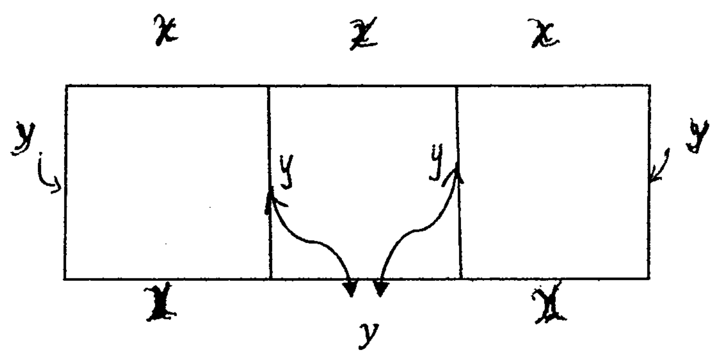
More Examples of Optimization Problems Lecture 13

1. Area Maximization

   An ecologist is conducting a research project on breeding pheasants in captivity. She first must construct suitable pens. She wants a rectangular area with two additional fences across its width, as shown below. Find the maximum area she can enclose with 3600m of fencing.



From the perimeter of the pens, we have the condition or consistent equation: $4y + 6x = 3600$ or $y = 900 - \frac{3}{2}x \rightarrow$ ①

Area $= A = 3xy$ and with the help of ①, we have:

$$A(x) = 3\left(900x - \frac{3}{2}x^2\right), \qquad 0 < x < 600$$

$A'(x) = 3(900 - 3x)$; $A'(x) = 0$ when $3(900 - 3x) = 0$ or $x = 300$

$A''(x) = -9 < 0$, and so $A''(300) = -9 < 0$ and a relative maximum occurs at $x = 300$. Also, by the Critical Point Theorem, this relative maximum is the absolute maximum.

Hence, the maximum area that she can enclose is:

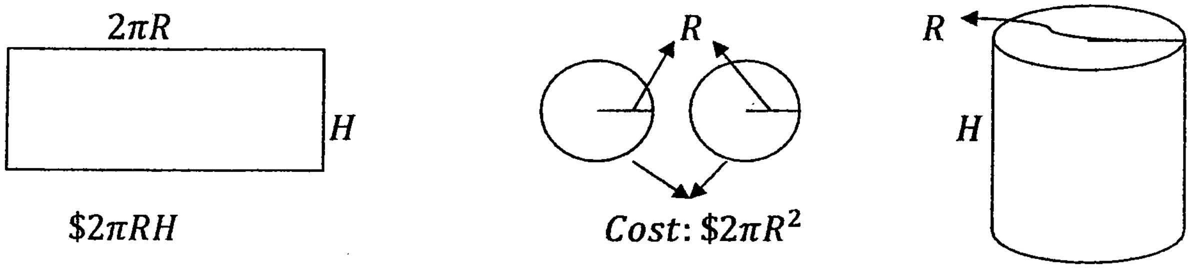
$A(300) = 3(300)\left(900 - \frac{900}{2}\right) = \frac{810{,}000}{2} = 405{,}000m^2$

This occurs when $x = 300$ and $y = 450$.

2. <u>Container Design with Cost Minimization</u>

A company needs to design cylindrical metal containers with a volume of $16m^3$. The top and bottom will made of a sturdy material that costs \$2 per m^2, while the material for the side costs \$1 per m^2. Find the radius, height, and cost of the least expensive container.

Volume: $\pi R^2 H = 16 \; or \; H = \frac{16}{\pi R^2} \rightarrow$ ①



$\$2\pi RH$ $\qquad$ $Cost\text{: }\$2\pi R^2$

Total Cost= $C = 4\pi R^2 + 2\pi RH$. Using ①, we have:

$$C(R) = 4\pi R^2 + \frac{32}{R}, \qquad R > 0$$

We want to minimize C(R).

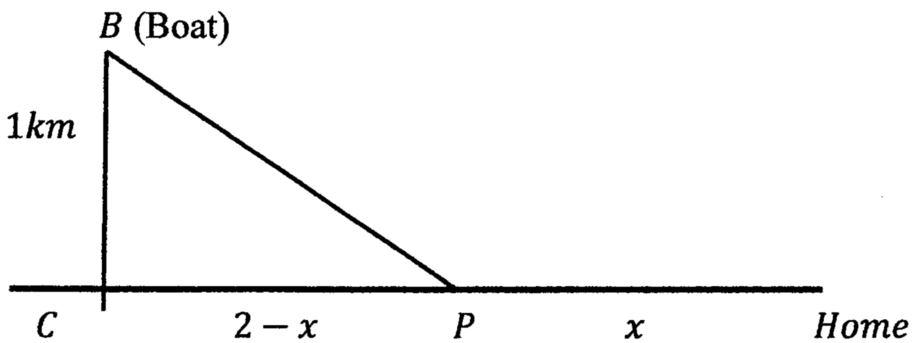
$$C'(R) = 8\pi R - \frac{32}{R^2}, \quad C'(R) = 0$$

$R = \left(\frac{4}{\pi}\right)^{1/3} \approx 1.08$, and $H = \frac{16}{\pi(1.08)^2} \approx 4.34$

$C'(R) = 8\pi + \frac{64}{R^3} > 0$, for $R > 0$ and $C''(1.08) > 0$ and therefore, a relative minimum occurs at $R = 1.08$, which is also the absolute minimum by the Critical Point Theorem. The minimum cost is then $C(1.08) \approx \$44.11$ and the radius is $1.08m$ and the height is $4.34m$.

3. Pigeon Flight with Minimum Energy

a) A pigeon is released from a boat $1km$ from the shore of a lake and it flies first to a point on the shore and then along the straight edge of the lake to reach its home. If its home is $2km$ from a point on the shore closest to the boat, and if a pigeon needs $4/3$ as much energy per km to fly over water as over land, find the location of the point on the shore that it reaches to minimize energy for the flight home from the boat.



b) In part a, if a pigeon needs $10/9$ as much energy to fly over water as over land, find the location of the point on the shore that it reaches to minimize energy for the flight home from the boat.

Solution:

a) Let energy used over land be 1 $unit/km$ and over water $4/3$

Let x be the distance of P from H

$$E(x) = \left(\frac{4}{3}\right) BP + (1)PH = \frac{4}{3}\sqrt{1+(2-x)^2} + x, \quad 0 \le x \le 2$$

$$E'(x) = \frac{4}{3}\cdot\frac{1}{2}[1+(2-x)^2]^{-1/2}\cdot 2(2-x)\cdot(-1)+1$$

$$= -\frac{4}{3}\cdot\frac{(2-x)}{\sqrt{1+(2-x)^2}}+1$$

$$E'(x) = 0 \Rightarrow \frac{4}{3}\cdot\frac{(2-x)}{\sqrt{1+(2-x)^2}} = 1$$

$$\frac{(2-x)^2}{1+(2-x)^2}=\frac{9}{16} \quad OR \quad 2-x=\pm\sqrt{\frac{9}{7}}, \quad x=2\pm\sqrt{\frac{9}{7}}$$

Since $0 \le x \le 1, x = 2-\sqrt{\frac{9}{7}} = 0.8661$

x	$E(x)$
0	$\frac{4}{3}\cdot\sqrt{5} = 2.9814$
2	$\frac{10}{3} = 3.3333$
0.8661	2.8818

Hence, the absolute minimum energy for the flight occurs when the pigeon comes to shore 0.8661 km from home.

b) $E(x) = \frac{10}{9}\sqrt{1+(2-x)^2}+x, \quad 0 \le x \le 2$

$$E'(x) = -\frac{10}{9}\cdot\frac{(2-x)}{\sqrt{1+(2-x)^2}}+1$$

$$E'(x)=0 \Rightarrow \frac{2-x}{\sqrt{1+(2-x)^2}}=\frac{9}{10} \quad OR$$

$$(2-x)^2=\frac{81}{19}, \qquad x = 2\pm\frac{9}{\sqrt{19}} \approx 2\pm 2.0647$$

Since $0 \le x \le 2$, there are no critical numbers in the domain of $E(x)$.

x	$E(x)$
0	$\frac{10}{9}\cdot\sqrt{5} = 2.4845$
2	$2+\frac{10}{9} = 3.111$

Hence, in this case, the absolute minimum energy for the flight occurs when the pigeon flies entirely over water to reach its home. The pigeon comes to shore at its home.