STAT 217 homework 6

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1 6.7

1.1

Let X_1, X_2, X_3 denote the time Ben, Max and Yolanda need to wait before being served. Then we have

$$X_1 \sim Exp(1)$$
$$X_2 \sim Exp(2)$$
$$X_3 \sim Exp(3)$$

Let $M = min\{X_1, X_2, X_3\}$. Then

$$P(M = X_3) = \frac{3}{1+2+3} = \frac{1}{2}.$$

1.2

$$\begin{aligned} P(X_1 < X_3) &= \int_0^\infty P(X_1 < X_3 | X_1 = x) P(X_1 = x) dx \\ &= \int_0^\infty P(X_3 > x) e^{-x} dx \\ &= \int_0^\infty e^{-x} dx \int_x^\infty 3e^{-3y} dy \\ &= \int_0^\infty e^{-x} e^{-3x} dx \\ &= \frac{1}{4}. \end{aligned}$$

1.3

Let $M = min\{X_1, X_2, X_3\}$. Then

$$M \sim Exp(1+2+3) = Exp(6).$$

So

$$E(M) = \frac{1}{6}.$$

2 6.13

2.1

$$
\begin{aligned}
P(N_s = k | N_t = n) &= \frac{P(N_s = k, N_t = n)}{P(N_t = n)} \\
&= \frac{P(N_s = k) P(N_{t-s} = n - k)}{P(N_t = n)} \\
&= \frac{\frac{e^{-\lambda s}(\lambda s)^k}{k!} \frac{e^{-\lambda(t-s)}(\lambda(t-s))^{n-k}}{(n-k)!}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}} \\
&= \frac{n!}{k!(n-k)!} \frac{s^k (t-s)^{n-k}}{t^n} \\
&= \binom{n}{k} (\frac{s}{t})^k (1 - \frac{s}{t})^{n-k}.
\end{aligned}
$$

Thus

$$
(N_s | N_t = n) \sim Binomial(n, \frac{s}{t}).
$$

2.2

$$
P(N_s = k | N_t = n) = P(N_{s-t} = k - n) = \frac{e^{-(s-t)\lambda}((s-t)\lambda)^{k-n}}{(k-n)!}.
$$

3 6.14

$$
\begin{aligned}
P(X = x) &= \int_0^\infty P(N_T = k | T = t) f_t(t) dt \\
&= \int_0^\infty \frac{e^{bt}(bt)^k}{k!} r e^{-rt} dt \\
&= \frac{r}{r+b} (\frac{b}{r+b})^k.
\end{aligned}
$$

Thus, X has a geometric distribution.

4 6.21

$$
\begin{aligned}
Cov(N_s, N_t) &= E(N_t N_s) - E(N_t) E(N_s) \\
&= E(N_t N_s) - \lambda^2 st.
\end{aligned}
$$

$$
\begin{aligned}
E(N_t N_s) &= E(E(N_t N_s | N_s)) \\
&= E(N_s E(N_t | N_s)) \\
&= E(N_s(N_s + E(N_{t-s}))) \\
&= E(N_s^2 + (t-s)_s) \\
&= E(N_s^2) + (t-s)\lambda E(N_s) \\
&= (E(N_s))^2 + Var(N_s) + (t-s)\lambda E(N_s) \\
&= \lambda s + \lambda^2 st.
\end{aligned}
$$

Thus,

$$Cov(N_t, N_s) = \lambda s + \lambda^2 st - \lambda^2 st = \lambda s.$$

Thus

$$
\begin{aligned}
Corr(N_t, N_s) &= \frac{Cov(N_t, N_s)}{\sqrt{Var(N_t)Var(N_s)}} \\
&= \frac{\lambda s}{\sqrt{\lambda t \cdot \lambda s}} \\
&= \sqrt{\frac{s}{t}}.
\end{aligned}
$$

Proved.

5 6.24

5.1

Let X_1, X_2 denote the time first see a meadowlark and a sparrow. Then

$$
\begin{aligned}
X_1 &\sim Exp(\lambda) \\
X_2 &\sim Exp(\mu)
\end{aligned}
$$

Then

$$P(X_1 < X_2) = \frac{\lambda}{\lambda + \mu}.$$

5.2

Let $N_m(t)$ denote the poisson process with parameter λ and $N_s(t)$ denote the poisson process with parameter μ.
Let $N_{m+s}(t) = N_m(t) + N_s(t)$. Then N_{m+s} is a poisson process with parameter $(\lambda + \mu)$.

$$P(N_{m+s}(1) = 1) = (\lambda + \mu)e^{-(\lambda+\mu)}.$$

5.3

Since the two process are independent,

$$\begin{aligned}
P(N_s(2)=1, N_m(2)=2) &= P(N_s(2)=1)P(N_m(2)=2) \\
&= \frac{2\mu e^{(} - 2\mu)}{1!} \frac{(2\lambda)^2 e^{-2\lambda}}{2!} \\
&= 4\mu\lambda^2 e^{-2(\lambda+\mu)}.
\end{aligned}$$

6 3.56

An absorbing Markov chain is built on $\{\o, H, HT, HTT, HTTH, HTTHH\}$, and

$$P(H) = \frac{1}{3}$$
$$P(T) = \frac{2}{3}$$

Thus the transition matrix is

$$P = \begin{array}{c c}
\mathit{states} & \begin{array}{cccccc} \o & H & HT & HTT & HTTH & HTTHH \end{array} \\
\begin{array}{c} \o \\ H \\ HT \\ HTT \\ HTTH \\ HTTHH \end{array} &
\begin{pmatrix}
\frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 \\
\frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}.$$

Making HTTHH an absorbing state, then

$$\begin{aligned}
F &= (I-Q)^{-1} \\
&= \begin{pmatrix}
\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\
0 & \frac{2}{3} & -\frac{2}{3} & 0 & 0 \\
0 & -\frac{1}{3} & 1 & -\frac{2}{3} & 0 \\
-\frac{2}{3} & 0 & 0 & 1 & -\frac{1}{3} \\
0 & 0 & -\frac{2}{3} & 0 & 1
\end{pmatrix}^{-1} \\
&= \begin{pmatrix}
21 & 17.25 & 13.5 & 9 & 3 \\
18 & 17.25 & 13.5 & 9 & 3 \\
18 & 15.75 & 13.5 & 9 & 3 \\
18 & 15 & 12 & 9 & 3 \\
12 & 10.5 & 9 & 6 & 3
\end{pmatrix}.
\end{aligned}$$

The row sums are

$$\begin{array}{c} \o \\ H \\ HT \\ HTT \\ HTTH \end{array} \begin{pmatrix} 63.75 \\ 60.75 \\ 59.25 \\ 57 \\ 40.5 \end{pmatrix}.$$

Thus, the expected number of flips to get HTTHH is 63.75.