STAT 217 homework 5

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July 2019

1 3.25

1.1

Case: k=2
The transition matrix is

$$P = \begin{array}{cc} states & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

The stationary distribution satisfies

$$\pi P = \pi \tag{1}$$

Solving **(1)**, we have

$$\pi = (\frac{1}{6}, \frac{2}{3}, \frac{1}{6}).$$

Case: k=3
The transition matrix is

$$P = \begin{array}{cc} states & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

The stationary distribution satisfies

$$\pi P = \pi \tag{2}$$

Solving **(1)**, we have

$$\pi = (\frac{1}{20}, \frac{9}{20}, \frac{9}{20}, \frac{1}{20}).$$

1.2

From **Exercise 2.12**, we know that

$$P_{ij} = \frac{1}{k^2} \begin{cases} (k-j+1)^2 & i = j-1 \\ 2j(k-j) & i = j \\ (j+1)^2 & i = j+1. \end{cases}$$

If $\pi_j = \binom{k}{j}^2 / \binom{2k}{k}$, then we have

- $j \in [1, k-1]$

$$
\begin{aligned}
\pi_j &= \sum_{i=0}^{k} \pi_i P_{ij} \\
&= \frac{\binom{k}{j-1}^2}{\binom{2k}{k}} \frac{(k-j+1)^2}{k^2} + \frac{\binom{k}{j}^2}{\binom{2k}{k}} \frac{2j(k-j)}{k^2} + \frac{\binom{k}{j+1}^2}{\binom{2k}{k}} \frac{(j+1)^2}{k^2} \\
&= \frac{\binom{k}{j}^2}{\binom{2k}{k}} \left(\frac{j^2}{(k-j+1)^2} \frac{(k-j+1)^2}{k^2} + \frac{2j(k-j)}{k^2} + \frac{(k-j)^2}{(j+1)^2} \frac{(j+1)^2}{k^2} \right) \\
&= \frac{\binom{k}{j}^2}{\binom{2k}{k}} \frac{j^2 + 2j(k-j) + (k-j)^2}{k^2} \\
&= \frac{\binom{k}{j}^2}{\binom{2k}{k}} \frac{(j+k-j)^2}{k^2} \\
&= \frac{\binom{k}{j}^2}{\binom{2k}{k}} = \pi_j.
\end{aligned}
$$

- $j = 0$

$$
\begin{aligned}
\pi_0 &= \sum_{i=0}^{k} \pi_i P_{i0} = \pi_1 P_{10} \\
&= \frac{\binom{k}{1}^2}{\binom{2k}{k}} \frac{1}{k^2} \\
&= \frac{1}{\binom{2k}{k}} = \frac{\binom{k}{0}^2}{\binom{2k}{k}}
\end{aligned}
$$

- $j = k$

$$\begin{aligned}
\pi_k &= \sum_{i=0}^{k} \pi_i P_{ik} = \pi_{k-1} P_{k-1k} \\
&= \frac{\binom{k}{k-1}^2}{\binom{2k}{k}} \frac{1}{k^2} \\
&= \frac{1}{\binom{2k}{k}} = \frac{\binom{k}{k}^2}{\binom{2k}{k}}
\end{aligned}$$

Proved.

2 3.27

2.1

A Markov chain is called irreducible if it has exactly one communication class. That is, all states communicate with each other.
$\forall m, n \in N$, we can first go from m to 0, then from 0 to 1, and so on, finally to n.
Thus, for all $m, n \in N$, $\sum_i P_{mn}^i > 0$. So the chain is irreducible.
A Markov chain is called aperiodic if it is irreducible and all states have period equal to 1.
$\forall i \in N$, we can first go from i to $i+1$, then to 0, then to 1, and so on, finally to i. This costs $i+2$ steps. So $P_{ii}^{i+2} > 0$.
$\forall i \in N$, we can first go from i to $i+1$, then to $i+2$, then to 0, then to 1, and so on, finally to i. This costs $i+3$ steps. So $P_{ii}^{i+3} > 0$.
Thus, $d(i) = gcd\{n > 0 : P_{ii}^n > 0\} \leqslant gcd\{i+2, i+3\} = 1$, so $d(i) = 1$, for all i.
So the chain is aperiodic.

2.2

$$\begin{aligned}
P(T_0 = 1 | X_0 = 0) &= 0 \\
P(T_0 = 2 | X_0 = 0) &= \frac{1}{2} \\
P(T_0 = 3 | X_0 = 0) &= \frac{1}{2} \times \frac{1}{3} \\
P(T_0 = 4 | X_0 = 0) &= \frac{1}{2} \times \frac{2}{3} \times \frac{1}{4} = \frac{1}{3} \times \frac{1}{4} \\
&\cdots \\
P(T_0 = n | X_0 = 0) &= \frac{1}{n-1} \times \frac{1}{n} \\
&\cdots
\end{aligned}$$

So we have

$$
\begin{aligned}
f_0 &= P(T_0 < \infty | X_0 = 0) \\
&= \sum_{i=1}^{\infty} P(T_0 = i | X_0 = 0) \\
&= 0 + \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{4} + ... + \frac{1}{n-1} \times \frac{1}{n} + ... \\
&= \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + ... \\
&= 1.
\end{aligned}
$$

So the chain is recurrent.

2.3

$$
\begin{aligned}
E(T_0 | X_0 = 0) &= \sum_{n=1}^{\infty} n P(T_0 = n | X_0 = 0) \\
&= 2 \times \frac{1}{2} + 3 \times \frac{1}{2} \times \frac{1}{3} + 4 \times \frac{1}{3} \times \frac{1}{4} + ... \\
&= 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ... \\
&= \infty.
\end{aligned}
$$

So the chain is null recurrent.

3 3.30

3.1 If the walk is periodic

A Markov chain is called periodic if it is irreducible and all states have period greater than 1.
Assume the graph is not bipartite.
Then there must by an edge on the graph joining two vertices that are in the same color. Let E_1, E_2 denote the two vertices.
Consider the starting point to be E_1. Then going back to E_1 can cost at least 2 steps, i.e., $E_1 \to E_2 \to E_1$. And it can also cost an odd number of steps $(2k+1)$, i.e., going to another neighbour of E_1 and finally coming back to E_1 from E_2.
So $d(E_1) \leqslant gcd\{2, 2k+1\} = 1$. The chain can't be periodic.
So the graph must be bipartite.

3.2 If the graph is bipartite

Let E denotes the starting point.
Since the graph is bipartite, starting from E then coming back to E always costs an even number of steps. So $d(E) \geqslant 2$, for all Es.
So the chain is periodic.

4 3.36

4.1

$$\because \textit{The chain is in stationary distribution,}$$
$$\therefore P(X_m = i) = \pi_i \quad i \in \{1, 2, ..., k\}, \forall m \in N^+.$$

Therefore,

$$\begin{aligned}
Cov(X_m, X_{m+n}) &= E(X_m X_{m+n}) - E(X_m)E(X_{m+n}) \\
&= \sum_{i,j \in \{1,2,\ldots,k\}} ijP(X_m = i, X_{m+n} = j) - \sum_{i=1}^{k} i\pi_i \sum_{j=1}^{k} j\pi_j \\
&= \sum_{i,j \in \{1,2,\ldots,k\}} ijP(X_{m+n} = j | X_m = i)P(X_m = i) - \sum_{i=1}^{k} i\pi_i \sum_{j=1}^{k} j\pi_j \\
&= \sum_{i,j \in \{1,2,\ldots,k\}} ijP(X_n = j | X_0 = i)P(X_m = i) - \sum_{i=1}^{k} i\pi_i \sum_{j=1}^{k} j\pi_j \\
&= \sum_{i,j \in \{1,2,\ldots,k\}} ijP_{ij}^n \cdot \pi_i - \sum_{i=1}^{k} i\pi_i \sum_{j=1}^{k} j\pi_j \\
&= Cov(X_0, X_n).
\end{aligned}$$

4.2

$$\begin{aligned}
\lim_{n\to\infty} Cov(X_n, X_{m+n}) &= \lim_{n\to\infty} \sum_{i,j \in \{1,2,\ldots,k\}} ijP_{ij}^n \cdot \pi_i - \sum_{i=1}^{k} i\pi_i \sum_{j=1}^{k} j\pi_j \\
&= \sum_{i,j \in \{1,2,\ldots,k\}} ij \cdot \pi_j \cdot \pi_i - \sum_{i=1}^{k} i\pi_i \sum_{j=1}^{k} j\pi_j \\
&= \sum_{i=1}^{k} i\pi_i \sum_{j=1}^{k} j\pi_j - \sum_{i=1}^{k} i\pi_i \sum_{j=1}^{k} j\pi_j \\
&= 0.
\end{aligned}$$

5 3.42

From the information of the question, we can find the transition matrix of this process:

$$P = \begin{pmatrix} states & 0 & 1 & 2 & 3 & ... & k & ... \\ 0 & 0 & 1 & 0 & 0 & ... & 0 & ... \\ 1 & p & 0 & 1-p & 0 & ... & 0 & ... \\ 2 & 0 & p & 0 & 1-p & ... & 0 & \\ 3 & 0 & 0 & p & 0 & ... & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k-1 & 0 & 0 & 0 & 0 & ... & 1-p & ... \\ k & 0 & 0 & 0 & 0 & ... & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

We want the chain to be reversible, so it must satisfy:

$$\pi_i P_{ij} = \pi_j Pji, \quad \forall i, j \in 0, 1, 2, ...$$

So we have

$$\pi_0 P_{0j} = \begin{cases} \pi_0 & j = 1 \\ 0 & otherwise \end{cases} \quad \pi_j P_{j0} = \begin{cases} p\pi_j & j = 1 \\ 0 & otherwise \end{cases}$$

$$\pi_1 P_{1j} = \begin{cases} p\pi_1 & j = 0 \\ (1-p)\pi_1 & j = 2 \\ 0 & otherwise \end{cases} \quad \pi_j P_{j1} = \begin{cases} \pi_j & j = 0 \\ p\pi_j & j = 2 \\ 0 & otherwise \end{cases}$$

$$\pi_i P_{ij} = \begin{cases} p\pi_i & j = i-1 \\ (1-p)\pi_i & j = i+1 \\ 0 & otherwise \end{cases} \quad \pi_j P_{ji} = \begin{cases} (1-p)\pi_j & j = i-1 \\ p\pi_j & j = i+1 \\ 0 & otherwise \end{cases} \qquad for\ i \geqslant 2.$$

Solving the above equations, we can get

$$\begin{cases} \pi_0 & = p\pi_1, \\ \pi_i & = \frac{p}{1-p}\pi_{i+1} \quad i \geqslant 2. \end{cases}$$

So we have

$$1 = \sum_{i=0}^{\infty} \pi_i = \pi_0 + \frac{1}{p}\pi_0 + \frac{1-p}{p^2}\pi_0 + \frac{(1-p)^2}{p^3}\pi_0 + ... \tag{3}$$

$$= \lim_{n\to\infty} \pi_0 (1 + \frac{\frac{1}{p}(1 - (\frac{1-p}{p})^n)}{1 - \frac{1-p}{p}}). \tag{4}$$

So $\frac{1-p}{p} < 1$, thus $\frac{1}{2} < p < 1$.
Continue working on (4), we can get

$$\begin{aligned} 1 &= \lim_{n\to\infty} \pi_0 (1 + \frac{\frac{1}{p}(1-(\frac{1-p}{p})^n)}{1-\frac{1-p}{p}}) \\ &= \frac{2p}{2p-1}\pi_0. \end{aligned}$$

So

$$\begin{cases} \pi_0 &= \frac{2p-1}{2p}, \\ \pi_i &= \frac{(1-p)^{i-1}}{p^i}\pi_0, \quad i \geqslant 2. \end{cases}$$