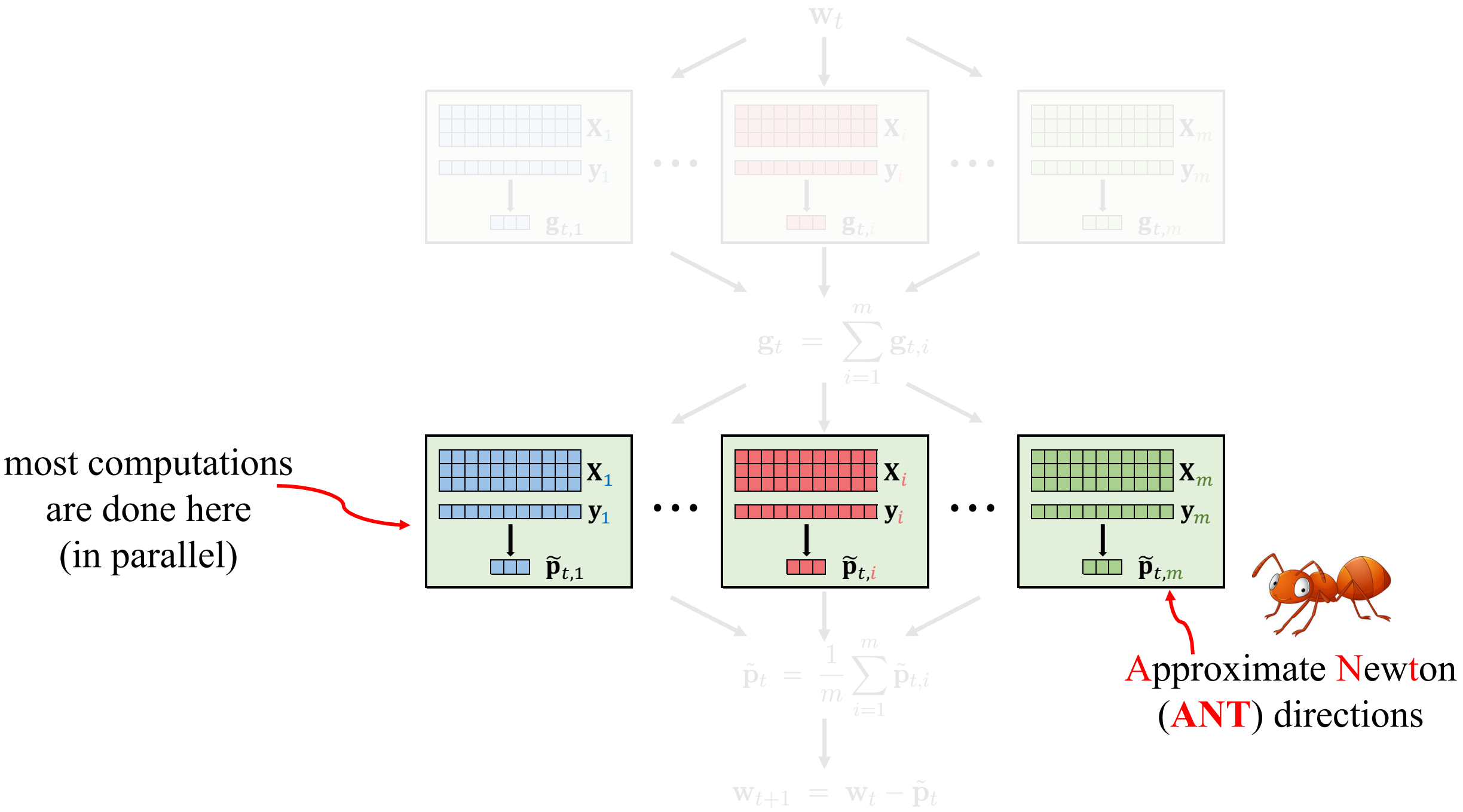
$\mathbf{w}_{t+1} = \mathbf{w}_t - \tilde{\mathbf{p}}_t$
Approximate Newton
(ANT) directions



Globally Improved ANT (**GIANT**) direction



Approximate Newton (**ANT**) directions


most computations
are done here
(in parallel)
$\mathbf{w}_t$
$\mathbf{X}_1$
$\mathbf{y}_1$
$\mathbf{g}_{t,1}$
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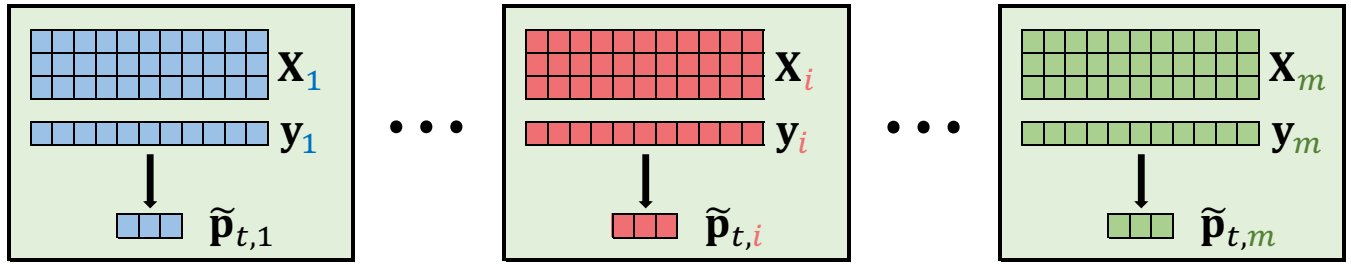
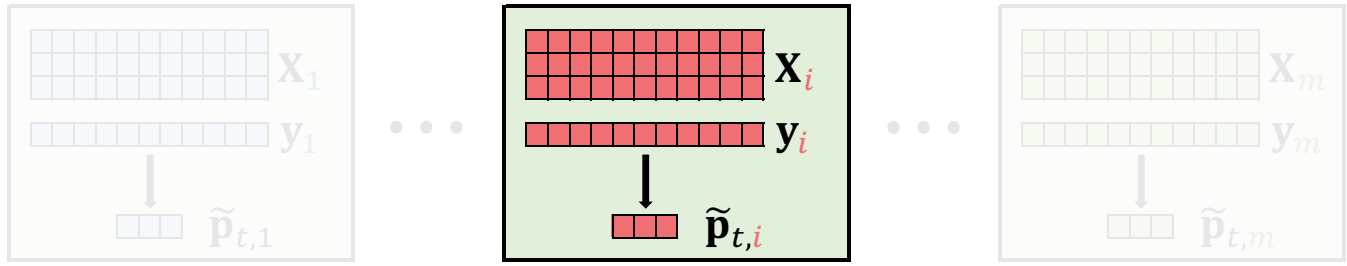
$\mathbf{g}_{t,m}$
$\mathbf{g}_t = \sum_{i=1}^{m} \mathbf{g}_{t,i}$
$\tilde{\mathbf{p}}_{t,1}$
$\tilde{\mathbf{p}}_{t,i}$
$\tilde{\mathbf{p}}_{t,m}$
$\tilde{\mathbf{p}}_t = \frac{1}{m} \sum_{i=1}^{m} \tilde{\mathbf{p}}_{t,i}$
$\mathbf{w}_{t+1} = \mathbf{w}_t - \tilde{\mathbf{p}}_t$
Approximate Newton
(ANT) directions

most computations
are done here
(in parallel)



most computations are done here (in parallel)



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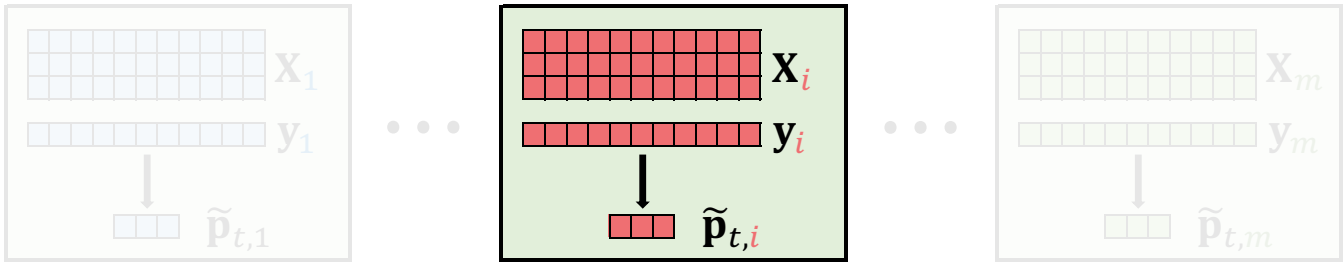


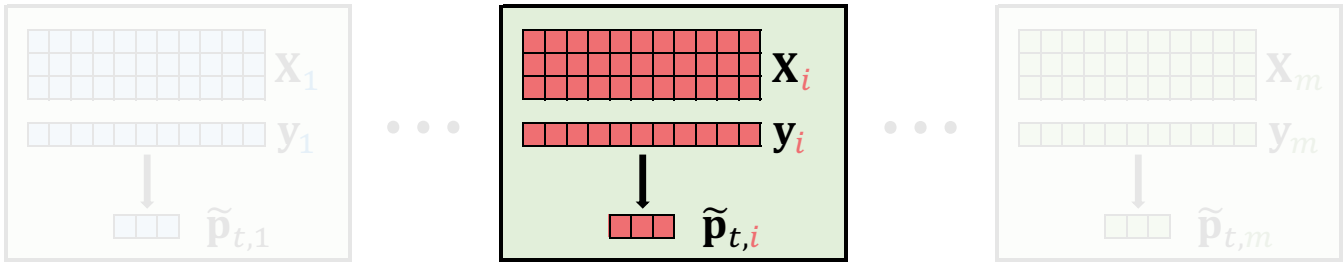
Naïve approach:

1. Form local Hessian $\widetilde{\mathbf{H}}_i \in \mathbb{R}^{d \times d}$
2. Invert $\widetilde{\mathbf{H}}_i$
3. Compute $\tilde{\mathbf{p}}_{t,i} = \widetilde{\mathbf{H}}_i^{-1} \mathbf{g}_t$

It is inefficient!

1. Multiply two matrices to form $\widetilde{\mathbf{H}}_i$
2. Invert the dense matrix $\widetilde{\mathbf{H}}_i$

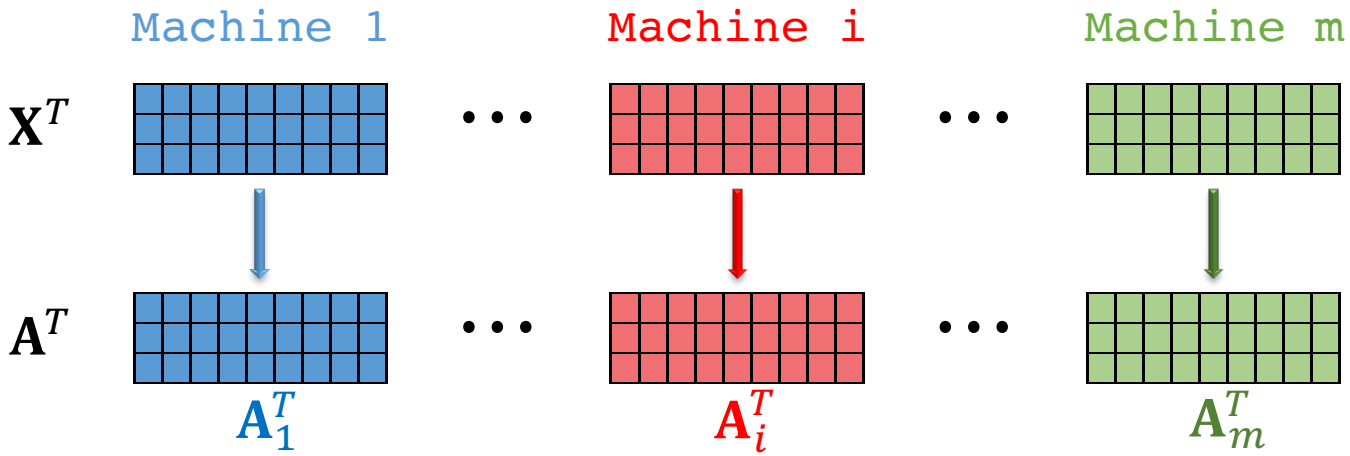
most computations are done here (in parallel)



Fact: For the problem

$$\min_{\mathbf{w}\in\mathbb{R}^d} \left\{ f(\mathbf{w}) \triangleq \frac{1}{n}\sum_{j=1}^{n} l(\mathbf{w}\,;\,\mathbf{x}_j, y_j) + \gamma\|\mathbf{w}\|_2^2 \right\},$$

the local Hessian can be written as $\tilde{\mathbf{H}}_i = \mathbf{A}_i^T\mathbf{A}_i + \gamma\mathbf{I}_d$.



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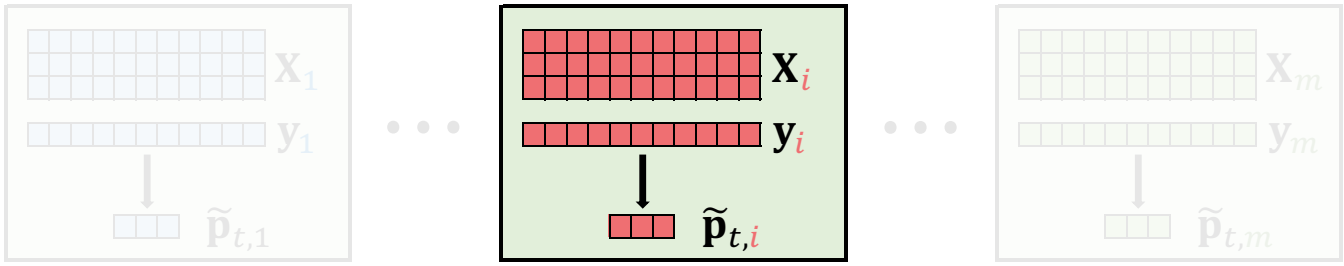

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the local Hessian can be written as $\tilde{\mathbf{H}}_i = \mathbf{A}_i^T\mathbf{A}_i + \gamma\mathbf{I}_d$.

Local solver:

- Inexactly solve $\left(\mathbf{A}_i^T\mathbf{A}_i + \gamma\mathbf{I}_d\right)\mathbf{p} = \mathbf{g}_t$ by taking q CG steps.
- Cost: $2q$ matrix-vector products.

most computations
are done here
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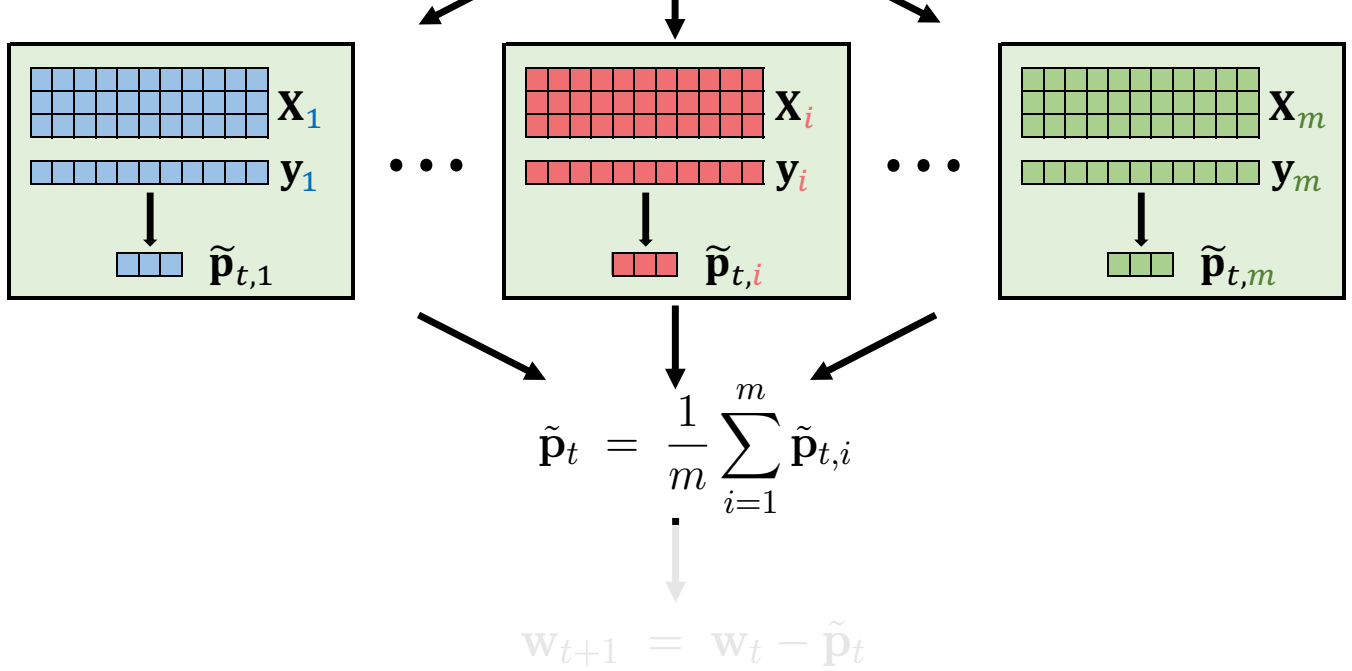
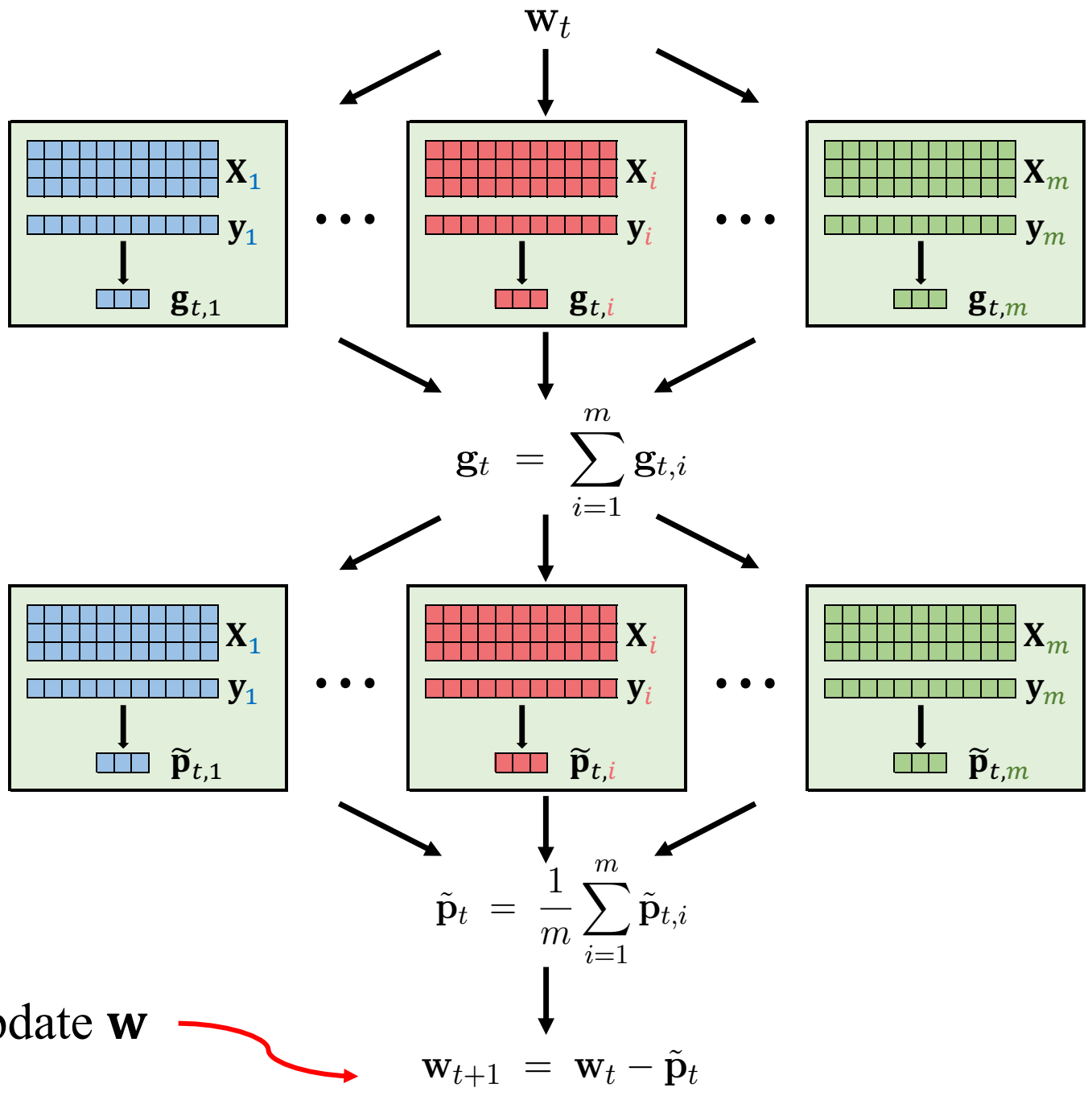




most computations are done here (in parallel)

update **w**

GIANT: Experiments

Experiment Environment

- Spark 2.1.1 + Scala 2.11.8

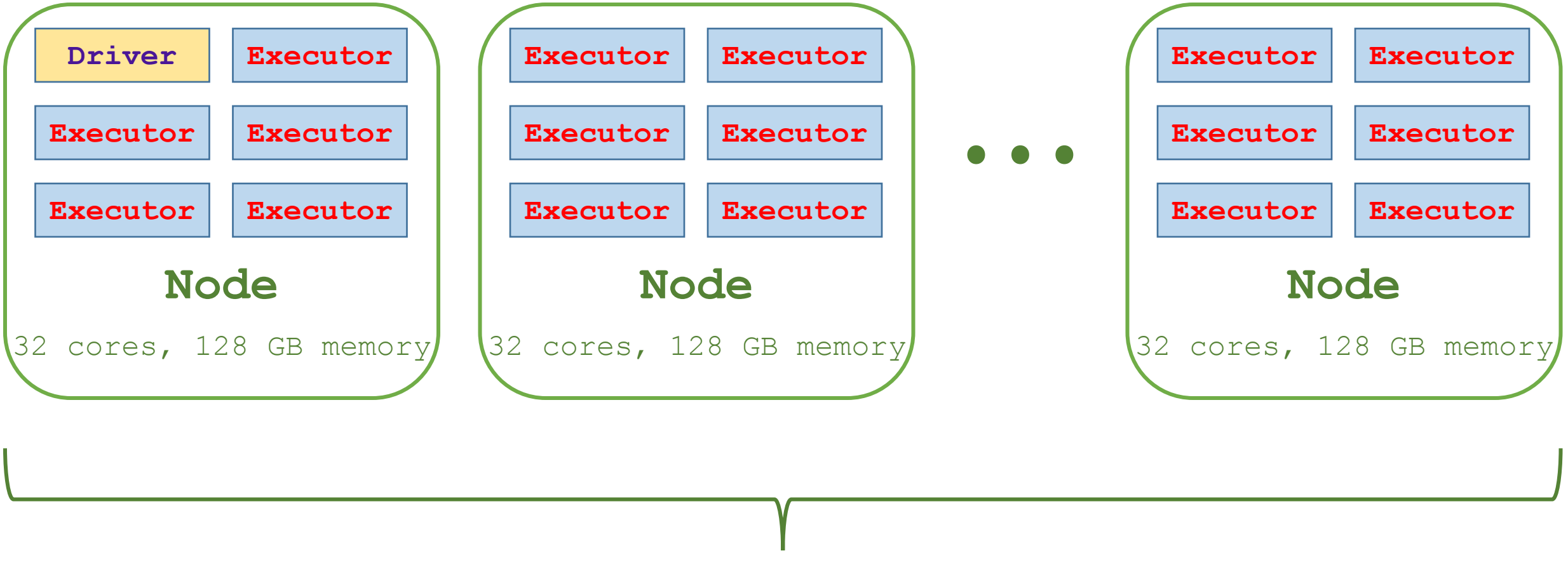
Experiment Environment

- Spark 2.1.1 + Scala 2.11.8

- Cori Supercomputer

National Energy Research
Scientific Computing Center

Experiment Environment



Settings

- Solve the ℓ_2-regularized logistic regression:

$$\min_{\mathbf{w}\in\mathbb{R}^d} \left\{ f(\mathbf{w}) \triangleq \frac{1}{n}\sum_{j=1}^{n} \log\left(1+e^{-y_j \mathbf{x}_j^T \mathbf{w}}\right) + \frac{\gamma}{2}\|\mathbf{w}\|_2^2 \right\}$$

- Split $\mathbf{X} \in \mathbb{R}^{n\times d}$ (by data) to $m = 59$ parts.
- Local sample size $s = \frac{n}{m}$

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- Split $\mathbf{X} \in \mathbb{R}^{n\times d}$ (by data) to $m = 59$ parts.
- Local sample size $s = \frac{n}{m}$ $\gtrsim$ number of features d.
- We use dense $\mathbf{X}$.

Compared Methods

- Accelerated gradient descent (AGD)
  - choose *step size* from {0.1, 1, 10, 100}
  - choose *momentum* from {0.5, 0.9, 0.95, 0.99, 0.999}

Compared Methods

- Accelerated gradient descent (AGD)
- Limited memory BFGS (a quasi-Newton method)
  - choose *number of history* from {30, 100, 300}
  - line search is used

Compared Methods

- Accelerated gradient descent (AGD)
- Limited memory BFGS
- DANE (another Newton-type method) [Shamir et al. 2014]
  - local solver: SVRG
  - choose *step size of SVRG* from {0.1, 1, 10, 100}
  - choose *max iteration of SVRG* from {30, 100, 300}

Reference:

Shamir, Srebro, & Zhang. Communication Efficient Distributed Optimization using an Approximate Newton-type Method. In *ICML*, 2014.

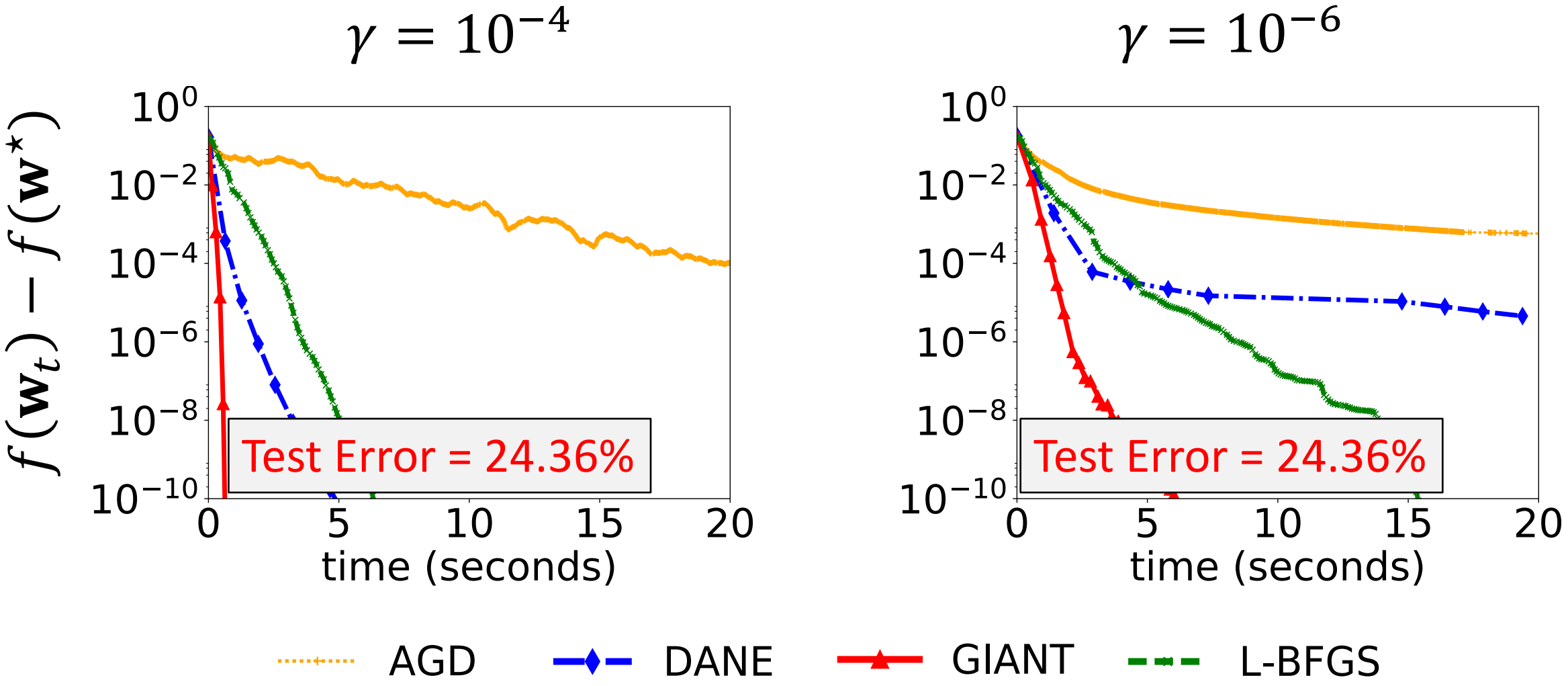
Compared Methods

- Accelerated gradient descent (AGD)
- Limited memory BFGS
- DANE (another Newton-type method)
- GIANT
  - local solver: conjugate gradient (CG)
  - choose *max iteration of CG* from {30, 100, 300}

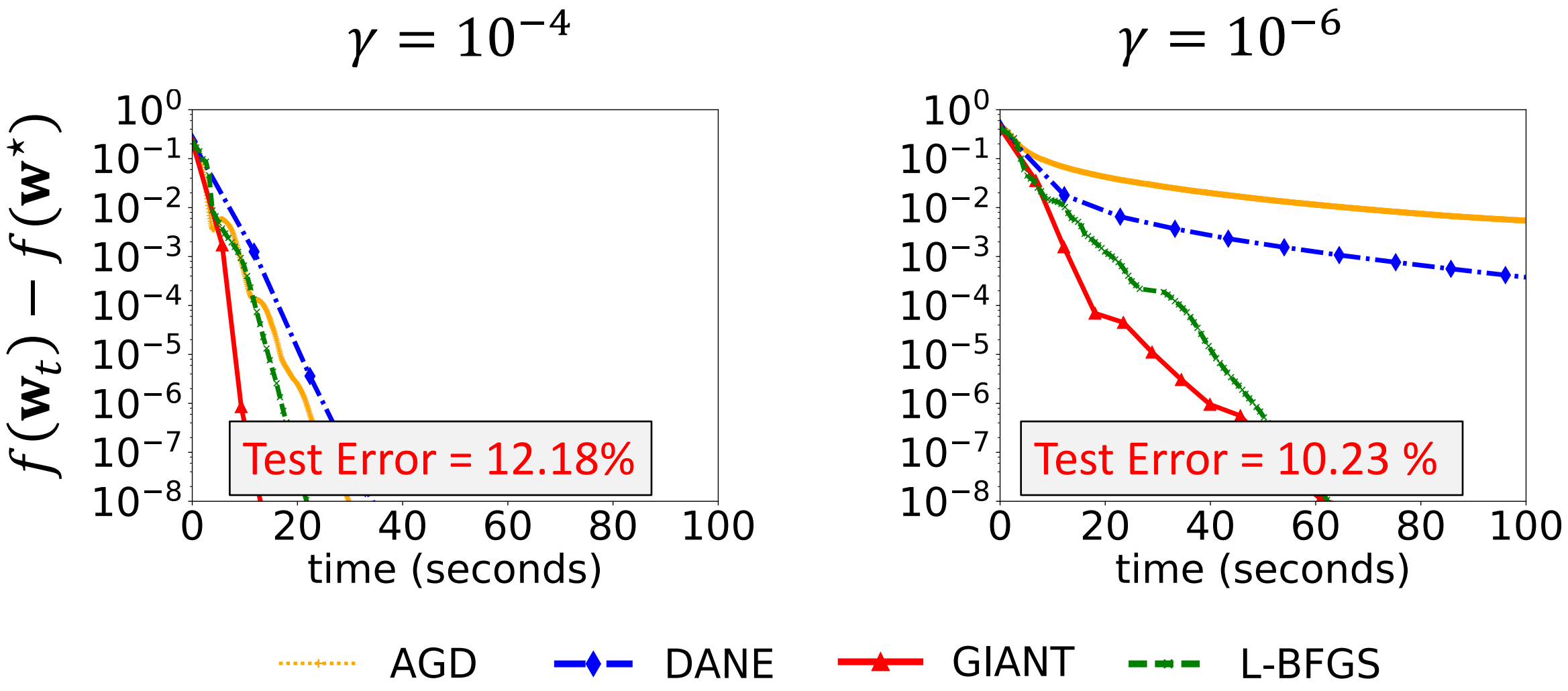
Compared Methods

- Accelerated gradient descent (AGD) — 2 Tuning Parameters
- Limited memory BFGS — 1 Tuning Parameter
- DANE (another Newton-type method) — 2 Tuning Parameters
- GIANT — 1 Tuning Parameter

Covtype (n=581K, d=54)



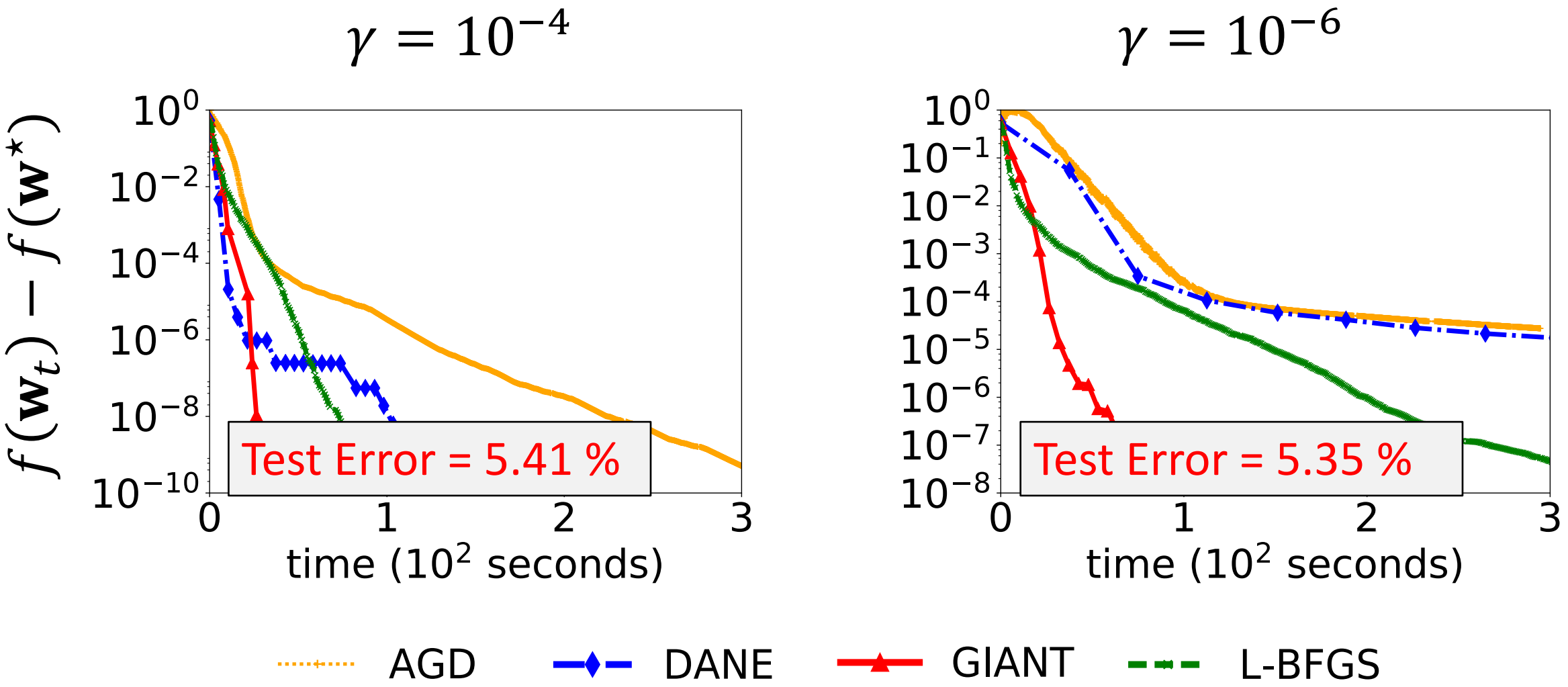
Epsilon (n=400K, d=2K)



MNIST8M (n=1.6M, d=784)

- Digits “4” versus “9”: 1.6M samples out of the total 8M samples

MNIST8M (n=1.6M, d=784)



How about Larger d?

- Split $\mathbf{X} \in \mathbb{R}^{n \times d}$ (by data) to $m = 59$ parts.
- Previously, local sample size $s = \frac{n}{m} \gg$ number of features d.
- Does GIANT work if $s \approx d$?

Random Feature Maps (RFM)

- Generate 10K *random Fourier features* *[Rahimi & Recht, 07]* of the *RBF kernel*

$$k(\mathbf{x}_i, \mathbf{x}_j) = \exp\left(-\frac{1}{2\sigma}\|\mathbf{x}_i - \mathbf{x}_j\|_2^2\right)$$

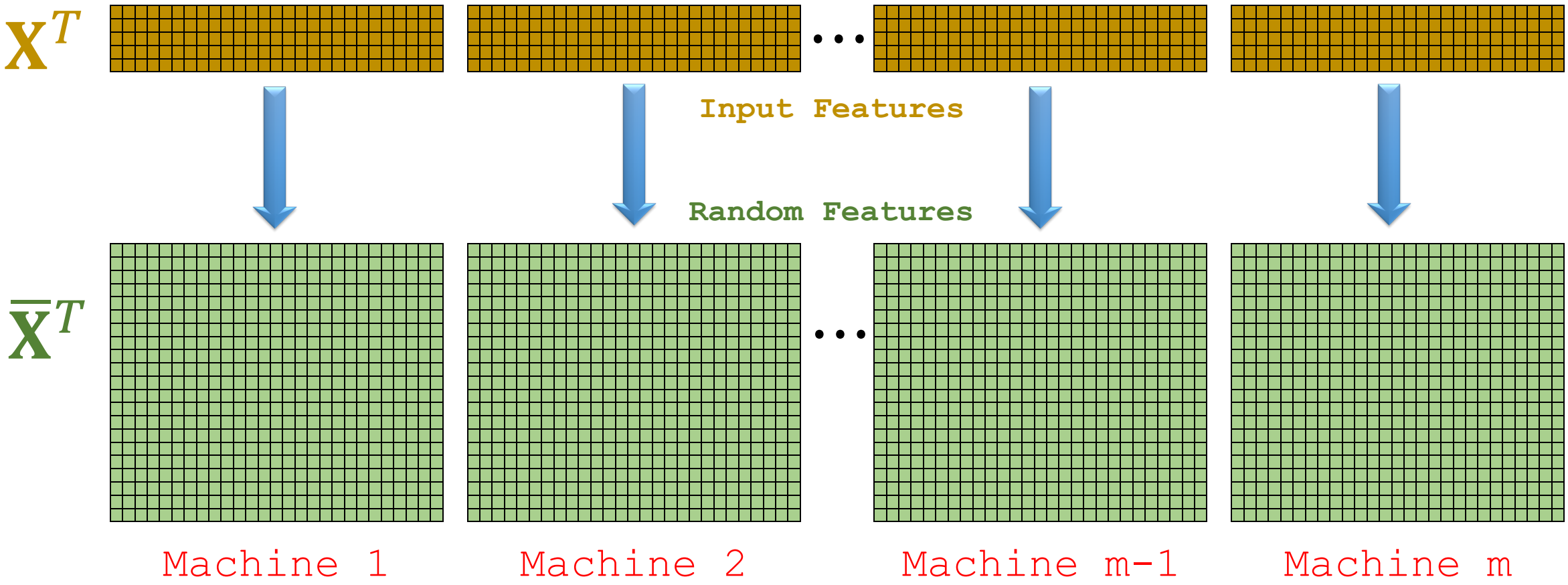
- Setting of RBF *kernel width parameter* σ:

$$\sigma = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\|\mathbf{x}_i - \mathbf{x}_j\|_2^2$$

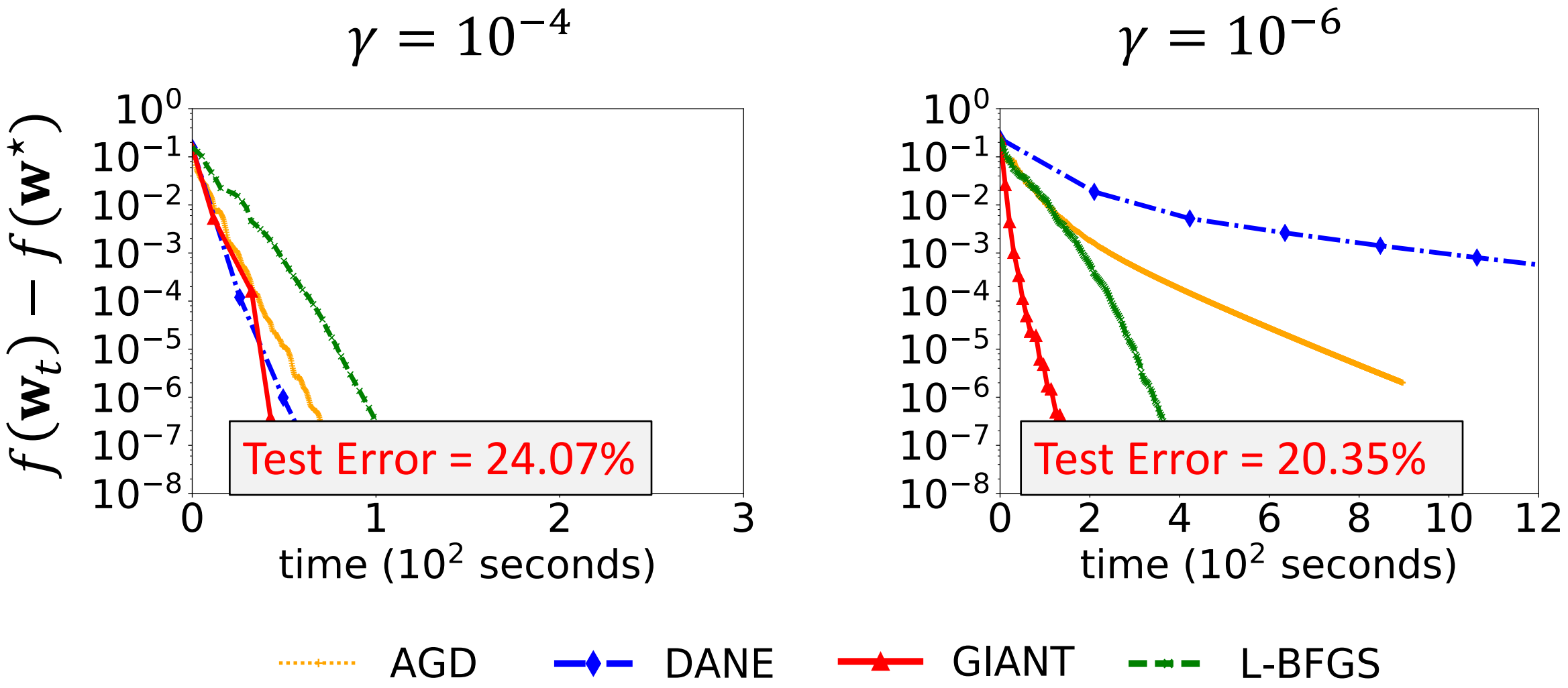
- Replace the original features by the higher-dim random features.

Reference:
Rahimi & Recht. Random Features for Large-Scale Kernel Machines. In *NIPS,* 2007.

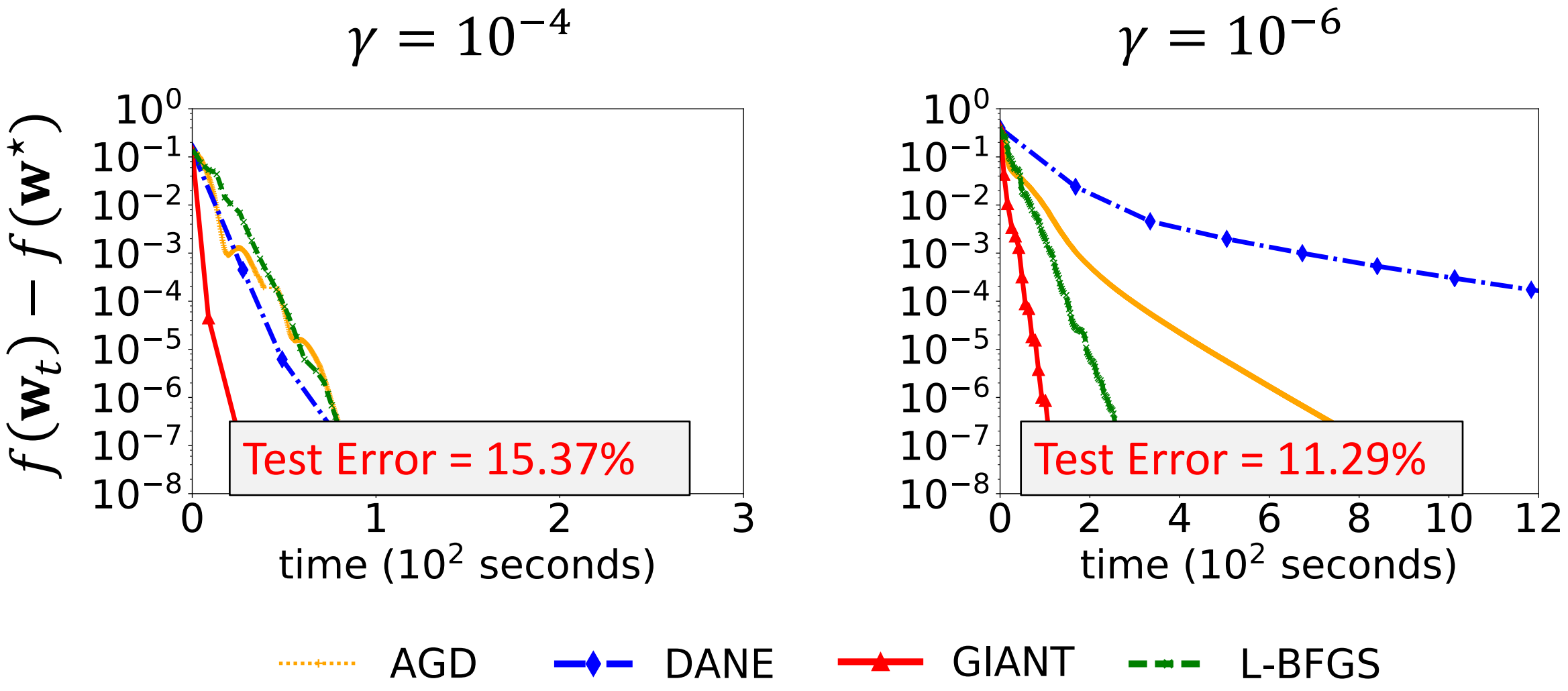
Random Feature Maps (RFM)



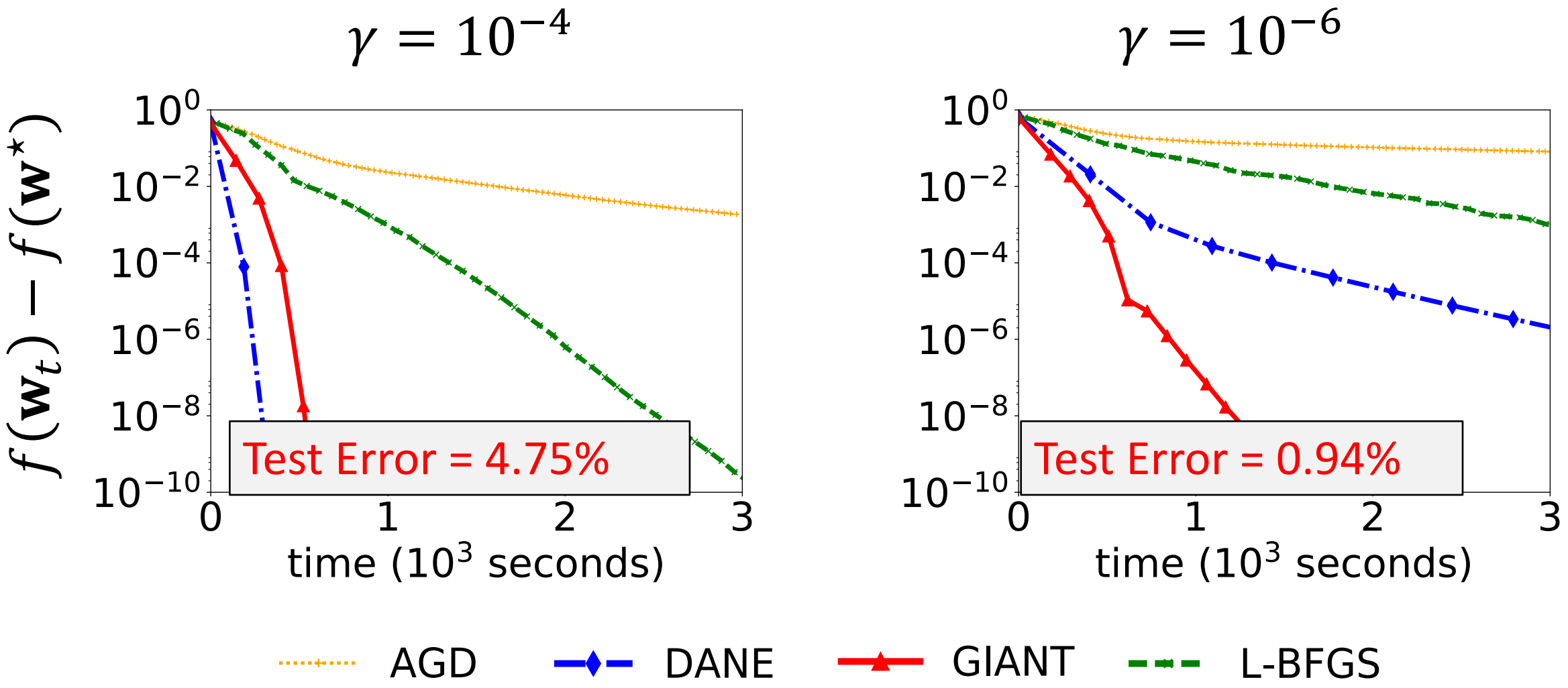
Covtype with RFM (n=581K, d=10K)



Epsilon with RFM (n=400K, d=10K)



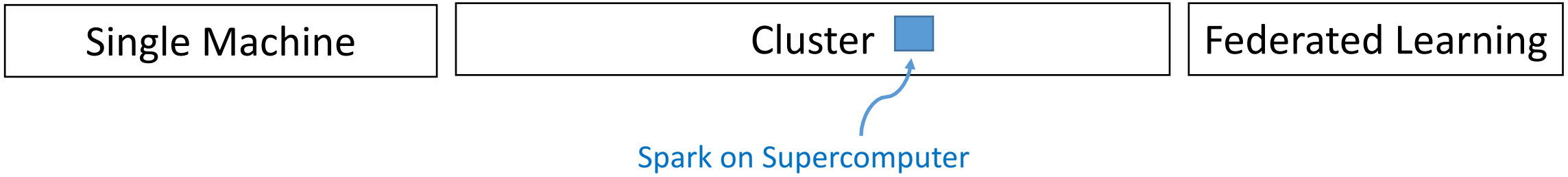
MNIST8M with RFM (n=1.6M, d=10K)



FLOPs versus Communication

In favor of

• SGD	• Mini-Batch SGD	• Full GD	• GIANT
• SCD	• Block SCD	• L-BFGS	• DANE



GIANT: Convergence Analysis

Quadratic Loss: Global Convergence

- Objective function: $f(\mathbf{w}) = \frac{1}{2n}\left\|\mathbf{X}\mathbf{w}-\mathbf{y}\right\|_2^2 + \frac{\gamma}{2}\|\mathbf{w}\|_2^2$

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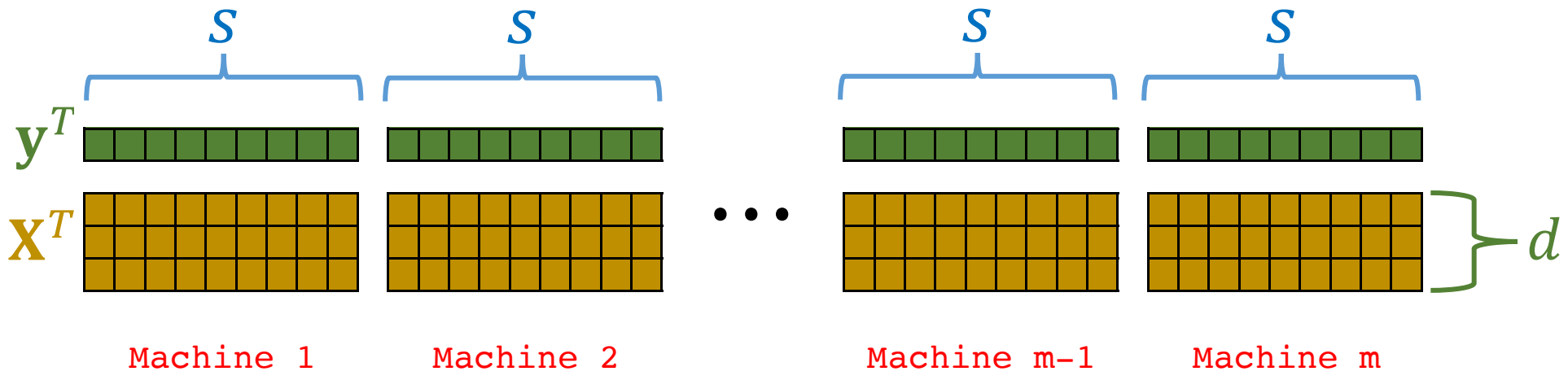
- Assume $\mathbf{X}$ is "incoherent" (information uniformly spread)
- Assume local sample size is $s = \Theta(\frac{d}{\epsilon^2}\log\frac{md}{\delta})$ for any $\epsilon, \delta \in (0,1)$



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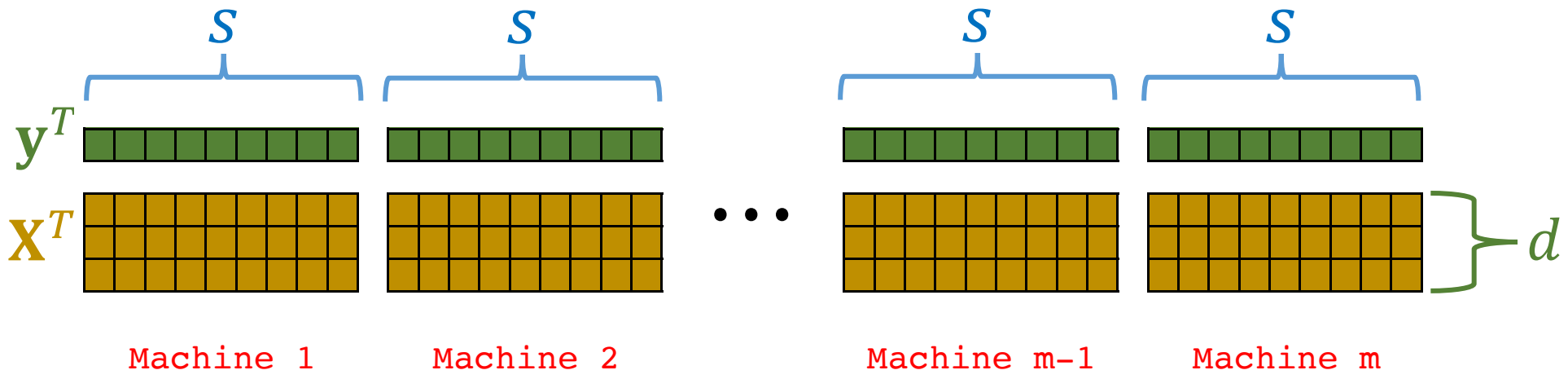
- With probability $1-\delta$,

$$\frac{\|\boldsymbol{\Delta}_t\|_2}{\|\boldsymbol{\Delta}_0\|_2} \leq \left(\frac{\epsilon}{\sqrt{m}} + \epsilon^2\right)^t \sqrt{\kappa}, \quad \text{where } \boldsymbol{\Delta}_t \triangleq \mathbf{w}_t - \mathbf{w}^\star.$$



General Smooth Loss: Local Convergence

- Denote $\mathbf{H}_t = \nabla^2 f(\mathbf{w}_t)$ and $\mathbf{H}^\star = \nabla^2 f(\mathbf{w}^\star)$
- Similar "incoherent" assumption (information uniformly spread)
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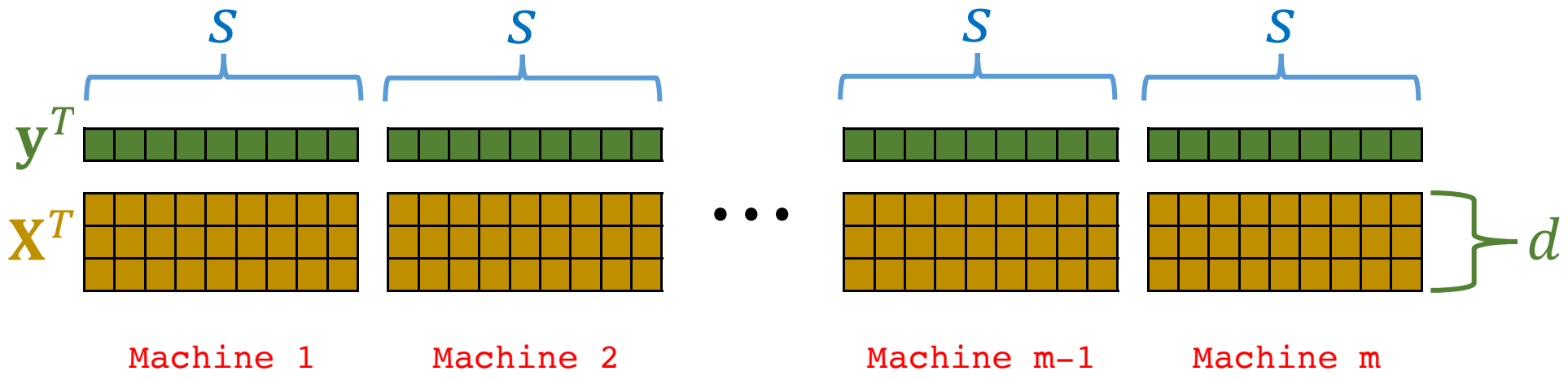

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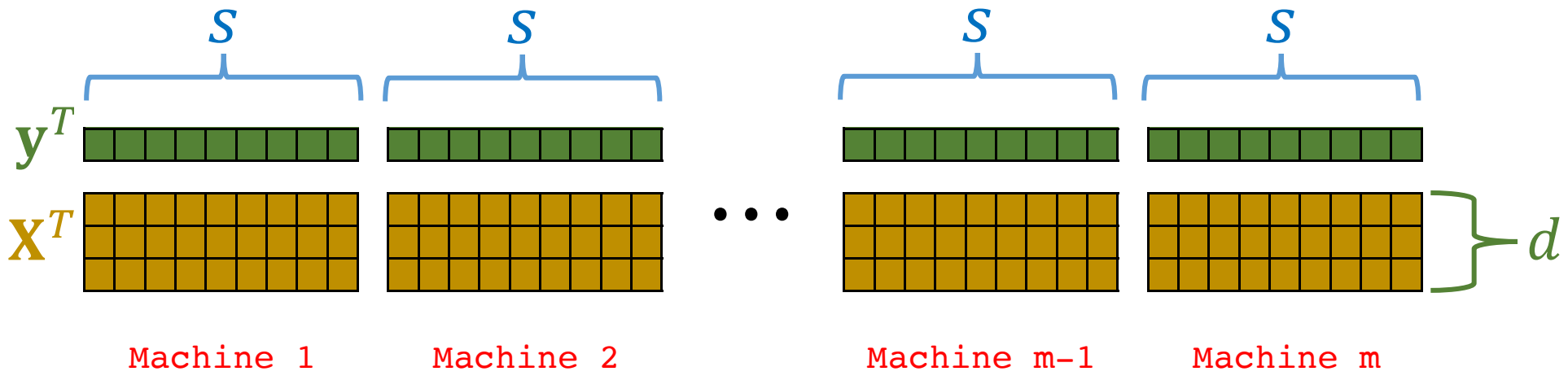
- Assume the Hessian is L-Lipchitz: $\|\mathbf{H}_t - \mathbf{H}^\star\|_2 \leq L\|\mathbf{w}_t - \mathbf{w}^\star\|_2$



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$$\|\boldsymbol{\Delta}_{t+1}\|_2 \leq \max\left\{\underbrace{(\tfrac{\epsilon}{\sqrt{m}} + \epsilon^2)\sqrt{\tfrac{\sigma_{\max}(\mathbf{H}_t)}{\sigma_{\min}(\mathbf{H}_t)}}\|\boldsymbol{\Delta}_t\|_2}_{\text{Linear}}, \underbrace{\tfrac{2L}{\sigma_{\min}(\mathbf{H}_t)}\|\boldsymbol{\Delta}_t\|_2^2}_{\text{Quadratic}}\right\}$$

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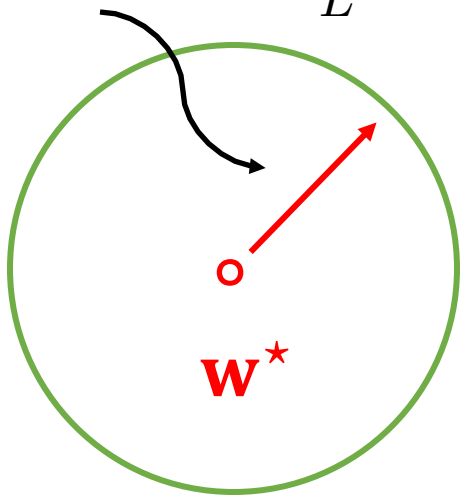
$$\text{radius} = \frac{\sigma_{\min}(\mathbf{H})}{L}$$



Only in this region, convergence is guaranteed.

General Smooth Loss: Local Convergence

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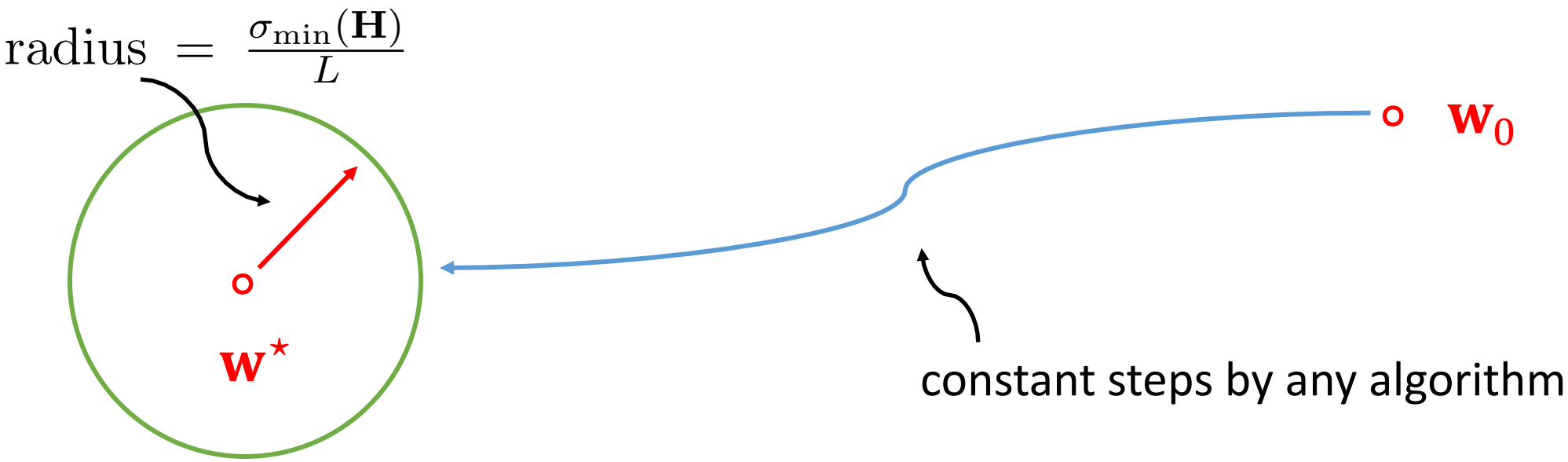
Linear

Quadratic



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Outline of Proof

Proof Techniques

Claim 1: Local Hessian $\widetilde{\mathbf{H}}_1, \cdots, \widetilde{\mathbf{H}}_m$ well approximate the true Hessian $\mathbf{H}$.

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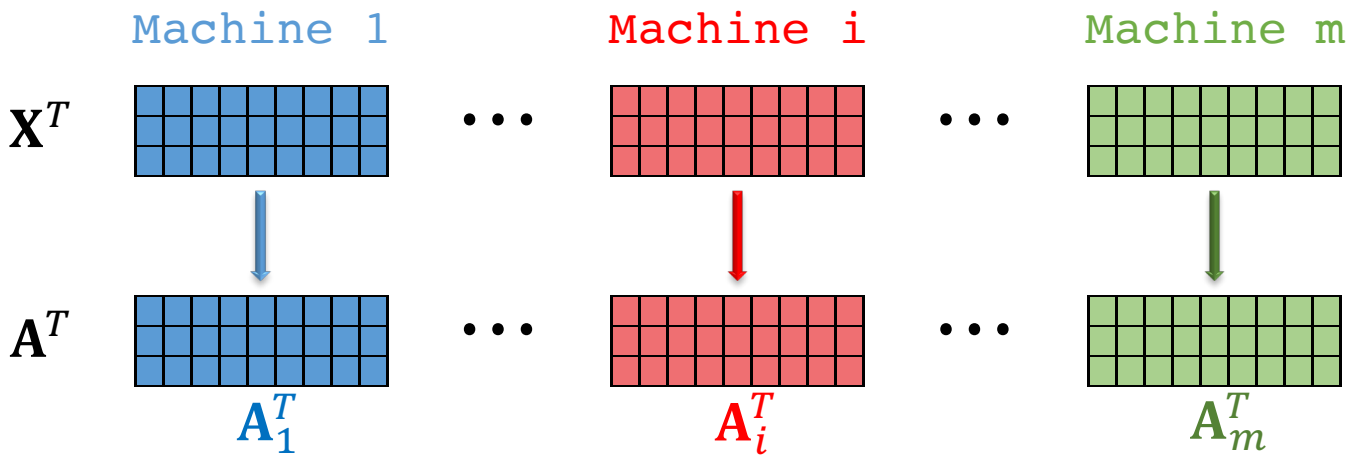

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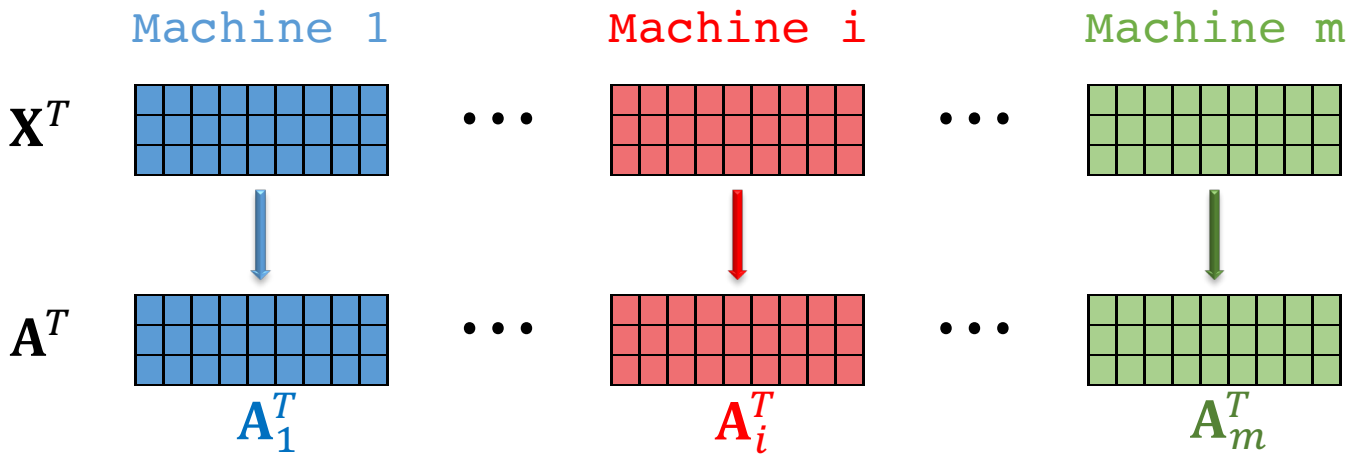
- Assume $\mathbf{A}$ is "incoherent".
- Uniformly partition $\mathbf{A}$ to form $\widetilde{\mathbf{A}}_i \in \mathbb{R}^{s\times d}$.



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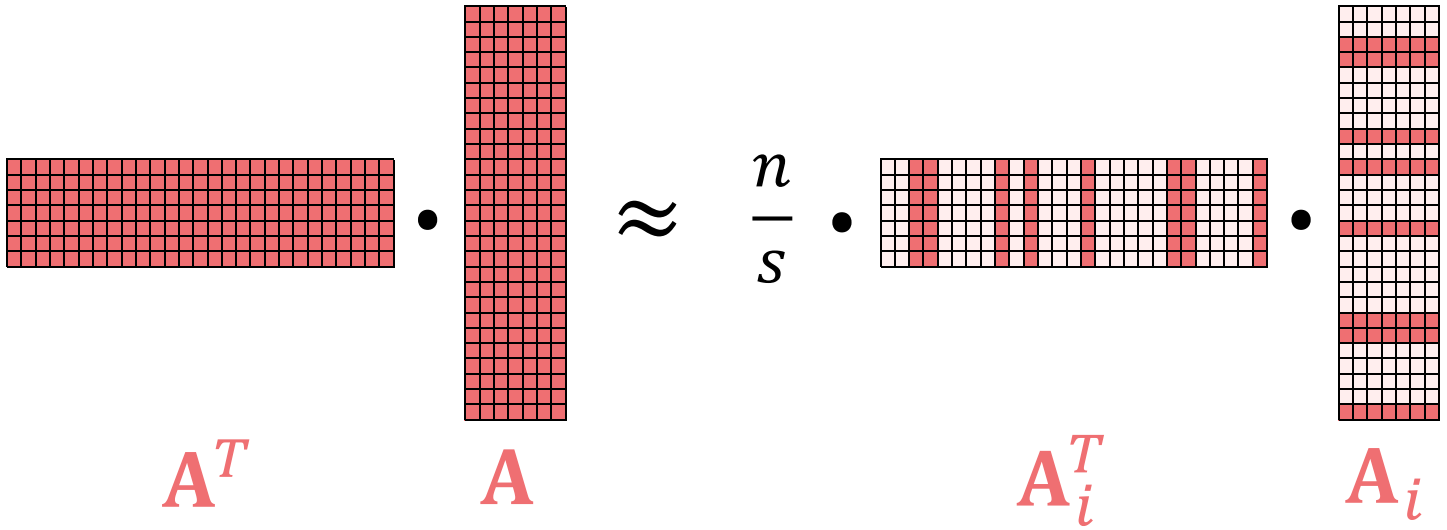

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- Sufficiently large samples size $s = \Theta(\frac{d}{\epsilon^2}\log\frac{d}{\delta})$

Proof Techniques

Claim 1: Local Hessian $\widetilde{\mathbf{H}}_1, \cdots, \widetilde{\mathbf{H}}_m$ well approximate the true Hessian $\mathbf{H}$.

- $\mathbf{H}$ can always be written as $\mathbf{H} = \mathbf{A}^T\mathbf{A} + \gamma\mathbf{I}_d$, for some $\mathbf{A} \in \mathbb{R}^{n\times d}$.
- Assume $\mathbf{A}$ is "incoherent".
- Uniformly partition $\mathbf{A}$ to form $\widetilde{\mathbf{A}}_i \in \mathbb{R}^{s\times d}$.
- Sufficiently large samples size $s = \Theta(\frac{d}{\epsilon^2}\log\frac{d}{\delta})$
- By matrix Bernstein (concentration inequality), with probability $1-\delta$,

$$(1-\epsilon)\mathbf{A}^T\mathbf{A} \preccurlyeq \frac{n}{s}\widetilde{\mathbf{A}}_i^T\widetilde{\mathbf{A}}_i \preccurlyeq (1+\epsilon)\mathbf{A}^T\mathbf{A}.$$

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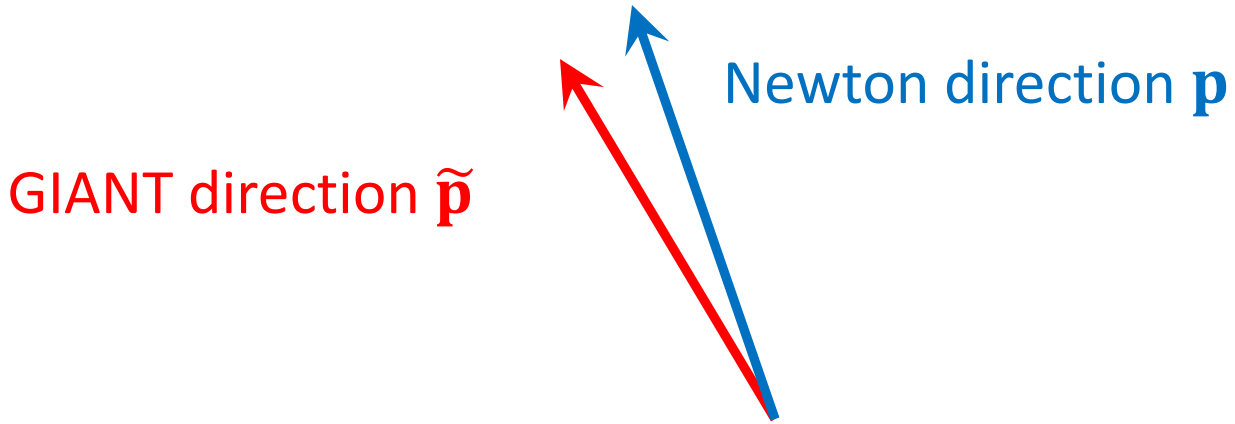
- Note that $\widetilde{\mathbf{H}}_i = \frac{n}{s}\widetilde{\mathbf{A}}_i^T\widetilde{\mathbf{A}}_i + \gamma\mathbf{I}_d \longrightarrow \widetilde{\mathbf{H}}_i$ well approximates $\mathbf{H}$.

Proof Techniques

Claim 2: The GIANT direction approximates $\mathbf{p}^\star = \mathbf{H}^{-1}\mathbf{g}$.

- Define the quadratic function $\phi(\mathbf{p}) \triangleq \frac{1}{2}\mathbf{p}^T\mathbf{H}\mathbf{p} - \mathbf{p}^T\mathbf{g} \quad (\leq 0)$



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- The GIANT directions is $\tilde{\mathbf{p}} \triangleq \frac{1}{m}\sum_{i=1}^{m}\tilde{\mathbf{p}}_i \triangleq \frac{1}{m}\sum_{i=1}^{m}\tilde{\mathbf{H}}_i^{-1}\mathbf{g}$
- Conditioning on **Claim 1** that $\widetilde{\mathbf{H}}_i$ well approximates $\mathbf{H}$, we get

$$\phi(\mathbf{p}^\star) \leq \phi(\tilde{\mathbf{p}}) \leq \left(1 - \alpha^2\right)\cdot\phi(\mathbf{p}^\star), \quad \text{where} \quad \alpha = \left(\frac{\epsilon}{\sqrt{m}} + \epsilon^2\right)$$

Reference:

W, Gittens, & Mahoney: Sketched Ridge Regression: Optimization Perspective, Statistical Perspective, and Model Averaging. In *ICML* 2017.

Proof Techniques

1. Use **Claim 2** that $\phi(\mathbf{p}^\star) \leq \phi(\tilde{\mathbf{p}}) \leq \left(1-\alpha^2\right) \cdot \phi(\mathbf{p}^\star), \quad \text{where } \alpha = \left(\frac{\epsilon}{\sqrt{m}} + \epsilon^2\right)$
2. Follow the standard convergence analysis of Newton's method.

⟹ Convergence of GIANT!

Conclusions & Future Work

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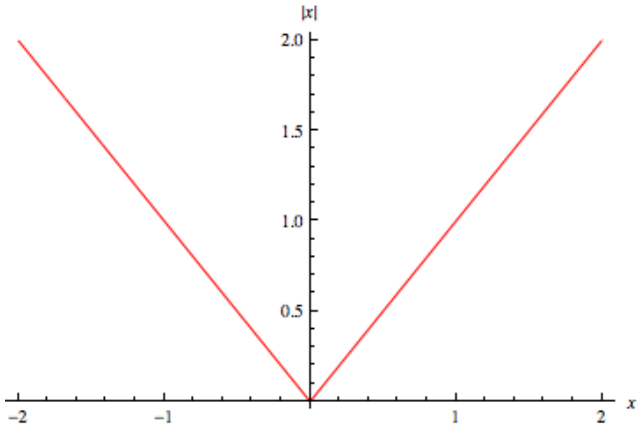
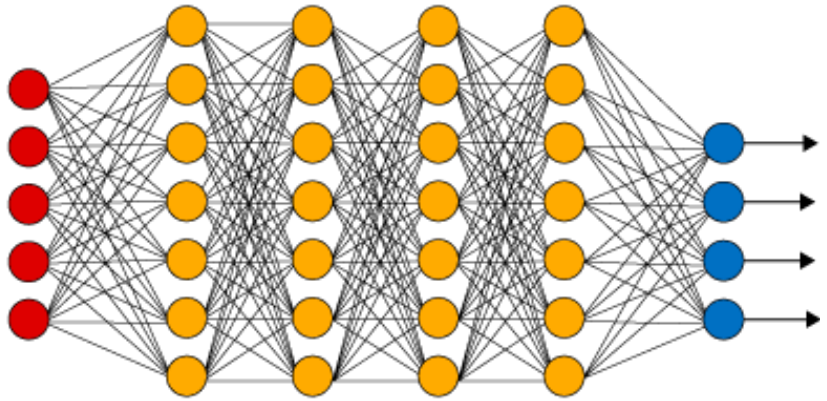
Counter-examples

LASSO

$$f(\mathbf{w}) = \frac{1}{2n}\sum_{j=1}^{n}\left(\mathbf{w}^T\mathbf{x}_j - y_j\right)^2 + \gamma\|\mathbf{w}\|_1$$



Neural Networks



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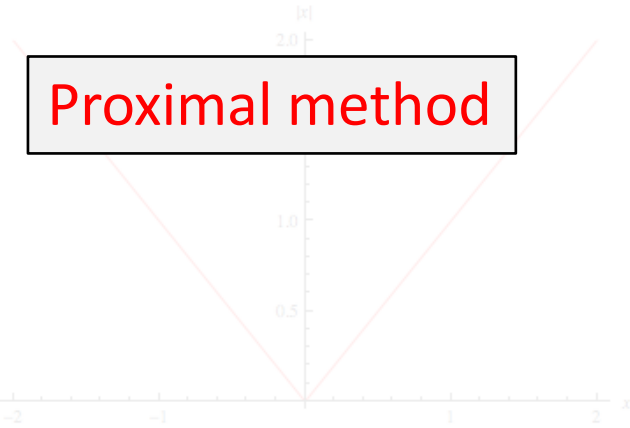
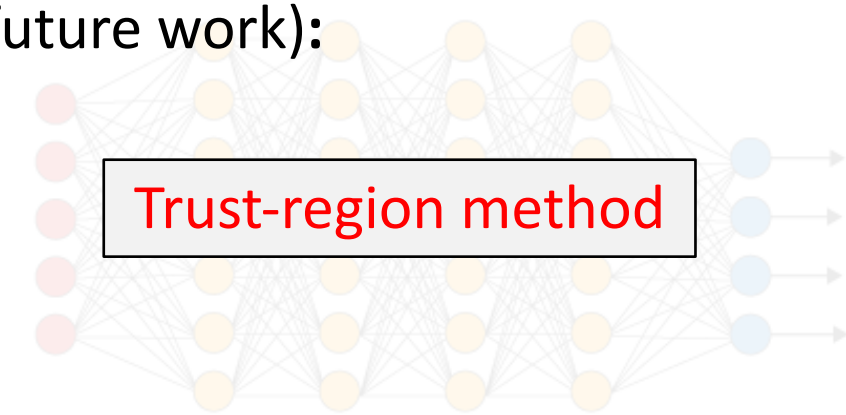
Extensions of GIANT (our future work)**:**

Proximal method





Trust-region method

Thank You!