Notes on developing a NEST class for a correlated Ornstein-Uhlenbeck process

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Our goal is to develop a class to produce multi-dimensional, correlated colored noise. An outline:

1. Statistics of the Ornstein-Uhlenbeck process
2. Gillespie's "exact" algorithm for simulating the OU process
3. Making the noises correlated
4. Tests we should perform on our NEST class

The Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process is a linear stochastic differential equation of the following form:

$$\tau\, dx_t \;=\; -x_t\, dt + \sigma\, dW_t \tag{1}$$

where $\tau, \sigma > 0$ are positive constants. W_t is a Wiener process; given a realization W_t (i.e. a Brownian path), the sample path x_t is uniquely determined. Without loss of generality, we have constructed this process to have zero mean. (There are other ways to set the parameters; here we choose the format that is most likely to be relevant to our application domain, i.e. neural models.) [AKB: something to think about: replace $\sigma \to \sigma\sqrt{\tau}$ i.e. make units come out cleaner?]

We can solve for the sample path by variation of parameters: defining $f(x_t, t) = x_t e^{t/\tau}$, then

$$\begin{aligned} df(x_t, t) &= x_t e^{t/\tau}/\tau\, dt + e^{t/\tau} dx_t \\ &= x_t e^{t/\tau}/\tau\, dt + e^{t/\tau}(-x_t\, dt + \sigma\, dW_t)/\tau \\ &= \frac{\sigma}{\tau} e^{t/\tau} dW_t \end{aligned}$$

Integrating from 0 to t,

$$\begin{aligned} f(x_t, t) - f(x_0, 0) &= \int_0^t \frac{\sigma}{\tau} e^{s/\tau}\, ds \Rightarrow \\ x_t &= x_0 e^{-t/\tau} + \int_0^t \frac{\sigma}{\tau} e^{-(t-s)/\tau}\, dW_s \end{aligned}$$

Using the Ito isometries, we conclude that

$$E[x_t] = x_0 e^{-t/\tau}$$

and

$$E[(x_t - E[x_t])^2] = E\left[\left(\int_0^t \frac{\sigma}{\tau} e^{-(t-s)/\tau}\, dW_s\right)^2\right] = E\left[\int_0^t \left(\frac{\sigma}{\tau} e^{-(t-s)/\tau}\right)^2 ds\right]$$

which is actually deterministic, i.e.

$$E[(x_t - E[x_t])^2] = \int_0^t \frac{\sigma^2}{\tau^2} e^{-2(t-s)/\tau}\, ds = \frac{\sigma^2}{\tau^2} \times \frac{\tau}{2}(1 - e^{-2t/\tau})$$

As $t \to \infty$, we recover an invariant measure:

$$E[x_t] = 0; \qquad E[x_t^2] = \frac{\sigma^2}{2\tau}$$

We can also compute the autocorrelation function

$$\begin{aligned}
C(s,t) &= E\left[(x_s - E[x_s])(x_t - E[x_t])\right] \\
&= E\left[\int_0^s \frac{\sigma}{\tau} e^{-(s-u)/\tau}\, dW_u \int_0^t \frac{\sigma}{\tau} e^{-(t-v)/\tau}\, dW_v\right] \\
&= \frac{\sigma^2}{\tau^2} e^{-(s+t)/\tau} E\left[\int_0^s e^{u/\tau}\, dW_u \int_0^t e^{v/\tau}\, dW_v\right] \\
&= \frac{\sigma^2}{\tau^2} e^{-(s+t)/\tau} \int_0^{\min(s,t)} e^{2u/\tau}\, du \\
&= \frac{\sigma^2}{\tau^2} e^{-(s+t)/\tau} \times \frac{\tau}{2}\left(e^{2\min(s,t)/\tau} - 1\right) \\
&= \frac{\sigma^2}{2\tau}\left(e^{-|t-s|/\tau} - e^{-(s+t)/\tau}\right)
\end{aligned}$$

Another way to describe the invariant measure is to take the initial time as t_0 rather than 0, and then let $t_0 \to -\infty$. Then we would find that:

$$E[x_t] = 0; \qquad E[x_t^2] = \frac{\sigma^2}{2\tau}; \qquad C(s,t) = \frac{\sigma^2}{2\tau} e^{-|t-s|/\tau} =: C(t-s)$$

Note that the autocorrelation function now only depends on the time lag $t - s$.

Gillespie's "exact" algorithm for the OU process

One option for simulating sample paths of an OU process, is to simply use the stochastic Euler method (ref here):

$$x_{t+\Delta t} = x_t + \Delta t \left(\frac{-x_t}{\tau}\right) + \frac{\sigma}{\tau}\sqrt{\Delta t}\, \eta_t \tag{2}$$

where η_t is chosen from a standard normal distribution.

Gillespie's algorithm improves on the Euler method, by exploiting the known statistics of the OU process:

$$x_{t+\Delta t} = x_t e^{-\Delta t/\tau} + \int_t^{t+\Delta t} \frac{\sigma}{\tau} e^{(t+\Delta t-s)/\tau}\, dW_s \tag{3}$$

$$E[x_{t+\Delta t}] = x_t e^{-\Delta t/\tau}; \qquad E\left[(x_{t+\Delta t} - E[x_{t+\Delta t}])^2\right] = \frac{\sigma^2}{2\tau}(1 - e^{-2\Delta t/\tau})$$

Alternatively, you could say that the increment $x_{t+\Delta t} - x_t$ is Gaussian with mean $x_t(e^{-\Delta t/\tau} - 1)$ and variance $\frac{\sigma^2}{2\tau}(1 - e^{-2\Delta t/\tau})$.

Therefore, the formula to follow in discrete time is:

$$x_{t+\Delta t} = x_t e^{-\Delta t/\tau} + \frac{\sigma}{\sqrt{2\tau}}\sqrt{1 - e^{-2\Delta t/\tau}}\,\eta_t \tag{4}$$

where η_t is drawn from a standard normal distribution.

By using Taylor's formula we can readily see that Eqns. (2) and (4) are equivalent up to $O(\Delta t)$.

Correlated noise

Now suppose you want a vector of correlated noises $\mathbf{x}_t \in \mathbb{R}^n$, say with covariance matrix $\mathbf{C}$; i.e.

$$E\left[\mathbf{x}_t \mathbf{x}_t^T\right] = \mathbf{C}$$

for $t \gg 1$ (i.e. we are close enough to the stationary measure) Assuming that all noises have the same time constant τ, here is what you do:

1. Use the Cholesky decomposition to find an upper triangular $\mathbf{L}$ such that $\mathbf{C} = \mathbf{L}^T\mathbf{L}$.
2. Generate a vector $\mathbf{y}$ of n independent OU processes with unit variance and time constant τ (that is, choose $\sigma = \sqrt{2\tau}$), as described in the previous section.
3. Use $\mathbf{x}_t = \mathbf{L}^T\mathbf{y}_t$ as your correlated processes.