STAT 217 homework 3

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1 2.6

1.1

Since

$$X_n = max\{R_0, R_1, ..., R_n\}$$
$$X_{n-1} = max\{R_0, R_1, ..., R_{n-1}\}$$

We have

$$X_n = max\{X_{n-1}, R_n\}$$

So

$$P(X_n = x_n | X_0 = x_0, X_1 = x_1, ..., X_{n-1} = x_{n-1}) \\ = P(X_n = x_n | X_{n-1} = x_{n-1}).$$

So $X_0, X_1, ..., X_n$ is a Markov chain, with the state space$= \{1, 2, 3, 4\}$. The transition matrix is as below:

$$P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

1.2

$$\begin{aligned} P(X_3 \geqslant 3) &= 1 - P(X_3 \leqslant 2) \\ &= 1 - P(max\{R_0, R_1, R_2, R_3\} \leqslant 2) \\ &= 1 - P(R_0 \leqslant 2)P(R_1 \leqslant 2)P(R_2 \leqslant 2)P(R_3 \leqslant 2) \\ &= 1 - (\frac{1}{2})^4 \\ &= \frac{15}{16} \end{aligned}$$

2 2.11

2.1

The state space is $\{0, 1, 2, 3, 4, 5\}$.
So the transition matrix is

$$P = \begin{pmatrix} states & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & (\frac{5}{6})^5 & 5(\frac{1}{6})(\frac{5}{6})^4 & 10(\frac{1}{6})^2(\frac{5}{6})^3 & 10(\frac{1}{6})^3(\frac{5}{6})^2 & 5(\frac{1}{6})^4(\frac{5}{6}) & (\frac{1}{6})^5 \\ 1 & 0 & (\frac{5}{6})^4 & 4(\frac{1}{6})(\frac{5}{6})^3 & 6(\frac{1}{6})^2(\frac{5}{6})^2 & 4(\frac{1}{6})^3(\frac{5}{6}) & (\frac{1}{6})^4 \\ 2 & 0 & 0 & (\frac{5}{6})^3 & 3(\frac{1}{6})(\frac{5}{6})^2 & 3(\frac{1}{6})^2(\frac{5}{6}) & (\frac{1}{6})^3 \\ 3 & 0 & 0 & 0 & (\frac{5}{6})^2 & 2(\frac{1}{6})(\frac{5}{6}) & (\frac{1}{6})^2 \\ 4 & 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2.2

From **2.1**, we can calculate P^3, which is

$$P^3 = \begin{pmatrix} states & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0.0649 & 0.2363 & 0.3440 & 0.2504 & 0.0912 & 0.0133 \\ 1 & 0 & 0.1122 & 0.3266 & 0.3567 & 0.1731 & 0.0315 \\ 2 & 0 & 0 & 0.1938 & 0.4233 & 0.3081 & 0.0748 \\ 3 & 0 & 0 & 0 & 0.3349 & 0.4876 & 0.1775 \\ 4 & 0 & 0 & 0 & 0 & 0.5787 & 0.4213 \\ 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

So $P^3_{0,5} = 0.0133$.

2.3

$$P^{100} = \begin{pmatrix} states & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

3 2.12

X_n has possible states $\{0, 1, 2, ..., k\}$ which are the number of blue balls in the left urn.
$P(X_{n+1} = k)$ depends only on the number of blue balls in left urn before the $(n+1)$th trial i.e. by the value X_n, so we have a Markov chain.
Suppose $X_n = j$ then the left urn has j blue balls and $k - j$ red balls in the left urn. Also, the right urn will have $k - j$ blue balls and j red balls (because total blue or red balls are k)
The possible outcomes from the next trial are:

- Blue ball from left urn and blue ball from right urn are selected with probability

$$\frac{j}{k} \cdot \frac{k-j}{k} = \frac{j(k-j)}{k^2}$$

- Blue ball from left urn and red ball from right urn are selected with probability

$$\frac{j}{k} \cdot \frac{j}{k} = \frac{j^2}{k^2}$$

- Red ball from left urn and blue ball from right urn are selected with probability

$$\frac{k-j}{k} \cdot \frac{k-j}{k} = \frac{(k-j)^2}{k^2}$$

- Red ball from left urn and red ball from right urn are selected with probability

$$\frac{k-j}{k} \cdot \frac{j}{k} = \frac{j(k-j)}{k^2}$$

So, the transition probabilities for $X_n = j$ are

$$\begin{aligned} P(X_{n+1} = j | X_n = j) &= \frac{2j(k-j)}{k^2} \\ P(X_{n+1} = j-1 | X_n = j) &= \frac{j^2}{k^2} \\ P(X_{n+1} = j+1 | X_n = j) &= \frac{(k-j)^2}{k^2} \end{aligned}$$

So the transition matrix is

$$P = \frac{1}{k^2} \begin{pmatrix} states & 0 & 1 & 2 & 3 & ... & k \\ 0 & 0 & k^2 & 0 & 0 & ... & 0 \\ 1 & 1 & 2(k-1) & (k-1)^2 & 0 & ... & 0 \\ 2 & 0 & 4 & 4(k-2) & (k-2)^2 & ... & 0 \\ 3 & 0 & 0 & 9 & 6(k-3) & ... & 0 \\ \vdots & ... & ... & ... & ... & ... & \\ k-1 & 0 & 0 & 0 & 0 & ... & 1 \\ k & 0 & 0 & 0 & ... & k^2 & 0 \end{pmatrix}$$

4 2.13

The state space is the set of all permutations.
So the transition matrix is

$$P = \begin{pmatrix} states & abc & acb & bac & bca & cab & cba \\ abc & P_a & P_c & P_b & 0 & 0 & 0 \\ acb & P_b & P_a & 0 & 0 & P_c & 0 \\ bac & P_a & 0 & P_b & P_c & 0 & 0 \\ bca & 0 & 0 & P_a & P_b & 0 & P_c \\ cab & 0 & P_a & 0 & 0 & P_c & P_b \\ cba & 0 & 0 & 0 & P_b & P_a & P_c \end{pmatrix}$$

5 2.14

5.1

The state of transition matrix be $\{0, 1, 2, ..., k\}$ which denotes the numbers q unique songs played.
The transition probability from state m to state $m+1$ is $\frac{k-m}{k}$; the transition probability from state m to state m is $\frac{m}{k}$.
So, we have

$$\begin{aligned} &P(X_{m+1} = j | X_m = x_m, X_{m-1} = x_{m-1}, ..., X_0 = 0 \\ &= P(X_{m+1} = j | X_m = x_m), \end{aligned}$$

so $X_0, X_1, ...$ is a Markov chain. the transition matrix is

$$P = \begin{pmatrix} states & 0 & 1 & 2 & ... & k-1 & k \\ 0 & 0 & 1 & 0 & ... & 0 & 0 \\ 1 & 0 & \frac{1}{k} & \frac{k-1}{k} & ... & 0 & 0 \\ 2 & 0 & 0 & \frac{2}{k} & ... & 0 & 0 \\ \vdots & ... & ... & ... & ... & 0 & 0 \\ k-1 & 0 & 0 & 0 & ... & \frac{k-1}{k} & \frac{1}{k} \\ k & 0 & 0 & 0 & ... & 0 & 1 \end{pmatrix}$$

5.2

For $k = 4$, the transition matrix is

$$P = \begin{pmatrix} states & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 2 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 4 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

then, we can calculate that

$$P^6 = \begin{pmatrix} states & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0.00098 & 0.09082 & 0.52734 & 0.38086 \\ 1 & 0 & 0.00024 & 0.04614 & 0.44092 & 0.51270 \\ 2 & 0 & 0 & 0.01562 & 0.32471 & 0.65967 \\ 3 & 0 & 0 & 0 & 0.17798 & 0.82202 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

therefore,

$$P(X_6 = 4 | X_0 = 0) = P^6_{0,4} = 0.38086.$$