# Iterative Hessian Sketch for Constrained Least-Squares

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# Problem description

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- Idea: "sketch" $A$ using $SA$;
- $S \in \mathbb{R}^{m \times n}$: sketching matrix; $\mathbb{E} S^\top S = I_n$.

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- random row sampling: randomly select $m$ rows from $I_n$ (rescaled by $\sqrt{\frac{n}{m}}$);
- randomized orthogonal system: randomly select $m$ rows from an orthogonal matrix $H$ (rescaled by $\sqrt{\frac{n}{m}}$);

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- We expect the optimal method satisfies $m \approx O(p)$ Think about this part more carefully?;

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- *An iterative idea for improvement*: "$\|x^{t+1} - x_{ls}\| \leq \rho \|x^t - x_{ls}\|$"?

A detailed idea:

- Given $x^t$, a modified least square with $x_{ls} - x^t$ as minimizer:

$$\min_u \frac{1}{2}\|A(u + x^t)\|_2^2 - \langle A^\top y, (u + x^t)\rangle$$

# Iterative Hessian Sketch

A detailed idea:

- Given $x^t$, a modified least square with $x_{ls} - x^t$ as minimizer:

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- Property of Hessian sketch: $\|u^{t+1} - (x_{ls} - x^t)\| \leq \rho\|x_{ls} - x^t\|$;
- Define $x^{t+1} = x^t + u^{t+1}$, then:

$$\|x^{t+1} - x_{ls}\| \leq \rho\|x^t - x_{ls}\|, \quad \text{w.h.p.}$$

# Iterative Hessian Sketch

## Algorithm

1. Initialize $x^0 = 0$.
2. For $t = 1, ...N$, generate independent sketches $S^t \in \mathbb{R}^{m \times n}$ and

$$x^t = \arg\min_{x \in C} \frac{1}{2} \|S^t A(x - x^{t-1})\|_2^2 - \langle A^T(y - Ax^{t-1}), x - x^{t-1} \rangle$$

3. Return $\hat{x} = x^N$.

# Iterative Hessian Sketch

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- Optimality:
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- Optimality:
  - To achieve the same bound as $\|x_{ls} - x_*\| \approx O(\frac{1}{\sqrt{n}})$, we need $N = log(n)$ iterations;
  - Union bound: $\|x^N - x_{ls}\| \leq \rho^N \|x_{ls}\|$, *w.h.p.*.

# Simulations in the paper

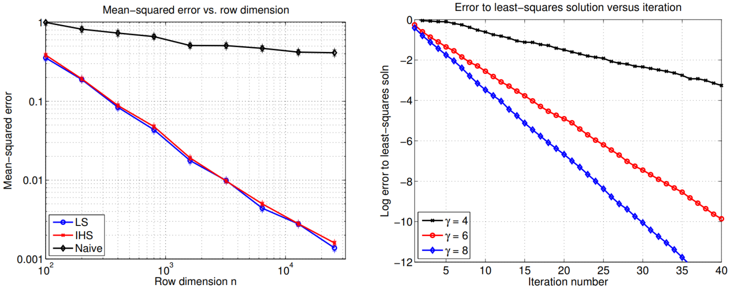


Figure : Left: classical sketch vs. iterative hessian sketch; Right: different choices of $m = \gamma p$.

# Simulations - different sketch matrices

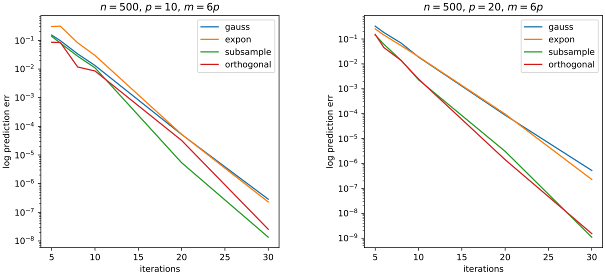


Figure : Using different sketch matrices, under Gaussian design. “expon” means Laplace design.

# Simulations - model mis-specifications

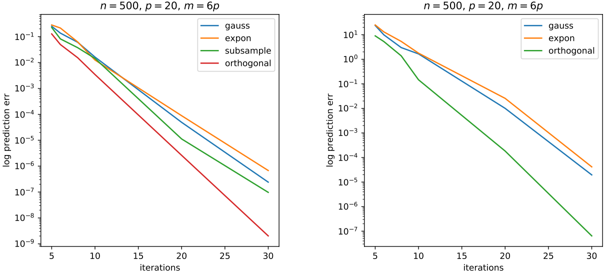


Figure : Left: exponential design; Right: Laplace design.

# Simulations - model mis-specifications cont.

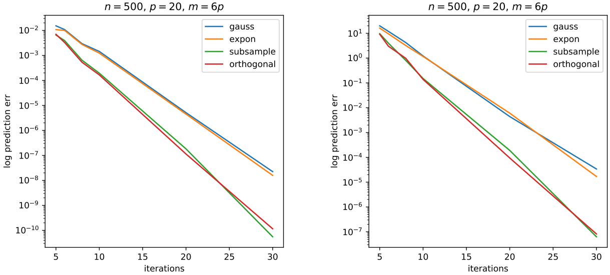


Figure : Left: $y = \sin(Ax) + w$; Right: $y = (Ax)^3 + w$. Both under Gaussian design. Large noise $w$ behabves similarly.

Thank you!

# Evaluation for Sketch Performance

## Cost Approximation

In terms of $f$-cost, the approximated $\tilde{x}$ is said to be $\epsilon$-optimal if

$$f(x^{LS}) \leq f(\tilde{x}) \leq (1+\epsilon)^2 f(x^{LS}).$$

## Solution Approximation

In terms of $f$-cost, the prediction norm defined as

$$\|\tilde{x} - x^{LS}\|_A := \frac{1}{\sqrt{n}} \|A(\tilde{x} - x^{LS})\|_2.$$