STATS 217 homework 2

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1

$$
\begin{aligned}
P(X=1) &= \frac{P^{(1)}(0)}{1!} = \frac{(\frac{1}{2-s})'\big|_{s=0}}{1} = \frac{1}{4} \\
P(X=2) &= \frac{P^{(2)}(0)}{1!} = \frac{(\frac{1}{2-s})''\big|_{s=0}}{2} = \frac{1}{8} \\
V(X) &= P''(1) + P'(1) - [P'(1)]^2 \\
&= \frac{2}{(2-s)^3}\bigg|_{s=1} + \frac{1}{(2-s)^2}\bigg|_{s=1} - \left(\frac{1}{(2-s)^2}\right)^2\bigg|_{s=1} = 2
\end{aligned}
$$

2

Since Y is binomial (n, U) where U is uniform $[0, 1]$,
We have

$$
\begin{aligned}
P(Y=k|U) &= \binom{n}{k} U^k (1-U)^{n-k}, \\
f_U(u) &= 1.
\end{aligned}
$$

So

$$
\begin{aligned}
P(Y=k) &= \int_0^1 \binom{n}{k} u^k (1-u)^{n-k} f_U(u) du \\
&= \binom{n}{k} \int_0^1 u^k (1-u)^{n-k} du \\
&= \frac{n!}{k!(n-k)!} \int_0^1 u^k (1-u)^{n-k} du \\
&= \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} B(k+1, n-k+1) \\
&= \frac{1}{n+1} \frac{B(k+1, n-k+1)}{B(k+1, n-k+1)} \\
&= \frac{1}{n+1}.
\end{aligned}
$$

$\because$ The range of Y is $[0, n]$ and Y is an integer
$\therefore$ Y is uniform on $0, 1, 2, 3, ..., n$.

3

Let $X_{n,i}$ be a random variable denoting the number of direct successors of member i in period n, where $X_{n,i}$ are i,i,d random variables over all $n \in \{0, 1, 2, ...\}$ and $i \in \{1, ..., Z_n\}$. Then we have

$$Z_{n+1} = \sum_{i=1}^{Z_{n-1}} X_{n-1,i}.$$

$\because Z_0 = 1$
$\therefore Z_1 = X_{0,1}$
$\therefore E(X_{n,i}) = E(X_{0,1}) = E(Z_1) = \mu$, for all $n \in \{0, 1, 2, ...\}$ and $i \in \{1, ..., Z_n\}$.
$\therefore$

$$\begin{aligned} E(Z_m Z_n) &= E((\sum_{i=1}^{Z_{m-1}} X_{m-1,i})(\sum_{i=1}^{Z_{n-1}} X_{n-1,i})) \\ &= E(\sum_{i=1}^{Z_{m-1}} \sum_{j=1}^{Z_{n-1}} X_{m-1,i} X_{n-1,j}) \\ &= E(Z_{m-1} Z_{n-1}) \cdot \mu^2 \end{aligned}$$

Similarly,

$$E(Z_m^2) = E((\sum_{i=1}^{Z_{m-1}} X_{m-1,i})^2) = E(Z_{m-1}^2) \cdot \mu^2.$$

$\therefore$

$$\begin{aligned} \frac{E(Z_m Z_n)}{E(Z_m^2)} = \frac{E(Z_{m-1} Z_{n-1})}{E(Z_{m-1}^2)} = ... &= \frac{E(Z_{n-m} Z_0)}{E(Z_0^2)} \\ = E(Z_{n-m}) &= E\big((\sum_{i=1}^{Z_{n-m-1}} X_{n-m-1,i})\big) \\ = \mu E(Z_{n-m-1}) &= ... = \mu^{n-m}. \end{aligned}$$

Therefore,

$$E(Z_n Z_m) = \mu^{n-m} E(Z_m^2).$$

4

Let X_i denotes the number of chicks hatched by the i^{th} egg.
Then $X_i \sim Bernoulli(p)$.
Therefore,

$$P_{X_i}(s) = P_X(s) = 1 - p + ps.$$

Since $N \sim Poisson(\lambda)$,

$$P_N(s) = e^{\lambda(s-1)}.$$

Since

$$K = \sum_{i=1}^{N} X_i,$$

we have

$$P_K(s) = P_N(P_X(s)) = e^{\lambda(1-p+ps-1)} = e^{p\lambda(s-1)}.$$

According to the uniqueness of pgf, K has a poisson distribution with parameter $= p\lambda$.

5

5.1

$$\begin{aligned} P_X(s) &= \sum_{k=0}^{\infty} P(X=k) \cdot s^k = p \sum_{k=0}^{\infty} (qs)^k \\ &= \frac{p}{1-qs} \end{aligned}$$

where $|s| < \frac{1}{q}$.

5.2

According to the information, we have

$$E(s^X|\Lambda) = \sum_{n=0}^{\infty} \frac{(s\Lambda)^n}{n!} = e^{\Lambda(s-1)}.$$

Therefore

$$\begin{aligned} P_X(s) &= E(s^X) = E(E(s^X|\Lambda)) \\ &= \int_0^{\infty} e^{\lambda(s-1)} \mu e^{-\mu\lambda} d\lambda \\ &= \frac{\mu}{s-\mu-1} e^{\lambda(s-\mu-1)}\Big|_0^{\infty} \\ &= \frac{\mu}{\mu+1-s} \\ &= \frac{\frac{\mu}{\mu+1}}{1-\frac{1}{\mu+1}s}. \end{aligned}$$

Let $p = \frac{\mu}{\mu+1}$, then

$$P_X(s) = \frac{p}{1-(1-p)s}.$$

So X has the distribution described in **5.1**.