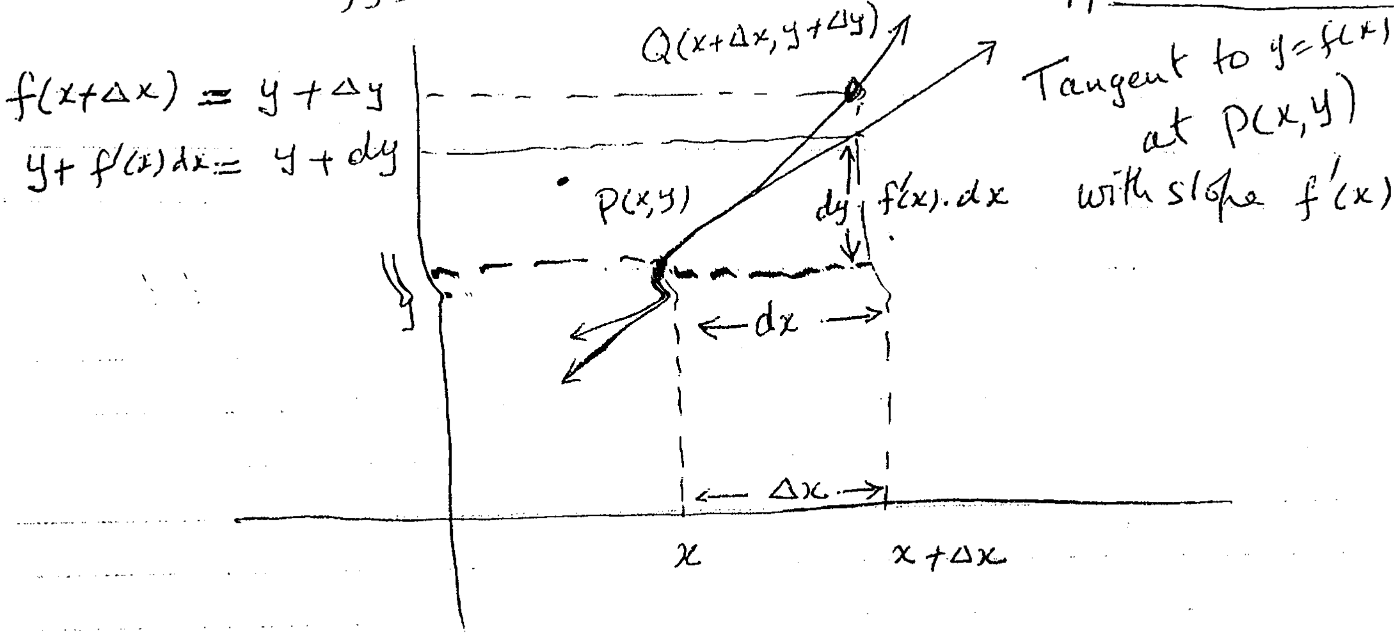
Lecture 15

Differentials and Linear Approximation 6.6



For a function $y = f(x)$ whose derivative exists, we define the following objects. Let P and Q be two points on the graph of $y=f(x)$ as shown above in the figure.

1. The differential of the independent variable x is
$$\underline{dx = \Delta x}$$
where Δx denotes the change in x as we move from P to Q.

2. The differential of the dependent variable y is
$$\underline{dy = f'(x)\,dx}.$$

Thus dy is the rise of the tangent line to $y=f(x)$ at $P(x,y)$ with a run of dx.

If Q is near P, then we have
$$\underline{\Delta y \approx dy}$$
$$f(x+\Delta x) = f(x) + \Delta y = y + \Delta y \approx f(x) + dy = y + dy = f(x) + f'(x)\,dx$$

The last line is called the Linearization of $f(x)$ at P and we write:

$$f(x+\Delta x) \approx L(\Delta x) = \underline{f(x) + dy}$$

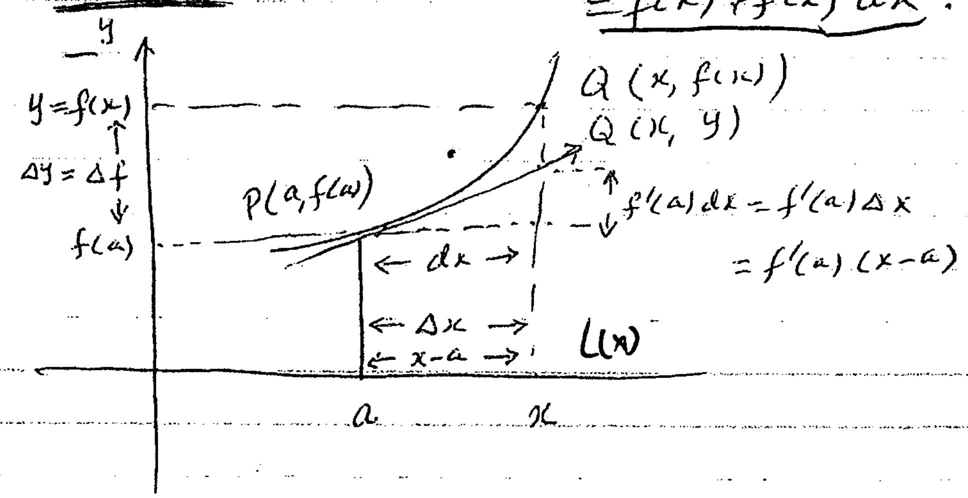
$$f(x+dx) \approx L(dx) = \underline{f(x) + f'(x)\, dx}$$

$$f(x+\Delta x) \approx L(\Delta x) = \underline{f(x) + f'(x)\, \Delta x}$$

since $dx = \Delta x$.

In other words at $P(x, y) = P(x, f(x))$, we approximate the function $y = f(x)$ by the tangent line at x, $\underline{L(\Delta x)} = L(dx) = f(x) + f'(x)\,\Delta x$
$= \underline{f(x) + f'(x)\, dx}$.

Linearization of $f(x)$ at $x=a$



$$f(x) \approx L(x) = \underline{f(a) + f'(a) \cdot (x-a)}$$

Error Estimation

The change in $y = f(x)$ from point $P(a, f(a))$ to a nearby point $Q(a+\Delta x, f(a+\Delta x))$ can be described in following three ways.

	True Value	Approximate Value
Total change	$\Delta y = f(a+\Delta x) - f(a)$	$dy = f'(a)\,dx = f'(a)\,\Delta x$
Relative change	$\dfrac{\Delta y}{y} = \dfrac{\Delta y}{f(a)}$ $\Delta y = f(a+\Delta x) - f(a)$	$\dfrac{dy}{y} = \dfrac{f'(a)\,dx}{f(a)} = \dfrac{f'(a)\,\Delta x}{f(a)}$
Percentage change	True Relative change times 100	Relative change times 100

Examples

1. Find dy for $y = f(x) = \sqrt{3x+15}$.

$$dy = f'(x)\,dx = \frac{1}{2}(3x+5)^{-1/2} \cdot 3 \quad dx$$

$$= \frac{3}{2}\frac{1}{\sqrt{3x+15}}\,dx$$

2. Find dy for $y = f(x) = \sqrt{3x+5}$ for $x = 7$, $\Delta x = 0.08$.

$$dy(x=7,\ dx=\Delta x=0.08) = \frac{3}{2}\frac{1}{\sqrt{36}} \cdot 0.08$$

$$= 0.02$$

3. Use the differential to approximate $\sqrt{145}$ and then use a calculator to approximate the quantity and give the absolute value of the difference in the two results to 4 decimal places.

$f(x) = \sqrt{x}$, we know $f(144) = \sqrt{144} = 12$

$$f(145) = f(144+1) = f(144) + \Delta y$$

$$= \sqrt{144} + \Delta y \approx 12 + \frac{1}{24} \approx 12.0417$$

$$\Delta y \approx f'(x)\,dx \Big|_{\substack{x=144 \\ dx=\Delta x=145-144=1}}$$

$$= \frac{1}{2}\frac{1}{\sqrt{x}}\,dx \Big|_{\substack{x=144 \\ dx=\Delta x=1}} = \frac{1}{2}\frac{1}{\sqrt{144}} \cdot 1$$

$$= \frac{1}{24}$$

By calculator $\sqrt{145} \approx 12.0416$
The absolute value of the difference is

$$|12.0417 - 12.0416|$$

$$= 0.0001$$

4. For the profit function with demand x is

$$P(x) = 12000 \ln(0.01x+1) - 75x - 150.$$

Find the approximate change in profit for a 1-unit change in demand when demand is at a level of 100.

$$\Delta P \approx dP = P'(x)\,dx \Big|_{\substack{x=100 \\ dx=\Delta x=1}}$$

$$= \left[12000 \cdot \frac{1}{.01x+1} \cdot (0.01) - 75\right] dx \Big|_{\substack{x=100 \\ dx=\Delta x=1}}$$

$$= \left[\frac{12000}{1+1} \cdot \frac{1}{100} - 75\right][1]$$

$$= 60 - 75$$

$$= -15$$

Thus the change in profit is a loss of about \$15.