拉普拉斯算子的研究

傅秋桃
(郧阳师范高等专科学校 数学系，湖北 丹江口 442700)

[摘 要]在许多的教科书中，通过利用直角坐标和极坐标的转换，然后再应用复合函数的求导法则，得出了拉普拉斯算子的表达式，而本文则用变分原理导出了柱面坐标下的拉普拉斯算子的表达式.

[关键词]拉普拉斯算子；变分法；格林函数
[中图分类号]O13 [文献标识码]A [文章编号]1008—6072(2008)03—0013—02

首先介绍后面用到的两个公式：

格林第一公式：

$$\iiint_{\Omega} u\triangle v d\Omega=\iint_{\Gamma} u\frac{\partial v}{\partial n}ds-\iiint_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right)d\Omega$$

格林第二公式：

$$\iiint_{\Omega}(u\triangle v-v\triangle u)d\Omega=\iint_{\Gamma}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds$$

调和方程(又称拉普拉斯方程)为：

$$\triangle u\equiv\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

其中拉普拉斯算子在柱面坐标(r,θ,z)下可以写成

$$\triangle u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial z^2}.$$

下面用变分法求出柱面坐标下的拉普拉斯算子的表达式：

证明：令 $E(u)=\frac{1}{2}\iiint_{\Omega}|\nabla u|^2 dxdydz$

则 $\left.\frac{dE(u+\varepsilon w)}{d\varepsilon}\right|_{\varepsilon=0}=-\iiint_{\Omega}\triangle u\cdot w dxdydz$

$$\nabla u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$$

取$\vec{e_1},\vec{e_2},\vec{e_3}$为$R^3$中相互垂直的单位向量

$$\nabla u=\frac{\partial u}{\partial\vec{e_1}}\vec{e_1}+\frac{\partial u}{\partial\vec{e_2}}\vec{e_2}+\frac{\partial u}{\partial\vec{e_3}}\vec{e_3}$$

$$\begin{cases}\vec{e_1}=a_{11}\vec{i}+a_{12}\vec{j}+a_{13}\vec{k}\\ \vec{e_2}=a_{21}\vec{i}+a_{22}\vec{j}+a_{23}\vec{k}\\ \vec{e_3}=a_{31}\vec{i}+a_{32}\vec{j}+a_{33}\vec{k}\end{cases}$$

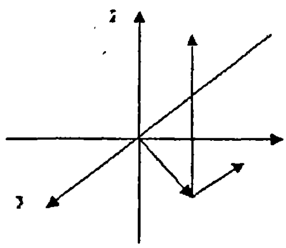
$$A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$$

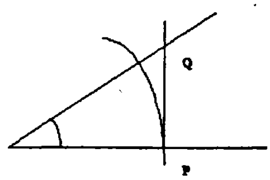
$$AA^T=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\xi=a_{11}x+a_{12}y+a_{13}z$$
$$\eta=a_{21}x+a_{22}y+a_{23}z$$
$$\zeta=a_{31}x+a_{32}y+a_{33}z$$

$$\text{右边}=\frac{\partial u}{\partial\xi}\vec{e_1}+\frac{\partial u}{\partial\eta}\vec{e_2}+\frac{\partial u}{\partial\zeta}\vec{e_3}$$

$$=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)A^TA\begin{pmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{pmatrix}$$





$$\frac{\partial u}{\partial\vec{e_1}}=\frac{\partial u}{\partial r},\frac{\partial u}{\partial\vec{e_3}}=\frac{\partial u}{\partial z}$$

$$\frac{\partial u}{\partial\vec{e_2}}=\lim_{Q\to P}\frac{u(Q)-u(P)}{|PQ|}=\lim_{Q\to P}\frac{u(Q)-u(P)}{|PQ|}$$

$$=\lim_{P\to Q}\frac{u(r,\theta+\triangle\theta,z)-u(r,\theta,z)}{r\triangle\theta}=\frac{1}{r}\frac{\partial u}{\partial\theta}$$

[收稿日期]2008—04—13
[作者简介]傅秋桃(1968—)，男，湖北汉川人，郧阳师范高等专科学校数学系讲师，主要从事应用数学研究.

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$$\nabla u=\frac{\partial u}{\partial\vec{e_1}}\vec{e_1}+\frac{\partial u}{\partial\vec{e_2}}\vec{e_2}+\frac{\partial u}{\partial\vec{e_3}}\vec{e_3}$$

$$|\nabla u|^2=\left|\frac{\partial u}{\partial\vec{e_1}}\right|^2+\left|\frac{\partial u}{\partial\vec{e_2}}\right|^2+\left|\frac{\partial u}{\partial\vec{e_3}}\right|^2$$

$$=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2$$

$$E(u)=\frac{1}{2}\iiint_\Omega|\nabla u|^2dxdydz$$

$$=\frac{1}{2}\iiint_\Omega\left(\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right)rdrd\theta dz$$

$$\frac{d}{d\varepsilon}E(u+\varepsilon w)\Big|_{\varepsilon=0}$$

$$=\iiint_\Omega\left(\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial u}{\partial\theta}\frac{\partial w}{\partial\theta}+\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right)rdrd\theta dz$$

$$=\iiint_\Omega\left(\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}w\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}w\right)+\frac{\partial}{\partial z}\left(r\frac{\partial u}{\partial z}w\right)\right)$$

$$drd\theta dz-\iiint_\Omega\left(\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r}\frac{\partial^2u}{\partial\theta^2}+r\frac{\partial^2u}{\partial^2z}\right)wdrd\theta dz$$

其中

$$\iiint_\Omega\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}w\right)drd\theta dz$$

$$=\iint d\theta dz\int_0^{r(\theta,z)}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}w\right)dr=0$$

$$\iiint_\Omega\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}w\right)drd\theta dz$$

$$=\iint drdz\int\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}w\right)d\theta=0\text{（周期函数）}$$

$$\iiint_\Omega\frac{\partial}{\partial z}\left(r\frac{\partial u}{\partial z}w\right)drd\theta dz$$

$$=\iint drd\theta\int_0^{r(\theta,z)}\frac{\partial}{\partial z}\left(r\frac{\partial u}{\partial z}w\right)dz=0$$

所以

$$\frac{d(E+\varepsilon w)}{d\varepsilon}\Big|_{\varepsilon=0}$$

$$=-\iiint_\Omega\left(\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r}\frac{\partial^2u}{\partial\theta^2}+r\frac{\partial^2u}{\partial z^2}\right)wdrd\theta dz$$

$$=-\iiint_\Omega\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+r\frac{\partial^2u}{\partial z^2}\right)wdrd\theta dz$$

因此

$$-\iiint_\Omega\triangle u\cdot wdxdydz$$

$$=-\iiint_\Omega\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+r\frac{\partial^2u}{\partial z^2}\right)wdrd\theta dz$$

对任意的 w 成立，只要 w 有各阶导数，并且在 $\partial\Omega$ 附近等于零，所以有

$$\triangle u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\partial^2u}{\partial z^2}$$

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【编校：胡军福】

A Research on Laplacian Operator

FU Qiu－tao

(Maths Department, Yunyang Teachers' College, Danjiangkou 442700, China)

Abstract: In many textbooks, the formula of Laplacian Operator is obtained by the change of rectangular coordinates and polar coordinates and the derivation rule of compound function. In this paper, the formula of Laplacian Operator in cylindrical coordinates is obtained by the variational principle.

Key words: Laplacian operator; variational method; green function

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