Assignment 1

## 1.1 (16, b)

want $\vec{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ Parallel to $\begin{bmatrix} 8t \\ 2t \end{bmatrix} \longrightarrow \begin{bmatrix} 4 \\ -1 \end{bmatrix} = k\begin{bmatrix} 8t \\ 2t \end{bmatrix} = \ell \begin{bmatrix} 8 \\ 2 \end{bmatrix}$ ($\ell = kt \in \mathbb{R}$).

$4 = 8\ell$ & $-1 = 2\ell$

$\frac{1}{2} = \ell$ $\quad -\frac{1}{2} = \ell$ So No solution.

However $\vec{0}$ is parallel to every vector so $t = 0$ will work.

## 1.1 (D9)

**Remark:** To answer T/F questions:

- If a statement is true, you must give a proper (formal) justification. An example or a drawing is NOT considered formal.

- " " " " false, " " " an explicit counterexample that clearly shows why the given statement is false.

Here $\vec{x}, \vec{y}, \vec{z}, \vec{u}, \vec{v} \in \mathbb{R}^n$ (unless specified otherwise).

a) If $\vec{x} + \vec{y} = \vec{x} + \vec{z} \longrightarrow \vec{y} = \vec{z}$.

True by properties of vector addition:

$\vec{x} + \vec{y} = \vec{x} + \vec{z}$ add $-\vec{x}$ to both sides:

$$\underbrace{\underbrace{-\vec{x} + \vec{x}}_{\vec{0}} + \vec{y}}_{\vec{y}} = \underbrace{\underbrace{-\vec{x} + \vec{x}}_{\vec{0}} + \vec{z}}_{\vec{z}} \checkmark$$

(Note: I did NOT give an example!)

b) $\vec{u} + \vec{v} = \vec{0} \longrightarrow a\vec{u} + b\vec{v} = \vec{0}$ for all $a, b \in \mathbb{R}$.

False: ex: $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ then $\vec{u} + \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

but $2\vec{u} + \vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(a counterexample to the given statement).

c) Parallel vectors w/ same length are equal.

False: $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ are parallel b/c $\vec{u} = -\vec{v}$

& $\|\vec{u}\| = \|\vec{v}\| = 1$

But $\vec{u} \neq \vec{v}$ (different first coordinates)

d) If $a\vec{x} = \vec{0}$, then either $a = 0$ or $\vec{x} = \vec{0}$.

True let $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$. so that $a\vec{u} = \begin{bmatrix} au_1 \\ \vdots \\ au_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

then.
$$au_1 = 0$$
$$au_2 = 0$$
$$\vdots$$
$$au_n = 0.$$

Now: if $a \neq 0$, then divide by $a$ to get

$$\left.\begin{matrix} u_1 = 0 \\ \vdots \\ u_n = 0 \end{matrix}\right\} \longrightarrow \vec{u} = \vec{0}.$$

e) if $a\vec{u} + b\vec{v} = \vec{0}$ then $\vec{u}$ & $\vec{v}$ are Parallel.

False:

$a = 0$ $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
$b = 0$

then $a\vec{u} + b\vec{v} = \vec{0}$ but $\vec{u} \neq t\vec{v}$.

(need more info about $a, b, \vec{u}, \vec{v}$ to conclude they are parallel)

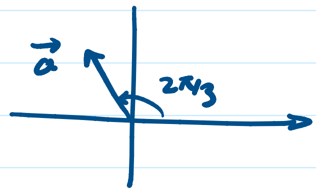
f) $\vec{u} = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1/\sqrt{2} \\ \sqrt{3}/2 \end{bmatrix}$ are equivalent.

False (see def. of equivalent / equal)

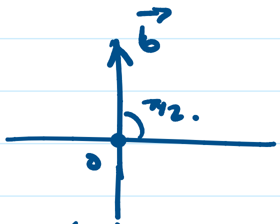
## 1.2 (15)

$\|\vec{a}\| = 9$

$120°$ ccw from + x-axis $\longrightarrow$




$\|\vec{b}\| = 5$




$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos(\theta)$. $\quad \theta$ angle b/w $\vec{a}$ & $\vec{b}$

$= \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$.

$= 9 \cdot 5 \cdot \cos(\pi/6)$

$= \frac{45}{2} \cdot \sqrt{3}$

## 1.2 (24)

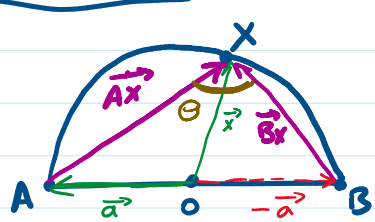
$\vec{v_1} \cdot \vec{v_2} = 0 - \frac{2}{6} + \frac{1}{3} = 0$ $\quad \|\vec{v_1}\| = 1$

$\vec{v_1} \cdot \vec{v_3} = -\frac{1}{2} + \frac{1}{6} + \frac{1}{3} = 0.$ $\quad \|\vec{v_2}\| = 1$

$\|\vec{v_3}\| = 1.$

$\vec{v_2} \cdot \vec{v_3} = 0 - \frac{2}{6} + \frac{1}{3} = 0$ $\quad$ (orthonormal means orth. & unit)

## 1.2 (P9)




$\vec{a} + \vec{AX} = \vec{X} \longrightarrow \vec{AX} = \vec{X} - \vec{a}$

$-\vec{a} + \vec{BX} = \vec{X} \longrightarrow \vec{BX} = \vec{X} + \vec{a}.$

$\theta$ is $\frac{\pi}{2}$ $\xleftrightarrow[\text{only if}]{\text{if and}}$ $\vec{AX} \cdot \vec{BX} = 0$

$(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a})$

$= \vec{x} \cdot \vec{x} + \underbrace{\vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{x}}_{0} - \vec{a} \cdot \vec{a}$

$= \|\vec{x}\|^2 - \|\vec{a}\|^2$

But $\|\vec{a}\| = \|\vec{x}\|$ as these are different radii inside the given semi circle.

$\therefore \quad \|\vec{x}\|^2 - \|\vec{a}\|^2 = 0$ ✓.

## 1.3 (D3):

$\vec{v}, \vec{w_1}, \vec{w_2} \in \mathbb{R}^n$

a) Given $\vec{v}$ ⊥ to $\vec{w_1}, \vec{w_2}$ so $\vec{v}\cdot\vec{w_1} = 0$ & $\vec{v}\cdot\vec{w_2} = 0$.
then let $k_1, k_2 \in \mathbb{R}$. by properties of the dot product.

$$\vec{v}\cdot(k_1\vec{w_1} + k\vec{w_2}) = k_1(\vec{v}\cdot\vec{w_1}) + k_2(\vec{v}+\vec{w_2}) = k_1\cdot 0 + k_2\cdot 0 = 0$$

so $\vec{v}$ is orth. to any linear comb. of $\vec{w_1}, \vec{w_2}$.

b) Note: any vector of form $k_1\vec{w_1} + k_2\vec{w_2}$ lies on the plane (in $\mathbb{R}^n$, provided $\vec{w_1}, \vec{w_2}$ are not parallel) & through $\vec{0}$ formed (parallel to) by $\vec{w_1}, \vec{w_2}$. Part a says that if $\vec{v}$ is orth. to $\vec{w_1}$ & $\vec{w_2}$, then $\vec{v}$ is orth. to the entire plane formed by $\vec{w_1}, \vec{w_2}$. If $\vec{w_1}, \vec{w_2}$ are parallel, Then $k_1\vec{w_1} + k\vec{w_2} = t\vec{w_1}$ for some $t \in \mathbb{R}$ (why?) so Part a says that if $\vec{v}$ is orth. to $\vec{w_1}$, then $\vec{v}$ is orth. to the line that is parallel to $\vec{w_1}$ and through $\vec{0}$.

## Extra 3

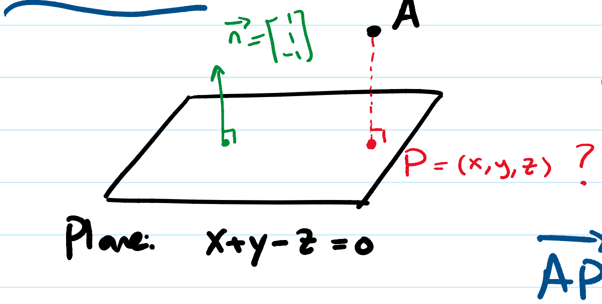
Angle between planes = angle b/w their normal vectors

$$x - y + z = 5 \longrightarrow \vec{n_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \|\vec{n_1}\| = \sqrt{3}$$

$$x + y + 2z = -3 \longrightarrow \vec{n_2} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}. \qquad \|\vec{n_2}\| = \sqrt{6}$$

$$\theta = \cos^{-1}\left(\frac{\vec{n_1}\cdot\vec{n_2}}{\|\vec{n_1}\|\cdot\|\vec{n_2}\|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{18}}\right) \approx 61.8^\circ$$

## Extra 4




want $\overrightarrow{AP}$ & $\vec{n}$ to be Parallel to ensure $\overrightarrow{AP}$ is perp. to the plane.

$\therefore$ Shortest length.

$$\overrightarrow{AP} = \begin{bmatrix}x-1\\y+2\\z-1\end{bmatrix} = t\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}t\\t\\-t\end{bmatrix} \quad \text{for some } t.$$

$\therefore$ $x-1=t$
$y+2=t$
$z-1=-t$

Now, $(x,y,z)$ is on the plane so it satisfies $x+y-z=0$.

$\therefore$ $(x-1)+(y+2)-(z-1) = t+t-(-t)$

$$\underbrace{x+y-z}_{0}+2 = 3t$$

$$\boxed{t = 2/3}$$

$\therefore$ $x-1 = 2/3 \longrightarrow x = 5/3$
$y+2 = 2/3 \qquad y = -4/3$
$z-1 = -2/3 \qquad z = 1/3$

$$P = (5/3, -4/3, 1/3)$$

## Extra 5

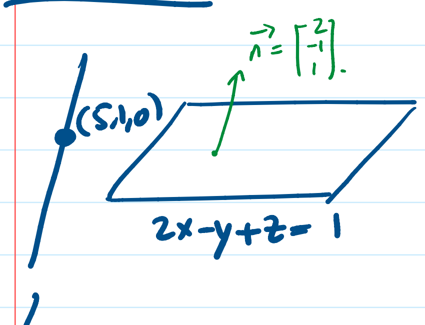
$2x+z=1. \longrightarrow$ general eq.

want Param. eqs: $\quad 2\overset{t}{x} + 0\cdot\overset{s}{y} + z = 1$

(you can also solve for $x$ in terms of $y, z$, etc.).

$x = t$
$y = s$ $\qquad t, s \in \mathbb{R}.$
$z = 1-2t$

## Extra 7:




want $L$ Perp. to $2x-y+z=1$

so $\vec{n}$ is parallel to $L$.

so $\vec{x} = \begin{bmatrix}5\\1\\0\end{bmatrix} + t\begin{bmatrix}2\\-1\\1\end{bmatrix}$. $\quad$ $x = 5 + 2t$, $y = 1 - t$, $z = t$ $\qquad t \in \mathbb{R}.$