Lecture 12

Absolute Extrema 6.1

Let f be a function defined on some interval. Let c be a number in the interval. Then $f(c)$ is the absolute maximum of f on the interval if

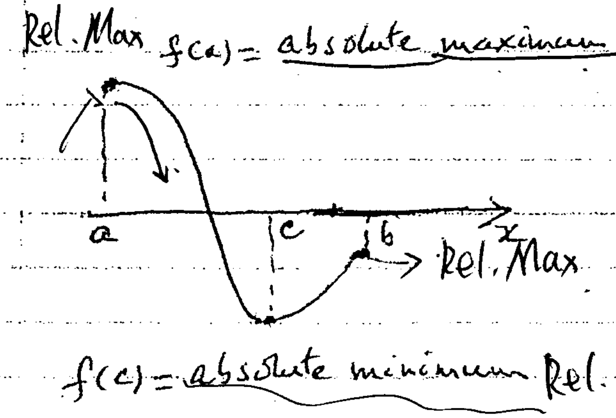
$$f(x) \leq f(c)$$

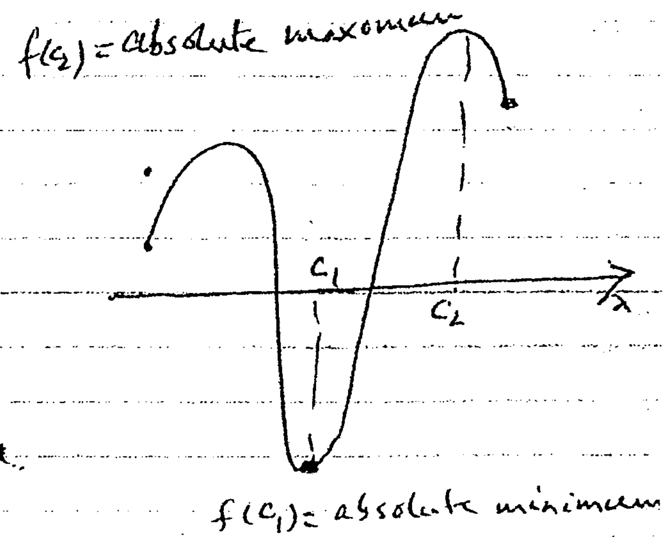
for every x in the interval, and $f(c)$ is the absolute minimum of f on the interval if

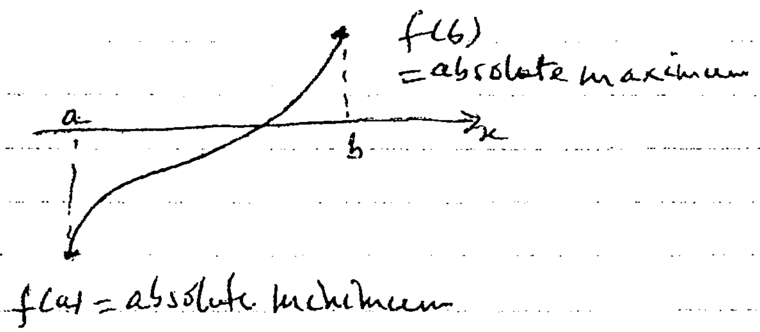
$$f(x) \geq f(c)$$

for every x in the interval.

A function has an absolute extremum (plural: extrema) at c if it has either an absolute maximum or an absolute minimum there.







Extrema Value Theorem 定理 (EVT)

This theorem gives the existence of absolute extrema under some conditions.

A function f that is continuous on a closed interval $[a, b]$ will have both an absolute maximum and an absolute minimum on the interval.

Finding Absolute Extrema

Let f be a continuous function on a closed interval $[a, b]$.

1. Find all critical numbers for f in the open interval (a, b).
2. Evaluate f for all critical numbers in (a, b)
3. Evaluate f for the endpoints a and b of the closed interval $[a, b]$.
4. The largest value found in Step 2 or 3 is the absolute maximum for f on $[a, b]$ and the smallest value found is the absolute minimum for f on $[a, b]$.

$f(x) = x^3 - 6x^2 + 1 \quad [-1, 5]$

$f'(x) = 3x^2 - 12x$

$f'(0) = \quad x = 0$

$x = 4$

x	$f(x)$	
0	1	→ abs max
4	−31	→ Abs Min.
−1	−6	
5	−24	

Critical Point Theorem (CPT)

Suppose a function f is continuous on an interval I and that f has exactly one critical number in the interval I, located at $x=c$.

If f has a relative maximum at $x=c$, then this relative maximum is the absolute maximum of f on the interval I.

If f has a relative minimum at $x=c$, then this relative minimum is the absolute minimum of f on the interval I.

Note: I may be an open or closed interval.

Examples

Question 1. Find the absolute extrema, if they exist for the following functions defined on the given interval.

a) $f(x) = x^3 - 6x^2 + 1, \quad -1 \le x \le 5$

$f'(x) = 3x^2 - 12x \qquad f'(x) = 0 \qquad 3x(x-4) = 0, \; x = 0, 4.$

x-value (at critical pts. & end-points)	$f(x)$
0	1
4	-31
-1	-6
5	-24

The absolute minimum, -31, occurs when $x=4$, and the absolute maximum, 1, occurs when $x=0$.

b. $f(x) = 10 + |x-3| \qquad -1 \le x \le 5$

$$= \begin{cases} 7+x & \text{for } 3 \le x \le 5 \\ 13 - x & \text{for } -1 \le x < 3 \end{cases}$$

f' is never zero on $[-1,5]$. f' does not exist at $x=3$.

x-value (at critical point & endpoints)	$f(x)$
3	10
-1	14
5	12

The absolute minimum, 10, occurs at $x=3$, and absolute maximum, 14, occurs at $x=-1$.

c. $f(x) = 5x^{2/3} - x^{5/3}, \qquad -1 \le x \le 4$

$f'(x) = \frac{10}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{5}{3}\cdot\frac{1}{x^{1/3}}(2-x)$.

Critical numbers are 0, 2.

x	$f(x)$
0	0
2	$5\cdot 2^{2/3} - 2^{5/3} \approx 4.76$
-1	6
4	$5\cdot 4^{2/3} - 4^{5/3} \approx 2.52$

The absolute minimum, 0, occurs at $x=0$, and the absolute maximum, 6, occurs at $x=-1$.

Q2 The demand equation for a manufacturer's product is $p = D(q) = \frac{1}{4}(80-q)$ with $0 \le q \le 80$, where q is the number of units and p is the price per unit in dollars.

a) At what value of q will there be maximum revenue?

b) What is the maximum revenue?

Solution

$R(q) = q \cdot D(q) = -\frac{1}{4}q^2 + 20q \qquad 0 \le q \le 80$

$\frac{dR}{dq} = -\frac{1}{2}q + 20$

$R'(q) = 0$ when $q = 40$.

q	$R(q)$
40	10
0	0
80	0

a) There will be maximum revenue at $q = 40$.

b) The maximum revenue is \$10.

Application of Extrema 6-2

Guidelines for Solving An Applied Extrema Problem

1. Read the problem carefully. Make sure you understand what is given and what is unknown
2. If possible, sketch a diagram. Label the various parts.
3. Decide on the variables that must be maximized or minimized. Express that variable as a function of one other variable.
4. Find the domain of the function in step 3.
5. Find the critical points for the function in step 3.
6. Find the absolute extrema as discussed.

Question 1 Minimize Average cost

For the total cost function

$$C(x) = 10 + 20x^{1/2} + 16x^{3/2}$$

find where the average cost is minimum.

solution.

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{10}{x} + \frac{20}{\sqrt{x}} + 16\sqrt{x}$$

Domain: $x > 0$.

$$\overline{C}'(x) = -\frac{10}{x^2} - \frac{10}{x^{3/2}} + \frac{8}{x^{1/2}}$$

$$= -\frac{10 + 10x^{1/2} - 8x^{3/2}}{x^2}$$

Since there is only one critical point at $x \approx 2.110$, the average cost is minimum at $x \approx 2.110$.