Iterative Hessian Sketch for Constrained Least-Squares

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Problem description

- Consider the **constrained least-squares**

$$x_{ls} := \arg\min_{x \in C} \; \frac{1}{2}\|Ax - y\|_2^2$$

where C: constraint; $A \in \mathbb{R}^{n \times p}$: design matrix; y: response.

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- $S \in \mathbb{R}^{m \times n}$: sketching matrix; $\mathbb{E}S^\top S = I_n$.

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- random row sampling: randomly select m rows from I_n (rescaled by $\sqrt{\frac{n}{m}}$);
- randomized orthogonal system: randomly select m rows from an orthogonal matrix H (rescaled by $\sqrt{\frac{n}{m}}$);

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- We expect the optimal method satisfies $m \approx O(p)$ Think about this part more carefully?;

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- Only sketch A, but not y:

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$$x_{hs} := \arg\min_{x \in C} \ \frac{1}{2}\|SAx\|_2^2 - \langle A^\top y, x \rangle.$$

- Still **sub-optimal** in the same sense;
- The above serves as the building block for an iterative method

- Idea: construct a new least-squares problem for which the optimal solution is $x^{LS} - x^t$. Then applying HS to the problem will produce a new x^{t+1} whose distance to x^{LS} has been reduced by a factor of ρ.
- $\hat{u} := argmin_{u \in \{C - x^t\}} \{ \frac{1}{2} \|Au\|_2^2 - \langle A^T(y - Ax^t), u \rangle \}$.
  By construction, $\hat{u} = x^{LS} - x^t$.
- Advantages: sample size efficiency and computational time saving

Iterative Hessian Sketch Algorithm

Algorithm

1. Initialize $x^0 = 0$.
2. For iterations $t = 0, 1, 2, ... N - 1$, generate independent sketches $S^{t+1} \in \mathbb{R}^{m \times n}$, and perform the updates
$$x^{t+1} = argmin_{x \in C} \{ \frac{1}{2m} \| S^{t+1} A(x - x^t) \|_2^2 - \langle A^T(y - Ax^t), x \rangle \}.$$
3. Return $\hat{x} = x^N$.

References

📄 Sarlos (2006)

Title of the publication

Journal Name 12(3), 45 – 678.

Evaluation for Sketch Performance

Cost Approximation

In terms of f-cost, the approximated $\tilde{x}$ is said to be ϵ-optimal if

$$f(x^{LS}) \leq f(\tilde{x}) \leq (1 + \epsilon)^2 f(x^{LS}).$$

Solution Approximation

In terms of f-cost, the prediction norm defined as

$$\|\tilde{x} - x^{LS}\|_A := \frac{1}{\sqrt{n}}\|A(\tilde{x} - x^{LS})\|_2.$$

Thank you!