The SIMPLE Algorithm
for Pressure-Velocity Coupling

ME 448/548 Notes

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Overview

1. Basic ideas
2. Mesh ideas: structured versus unstructured, and staggered
3. SIMPLE: a classic segregated solution method for incompressible flows
4. Convergence checking
5. Summary and recommendations

Basic Ideas: High-level View of CFD

CFD modeling involves these steps

1. Define the geometry of the fluid volume
2. Locate boundary regions and assign values to boundary condition parameters
3. Specify thermophysical properties and physics, e.g., buoyancy, turbulence, radiation
4. Create the mesh
5. Define parameters that control the solution
6. Solve the flow model
7. Vizualize the results

Commercial CFD packages have preprocessors (graphical user interfaces) for setting up the model and viewing the results. The order of preprocessing steps may be different for different CFD packages.

Users control step 6 by specifying parameters the affect how the solver updates the velocity, pressure and other scalar fields. That control is indirect, at best.
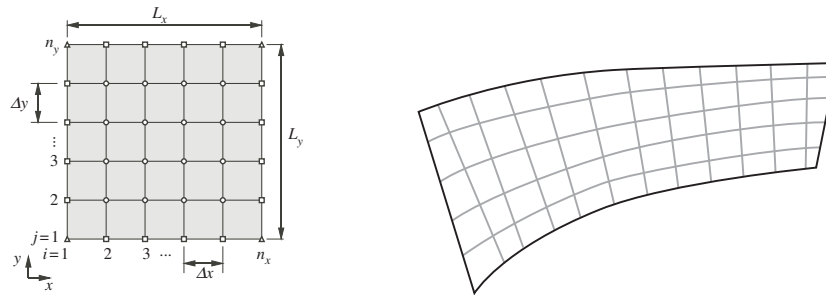
Meshes:

Structured,

Unstructured,

and Staggered

Structured and Unstructured Meshes

The SIMPLE algorithm can be developed for non-staggered, unstructured meshes. We will use the case of a staggered, structured mesh because the nomenclature is simpler. It is also the historical origin of the method.
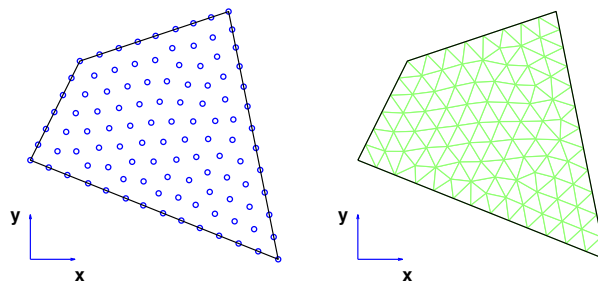
Structured Meshes: Cells (nodes) are arranged in rows and columns

Each cell has the same number of neighbors. The connectivity is regular and uniform.
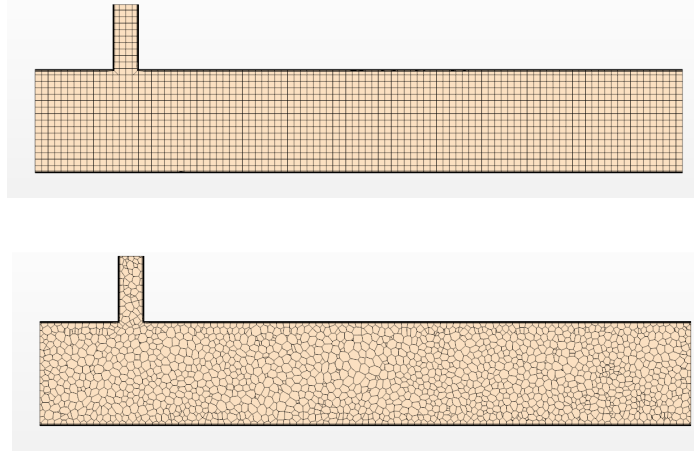

Structured and Unstructured Meshes

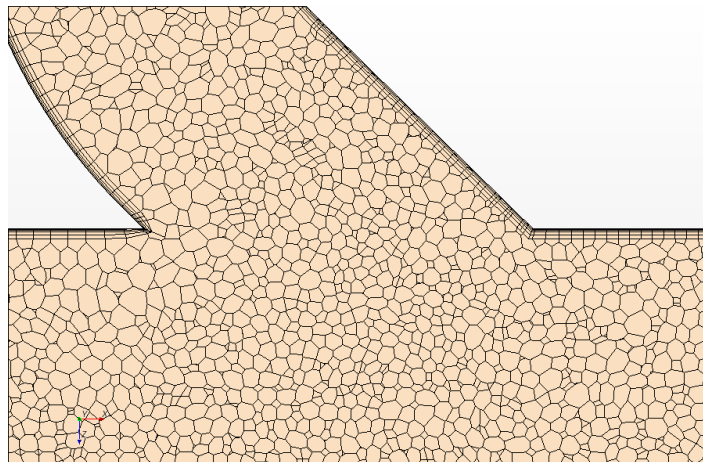**Unstructured Meshes**: Cells and nodes are not arranged in rows and columns.

Structured and Unstructured Meshes
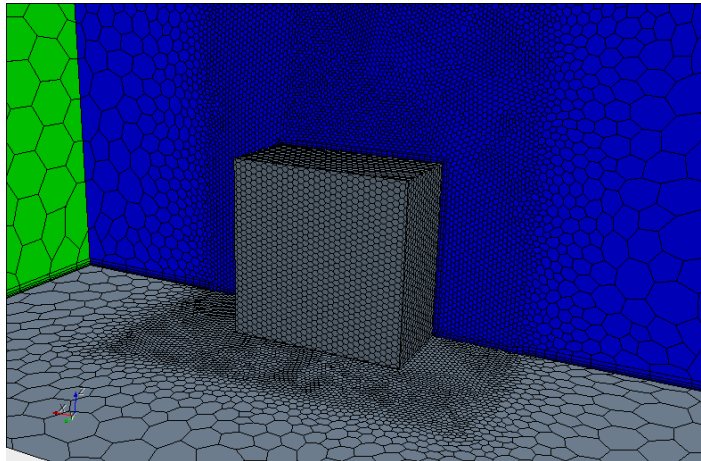
Trimmer and Polyhedral volume meshes in STAR-CCM+

Structured and Unstructured Meshes

Unstructured Meshes: Cells and nodes can have arbitrary connectivity.

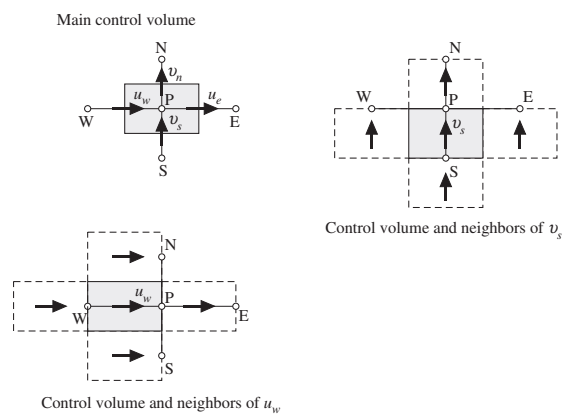# Structured and Unstructured Meshes

Unstructured Meshes: Connectivity can vary with position in the domain.
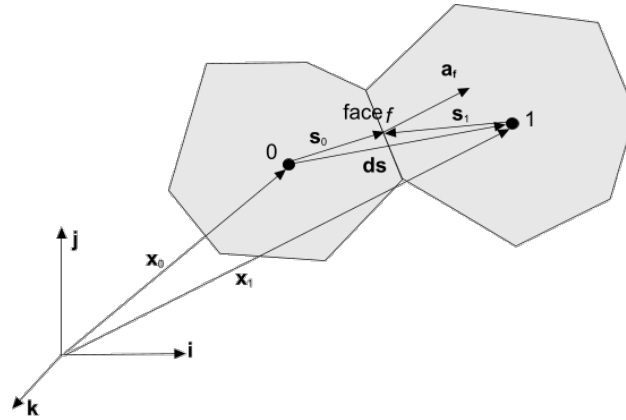

Staggered Mesh

Main characteristics

- Velocities are computed at cell faces
- Pressure is computed at cell centers

Main control volume

Control volume and neighbors of v_s

Control volume and neighbors of u_w

General Polyhedral Mesh Elements in STAR-CCM+

See Theory $\rightarrow$ Numerical Flow Solution $\rightarrow$ Finite Volume Discretization in the on-line STAR-CCM+ documentation ("Help" menu).

Basic Discretization – 2D (1)

For a two-dimensional, incompressible flow, applying the finite volume method on a staggered mesh leads to a set of three equations for each interior control volume.
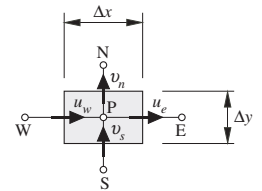
$$-a_S^u u_{i,j-1} - a_W^u u_{i-1,j} + a_P^u u_{i,j} - a_E^u u_{i+1,j} - a_N^u u_{i,j+1} = \frac{p_{i-1,j} - p_{i,j}}{\delta x_w} + b_o^{(u)}$$

$$-a_S^v v_{i,j-1} - a_W^v v_{i-1,j} + a_P^v v_{i,j} - a_E^v v_{i+1,j} - a_N^v v_{i,j+1} = \frac{p_{i,j-1} - p_{i,j}}{\delta y_s} + b_o^{(v)}$$

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta x} + \frac{v_{i,j+1} - v_{i,j}}{\Delta y} = 0$$

where $b_o^{(u)}$ and $b_o^{(v)}$ are source terms due to non-uniform viscosity, gravity, and other effects.

Basic Discretization – 2D (2)

These equations are nonlinear. For example, if the central difference scheme is used,

$$a_E^u = \mathcal{F}\left(u_{i,j},\, u_{i+1,j},\, \nu,\, \text{geometry}\right)$$

The solution process – even when a coupled solver is used – requires these basic steps

1. Make a guess at the current flow field
2. Compute the coefficients of the (now) linearized momentum and continuity equations
3. Solve the equations

These steps are repeated until the guessed flow field does not result in a change in the coefficients.
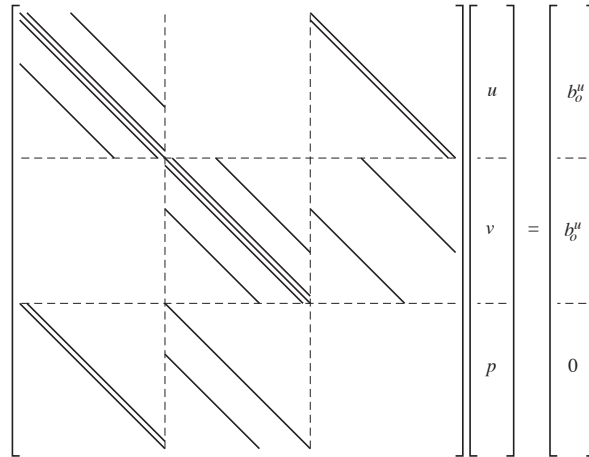
Basic Discretization – 2D (3)

Recall the structure of the coefficient matrix for the finite-volume approximation to the 2D Poisson equation.
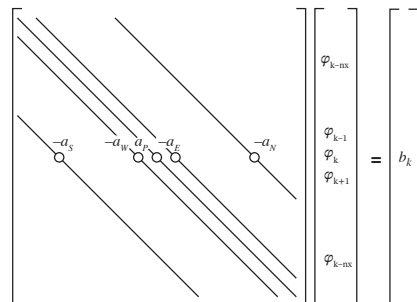
Applying the finite-volume method to the 2D incompressible flow equations give three matrices with this structure. However, the unknowns for each of these matrices are coupled. (Actually, there's a bit of a problem with the pressure equation.)

Basic Discretization – 2D (4)

The coupled system of equations looks like this for a staggered, structured, 2D mesh.
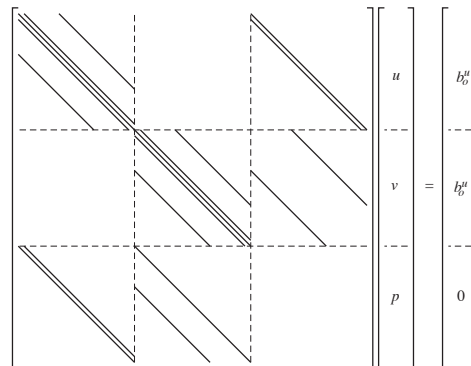
$$
\begin{bmatrix} \ddots \end{bmatrix}
\begin{bmatrix} u \\ v \\ p \end{bmatrix}
=
\begin{bmatrix} b_o^u \\ b_o^u \\ 0 \end{bmatrix}
$$

Diagonal lines in the coefficient matrix represent the position of non-zero entries.

Coupled or Segregated Solution

We have a two basic choices

1. Solve the large sparse system as a single, linearized system

2. Use sequential, a.k.a. *segregated* method to *separately* solve each linearized system for the dependent vectors u, v, and p.

$$
\begin{bmatrix} \ddots \end{bmatrix}
\begin{bmatrix} u \\ v \\ p \end{bmatrix}
=
\begin{bmatrix} b_o^u \\ b_o^u \\ 0 \end{bmatrix}
$$

SIMPLE uses the segregated approach

Refer to the STAR-CCM+ **documentation**

Theory $\rightarrow$ Numerical Flow Solution
$\quad\quad\rightarrow$ Finite Volume Discretization
$\quad\quad\quad\rightarrow$ Segregated Flow Solver
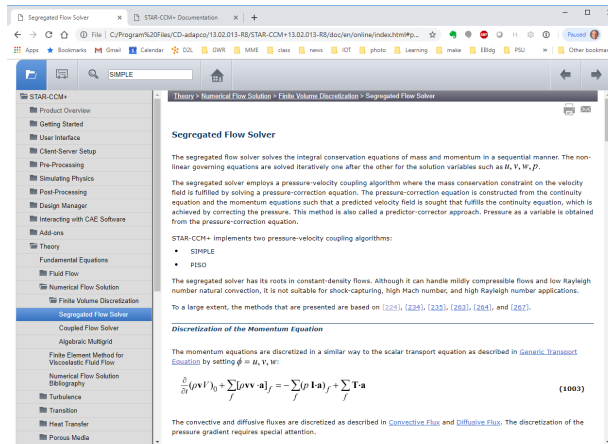
Segregated Solution Procedure (1)

SIMPLE: **S**emi-**I**mplicit **M**ethod for **P**ressure-**L**inked **E**quations

Preview: For laminar flow in a 2D domain

Unknowns: u, v, p: x-direction velocity, y-direction velocity and pressure

1. Solve u equation assuming v and p are frozen
2. Solve v equation assuming u and p are frozen
3. Solve a *pressure correction* equation for p that enforces continuity
4. Update u, v and p from the pressure correction

References: Ferziger and Perić, §,7.5.1, pp. 188–196

Patankar

Tu, Yeoh, and Liu, §4.3.3

Segregated Solution Procedure (1)

Apply the finite-volume discretization separately to each momentum equation

- For the u equation, v and p are frozen

$$A^{(u)}u = b^{(u)}$$

- For the v equation, u and p are frozen

$$A^{(v)}v = b^{(v)}$$

Each of these matrix equations is a linearized version of the finite-volume model.

The momentum source terms, $b^{(u)}$ and $b^{(v)}$, consist of pressure terms and other terms due to velocity gradients and non-uniform viscosity.

$$b^{(u)} = b_o^{(u)} - b_p^{(u)} \qquad b^{(v)} = b_o^{(v)} - b_p^{(v)}$$

$b_p^{(u)}$ and $b_p^{(v)}$ are contributions due to the frozen pressure field (p^*).

Segregated Solution Procedure (2)

Discretization of momentum equations results in a non-linear, coupled system of equations for u and v (in 2D).

$$A^{(u)}u = b^{(u)} \qquad A^{(v)}v = b^{(v)}$$

- $A^{(u)}$ is the coefficient matrix, and $b^{(u)}$ is the source term (vector) for the u equation when v and p are frozen.
- $A^{(v)}$ is the coefficient matrix, and $b^{(v)}$ is the source term (vector) for the v equation when u and p are frozen.
- Momentum equations are nonlinear, so $A^{(u)}$ and $A^{(v)}$ depend on u and v.
- Nonlinearity requires iterative solution: $A^{(u)}$ and $A^{(v)}$ are updated with new values of u and v, and the systems are solved again.

What about the continuity equation?

Where is the equation for pressure?

Pressure in Momentum Source Terms

Pressure is the dominant source term in the momentum equations

$$A^{(u)}u = b^{(u)} \quad \text{where} \quad b^{(u)} = b_P^{(u)} + b_o^{(u)} \sim d_u(p_W - p_P) + \ldots + b_o^{(u)}$$

$$A^{(v)}v = b^{(v)} \quad \text{where} \quad b^{(v)} = b_P^{(v)} + b_o^{(v)} \sim d_v(p_S - p_P) + \ldots + b_o^{(v)}$$

The d_u and d_v are coefficients that depend on the geometry of of the control volume.

Note: The $\sim$ symbol is meant to convey the basic features of the terms, and allows a degree of notational abuse. To make these expressions more precise, read the $\sim$ symbol as "contains terms like".

<div align="center">

**Pressure does not have its own *natural*
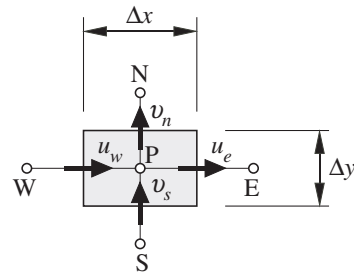equation for incompressible flow.**

</div>

Pressure Correction and Continuity (1)

Solution to the linearized momentum equations

$$A^{(u)}u = b^{(u)} \qquad A^{(v)}v = b^{(v)}$$

gives updated u and v fields.

In general these new u and v fields will not satisfy the continuity equation for each cell

$$u_w\Delta y - u_e\Delta y + v_s\Delta x - v_n\Delta x \neq 0$$

Pressure Correction and Continuity (2)

Assume that we can find a *velocity correction* at each node that adjusts the velocity field so that it satisfies the continuity equation.

$$u_w = \tilde{u}_w + u'_w \tag{1}$$

$$v_s = \tilde{v}_s + v'_s \tag{2}$$

- u_w and v_s are the velocity components that satisfy the continuity equation.
- $\tilde{u}_w$ and $\tilde{v}_s$ are the velocity components obtained by satisfying the momentum equations.
- u'_w and v'_s are the corrections to the velocity components needed to satisfy the continuity equation.

Equations (1) and (2) define u'_w, and v'_s values *for each cell* in the domain. The $\tilde{u}_w$ and $\tilde{v}_s$ fields are obtained by solving the momentum equations. We need to develop another procedure for computing the u'_w and v'_s fields. Ultimately the u'_w and v'_s fields are obtained by requiring each cell to satisfy the continuity equation.

Pressure Correction and Continuity (3)

Assume that the velocity corrections can be derived from suitable *pressure corrections*

$$p_P = \tilde{p}_P + p'_P \tag{3}$$

- p_P is the desired pressure at the cell center, i.e., the one that brings the velocity field into balance so that continuity equation is satisfied for each cell.
- $\tilde{p}_P$ is the current guess at the pressure at the cell center.
- p'_P is the change in $\tilde{p}_P$ needed to satisfy the continuity equation.

Pressure Correction and Continuity (4)

Unifying assumption:

Assume that the velocity corrections are uniquely determined by the pressure corrections.

$$u'_w \approx c_w(p'_W - p'_P) \tag{4}$$

$$v'_s \approx c_s(p'_S - p'_P) \tag{5}$$

where c_w and c_s are coefficients that depend on terms from the momentum equation.

(We've skipped a couple of steps here.)

Pressure Correction and Continuity (5)

Tying the velocity corrections to the pressure corrections allows us to close the system of equations. An equation for the p' is obtained by substituting the Equations (1), (2), (4), and (5) into the continuity equation. The result is

$$A^{(p')}p' = b^{(p')}$$

where

$$b^{(p')} \sim \tilde{u}_w \Delta y - \tilde{u}_e \Delta y + \tilde{v}_s \Delta x - \tilde{v}_n \Delta x$$

In words: *The source term for the p' field is the* local *error in the continuity equation that results from updating the u and v fields from the previous iteration to get to $\tilde{u}$ and $\tilde{v}$.*

The Pieces (in 2D)

The momentum equations

$$A^{(u)}u = b^{(u)} \qquad A^{(v)}v = b^{(v)} \qquad (\star)$$

The pressure correction equation

$$A^{(p')}p' = b^{(p')} \qquad (\star\star)$$

Momentum and pressure *corrections*

$$u = \tilde{u} + u' \qquad v = \tilde{v} + v' \qquad p = \tilde{p} + p'$$

$\tilde{u}$ and $\tilde{v}$ are the solutions to Equations ($\star$) obtained when the previous field variables (u, v, and p) are used to compute $A^{(u)}$, $b^{(u)}$, $A^{(v)}$ and $b^{(v)}$. $\tilde{p}$ is the pressure field from the previous global iteration.

p' is the solution to Equation ($\star\star$) when $b^{(p')}$ is computed from the newly obtained $\tilde{u}$ and $\tilde{v}$.

SIMPLE Algorithm

Repeat the following steps until convergence:

1. Compute $A^{(u)}$ and $b^{(u)}$, and solve $A^{(u)}u = b^{(u)}$ to obtain $\tilde{u}$, a tentative guess at the x-direction velocity field.
2. Compute $A^{(v)}$ and $b^{(v)}$, and solve $A^{(v)}v = b^{(v)}$ to obtain $\tilde{v}$, a tentative guess at the y-direction velocity field. Note that $A^{(v)}$ and $b^{(v)}$ use the most recently available $\tilde{u}$ field, and the v field from the previous iteration
3. Compute $A^{(p')}$ and $b^{(p')}$, and solve $A^{(p')}p' = b^{(p')}$ to obtain the pressure correction field p'.
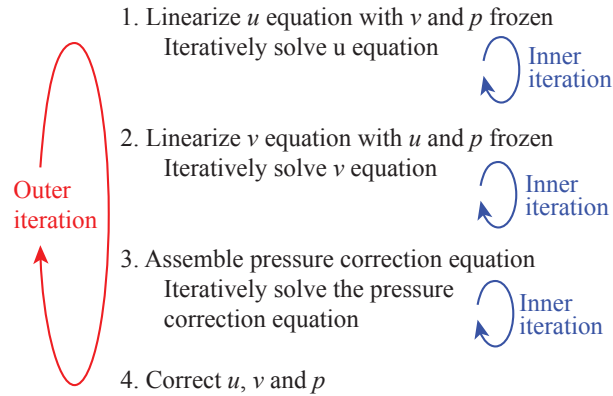4. Correct the velocities and pressures.

$$p_P \leftarrow \tilde{p}_P + p'_P \qquad u_w \leftarrow \tilde{u}_w + u'_w \qquad v_s \leftarrow \tilde{v}_s + v'_s$$

5. Test for convergence, and return to Step 1 if necessary.

The SIMPLE algorithm is described by Patankar [2], and Ferzier and Perić [1, § 7.5.1], and Tu et al.[3, §4.3.3].

Inner and Outer Iterations

The iteration number in the STAR-CCM+ residual plot is the number of outer iterations.

1. Linearize u equation with v and p frozen
   Iteratively solve u equation — Inner iteration

2. Linearize v equation with u and p frozen
   Iteratively solve v equation — Inner iteration

3. Assemble pressure correction equation
   Iteratively solve the pressure correction equation — Inner iteration

4. Correct u, v and p

Outer iteration

Solution of the linearized equations in steps 1, 2 and 3 also use iterative methods (as opposed to direct matrix solution). Those iterations are called "inner iterations".

Convergence Checking (1)

Each outer iteration of the SIMPLE loop, $A^{(u)}$ and $b^{(u)}$ are updated with the newest information about the u, v and p fields.

Before solving

$$A^{(u)}u = b^{(u)} \tag{6}$$

compute the residual

$$r^{(u)} = b^{(u)} - A^{(u)}u \tag{7}$$

Note that the u in these equations is from the previous iteration.

If $r^{(u)}$ is small, then the old u is a good approximation to the system of equations defined by the newly updated coefficients in $A^{(u)}$ and $b^{(u)}$.

Convergence Checking (2)

Similarly,

$$r^{(v)} = b^{(v)} - A^{(v)}v$$

is a measure of how well the old v satisfies the linearized equations based on the newly updated coefficients in $A^{(v)}$ and $b^{(v)}$

Each of the unknown fields, u, v, p (in 2D) will have a residual.

The residual plot in StarCCM+ shows the residuals of each of the dependent vector variables.

Convergence Checking (3)

$r^{(u)}$ and $r^{(v)}$ are residual vectors: there is a residual for each interior cell in the domain.

To check convergence, we need to look at a scalar value, not an entire vector.

Let σ be a normalized scalar value of the residual.

$$\sigma_u = \frac{\|r_u\|}{\|r_{u,\mathrm{ref}}\|}, \qquad \sigma_v = \frac{\|r_v\|}{\|r_{v,\mathrm{ref}}\|}$$

The values of $\|r_{u,\mathrm{ref}}\|$ and $\|r_{v,\mathrm{ref}}\|$ are chosen so that $\sigma_u \sim 1$ and $\sigma_v \sim 1$ on the first iteration, or close to the first iteration.

Convergence Checking (4)

For the pressure correction equation, the residuals are not as informative as the source term

$$b^{(p')} \sim \tilde{u}_w \Delta y - \tilde{u}_e \Delta y + \tilde{v}_s \Delta x - \tilde{v}_n \Delta x$$
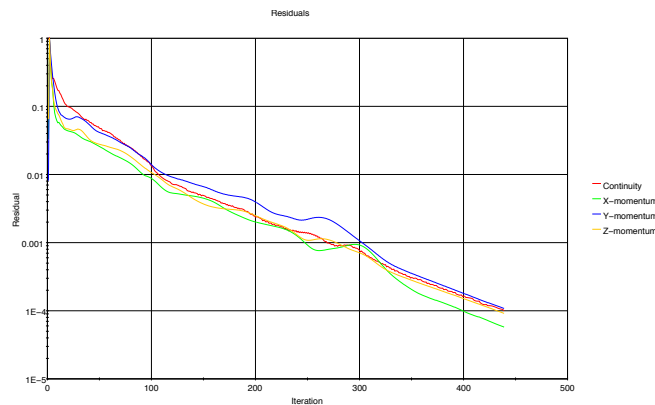
If $b^{(p')}$ is small, then the current guess at the velocity field comes close to satisfying the continuity equation *before* the pressure and velocity corrections are applied.

To check convergence of the continuity equation, monitor σ_c

$$\sigma_c = \frac{\|b^{(p')}\|_\infty}{\|b^{(p')}\|_{\infty,\text{ref}}}$$
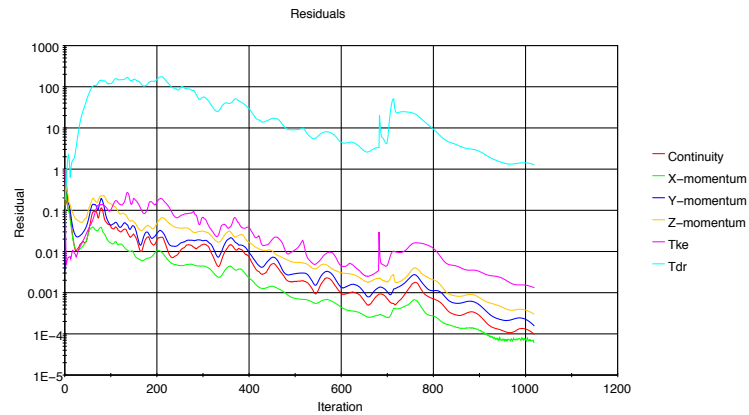
Monitoring Residuals

Convergence history for a well-behaved, laminar flow simulation with STAR-CCM+:

Remember: The iteration number in this plot refers to the *outer iterations*.
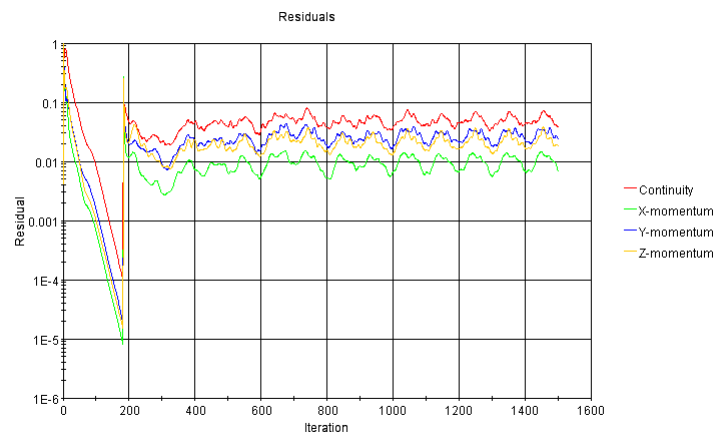
Monitoring Residuals

Convergence history for a *less* well-behaved, turbulent flow simulation with STAR-CCM+:

Monitoring Residuals

Non-convergence for a laminar flow that is unsteady, but the user is trying to use a steady solution method. A flow with lower inlet velocity was used for an initial guess.

Summary

- SIMPLE is a *segregated* solution method: The u, v, w, and p fields are solved separately. Coupling between these field variables is achieved via velocity and pressure corrections.
- Convergence occurs when the residuals for each of the equations is reduced to a value below a tolerance.
- The "residuals" are normalized scalars. The normalization is chosen so that $\sigma \sim 1$ on the first iteration.
- σ_c is the largest local (cell-wise) error in the continuity equation *before* the pressure-based corrections to the velocity fields are applied. After the velocity field is corrected, local mass conservation is obtained for all cells. Thus, at the end of each iteration of SIMPLE, both global *and* local mass conservation is guaranteed.
- By default, StarCCM+ does not use a stopping criterion other than limiting the simulation 10 1000 outer iterations. Turning on convergence checking requires user action.

STAR-CCM+ **Gives You a Choice**

When specifying the Physics Continuum:
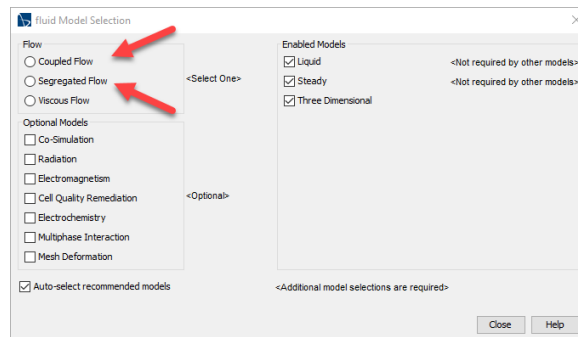
Coupled flow solves the large system of equations for u, v, w and p simultaneously (with an iterative method).

Segregated flow uses SIMPLE for steady problems or PISO for transient problems.

Adding a Convergence Criterion in Star-CCM+

To add a convergence criterion in Star-CCM+

1. Right click on Stopping Criteria
2. Select Create New Criterion→From Monitor. . . and select one of the field variables.

I recommend using *Continuity* as a baseline convergence criterion.

For hard problems, one of the turbulence field variables or temperature might also be used as a convergence criterion.

The same convergence criteria can be used with coupled and segregated solvers.
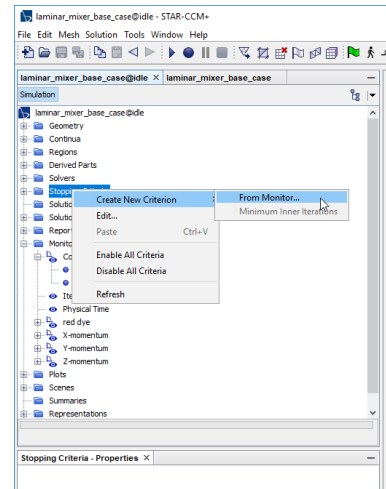
Recommendations

- *Always* inspect the residual plot after each run.
- If the residual tolerances have not been met, continue the solution by increasing the maximum allowable number of steps. This approach continues the solution from the last iteration.
- For some hard problems, other adjustments to the solution parameters might be necessary in order obtain convergence.
- STAR-CCM+ also has a *coupled flow solver*, which may *or may not* converge better (and faster) than the segregated solver. The coupled flow solver uses more memory than the segregated solver.

References

[1] Joel H. Ferziger and Milovan Perić. *Computational Methods for Fluid Dynamics*. Springer-Verlag, Berlin, third edition, 2001.

[2] S.V. Patankar. *Numerical Heat Transfer and Fluid Flow*. Hemisphere, Washington D.C., 1980.

[3] Jiyuan Tu, Guan Heng Yeoh, and Chaoqun Liu. *Computational Fluid Dynamics: A Practical Approach*. Butterworth-Heinemann, 2008.