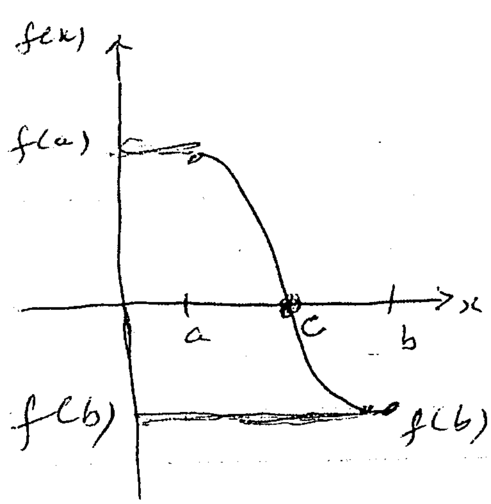
# Lecture 16

Newton's Method to solve $f(x)=0$ 12.6.

Intermediate Value Theorem (IVT)

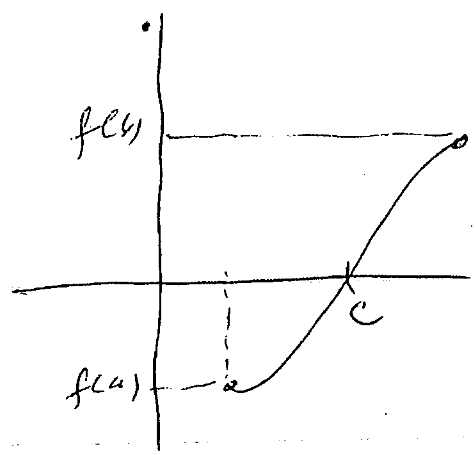
If a function $f$ is continuous on the closed interval $a \le x \le b$, then the function takes on every value between $f(a)$ and $f(b)$ on the open interval $a < x < b$.

Suppose in addition $f(a)$ and $f(b)$ are of opposite signs. Then $0$ lies between them and the IVT says that there is a number $c$ in the open interval such that $f(c)=0$.




$f(a)>0$, $f(b)<0$

$f(c)=0$




$f(b)>0$, $f(a)<0$

$f(c)=0$

let $x = \sqrt{145} > 0$

$f(x) = x^2 - 145 = 0, \ x > 0.$

$\downarrow$

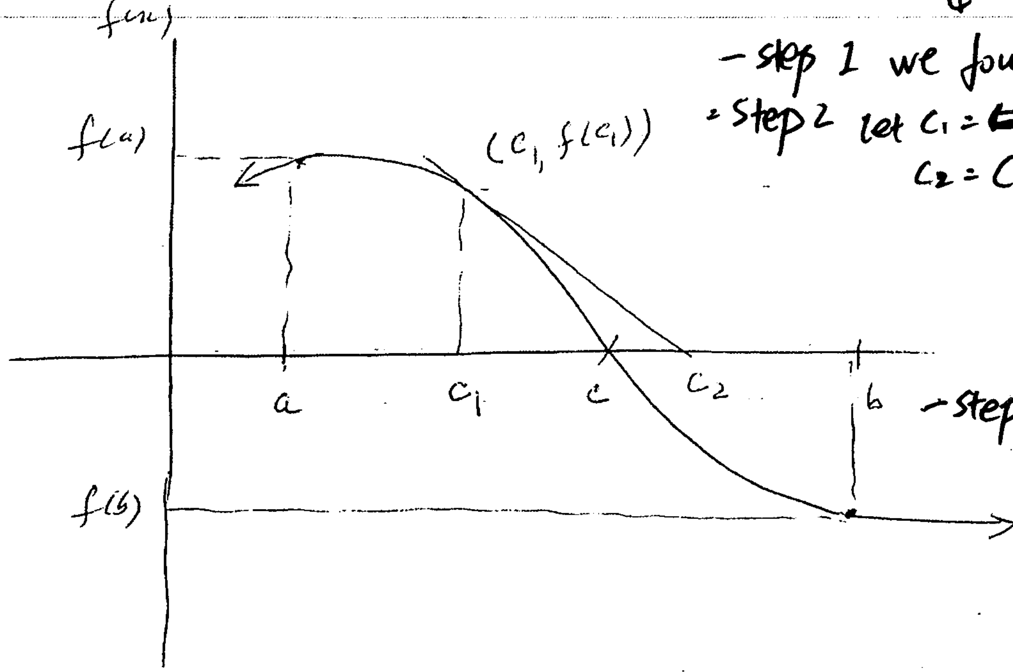
- step 1 we found $12 < \sqrt{145} < 13$.
- Step 2 let $c_1 = 12.1$

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 12.1 - \frac{f(12.1)}{2(12.1)}$$

$$= 12.1 - \frac{12.1^2 - 145}{2(12.1)}$$

$$= 12.041736$$

- step 3 $c_3 = c_2 - \frac{f(c_2)}{f'(c_2)}$




Problem Solve $f(x) = 0$

Step I Find $a$ and $b$ such that $f(a)$ and $f(b)$ are of opposite signs.

Then, there is a root or zero of $f(x)$, $c$, such that $a < c < b$.

Step II Choose an initial approximation for $c$ in $(a, b)$, call it $c_1$. Then $c_2$ is a better approximation to $c$ in $(a, b)$ given by

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)}$$

[ Tangent at $(c_1, f(c_1))$ is

$$y - f(c_1) = f'(c_1)(x - c_1)$$

and its $x$-intercept $c_2$ is found by putting $y = 0$:

$$-f(c_1) = f'(c_1)(x - c_1) \text{ or}$$

$$x - c_1 = -\frac{f(c_1)}{f'(c_1)}$$

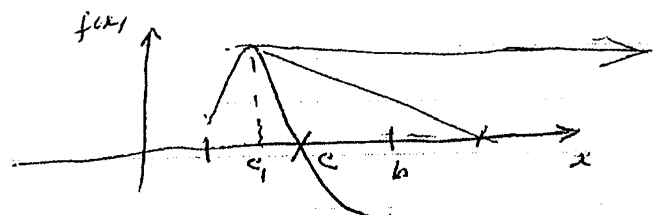
$$x = c_1 - \frac{f(c_1)}{f'(c_1)} \text{ ]}$$

Step III Repeat step II till desired approximation is reached.

Summary

If $c_1$ is the initial approximation of a solution to $f(x)=0$, then nth approximation is given by

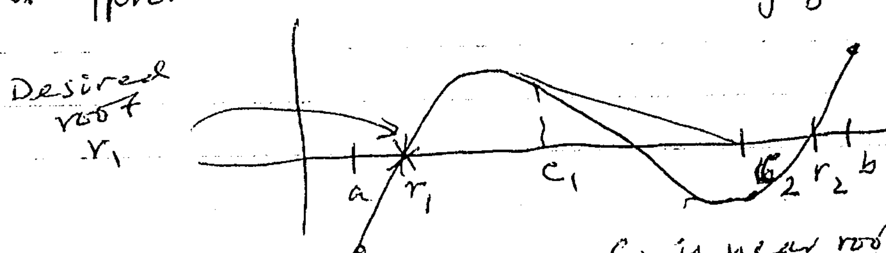
$$c_{n+1} = c_n - \frac{f(c_n)}{f'(c_n)} \quad , n=1,2,3,\ldots$$




Notes

1. Newton's method does not give a better approximation if $f'(c_1)$ is nearly zero and of course does not work at all if $f'(c_1)=0$. In such situations we need to choose a better initial value for $c_1$ in $(a,b)$. See the above figure.

2. If the initial estimate is not close enough to the desired root, the Newton's method may give a different root approximations as shown in the figure below.




$c_2$ is near root $r_2$.

3. Sometimes the successive estimates of the Newton's method may converge either too slowly or not all. In such situations, we have to try some alternative strategies.

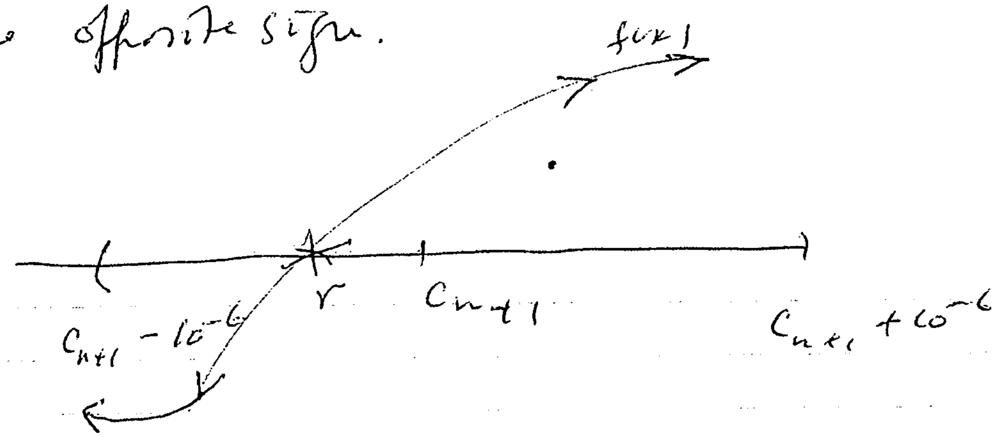
4. Inspite of the above 1 – 3 notes, Newton's method is extremely efficient in approximating a root provided the initial estimate is close enough.

5. When the Newton's method is working well, the number of correct decimal places roughly doubles with each iteration.
If we want to compute a root correct to 4 decimal places, a good rule to follow is to compute until two successive iteration agree to 4 decimal places.

6. Accuracy determination: If after $n$ iterations our estimate of the root $r$ is $c_{n+1}$, then it is within $10^{-6}$ if

$$f(c_{n+1} - 10^{-6}) \text{ and } f(c_{n+1} + 10^{-6})$$

are of the opposite sign.




clearly $|r - c_{n+1}| < 10^{-6}$, follows from the figure above.

## Examples

1. Use the Newton's method to find $\sqrt{3}$ within $10^{-4}$.

Solution.

$$f(x) = x^2 - 3 = 0$$

Since $f(0) < 0$ and $f(2) > 0$, there is a root in $(0, 2)$.
Let $c_1 = 1$

$$c_{n+1} = c_n - \frac{f(c_n)}{f'(c_n)}, \quad n = 1, 2, 3, \ldots$$

$$= c_n - \frac{c_n^2 - 3}{2c_n}$$

$$= \frac{2c_n^2 - c_n^2 + 3}{2c_n} = \frac{c_n^2 + 3}{2c_n}$$

$$= \frac{1}{2}\left(c_n + \frac{3}{c_n}\right)$$

$$\begin{cases} c_2 = \frac{1}{2}\left(c_1 + \frac{3}{c_1}\right) \\ c_2 = \frac{1}{2}\left(1 + \frac{3}{1}\right) = 2 \end{cases}$$

$$c_3 = \frac{1}{2}\left(c_2 + \frac{3}{c_2}\right) = \frac{1}{2}\left(2 + \frac{3}{2}\right) = 1.7500$$

$$c_4 = \frac{1}{2}\left(c_3 + \frac{3}{c_3}\right) = \frac{1}{2}\left(1.7500 + \frac{3}{1.7500}\right) = 1.732142857$$

$$c_5 = \frac{1}{2}\left(c_4 + \frac{3}{c_4}\right) = 1.73205081$$

$$c_6 = \frac{1}{2}\left(c_5 + 3/c_5\right) = 1.732050808\,0$$

Since $c_5$ and $c_6$ agree to 4 decimal places, hence $\sqrt{3} = 1.7320$ within $10^{-4}$ of its true value.

CHECK $f(1.7319) = -5.224 \times 10^{-4} < 0$

$f(1.7321) = +1.7041 \times 10^{-4} > 0$


Hence, $|\sqrt{3} - 1.7320| < 10^{-4}$.

2. A new manufacturing process produces savings of

$$S(x) = x^2 + 40x + 20$$

dollars after $x$ years, with increased costs of

$$C(x) = x^3 + 5x^2 + 9$$

dollars. For how many years, to the nearest hundredth, should the process be used?

Solution. The process should be used until $S(x) = C(x)$

i.e. we solve $f(x) = -S(x) + C(x) = 0$

or $f(x) = -x^2 - 40x - 20 + x^3 + 5x^2 + 9 = 0$

$= x^3 + 4x^2 - 40x - 11$

$f(x) = x^3 + 4x^2 - 40x - 11$

$f'(x) = 3x^2 + 8x - 40$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $f(x)$ | −11 | −46 | −67 | −68 | −43 | 14 |

change of sign

We seek a solution in $(4, 5)$..

Let $c_1 = 4$ $\qquad c_2 = 4 - \dfrac{-43}{40} = 5.08$

$$c_3 = 5.08 - \frac{20.122}{78.059} = 4.82$$

$$c_4 = 4.82 - \frac{1.1098}{68.257} = 4.80$$

$$c_5 = 4.80 - \frac{-0.248}{67.32} = 4.80$$ to the nearest hundredth.

Till $x = 4.80$ $\quad f(x) < 0$ or $-S + C < 0$ or $C < S$.

The process should be used for 4.80 years.