PRACTICE FINAL

21/9

1. Differentiate the following functions as indicated. [16 marks]

a) $f(x) = 5x^4 - \cos(x) + \ln|2x - 1| + e^x$, $find\ f'(0)$.

$f'(x) = 20x^3 + \sin(x) + \frac{2}{|2x-1|} + e^x$

$f'(x) = 0 + 0 + 2 + 1$

$f'(x) = 3$

b) $g(x) = \frac{\sin\pi x}{x^2+x+1}$, $find\ g'(1)$.

$g'(x) = \frac{\pi\cos\pi x\,(x^2+x+1) - (2x+1)(\sin\pi x)}{(x^2+x+1)^2}$

$= \frac{\pi(-1)(3) - (3)(0)}{(3)^2}$

$= \frac{-3\pi - 0}{\cancel{9}\ 3}$

$= \frac{-\pi}{3}$

c) $h(x) = (2x+1)^2 e^{-x^2}$, *find* $h'(0)$

$h'(x) = 2(2x+1)\cdot 2 \cdot e^{-x^2} + (2x+1)^2 e^{-x^2} \cdot -2x$

$= 4(2x+1)\cdot e^{-x^2} + -2x(2x+1)^2 e^{-x^2}$

$= 4(1)\cdot e^{0} + 0$

$= 4$

d) $y = (x+2)^x$, *find* $\frac{dy}{dx}$ *when* $x = -1$.

$y' = \ln(x+2)\cdot(x+2)^x$

$y'(1) = \ln(1)\cdot(1)^{-1}$

$= 0$

2. [2+4 =6 marks]

a) State the limit ~~definition of~~ the derivative of a function f at $x = a$.

The slope of the tangent line to the graph of

x equal to a.

b) Use the limit definition to find $f'(a)$ where $f(x) = 2x^2 - 4x + 1$

$f(x) = 2x^2 - 4x + 1$

$f'(x) = 4x - 4$

$f'(a) = 4a - 4$

3. Find the horizontal and vertical asymptotes of the graphs of the following functions [6 marks]

a) $f(x) = \frac{2x^2-x-3}{x(x+1)}$

$X(x+1) \neq 0$

$x \neq 0$, $x \neq -1$, $\therefore$ vertical asymptotes is $x=0$, $x=-1$.

$\lim_{x\to+\infty} \frac{2x^2-x-3}{x^2+x}$

$= \lim_{x\to+\infty} \frac{4x-1}{2x+1}$

$= \lim_{x\to+\infty} \frac{4}{2}$

$=2$

$\lim_{x\to-\infty} \frac{2x^2-x-3}{x^2+x}$

$= \lim_{x\to-\infty} \frac{4x-1}{2x+1}$

$= \lim_{x\to+\infty} \frac{4}{2}$

$=2$

$\therefore$ H.A is $y=2$.

$f(x) = \frac{1}{1-e^x}$

$1-e^x \neq 0$

$1 \neq e^x$

$x \neq 0$

$\therefore$ V.A. is $x=0$.

$\lim_{x\to+\infty} \frac{1}{1-e^x}$

$= \lim_{x\to+\infty} \frac{\frac{1}{e^x}}{\frac{1}{e^x}-1}$

$=0$

$\therefore$ H.A. is $y=0$.

4. The price p (in dollars) and demand q (in thousands of units) for a product are related by:[3+3+2=8 marks]

$$q^3 + 10pq + p^3 = 1625$$

a) Find $\frac{dq}{dp}$ when the price $p = \$5$ and $q = 10$.

b) Find the elasticity of demand when $p = 5$ *and* $q = 10$. If the price were to decrease by 12 %, estimate the resulting percent change in demand.

c) Should the price be raised or lowered from its current \$5 level to increase the revenue. Justify your answer.

(a)

$3q^2 \cdot dq + 10q + p \cdot dq + 3p^2 = 0$

$3q^2 \cdot \frac{dq}{dp} + 10q + 10p \cdot \frac{dq}{dp} + 3p^2 = 0$

$3(10)^2 \cdot dq + 10 \cdot 10 + 10 \cdot 5 \cdot dq + 3(5)^2 = 0$

$300dq + 100 + 50dq + 75 = 0$

$350dq + 175 = 0$

$\frac{dq}{dp} = -\frac{1}{2}$

(b)

$E(p) = \frac{p}{q} \cdot q'$

$= \frac{5}{10} \cdot -\frac{1}{2}$

$= \frac{1}{4} < 1$

$\therefore$ It is inelastic.

$3q^2 \cdot dq + 10q \cdot dp + 10p \cdot dq + 3p^2 \cdot dp = 0$

$300dq + 100(-0.12) + 10 \cdot 5 \cdot dq + (-9) = 0$

$300dq - 6 + 50dq + (-9) = 0$

$350dq = 15$

$dq = \frac{3}{70}$

$= 4.2857\%$

$\therefore$ increasing 4.2857%

(c)

Because it is inelastic demand, this the increase in the price results in an increase respectively in the revenue.

$\frac{1}{4} \approx \frac{\%\ change\ q}{-12}$

$\%\ change\ q \approx 3\%$.

(-1, -36)
(1, 16)
(4, 64)

5. Given $f(x) = x^3(x-5)^2$, $f'(x) = 5x^2(x-5)(x-3)$ and $f''(x) = 10x(2x^2 - 12x + 15)$, sketch a complete graph of f. Be sure to clearly indicate all axis intercepts, relative extrema, concavity, and inflection points. [12 marks]

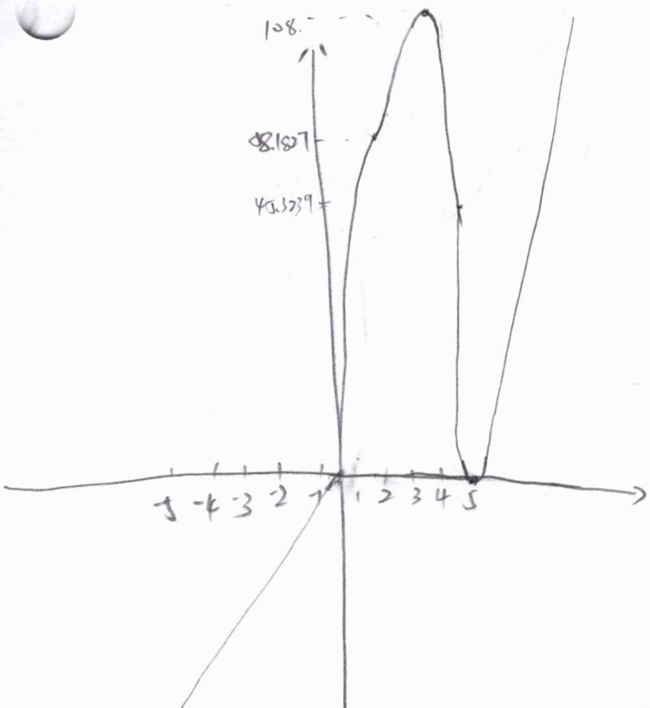
① domain.
$(-\infty, \infty)$.

②
$0 = x^3(x-5)^2$

$\therefore x = 0$ or $x = 5$.

$y = 0(0-5)^2 \quad \therefore (0,0)\ (5,0).$
$y = 0$

③. $f(x) \neq f(-x)$ it is odd.



$= \lim\limits_{x \to +\infty} \dfrac{7x^6 - 280x^4 + 1875x^2}{2x+10}$

$= \lim\limits_{x \to +\infty} \dfrac{42x^5 - 1000x^3 + 3750x}{2}$

$= \infty$.

$\therefore$ No y asymptote.

⑤ $0 = 5x^2(x-5)(x-3)$

$x = 0$ or $x = 5$ or $x = 3$.

$f'(-1),\ f'(1),\ f'(4),\ f'(6)$

+Ve 0 +Ve 3 -Ve 5 +Ve

$f(0) = 0$.

$f(3) = 108$.

$f(5) = 0$.

⑥ $0 = 10x(2x^2 - 12x + 15)$

$x = 0$ or $x = 4.2247$ or $x = 1.7753$.

$f''(-1),\ f''(1),\ f''(2),\ f''(5)$

-Ve 0 +Ve 1.7753 -Ve 4.2247 +Ve.

$f(0) = 0$

$f(1.7753) = 58.1827$

$f(4.2247) = 45.3239$

④. No x asymptote.

$\lim\limits_{x \to +\infty} x^3(x-5)^2$

$= \lim\limits_{x \to +\infty} \dfrac{x^3(x-5)^2(x+5)^2}{(x+5)^2}$

$= \lim\limits_{x \to +\infty} \dfrac{x^3(x^2-10x+25)(x^2+10x+25)}{(x+5)^2}$

$= \lim\limits_{x \to +\infty} \dfrac{x^3(x^4 \cancel{+10x^3} + 25x^2 \cancel{-10x^3} - 100x^2 \cancel{-250x} + 25x^2 \cancel{+250x} + 625)}{x^2+10x+25}$

$= \lim\limits_{x \to +\infty} \dfrac{x^3(x^4 - 50x^2 + 625)}{x^2+10x+25}$

$= \lim\limits_{x \to +\infty} \dfrac{x^7 - 50x^5 + 625x^3}{x^2+10x+25}$

6. For a particular product, the revenue and cost functions are:

$$R(x) = x^3 \text{ and } C(x) = 30x + 500$$

Use Newton's method to approximate the break-even point to the nearest hundredth. [6 marks]

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$P(x) = x^3 - 30x - 500 = 0$

$x \approx 9$

$P'(x) = 3x^2 - 30.$

$x_1 = x_0 - \frac{x^3 - 30x - 500}{3x^2 - 30}$

$x_1 = 9 - \frac{-41}{213}$

$x_1 = 9.2$

$x_2 = 9.2 - \frac{2.688}{223.92}$

$= 9.19.$

$x_3 = 9.19 - \frac{0.451549}{223.3683}$

$= 9.188$

$x_1 = 9.1$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = N(x_1)$

$= 9.1 - \frac{f(9.1)}{f'(9.1)}$

$= 9.1889484$

$x_3 = 9.187978.$

$\therefore$ the break-even point approximate in 9.19.

~~7.~~ Let $f(x) = x^{\frac{2}{3}}$ [3+2+1=6 marks] $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

a) Find the linear approximation to $f(x)$ at $x = -27$.

b) Approximate $(-26.5)^{\frac{2}{3}}$ using your answer in part (a).

c) Without actually evaluating $(-26.5)^{\frac{2}{3}}$, use the shape of the graph of $f(x)$ to determine whether your approximation in (b) is too large or too small.

(a). $L(x) = f'(a)(x-a) + f(a)$

$= \frac{2}{3\sqrt[3]{a}}(x-a) + a^{\frac{2}{3}}$

$= \frac{2}{3\sqrt[3]{27}}(x-27) + \sqrt[3]{27^2}$

$= \frac{2}{3\cdot 3}(x-27) + 9$

$= \frac{2}{9}(x-27) + 9$

$= \frac{2x}{9} - 6 + 9$

$= \frac{2x}{9} + 3$.

$f(x) = x^{\frac{2}{3}}$

Linearization at $x = -27$.

$L(x) = f(a) + f'(a)(x-a)$

$= f(-27) + f'(-27)(x+27)$

$f(-27) = 9$

$f'(x) = \frac{2}{3} \cdot \frac{1}{x^{\frac{1}{3}}}$

$= -\frac{2}{9}$

$(-26.5)^{\frac{2}{3}} \approx f(-26.5) \approx L(-26.5) - 9 - \frac{2}{9}(0.5) = 9 - \frac{1}{9}$

(b)

$L(x) = \frac{2(26.5)}{9} + 3$

$= 8.89$

(c) ① domain $[0, \infty)$

② $0 = x^{\frac{2}{3}}$

$x = 0$ ∴ $(0, 0)$

③ $f(x) = f(-x)$ ∴ It is asymetry.

④ No x-asymptote

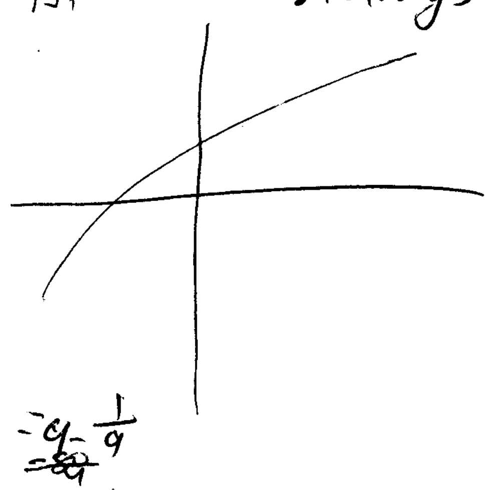
$\lim_{x \to \infty} = \infty$ ∴ No y-asymptote.

⑤ $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

$= \frac{2}{3\sqrt[3]{x}} > 0$ ∴ It always increasing.

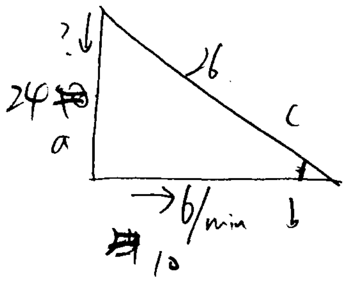
⑥ $f''(x) = -\frac{2}{9}x^{-\frac{4}{3}}$

$= -\frac{2}{9\sqrt[3]{x^4}} < 0$ ∴ It always decreasing



倚靠

8. A 26-foot ladder ~~is~~ placed against a building. The base of the ~~ladder is~~ slipping away from the building at a rate of 6 feet per minute. Find the rate at which the top of the ladder is sliding down the building at the instant when the bottom of the ladder is 10 feet from the base of the building. [6 marks]



$a^2+b^2=26^2$

$10^2+b^2=26^2$

$b=24$

$2a\cdot\frac{da}{dt}+2b\cdot\frac{db}{dt}=0$

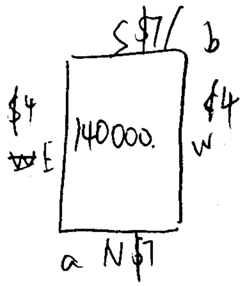
$2\cdot10\cdot6+2\cdot24\cdot\frac{da}{dt}=0$

$x=-2.5$ f/min

∴ the top sliding down is 2.5 per minute.

篱笆 围住 长方形的

9. A fence must be built to enclose a rectangular area of $140{,}000\ m^2$. Fencing material cost $7 per metre for the two sides facing north and south, and $4 per metre for the other two sides. Find the cost of the least expensive fence. Justify the result. [6 marks]



$a \cdot b = 140000$

$b = \frac{140000}{a}$

All cost : $7 \cdot 2a + 4 \cdot 2 \cdot \frac{140000}{a}$

$= 14a + 8 \cdot \frac{140000}{a}$

$= 14a + \frac{1120000}{a}$

$f'(a) = 14 - \frac{1120000}{a^2}$

$0 = 14 - \frac{1120000}{a^2}$

$\frac{1120000}{a^2} = 14$

$1120000 = 14a^2$

$a = 282.8427$

$C'' = \frac{(28)(140000)}{x^3} > 0$

$b = \frac{140000}{282.8427}$

$= 494.9748$

$\therefore$ All cost $= 14 \cdot 282.8427 + \frac{1120000}{282.8427}$

$= 7919.5959$

9.7
13.5005
24.666
58
3.
8.
3.3
14.58
17.5
11
9.46
11.4
40-
63.93
0.4
37
70.
2.925
3.3
3.3
3.7
2.7
3

MATH 3.3
PSYC 2.7
LING 3.

7.

10. A company is developing a new soft drink. The cost in dollars to produce a batch of the drink is given by

$$C(x,y) = 27x^3 - 72xy + 8y^3 + 1442,$$

where x is the number of kilograms of sugar per batch and y is the number of grams of flavoring per batch. Find the amounts of sugar and flavoring that result in the minimum cost per batch. Use the Second Derivative Test to justify your result. [8 marks]

$C_x'(x,y) = 81x^2 - 72y$

$0 = 81x^2 - 72y$

$72y = 81x^2$ — ①

$C_y'(x,y) = -72x + 24y^2$

$24y^2 = 72x$ — ②

① ÷ ②

$\frac{3}{y} = \frac{9x}{8}$

$\frac{3}{y} \cdot \frac{8}{9_3} = x$

$\frac{8}{3y} = x$

$24y^2 = 72 \cdot \frac{8}{3y}$

$24y^2 = \frac{192}{y}$

$24y^3 = 192$

$y^3 = 8$

$y = 2$

$\therefore \frac{8}{3 \cdot 2} = x$

$\frac{4}{3} = x$

$C_{xy} = -72 = C_{yx}$

$C_{xx} = 162x$

$C_{yy} = 48y$

$D(x,y) = C_{xx}C_{yy} - C_{xy}^2$

$= (162)(48)xy - (72)^2$

$D(\frac{4}{3},2) = (162)(48)(\frac{8}{3}) - (72)^2 > 0$

$C_{xx}(\frac{4}{3},2) > 0$

$\therefore$ By SDT Rel min abs. min (CPT)

2kg of sugar.

$\frac{4}{3}$g of flavoring per batch