7:30

Define.

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ if the meaning exist.

$y = f(x)$

$\frac{dy}{dx} = f'(a)$

Derivative of f with respect to "x" at "a"

$y = f[g(x)]$

$\frac{dy}{dx} = \frac{df}{dg}\cdot\frac{dg}{dx}$

$= f'(g)\cdot g'(x)$

eg. $y = (2x+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{\sqrt{2x+1}}\cdot 2$

y	$\frac{dy}{dx}$
k	0
x^n	nx^{n-1}
e^x	e^x
a^x	$\ln a\cdot a^x$
$\ln\lvert x\rvert$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{\ln a}\cdot\frac{1}{x}$

y	$\frac{dy}{dx}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x\cdot\cot x$
$\sec x$	$\sec x\cdot\tan x$
$\cot x$	$-\csc^2 x$

y	$\frac{dy}{dx}$	
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$	$x\in(-1,1)$
$\cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}$	$x\in(-1,1)$
$\tan^{-1}x$	$\frac{1}{1+x^2}$	$x\in(-\infty,\infty)$
$\sec^{-1}x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$x\in(-\infty,-1)\cup(1,\infty)$
$\csc^{-1}x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$x\in(-\infty,-1)\cup(1,\infty)$
$\cot^{-1}x$	$-\frac{1}{1+x^2}$	$x\in(-\infty,\infty)$

$[f^{-1}]'(a) = \frac{1}{f'[f^{-1}(a)]}$

k	0
x^n	nx^{n-1}
e^x	e^x
a^x	$\ln a\cdot a^x$
$\ln\lvert x\rvert$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{\ln a}\cdot\frac{1}{x}$

y	$\frac{dy}{dx}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x\cdot\cot x$
$\sec x$	$\sec x\cdot\tan x$
$\cot x$	$-\csc^2 x$

y	$\frac{dy}{dx}$
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}x$	$\frac{1}{-\sqrt{1-x^2}}$
$\tan^{-1}x$	$\frac{1}{1+x^2}$
$\sec^{-1}x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\csc^{-1}x$	$\frac{1}{-\lvert x\rvert\sqrt{x^2-1}}$
$\cot^{-1}x$	$\frac{1}{-1+x^2}$

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4.8 Derivatives (导数) of Inverse Function

Theorem (原理) 4.8 : Derivative of Inverse Function.

Given an invertible (可逆的) function $f(x)$, the derivative of its inverse function $f^{-1}(x)$ evaluated at $x=a$ is:

$$[f^{-1}]'(a) = \frac{1}{f'[f^{-1}(a)]}$$

To see why this is true, start with the function $y=f^{-1}(x)$. Write this as $x=f(y)$ and differentiate (求…的微分) both sides implicitly (隐式的) with respect to x using the Chain Rule:

$$1 = f'(y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{f'(y)}$$

but $y=f^{-1}(x)$ thus

$$\frac{dy}{dx} = \frac{1}{f'[f^{-1}(x)]}.$$

at the point $x=a$ this becomes:

$$[f^{-1}]'(a) = \frac{1}{f'[f^{-1}(a)]}.$$

Example 4.81 Derivatives of Inverse Functions.

Suppose $f(x)=x^5+2x^3+7x+1$. Find $[f^{-1}]'(1)$.

$1=x^5+2x^3+7x+1$

$x=0$. $\quad f^{-1}(1)=0$

$f(0)=1$. $\qquad f'(x)=5x^4+6x^2+7$

$$[f^{-1}]'(1)=\frac{1}{f'[f^{-1}(1)]} = \frac{1}{f'(0)} = \frac{1}{7}$$

$f'(0)=0+0+7$
$=7$

Solution: First we should show that f^{-1} exists. (i.e. the f is one-to-one) In this case the derivative $f'(x)=5x^4+6x^2+7$ is strictly (完全的) greater (较大的) than 0 for all x. (Because here have 7, and the degree are all even), so f is strictly increasing and thus one-to-one.

It's difficult to find the inverse of $f(x)$ (and then take the derivative). Thus, we use the above formula ~~evalu~~ evaluated at 1:

$$[f^{-1}]'(a) = \frac{1}{f'[f^{-1}(a)]}$$

$$[f^{-1}]'(1) = \frac{1}{f'[f^{-1}(1)]}$$

Note that to use this formula we need to ~~who~~ know what $f^{-1}(1)$ is, and the derivative $f'(x)$. To find $f^{-1}(1)$ we make a table of values (plugging in(代入) $x=-3,-2,-1,0,1,2,3$ into $f(x)$) and see what value of x gives 1. We omit(省去) the table and simply observe that $f(0)=1$. Thus,

$$f^{-1}(1) = 0$$

Now we have:

$$[f^{-1}]'(1) = \frac{1}{f'(0)}$$

And so, $f'(0)=7$. Therefore,

$$[f^{-1}]'(1) = \frac{1}{7}$$

Example 4.82 Tangent line of Inverse Functions.

Find the equation(方程式) of the tangent line to the inverse of $f(x) = \frac{e^{-3x}}{x^2+1}$, at $(-1, 0)$

$$f^{-1}(-1) = 0.$$

$$f'(x) = \frac{e^{-3x}\cdot(x^2+1) - e^{-3x}\cdot(2x)}{(x^2+1)^2}$$

$$[f^{-1}]'(\cancel{x}) = \frac{1}{f'[0]}$$

$$= \frac{1}{\frac{e^0\cdot(1) - e^0\cdot(2\cdot 1)}{(0^2+1)^2}}$$

$$= \frac{1}{\frac{1-2}{1}}$$

$$= \frac{1}{-1}$$

$$= -1.$$

X.

$$y - 0 = -1(x+1)$$

$$y = -x - 1.$$

$$\frac{-3e^{-3x}\cdot(x^2+1) - e^{-3x}\cdot(2x)}{(x+1)^2}$$

$$-3x^2e^{-3x} - 3e^{-3x} - 2xe^{-3x}$$

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4.8.1 Derivatives of ~~Inv~~ Inverse Trigonometric (三角学函数) Function.

Exercise 4.8.1 Find the derivative of the function.

(a) $f(x) = \csc^{-1}(5x^2+1)$

$$f'(x) = \frac{1}{-|5x^2+1|\sqrt{(5x^2+1)^2-1}} \cdot 10x$$

$$= \frac{10x}{-|5x^2+1|\sqrt{(5x^2+1)^2-1}}$$

(b) $f(x) = (\tan^{-1}(2x))^3$

$$= 3(\tan^{-1}(2x))^2 \cdot \frac{1}{1+(2x)^2} \cdot 2$$

$$= \frac{6(\tan^{-1}(2x))^2}{1+4x^2}$$

(c) $g(x) = \sqrt{e^{\cos^{-1}(x)}}$.

$$= \frac{1}{2}(e^{\cos^{-1}(x)})^{-\frac{1}{2}} \cdot e^{\cos^{-1}(x)} \cdot -\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{\sqrt{e^{\cos^{-1}(x)}}}{-2\sqrt{1-x^2}}$$

(d) $f(t) = \ln(\sin^{-1}t)$

$$= \frac{1}{\sin^{-1}t} \cdot \frac{1}{\sqrt{1-t^2}}$$

$$= \frac{1}{\sin^{-1}t(\sqrt{1-t^2})}$$

(e) $f(x) = \sec^{-1}(x^{\frac{3}{2}})$

$$= \frac{1}{(x^{\frac{3}{2}})\sqrt{x^3-1}} \cdot \frac{3}{2}x^{\frac{1}{2}}$$

$$= \frac{3}{2x\sqrt{x^3-1}}$$

(f) $h(s) = \cos^{-1}(\log_2 s)$

$$= \frac{1}{-\sqrt{1-(\log_2 s)^2}} \cdot \frac{1}{\ln 2} \cdot \frac{1}{s} = \frac{1}{-\sqrt{1-(\log_2 s)^2}\,\ln 2 \cdot s}$$

(g) $f(x) = (\cot^{-1} x)^{\frac{1}{3}}$

$$= \frac{1}{3}(\cot^{-1} x)^{-\frac{2}{3}} \cdot -\frac{1}{1+x^2}$$

$$= \frac{1}{3\sqrt[3]{(\cot^{-1} x)^2} \cdot -(1+x^2)}$$

(h) $g(t) = \sin^{-1}(3^t)$

$$= \frac{1}{\sqrt{1-(3^t)^2}} \cdot \ln 3 \cdot 3^t$$

$$= \frac{\ln 3 \cdot 3^t}{\sqrt{1-3^{2t}}}$$

4.8.2. Find $\frac{dy}{dx}$ by implicit differentiation.

(a) $\sin^{-1}(xy) + xy = x$

$$\frac{y + xy'}{\sqrt{1-x^2y^2}} + y + xy' = 1$$

$$\frac{y + xy' + (\sqrt{1-x^2y^2})(y+xy')}{\sqrt{1-x^2y^2}} = 1$$

$$\frac{(y+xy')(1+\sqrt{1-x^2y^2})}{\sqrt{1-x^2y^2}} = 1$$

$$(y+xy')(1+\sqrt{1-x^2y^2}) = \sqrt{1-x^2y^2}$$

$$y + xy' = \frac{\sqrt{1-x^2y^2}}{1+\sqrt{1-x^2y^2}}$$

$$xy' = \frac{\sqrt{1-x^2y^2} - (y + y\sqrt{1-x^2y^2})}{1+\sqrt{1-x^2y^2}}$$

$$y' = \frac{(1-y)\sqrt{1-x^2y^2} - y}{x + x\sqrt{1-x^2y^2}}$$

(b) $\tan^{-1}(x-y) = xy$.

$$\frac{1}{1+(x-y)^2} \cdot 1-y' = y + xy'$$

$$\frac{1-y'}{1+(x-y)^2} = y + xy'$$

$$1-y' = (y+xy')(1+(x-y)^2)$$

$$1-y' = (y+xy')(1+x^2-2xy+y^2)$$

$$1-y' = y + x^2y - 2xy^2 + y^3 + xy' + x^3y' - 2x^2yy' + xy^2y'$$

$$-xy' - x^3y' + 2x^2y\,y' - xy^2y' - y' = y + x^2y - 2xy^2 + y^3 - 1$$
$$y'(-x - x^3 + 2x^2y - xy^2 - 1) = y + x^2y - 2xy^2 + y^3 - 1$$
$$y' = \frac{y + x^2y - 2xy^2 + y^3 - 1}{-x - x^3 + 2x^2y - xy^2 - 1}$$

4.8.3.
Given $f(x) = 1 + \ln(x-2)$, first show that f^{-1} exists, then compute $[f^{-1}]'(1)$

$$[f^{-1}]'(1) = \frac{1}{f'[f^{-1}(1)]} = \frac{1}{3-2} = 1$$

$1 = 1 + \ln(x-2)$
$0 = \ln(x-2)$
$x = 3$
$f'(x) = \frac{1}{x-2}$

4.8.4. The inverse ~~cot~~ cotangent function, denote by $\cot^{-1}(x)$, is defined to be the inverse of the restricted cotangent function: $\cot(x)$, $0 < x < \pi$, Find the derivative of $\cot^{-1}(x)$.

4.7.1 Find a formula for the derivative y' at the point (x,y)

(a) $y^2 = 1 + x^2$

$2\ln y = 0 + 2\ln x$

$2\frac{y'}{y} = \frac{2}{x}$

$2y' = \frac{2y}{x}$

$y' = \frac{2y}{2x}$

$y' = \frac{y}{x}$

(b) $x^2 + xy + y^2 = 7$

$\frac{dy}{dx}$ $2x + y + xy' + 2y\cdot y' = 0$

$2x + y + y'(x+2y) = 0$

$y'(x+2y) = 0 - 2x - y$

$y' = \frac{0-2x-y}{x+2y}$

$y' = \frac{-2x-y}{x+2y}$

(c) $x^3 + xy^2 = y^3 + yx^2$

$3x^2 + y^2 + 2xy\cdot y' = 3y^2\cdot y' + y'x^2 + 2yx$

$3x^2 + y^2 - 2xy = 3y^2\cdot y' + x^2y' - 2xy\cdot y'$

$3x^2 + y^2 - 2xy = (3y^2 + x^2 - 2xy)\,y'$

$$y' = \frac{3x^2 + y^2 - 2xy}{3y^2 + x^2 - 2xy}$$

(d) $\cancel{\cos}$ $4\cos x \sin y = 1$

$4(-\sin x \sin y + \cos x \cos y \cdot y') = 0$

$-\sin x \sin y + \cos x \cos y \cdot y' = 0$

$\cos x \cos y \cdot y' = \sin x \sin y$

$y' = \dfrac{\sin x \sin y}{\cos x \cos y}$

(e) $\sqrt{x} + \sqrt{y} = 9$

$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 9$

$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}} \cdot y' = 0$

$\dfrac{1}{2\sqrt{x}} + \dfrac{y'}{2\sqrt{y}} = 0$

$\dfrac{y'}{2\sqrt{y}} = -\dfrac{1}{2\sqrt{x}}$

$y' = -\dfrac{2\sqrt{y}}{2\sqrt{x}}$

$y' = \dfrac{-\sqrt{y}}{\sqrt{x}}$

(f) $\tan\left(\frac{x}{y}\right) = x + y$

$\cancel{\sec^2\left(\frac{x}{y}\right) =}$

$\sec^2\left(\frac{x}{y}\right) \cdot \dfrac{y - x \cdot y'}{y^2} = 1 + y'$

$\dfrac{\sec^2\left(\frac{x}{y}\right)(y - xy')}{y^2} = 1 + y'$

$\sec^2\left(\frac{x}{y}\right)(y - xy') = y^2 + y^2 y'$

$y \sec^2\left(\frac{x}{y}\right) - xy' \sec^2\left(\frac{x}{y}\right) = y^2 + y^2 y'$

$-y^2 y' - xy' \sec^2\left(\frac{x}{y}\right) = y^2 - y \sec^2\left(\frac{x}{y}\right)$

$y'\left(-y^2 - x\sec^2\left(\frac{x}{y}\right)\right) = y^2 - y\sec^2\left(\frac{x}{y}\right)$

$y' = \dfrac{y\left(y - \sec^2\left(\frac{x}{y}\right)\right)}{-y^2 - x\sec^2\left(\frac{x}{y}\right)}$

(g) $\sin(x + y) = xy$

$\cos(x+y) \cdot (1 + y') = y + xy'$

$\cos(x+y) + \cos(x+y) y' = y + xy'$

$\cos(x+y) y' - xy' = y - \cos(x+y)$

$(\cos(x+y) - x)\, y' = y - \cos(x+y)$

$y' = \dfrac{y - \cos(x+y)}{(\cos(x+y) - x)}$

(h) $\frac{1}{x} + \frac{1}{y} = 7$

$x^{-1} + y^{-1} = 7$

$-x^{-2} - y^{-2} \cdot y' = 0$

$-\dfrac{1}{x^2} - \dfrac{y'}{y^2} = 0$

$-\dfrac{1}{x^2} = \dfrac{y'}{y^2}$

$-\dfrac{y^2}{x^2} = y'$

(i) $\cos(xy) - \sin x = 1$

$-\sin(xy) \cdot (y + xy') - \cos x = 0$

$-\sin(xy) \cdot (y + xy') = \cos x$

$y + xy' = \dfrac{\cos x}{-\sin xy}$

$xy' = \dfrac{\cos x}{-\sin xy} - y$

$y' = \dfrac{\cos x}{-x \sin xy} - \dfrac{y}{x}$

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j) $x\sec(y) = \ln(\sin x)$

$\sec(y) + x\cdot\sec(y)\cdot\tan(y)\cdot y' = \frac{\cos x}{\sin x}$

$x\cdot\sec(y)\cdot\tan(y)\cdot y' = \frac{\cos x}{\sin x} - \sec(y)$

$= \frac{\cos x - \sin x \sec y}{\sin x}$

$y' = \frac{\cos x - \sin x \sec y}{\sin x \cdot x\cdot\sec(y)\cdot\tan y)}$

$y = \frac{\cot x - \sec y}{x\sec(y)\tan(y)}$

k) $\sin(x+y) - \sin^{-1} y = 0$

$\cos(x+y)\cdot(1+y') - \frac{y'}{\sqrt{1-y^2}} = 0$

$\sqrt{1-y^2}\cos(x+y)\cdot(1+y') - y' = 0$

$(\sqrt{1-y^2}\cos(x+y)) + (\sqrt{1-y^2}\cos(x+y))y' - y' = 0$

$(\sqrt{1-y^2}\cos(x+y) - 1)y' = -\sqrt{1-y^2}\cos(x+y)$

$y' = \frac{-\sqrt{1-y^2}\cos(x+y)}{\sqrt{1-y^2}\cos(x+y) - 1}$

l) $(x^2-y^2)\tan(y) = \sqrt{y}$

$(2x - 2y\cdot y')\tan(y) + (x^2-y^2)\sec^2(y)\,y' = \frac{1}{2}y^{-\frac{1}{2}}\cdot y'$

$2x\tan(y) - 2yy'\tan(y) + (x^2-y^2)\sec^2(y)\,y' = \frac{1}{2}y^{-\frac{1}{2}}\cdot y'$

$2x\tan(y) = \frac{1}{2}y^{\frac{1}{2}}\cdot y' + 2y\tan(y)y' - (x^2-y^2)\sec^2(y)\,y'$

$2x(\tan(y)) = (\frac{1}{2}y^{\frac{1}{2}} + 2y\tan(y) - (x^2-y^2)\sec^2(y))\,y'$

$\frac{2x(\tan(y))}{\frac{1}{2}y^{\frac{1}{2}} + 2y\tan(y) - (x^2-y^2)\sec^2(y)} = y'$