STATS 217 homework 1

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1 1.10

Let E_1 denotes the event that the number of rolling ends up with an odd number.
Let E_2 denotes the event that the number of rolling ends up with an even number.
Let e_1 denotes the outcome number of the first roll.

Then, we have:

$$P(E_1) + P(E_2) = 1 \tag{1}$$

and

$$P(E_1) = P(e_1 = 3) \times P(E_1|e_1 = 3) + P(e_1 \neq 3) \times P(E_1|e_1 \neq 3).$$

Since that

$$P(E_1|e_1 \neq 3) = P(E_2)$$

and

$$P(E_1|e_1 = 3) = 1,$$

we have

$$P(E_1) = \frac{1}{6} \times 1 + \frac{5}{6} \times P(E_2) \tag{2}$$

Solving (1) and (2), we can get that

$$\begin{cases} P(E_1) = \frac{6}{11} \\ P(E_2) = \frac{5}{11} \end{cases}$$

So the probability that an even number of rolls is needed is $\frac{5}{11}$.

2 1.17

$$
\begin{aligned}
E(X|X>2) &= \sum_x xP(X=x|X>2) \\
&= \sum_x \frac{P(X=x, X>2)}{P(X>2)} \\
&= \frac{\sum\limits_{x=3}^{\infty} xP(X=x)}{\sum\limits_{x=3}^{\infty} P(X=x)} \\
&= \frac{E(X) - P(X=1) - 2P(X=2)}{1 - P(X=0) - P(X=1) - P(X=2)}.
\end{aligned}
$$

Since that

$$
\begin{aligned}
E(X) &= \lambda = 3, \\
P(x=0) &= \frac{e^{-3}}{0!} = e^{-3}, \\
P(x=1) &= \frac{3 \times e^{-3}}{1!} = 3 \times e^{-3}, \\
P(x=2) &= \frac{3^2 \times e^{-3}}{2!} = \frac{9 \times e^{-3}}{2},
\end{aligned}
$$

we have

$$
\begin{aligned}
E(X|X>2) &= \frac{3 - 12e^{-3}}{1 - \frac{17}{2}e^{-3}} \\
&\approx 4.16525.
\end{aligned}
$$

3 1.22

3.1 (b)

$$
\begin{aligned}
E(Y|X) &= \int_0^x y f_{Y|X}(y|X=x)dy \\
&= \int_0^x y \frac{f_{xy}(x,y)}{f_X(x)} dy \\
&= \int_0^x y \frac{f_{xy}(x,y)}{\int_0^x f_{xy}(x,y)dy} dy.
\end{aligned}
$$

Since that (X, Y) is uniformly distributed on the triangle,

$$
f_{xy}(x,y) = \frac{1}{S_\Delta} = \frac{1}{\frac{1}{2}} = 2.
$$

Then

$$\begin{aligned}
E(Y|X) &= \int_0^x y \frac{2}{\int_0^x 2dy} dy \\
&= \int_0^x y \frac{2}{2x} dy \\
&= \frac{1}{x} \times \frac{1}{2} x^2 \\
&= \frac{x}{2}.
\end{aligned}$$

3.2 (c)

$$\begin{aligned}
E(Y|X) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y f_{Y|X}(y|X=x) dy \\
&= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y \frac{f_{xy}(x,y)}{f_X(x)} dy \\
&= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y \frac{f_{xy}(x,y)}{\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{xy}(x,y) dy} dy.
\end{aligned}$$

Since that (X, Y) is uniformly distributed on the disc,

$$f_{xy}(x,y) = \frac{1}{S_O} = \frac{1}{\pi}.$$

Then

$$\begin{aligned}
E(Y|X) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y \frac{\frac{1}{\pi}}{\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy} dy \\
&= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y \frac{1}{2\sqrt{1-x^2}} dy \\
&= \frac{1}{2\sqrt{1-x^2}} \times 0 \\
&= 0.
\end{aligned}$$

4 1.26

$$\begin{aligned}P(Y<2) &= \int_0^2 f_Y(y)dy \\ &= \int_0^2 dy \int_y^\infty f_{xy}(x,y)dx \\ &= \int_0^2 dy \int_y^\infty f_X(x) f_{Y|X}(Y|X=x)dx \\ &= \int_0^2 dy \int_y^\infty xe^{-x}\frac{1}{x}dx \\ &= \int_0^2 dy \int_y^\infty e^{-x}dx \\ &= \int_0^2 \frac{1}{e^y}dy \\ &= 1-\frac{1}{e^2} \approx 0.865.\end{aligned}$$

5 1.31

Let e_1 denotes the result of the first trail. That is, $e_1 = 1$ means success, otherwise $e_1 = 0$. Since that

$$X \sim Ge(p),$$

we have

$$\begin{aligned}E(X) &= P(e_1=1)\times E(X|e_1=1) + P(e_1=0)\times E(X|e_1=0) & (3)\\ &= pE(X|e_1=1) + (1-p)E(X|e_1=0) & (4)\\ &= p\times 1 + (1-p)(E(X)+1). & (5)\end{aligned}$$

(5) solves that

$$E(X) = \frac{1}{p}.$$

Since that

$$\begin{aligned}E(Var(X|e_1)) &= (1-p)Var(X|e_1=0) + pVar(X|e_1=1)\\ &= (1-p)Var(X),\end{aligned}$$

and

$$
\begin{aligned}
Var(E(X|e_1)) &= E((E(X|e_1))^2) - (E(E(X|e_1)))^2 \\
&= E((E(X|e_1))^2) - \frac{1}{p^2} \\
&= (1-p)E((E(X|e_1=0))^2) + pE((E(X|e_1=1))^2) - \frac{1}{p^2} \\
&= (1-p)E((E(X)+1)^2) + p - \frac{1}{p^2} \\
&= (1-p)(\frac{1}{p}+1)^2 + p - \frac{1}{p^2} \\
&= \frac{1-p}{p},
\end{aligned}
$$

we have

$$
\begin{aligned}
Var(X) &= E(Var(X|e_1)) + Var(E(X|e_1)) && (6) \\
&= (1-p)Var(X) + \frac{1-p}{p}. && (7)
\end{aligned}
$$

(7) solves that

$$Var(X) = \frac{1-p}{p^2}.$$