

Lecture 3: Logistic Regression

Week 3

Lecturer: Tianyu Wang

1 Surrogate Functions for the Classification Objective

Recall the goal of classification is to find a function f in a hypothesis class \mathcal{H} to fit a given dataset $\{(x_i, y_i)\}_{i=1}^n$ ($x_i \in \mathbb{R}^d, y_i \in \{+1, -1\}$). The optimization objective (empirical risk) for this problem is

$$\min_{f \in \mathcal{H}} \sum_{i=1}^n \mathbb{I}_{[f(x_i) \neq y_i]},$$

which can be written as

$$\min_{f \in \mathcal{H}} \sum_{i=1}^n \mathbb{I}_{[y_i f(x_i) < 0]}.$$

Minimizing this empirical risk is not easy, since it is highly discontinuous and we cannot do much better than trial-and-error. Luckily, we can use surrogate functions, that are more friendly to work with, to approximate this objective. See Figure 1 for an illustration.

2 Logistic Regression

2.1 The surrogate objective

Consider a linear classifier $f(x) = \begin{cases} +1, & \text{if } w^\top x + b \geq 0 \\ -1, & \text{otherwise.} \end{cases}$ Here we use a positive tie-breaking.

The empirical risk for this classifier is

$$\sum_{i=1}^n \mathbb{I}_{[y_i(w^\top x_i + b) < 0]}.$$

If we use the logistic loss, the surrogate objective is

$$l(w, b) = \sum_{i=1}^n \log(1 + \exp(-y_i(w^\top x_i + b))).$$

With this surrogate objective, one can solve for w and b using gradient-based methods, such as gradient descent. We will briefly discuss some optimization methods later in this course.

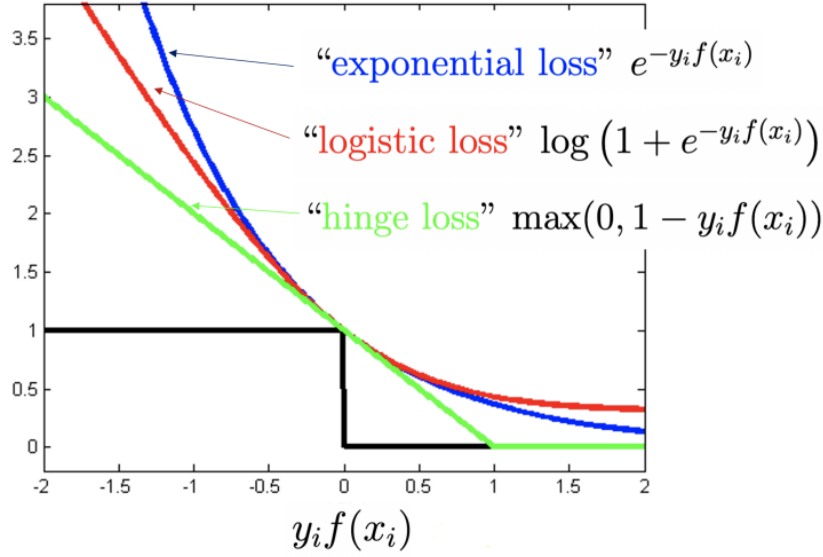


Figure 1: Surrogate functions for the objective. Picture Credit: C. Rudin.

2.2 A probabilistic perspective

Let's say the labels (y_i 's) are generated from a Bernoulli distribution:

$$y \sim \text{Bernoulli}(\mathbb{P}(y = 1|x, w, b)).$$

The Bernoulli distribution is fully determined by the success probability, which is a number in $[0, 1]$. This means we cannot directly impose a linear model on the success probability, since a non-trivial linear function takes values outside of $[0, 1]$. To handle this, we consider the following equation

$$\log \frac{\mathbb{P}(y = 1|x, w, b)}{1 - \mathbb{P}(y = 1|x, w, b)} = w^\top x + b.$$

Note that both sides of the above equation can take values in $(-\infty, \infty)$. Solving for $\mathbb{P}(y = 1|x, w, b)$ in the above equation gives

$$\mathbb{P}(y = 1|x, w, b) = \frac{\exp(w^\top x + b)}{1 + \exp(w^\top x + b)}.$$

For both $y_i = 1$ and $y_i = -1$, the probability can be expressed as

$$\mathbb{P}(y|x, w, b) = \frac{1}{1 + \exp(-y(w^\top x + b))}.$$

Exercise. Verify the above fact.

With this probability, the likelihood is

$$L(w, b | \{(x_i, y_i)\}_{i=1}^n) = \prod_{i=1}^n \mathbb{P}(y_i | x_i, w, b) = \prod_{i=1}^n \frac{1}{1 + \exp(-y_i(w^\top x_i + b))}.$$

The log likelihood is

$$\log L(w, b | \{(x_i, y_i)\}_{i=1}^n) = - \sum_{i=1}^n \log(1 + \exp(-y_i(w^\top x_i + b))).$$

Again, maximizing the log-likelihood is equivalent to minimizing the logistic loss.

3 ROC Curve and Area Under Curve

We will use slides by Rudin for this part. Recall the confusion matrix.

Acknowledgement

TW used lecture notes by Cynthia Rudin to compile this notes.