Multivariable Calculus or Calculus of Several Variables

Functions of Two or More Variables

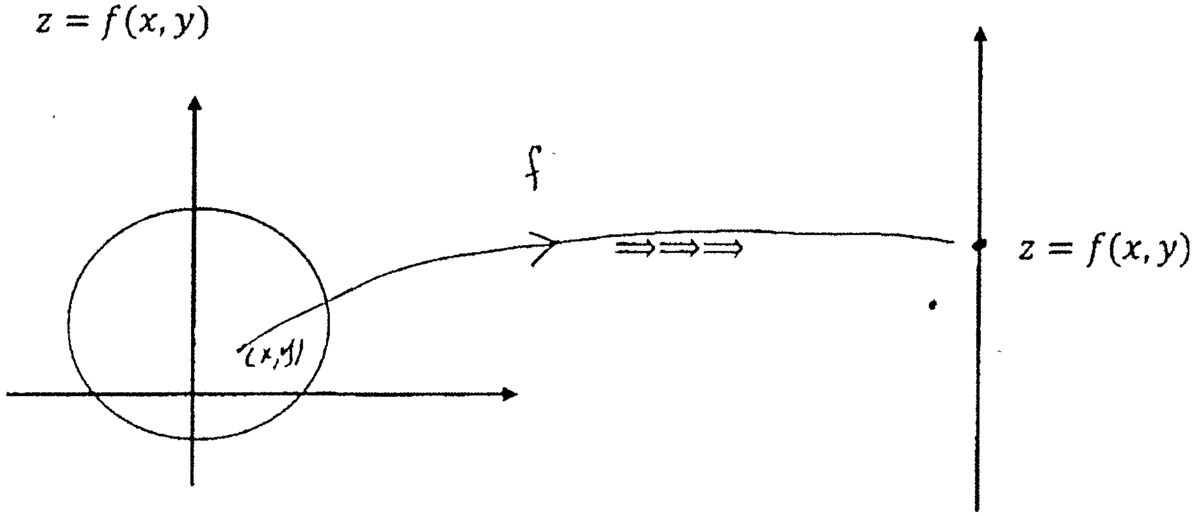
The expression $z = f(x, y)$ is a function of two variables if a unique value of z is obtained from each pair of real numbers (x, y). The variables x and y are independent variables, and z is the dependent variable. The set of all ordered pairs (x, y) such that $f(x, y)$ exists is the domain of f; the set of all values of $f(x, y)$ is the range. Similar definitions are given for functions of three, four, or more independent variables.

$W = f(x, y, z)$

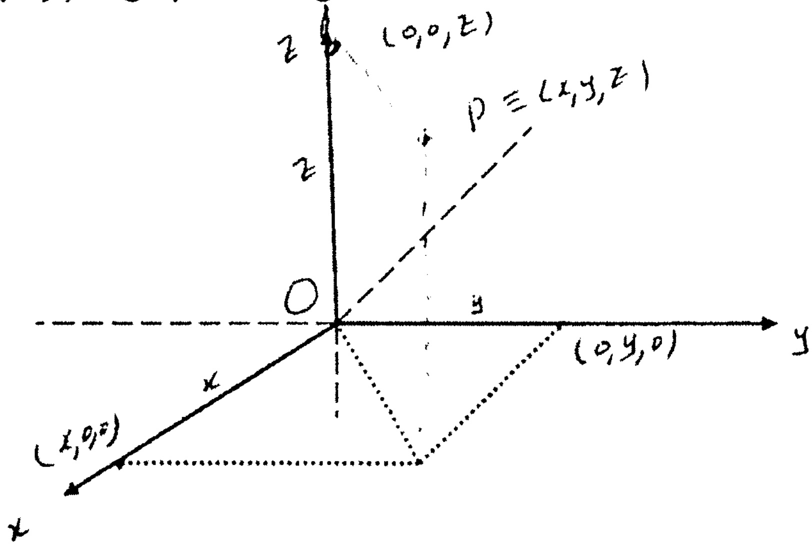
$W = f(x_1, x_2, x_3, x_4)$

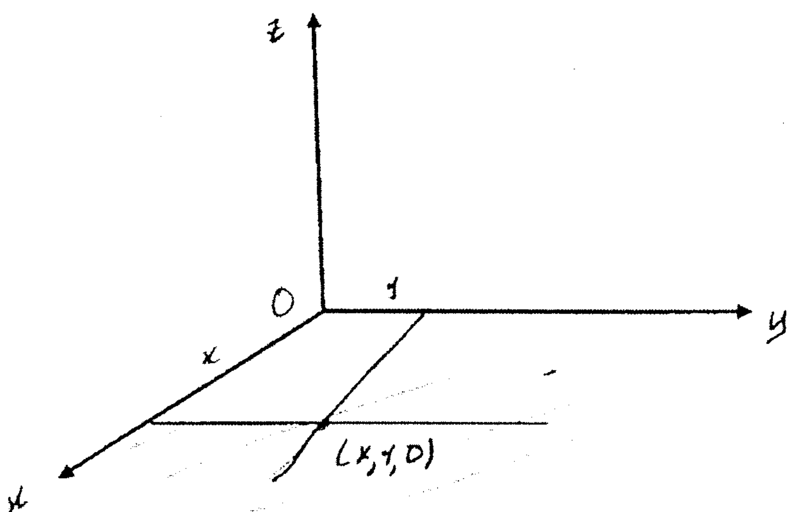
$W = f(x_1, x_2, x_3, x_4, x_5)$

$z = f(x, y)$

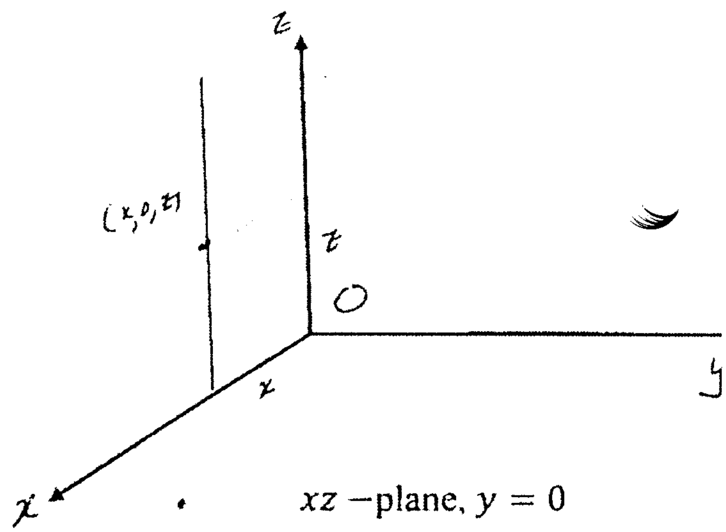


$z = f(x, y)$ is graphed using a three-dimensional coordinate system.

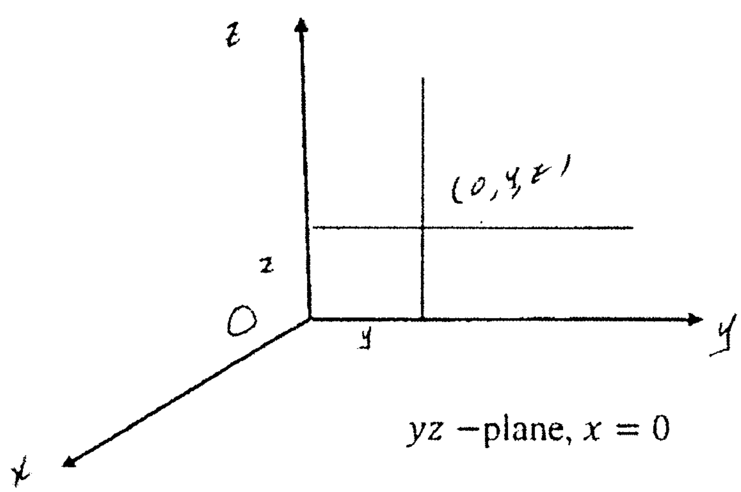




xy −plane, $z = 0$



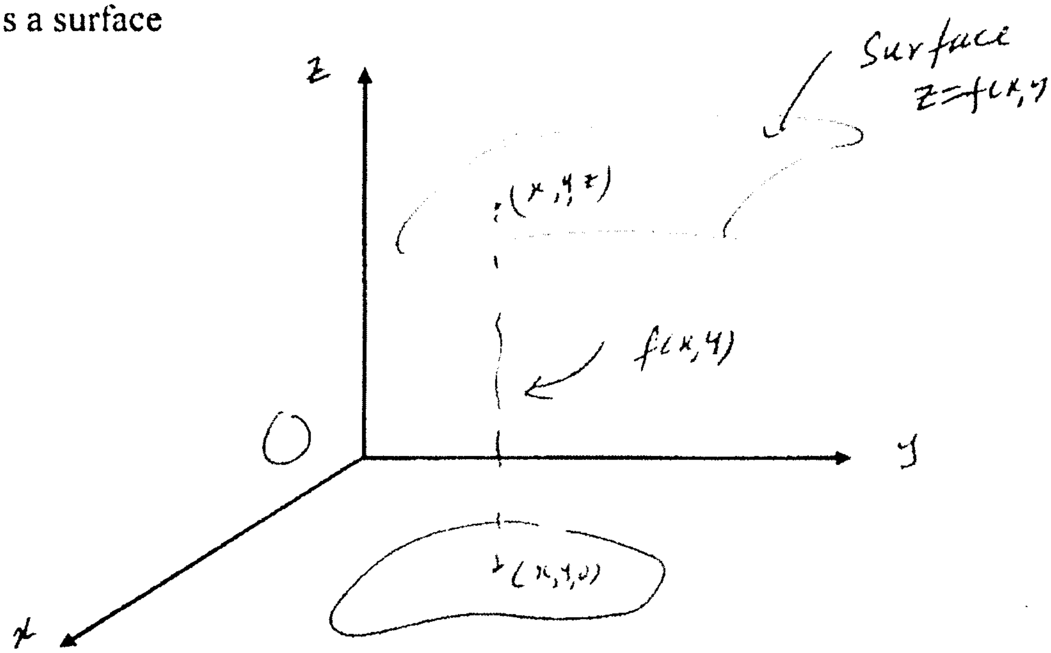
xz −plane, $y = 0$



yz −plane, $x = 0$

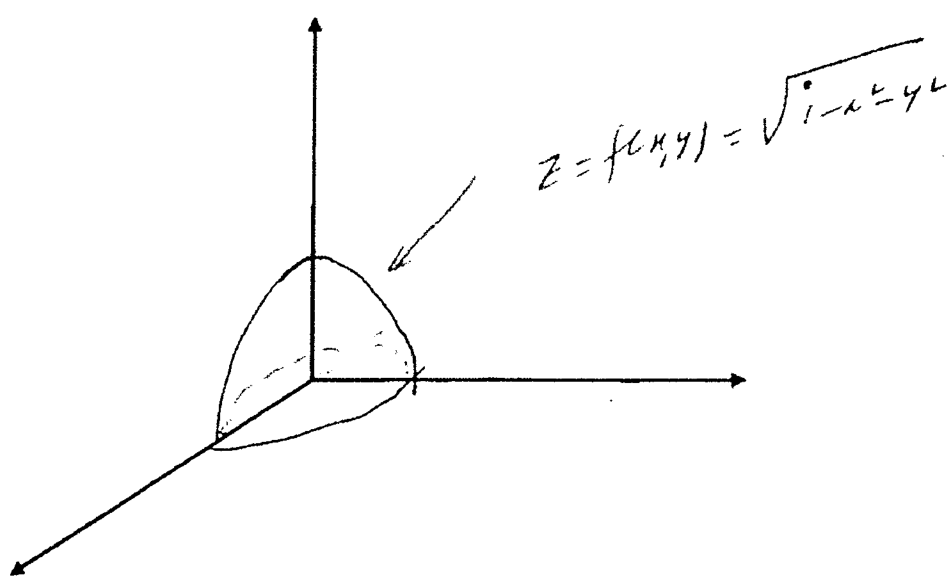
Graph of $z = f(x, y)$

is a surface



e.g. $z = f(x, y) = \sqrt{1 - x^2 - y^2}$

represents the surface of a hemisphere



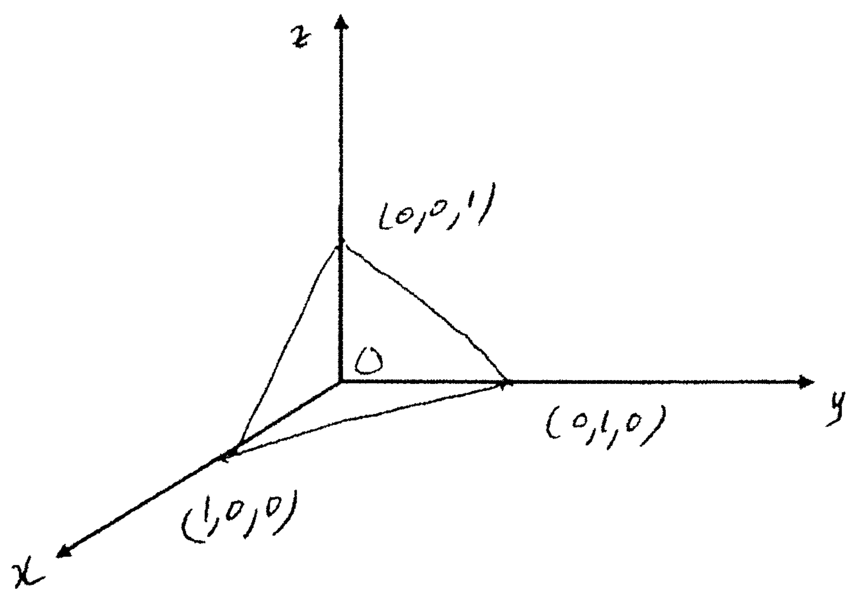
Plane

The graph of $ax + by + cz = d$ or:

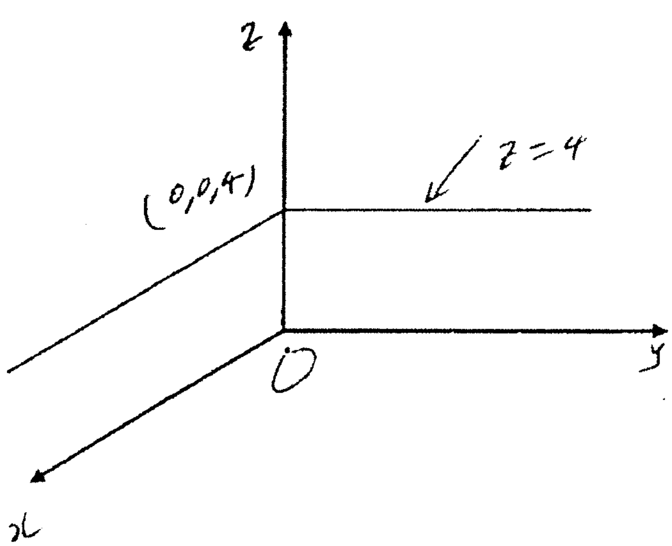
$$z = f(x,y) = \frac{d - ax - by}{c} = \frac{d}{c} - \frac{a}{c} \cdot x - \frac{b}{c} \cdot y$$

Is a plane if a, b, and c are not all zero.

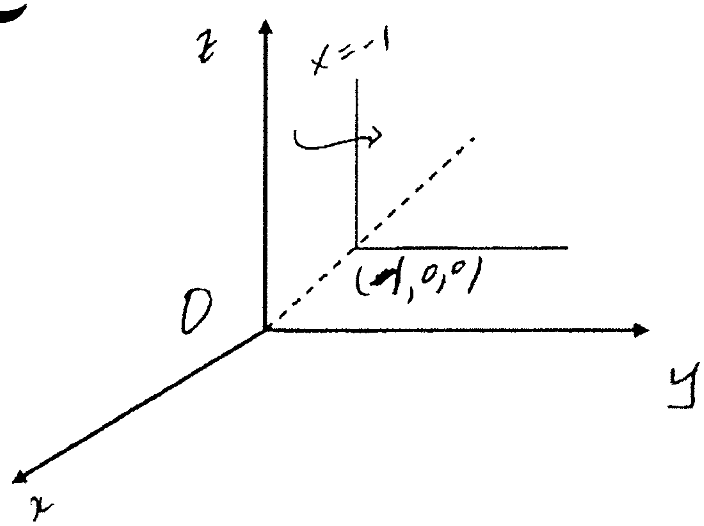
Example: $x + y + z = 1$



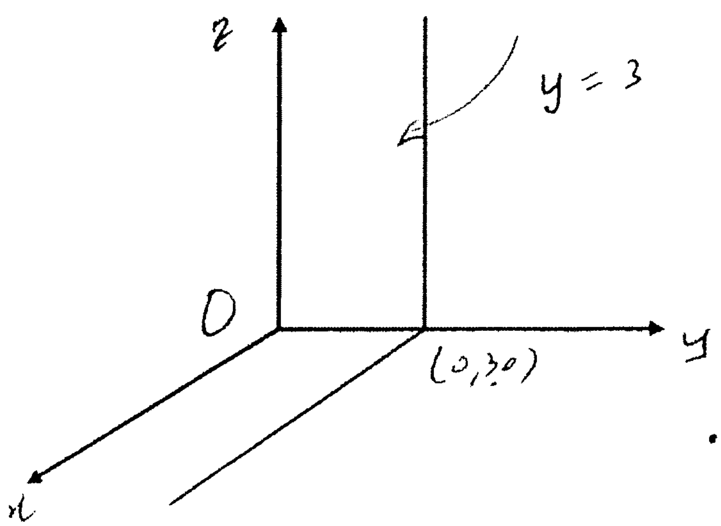
Plane $z = 4$ is parallel to the xy −plane and passes through (0,0,4).



Plane $x = -1$ is parallel to the yz −plane and passes through $(-1,0,0)$.



Plane $y = 3$ is parallel to the xz −plane and passes through $(0,3,0)$.

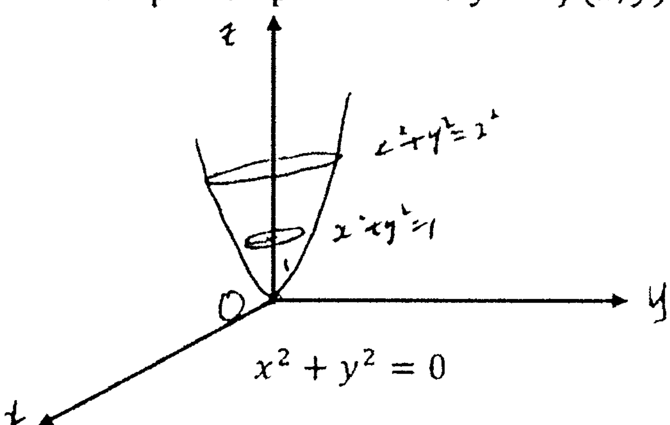


Example: Let $f(x, y, z) = 4xz - 3x^2y + 2z^2$, then

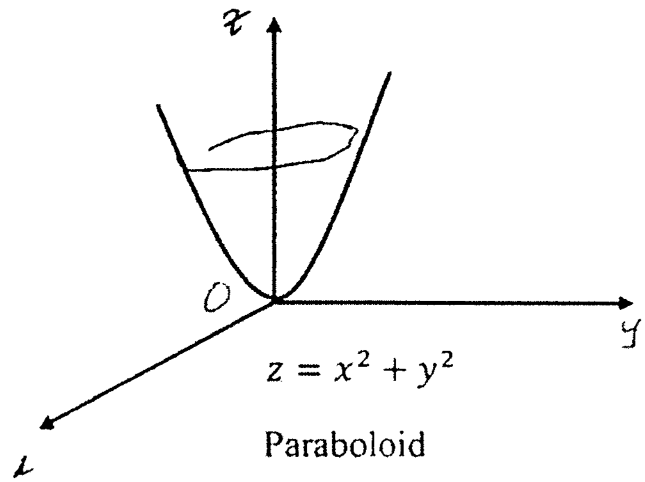
$$f(2,-3,1) = 4(2)(1) - 3(2^2)(-3) + 2(1^2) = 8 - (-36) + 2 = 46$$

Graphing $z = f(x, y)$ is done by using xy −trace or yz −trace or xz −trace. For example the xy −plane cuts the surface $z = f(x, y)$ in a xy −trace. Planes parallel to the xy −plane produce level curves in planes parallel to the xy −plane.

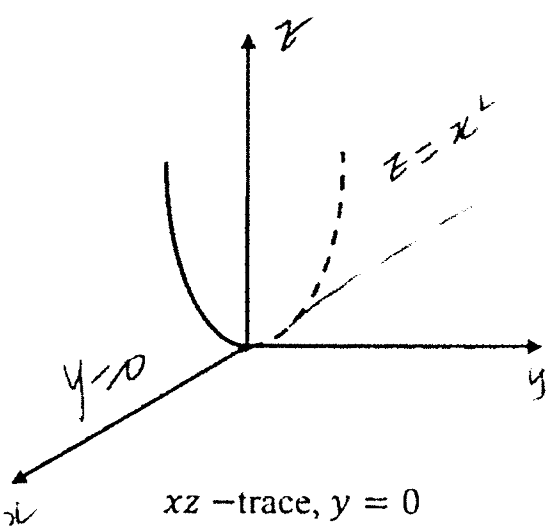
Example: Graph $z = x^2 + y^2 = f(x, y)$



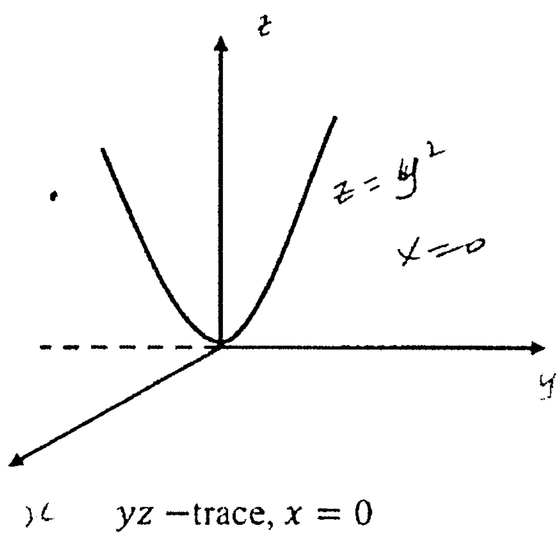
xy −traces $z = 0, z = 1, z = 4$



Paraboloid



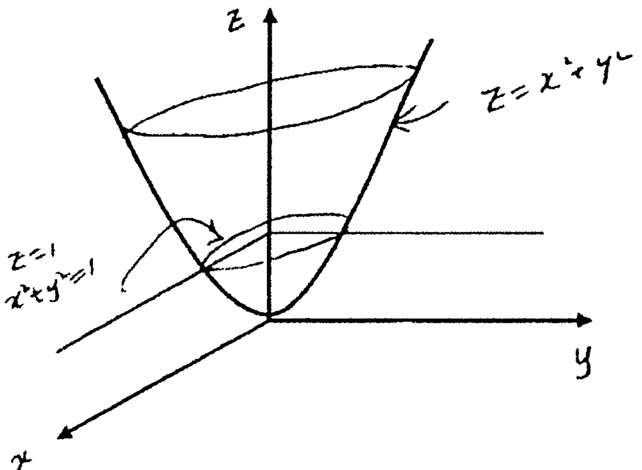
xz −trace, $y = 0$

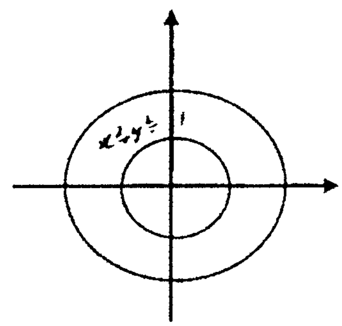


yz −trace, $x = 0$

Level Curves

For surface $z = f(x,y)[e.g. z = f(x,y) = x^2 + y^2]$, we get level curves by setting $z = constant = k$. [For $z = f(x,y) = x^2 + y^2$, the level curves are $x^2 + y^2 = k, k \geq 0$. These are concentric circles with centre $(0,0,k)$ and radius $\sqrt{k}$ in plane $z = k$, which is parallel to the xy −plane.





Level curves help us visualize surfaces.

Cobb-Douglas Production Function

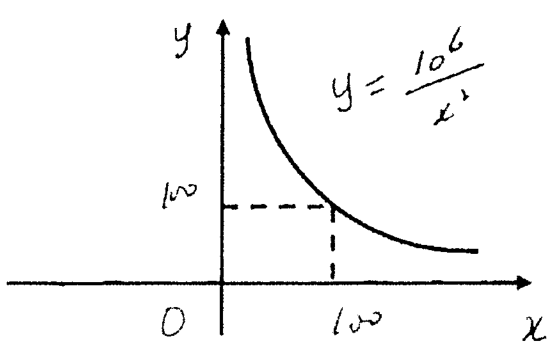
A production formula $z = f(x,y)$ is a function that gives the quantity z of an item produced as a function of x and y, where x is the amount of labour and y is the amount of capital needed to produce z units.

A production function of the form:

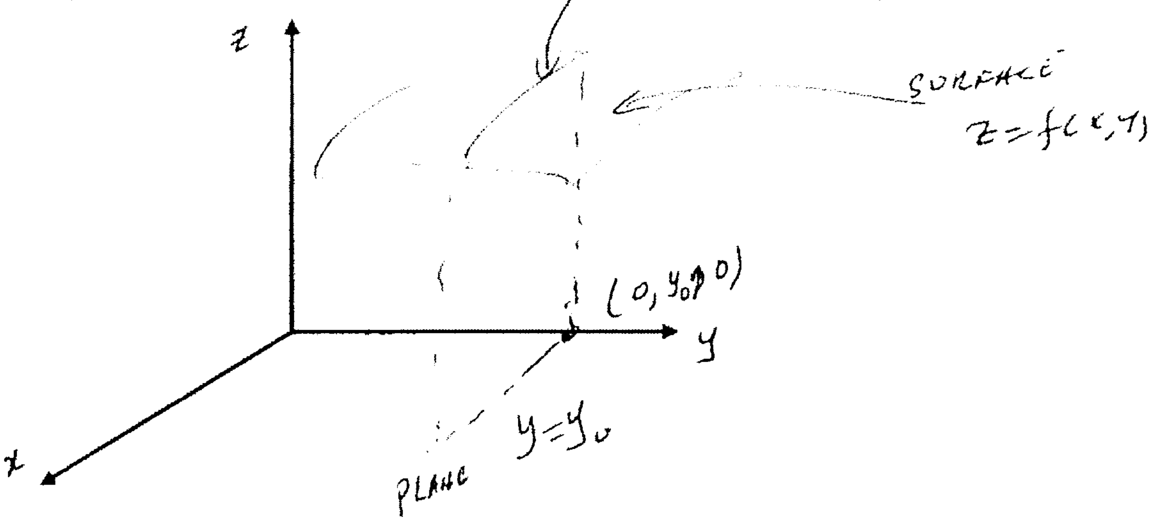
$z = f(x,y) = Ax^a y^{1-a}$, where A is a constant and a is a constant with $0 < a < 1$.

Example: Find the level curve at a production level of 100 items for the Cobb-Douglas production function $z = x^{2/3}y^{1/3}$

Solution: $z = 100 \Rightarrow 100 = x^{2/3}y^{1/3}$ or $y = \frac{10^6}{x^2}$



Partial Derivatives



Intersection of the surface $z = f(x, y)$ by the plane $y = y_0$ gives the curve:

$$z = z(x) = f(x, y_0)$$

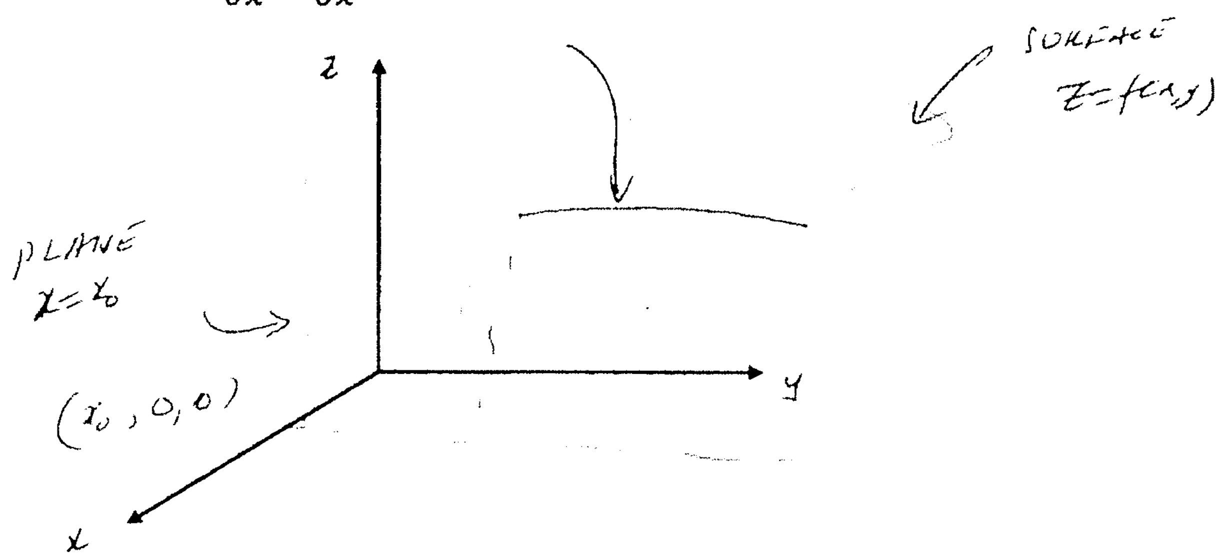
The partial derivative of z or f with respect to x is:

$$z_x = f_x(x, y) = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

provided that the limit exists.

e.g. $z = f(x, y) = x^2 + y^2$

$$z_x = f_x(x, y) = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = 2x$$



Intersection of the surface $z = f(x, y)$ by the plane $x = x_0$ gives the curve:

$$z = z(y) = f(x_0, y)$$

The partial derivative of z or f with respect to y is:

$$z_y = f_y = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

provided that the limit exists.

e.g. $z = f(x, y) = Ax^a y^{1-a}$

$$z_y = f_y = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = A(1-a)\, x^a y^{1-a}$$

Note: $\frac{\partial f}{\partial x}(x, y)$ gives the rate of change of f with respect to x, if y is held constant.

$\frac{\partial f}{\partial y}(x, y)$ gives the rate of change of f with respect to y, if x is held constant.

Higher-Order Partial Derivatives

Just as in the one-variable case, differentiating a two-variable function more than once leads to higher-order derivatives. In the case of two-variable functions, there are four possible second-order partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = z_{xx} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = z_{yy} = f_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = z_{xy} = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = z_{yx} = f_{yx}$$

Note: If $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ are continuous, then they are equal, i.e. the order of differentiation does not matter.

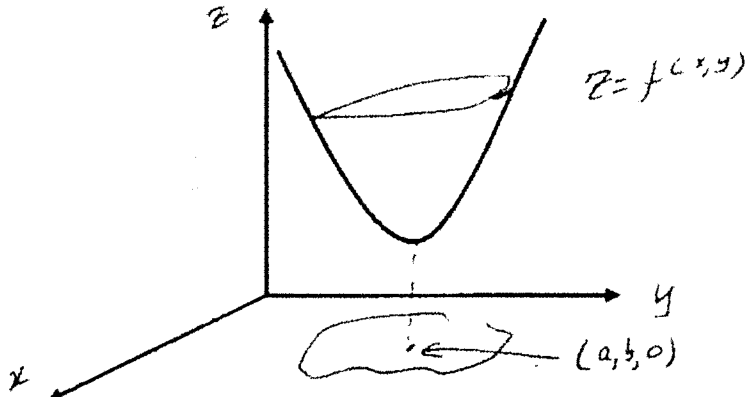
e.g. $z = f(x,y) = e^x e^{-y} - x^3 y^2$

$f_x = e^x e^{-y} - 3x^2 y^2$ $\qquad\qquad$ $f_{xy} = -e^x e^{-y} - 6x^2 y$

$f_y = -e^x e^{-y} - 2x^3 y$ $\qquad\qquad$ $f_{yx} = -e^x e^{-y} - 6x^2 y$

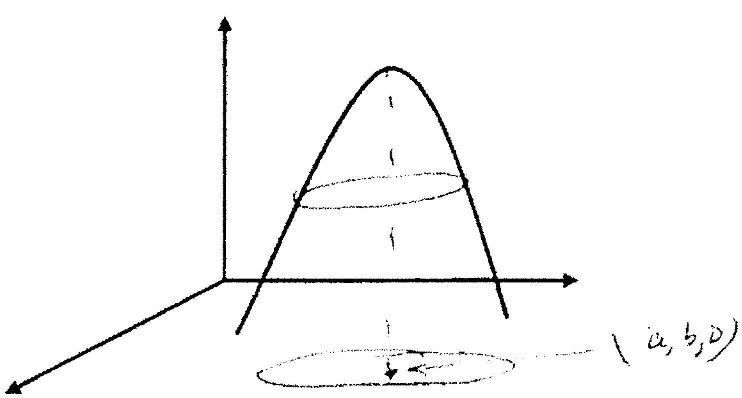
$$\therefore f_{xy} = f_{yx}$$

Maxima and Minima in Two Variables



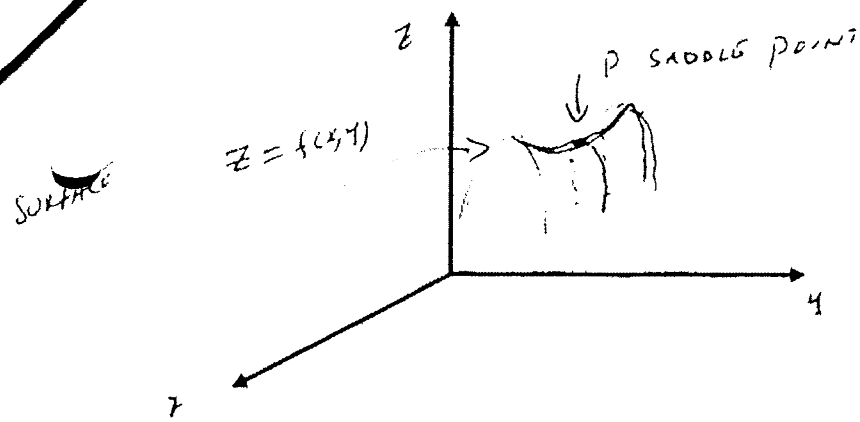
Relative or local minimum occurs at (a,b) and this minimum value is $f(a,b)$

e.g. $z = f(x,y) = x^2 + y^2 + 1$



Relative or local maximum occurs at (a,b) and this maximum value is $f(a,b)$

e.g. $z = f(x,y) = 1 - x^2 - y^2$



P is local minimum and local maximum depending on how we approach it

P is local minimum and P is local maximum, depending on how we approach it

e.g. $z = f(x, y) = x^2 - y^2$

Location of Relative (Local) Extrema

For $z = f(x, y)$, $f(a, b)$is the local extrema if:

$f_x(a, b) = 0$ and $f_y(a, b) = 0$. (a, b) is called its critical point

Test for Relative Extrema for the Critical Point (a, b)

Let $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$
Then,

1. $f(a, b)$is a relative maximum if $D > 0$ and $f_{xx}(a, b) < 0$
2. $f(a, b)$is a relative minimum if $D > 0$ and $f_{xx}(a, b) > 0$
3. $f(a, b)$is a saddle point (neither a maximum nor a minimum if $D < 0$
4. If $D = 0$, the test fails or gives us no information and we have to use some other method.

Examples: Let $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2$. Find the critical points and use the second derivative test to determine, if possible, the nature of $f(x, y)$ at each of these points.

Solution: For critical points, we solve:

$f_x = 3x^2 + 3y^2 - 6x = 0$ or $\quad x^2 + y^2 - 2x = 0$ (1)

$f_y = 6xy - 6y = 0$ or $\quad y(x-1) = 0$ (2)

From (2): $y = 0$ or $x = 1$

$\underline{y = 0}$ Substituting this into (1) gives: $x(x-2) = 0;\ x = 0$ or 2

$\therefore P(0,0)$ and $Q(2,0)$ are critical points.

$\underline{x = 1}$ Substituting this into (1) gives: $y^2 - 1 = 0;\ y = \pm 1$

$\therefore R(1,1)$ and $S(1,-1)$ are critical points.

$f_{xx} = 6(x-1);\ f_{yy} = 6(x-1);\ \therefore\ f_{xy} = f_{yx} = 6y$

$D(x,y) = f_{xx} \cdot f_{yy} - f_{xy}^2 = 36[(x-1)^2 - y^2]$

We next apply the second derivative test to each of these critical points.

$P(0,0)$: $D(0,0) = 36 > 0$ and $f_{xx}(0,0) = -6 < 0$. Therefore, f has a relative maximum at $(0,0)$ and its value is $f(0,0) = 0$.

$Q(2,0)$: $D(2,0) = 36 > 0$ and $f_{xx}(2,0) = 6 > 0$. Therefore, f has a relative minimum at $(2,0)$ and its value is -4.

$R(1,1)$: $D(1,1) = -36 < 0$. Therefore, f has a saddle point at $(1,1)$ and its value is -2.

$S(1,-1)$: $D(1,-1) = -36 < 0$. Therefore, f has a saddle point at $(1,-1)$ and its value is -2.

Global Maxima and Minima for Quadratic Functions

If a quadratic function $[f(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F]$ has a relative maximum (minimum) at a point, then it also has a global maximum (minimum) there.

2. Example: A company is developing a new soft drink. The cost in dollars to produce a batch of the drink is approximated by:

$$C(x,y) = 2200 + 27x^3 - 72xy + 8y^2, \quad x > 0, \ y > 0$$

Where x is the number of kilograms of sugar per batch and y is the number of grams of flavouring per batch. Find the amounts of sugar and flavouring that result in the minimum cost per batch. What is the minimum cost?

Solution:

$$C_x = 81x^2 - 72y = 0 \text{ or } y = \frac{9}{8}x^2 \qquad (1)$$

$$C_y = -72x + 16y = 0 \text{ or } y = \frac{9}{2}x \qquad (2)$$

From (1) and (2), we have:

$\frac{9}{8}x^2 = \frac{9}{2}x$ or $x = 0,4$

For $x = 4, y = 18$

Since (4,18) is the only critical point, then it gives the minimum cost, $C(4{,}18) = 1336$.

Also $D(4{,}18) = 5184 > 0, C_{xx}(4{,}18) = 162x\big|_{(4,18)} = (162)(4) > 0$

∴ Relative minimum at (4,18)

3. Example: The profit function of a company selling x units of one product and y units of another product is:

$$P(x,y) = 12x + 9y - 450 - 0.01(4x^2 + xy + y^2)$$

Find the pair (x, y) that maximizes the company's profit and determine the maximum profit.

Solution:

$P_x = 0 \Longrightarrow 8x + y = 1200$

$P_y = 0 \Longrightarrow x + 2y = 900$

$\Longrightarrow (x, y) = (100{,}400)$.

$D(100{,}400) = P_{xx} \cdot P_{yy} - P_{xy}^2 = 0.0015 > 0$

$P_{xx} = -0.08,\ P_{yy} = -0.02;\ P_{xy} = -0.01$

Since $P_{xx} < 0$, we have the maximum profit for $(x, y) = (100{,}400)$.