Fraser International College

Math 152 Lecture Notes

(to be used in conjunction with **Calculus: Early Transcendentals, 9th Edition** by James Stewart ISBN-10: 1337624187 ISBN-13: 978-1337624183)

Calculus II

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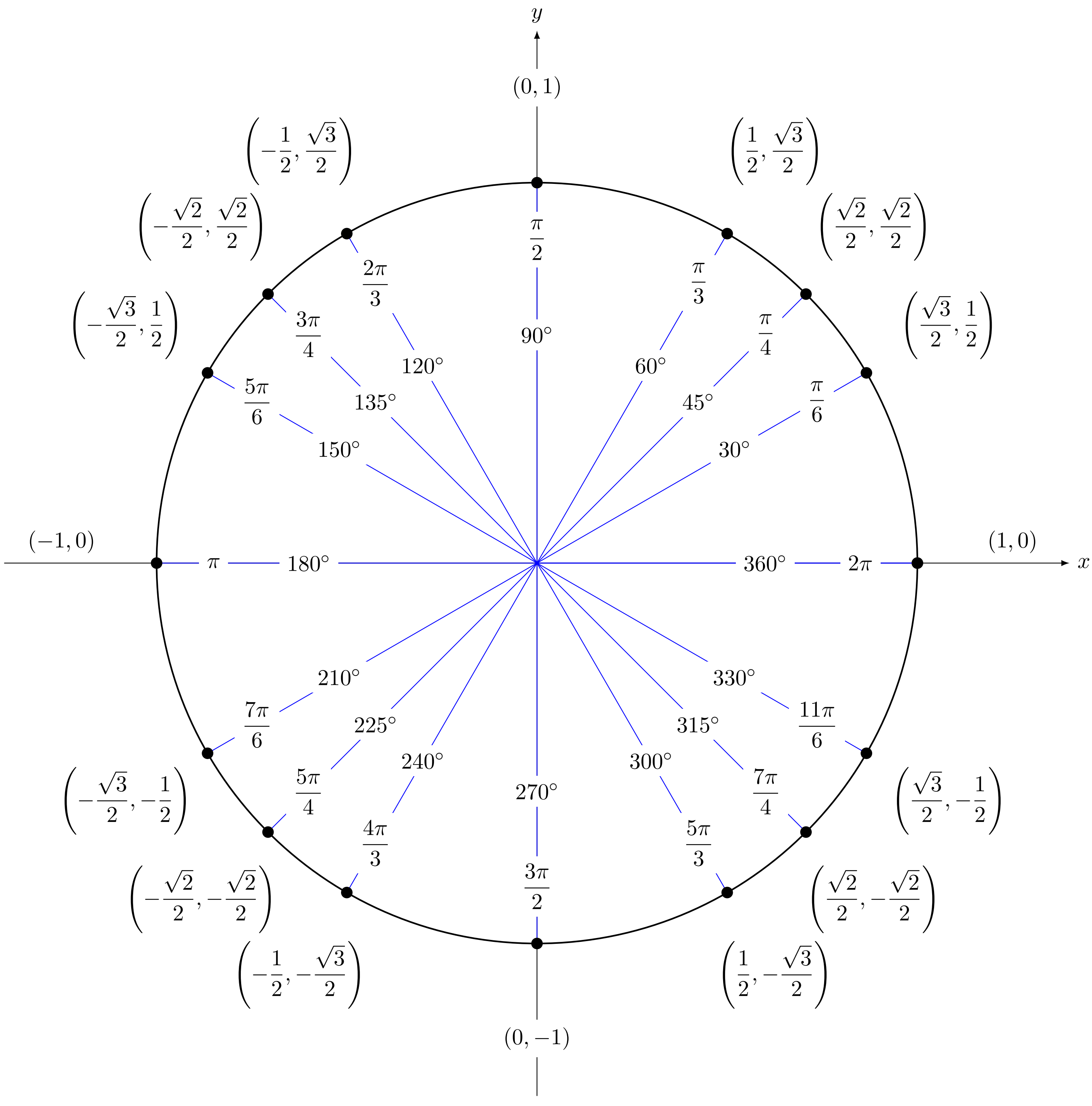
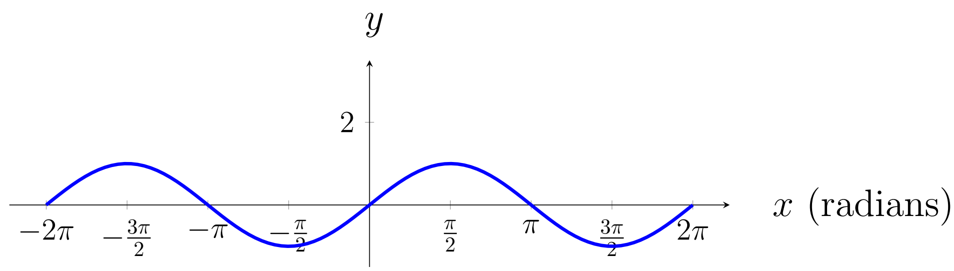
The Unit Circle

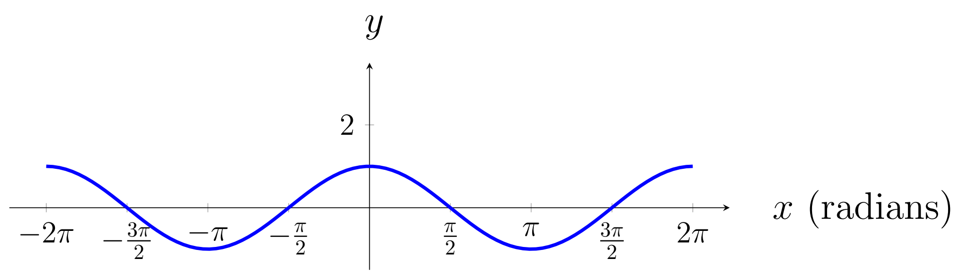


Graphs of Basic Trigonometric (and Inverse Trigonometric) Functions

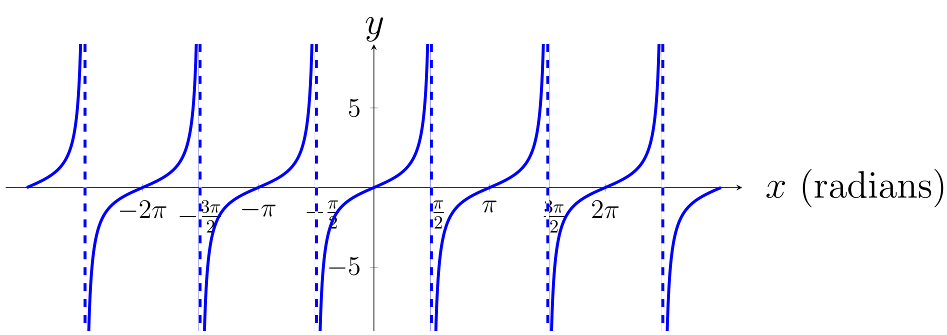
a) $y = \sin x$



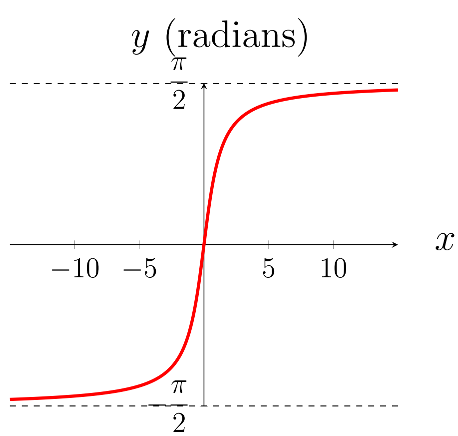
b) $y = \cos x$



c) $y = \tan x$



d) $y = \arctan x \, (= \tan^{-1} x)$



Differentiation Rules (Memorize!)

- $\dfrac{d}{dx}(c) = 0$, where $c \in \mathbb{R}$
- $\dfrac{d}{dx}(x^n) = nx^{n-1}$, where $n \in \mathbb{R}\backslash\{-1\}$
- $\dfrac{d}{dx}(e^x) = e^x$
- $\dfrac{d}{dx}(a^x) = a^x \ln a$, where a is a positive real number, and not equal to 1
- $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}$
- $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$
- $\dfrac{d}{dx}(\sin x) = \cos x$
- $\dfrac{d}{dx}(\cos x) = -\sin x$
- $\dfrac{d}{dx}(\tan x) = \sec^2 x$
- $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$
- $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
- $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
- $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

Contents

Chapter 5. Integrals

§4.9 Antiderivatives; §5.4 (Part 1) Indefinite Integrals

Given a function f, can we find a function F whose derivative is f? If such a function F exists, it is called an *antiderivative* of f.

Definition. *The function F is an* ***antiderivative*** *of f on some interval I if*

for all x in the interval I.

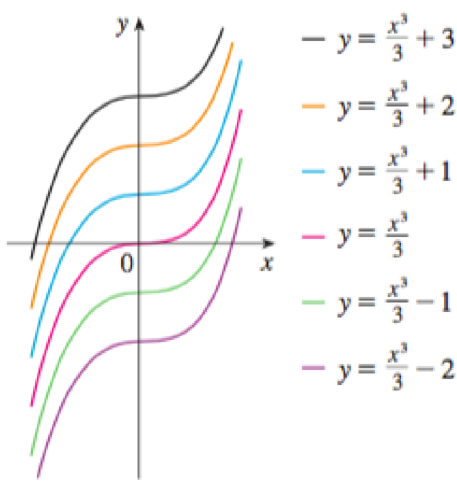
Example. Suppose that $f(x) = x^2$. Then an antiderivative of f is

$F(x) =$

In fact, any function of the form ____________ (where $C \in \mathbb{R}$) is an antiderivative of f.

Check:

Intuitively this makes sense; adding a constant C to a function simply vertically shifts the graph of the function by C units, which preserves the value of the derivative (i.e. ________ of the tangent line) at every point on the graph.



In general, we have the following theorem:

Theorem. *If F is an antiderivative of f on an open interval I, then the most general antiderivative of f on I is* $\qquad\qquad\qquad\qquad$ *where $C \in \mathbb{R}$.*

Example. Find the most general antiderivative of each function.

a) $f(x) = x \quad \Rightarrow F(x) =$

b) $f(x) = x^2 \quad \Rightarrow F(x) =$

c) $f(x) = \dfrac{1}{x^\pi} = \qquad \Rightarrow F(x) =$

d) $f(x) = x^{-1} \qquad \Rightarrow F(x) =$

(hint: the domain of f is ________. The domain of F should be ________.)

Based on your answers above, answer the following:

e) $f(x) = x^n \quad (n \in \mathbb{R}\backslash\{-1\}) \quad \Rightarrow F(x) =$

Definition. *The process of finding antiderivatives is called **antidifferentiation** or **integration**.*

<u>Notation:</u>

$$\boxed{\frac{d}{dx}[F(x)] = f(x) \Leftrightarrow \qquad\qquad\qquad\qquad\qquad\qquad}$$

where C is a constant.
(read "$\int f(x)\,dx$" as "the integral of $f(x)$ with respect to x")

- The expression $\int f(x)\,dx$ is called an **indefinite integral**.
- The symbol $\int$ is called an **integral sign**.
- The function $f(x)$ is called the **integrand**.
- The constant C is called the **constant of integration**.
- The symbol dx indicates that the independent variable is x.

Example.

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2 \Leftrightarrow \int \qquad\qquad dx =$$

Properties of the Indefinite Integral

a) $\displaystyle\int c \cdot f(x)\,dx =$ where c is a constant.

b) $\displaystyle\int (f(x) \pm g(x))\,dx =$

Example. Evaluate.

a) $\displaystyle\int \frac{4x \cdot \csc^2 x - 1}{x}\,dx =$

b) $\displaystyle\int \frac{2^x}{3}\,dx =$

c) $\displaystyle\int \frac{x^2}{x^2+1}\,dx =$

Caution.

- $\int f(x) \cdot g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx$;
- $\displaystyle\int \frac{f(x)}{g(x)}\,dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}$

Integration Formulas (Memorize)

- $\int 1 dx = x + C$
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
- $\int \cos x \, dx = \sin x + C$
- $\int \sin x \, dx = -\cos x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \csc^2 x \, dx = -\cot x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \csc x \ \cot x \, dx = -\csc x + C$
- $\int e^x \, dx = e^x + C$
- $\int b^x \, dx = \frac{b^x}{\ln b} + C,\ 0 < b, b \neq 1$
- $\int \frac{1}{x} dx = \ln |x| + C$
- $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$
- $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$

Application: Rectilinear Motion

Recall: Suppose that an object is moving in a straight line. If the object can be described by the position function $s = s(t)$, then:

- the velocity function $v(t) = s'(t)$. That is, $\boxed{s(t) = \quad\quad\quad\quad}$
- the acceleration function $a(t) = v'(t)$. That is, $\boxed{v(t) = \quad\quad\quad\quad}$

Example. A particle moves in a straight line and has acceleration given by

$$a(t) = 6t + 4$$

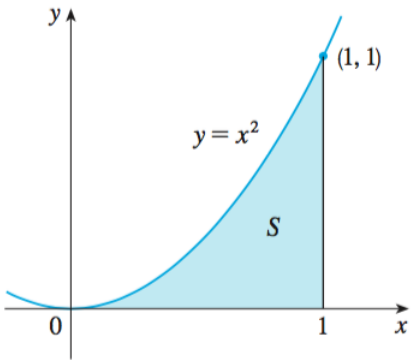
Its initial velocity is $v(0) = -6$cm/s and its initial displacement is $s(0) = 9$cm. Find its position function $s(t)$.

§4.9: # 5 – 21 (odd), 31, 47, 55, 65, 83
§5.4: # 9, 13, 15, 21, 23

§5.1 Areas

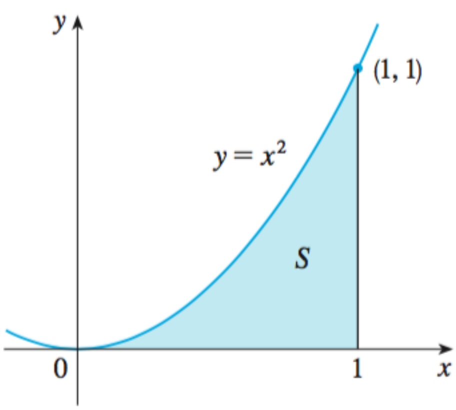
The Area Problem

How can we find the area A of the region S that lies under the curve $y = x^2$ from $x = 0$ to $x = 1$?



We don't know how to find the exact area of S (yet) but we can ________________ it using rectangles.

Method 1. Let's divide up the region into ____________ equal subintervals then create a rectangle for each subinterval by taking the height of the rectangle to be the value of the function at the **right endpoint** on the curve of each subinterval:



Each rectangle has width _____. We call this width _____.

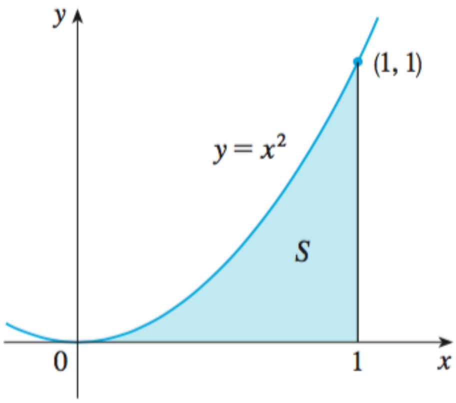
- The rectangle in the 1st subinterval has height __________.
- The rectangle in the 2nd subinterval has height __________.
- The rectangle in the 3rd subinterval has height __________.
- The rectangle in the 4th subinterval has height __________.

Thus the sum of the areas of the four rectangles, R_4, is:

$$R_4 =$$

Hence the area of S is approximately ________. Note that this is an ________________ of A, the area of S.

Method 2. Let's divide up the region into four equal subintervals then create a rectangle for each subinterval by taking the height of the rectangle to be the value of the function at the **left endpoint** on the curve of each subinterval:



Each rectangle has width _____. That is, _______________.

- The rectangle in the 1st subinterval has height __________.
- The rectangle in the 2nd subinterval has height __________.
- The rectangle in the 3rd subinterval has height __________.
- The rectangle in the 4th subinterval has height __________.

Thus the sum of the areas of the four rectangles, L_4, is:

$L_4 =$

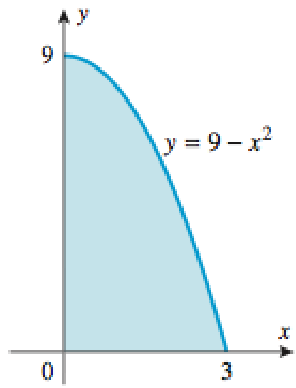
Hence the area of S is approximately ________________. Note that this is an ________________ of A, the area of S. In other words,

$$L_4 = 7/32 < A < R_4 = 15/32.$$

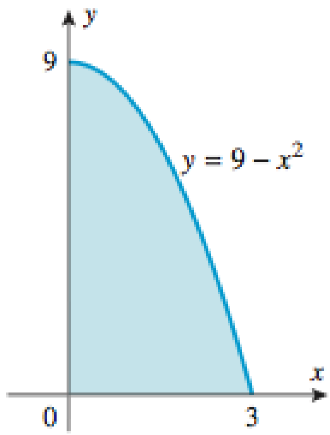
Remark. For this example, we got an *overestimate* of the area A using right endpoints and an *underestimate* using left endpoints. However, this is not always the case (it depends on the shape of the curve - see next example).

Example.

Let A be the area of the region that lies under the graph of $f(x) = 9 - x^2$ in the interval $[0, 3]$. Approximate A by using rectangles with **3 subintervals** of the same width. In each case, draw the rectangles.

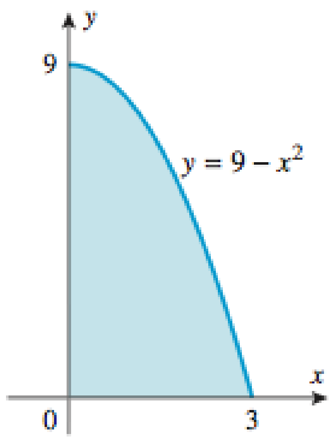


a) right endpoints



$R_3 =$

b) midpoints (i.e. still have three subintervals of equal width, but the height of each rectangle is the midpoint on the curve of each subinterval)

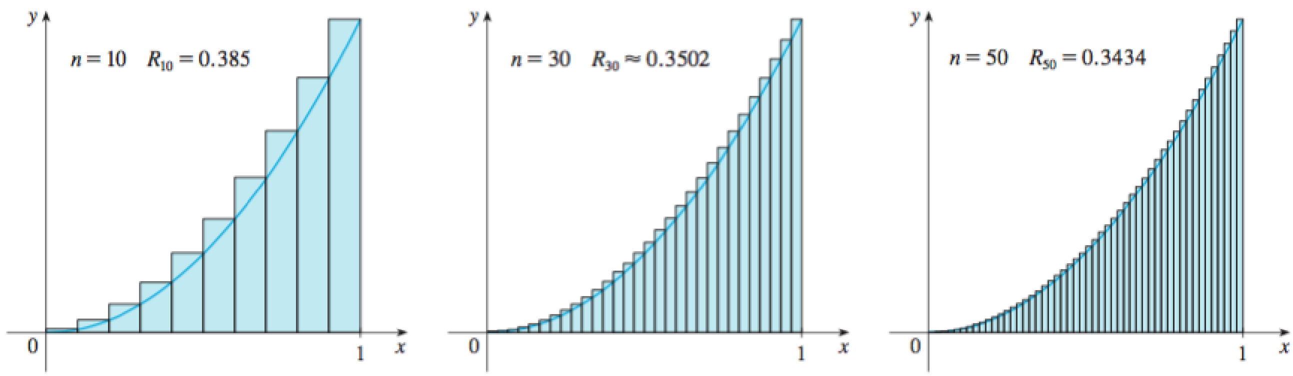


$M_3 =$

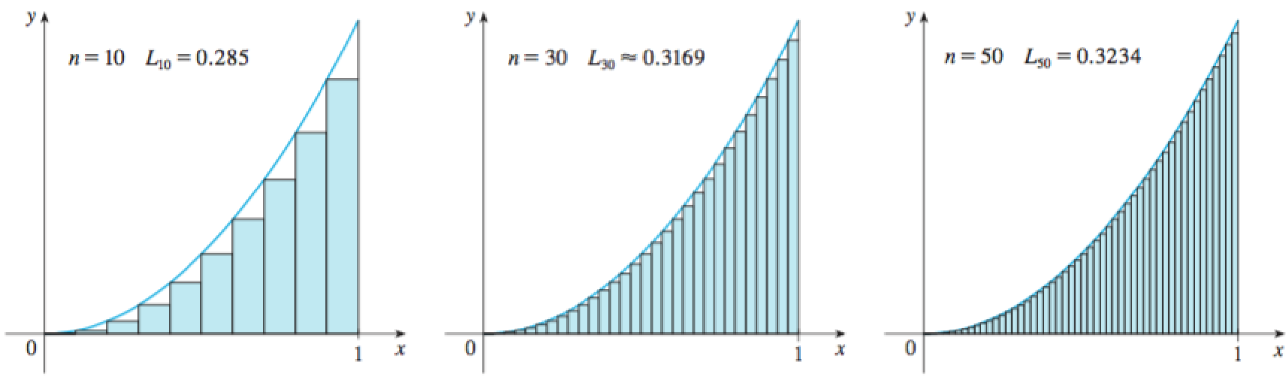
Remark. It turns out that the actual area of this region is 18. So it is an overestimate, but it is hard to tell from the rectangles.

We can estimate the area from Page 8 more accurately by using more subintervals. In what follows n = number of subintervals.

Using right endpoints:



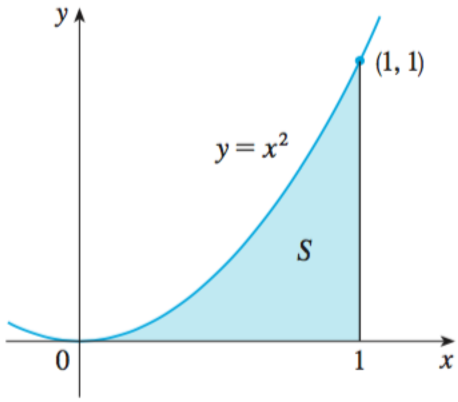
Using left endpoints:

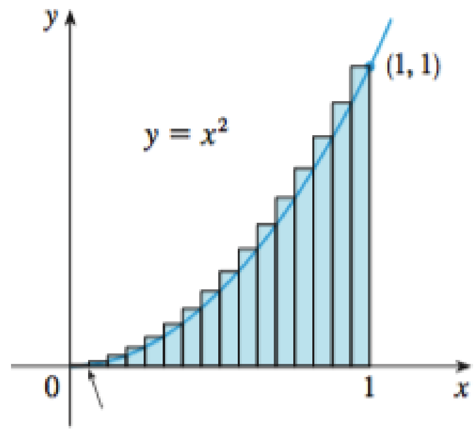


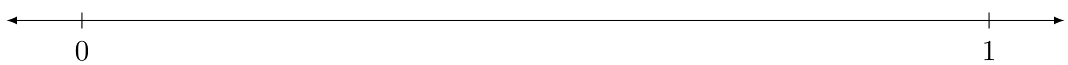
n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

It seems that R_n (and L_n) is approaching ________________ as n increases. The next example shows that this is in fact the case.

Example. Show that $\lim_{n\to\infty} R_n = \dfrac{1}{3}$ for the region S.







Let's first express R_n as a function of n. Each rectangle has width $\Delta x =$ ____.

- The rectangle in the 1st subinterval has height ________.
- The rectangle in the 2nd subinterval has height ________.
- The rectangle in the 3rd subinterval has height ________.

  $\vdots$
- The rectangle in the nth subinterval has height ________.

Thus

$$R_n =$$

The function in the limit is in *open form*, so one can't evaluate the limit as it is...

One can show that

$$\boxed{1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}}$$

(See Ex. 5 in Appendix E for proof – no need to memorize)

Therefore,

$$R_n = \frac{1}{n^3}(1^2 + 2^2 + \cdots + n^2) =$$

And so

$$\lim_{n\to\infty} R_n =$$

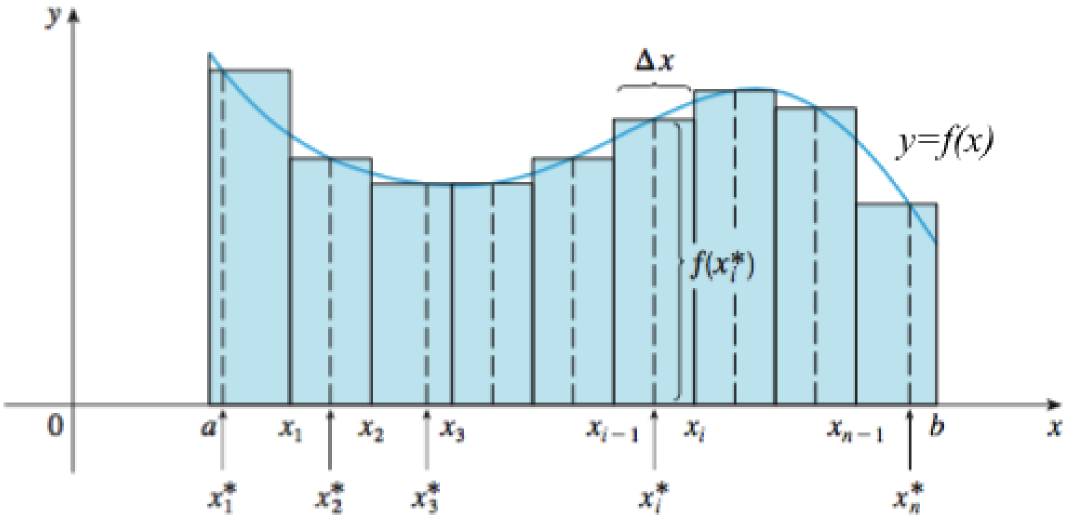
One can similarly show that (try!)

$$\lim_{n\to\infty} L_n = \frac{1}{3}.$$

As n increases, both L_n and R_n become better and better approximations to A. So we *define* the area A to be

$$A =$$

In general, we can take the height of the rectangle to be the value of f at ________ number x_i^* in the i-th subinterval $[x_{i-1}, x_i]$ and get the same area A as $n \to \infty$.



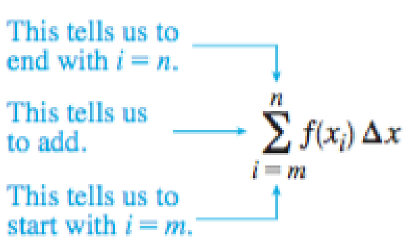
So the **area** A of the region under the graph of the continuous function f is defined as:

$$\boxed{\begin{aligned} A &= \lim_{n\to\infty} [f(x_1^*) \cdot \Delta x + f(x_2^*) \cdot \Delta x + \cdots + f(x_n^*) \cdot \Delta x] \\ &= \lim_{n\to\infty} [(f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)) \cdot \Delta x] \\ &= \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i^*) \Delta x \right) \end{aligned}} \tag{1}$$

where x_i^* is *any* number in the subinterval $[x_{i-1}, x_i]$ [1] .

Remark. If we set $\boxed{x_i^* = \qquad}$, then we have rectangles using the right endpoints.

If we set $\boxed{x_i^* = \qquad}$, then we have rectangles using the left endpoints.

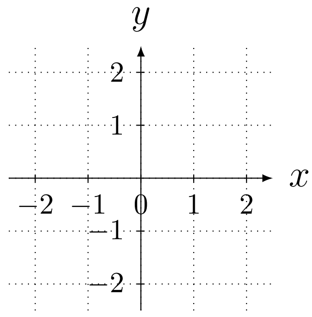


[1] If you need practice with sigma notation, look at the examples and try some of the exercises in Appendix E.

Example. Find the exact area under the graph of $f(x) = x - 1$ and the interval $[1, 2]$ using **right endpionts**.

Hint: $\boxed{\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}}$ (no need to memorize)

Note: we can get the answer to this question using simple geometry:



Area =

$\Delta x =$

The exact area under the curve using **right endpoints** is:

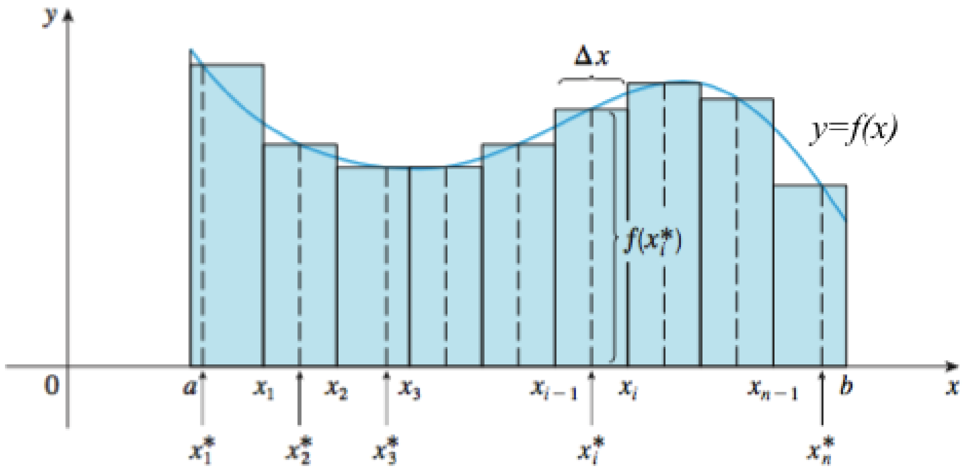
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) \cdot \Delta x] = \lim_{n\to\infty} [(f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)) \cdot \Delta x]$$

Remark. We would get the same answer if we use left-endpoints or midpoints, or any other point in each subinterval.

Homework. **5.1**: # 3,5 + review Sigma Notations (see .pdf in the Preliminaries folder in Moodle)

§5.2 The Definite Integral

Recall: To find an area under the graph of $y = f(x)$ on $[a, b]$, we can take the height of the rectangle to be the value of f at any number x_i^* in the i-th subinterval $[x_{i-1}, x_i]$ and get the exact area as $n \to \infty$.



We found in §5.1 that the formula for the *exact* area under the curve is

$$\boxed{A \;= \lim_{n\to\infty} [f(x_1^*) \cdot \Delta x + f(x_2^*) \cdot \Delta x + \cdots + f(x_n^*) \cdot \Delta x] = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x \right)}$$

where x_i^* is *any* number in the subinterval $[x_{i-1}, x_i]$. We give this limit a name:

Definition. *The* ***definite integral of*** f ***from*** a ***to*** b *is*

$$\boxed{\qquad\qquad\qquad = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x \right)}$$

provided that this limit exists and gives the same value for all possible choice of the sample points in the subintervals. If the limit does exist, we say that f is ________________ *on* $[a, b]$.

Notation:

1) a and b are called the ________________ **of integration**.

2) a is the **lower limit** and b is the **upper limit**.

3) The sum $\displaystyle\sum_{i=1}^{n} (f(x_i^*) \cdot \Delta x)$ is called a ________________________.

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i^*)\cdot\Delta x)}$$

Example. Write the following limit as a definite integral on the interval $[1,2]$.

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^*)^3\ \Delta x =$$

(Replace $\lim\sum$ by ____, replace x_i^* by ____, and replace Δx by ____.)
The above definite integral represents the ________ under the curve $y = x^3$ between $x = 1$ and $x = 2$.

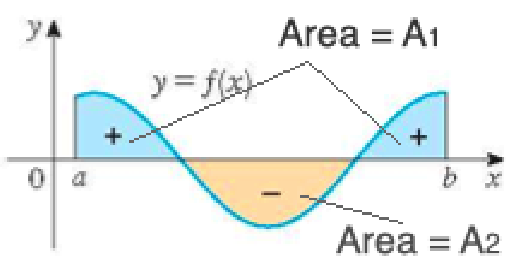
Remark.

1. The definite integral $\int_a^b f(x)\,dx$ is a ____________ (if it exists), since it represents an ________.

2. The function f appearing in the definition need not be positive.

   (a) If f is positive, then the definite integral can be interpreted as the ________ under the curve $y = f(x)$ from a to b.

   (b) If f is continuous on $[a,b]$ and takes on both positive and <u>negative</u> values, then the definite integral $\displaystyle\int_a^b f(x)\,dx$ can be interpreted as a **net area**:

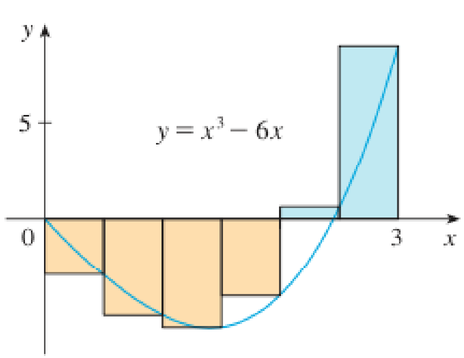


$$\int_a^b f(x)\,dx = \text{____________ where}$$

A_1 is the area above the $x-$axis, and

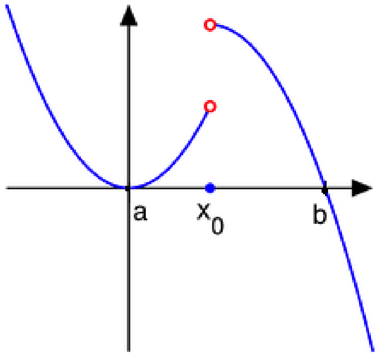
A_2 is the area below the $x-$axis.
(Note: $A_1, A_2 > 0$)

Why?

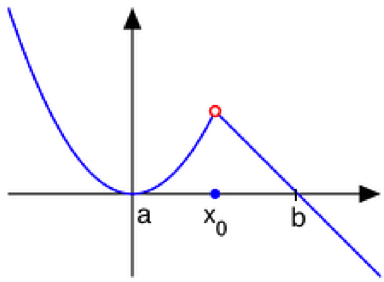


Question: When is a function integrable?

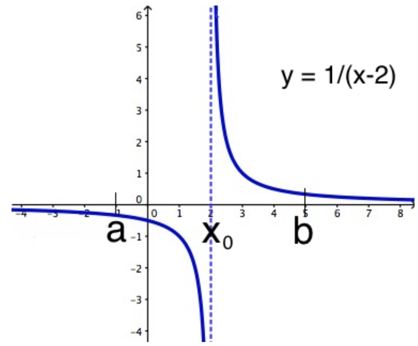
Recall: (types of discontinuities)



jump disc. at $x = x_0$



removable disc. at $x = x_0$

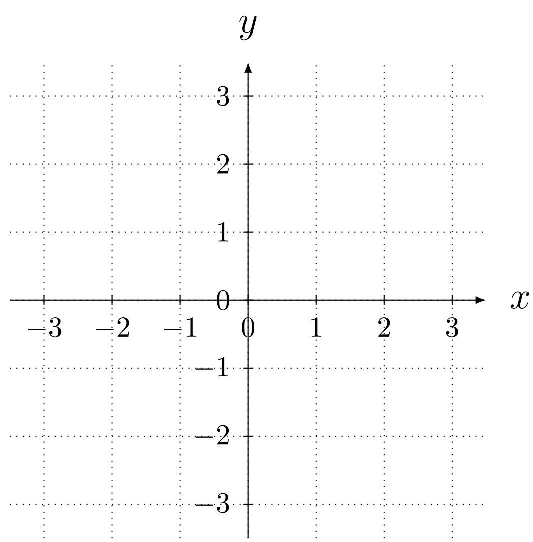


infinite disc. at $x = x_0$

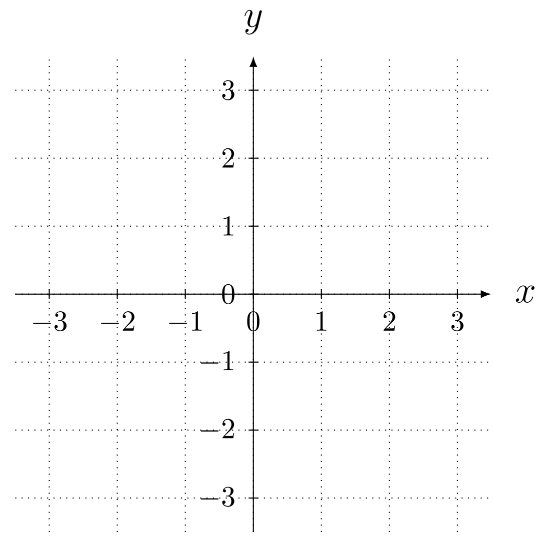
Theorem. *If f is continuous on $[a, b]$, or if f has only a finite number of of* ***jump*** *discontinuities or* ***removable*** *discontinuities, then f is* ***integrable*** *on $[a, b]$; that is, the definite integral* $\int_a^b f(x)\,dx$ ____________.

Example. Evaluate each definite integral **geometrically**, that is, by interpreting each in terms of **areas**.

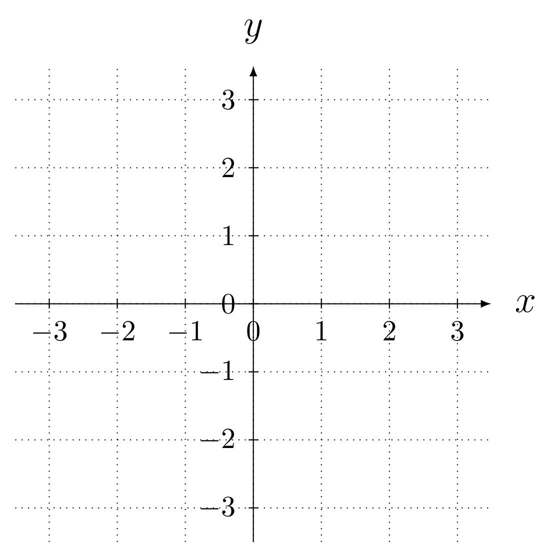
a) $\displaystyle\int_0^3 (x-1)\,dx$



b) $\displaystyle\int_1^3 f(x)\,dx$ where $f(x) = \begin{cases} 2, & 1 \leq x \leq 2 \\ -x, & 2 < x \leq 3 \end{cases}$



c) $\displaystyle\int_0^1 \sqrt{1 - x^2}\,dx$



When we defined the definite integral $\int_a^b f(x)\,dx$ we assumed that $a < b$. We now extend this definition to allow for a more general case of when $a \geq b$:

Definition. *a) If a is in the domain of f, then we define*

$$\boxed{\int_a^a f(x)\,dx = \qquad\qquad\qquad}$$

b) If f is integrable on $[a, b]$ (i.e. $a < b$) then we define

$$\boxed{\int_b^a f(x)\,dx = \qquad\qquad\qquad\qquad\qquad}$$

Example. $\displaystyle\int_1^0 \sqrt{1-x^2}\,dx =$

(as we computed from Page 19)

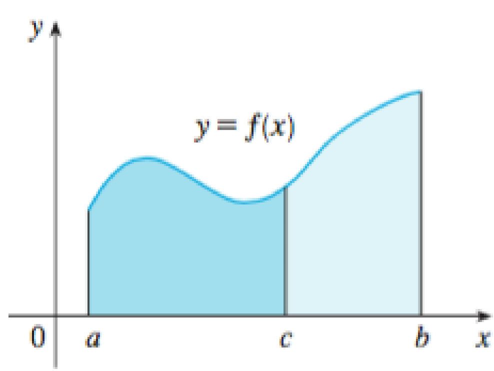
Properties of the Definite Integral

Suppose that f and g are continuous functions. Then:

a) $\displaystyle\int_a^b cf(x)\,dx =$, where c is a constant.

b) $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx =$

c) $\displaystyle\int_a^b f(x)\,dx = \int f(x)\,dx + \int f(x)\,dx$ (no matter how a, b, c are ordered)



Homework. §5.2: # 1, 3, 11, 21, 25, 35a-d, 41, 43, 45, 51, 57, 59, 63

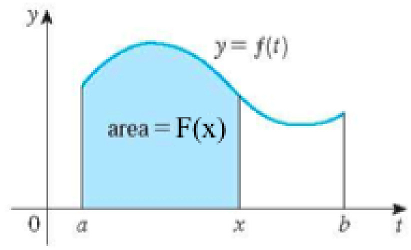
§5.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus establishes a connection between differential calculus (Math 151) and integral calculus (Math 152). This theorem allows us to compute areas and integrals easily without having to compute them as limits of sums.

Suppose that $f(x)$ is nonnegative and continuous on $[a, b]$ (and so it is integrable – see §5.2). Consider the definite integral

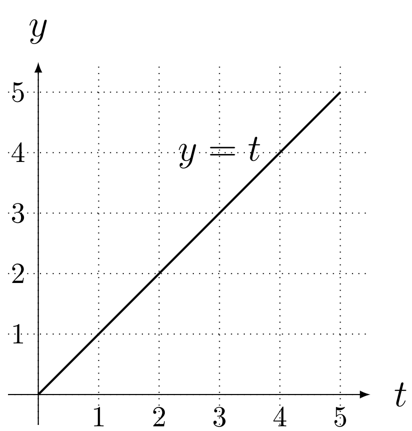
$$F(x) = \int_a^x f(t)\,dt \tag{1}$$

If we let x vary ($a \le x \le b$), $\int_a^x f(t)\,dt$ is *not* a number, but a ________________________.
In this case, we can think of F as the "area so far" function from a to x:



Note that F is a function that depends on ____, not on t.

Example. Let $f(t) = t$ and $a = 0$. Then $F(x) = \int_0^x f(t)dt$ (for $x \ge 0$) gives us the area of a triangle:



- $F(0) = \int_0 f(t)dt =$
- $F(1) = \int_0 f(t)dt =$
- $F(x) = \int_0 f(t)dt =$

Note: $F'(x) =$ ____(= ________)

That is, the derivative of the $F(x) = \int_a^x f(t)\,dt$ seems to be equal to $f(x)$...

(Fundamental Theorem of Calculus Part 1 (FTC1))

If f is ________________ on $[c, d]$, $a \in [c, d]$ then the function F defined by

is continuous on $[c, d]$, differentiable on (c, d) and

$F'(x) =$ ________ for each $x \in (c, d)$.
In alternate notation,

Roughly speaking, FTC1 says that if we first ________________ f and then ________________ the result, we get back to the original function f.

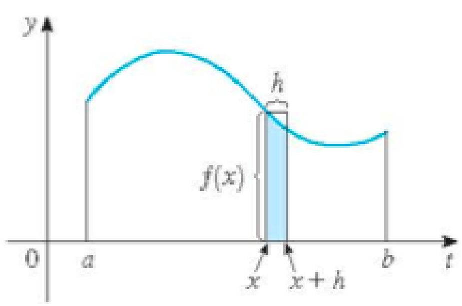
To see why this might be true (i.e. $F'(x) = f(x)$), let $F(x) = \int_a^x f(t)\, dt$ and let f be a non-negative function. From the definition of the derivative,

$$F'(x) = \tag{2}$$

Note that:

- $F(x + h)$ = area under the curve $y = f(t)$ from $t = a$ to $t =$ ________, and
- $F(x)$ = area under the curve $y = f(t)$ from $t = a$ to $t =$ ________.

Hence $F(x + h) - F(x) \approx$ ____________ (i.e. the area of the rectangle below)



Therefore, Eq. (2) becomes

$$F'(x) \approx$$

Intuitively, we thus expect

$$F'(x) =$$

In fact, it can be shown that this is true (and furthermore f need not be restricted to non-negative functions), and this is the FTC1 (for formal proof, see pp. 394 - 395).

Example. Find the *derivative* of $F(x) = \displaystyle\int_0^x \sqrt{1+t^2}\,dt$.

To use FTC1 on this question, note that the integrand $f(t) = \sqrt{1+t^2}$ is ________________ on an interval containing ____.

Hence $\dfrac{d}{dx}\left(\displaystyle\int_0^x \sqrt{1+t^2}\,dt\right) =$

Example. Find $\dfrac{d}{dx}\left(\displaystyle\int_3^x e^{t^2-t}\,dt\right)$.

Note that e^{t^2-t} is ________________ on an interval containing ____.

Hence $\dfrac{d}{dx}\left(\displaystyle\int_3^x e^{t^2-t}\,dt\right) =$

Recall: (§2.5- Math 151) To determine whether a function is continuous or not, one can use the following theorem(s):

1. The following types of functions are continuous at every number **in their domains**:
   - polynomials
   - rational functions
   - root functions
   - trig functions
   - inverse trig functions
   - exponential functions
   - logarithmic functions
2. If f and g are continuous at $x = a$ and $c \in \mathbb{R}$, then the following functions are continuous at $x = a$:
   - $f \pm g$
   - cf
   - fg
   - $\dfrac{f}{g}$ if $g(a) \neq 0$
3. If g is continuous at $x = a$ and f is continuous at $g(a)$, then $f(g(x))$ is continuous at $x = a$.
4. If function is differentiable at $x = a$ (i.e. derivative exists at $x = a$), then it is continuous at $x = a$.

Integrals with Functions as Limits of Integration

Question. How can we differentiate integrals in which at least one of the limits of integration is a function of x?

Example. $\dfrac{d}{dx}\left(\displaystyle\int_1^{\sin x}(1-t^2)\,dt\right)$, $\quad\dfrac{d}{dx}\left(\displaystyle\int_{\ln x}^{\pi}\frac{dt}{t^7-8}\right)$, $\quad\dfrac{d}{dx}\left(\displaystyle\int_{3x}^{x^2}\frac{t-1}{t^2+1}\,dt\right)$

To differentiate an integral of the form

$$\int_a^{g(x)} f(t)\,dt \tag{3}$$

where a is a constant, start with

$$F(x)=\int_a^x f(t)dt \tag{4}$$

Replacing x by $g(x)$ in Eq. (4) we get

Differentiating both sides with respect to x (by applying the Chain Rule), we get

Since $F'(x)=f(x)$, the above equation can be written as

$$\boxed{\frac{d}{dx}\left(\int_a^{g(x)} f(t)dt\right)=\qquad\qquad\qquad\qquad}$$

$$\boxed{\frac{d}{dx}\left(\int_a^{g(x)} f(t)\,dt\right) = f(g(x)) \cdot g'(x)}$$

Example. Evaluate $\dfrac{d}{dx}\left(\displaystyle\int_{3x}^{x^2} \frac{t-1}{t^2+1}\,dt\right)$.

$\dfrac{t-1}{t^2+1}$ is a ____________ function, so it is continuous on its domain, which is ________. Hence FTC1 can be used.

So $\dfrac{d}{dx}\left(\displaystyle\int_{3x}^{x^2} \frac{t-1}{t^2+1}\,dt\right) =$

(Fundamental Theorem of Calculus Part 2 (FTC2))

If f is continuous on $[a,b]$ and F is *any* antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx =$$

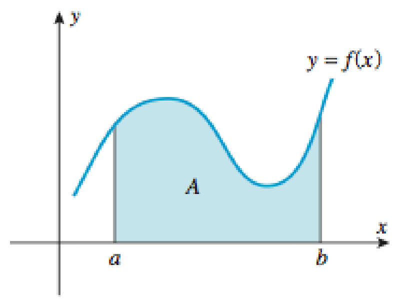
In alternate notation,

FTC2 says that if we _________________________ F then _________________ the result, we arrive back at the original function F, but in the form _____________.

Informal Proof (for the case when $f(x) \geq 0$ on $[a,b]$)

Suppose that $A = \displaystyle\int_a^b f(x)\,dx$ (see figure).

(We want to show that $F(b) - F(a) = A$.)



Let $F(x) = \displaystyle\int_a^x f(t)\,dt$ on $[a,b]$. By FTC1, we know that F is an antiderivative of f. Observe also that

- $F(a) = \displaystyle\int_a f(t)\,dt =$ ____
- $F(b) = \displaystyle\int_a f(t)\,dt =$ ____

Hence $F(b) - F(a) = A - 0 = A = \displaystyle\int_a^b f(x)\,dx$, as required.

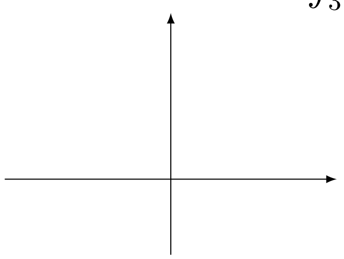
Notation: FTC2 can be stated using the following alternate notation:

$$\boxed{\int_a^b f(x)\,dx \qquad\qquad\qquad\qquad\qquad\qquad}$$

FTC 2: If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Example. Evaluate $\displaystyle\int_3^6 \frac{1}{x}\,dx$.



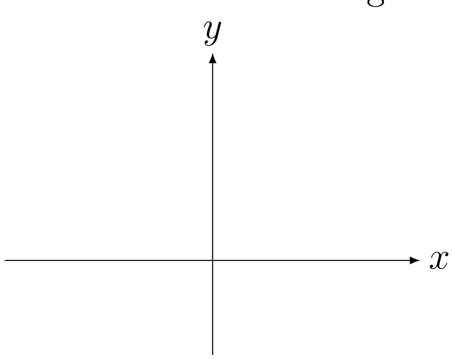
Since $f(x) = \dfrac{1}{x}$ is ________________ on $[3,6]$ (see graph), FTC2 can be applied.

Since $F(x) =$ ________________ is *an* antiderivative of f, FTC2 implies that

$$\int_3^6 \frac{1}{x}\,dx =$$

Example. Evaluate $\displaystyle\int_{-1}^3 \frac{1}{x^2}\,dx$

Since $f(x) = \dfrac{1}{x^2}$ is ________________________ on $[-1,3]$ (there is an ____________ discontinuity at $x =$ ____), FTC2 ________ be applied.
That is, $\displaystyle\int_{-1}^3 \frac{1}{x^2}\,dx$ ________________________ (for now). Geometrically, we can think of this definite integral as the value of the following area:



Remark. If you did not recognize that FTC2 could not be used, you would have done the following **erroneous** calculation:

$$\int_{-1}^3 \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Big|_{-1}^3 = -\frac{1}{3} - 1 = -\frac{4}{3}$$

Summary. (The Fundamental Theorem of Calculus)

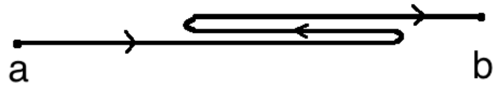
Let f be continuous on $[a, b]$ and let F be any antiderivative of f. Then:

1. $\dfrac{d}{dx}\left(\displaystyle\int_a^x f(t)\,dt\right) = f(x)$
2. $\displaystyle\int_a^b \left(\frac{d}{dx}(F(x))\right)\,dx = F(b) - F(a)$

The FTC tells us that differentiation and integration are __________________________.

Homework. §5.3: # 11, 13, 15, 17, 25, 35, 37, 41, 47, 49, 51, 53, 67, 73
§5.4: # 27 - 41 (odd), 45 - 53 (odd)

§5.4 (Part 2) Application of FTC2 in Physics



Suppose that a particle in rectilinear motion has velocity function $v(t)$ and position function $s(t)$.

displacement of a particle = ____________ position − ____________ position

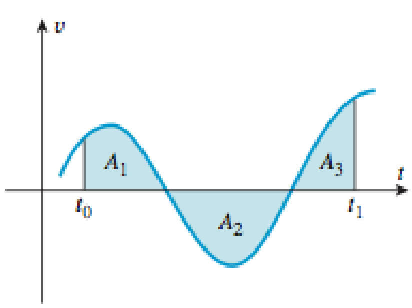
Recall: (**FTC2**) If f is continuous on $[a, b]$, then $\boxed{\int_a^b F'(t)\,dt = F(b) - F(a)}$

By setting $F(t) = s(t)$ (so $F'(t) =$ ________) with the interval $[a, b]$, we get:

$\int_a^b \quad dt =$ (= ________________________ over $[a, b]$)

Example. A particle moves on a coordinate line (i.e. rectilinear motion) so that its velocity at time t is $v(t) = 2t - 1$ m/s. Find the displacement of the particle during the time interval $0 \leq t \leq 3$.

Recall: $\int_a^b v(t)\,dt$ can be thought of as the **net area** of the graph of $y = v(t)$ in the interval $[a, b]$.



Hence, in the interval $[a, b]$:

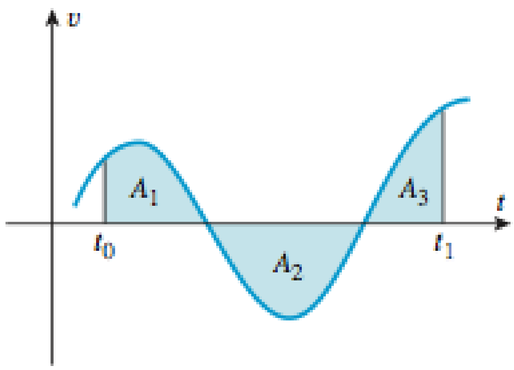
$$\text{Displacement} = \int_{t_0}^{t_1} v(t)\,dt =$$

Note: A_1, A_2, A_3 are areas, so they are > 0

Definition. ***Distance*** *traveled = distance travelled in the positive direction + distance travelled in the negative direction.*

Example. If a particle moves forward 5 m then moves backwards 5 m, its distance travelled is: ____ m. The displacement is ____ m.

Geometrically, the distance travelled by a particle can be found by adding up all the areas of the graph of $y = v(t)$ (NOT net area).



In the interval $[a, b]$:

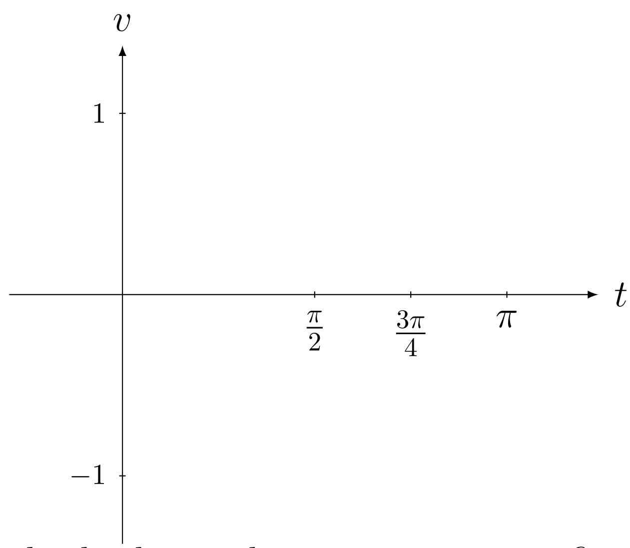
Distance travelled =

Note: A_1, A_2, A_3 are areas, so they are > 0

$$\boxed{\text{Distance traveled over time interval } [a, b] = \int_a^b \qquad\qquad dt}$$

$$\boxed{\text{Distance traveled over time interval } [a,b] = \int_a^b |v(t)|\,dt}$$

Example. A particle moves on a coordinate line (i.e. rectilinear motion) so that its velocity at time t is $v(t) = \cos t$ m/s. Find the distance traveled by the particle during the time interval $0 \le t \le 3\pi/4$.



We wish to find $\displaystyle\int \qquad\qquad dt$.

Note that $v(t) = \cos t$ is continuous on $[0, 3\pi/4]$, so FTC2 can be used.

To find an antiderivative of a function with absolute value signs, we must first re-express the function without the absolute value signs.

$$|\cos t| = \begin{cases} \qquad\qquad\qquad , & 0 \le t < \pi/2 \\ \qquad\qquad\qquad , & \pi/2 < t \le 3\pi/4 \end{cases}$$

Therefore,

$$\int_0^{3\pi/4} |\cos t| dt =$$

Homework. §5.4 (Part 2): # 69, 71

§5.5 The Substitution Rule

Motivation. How can we evaluate the integral

$$\int 2x \cdot \sqrt{1+x^2}\,dx?$$

We will see that we can evaluate this integral by applying the Chain Rule "backwards".

In general, suppose that we have an integral of the form

$$\boxed{\int f(g(x))g'(x)\,dx} \tag{1}$$

If we let ____________, then $\dfrac{du}{dx} =$ ________, so ____________. So (1) becomes:

$$\boxed{\int f(g(x))g'(x)\,dx = \qquad\qquad\qquad\qquad}$$

The process of evaluating an integral of the form $\int f(g(x))g'(x)\,dx$ using the substitution $u = g(x)$ (which converts it into $\int f(u)\,du$) is called the **Substitution Rule**.

Example. Evaluate.

a) $I = \displaystyle\int 2x \cdot \sqrt{1+x^2}\,dx$

b) $I = \displaystyle\int \cos(5x)\,dx$

c) $I = \displaystyle\int \sin^2 x \cos x \, dx$

$$\boxed{\text{Answer: } \frac{\sin^3 x}{3} + C}$$

d) $I = \displaystyle\int x^2 \cdot \sqrt{x-1} \, dx$

$$\boxed{\text{Answer: } \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C}$$

e) $I = \displaystyle\int \sqrt{e^x} \, dx$

$$\boxed{\text{Answer: } 2\sqrt{e^x} + C}$$

Evaluating Definite Integrals by Substitution

Question. How do we evaluate definite integrals in which a substitution is required?

Answer. Change the limits of integration when the variable is changed.

$$\boxed{\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \qquad\qquad\qquad\qquad}$$

Example. Evaluate.

a) $\displaystyle\int_0^{3/4} \frac{dx}{1-x}$ (note: this integral can also be written as $\displaystyle\int_0^{3/4} \frac{1}{1-x}\,dx$)

b) $\displaystyle\int_2^5 (2x-5)(x-3)^9\,dx$

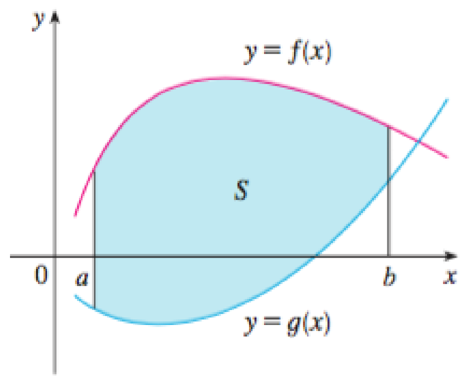
$$\boxed{\text{Answer: } \frac{2^{12}}{11} + \frac{2^{10}}{10} - \left(-\frac{2}{11} + \frac{1}{10}\right)}$$

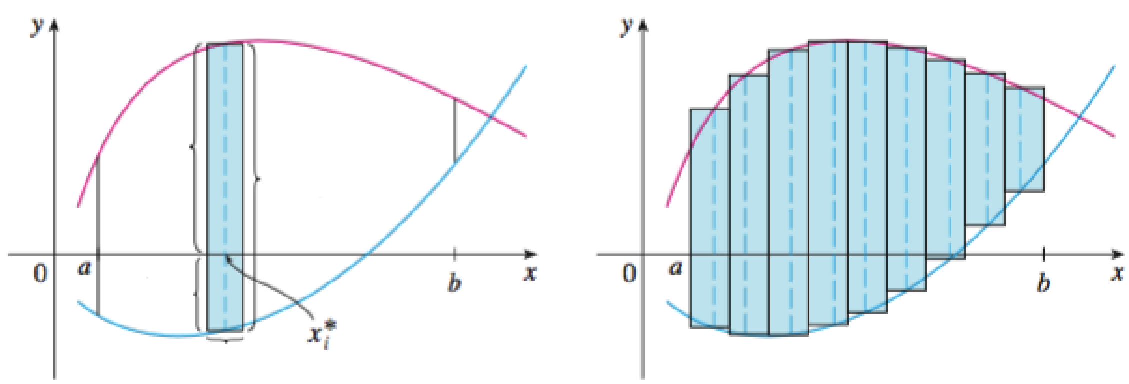
Homework. §5.5 # 1, 3, 5, 9 - 45 (odd), 51, 53, 59, 61, 63, 65, 69 - 79 (odd), 93, 97

Chapter 6. Applications of Integration

§6.1 Areas Between Curves

Let f and g be continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$. Let S be the region that lies between the curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$. How can we find the area of S?





(a) Typical rectangle (b) Approximating rectangles

Just as we did for areas under curves in §5.1, we divide S into n strips of equal width and approximate the area of the i-th strip by a ______________ with area (height) $\cdot$ (base)= ____________________________. (recall: x_i^* is *any* point in the interval $[x_{i-1}, x_i]$.)

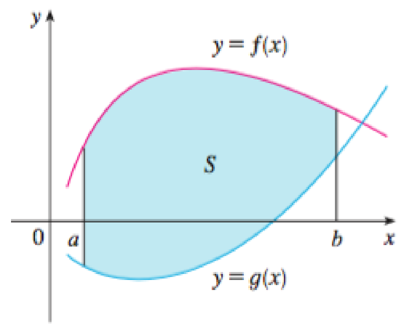
For the *exact* area A of the region S, we let _________ to get:

$$A = \qquad\qquad [f(x_i^*) - g(x_i^*)] \cdot \Delta x =$$

Hence we have the following result:

Let f and g be continuous and $f(x) \geq g(x)$ for all x in $[a, b]$. Then the area A of the region S bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and the vertical lines $x = a$ and $x = b$ is

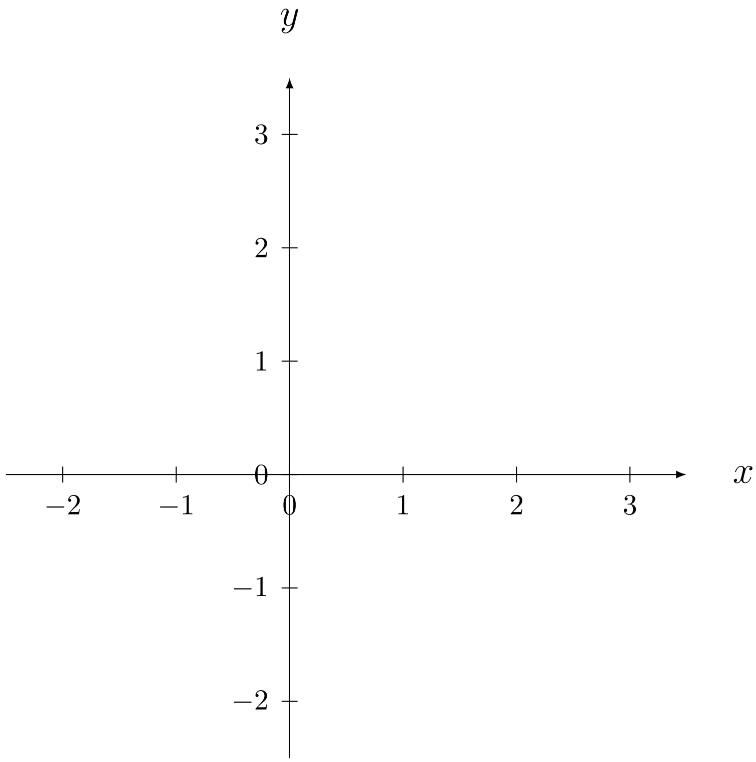
You can remember the integrand as (________ curve − ________ curve)



Remark. *** The area between two curves is NOT the net area (i.e. the area below the x-axis is NOT negative)! ***

Remark. In general, when we set up an integral for an area, it's helpful to sketch the region to identify the upper curve and the lower curve.

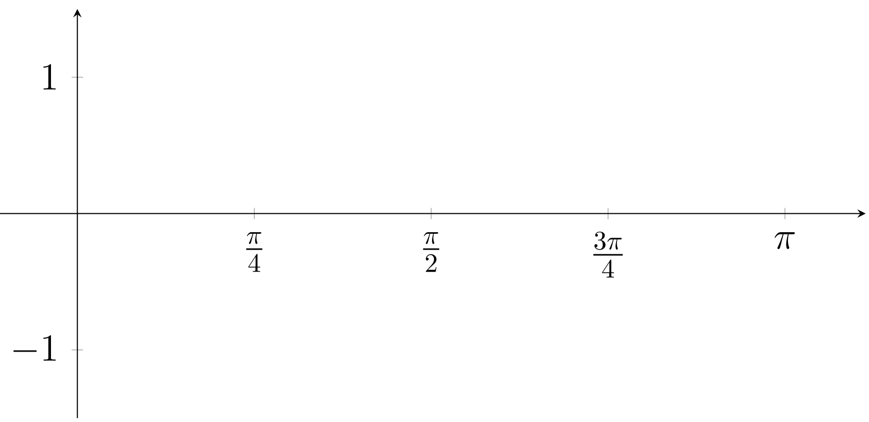
Example. Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.



The area of this region, A, is:

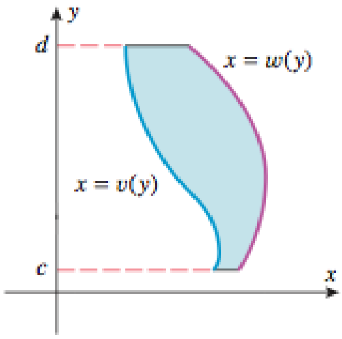
$$A = \int (\text{ upper curve } - \text{ lower curve }) \, dx$$

Example. Find the area of the regions bounded by the curves $y = \sin x$ and $y = \cos x$, $0 \leq x \leq \dfrac{\pi}{2}$. Hint: whenever given an interval for x, it may be helpful to draw vertical lines through the endpoints of the interval.



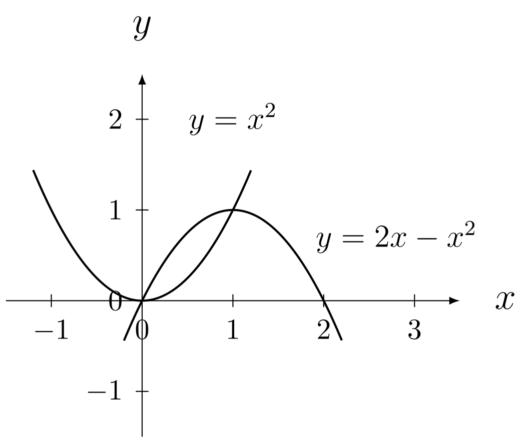
Some regions are best treated by regarding x as a function of y.

If w and v are continuous functions and $w(y) \geq v(y)$ for $c \leq y \leq d$ then the area of the region bounded on the left by the curve $x = v(y)$, on the right by the curve $x = w(y)$, and by the horizontal lines $y = c$ and $y = d$ is

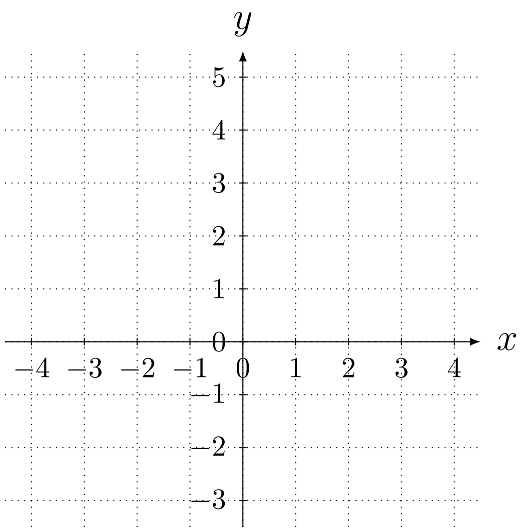


You can remember the integrand as (____________ curve − ____________ curve)

Example. **Set up, but do not evaluate**, the integral for the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$, by integrating with respect to y.



Example. Find the area of the region enclosed by $x = -y, \quad x = 2 - y^2$ using two methods:



The point of intersection is:

a) Integrate with respect to y.

$$A = \int (\text{ right curve } - \text{ left curve }) \_\_\_\_ =$$

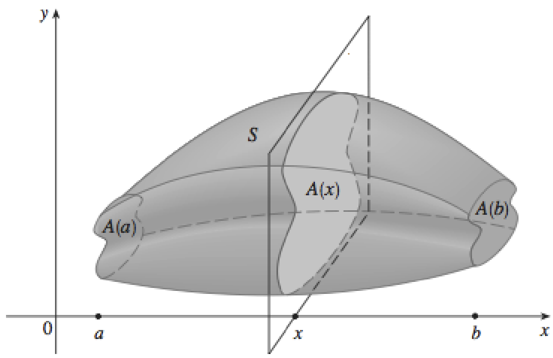
b) Integrate with respect to x.

Homework. 6.1: # 1, 3, 5, 7, 11 - 21 (odd), 27, 31, 35, 41, 65

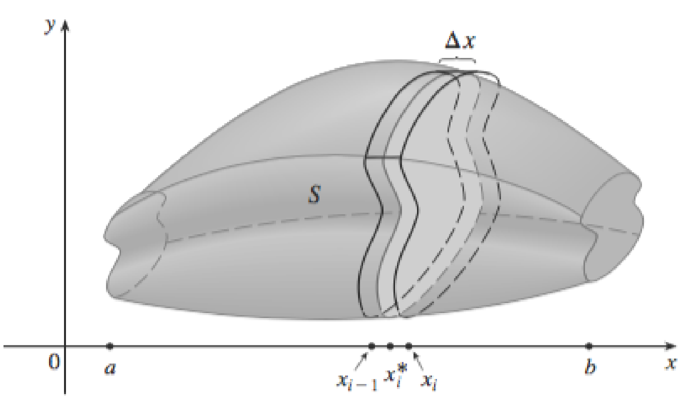
§6.2 Volumes by Slicing; Disks and Washers

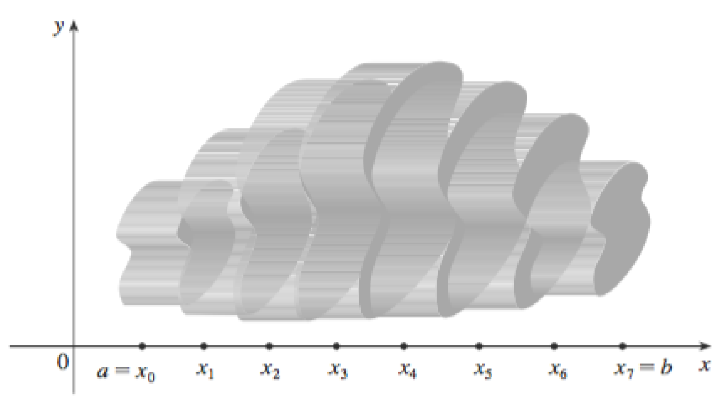
In trying to find the volume of a solid we face the same type of problem as finding areas. We have an intuitive idea of what volume means but we must make this idea precise by using Calculus to give an exact definition of volume.

Consider the problem of finding the volume of the following solid:



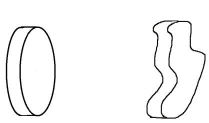
We begin by dividing the interval $[a, b]$ into n subintervals of equal width.
Let ________ denote the width of each subinterval, and let ________ be any point in the i-th interval $[x_{i-1}, x_i]$.
Suppose that the cross-section of the solid perpendicular to the x-axis at x_i^* has area ____________.





The volume of this slab can be *approximated* by a cylinder (or a **disk**[2]) of volume (base area) · (width) = ________________ for $i = 1, 2, \cdots, n$.

[2]A **disk** or a **cylinder** is a solid with two parallel sides (called "bases") with identical cross-sections when cut parallel to the bases:



The volume of a disk with base area A and width w is $A \cdot w$.

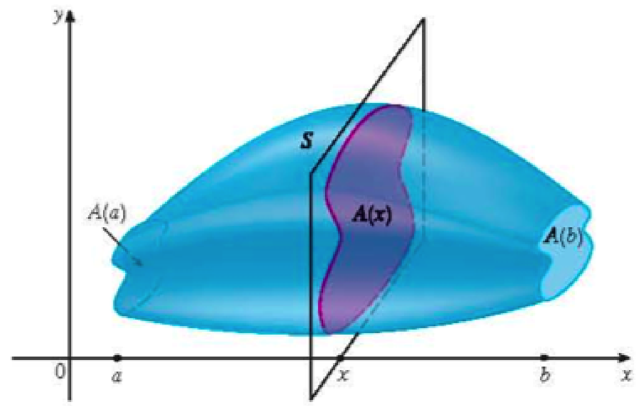
To get the *exact* volume V of the solid S, we let ________:

$$V =$$

To summarize:

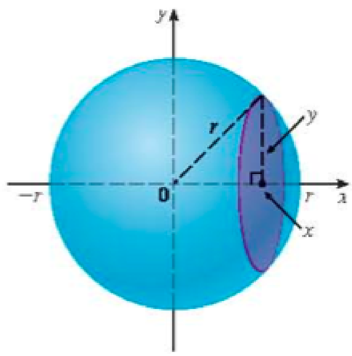
Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane perpendicular to the ____-axis (i.e. vertically) is $A(x)$ (where A is a continuous function), then the **volume** V of S is

$$V =$$

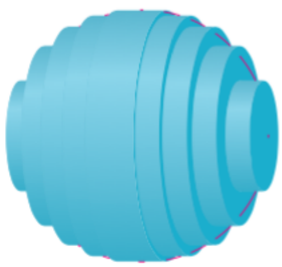


Example. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Let's place the sphere so that its centre is at the ____________.



Any plane perpendicular to the ____-axis intersects the sphere in a ____________ whose radius is ________. So the area of the cross-section is ________.



The width of each disk is ________. So we must express the area in terms of ____:

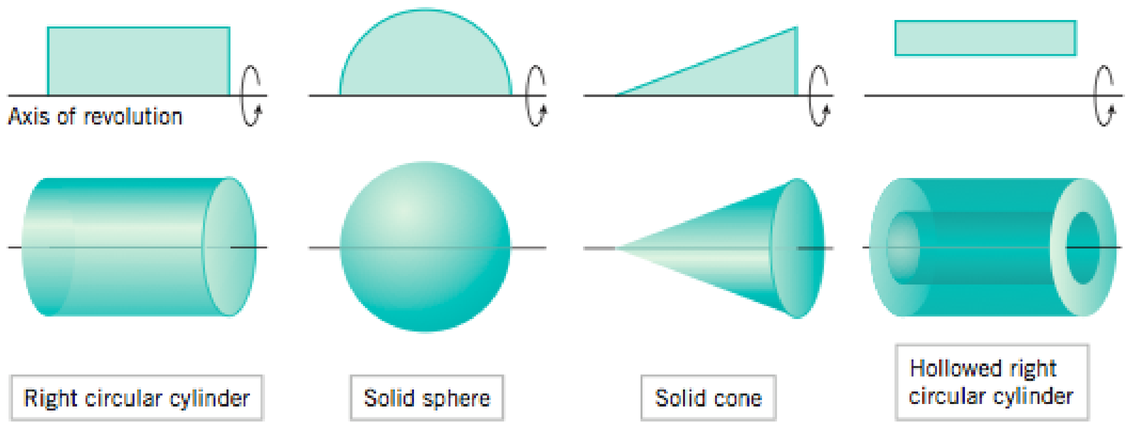
Hence the cross-sectional area at x (____ $\leq x \leq$ ____) is

$$A(x) =$$

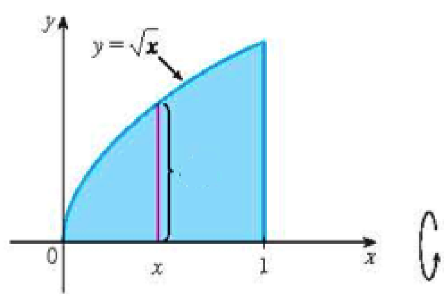
Thus the volume V of this sphere is:

$$V =$$

Definition. *A **solid of revolution** is a solid that is generated by revolving (rotating) a plane region about a line that lies in the same plane as the region. The line is called the **axis of revolution**.*



Example. Find the volume of the solid obtained by revolving the region under the curve $y = \sqrt{x}$ from $x = 0$ to 1 about the x-axis. (Remember: "a region under a curve" always implies "above the x-axis" as well).



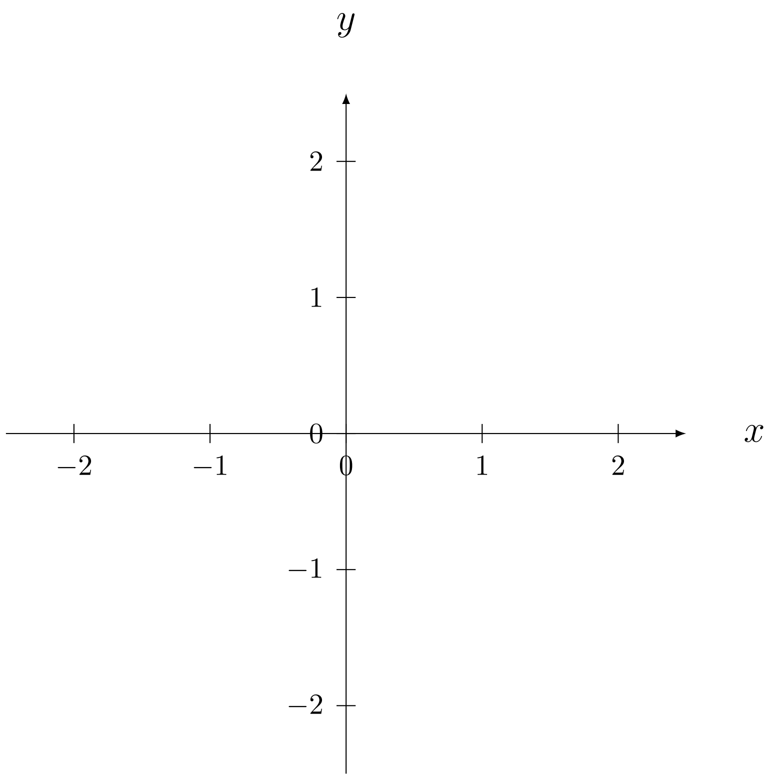
When we slice ________________ we get a ________ with radius ________ with width ____. So the area of this cross-section is:

$A(x) =$

Hence the volume V of this solid is:

$V =$

Example. Set up, but do not evaluate, the integral for the volume of the solid obtained by rotating the region $y = 2 - x^2$ bounded by the curve $y = 1$ about the line $y = 1$.

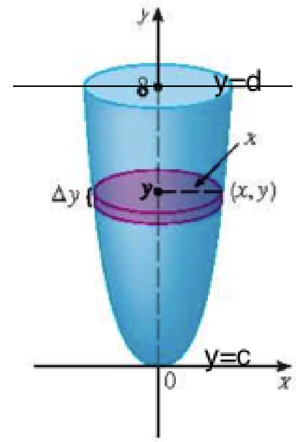


$\boxed{\text{Answer:} \dfrac{16\pi}{15}}$

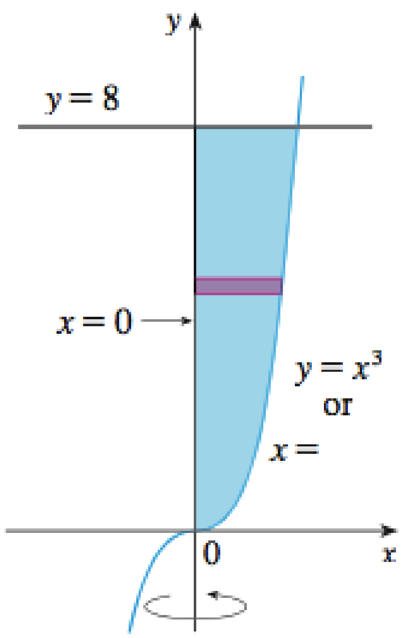
Similar result for cross sections perpendicular to the y-axis:

Let S be a solid that lies between $y = c$ and $y = d$. If the cross-sectional area of S in the plane perpendicular to the ____-axis (i.e. horizontally) is $A(y)$ (where A is a continuous function) then the **volume** V of S is

$$V =$$

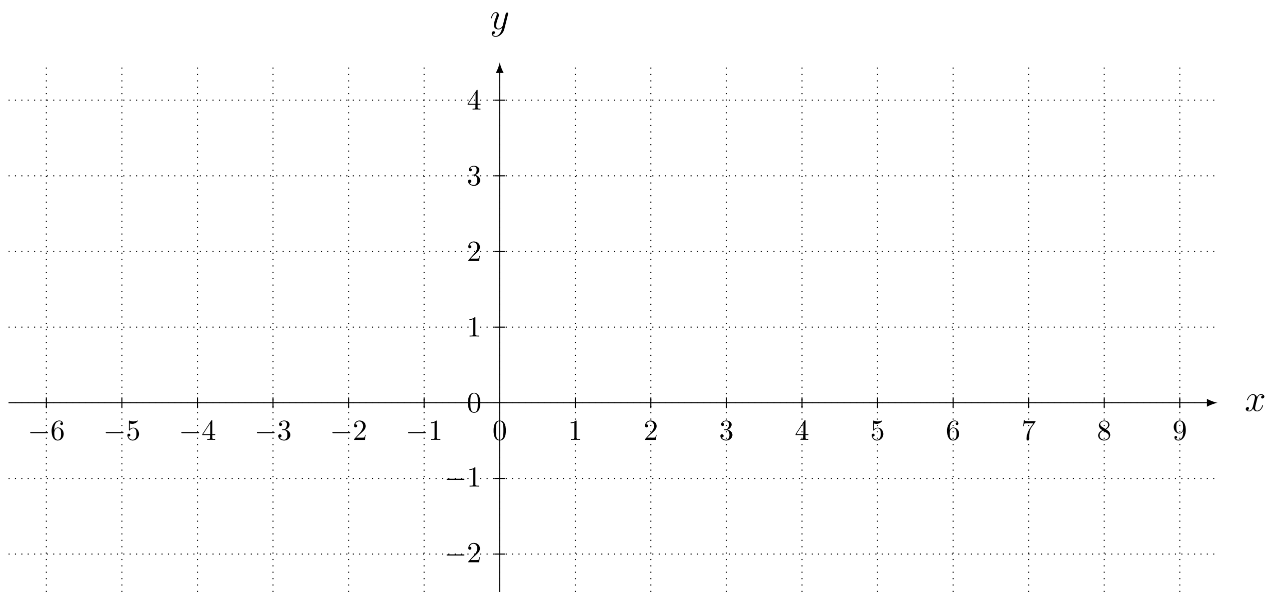


Example. Set up, but do not evaluate, the integral for the volume of the solid obtained by rotating the region bounded by $y = x^3, y = 8$ and $x = 0$ about the $y-$axis.



When we slice ________________ we get a ________ with radius ____________(the width of each disk is ____ so we must integrate with respect to ____).
Hence the volume of this solid can be found using the following integral:

Example. Set up, but do not evaluate, the integral for the volume of the solid obtained by rotating the region bounded by the curves $y = \ln x$ and $x = 1$ between $y = 1$ and $y = 2$ about the line $x = 1$. Hint: $e \approx 2.7$ and $e^2 \approx 7.3$.

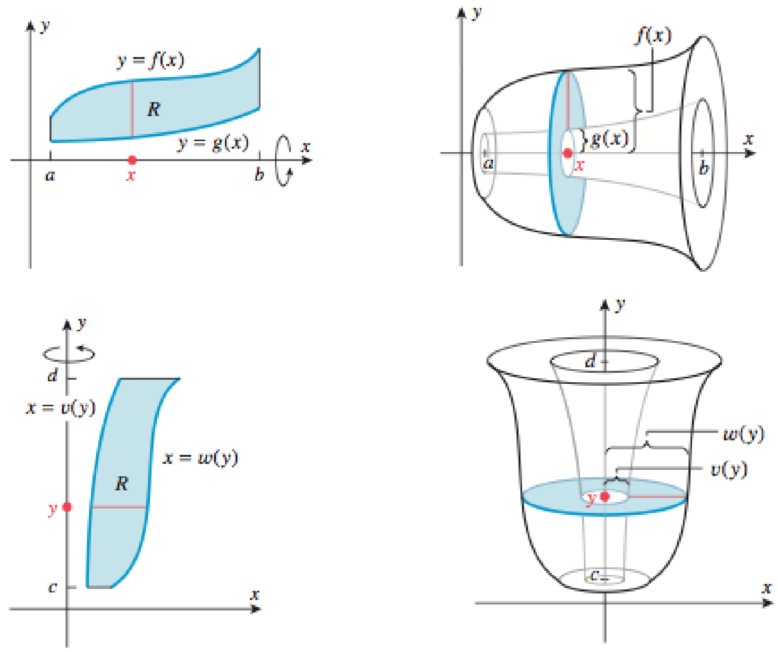


$$\boxed{\text{Answer:}\pi\left(\frac{e^4}{2} - \frac{5e^2}{2} + 2e + 1\right)}$$

Homework. 6.2 (part 1) # 11, 13, 15, 63, 69

Volume by Washers

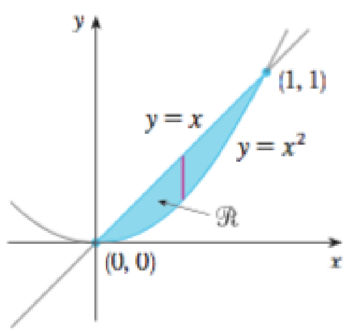
Not all solids of revolution have solid interiors:



When we have hollow interior as above, we are not approximating the volume of each slab by a disk, but by a **washer**[3].

[3] A **washer** is a thin plate used in conjunction with screws in, for example, some furniture.

Example. The region $\mathfrak{R}$ is enclosed by the curves $y = x$ and $y = x^2$. Set up, but do not evaluate, the integral for the volume of the solid obtained by rotating the region about the line $y = 2$.



If we slice ________________ we get a ____________ with width ____ (hence we integrate with respect to ____).
So the volume of the solid is

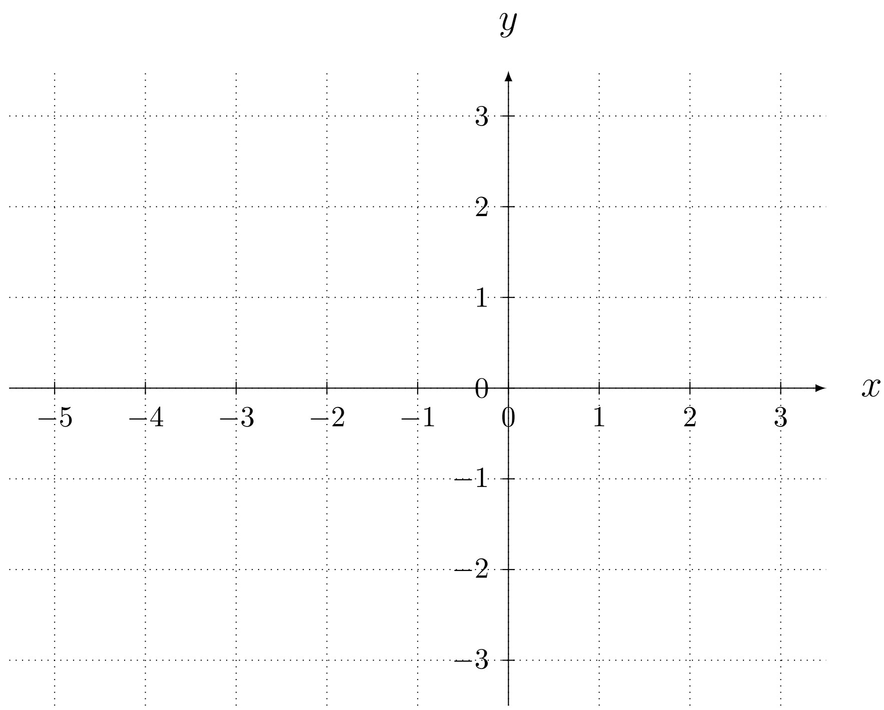
$$V = \int_{x=}^{x=} A(x)\,dx = \int_{x=}^{x=} \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2\,dx$$

$\boxed{\text{Answer:} \frac{8\pi}{15}}$

Example. Set up, but do not evaluate, the integral for the volume of the solid obtained by rotating the region bounded by the curves

$$y = x^2 - 1,\ y = 0,\ 1 \le x \le 2$$

about the line $x = -1$.



$$V = \int_{y=}^{y=} A(y)\,dy = \int_{y=}^{y=} \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2\,dy$$

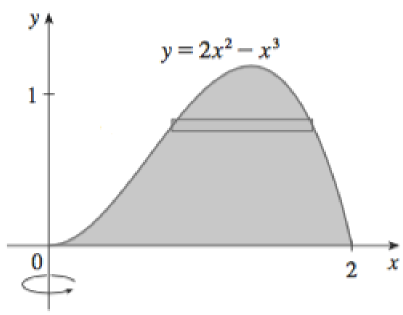
$\boxed{\text{Answer: } \frac{43\pi}{6}}$

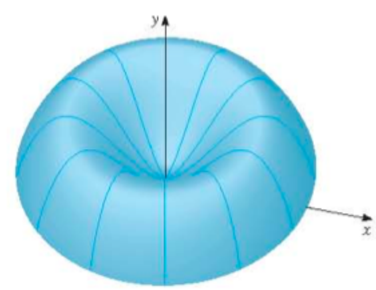
Homework. 6.2 (part 2): # 17, 19, 21, 25, 27, 29 - 39 (odd)

§6.3 Volumes by Cylindrical Shells

Some volume problems are very difficult to handle by the disk/washer method.

Example. Consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$:



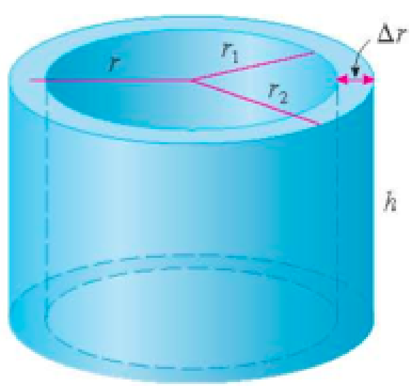


If we were to use the washer method, we would integrate with respect to ____. This means we need to solve $y = 2x^2 - x^3$ for ____, which is not easy to do (try!).

$$x = \frac{1}{3}\left(\sqrt[3]{-\frac{27y}{2} + \frac{3}{2}\sqrt{3}\sqrt{y(27y-32)} + 8} + \frac{4}{\sqrt[3]{-\frac{27y}{2} + \frac{3}{2}\sqrt{3}\sqrt{y(27y-32)} + 8}} + 2\right)$$

In such a case, a method called the **method of cylindrical shells** can be used instead.

Definition. *A* ***cylindrical shell*** *is a solid enclosed by two concentric right circular cylinders:*



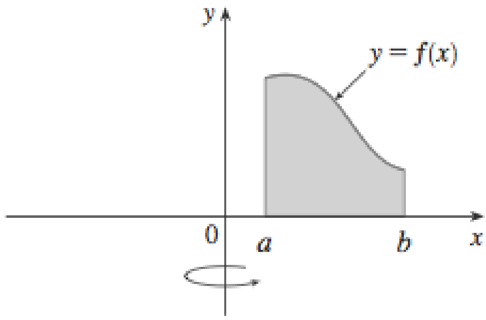
Let $\Delta r = r_2 - r_1$ (the thickness of the shell) and $r = \frac{r_2+r_1}{2}$ (the average radius of the shell) the volume of this cylindrical shell is:

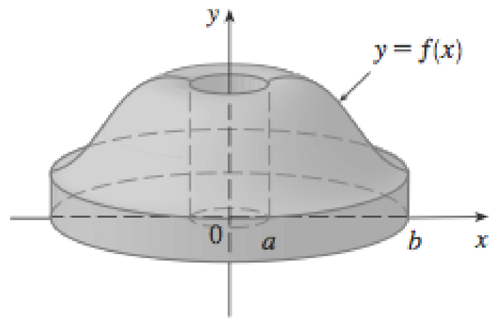
$$\boxed{V = 2\pi r \cdot h \cdot \Delta r}$$

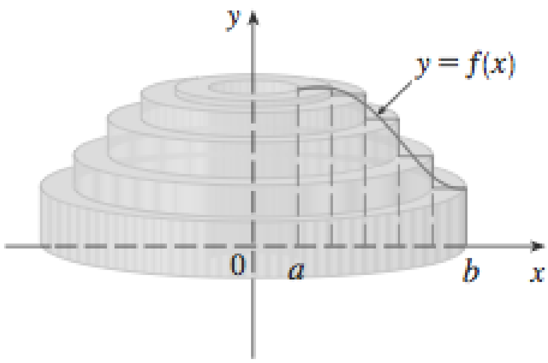
This formula can be remembered as

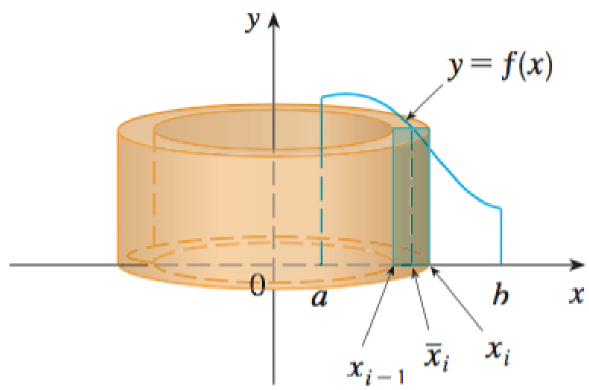
$$\boxed{V = [circumference][height][thickness]}$$

We will now show how we can use this formula to solve volume problems.
Suppose you are asked to find the volume of the solid (see right) generated by rotating the region (see left) about the y-axis:









(See http://mathdemos.org/mathdemos/shellmethod/gallery/gallery.html for demo)
To find the volume of the i-th cylindrical shell [using midpoints of each subinterval]:

- Let ________ be the midpoint of the i-th subinterval $[x_{i-1}, x_i]$.
- Construct a rectangle in the i-th subinterval with width Δx and the **midpoint** of the curve as the height of the rectangle.
- Rotating this rectangle gives a cylindrical shell with:
  - thickness ________,
  - average radius ________,
  - height ________

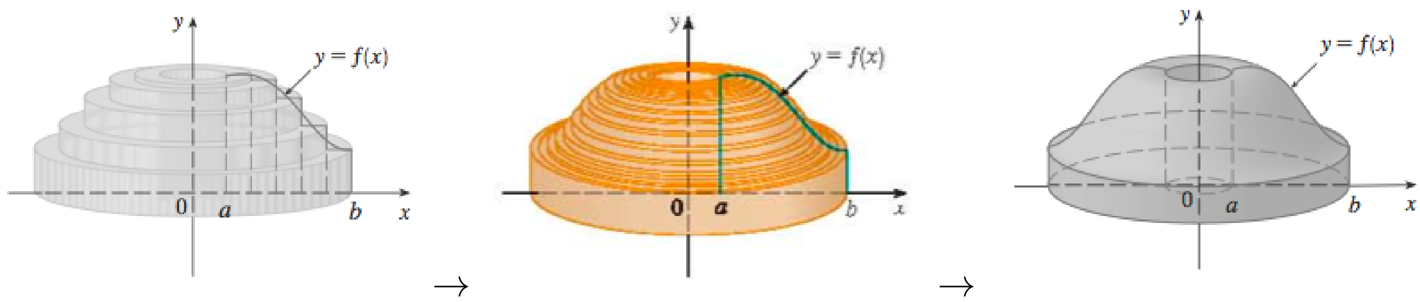
Hence V_i, the volume of the i-th cylindrical shell, is

$V_i =$ [circumference][height][thickness] $=$

Adding up the volumes of all the cylindrical shells, we get an approximate volume V of solid:

$$V \approx \qquad V_i = \qquad (2\pi \overline{x}_i) \cdot f(\overline{x}_i) \cdot \Delta x$$

This approximation becomes better as $n \to \infty$:



That is, the exact volume, V, of the solid is:

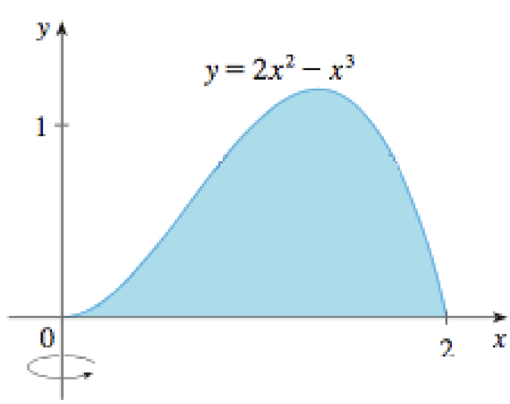
$$V = \qquad \sum_{i=1}^{n} 2\pi \cdot \overline{x}_i \cdot f(\overline{x}_i) \cdot \Delta x =$$

where the last equality follows from the definition of the definite integral (§5.2).

This volume formula does not apply to all solids. As such, you should not memorize this formula. Instead you should memorize the volume as:

$$\boxed{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

Example. Set up, but do not evaluate, the integral for the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.



A typical shell has:

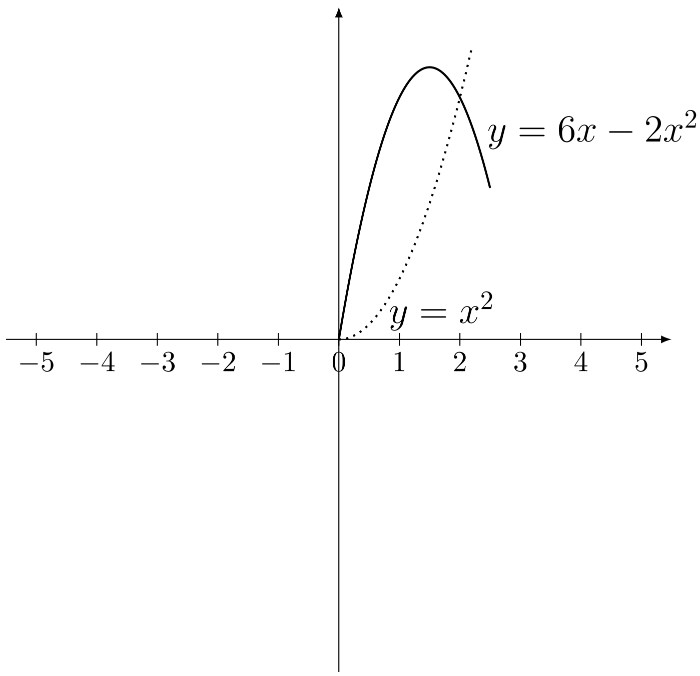
- thickness ________,
- radius ________,
- height ____________.

Thus by the shell method the volume is

$$\boxed{\frac{16\pi}{5}}$$

Example. Use cylindrical shells to set up, but do not evaluate, an integral for the volume of the solid generated when the region enclosed by the given curves is rotated about the line $x = -1$.

$$y = x^2, \quad y = 6x - 2x^2$$



A typical shell has:

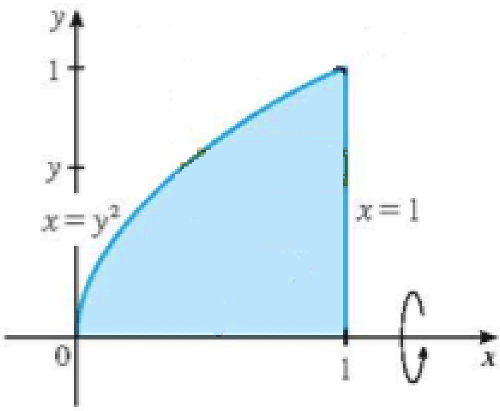
- thickness ________,
- radius ________,
- height ________________________________.

Hence the volume is: $V =$

Answer: $\boxed{16\pi}$

Remark. When rotating about the x-axis (or any horizontal line), the thickness of a shell will be ____, in which case you would integrate with respect to ____.

Example. Use cylindrical shells to set up, but do not evaluate, an integral for the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



A typical shell has:

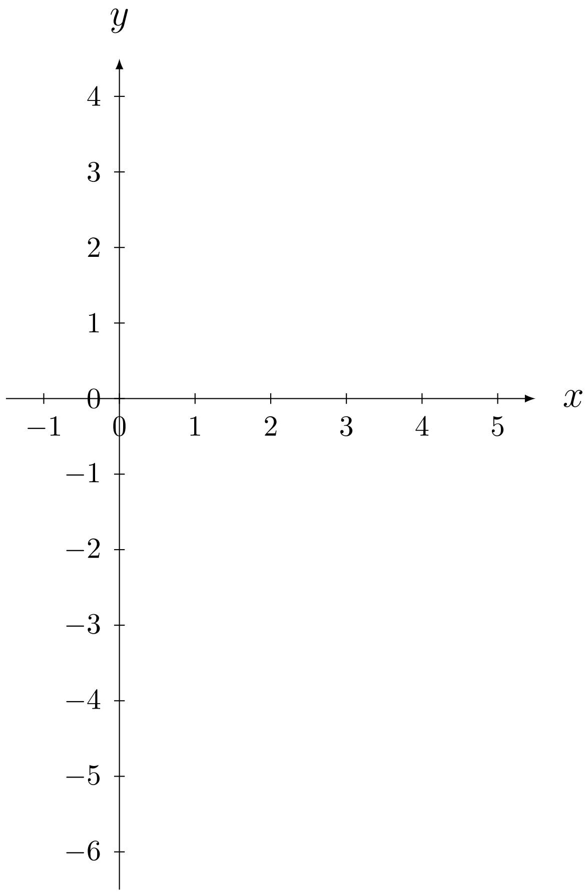
- thickness ________,
- radius ________,
- height ________________________.

Hence the volume is $V =$

Answer: $\boxed{\dfrac{\pi}{2}}$

Remark. The above example was done in §6.2 using the disk method (which was simpler to do) – for the disk method, we integrated with respect to x. In general, the cylindrical shell method and the disk method always have different variable in the integration.

Example. Use cylindrical shells to set up, but do not evaluate, an integral for the the volume of the solid generated when the region R under $y = x^2$ over the interval $[0, 2]$ is rotated about the line $y = -1$.



A typical shell has:

- thickness ________,
- radius ________,
- height ________________.

Hence the volume of this solid is

Answer: $\boxed{\dfrac{176\pi}{15}}$

Homework. 6.3: # 5, 9, 15, 19, 25, 27, 29, 53, 59 (If you do not have the student solutions manual, which gives full solutions to each exercise, (the back of the textbook gives the final answer only) and you want to check that your integral is correct before calculating the integral, go to WolframAlpha.com. Input your integral and see if the answer matches with the answer in the back of the book before you evaluate the integral to save time.)

Chapter 7. Techniques of Integration

In this chapter, we develop techniques for using the basic integration formulas to obtain indefinite integrals of more complicated functions.

§7.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For example, the **Substitution Rule** (§5.5) corresponds to the **Chain Rule** for differentiation. In this section, we study the rule for integration that corresponds to the **Product Rule** for differentiation.

<u>Recall:</u> The Product Rule for differentiation states that

$[f(x)g(x)]' =$

Stated in terms of integrals, we have

Rearranging we get

$$\qquad (1)$$

Formula (1) is called the **formula for integration by parts (IBP)**.

Let $u = f(x)$ and $v = g(x)$. Then $du =$ ____________ and $dv =$ ____________. So Formula (1) becomes:

$$\boxed{\qquad\qquad\qquad\qquad} \qquad (2)$$

One strategy for choosing u and dv that can be applied when the integrand is a product of two functions from different categories in the list LIATE:

Logarithmic, **I**nverse trigonometric, **A**lgebraic[4], **T**rigonometric, **E**xponential

In this cases, take u to be the function whose category appears earlier in the list and take dv to be the rest of the integrand. (caveat: this method does not always work)

Example. Evaluate $\displaystyle\int x \sin x \, dx$.

Let

$$u = \qquad\qquad \Rightarrow \qquad du =$$

$$dv = \qquad\qquad \Rightarrow \qquad v =$$

(for v choose any antiderivative of v').

Applying the Integration By Parts Formula $\boxed{\displaystyle\int udv = uv - \int vdu}$ we get

$$\int x \sin x \, dx =$$

Example. Evaluate $\displaystyle\int \ln x \, dx$.

Let

$$u = \qquad\qquad \Rightarrow \qquad du =$$

$$dv = \qquad\qquad \Rightarrow \qquad v =$$

[4] A function is **algebraic** if it can be constructed from polynomials by applying finitely many algebraic operations (addition/subtraction/multiplication/division/raising to a fractional power). Some examples of algebraic functions are: polynomials, rational functions (polynomial/polynomial), root functions, etc.

$$\boxed{\int u dv = uv - \int v du}$$

Example. Evaluate $\int e^x \sin x \, dx$. [HINT: you will need to do IBP twice]

Let

$$u = \qquad \Rightarrow \qquad du =$$

$$dv = \qquad \Rightarrow \qquad v =$$

$$\Rightarrow \int e^x \sin x \, dx =$$

Integration by Parts for Definite Integrals

For definite integrals, the formula for Integration by Parts is:

$$\boxed{\int u\,dv = uv \quad - \int v du}$$

Example. Evaluate $\displaystyle\int_0^1 \arctan x\,dx$ (hint: you will need IBP <u>and</u> substitution).

Let

$$u = \qquad\qquad \Rightarrow \qquad du =$$

$$dv = \qquad\qquad \Rightarrow \qquad v =$$

$$\Rightarrow \int_0^1 \arctan x\,dx =$$

Homework. 7.1: # 1, 7, 13, 15, 17, 19, 23, 25, 33, 37, 39, 45, 47, 67, 69

§7.2 Trigonometric Integrals

In this section, we learn how to integrate combinations of trigonometric functions of the form $\displaystyle\int \cos^n x \cdot \sin^m x \, dx$ (where $n, m \geq 0$).

Example. Evaluate $I = \displaystyle\int \cos^3 x \, dx$.

Step 1. Factor out one ________factor [in general, take out one factor of cosine or sine from whichever has an ________ power].

Step 2. Use the identity $\boxed{\cos^2 x + \sin^2 x = 1}$ to express the remaining trigonometric function in terms of ________.

Step 3. Substitution:

Example. Find $I = \displaystyle\int \sin^5 x \cos^2 x \, dx$.

Step 1. Factor out one ________ factor. [in general, take out one factor of cosine or sine from whichever has an odd power]

Step 2. Use the identity $\boxed{\cos^2 x + \sin^2 x = 1}$ to express the remaining trigonometric function in terms of ________.

Step 3. Substitution:

$$\boxed{\text{Answer: } -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C}$$

Example. Evaluate $\int \cos^4 x\, dx$.

Since the integrand contains both even powers of cosine and sine, the above method fails. In such cases, we use the half-angle identities:

$$\boxed{\sin^2 x = \underline{\hspace{3cm}}} \qquad \boxed{\cos^2 x = \underline{\hspace{3cm}}}$$

$$\int \cos^4 x\, dx =$$

Remark. In cases where the integrand contains even powers of cosine and sine, It will sometimes be helpful to use the identity

$$\boxed{\sin x \cos x = \frac{\sin(2x)}{2}}$$

ex. $\int \sin^2 x \cos^2 x\, dx = \int \left(\frac{\sin(2x)}{2}\right)^2 dx = \cdots$

Homework. 7.2: # 1- 13 (odd), 17, 69, 71 (for #69 and 71: draw the regions)

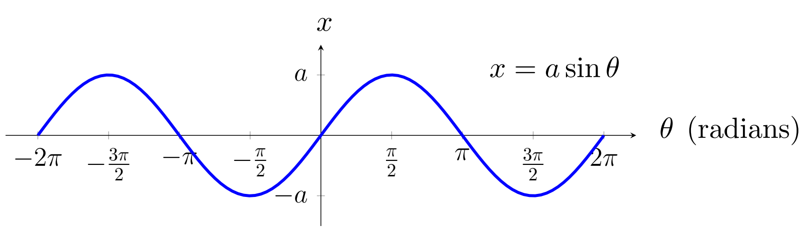
§7.3 Trigonometric Substitution

Let a be a positive number and suppose that we need to evaluate an integral that contains the expression $\sqrt{a^2 - x^2}$.

Idea: We evaluate such integrals by eliminating the radical via making an appropriate *trigonometric substitution.*

Example. To eliminate $\sqrt{a^2 - x^2}$, let $x =$ ____________ ($a > 0$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$).

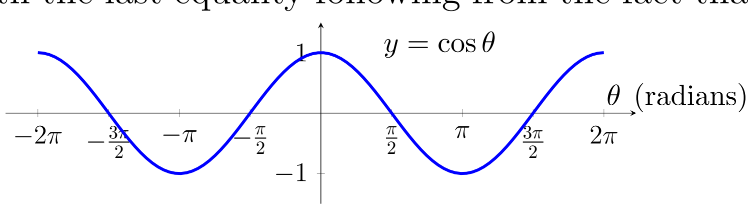
Remark. This restriction on θ is allowed, since the domain of $\sqrt{a^2 - x^2}$ is $x \in$ ____________, and in the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, the function $a \sin\theta \,(= x)$ takes on all values in $[-a, a]$.



Then

$$\sqrt{a^2 - x^2} =$$

with the last equality following from the fact that ____________ when $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.



Remark. In trigonometric substitutions the old variable is a function of the new variable (ex. $x = a\sin\theta$).

Example. Evaluate $I = \displaystyle\int \frac{\sqrt{9 - x^2}}{x^2}\, dx.$

Example. Evaluate $I = \displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-4x^2}}\,dx$.

We must first change the expression $\sqrt{1-4x^2}$ into the form $k \cdot \sqrt{a^2 - x^2}$ by factoring ____ out of the square root.

$$\boxed{Answer: \frac{\pi}{4}}$$

Integrals involving $ax^2 + bx + c$

Example. Evaluate $I = \displaystyle\int \frac{x}{\sqrt{3 - 2x - x^2}}\, dx.$

Integrals that involve a quadratic expression $ax^2 + bx + c$ where $a \neq 0$ and $b \neq 0$, can often be evaluated by first completing the square (i.e. turn $ax^2 + bx + c$ to $a(x - d)^2 + f$), then making an appropriate substitution.

Homework. 7.3: # 5, 17, 21, 27, 35

§7.5 Strategy for Integration

(This is a good review for Midterm 1!) In this section, we present a collection of integrals in random order and the main challenge is to recognize which integration technique/formula to use. No hard and fast rules can be given as to which method applies in a given situation, but we give some advice on strategy that you may find useful.

Recall that the integration techniques we learned are:

- Substitution
- Integration By Parts
- Trigonometric Integrals (and Trigonometric Substitutions)
- Partial Fractions (later)

Strategy

1. *Simplify the integrand, if possible.*

   Example.

   a) $\displaystyle\int \sqrt{x}\,(1+\sqrt{x})\,dx = \int \sqrt{x} + x\,dx = \cdots$

   b) $\displaystyle\int (\sin x+\cos x)^2\,dx = \int (\sin^2 x+2\sin x\cos x+\cos^2 x)\,dx = \int 1+2\sin x\cos x\,dx \cdots$

2. *Look for an obvious substitution.*

   Example. $\displaystyle\int \frac{x}{x^2-1}\,dx$: let $u = x^2-1 \Rightarrow du = 2x\,dx \cdots$

3. *Classify the integrand according to its form.*

   a) Trigonometric functions: If the integrand is a product of $\sin x$ and $\cos x$, then use the substitutions recommended in §7.2:

$\int \sin^m x \cos^{2k+1} x \, dx$	• Split off a factor of $\cos x$ • Express rest of cosine factors in terms of $\sin x$ (using $\cos^2 x = 1 - \sin^2 x$) • Use substitution $u = \sin x$
$\int \sin^{2k+1} x \cos^n x \, dx$	• Split off a factor of $\sin x$ • Express rest of sine factors in terms of $\cos x$ (using $\sin^2 x = 1 - \cos^2 x$) • Use substitution $u = \cos x$
$\int \sin^{2k} x \cos^{2r} x \, dx$	• Use the half-angle identities $\sin^2 x = \frac{1 - \cos(2x)}{2}$ or $\cos^2 x = \frac{1 + \cos(2x)}{2}$ to reduce the powers on $\sin x$ and $\cos x$. • It is sometimes helpful to use the identity $\sin x \cos x = \frac{\sin(2x)}{2}$

   b) Rational functions: Try partial fractions (§7.4)

   c) Integration by Parts (IBP): If the integrand is a product of a polynomial and a non-polynomial function (ex. trigonometric, exponential, logarithmic...) then try IBP (§7.1) (recall: in general, when deciding on a choice for u and dv, we usually try to choose u to be a function that becomes simpler when differentiated, as long as dv can be readily integrated to give v.)

   d) Radicals: When $\sqrt{a^2 - x^2}$ occurs, use trigonometric substitution $x = a \sin \theta$) (§7.3)

4. Try again...

Example. $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\,dx$

Simplify the integrand:

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \frac{\sin^3 x}{\cos^3 x}\frac{1}{\cos^3 x}\,dx = \int \frac{\sin^3 x}{\cos^6 x}\,dx = \int \frac{\sin^2 x \cdot \sin x}{\cos^6 x}\,dx$$
$$= \int \frac{(1-\cos^2 x)\cdot \sin x}{\cos^6 x}\,dx$$

Substitution: let $u = \cos x \Rightarrow du = -\sin x\,dx$

$$\int \frac{(1-\cos^2 x)\cdot \sin x}{\cos^6 x}\,dx = \int \frac{1-u^2}{u^6}(-du) = \int -u^{-6} + u^{-4}du = \cdots$$

$$\boxed{answer : \frac{1}{5\cos^5 x} - \frac{1}{3\cos^3 x} + C}$$

Example. $\displaystyle\int \frac{dx}{x\sqrt{\ln x}}$

Use substitution $u = \ln x$...

$$\boxed{answer : 2\sqrt{\ln x} + C}$$

Example. $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,dx$

$$\int \sqrt{\frac{1-x}{1+x}}\,dx = \int \sqrt{\frac{1-x}{1+x}}\frac{\sqrt{1-x}}{\sqrt{1-x}}\,dx = \int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx$$
$$= \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Do substitution $u = 1 - x^2$ for second integral.

$$\boxed{answer : \sin^{-1} x + \sqrt{1-x^2} + C}$$

Homework. 7.5: #1, 3, 7, 17 – 25 (odd), 31, 33, 41, 43, 45, 47, 57, 61, 65, 67, 71, 73, 75

§7.4 Integration of Rational Functions by Partial Fractions

Recall: A **rational function** is a ratio of two ________________.

Example. $\dfrac{x+5}{3x-2}$, $\dfrac{x^3+x}{x-1}$, etc.

Motivation: How do we integrate rational functions such as $\displaystyle\int \frac{4x}{x^3-x^2-x+1}\,dx$?

In this section, we show how to integrate some rational functions by expressing them as a sum of simpler fractions, called **partial fractions**, that we already know how to integrate.

Definition. *The* ***degree of the polynomial*** $p(x)$*, written* $\deg(p)$*, is the highest power of* x *in* $p(x)$*.*

Example. Let $p(x) = 1 + x + x^3$. Then $\deg(p) =$____.

Definition. *Let*

$$f(x) = \frac{p(x)}{q(x)}$$

where p *and* q *are polynomials. We say that* f *is a* ***proper*** *rational function if* ________________________. *We say that* f *is an* ***improper*** *rational function if* ________________________

Example.

- $f(x) = \dfrac{x+1}{x^2-3}$ is a ________________ rational function.
- $f(x) = \dfrac{x^3+1}{x^2-x+3}$ is a ________________ rational function.

In what follows, we assume that $f(x) = \dfrac{p(x)}{q(x)}$ is a ____________ rational function (If improper, we must rewrite the function as a polynomial plus a proper rational function first).

To express the **proper** rational function $f(x) = \dfrac{p(x)}{q(x)}$ as a sum of simpler fractions, we follow the steps below:

Step 1. Factor $q(x)$ **completely** [*any* polynomial can be factored as a product of *linear factors*, $ax + b$, and/or *irreducible quadratic factors*, $ax^2 + bx + c$ where $b^2 - 4ac$ ____ 0 (i.e. $ax^2 + bx + c$ cannot be factored any further)].

Example.

$$x^4 - 16 =$$

Here, $x-2$ and $x+2$ are ________ factors, and x^2+4 is an ____________ ____________ factor.

Step 2. Decompose $\dfrac{p(x)}{q(x)}$ into partial fractions

Each factor of q falls into one of the following cases:

1. a **non-repeated linear factor**, $ax + b$. ex. $3x + 7$
2. a **repeated linear factor**, $(ax + b)^r$, $r > 1$. ex. $(3x + 7)^3$
3. a **non-repeated irreducible quadratic factor**, $ax^2 + bx + c$. ex. $3x^2 + 7$
4. a **repeated irreducible quadratic factor**, $(ax^2 + bx + c)^r$, $r > 1$. ex. $(3x^2 + 7)^2$

- Case 1: The denominator $q(x)$ contains a **non-repeated linear factor**, $ax+b$.

  A non-repeated linear factor $(ax+b)$ will give one partial fraction of the form

  where A is a constant to be determined.

  Example.

  $\dfrac{1}{x^2+x-2} = \dfrac{1}{(x-1)(x+2)} =$

  Here, the denominator is a product of two non-repeated linear factors, ________ and ________, and A and B are constants to be determined.

  To find A and B, multiply the above equation by (to clear all the denominators) to get

  $$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

  Hence $\dfrac{1}{x^2+x-2} =$

  Now we can readily integrate the right hand side.

- <u>Case 2</u>: The denominator $q(x)$ contains a **repeated linear factor, ____________, where** $r > 1$**,** r is an integer.

  If $(ax+b)^r$ (with $r > 1$) occurs in the factorization of $q(x)$, then we we will have in the decomposition

  where $A_1, A_2, \cdots, A_r$ are constants to be determined.

  Example.

  $$\frac{x^3 - x + 1}{4x(x-1)^3} =$$

  where A, B, C, D are constants to be determined.

- <u>Case 3</u>: The denominator $q(x)$ contains a distinct (i.e. non-repeated) **irreducible quadratic factor.**

  In this case, for each non-repeated irreducible quadratic factor $ax^2 + bx + c$, the decomposition will have a term of the form

  where A and B are constants to be determined.

Example. Decompose $\dfrac{3x^3+4x-3}{3x^3-x^2+3x-1}$ into partial fractions.

First note that this is an ____________ rational function. So we must first write it as a polynomial plus a proper rational function:

$$\frac{3x^3+4x-3}{3x^3-x^2+3x-1} =$$

Now let's factor the denominator:

$$3x^3-x^2+3x-1 =$$

There is a ____________ factor (not repeated) [case 1] and an ________________ ________________ factor (not repeated) [case 3].

Hence

$$\frac{3x^3+4x-3}{3x^3-x^2+3x-1} = 1+\frac{x^2+x-2}{(3x-1)(x^2+1)} = 1+$$

where A, B, C are constants to be determined.

- Case 4: The denominator $q(x)$ contains a **repeated irreducible quadratic factor** $(ax^2+bx+c)^r$**, where** $r>1$**,** r is an integer.

  In such cases, for each repeated irreducible quadratic factor $(ax^2+bx+c)^r$, the decomposition of $p(x)/q(x)$ will contain terms of the form

  where $A_1, A_2, \cdots, A_r, B_1, B_2, \cdots, B_r$ are constants to be determined.

  Example.

  $$\frac{x^3+x^2+1}{(x^2+1)^3} =$$

  where A, B, C, D, E, F are constants to be determined.

Example. Evaluate $\displaystyle\int \frac{4x}{x^3 - x^2 - x + 1}\, dx.$

It is a proper rational function. We first factor the denominator $x^3 - x^2 - x + 1$:

$x^3 - x^2 - x + 1 =$

(Hint: You should have one non-repeated linear factor and a repeated linear factor)

Hence $\dfrac{4x}{x^3 - x^2 - x + 1} =$

Now we solve for the constants.

Thus $\dfrac{4x}{x^3 - x^2 - x + 1} =$

So $\displaystyle\int \frac{4x}{x^3 - x^2 - x + 1}\, dx =$

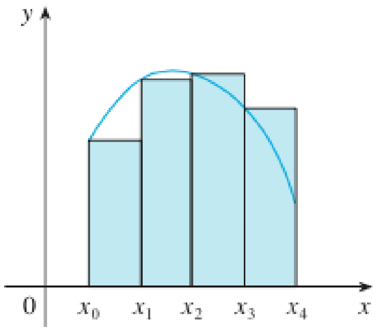
There are some cases in which the method of partial fractions is inappropriate.

Example. $\displaystyle\int \frac{3x^2+2}{x^3+2x-8}\,dx$ can be integrated using substitution ________________.

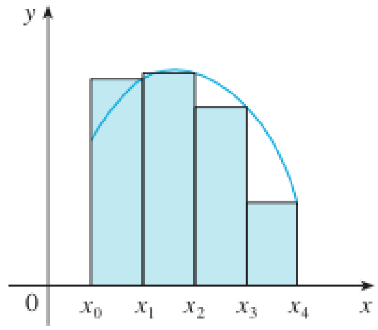
Homework. 7.4: # 1 - 13 (odd), 19, 21, 25, 27, 41 (note: if using the Partial Fractions method, always check first that the integrand is proper before decomposing!)
§7.5: # 17, 67

§7.7 Approximate Integration

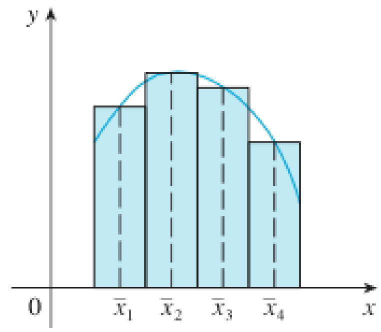
Not all definite integrals can be evaluated (ex. $\displaystyle\int_0^2 e^{x^2}\,dx$, $\displaystyle\int_{-1}^1 \sqrt{1+x^3}\,dx$, etc.) — there are no antiderivatives in terms of elementary functions. In such cases, we find a numerical approximation of the integral. We've already seen three such approximations in the context of areas:



(a) Left endpoint approximation



(b) Right endpoint approximation



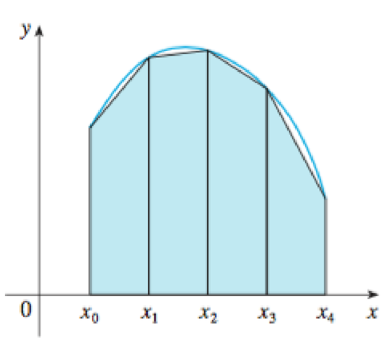
(c) Midpoint approximation

<u>Recall:</u> **§5.1 Midpoint Rule (using n subintervals):**

$$\boxed{\begin{gathered}\int_a^b f(x)\,dx \approx M_n = \Delta x[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)] \\ \text{where } \Delta x = \frac{b-a}{n} \quad \text{and} \\ \overline{x}_i = \text{ midpoint of } [x_{i-1}, x_i] = \frac{x_{i-1}+x_i}{2}\end{gathered}} \tag{1}$$

In this section we will develop some new methods that often provide more accuracy with less computation.

Another approximation can be obtained by joining the endpoints of each subinterval and adding up the areas of the trapezoids that result. This approximation is called the **Trapezoidal Rule**, denoted T_n (n is the # of subintervals (or "strips")).



The Trapezoidal Rule using n subintervals has the following formula:

$$\boxed{\begin{gathered}\int_a^b f(x)\,dx \approx T_n = \left(\frac{\Delta x}{2}\right) \cdot [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] \\ \text{where where } \Delta x = \frac{b-a}{n}.\end{gathered}}$$

Comparison of the Midpoint and Trapezoidal Rules

Let M_n be the value of the Midpoint Rule using n subintervals, and let T_n be the value of the Trapezoidal Rule using n subintervals.

The following table shows the results for the **errors** in approximating $\int_1^2 \frac{1}{x}\,dx$ using $n = 5, 10, 20$ for the left- (L_n), right- (R_n), Midpoint Rules (M_n), as well as the Trapezoidal Rule (T_n).

n	$\int_1^2 \frac{1}{x}\,dx - L_n$	$\int_1^2 \frac{1}{x}\,dx - R_n$	$\int_1^2 \frac{1}{x}\,dx - T_n$	$\int_1^2 \frac{1}{x}\,dx - M_n$
5	-0.052488	0.047512	-0.002488	0.001239
10	-0.025624	0.024376	-0.000624	0.000312
20	-0.012656	0.012344	-0.000156	0.000078

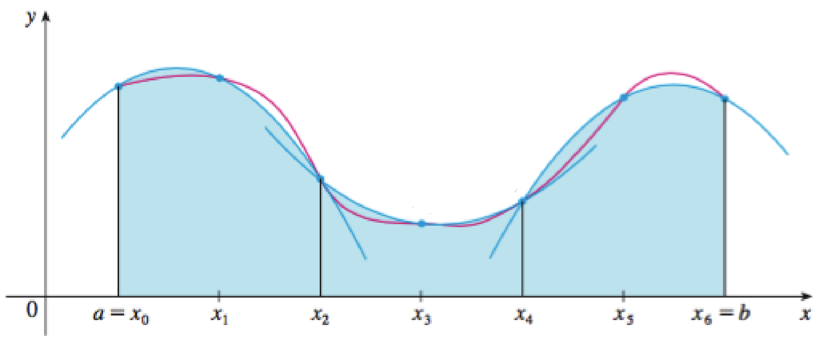
Remark.

- In all of the approximation methods we get more accurate approximations when we ________________ the value of n.
- The Trapezoidal and the Midpoint Rules are more accurate than the left- and right- endpoint approximations.
- The errors in the left- and right- endpoint approximations decrease by a factor of $\approx$ ____ when we we double the value of n.
- The errors in the Trapezoidal and Midpoint Rules decrease by a factor of $\approx$ ____ when we double the value of n.
- The size of the error in the Midpoint Rule is approximately ________ the size of the error in the Trapezoidal Rule.

Another rule for approximate integration results from using ______________ instead of straight line segments to approximate a curve. This is Simpson's Rule:

<u>Simpson's Rule</u>

We divide $[a, b]$ into n subintervals of length $\Delta x = (b-a)/n$, where n is even.



Simpson's Rule (For proof, see textbook p. 519 - 520)

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3} \cdot [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where $\Delta x = \dfrac{b-a}{n}$ and n is even.

<u>Errors for Midpoint Rule (M_n) and Simpson's Rule (S_n)</u> (for estimating the value of $\displaystyle\int_1^2 \frac{1}{x}\,dx$)

n	$\int_1^2 \frac{1}{x}\,dx - M_n$	$\int_1^2 \frac{1}{x}\,dx - S_n$
4	0.00192729	-0.00000735
8	0.00048663	-0.00000047
16	0.00012197	-0.00000003

Simpson's Rule provides with a much better approximation than the Midpoint Rule. Furthermore, the errors in using Simpson's Rule decrease by a factor of $\approx$ ____ when we we double the value of n.

Homework. 7.7 (optional): # 1, 5, 11 You will not be tested on applying (i.e. using the formulas of) the Trapezoidal or the Simpson's Rule. However, you will be tested on roughly how these rules work and how good of approximations they produce in comparison to each other.

§7.8 Improper Integrals

In this section, we extend the concept of a definite integral to the cases where:

1. the **interval of integration is infinite** $\left(\int_1^\infty \frac{dx}{x}, \int_{-\infty}^\infty \frac{dx}{1+x^2}, \cdots\right)$, and/or
2. f has an **infinite discontinuity** (i.e. vertical asymptote) in the interval of integration $\left(\int_{-3}^3 \frac{dx}{x^2}, \int_0^\pi \tan x\, dx, \cdots\right)$

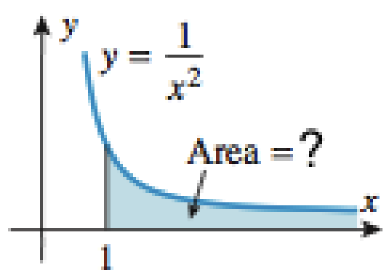
Such integrals are called **improper integrals**.

1. Integrals over Infinite Intervals

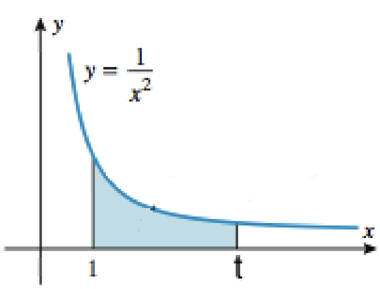
To evaluate $\int_1^\infty \frac{dx}{x^2}$, we cannot use FTC2, since the upper limit of integration (∞) is **not** in the domain of $\frac{1}{x^2}$. i.e. the following calculation is **wrong**:

$$\int_1^\infty \frac{dx}{x^2} = \frac{x^{-1}}{-1}\Big|_1^\infty = \frac{-1}{\infty} + 1 \text{ [\textbf{wrong}]}$$

Instead, let's think of it as an area problem:



Consider instead the area of the part of region in the interval $[1, t]$ where $t > 1$:



This area, which is a function of ____, is:

$A(t) =$

Now letting $t \to$ ________, we get:

Using this example as a guide, we make the following definitions:

Definition.

- *If* $\int_a^t f(x)\,dx$ *exists for every number* $t \geq a$ *then* $\boxed{\int_a^\infty f(x)\,dx = \qquad\qquad\qquad\qquad}$

  provided this limit exists (as a finite number).

- *If* $\int_t^b f(x)\,dx$ *exists for every number* $t \leq b$ *then* $\boxed{\int_{-\infty}^b f(x)\,dx = \qquad\qquad\qquad\qquad}$

  provided this limit exists (as a finite number).

These improper integrals $\int_a^\infty f(x)\,dx$ *and* $\int_{-\infty}^b f(x)\,dx$ *are said to be* __________________ *if the corresponding limit exists, and* __________________ *if the limit does not exist.*

- *If both* $\int_a^\infty f(x)\,dx$ *and* $\int_{-\infty}^a f(x)\,dx$ *are convergent, then we define*

$$\boxed{\int_{-\infty}^\infty f(x)\,dx = \int f(x)\,dx + \int f(x)\,dx}$$

  where a *is any real number.* $\int_{-\infty}^\infty f(x)\,dx$ *is said to* ***converge*** *if* <u>*both*</u> *integrals on the right side converge, and* ***diverge*** *If* <u>*either*</u> *term on the right diverges.*

Caution. In general, $\int_{-\infty}^\infty f(x)\,dx \neq \lim_{t\to\infty} \int_{-t}^t f(x)\,dx$. Why?

The right side is much more specific than the left side. On the right side we have a *single* variable that controls both the approach to $+\infty$ and $-\infty$, rather than one variable going off and reaching ∞ first before we start moving the other variable. In #61, you will show that

$$\lim_{t\to\infty} \int_{-t}^t x\,dx = 0 \tag{1}$$

but

$$\int_{-\infty}^\infty x\,dx = \lim_{t\to-\infty} \int_t^0 x\,dx + \lim_{t\to\infty} \int_0^t x\,dx = -\infty + \infty \text{ is divergent.}$$

WHY? In (1), t is approaching $\pm\infty$ in tandem, so they cancel each other out. But if t is approaching $\pm\infty$ slightly differently, then we get a different limit.

Example. Is the improper integral $I = \displaystyle\int_1^\infty \frac{1}{x}\,dx$ convergent or divergent?

This limit does not exist as a number, so $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is __________________.

We found that $\displaystyle\int_1^\infty \frac{1}{x^2} = 1$ (i.e. finite area under the curve) but $\displaystyle\int_1^\infty \frac{1}{x} = \infty$ (i.e. infinite area under the curve). One explanation for this discrepancy is that $1/x^2$ approaches zero more rapidly than $1/x$ as $x \to \infty$, so that the area over the interval $[1, t]$ accumulates less rapidly under the curve $y = 1/x^2$ than under the curve $y = 1/x$ as $t \to \infty$, and the difference is enough that the first area is finite, but the second area is infinite.

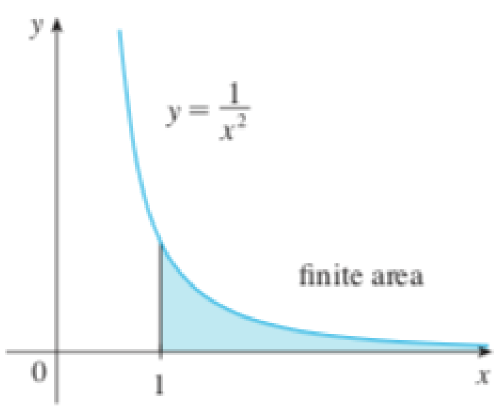


FIGURE 4 $\int_1^\infty (1/x^2)\,dx$ converges

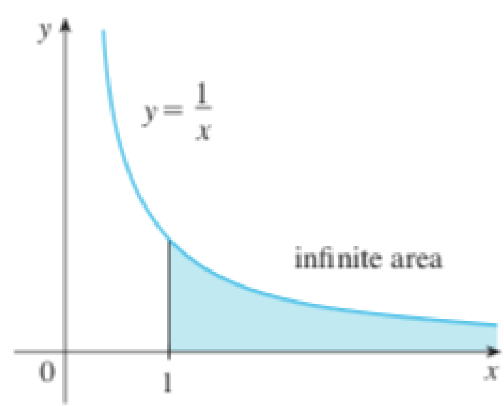


FIGURE 5 $\int_1^\infty (1/x)\,dx$ diverges

Example. Determine whether $I = \displaystyle\int_{-\infty}^{0} xe^x\,dx$ converges or diverges. If it converges, determine what it converges to.

$$I = \int_{-\infty}^{0} xe^x\,dx =$$

Now, $\displaystyle\lim_{t\to-\infty}(-te^t)$ is an indeterminate form[5]: of type $[\infty \cdot 0]$. So we change it to an indeterminate form of type $[\infty/\infty]$ and apply l'Hôpital's Rule[6]

$$\lim_{t\to-\infty}(-te^t) =$$

Thus

$$\int_{-\infty}^{0} xe^x\,dx =$$

[5]Indeterminate forms: $[0/0], [\infty/\infty], [0\cdot\infty], [\infty-\infty], [0^0], [\infty^0], [1^\infty]$

[6]L'Hôpital's Rule (§4.4): If $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \qquad \text{if the limit on the right side exists (or is } \infty \text{ or } -\infty).$$

Example. Determine whether the integral $I = \displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ converges or diverges.

If the two improper integrals on the right are convergent, then we can evaluate I as follows:

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx \stackrel{?}{=} \int \frac{1}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx$$

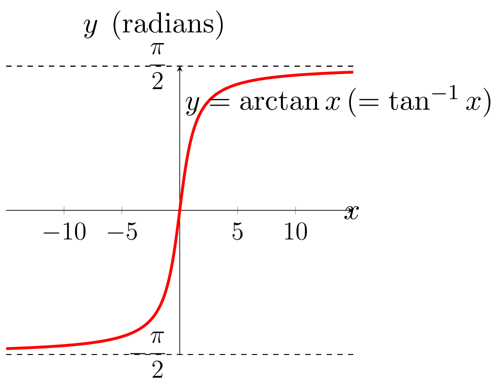
Let's evaluate the integrals on the right side:

$$\int \frac{1}{1+x^2}\,dx =$$

$$\int \frac{1}{1+x^2}\,dx =$$

Thus

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx =$$



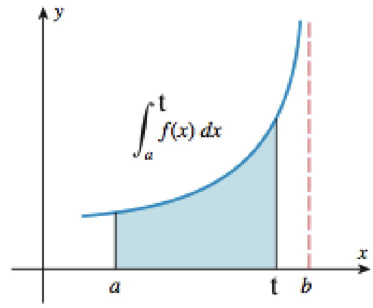
2. Discontinuous Integrands

Suppose that we have a finite interval of integration $[a, b]$ and the infinite discontinuity (i.e. vertical asymptote) occurs at $x = b$.

The area of the region in the interval $[a, t]$ (where $a < t < b$, see figure) is

$$A(t) =$$

If we let t approach ____ from the ________ side of b, we can fill out the area of the entire region.



Definition.

- *If f is continuous on $[a, b)$ and has an infinite discontinuity at b, then*

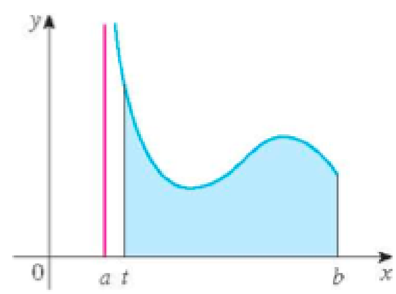
$$\boxed{\int_a^b f(x)\,dx = \qquad\qquad\qquad}$$

provided this limit exists as a finite number.

- *If f is continuous on $(a, b]$ and has an infinite discontinuity at a, then*

$$\boxed{\int_a^b f(x)\,dx = \qquad\qquad\qquad}$$

provided this limit exists as a finite number.

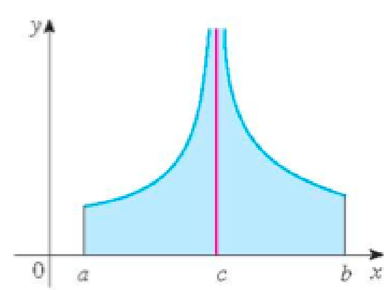


The improper integral $\int_a^b f(x)\,dx$ is called ***convergent*** *if the corresponding limit exists, and* ***divergent*** *if the limit does not exist.*

- *If f has an infinite discontinuity at $c \in (a, b)$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent, then we define*

$$\boxed{\int_a^b f(x)\,dx = \int f(x)\,dx + \int f(x)\,dx}$$

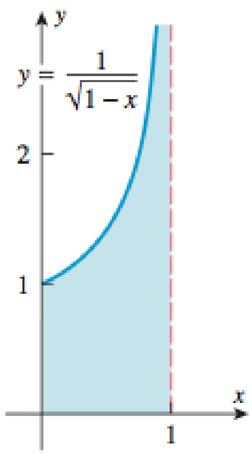
$\int_a^b f(x)\,dx$ is said to ***converge*** *if both integrals on the right side converge, and* ***diverge*** *If either term on the right diverges.*



Example. Determine whether the integral $I = \displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$ converges or diverges.

First, recognize that this is an improper integral since it has a vertical asymptote (i.e. infinite discontinuity) at $x =$ ____. Hence

$$I = \int_0^1 \frac{dx}{\sqrt{1-x}} =$$



Example. Determine whether the integral $I = \displaystyle\int_0^3 \frac{dx}{x-1}$ converges or diverges.

There is a vertical asymptote at $x =$ ____, so this is an improper integral.

Caution. If we had not noticed the asymptote, we may have made the following erroneous calculation:

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big|_0^3 = \ln 2 - \ln 1 = \ln 2 \text{ [WRONG]}$$

This is wrong because the integral is improper.

We must break up the integral as follows:

$$I = \int_0^3 \frac{dx}{x-1} \stackrel{?}{=} \int \frac{dx}{x-1} + \int \frac{dx}{x-1} \tag{2}$$

$$\int \frac{dx}{x-1} =$$

$$\int \frac{dx}{x-1} =$$

Hence $I = \displaystyle\int_0^3 \frac{dx}{x-1}$ ____________.

If one of the improper integrals on the right in (2) diverges, then so does the improper integral on the left. So for this question, you could just evaluate one of the improper integrals on the right and conclude that I diverges. But the reason I wanted you to evaluate both these integrals is for you to note that $-\infty + \infty + \ln 2 \neq \ln 2$, since ∞ is not a number, so they cannot cancel out.

From now on, whenever you are asked to evaluate $\displaystyle\int_a^b f(x)\,dx$ *you must first decide whether it is an ordinary definite integral or an improper integral. If it is improper, you must use limit(s).*

Homework. 7.8: # 1, 11, 13, 19, 23 - 39 (odd)

Chapter 11. Infinite Sequences and Series

§11.1 Sequences

Definition. *A* ***sequence****, is an unending succession of numbers, called* ***terms****, written in a definite order:*

The number a_1 is called the ***first term****,*
the number a_2 is called the ***second term****, and*
in general a_n ($n \geq 1$) is called the ***n-th term*** *or the* ***general term*** *of the sequence.*

Remark. n need not start at 1.

Example. Find the general term a_n of the following sequence (assume n starts at 1).

$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \cdots$ $\qquad\qquad$ $a_n =$

Notation: The sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ can also be written as

$\boxed{}$ or $\boxed{}$

Example. The sequence in example above can be written as:

Graphs of Sequences

For every positive integer n there is a corresponding number a_n, and so a sequence can be defined as a ____________ of n.

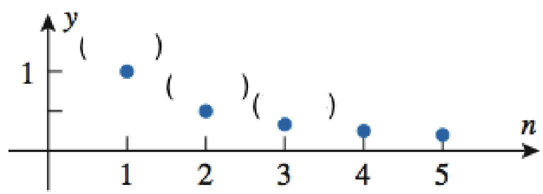
Remark. This function has domain n positive integers only. Hence it is **not** a differentiable function.

The graph of the sequence $\{a_n\}_{n=1}^{\infty}$ consists of the points:

$$(\qquad , \qquad)$$

Graph of the sequence

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$$



From the graph above, it appears that the terms of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ are approaching ____ as n becomes large. To prove this formally, we need to use limits.

The notation

$$\lim_{n\to\infty} a_n = \text{\_\_\_\_}$$

means that the terms of the sequence $\{a_n\}$ approach ____ as n becomes large.
If $\lim_{n\to\infty} a_n$ exists, we say that the sequence ________________ (or is **convergent**) to ____. Otherwise we say the sequence ________________ (or is **divergent**).

Example. For the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ above, since $\lim_{n\to\infty} \frac{1}{n} =$ ____, we say that this sequence ________________ to ____.

Example. Determine whether the sequence converges or diverges.

a) $\left\{ \dfrac{n}{2n+1} \right\}_{n=1}^{\infty}$

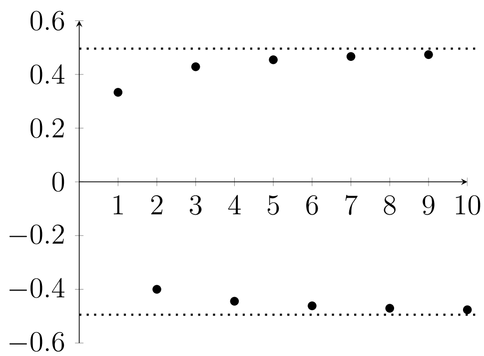
b) $\left\{ (-1)^{n+1} \dfrac{n}{2n+1} \right\}_{n=1}^{\infty}$

We can divide up this sequence into odd-numbered terms and even-numbered terms.

- When n is odd, the n-th term is ________, so $\lim\limits_{n\to\infty}$ ________ $=$ ____.

- When n is even, the n-th term is ________, so $\lim\limits_{n\to\infty}$ ________ $=$ ____.

Since the even- and odd- numbered terms approach different values as $n \to \infty$, the original sequence has ____________– it ________________.



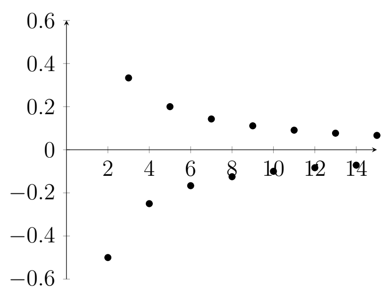
c) $\left\{(-1)^{n+1}\dfrac{1}{n}\right\}_{n=1}^{\infty}$

Solution 1. We can divide up this sequence into odd-numbered terms and even-numbered terms.

- When n is odd, the n-th term is ________, so $\lim\limits_{n\to\infty}$ ________ $=$ ____.

- When n is even, the n-th term is ________, so $\lim\limits_{n\to\infty}$ ________ $=$ ____.

Thus $\lim\limits_{n\to\infty}(-1)^{n+1}\dfrac{1}{n} =$ ____. So this sequence ________________ to ____.



Solution 2. The following theorem is useful for finding the limit of a sequence with both positive and negative terms:

Theorem. *If $\lim\limits_{n\to\infty}|a_n| = 0$ then $\lim\limits_{n\to\infty} a_n =$ ____ i.e., the sequence $\{a_n\}$ ________________ to ____.*

So for this example, $\lim\limits_{n\to\infty}|a_n| =$

Hence the sequence ________________ to ____.

Remark. If $\lim\limits_{n\to\infty}|a_n| \neq 0$,then the sequence might converge or diverge.

Example. Give an example of:

a) a sequence $\{a_n\}$ for which $\lim_{n\to\infty}|a_n| \neq 0$ and $\{a_n\}$ converges:

b) a sequence $\{a_n\}$ for which $\lim_{n\to\infty}|a_n| \neq 0$ and $\{a_n\}$ diverges:

d) $\left\{ \dfrac{n}{e^n} \right\}_{n=1}^{\infty}$

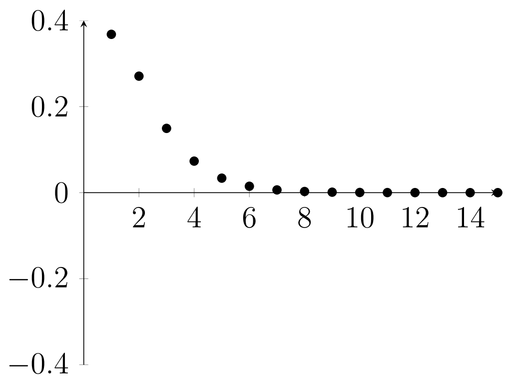
Solution 1. $\displaystyle\lim_{n\to\infty} \frac{n}{e^n}$ is an indeterminate form[7] of type ____. However, we cannot apply L'Hôpital's rule[8] directly to $\frac{n}{e^n}$ because the functions n and e^n are defined here only for positive integers n, so are ________ ________________ functions. But we can apply L'Hôpital's rule to the related function $f(x) =$ whose domain is [____, ____) (so that x and e^x are differentiable on the domain) to get

$$\lim_{x\to\infty} \frac{x}{e^x} =$$

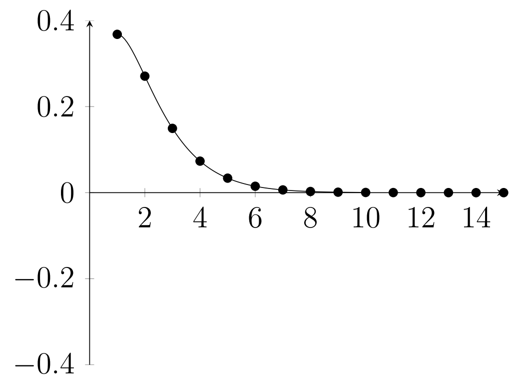
Thus

$$\lim_{n\to\infty} \frac{n}{e^n} =$$

So this sequence ________________ to ____.



$\boxed{y = \dfrac{n}{e^n}, \quad n = 1, 2, 3, \cdots}$



$\boxed{y = \dfrac{x}{e^x}, \quad x \geq 1}$

[7]Indeterminate forms: $[0/0], [\infty/\infty], [0 \cdot \infty], [\infty - \infty], [0^0], [\infty^0], [1^\infty]$

[8]L'Hôpital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). If $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \qquad \text{if the limit on the right side exists (or is } \infty \text{ or } -\infty).$$

Theorem. *Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *converge (i.e.* $\lim\limits_{n\to\infty} a_n$ *and* $\lim\limits_{n\to\infty} b_n$ *exist) and* c *is a constant. Then:*

a) $\lim\limits_{n\to\infty} (c \cdot a_n) = c \cdot \lim\limits_{n\to\infty} a_n$

b) $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

c) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

d) $\lim\limits_{n\to\infty} ((a_n)^m) = \left(\lim\limits_{n\to\infty} a_n\right)^m$

e) $\lim\limits_{n\to\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ *(provided* $\lim\limits_{n\to\infty} b_n \neq 0)$

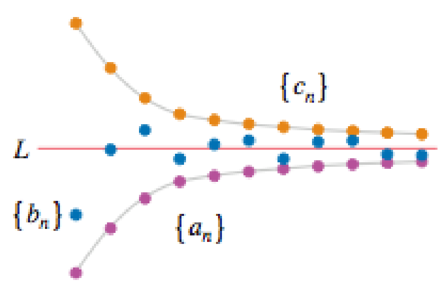
Theorem (Squeeze Theorem for Sequences)**.** *Let* $\{a_n\}, \{b_n\},$ *and* $\{c_n\}$ *be sequences such that*

$$a_n \leq b_n \leq c_n$$

for $n \geq n_0$ *(where* n_0 *is some integer greater than or equal to 1) and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

Then $\lim\limits_{n\to\infty} b_n =$ _____.

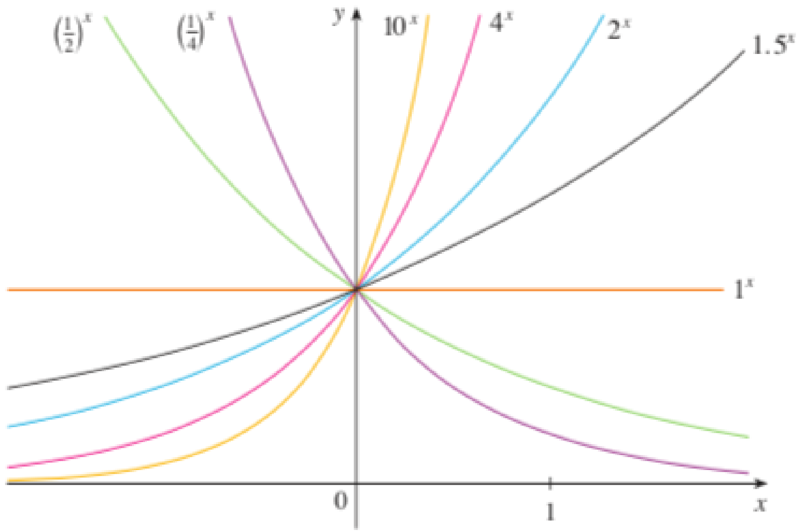


Example. Does the sequence $\left\{\dfrac{\cos^2 n}{2^n}\right\}_{n=1}^{\infty}$ converge or diverge? If it converges, find the limit.

Example. For what values of r is the sequence $\{r^n\}$ convergent?

- $r = 1$: $\{r^n\} = \{1^n\} =$ ____________. So the sequence ____________ to ____.
- $r = 0$: $\{r^n\} = \{0^n\} =$ ____________. So the sequence ____________ to ____.
- $r = -1$: $\{r^n\} = \{(-1)^n\} =$ ____________. So the sequence ____________.

Recall from graphs of exponential functions:



- $r > 1$: $\lim_{n\to\infty} r^n =$ ____. So the sequence ____________.
- $0 < r < 1$: $\lim_{n\to\infty} r^n =$ ____. So the sequence ____________ to ____.
- $-1 < r < 0$: $\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n =$ ____. So the sequence ____________ to ____.
- $r < -1$: When n is odd, $\lim_{n\to\infty} r^n =$ ____.

  When n is even, $\lim_{n\to\infty} r^n =$ ____, so the sequence ____________.

In summary:

The sequence $\{r^n\}$ converges if ____________ and diverges otherwise.

$$\lim_{n\to\infty} r^n = \begin{cases} \underline{\qquad}, & -1 < r < 1 \\ \underline{\qquad}, & r = 1 \end{cases}$$

Definition. *A sequence* $\{a_n\}_{n=1}^{\infty}$ *is called:*

- ***increasing*** *if* $a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_n \quad \cdots$ *(i.e.* $\boxed{a_n \qquad a_{n+1}}$ $\forall\, n \geq 1$*)*
- ***decreasing*** *if* $a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_n \quad \cdots$ *(i.e.* $\boxed{a_n \qquad a_{n+1}}$ $\forall\, n \geq 1$*)*

A sequence that is either increasing or decreasing is said to be **monotonic**.

Example. Is the following sequence increasing or decreasing?

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

Let $a_n = \dfrac{n}{n+1}$. Then $a_{n+1} =$

We conjecture that this sequence is ________________ (i.e. a_n ____ a_{n+1} for all positive integers n)

[9]

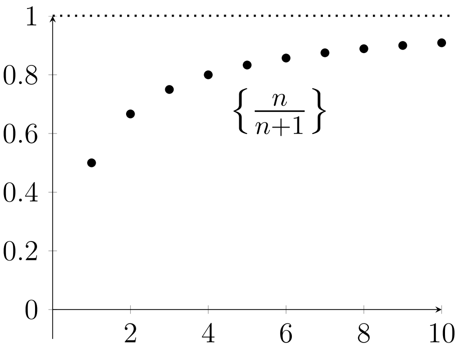
[9] Alternatively, one could show that this sequence is increasing by taking the related function $f(x) = \dfrac{x}{x+1}$ with domain $[1, \infty)$ and showing that $f'(x) > 0$ in $[1, \infty)$. Try as homework!

Definition.

- *A sequence* $\{a_n\}$ *is* ***bounded above by some number*** M *if*

$$a_n \qquad\qquad \text{for all } n \geq 1.$$

That is, no term in the sequence $\{a_n\}$ *is larger than* M.

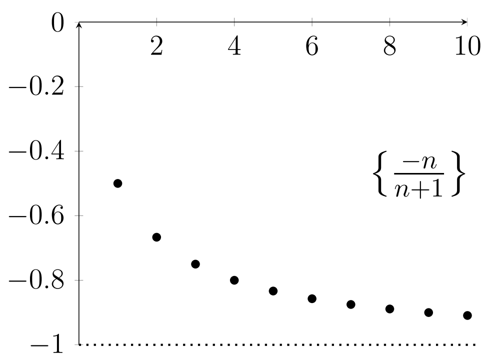


This sequence is bounded above by, for example, ____.

- *A sequence is* ***bounded below by some number*** m *if*

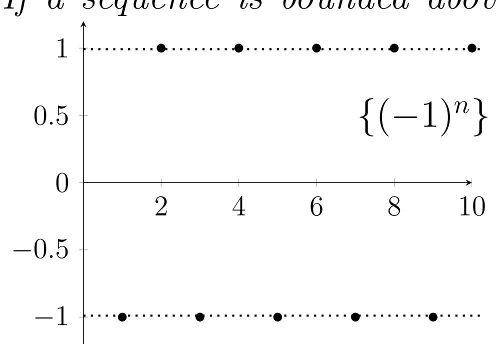
$$a_n \quad \text{for all } \ n \geq 1.$$

That is, no term in the sequence $\{a_n\}$ *is less than* m.



This sequence is bounded below by, for example, ____.

- *If a sequence is bounded above and below, then* $\{a_n\}$ *is said to be* ***bounded****.*



This sequence is bounded above by, for example, ____ and below by, for example, ____. So it is a bounded sequence.

The following theorem tells us what types of bounded sequences are convergent:

Theorem (Monotonic Sequence Theorem)**.**

If a sequence is both ________________ *AND* ________________ *then it is convergent.*

Example. Use the Monotonic Sequence Theorem to show that the sequence

$\left\{\dfrac{4n-3}{e^n}\right\}_{n=3}^{\infty}$ converges.

- **Monotonic:** Let's use the derivative test (recall: if $f' < 0$, then it is decreasing; if $f' > 0$ then it is increasing).

- **Bounded:** All the terms in this sequence are positive, so it is bounded below by ____. Since it is a decreasing sequence, it is bounded above by the first term, which is . In other words, this is a bounded sequence.

By the Monotonic Sequence Theorem, this sequence ________________.

Homework. §11.1: # 1, 7, 19, 29 - 41 (odd), 45 - 57 (odd), 77, 79, 81

§11.2 Series

In this section, we discuss sums that contain infinitely many terms.

Definition. *An **infinite series**, or simply a **series**, is an expression in the form*

$$\sum_{n=1} a_n =$$

Example.

$$\sum_{n=1}^{\infty} \frac{3}{10^n} =$$

Since $0.33333\cdots$ is a decimal expansion of ____, we would like to obtain a definition for $\displaystyle\sum_{n=1}^{\infty} \frac{3}{10^n}$, which will yield ____.

Consider the following ________________ of (finite) sums:

$$s_1 = \frac{3}{10} =$$

$$s_2 = \frac{3}{10} + \frac{3}{10^2} =$$

$$s_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} =$$

$$\vdots$$

$$s_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} =$$

The numbers $s_1, s_2, \cdots$ are called **partial sums**.
The ________________ of numbers

$$\{s_n\}_{n=1}^{\infty} = \{s_1, s_2, s_3, s_4, \cdots\}$$

(called the **sequence of partial sums**) can be viewed as a succession of approximations to the "sum" of the infinite series, which we want to be $1/3$ (i.e. we expect   ).

To evaluate

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{3}{10}+\frac{3}{10^2}+\cdots+\frac{3}{10^n}\right)$$

we first write s_n in a closed form. Note that

$$\begin{aligned} s_n &= \frac{3}{10}+\frac{3}{10^2}+\cdots+\frac{3}{10^{n-1}}+\frac{3}{10^n} \\ \Rightarrow\ \frac{1}{10}s_n &= \\ \hline \Rightarrow\ s_n-\frac{1}{10}s_n &= \end{aligned}$$

That is,

Thus

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}$$

So we say that

$$\sum_{n=1}^{\infty}\frac{3}{10^n} = \frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+\frac{3}{10^4}+\cdots =$$

In general, we have the following definition:

Definition. *Consider a general series*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

and consider its partial sums

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \end{aligned}$$

If the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ $(= \{s_1, s_2, s_3, \cdots\} = \{a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3, \cdots\})$ *converges to a limit* S *(that is,* $\lim_{n\to\infty} s_n =$ *) where* S *is a finite number, then the series* $\sum_{n=1}^{\infty} a_n$ *is said to* ________________ *to* S*, and* S *is called the* ________ *of the series. We denote this by writing*

$$S =$$

If the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ *diverges, then the series is said to* ________________*. A divergent series has* ____________*.*

Example. Determine whether the series

$$1 - 1 + 1 - 1 + 1 - \cdots$$

converges or diverges. If it converges, find the sum.
Let's consider the partial sums:

$$\begin{aligned} s_1 &= \\ s_2 &= \\ s_3 &= \\ s_4 &= \\ &\vdots \end{aligned}$$

Hence the sequence of partial sums $\{s_n\}_{n=1}^{\infty} = \{s_1, s_2, s_3, \cdots\}$ is

Since this sequence of partial sums is ________________ $\left(\lim_{n\to\infty} s_n \text{\_\_\_\_\_\_\_\_}\right)$, the given series is ________________, i.e. it has ________________.

Telescoping Sums

Example. Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$. Note that

$s_n =$

Hence $\lim_{n\to\infty} s_n = \lim_{n\to\infty}$ That is, the series converges to ____.

Geometric Series

An important example of an infinite series is the **geometric series**

$$\boxed{\phantom{a + ar + ar^2 + \cdots + ar^{n-1} + \cdots}} \tag{1}$$

The first term is ____. All the other terms are obtained by multiplying the preceding term by the **common ratio** ____.

Remark. One can write the above geometric series in summation notation in many ways:

Example. (the following are all geometric series)

- $\frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+\cdots+\frac{3}{10^n}+\cdots \qquad (a = \_\_\_\_, r = \_\_\_\_)$

  [we saw in an earlier example that this series converges to 1/3]

- $1-1+1-1+\cdots \qquad (a = \_\_\_\_, r = \_\_\_\_)$

  [we saw in an earlier example that this series diverges]

The geometric series

$$a + ar + ar^2 + \cdots \qquad (a \neq 0)$$

converges if __________ and diverges otherwise. If the series converges, then the sum is

$$a + ar + ar^2 + \cdots =$$

Example. Determine whether the series converges, and if so find its sum.

a) $\displaystyle\sum_{n=1}^{\infty} 2^{2n}\, 3^{1-n} =$

This is a geometric series with $a =$ ____ and $r =$ ____. Since $|r|$________, this series ____________.

b) $5 - \dfrac{10}{3} + \dfrac{20}{9} - \dfrac{40}{27} + \cdots$

This is a geometric series with $a =$ ____ and $r =$ ____. Since $|r|$________, this series ____________. Its sum is:

<u>Proof of convergence of Geometric Series when $|r| < 1$ and divergence otherwise:</u>

Case 1: $r = 1 \quad \Rightarrow \quad s_n = a + a + a + \cdots + a = na.$

$$\Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} na = DNE \text{ (will be } +\infty \text{ or } -\infty \text{ depending on the sign of } a)$$

Hence the geometric series diverges if $r = 1$.

Case 2: $r \neq 1 \quad \Rightarrow$

$$\begin{array}{lll} & s_n & = a + ar + ar^2 + \cdots + ar^{n-1} \\ \Rightarrow & rs_n & = \phantom{a+{}} ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ \hline & (1-r)s_n & = a \qquad\qquad\qquad\qquad\quad - ar^n \end{array}$$

$$\Rightarrow s_n = \frac{a - ar^n}{1-r} = \frac{a}{1-r}\left(1 - r^n\right)$$

- If $|r| < 1$ then as $n \to \infty$, $r^n \to 0$. Thus

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a}{1-r}\left(1 - r^n\right) = \frac{a}{1-r}$$

That is, the geometric series converges to $\dfrac{a}{1-r}$ when $|r| < 1$.

- If $r > 1$ then as $n \to \infty$, $r^n \to \infty$. Thus

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a}{1-r}\left(1 - r^n\right) = \pm\infty \text{ (depending on the sign of } a)$$

That is, the geometric series diverges when $r > 1$.

- If $r < -1$ then r^n oscillates between positive and negative values that grow in magnitude as $n \to \infty$. So

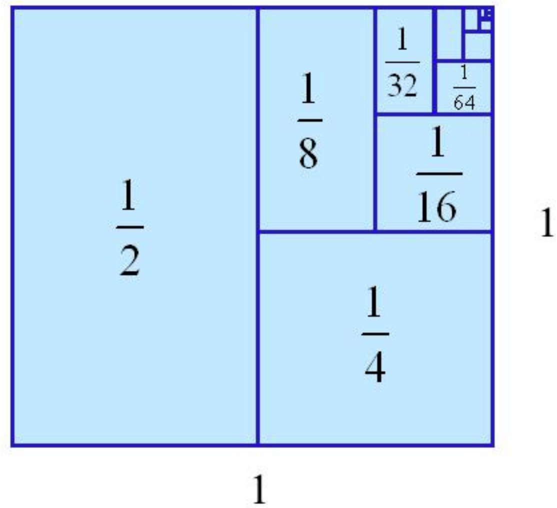
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a}{1-r}\left(1 - r^n\right) \text{ diverges}$$

That is, the geometric series diverges when $r < -1$.

Start with a square one unit by one unit:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots=1$$

This is an example of an <u>infinite series</u>.



This series <u>converges</u> (approaches a limiting value.)

Many series do not converge: $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots=\infty$

Example. Show that the **harmonic series**:

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

diverges.

For this series it's convenient to consider the partial sums $s_4, s_8, s_{16}, s_{32}, \cdots$ and show that they become large.

$$s_{2^2} = s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) =$$

$$s_{2^3} = s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$=$$

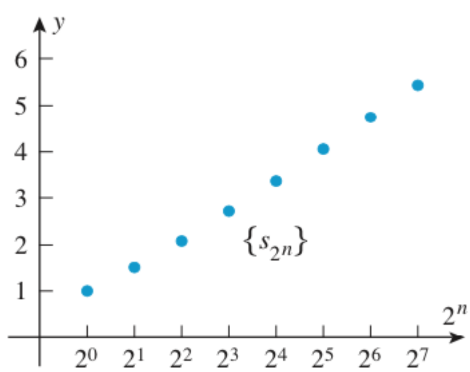
$$s_{2^4} = s_{16} = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \tfrac{1}{10} + \cdots + \tfrac{1}{16}\right)$$

$$> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \tfrac{1}{16} + \cdots + \tfrac{1}{16}\right)$$

$$=$$

Similarly, $s_{2^5} = s_{32} >$ $\qquad\qquad$, $s_{2^6} = s_{64} >$ $\qquad\qquad$, and in general

$$s_{2^n} >$$



That is, $s_{2^n} \to$ ____ as $n \to \infty$, so $\{s_n\}_{n=1}^{\infty}$ is ________________.

That is, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ ________________.

Theorem 1.

Let c be a constant.

a) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then $\sum_{n=1}^{\infty} (c \cdot a_n)$ and $\sum_{n=1}^{\infty}(a_n \pm b_n)$ also ____________, and:

- $\sum_{n=1}^{\infty} (c \cdot a_n) =$
- $\sum_{n=1}^{\infty}(a_n \pm b_n) =$

b) If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} (c \cdot a_n)$ (where $c \neq 0$) diverges too.

c) Convergence or divergence is unaffected by deleting a ____________ number of terms from a series.

Example. Determine whether the following series converge or diverge. If a series converges, find its sum.

a) $\sum_{n=1}^{\infty} \left(\frac{3}{10^n} - \frac{2}{5^{n-1}} \right)$

(We will first conjecture that $\sum \frac{3}{10^n}$ and $\sum \frac{2}{5^{n-1}}$ are convergent and find their sums.) We have already seen earlier that the series $\sum_{n=1}^{\infty} \frac{3}{10^n}$ converges and the sum of this series is $\frac{1}{3}$.

The series $\sum_{n=1}^{\infty} \frac{2}{5^{n-1}} =$ is also a geometric series with $a =$ ____ and $r =$ ____ so it converges and the sum of this series is

Hence by Theorem 1 a) the given series ____________, and

$$\sum_{n=1}^{\infty} \left(\frac{3}{10^n} - \frac{2}{5^{n-1}} \right) =$$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n}$

Note that $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n} =$

We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent ____________ series. The given series is obtained by multiplying each term of the series by ____. Hence by Theorem 1 b) we conclude that the given series ____________.

c) $\displaystyle\sum_{n=1001}^{\infty} \frac{1}{n}$

This series can be obtained by deleting the first ____ terms from the divergent ____________ series, so this series ____________ by Theorem 1 c).

Test for Divergence: If $\lim_{n\to\infty} |a_n|$ ____ 0 then the series $\sum_{n=1}^{\infty} a_n$

Example. Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \qquad =$$

Hence this series ____________ by the Test for Divergence.

Remark.

- If $\lim_{n\to\infty} |a_n| = 0$ then $\sum_{n=1}^{\infty} a_n$ may or may not converge. (i.e. **no** conclusion)
- If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} |a_n| =$ ____. (this is the *contrapositive* of the Test for Divergence)

Example. Can we make any conclusions about the convergence or the divergence of these series using the Test for Divergence?

- $\sum_{n=1}^{\infty} \frac{e^n}{n}$

  $\lim_{n\to\infty} |a_n| =$

- $\sum_{n=1}^{\infty} \frac{3}{10^n}$ $\qquad \lim_{n\to\infty} |a_n| =$

  (We already saw that this is a *convergent* geometric series.)

- $\sum_{n=1}^{\infty} \frac{1}{n}$ $\qquad \lim_{n\to\infty} |a_n| =$

  (We already saw that this is the *divergent* harmonic series.)

Homework. 11.2: # 1, 3, 15, 17, 19, 27, 29, 31, 37 - 47 (odd), 59 - 65 (odd), 77

§11.3 The Integral Test and Estimates of Sums

In general, it is difficult to find the exact sum of a series (i.e. $\lim_{n\to\infty} s_n$) if it is not easy to find a simple formula for the nth partial sum s_n.

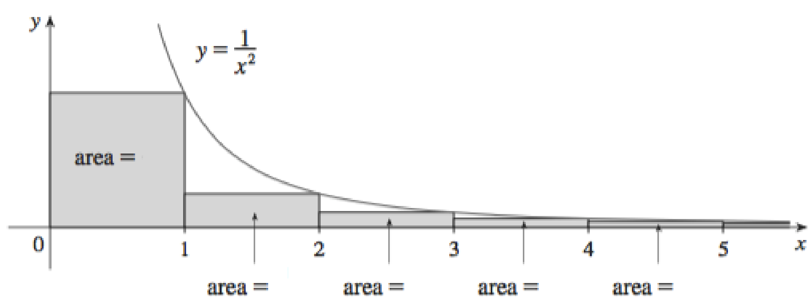
In the next few sections, we will develop various tests that can be used to determine whether a given series converges or diverges without explicitly finding its sum.

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ converge or diverge?

There is no simple formula for the sum s_n of the first n terms, but the computer-generated table of approximate values (see right) suggests that the partial sums are approaching ≈ 1.64 as $n \to \infty$. That is, the series seems to be ________________.

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

We can confirm this impression with a geometric argument:



The sum of the areas of the rectangles is exactly the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

If we exclude the first rectangle, the total area of the remaining rectangle is smaller than the area under the curve $y = \dfrac{1}{x^2}$ for $x \geq 1$, which is equivalent to:

So all the partial sums must be less than

Hence we suspect that the sum of the areas of all the rectangles must be less than ____ (i.e. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ must be ________________.)

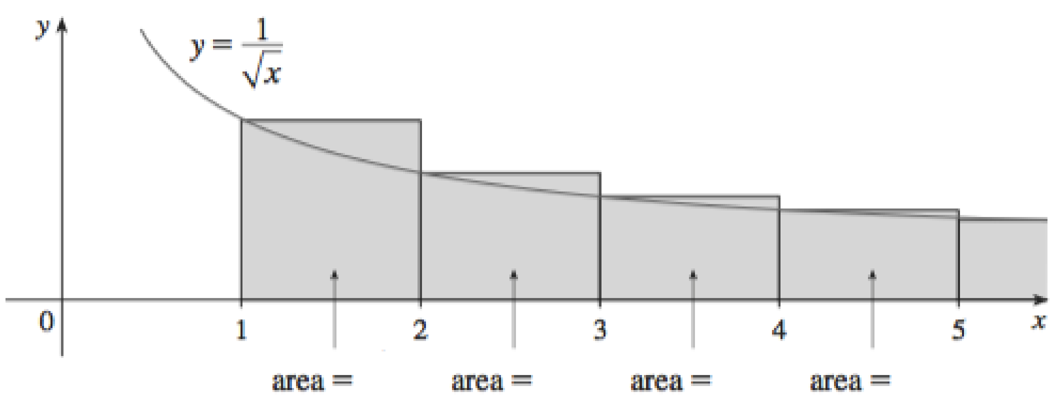
Example. Determine the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \,(= \qquad\qquad\qquad\qquad\qquad\qquad)$$

The table of values (see right) suggests that the partial sums are not approaching a finite number, so this series seems to be ____________.

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

Again, we confirm this with a geometric argument:



The sum of the areas of all the rectangles is exactly the sum of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

The total area of these rectangles is greater than the area under the curve $y = \dfrac{1}{\sqrt{x}}$ for $x \geq 1$, which is equal to

That is, the area under the curve is ____________. So the sum of the areas of all the rectangles must be ________________; that is, this series is ________________.

The same sort of geometric reasoning can be used to prove the following test:

The Integral Test

Let $a_n = f(n)$ for all $n \geq k \geq 1$. Suppose that $f(x)$ is a ________________, ________________ and ________________ function on $[k, \infty)$. Then

$$\text{and} \qquad \sum_{n=k}^{\infty} a_n$$

both converge or both diverge.

Example. Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge or diverge?

Let $f(x) = \dfrac{1}{x^2+1}, x \in [1, \infty)$.

First check that we can apply the Integral Test.

1. Is f decreasing on $[1, \infty)$? Let's check if $f' < 0$ on $[1, \infty)$:

   $$\frac{d}{dx}\left(\frac{1}{x^2+1}\right) =$$

2. Is f continuous on $[1, \infty)$?

   Recall that if a function is differentiable on (a, b) then the function is continuous on (a, b). We found above that the derivative exists for all $x \in [1, \infty)$. That is, f is ________________ on $[1, \infty)$. So f is ________________ on $[1, \infty)$.

3. Is f positive on $[1, \infty)$?

Hence f is continuous, positive and decreasing on $[1, \infty)$, so we can apply the Integral Test:

Since this integral is ______________, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ ______________, by the Integral Test.

Remark. The above result does NOT imply that the $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges to $\pi/4$. The Integral Test does not give us any information about what number the series converges to. In fact, the sum of the series is: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi \coth(\pi) - 1}{2}$

p-Series

Definition. *Let $p > 0$. A p-**series** is an infinite series of the form*

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} = \qquad\qquad\qquad\qquad\qquad\qquad}$$

Example. The following are examples of p-series:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ ($p =$ ____) - recall: this is the ________________ series (we saw that this series is divergent)

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ ($p =$ ____) (we saw that this series is convergent)

c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$ ($p =$ ____) (we saw that this series is divergent)

Example. For what values of p is the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Case 1. $p = 1$: This is the ____________ series, so it ____________.

Case 2. $p \neq 1$: Let $f(x) = \frac{1}{x^p}$, $x \in [1, \infty)$. Since f is continuous, positive and decreasing (check!), the Integral Test can be applied:

$$\int_1^{\infty} \frac{1}{x^p}\, dx =$$

- If $p > 1$, then $-p + 1$________, so

  $$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{t \to \infty} \left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} \right) =$$

  That is, when $p > 1$, the integral (and hence the series) ____________ by the Integral Test.

- If $0 < p < 1$, then $-p + 1$________, so

  $$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{t \to \infty} \left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} \right) =$$

  That is, when $0 < p < 1$, the integral (and hence the series) ____________ by the Integral Test.

In summary,

$$\boxed{\text{The } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if \_\_\_\_\_\_\_\_ and diverges otherwise.}}$$

Example. Does the following series converge or diverge?

$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

This series is equivalent to .

This is a p-series with $p =$ ____________. Hence this series ____________.

Estimating the Sum of a Series

Suppose that we used the *Integral Test* to show that the series $\sum a_n$ is convergent. How can we get an approximation to the sum s of the series?

Using the same notation as the Integral Test, suppose that $f(n) = a_n$ and f is decreasing on $[n, \infty)$.

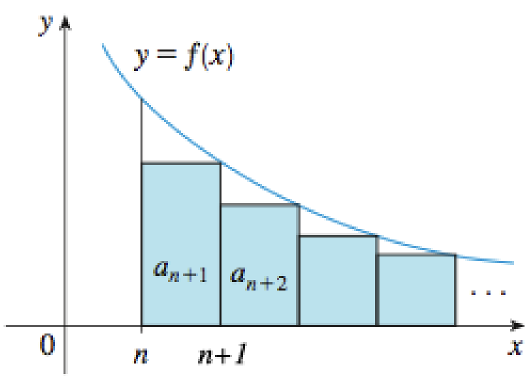
Recall:

- $s = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$
- $s_n = a_1 + a_2 + a_3 + \cdots a_n$

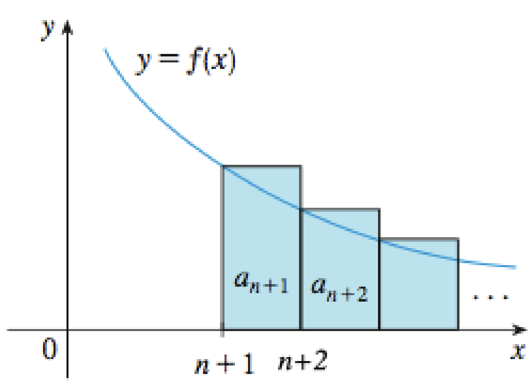
Definition. *The* ***remainder*** R_n *denotes the error made when* s_n *(the sum of the first* n *terms) is used to approximate the sum* s *of a series. That is,*

$$R_n =$$

There are two ways to express R_n using rectangles:



$$R_n = a_{n+1} + a_{n+2} + \cdots \qquad \int_n^{\infty} f(x)\,dx$$



$$R_n = a_{n+1} + a_{n+2} + \cdots \qquad \int_{n+1}^{\infty} f(x)\,dx$$

Based on the figures, we have the following error estimate.

Suppose $f(n) = a_n$, where f is a continuous, positive, decreasing function on $[n, \infty)$ and $\sum a_n$ converges to s. If $R_n = s - s_n$, then

$$\int^{\infty} f(x)\,dx < R_n < \int^{\infty} f(x)\,dx$$

Example.

a) Estimate the error involved in approximating the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first 10 terms, s_{10}.
Note: $s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3} + \frac{1}{8^3} + \frac{1}{9^3} + \frac{1}{10^3} \approx 1.1975$

One can show that $f(x) = \frac{1}{x^3}$ is continuous, positive, and decreasing on $[10, \infty)$ (check!). Now,

b) How many terms are required to ensure that the sum is accurate to within 0.0005?
Hint: $\sqrt{1000} \approx 31.6$

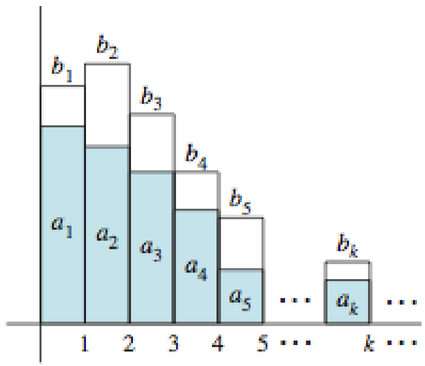
Hence, we need ____ terms to ensure accuracy to within 0.0005.

Homework. 11.3:# 3, 5, 7, 23, 25, 27, 29, 31, 37a, 39ad

§11.4 The Comparison Tests

Idea: Compare a given series with a series that is known to be convergent / divergent.

For each rectangle, a_n denotes the area of the dark portion and b_n denotes the combined area of the white and dark portions.



- if the total area $\sum b_n$ is finite, then the area $\sum a_n$ must also be ____________.

- If the total area $\sum a_n$ is infinite, then the total area $\sum b_n$ must also be ____________.

The Comparison Test

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ be series and suppose that

$$0 \le a_n \le b_n \quad \text{for all} \quad n \ge 1.$$

a) If the "bigger series" $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, the "smaller series" $\displaystyle\sum_{n=1}^{\infty} a_n$ ____________.

b) If the "smaller series" $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, the "bigger series" $\displaystyle\sum_{n=1}^{\infty} b_n$ ____________.

In using the Comparison Test, we need some series whose convergence we know for the purpose of comparison. We usually use one of these series:

- **a p-series** ($\sum \frac{1}{n^p}$ converges if ____________ and diverges otherwise)
- **a geometric series** (converges if ________ and diverges otherwise)

Remark. In testing many series $\sum a_n$, we may find a suitable comparison series by keeping only the ________________________ (i.e. the terms that "grow" the fastest as $n \to \infty$) in the numerator and denominator of a_n.

Example. Determine whether the following series converges or diverges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2 + n + 1}$

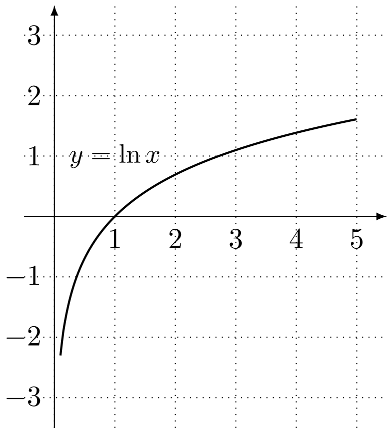
For large n, the dominant term in the denominator is ________, so we compare the given series with the series $\displaystyle\sum_{n=1}^{\infty}$ $\qquad$, which is a constant times a ________________ series (________), so it ________________. Note that

We conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2 + n + 1}$ ____________ by the Comparison Test.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{7}{\sqrt{n} - \frac{1}{2}}$

So the given series ____________ by the Comparison Test.

Example. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.



$\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ is the ______________________________ series with the first ____ terms removed, so it is still divergent. Hence by the Comparison Test, the "larger" series $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ must ___________. Since $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ has two more terms added to the divergent series $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$, it must also _______________.

The Limit Comparison Test

The following comparison is usually easier to apply than the Comparison Test.

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. Let $\rho =$ If ____ $< \rho <$ ____ then the series both converge or both diverge.

Example. Test the series for convergence or divergence.

a) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

Take the comparison series of $\sum_{n=1}^{\infty} b_n =$, which is a __________________ __________________ series (______________).

Remark. The Comparison Test will NOT work with this comparison series, since $\frac{1}{2^n - 1}$ $\quad$ $\frac{1}{2^n}$ and the "smaller series" converges.

$$\rho = \lim_{n\to\infty} \frac{a_n}{b_n} =$$

Since $0 < \rho < \infty$, we can use the Limit Comparison Test. Since $\sum_{n=1}^{\infty} b_n$ is ______________, we conclude that the original series is also __________________ by the Limit Comparison Test.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$

We use the comparison series of $\displaystyle\sum_{n=1}^{\infty} b_n =$, which is a ______________________ ______________________ series (____________).

Now,

$$\rho = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}$$

Since $0 < \rho < \infty$, we can use the Limit Comparison Test.

Since $\displaystyle\sum_{n=1}^{\infty} b_n$ is ______________________, we conclude that the original series is also ______________________ by the Limit Comparison Test.

Remark. Comparison Test with the above comparison series does NOT work ($\displaystyle\frac{n+4^n}{n+6^n} > \frac{4^n}{6^n}$)

Homework. 11.4: # 1, 7 - 33 (odd), 37

§11.5 Alternating Series

Up to now we only focused on series with non-negative terms. In this section, we discuss how to deal with series whose terms are not necessarily all positive.

Definition. *An **alternating series** is a series whose terms alternate between positive and negative numbers.*

Example.

- $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$
- $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$

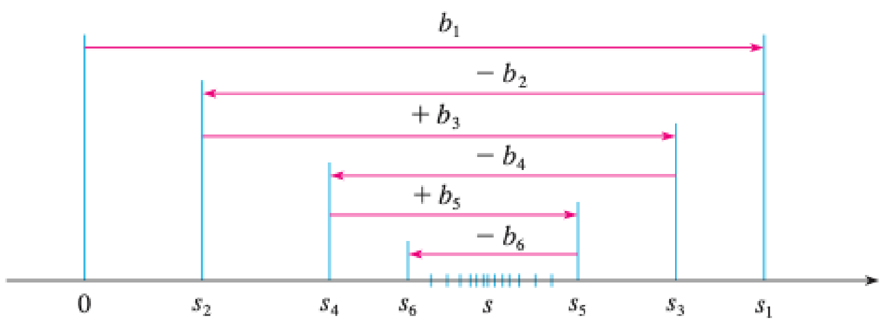
Alternating Series Test (AST)

The **alternating series** $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if:

(i) $\displaystyle\lim_{n\to\infty} |a_n| =$ and

(ii) $|a_1| \quad |a_2| \quad |a_3| \quad \cdots \quad |a_n| \quad \cdots$ (i.e. $|a_n| \quad |a_{n+1}|$ for all $n \geq 1$)

Idea behind proof: (for the case when we have an alternating series of the form $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ where b_n's are positive)



Example. Test the following alternating series for convergence.

a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} =$

This is called an **alternating harmonic** series.

(i) $\displaystyle\lim_{n\to\infty}|a_n| =$

(ii) $|a_n| =$

Hence the given series ________________ by the Alternating Series Test.

b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 3n}{4n-1}$

This is an alternating series. So we try the Alternating Series Test:

Condition (i): $\displaystyle\lim_{n\to\infty}|a_n| =$

That is, this series ________________ by the *Test for Divergence*.

Remark. In general, if condition (i) in the Alternating Series Test fails then the series must diverge by the Test for Divergence[10].

[10] Recall: The Test for Divergence (§11.2) says: If $\lim_{n\to\infty}|a_n| \neq 0$ then $\sum a_n$ diverges.

c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n(n+1)}$

Check condition (i):

$$\lim_{n\to\infty}|a_n| =$$

Check condition (ii):

$|a_n| \geq |a_{n+1}|$ FOR ALL $n \geq 1 \Leftrightarrow$

Hence the series ________________ by the Alternating Series Test.

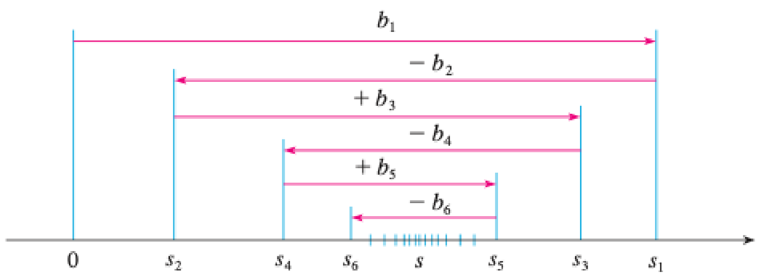
Remark. Instead of using inequalities, we can check condition (ii) by looking at the related (differentiable) function $f(x) = \dfrac{x+3}{x(x+1)}$ for $x \in [1,\infty)$ and showing that $f'(x) \leq 0$ for $x \in [1,\infty)$.

Estimating Sums

Suppose that we estimate the sum of the convergent alternating series $\sum_{n=1}^{\infty} a_n$ by the n-th partial sum, $s_n = a_1 + a_2 + \cdots + a_n$. We'd like to know how good this estimate is.

> If $\sum_{n=1}^{\infty} a_n$ is a **convergent alternating series** and if $s = \sum_{n=1}^{\infty} a_n$, then
>
> $$|R_n| = |s - s_n| \le |a_{n+1}|$$

Proof. Assume that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ where $b_n > 0$:



In general, $|s - s_n| \le b_{n+1}$ for all n. s is always between two consecutive partial sums, s_n and s_{n+1}.

Example. It can be shown that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ is convergent (try as homework – note that $n! = n(n-1)(n-2)\cdots 2 \cdot 1$). How many terms do we need to add to estimate the sum of this series with error less than $1/300$?

$$s \;= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} + \cdots$$

The error in using s_5 to estimate the series is at most $|a_{\quad}| =$

Hence we must add at least the first ________ terms of the series to estimate the sum with error less than $1/300$.

Homework. 11.5: # 3- 17 (odd), 37, 39, 41, 47

§11.6 Absolute Convergence and the Ratio and Root Tests

Absolute Convergence

Consider the following series

$$1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$$

This series does not fit any of the categories studied so far – it has mixed signs but is not alternating. We don't yet have a test for convergence/divergence of such series, which we now develop.

Definition. *The series* $\sum_{n=1}^{\infty} a_n$ *is said to* ***converge absolutely*** *(or is* ***absolutely convergent*** *(abbreviated ________)) if* $\sum_{n=1}^{\infty}$ ____ *converges.*

Example. Determine whether the following series converge absolutely.

a) $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$

Taking the absolute value of each term, we get the following series:

which is a ____________ ____________ series ($r =$ ____). So the given series is ________.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Taking the absolute value of each term, we get the series $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| =$

which is the ____________ series, which we know to ____________, so the given series is ________________.

Theorem.

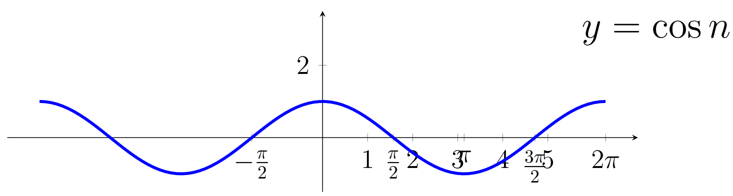
If the series $\sum_{n=1}^{\infty} |a_n|$ *is convergent, then the series* $\sum_{n=1}^{\infty} a_n$ *is also convergent. In other words, if a series is ________, then it is ________________.*

Example. Determine the convergence of each series.

a) $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$

We saw in a previous example that this series was absolutely convergent. Thus by the above Theorem, this series is ________________.

b) $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$



$$\cos 1 > 0, \qquad \cos 2 < 0, \qquad \cos 3 < 0, \qquad \cos 4 < 0, \qquad \cos 5 > 0 \cdots$$

This series has both positive and negative terms (but it is not alternating). Let's check if the series is absolutely convergent by testing $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ for convergence. We will use the Comparison Test:

Since the "larger series" $\sum_{n=1}^{\infty}$ is a ________________________ series (________), the "smaller series" $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ must ________________ by the Comparison Test.

That is, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is ________. So $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ must ________________.

Definition. *A series $\sum a_n$ is said to **converge conditionally** or (is **conditionally convergent**(abbreviated ________)) if:*

- *$\sum |a_n|$ is <u>not</u> convergent (i.e. is ____________), AND*
- *$\sum a_n$ is ____________.*

Example.

a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

- The series of absolute values is the ____________ series, which we know to ____________, so it is ____________.
- The original series is the alternating harmonic series (§11.5), so it ____________ .

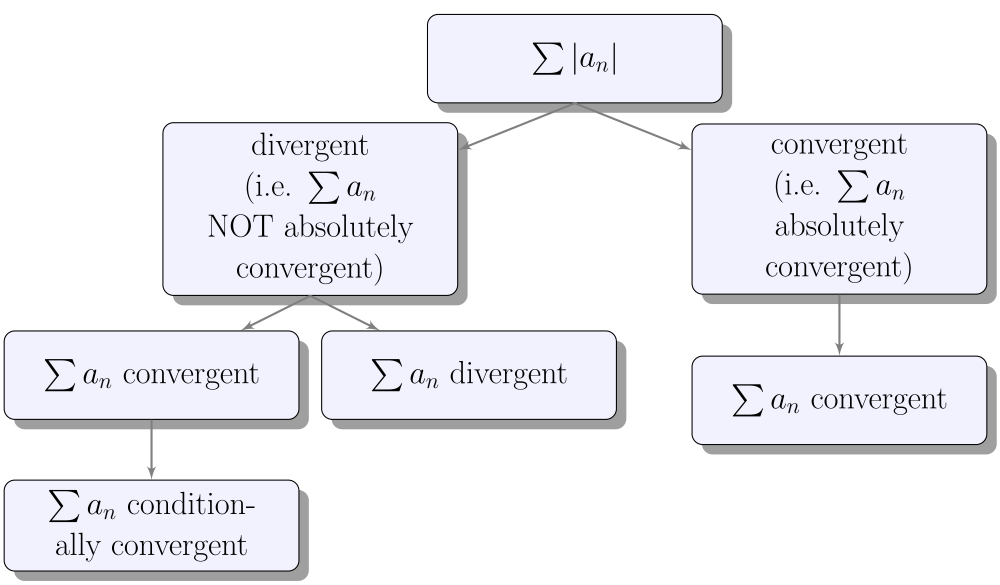
Hence the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is ________.

b) $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$

- The series of absolute values is the ____________ series, which we know to ____________, so it is ____________.
- Since the original series is a constant (-1) times a divergent harmonic series, it is also ________________(so it is ____________).

You can see from the examples above that:

$\boxed{\text{If a series does NOT converge absolutely, then the series may converge or diverge.}}$



Example. Is the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n(n+1)}$ absolutely convergent, conditionally convergent, or divergent?

Let $a_n = (-1)^{n+1}\dfrac{n+3}{n(n+1)}$. *First check if it is absolutely convergent:*

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{n+3}{n(n+1)}\right| = \sum_{n=1}^{\infty}$$

We use the Limit Comparison Test with the comparison series $\displaystyle\sum_{n=1}^{\infty} b_n =$ __________, which is the ______________ series, which ______________.

$$\rho = \lim_{n\to\infty}\frac{|a_n|}{b_n} =$$

Since _____ $< \rho <$ _____ and $\sum b_n$ is ______________, $\sum |a_n|$ must also ______________ by the Limit Comparison Test. Hence the original series is __________________.

Since this series is not AC, we need to check if it is CC:

Since we saw from §11.5 that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n(n+1)}$ is convergent (using the Alternating Series Test), we conclude that the series is __________________.

The following test is very useful in determining whether a given series is absolutely convergent (hence convergent).

Ratio Test

Let $\sum a_n$ be a series with non-zero terms and suppose that

$$\rho =$$

(i) If $\rho < 1$ the series $\sum a_n$ converges absolutely (hence ____________).

(ii) If $\rho > 1$ or $\rho = \infty$ the series $\sum a_n$ ________________.

(iii) If $\rho = 1$, no conclusion can be made about the convergence or the divergence; another test must be used.

(See p.739 for proof)

Remark. The following series will give **no conclusion** when the Ratio Test is applied:

- $\displaystyle\sum_{n=1}^{\infty} 1$: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{1} = 1$ (but this series is obviously *divergent*)
- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1/(n+1)^2}{1/n^2} = 1$ (but this is a p-series with $p = 2$, so it is *absolutely convergent*)
- $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = 1$ (but this is the alternating harmonic series, which we know to be *conditionally convergent*)

Example. Determine whether each of the following series converges.

a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n!}$ $\boxed{n! = n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot 2\cdot 1 \text{ (where } n \text{ is a positive integer)}}$

$\boxed{\text{We define } 0!=1}$

Note: $\boxed{\dfrac{(n+1)!}{n!} = \dfrac{(n+1)(n)(n-1)(n-2)\cdots 2\cdot 1}{n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot 2\cdot 1} = \dfrac{(n+1)(n!)}{n!} = n+1}$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| =$$

Hence by the Ratio Test the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n!}$ is ________, and therefore ____________.

b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\ \frac{3^n}{n^3}$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| =$$

Hence by the Ratio Test the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\ \frac{3^n}{n^3}$ is ___________.

c) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^n}$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}$$

Remark. For this limit at infinity, considering only the dominant term inside the brackets will **not** work! (yields the wrong answer) In general, you cannot take only the dominant term of a base that is raised to a power.

Hint: $\boxed{\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$ (Math 151/see below)

Hence by the Ratio Test the series is ________, and therefore ________________.

$\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ is an indeterminate form of type $[1^\infty]$. Let $y=\left(1+\frac{1}{x}\right)^x$. Then $\ln y = x\ln\left(1+\frac{1}{x}\right)$.

$$\begin{aligned}
\lim_{x\to\infty}\ln y &= \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right] \qquad [\infty\cdot 0]\\
&= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x} \qquad \left[\frac{0}{0}\right]\\
&=^H \lim_{x\to\infty}\frac{\frac{1}{1+1/x}(-1/x^2)}{-1/x^2} = \lim_{x\to\infty}\frac{1}{1+1/x} = 1
\end{aligned}$$

That is, $\displaystyle\lim_{n\to\infty}\ln y = 1$. Hence $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} y = \lim_{n\to\infty} e^{\ln y} = e^{\left(\lim_{n\to\infty}\ln y\right)} = e^1 = e$

The Root Test

The following test is convenient to apply when nth powers occur in the terms.

> Let $\sum a_n$ be a series and suppose that
>
> (i) If $\rho < 1$ the series $\sum a_n$ converges absolutely (hence ____________).
>
> (ii) If $\rho > 1$ or $\rho = \infty$ the series $\sum a_n$ ________________.
>
> (iii) If $\rho = 1$, no conclusion can be made about the convergence or the divergence; another test must be used.

Remark. If $\rho = 1$, another test may be used. In such a case, do not try the Ratio Test (because ρ in the Ratio Test will also be 1). Similarly, if $\rho = 1$ in the Ratio Test, do not try the Root Test (ρ in the Root Test will also be 1).

Example. Test the following series for convergence.

a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$

$$\rho = \lim_{n\to\infty}$$

so the series ________________ by the Root Test.

b) $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$ [hint: use the result from Page 132]

so the series ________________ by the Root Test.

c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\left(\frac{\ln n}{n}\right)^{n}$

So this series ________________ by the Root Test.

Homework. 11.6: # 1, 3 - 15 (odd), 19, 21, 23, 27 - 33 (odd), 39, 41

§11.7 Strategy for Testing Series

Testing series for convergence is similar to integrating functions in that there are no hard and fast rules about which test to apply to a given series. As with integration, the main strategy would be to classify the series according to its form:

1. If a series can be written in the form $\sum \frac{1}{n^p}$ then it is a p**-series** (converges if $p > 1$ and diverges otherwise).

2. If a series can be written in the form $\sum ar^{n-1}$ or $\sum ar^n$ it is a **geometric series** (converges if $|r| < 1$ and diverges otherwise).

3. If a series is similar in form to a p-series or a geometric series, then try the **Comparison Test** or the **Limit Comparison Test**. The comparison tests only apply to series with positive terms, but if $\sum a_n$ has some negative terms, you can apply the (Limit) Comparison Test to $\sum |a_n|$ and test for absolute convergence.

4. If you can see at a glance that $\lim\limits_{n\to\infty} |a_n| \neq 0$ then use the **Test for Divergence**.

5. If a series is of the form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$ then try the **Alternating Series Test**.

6. Series that involve factorials (!) or other products can often be tested using the **Ratio Test**.

7. If a_n is of the form $(b_n)^n$, then try the **Root Test**.

8. If $a_n = f(n)$ where it is easy to evaluate $\displaystyle\int_1^\infty f(x)\,dx$, then try the **Integral Test**. Make sure the hypotheses of this test are satisfied first!

Test for Divergence (§11.2)	If $\lim\limits_{n\to\infty} \lvert a_n\rvert \neq 0$ then $\sum a_n$ diverges. (**Remark.** If $\lim\limits_{n\to\infty} a_n = 0$ then $\sum a_n$ may or may not converge.)
Integral Test (§11.3)	Let $\sum a_n$ be a series with positive terms. If f is a function that is decreasing and continuous on $[k, \infty)$ and such that $a_n = f(n)$ for all $n \geq k$, then $\displaystyle\sum_{n=k}^{\infty} a_n$ and $\displaystyle\int_k^{\infty} f(x)\,dx$ both converge or both diverge.
Comparison Test (§11.4)	Let $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ be series with non-negative terms such that $$a_1 \leq b_1, a_2 \leq b_2, \cdots, a_n \leq b_n, \cdots$$ If $\sum b_n$ converges then $\sum a_n$ converges. If $\sum a_n$ diverges then $\sum b_n$ diverges.
Limit Comparison Test (§11.4)	Let $\sum a_n$ and $\sum b_n$ be series with positive terms and let $$\rho = \lim_{n\to\infty} \frac{a_n}{b_n}$$ If $0 < \rho < \infty$ then the series both converge or both diverge.
Alternating Series Test (§11.5)	The alternating series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if: $\lvert a_n\rvert \geq \lvert a_{n+1}\rvert\ \forall\, n \geq 1$ AND $\lim\limits_{n\to\infty} \lvert a_n\rvert = 0$.

Ratio Test (§11.6)	Let $\sum a_n$ be a series with nonzero terms only and suppose that $$\rho = \lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\|$$ The series converges if $\rho < 1$. The series diverges if $\rho > 1$ or $\rho = \infty$. The test is inconclusive if $\rho = 1$.
Root Test (§11.6)	Let $\sum a_n$ be a series and suppose that $$\rho = \lim_{n\to\infty} \sqrt[n]{\|a_n\|}$$ The series converges if $\rho < 1$. The series diverges if $\rho > 1$ or $\rho = \infty$. The test is inconclusive if $\rho = 1$.

Homework. §11.7: # 9 - 35 (odd), 41, 43, 47

§11.8 Power Series

Definition. *A series of the form*

$$\boxed{\sum_{n=0}^{\infty} c_n(x-a)^n = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , x \in \mathbb{R}}$$

where a is a constant, is called a ***power series in*** *$x-a$, or a* ***power series centred at*** *a, or a* ***power series about*** *a.Here, x is a variable and c_n's are constants called the* ***coefficients*** *of the series.*

Notice that a power series resembles a polynomial, but it is not a polynomial because it has infinitely many terms.

Example. $\displaystyle\sum_{n=0}^{\infty} x^n =$

A power series may converge for some values of x and diverge for other values of x. For example, the series in this example is a ________________ series with $r =$ ____, so it converges for ________________ and diverges otherwise.

Example. For what values of x is the power series $\displaystyle\sum_{n=0}^{\infty} n!x^n$ convergent?

It is easy to see that when $x =$ ____ this series converges (note: $0^0 = 1$).
So suppose that $x \neq$ ____ and let us use the Ratio Test.

$\rho =$

Thus by the Ratio Test this series ________________ when $x \neq 0$. Hence this series converges only when $x =$ ____.

Example. For what values of x is the power series $\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ convergent?

It is easy to see that when $x =$ ____ this series converges. So suppose that $x \neq$ ____ and let us use the Ratio Test:

So by the Ratio Test the series converges when $x \in$ ________.

Example. For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

When $x =$ ____ the series converges. Suppose that $x \neq$ ____ and use Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| =$$

By the Ratio Test, this series converges when ____________. That is, the series converges when

The Ratio Test gives no information when $\rho =$, so we must consider the case when $x =$ ____ and $x =$ ____ separately.

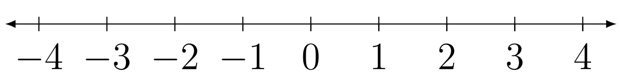
- $x =$ ____: $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} =$

  This is the ________________________________ series, which we know to be ________________(see §11.5).

- $x =$ ____: $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} =$

  This is the ____________ series, which ____________.

Therefore the given power series converges for ____________.

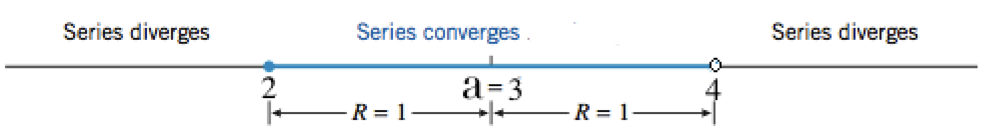


Definition. *The **interval of convergence (or IOC)** of a power series is the interval that consists of all values of x for which the series converges.*

For the above example, the IOC is ________.

Definition. *The **radius of convergence (or ROC)** of a power series is ________ the length of the interval of convergence (if it is finite).*

For the above example, the ROC is ____________.



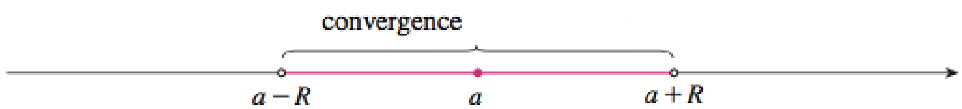
Theorem.

For a given power series $\sum_{n=1}^{\infty} c_n(x-a)^n$, *exactly one of the following is true:*

a) The series converges only for ____________.
(In this case, by convention we say the ROC is ____*)*

b) The series converges for ________________.
(In this case, by convention we say the ROC is ____*)*

c) The series converges in the interval $(a-R, a+R)$ *(where* R *is some positive number). At the endpoints of the interval (i.e.* $x =$ ________*), the series may converge or diverge, depending on the series. Note that the* ***centre of the interval of convergence*** *is* ____.



Example. Find the radius of convergence and interval of convergence of $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \cdot \sqrt{n}}$.

When $x =$ ____, the series converges, so suppose $x \neq$ ____.

Let $a_n = \frac{(2x-1)^n}{5^n \cdot \sqrt{n}}$. Using the Ratio Test, we get

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| =$$

Thus this series converges if $\qquad\qquad <$ ____ $\Rightarrow$

(continue on the next page)

We now check for convergence at the endpoints of the interval $x =$ ____ and $x =$ ____:

- $x =$ ____:

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \cdot \sqrt{n}} =$$

- $x =$ ____:

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \cdot \sqrt{n}} =$$

Therefore the interval of convergence is ____________, and the radius of convergence is ____.

Homework. 11.8: # 1, 3, 9, 13, 15, 19, 21, 23, 25, 33, 37

§11.9 Representations of Functions as Power Series

Recall: A series of the form

$$\boxed{\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots}$$

where a is a constant, is called a **power series centred at** a. Note that x is a variable.

In this section, we learn how to represent certain **functions** as **sums of power series.**

First, let's see what function has the representation as the power series $\sum_{n=0}^{\infty} x^n$.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

is a geometric series with $r =$ ____ and $a =$ ____.
If $|r| =$ ____ $<$ ____, then we know from §11.2 that the sum is equal to .

Hence

$$\boxed{\qquad = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1 \ (\text{ ROC} = 1)} \tag{1}$$

So we have expressed the *rational function* $\dfrac{1}{1-x}$ as a *sum of a power series* (for $|x| < 1$).

Example. Express $\dfrac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

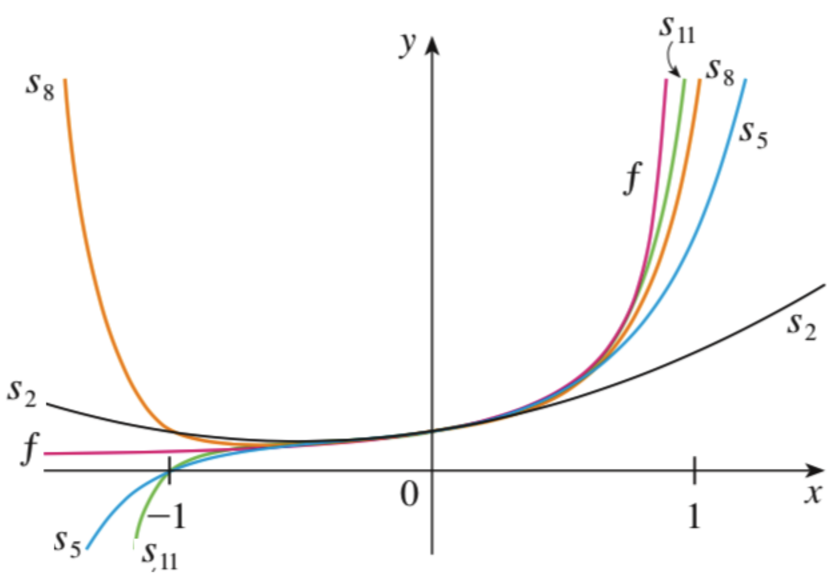
$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots\,, \quad |x| < 1}$$

Replacing x by ____ in the above equation, we get

$$\frac{1}{1+x^2} = \frac{1}{1-(\qquad)} = \sum_{n=0}^{\infty} (\qquad)^n =$$

Note that this is a ________________ series with $r =$ ____, so it converges when

Hence the interval of convergence is ________.



Note that the n-th partial sum, s_n, resembles the function f more and more as n increases.

Example. Find a power series representation for $\dfrac{x^3}{2+x}$. Also find its IOC.

We first put this function in the form of the left side in the equation:

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1}$$

$$\frac{x^3}{2+x} = \left(-\right) \frac{1}{1-\qquad} \tag{2}$$

So if we multiply the left side of equation in the box by ____ then replace x by ________ we would get $\dfrac{x^3}{2+x}$. Hence

$$\frac{x^3}{2+x} = \left(-\right) \frac{1}{1-\qquad} = \left(-\right) \sum_{n=0}^{\infty} (\qquad)^n$$

This is a ____________ series with ____ = ____, so the series converges if

____________ $\Rightarrow$

Thus the interval of convergence is ________.

Remark. If we use another approach:

$$\frac{x^3}{x+2} = x^3 \cdot \left(\frac{1}{1-(-x-1)}\right) = \sum_{n=0}^{\infty} x^3 \cdot (-x-1)^n = \sum_{n=0}^{\infty} x^3 (-1)^n (x+1)^n$$

This is not a power series expression.

Differentiation and Integration of Power Series

The following theorem tells us that we can differentiate and integrate the sum of a power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ term-by-term.

Theorem.

Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then

- $\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = \qquad = \frac{d}{dx}(c_0) + \frac{d}{dx}(c_1(x-a)) + \frac{d}{dx}(c_2(x-a)^2) + \cdots$
- $\int \left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) dx = \qquad = \int c_0\, dx + \int c_1(x-a)\, dx + \int c_2(x-a)^2\, dx + \cdots$

The radii of convergence of the power series after the differentiation and integration stay the ________.

Example. Express $\frac{1}{(1-x)^2}$ as a power series. Find the radius of convergence.

Recall:

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \ \ (\text{ROC} = 1)}$$

Note that $\frac{d}{dx}\left(\frac{1}{1-x}\right) =$. So differentiating both sides of the equation in the box above, we get:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) =$$

$\Rightarrow$

Radius of Convergence:

Example. Find a power series representation for $f(x) = \ln(1+x)$ and find its radius of convergence.
<u>Recall:</u>

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \text{ (ie. R.O.C} = 1)}$$

Hint:

$\ln(1+x) =$

Let's find the value of C: Plugging in $x = 0$ into the above equation, we get

<u>Radius of Convergence:</u>
Rather than finding the R.O.C of the above series, it is easier to find the R.O.C. of the series before we integrated, i.e. $\sum_{n=0}^{\infty}(-1)^n x^n$, since the R.O.C does not change after integration. Now, $\sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$
This is a geometric series with $r =$ ________, so the series converges when

So the R.O.C. of the series $\sum_{n=0}^{\infty}(-1)^n x^n$ is: ____, which means that the R.O.C. of $\ln(1+x)$ must also be ____.

Example. Find a power series representation for $f(x) = \tan^{-1} x$ and find the R.O.C.

Hint: start with $\boxed{\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \;\; |x| < 1 \;\; (\text{ROC} = 1)}$

Also use the fact that $\dfrac{d}{dx}(\tan^{-1} x) =$

Homework. 11.9: # 1 - 21 (odd), 27 , 29, 45

§11.10 Taylor and Maclaurin Series

Not every function has a power series representation. In this section we examine functions that do have a power series representation, and what their power series representations look like.

Suppose that a function f can be represented by a power series. That is,

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots, \quad |x-a| < R}$$

Let's determine the coefficients c_n in terms of the function f. When $x = a$, the above equation becomes:

Next, differentiate both sides of the boxed equation to get:

$$f'(x) = \tag{1}$$

When $x = a$, Eq. (1) becomes:

Next, differentiate Eq. (1) to get:

$$f''(x) = \tag{2}$$

When $x = a$, Eq. (2) becomes: $\Rightarrow$

Next, differentiate Eq. (2) to get:

$$f'''(x) = \tag{3}$$

When $x = a$, Eq. (3) becomes: $\Rightarrow$

In general, we have

So we have the following theorem:

If f has a power series representation at a, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

That is, $c_n = \dfrac{f^{(n)}(a)}{n!}$ for $n = 0, 1, 2, \cdots$ (by definition, $f^{(0)}(a) = f(a)$)

This series is called the **Taylor series of f centred at a**.

$$\boxed{\text{In other words: } \textbf{if} \text{ a function has a power series representation at } x = a \text{ then it is always its Taylor series centred at } x = a.}$$

Example. Find the Taylor series for $f(x) = \dfrac{1}{1-x}$ centred at $a = 0$.

Recall: The Taylor series of f centred at a is:

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots}$$

$$f(x) = \frac{1}{1-x} \qquad \Rightarrow f(0) =$$

$$f'(x) = \frac{1}{(1-x)^2} \qquad \Rightarrow f'(0) =$$

$$\Rightarrow \quad f''(x) = \frac{2}{(1-x)^3} \qquad \Rightarrow f''(0) =$$

$$\Rightarrow \quad f^{(3)}(x) = \frac{2(3)}{(1-x)^4} \qquad \Rightarrow f^{(3)}(0) =$$

$$\Rightarrow \quad f^{(4)}(x) = \frac{2(3)(4)}{(1-x)^5} \qquad \Rightarrow f^{(4)}(0) =$$

$$\vdots$$

$$\Rightarrow f^{(n)}(0) =$$

Hence the Taylor series for $\dfrac{1}{1-x}$ at $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n =$$

We already saw in §11.9 that $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n, \quad |x| < 1$. That is, we already knew that $\dfrac{1}{1-x}$ has a power series representation centred at 0 (when $|x| < 1$), so the series **must be** the Taylor series for the function at centred $x = 0$.

Example. Write the Taylor series of $f(x) = e^x$ centered at $a = 1$.

Recall: The Taylor series for f centred at a is:

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots}$$

$$\begin{array}{ll} f'(x) = & \Rightarrow f'(1) = \\ \Rightarrow\ f''(x) = & \Rightarrow f''(1) = \\ \Rightarrow\ f^{(3)}(x) = & \Rightarrow f^{(3)}(1) = \\ \vdots & \vdots \\ & \Rightarrow f^{(n)}(1) = \end{array}$$

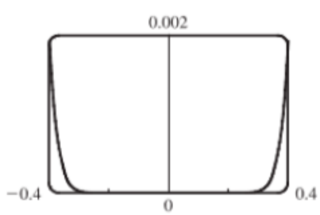
Hence the Taylor series of $f(x) = e^x$ centred at $a = 1$ is:

Remark. We do **not** know if $f(x) = e^x$ is equal to the above Taylor series! We only know that **if** $f(x) = e^x$ **does** have a power series representation at $a = 1$ then it **must** be the above Taylor series.

Remark. There exist functions that are not equal to the sum of their Taylor series. For example, one can show that the function defined by

$$f(x) = \begin{cases} e^{1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is not equal to its Taylor series centred at $a = 0$.



Note: $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x} = \lim_{x\to 0} \frac{1/x}{e^{1/x^2}} = \lim_{x\to 0} \frac{x}{2e^{-1/x^2}} = 0$ Similarly, using the definition of the derivative and l'Hopital's Rule, we can show that

$$f''(0) = f'''(0) = \cdots = f^{(n)}(0) = 0$$

so that all the terms of Taylor Series at $a = 0$ will be zero. But since $f(x) \neq 0$ except at $x = 0$, it is clear that f is not equal to its Taylor series at $x = 0$.

Definition. *In the special case where $a = 0$, this Taylor series becomes*

$$\boxed{\phantom{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad}}$$

This series is called the ***Maclaurin series of*** *f.*

Example.

a) Find the Maclaurin series for $f(x) = e^x$.

$$f^{(n)}(x) = \underline{\qquad\qquad} \qquad \text{for all} \quad n \geq 0.$$

Thus

$$f^{(n)}(0) = \underline{\qquad\qquad} \qquad \text{for all} \quad n \geq 0.$$

Thus the Maclaurin series for $f(x) = e^x$ is

$$\sum_{n=0}^{\infty}$$

b) Find the radius of convergence for the Maclaurin series you found in a).
When $x =$ ____, the series converges, so assume $x \neq$ ____ and let's use the Ratio Test [11]:

$$\left| \frac{a_{n+1}}{a_n} \right| = \qquad\qquad\qquad\qquad\qquad\qquad \text{as } n \to \infty$$

Hence the series converges for ____________ and the radius of convergence is _____.

We will not prove that e^x has a power series representation in this course, but it can be shown that it does have a power series representation, which means that

$$\boxed{e^x = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

In particular, when $x = 1$ we get a new expression for the number e as an infinite series:

$$\boxed{e = \qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

[11]**Ratio Test** (§11.6): Let $\sum a_n$ be a series with nonzero terms only and suppose that $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. Σa_n converges if $\rho < 1$ and diverges if $\rho > 1$ or $\rho = \infty$. The test is inconclusive if $\rho = 1$.

Example. Find the Maclaurin series for $f(x) = \sin x$ and find the radius of convergence.

Recall: The definition for the Maclaurin series for $f(x)$ is:

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots}$$

$$\begin{aligned}
f(x) &= \sin x && \Rightarrow f(0) = \\
f'(x) &= \cos x && \Rightarrow f'(0) = \\
f''(x) &= -\sin x && \Rightarrow f''(0) = \\
f'''(x) &= -\cos x && \Rightarrow f'''(0) = \\
f^{(4)}(x) &= \sin x && \Rightarrow f^{(4)}(0) = \\
& && \vdots
\end{aligned}$$

So the Maclaurin series for $\sin x$ is:

One can show using the Ratio Test (try!) that the radius of convergence for this Maclaurin series is ____.

We will not prove that $\sin x$ has a power series representation in this course, but it can be shown that it does have a power series representation, which means that

$\sin x =$	$R.O.C =$

Example. Find the Maclaurin series for the function $f(x) = \cos x$ and its radius of convergence.

Remark. We can use the definition of the Maclaurin series to do this question (like we did on the previous page for $\sin x$), or we can use the fact that $\frac{d}{dx}(\sin x) =$ ________ and the fact that

$$\boxed{\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (R.O.C = \infty)}$$

In other words, we can simply differentiate the both sides of the above equation to get:

$$\cos x = \frac{d}{dx}$$

Since the Maclaurin series for $\sin x$ converges for all x, we know (§11.9) that the differentiated series also converges for all x, i.e. ROC = ________.

Some Important Maclaurin Series & Their Radii of Convergence, ROC
(Do not memorize this; it will be provided for you if necessary on assessments)

- $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \;\; (ROC = 1)$
- $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \;\; (ROC = \infty)$
- $\sin x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \;\; (ROC = \infty)$
- $\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \;\; (ROC = \infty)$
- $\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \;\; (ROC = 1)$
- $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \;\; (ROC = 1)$

One reason that Taylor series are important is that they enable us to integrate functions that we couldn't previously handle.

Example. Evaluate $\displaystyle\int \frac{e^x}{x}\,dx$ as an infinite series.

Recall: : $\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad R.O.C = \infty}$

$$\frac{e^x}{x} =$$

Now we integrate both sides of the equation:

Homework. 11.10: # 5, 7 , 9, 13, 17, 23, 25, 29, 35, 43, 83 (hint for #83: Use the Maclaurin series for e^x), 85, 89

§11.11 Applications of Taylor Polynomials

Definition. *If a function f can be differentiated n times at $x = a$, then we define $T_n(x)$, the n^{th}* ***degree Taylor polynomial of*** *f* ***at*** *a, to be*

$$\boxed{\begin{aligned} T_n(x) &= \sum_{i=0} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}}$$

Remark.

1. $T_n(x)$ of f at a is just a ________________ of the Taylor series of f at a.
2. The degree of $T_n(x)$ is ____(or smaller than n if $\dfrac{f^{(n)}(a)}{n!} = 0$).

Example. Find the 2nd degree Taylor polynomial of e^x at $a = 0$.

$$f(0) = f'(0) = f''(0) = \_\_\_\_$$

$$T_2(x) =$$

Taylor polynomials are used to ________________ functions that have Taylor Series representations.

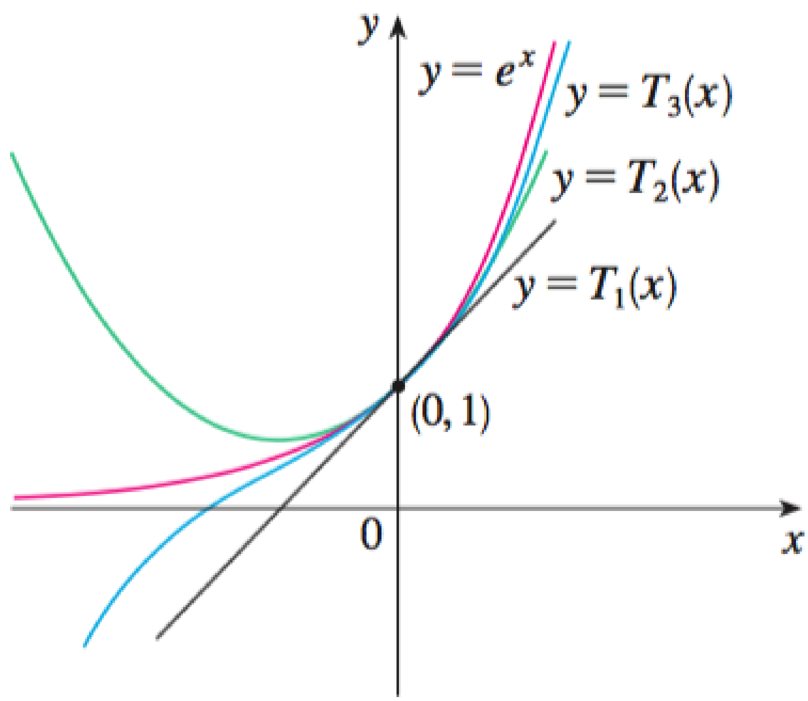
Example. We know that e^x has a power series representation at $x = 0$. So we can use the approximation $e^x \approx T_2(x) = 1 + x + \dfrac{x^2}{2!}$ to estimate the value of $e^{0.2}$:

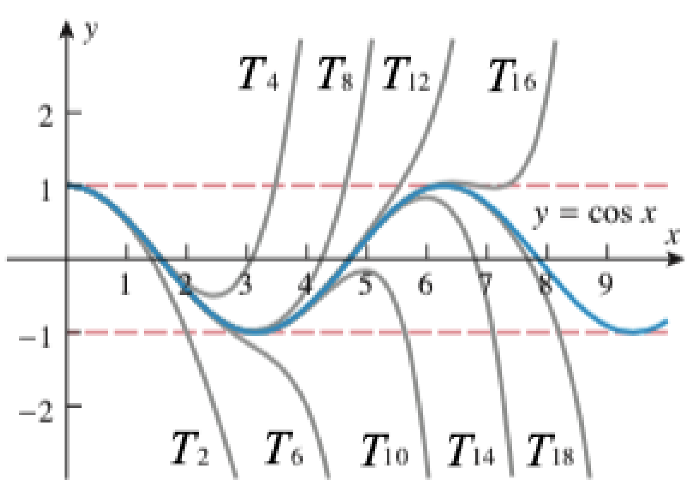
$$e^{0.2} \approx T_2(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2!} = 1.22$$

Using a calculator, we can show that $e^{0.2} = 1.221403...$, so the approximation is pretty close.

	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

- The approximation gets more accurate when we increase the degree, n, of the Taylor polynomial.
- In general, the farther away x is from the centre, the slower the convergence will be to the actual function value.





When using a Taylor polynomial T_n to approximate a function f, we would like to know how good of an approximation it is. The **error**, $|R_n(x)|$, in using $T_n(x)$ as an approximation to f is defined as:

$$\boxed{|R_n(x)| = |f(x) - T_n(x)|}$$

The following inequality, called **Taylor's Inequality**, can be used to bound the error, R_n (proof can be found in textbook p. 762):

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{1}$$

where:

- a is the centre of the Taylor polynomial, $T_n(x)$
- n is the degree of the Taylor polynomial, $T_n(x)$
- M is a number such that $|f^{(n+1)}(x)| \leq M$

Example. How accurate is the approximation $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $x \in [-0.3, 0.3]$?

Recall: $\boxed{\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$

Note that $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is the ____-th degree Taylor polynomial, ________, of $f(x) =$ ________ centred at ____. So by Taylor's Inequality, we have

$$|R_5(x)| \leq$$

where $|f^{(6)}(x)| \leq M$. Since $f^{(6)}(x) = -\sin x$, we have $|f^{(6)}(x)| \leq$ ____ $=$ ____. Hence

$$|R_5(x)| \leq \qquad\qquad \approx 0.000001012$$

So, for example, the estimate

$$\sin 0.1 \approx 0.1 - \frac{0.1^3}{3!} + \frac{(0.1)^5}{5!}$$

has a maximum error of ____________.

Recall: The nth degree Taylor polynomial of f centred at a is:

$$\boxed{T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Example.

a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2, $T_2(x)$, at $a = 8$.

$$f(x) = \sqrt[3]{x} = x^{1/3} \quad \Rightarrow f(8) =$$

$$f'(x) = \frac{1}{3}x^{-2/3} \quad \Rightarrow f'(8) =$$

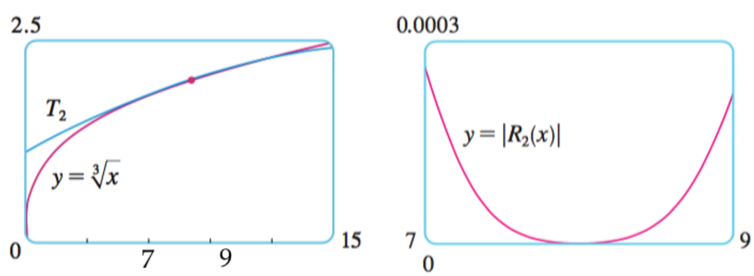
$$f''(x) = -\frac{2}{9}x^{-5/3} \quad \Rightarrow f''(8) =$$

Thus $T_2(x) =$

b) How accurate is the approximation when $7 \le x \le 9$?

Recall: $\boxed{|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}}$, $\qquad f'''(x) = \frac{10}{27x^{8/3}}$

That is, if $7 \le x \le 9$, the approximation in part a) is accurate to within ____________.



Remark. To show that the function $f(x)$ is equal to its Taylor series centred at $x = a$, i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is to show the following:

$$\boxed{f(x) = \lim_{n\to\infty}\left(\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i\right)}$$

One way to show this is by showing that

$$\lim_{n\to\infty}\left(f(x) - \left(\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i\right)\right) = 0$$

But this is equivalent to:

$$\lim_{n\to\infty}(f(x) - T_n(x)) = \lim_{n\to\infty} R_n = 0$$

Directly proving that $\lim\limits_{n\to\infty} R_n = 0$ is usually impossible, so we prove it indirectly by using the upper bound on $|R_n(x)|$ (i.e. $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$), then applying the Squeeze Theorem for sequences.

Example. Show that the Maclaurin series for $f(x) = \cos x$ is equal to $f(x) \forall x \in \mathbb{R}$.
We need to show that

$$\cos x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, x \in \mathbb{R}$$

In other words, we need to show that $R_n(x) \to 0$ for all x as $n \to \infty$. Since $0 \leq |R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$, we will show that $\dfrac{M}{(n+1)!}|x-0|^{n+1} \to 0$ for all x as $n \to \infty$.
Note that if $f(x) = \cos x$, then $f^{(n+1)}(x) = \pm\cos x$ or $\pm\sin x$. Hence

$$|f^{(n+1)}(x)| \leq 1 (= M)$$

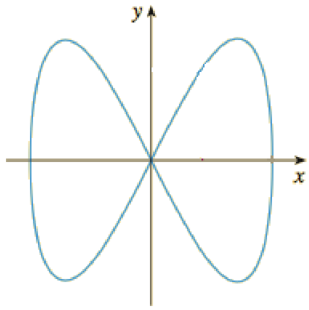
So we have $0 \leq |R_n(x)| \leq \dfrac{1}{(n+1)!}|x|^{n+1}$ for all x. Now, $\lim\limits_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)!} = 0 \ \forall x$, since $(n+1)!$ is a factorial function, which grows faster than any polynomial $|x|^{n+1}$. So by the Squeeze Theorem for sequences we conclude that $\lim\limits_{n\to\infty} |R_n(x)| = 0$ for all x.

Homework. 11.11: # [3, 5, 7, 9] $\leftarrow$ skip graphing, 13ab, 15ab, 19 ab + Review sections 10.1 and 10.3 (it is enough to review the material for these sections in this lecture book)

Chapter 10. Parametric Equations and Polar Coordinates

§10.1Parametric Equations & Curves (REVIEW as homework)

Imagine that a particle moves along the curve C shown in the following figure.



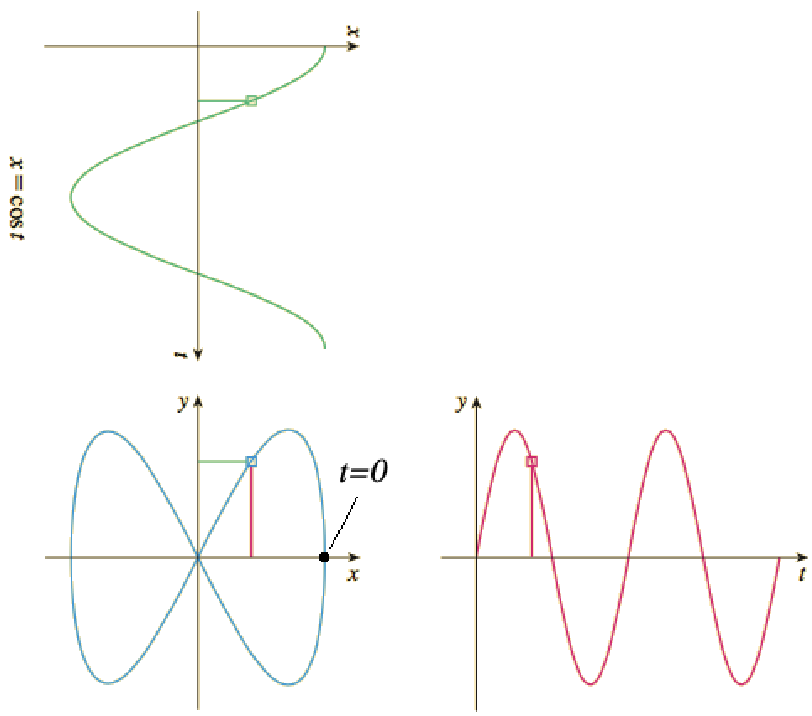
This curve is not the graph of any function $y = f(x)$ because it fails the Vertical Line Test.

But we can express the x- and y- coordinates of the particle *separately* as *functions* of t (if the curve C describes a movement of a particle, t can be thought of as time). That is,

$$\boxed{x = f(t) \qquad \text{and} \qquad y = g(t)}$$

We call these the **parametric equations**, and the variable t is called the **parameter** for these equations.

Each value of t determines a point (x, y) on the $xy-$plane. As t varies, $(x, y) = (f(t), g(t))$ varies and traces out the curve C, which we call a **parametric curve**.



$x = \cos t \quad y = \sin 2t$ $\qquad\qquad$ $y = \sin 2t$

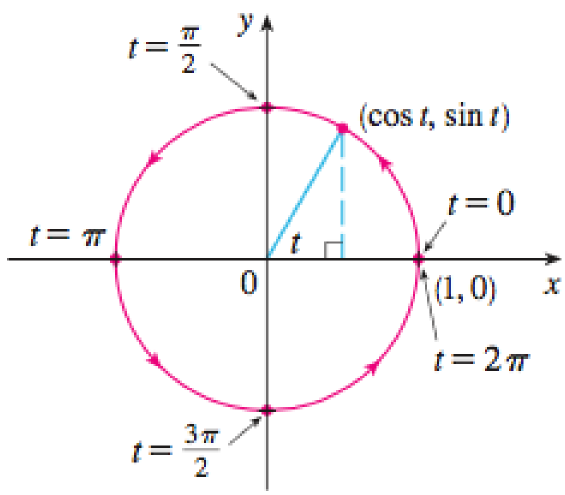
Example. What curve is represented by the following parametric equations?

a) $x = \cos t, \qquad y = \sin t, \quad 0 \le t \le 2\pi$

Let's **eliminate the parameter** t to find a Cartesian equation of the curve: observe that $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the Cartesian equation for this curve is

$$x^2 + y^2 = 1$$

which is the equation of a circle of radius 1 centred at $(0, 0)$.



In this example, the parameter t can be thought of as the angle (in radians). The particle starts $(x, y) = (\cos 0, \sin 0) = (1, 0)$, and as t increases, the point $(x, y) = (\cos t, \sin t)$ moves around the circle once in counterclockwise direction.

Remark. The equation in x and y (in this case $x^2 + y^2 = 1$) describes *where* the particle has been, but it does not tell us *when* the particle was at a particular point. The parametric equations have an advantage – they tell us *when* the particle was at a point. They also indicate the direction of motion.

b) $x = \sin(2t), \qquad y = \cos(2t), \quad 0 \le t \le 2\pi$

Observe that

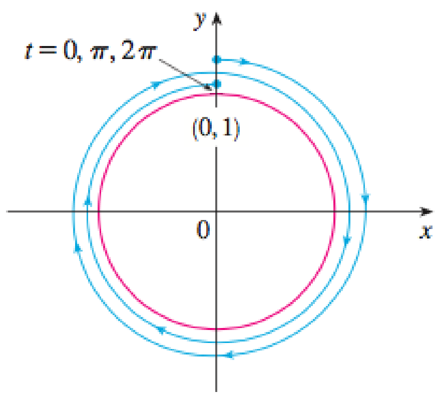
$$x^2 + y^2 = \sin^2(2t) + \cos^2(2t) = 1$$

So we again have a unit circle $x^2 + y^2 = 1$.

As t increases from 0 to 2π, the point $(x, y) = (\sin(2t), \cos(2t))$ starts at

$$(\sin(2 \cdot 0), \cos(2 \cdot 0)) = (0, 1)$$

To see how many times the circle is traced in the interval $0 \le t \le 2\pi$, we again think of $2t$ as an angle (in radians). Since $2t$ takes on all values from 0 to 4π, the parametric equations must trace the circle *twice*, in a counterclockwise direction.



Remark. As seen in the above example, two different sets of parametric equations can represent the **same curve**. But they represent **different parametric curves**.

Homework. §10.1 (optional): 1, 5, 9, 11, 13, 15, 25, 27

§10.3 Polar Coordinates (REVIEW as homework)

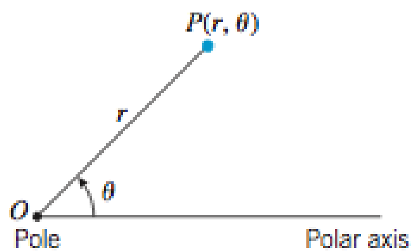
Up to now, we specified the location of a point in the plane by using the Cartesian plane (the $xy-$plane). However, a different kind of coordinate system is sometimes more useful. In this section, we introduce a new coordinate system called the **polar coordinate system**.

Polar Coordinate System

A polar coordinate system consists of a fixed point O, called the **pole** (or **origin**) and a ray coming out of the pole, called the **polar axis**. This axis is usually drawn horizontally to the right of O. (see diagram)

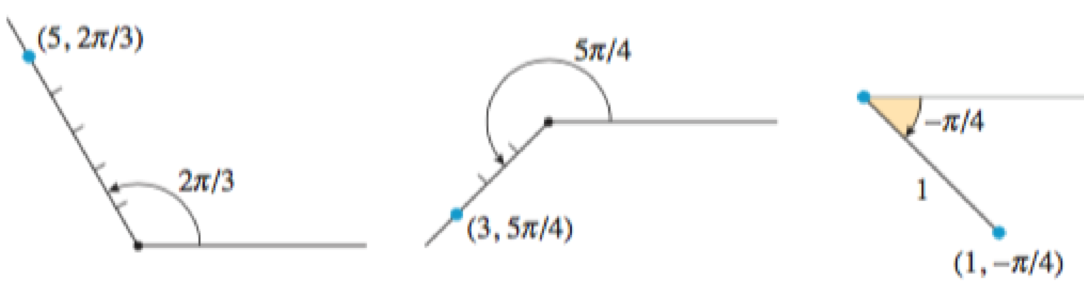
We associate with each point P in the plane a pair of **polar coordinates** (r, θ) where:

- r is the **distance** from P to the pole O, and
- θ is the **angle** (usually measured in radians) between the ray (OP) and the polar axis.

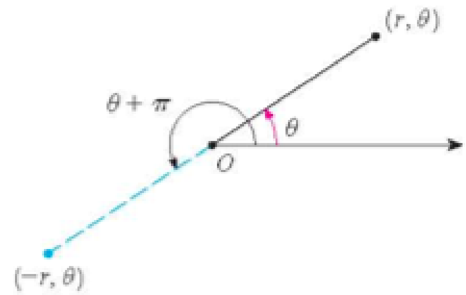


By convention:

(i) An angle is *positive* if measured counterclockwise from the polar axis, and *negative* otherwise. ex:



(ii) If $r > 0$ then $(-r, \theta)$ is defined as the point $(r, \theta + \pi)$.

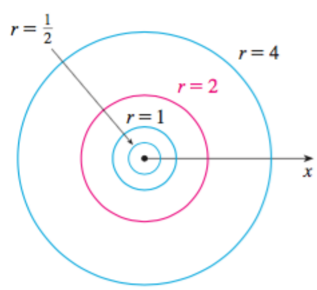


Polar Curves

We now consider the problem of graphing equations in r and θ.
Example. Sketch the given polar curve.

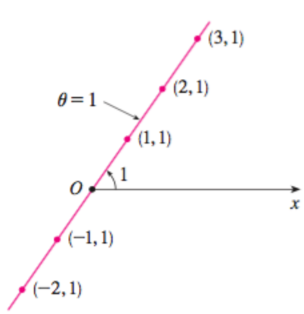
a) $r = 2$

The curve consists of all points $(r, \theta) = (2, \theta)$. Since r represents the distance from the point to the pole, the curve $r = 2$ represents the circle of radius 2 centred at the pole.



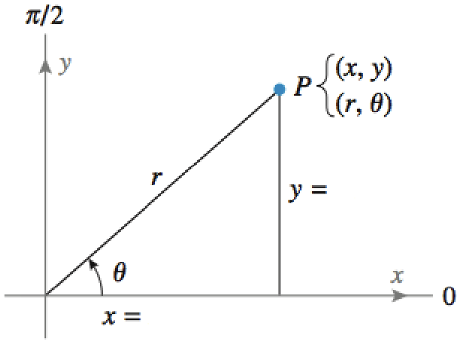
b) $\theta = 1$

The curve consists of all points $(r, \theta) = (r, 1)$. Since θ represents the angle from the polar axis, this curve is a line passing through the pole making an angle of 1 radian with the polar axis.



Relationship between polar and Cartesian coordinates

Often it is useful to superimpose the Cartesian coordinate system on top of the polar coordinate system, making the positive x-axis coincide with the polar axis.
Let's express x and y in terms of r and θ.



$$\cos\theta = \frac{x}{r} \Rightarrow \quad x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \Rightarrow \quad y = r\sin\theta$$

In summary:

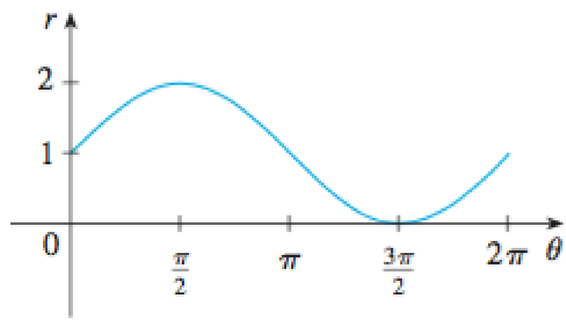
$$\boxed{x = r\cos\theta, \quad y = r\sin\theta} \tag{1}$$

Eq. (1) implies that:

$$\boxed{x^2 + y^2 = r^2, \quad \tan\theta = \frac{y}{x}} \tag{2}$$

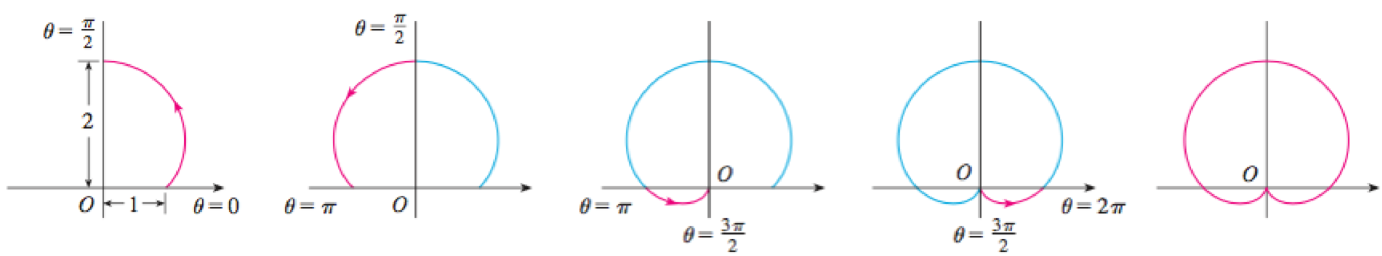
Example. Sketch the curve $r = 1 + \sin\theta$ by first sketching the graph of r as a function of θ in *Cartesian coordinates.*

Sketching the graph of $r = 1 + \sin\theta$ in *Cartesian coordinates*, we get:



- As θ increases from 0 to $\dfrac{\pi}{2}$, r (which represents the distance from the pole O) increases from 1 to 2.
- As θ increases from $\dfrac{\pi}{2}$ to π, r decreases from 2 to 1.
- As θ increases from π to $\dfrac{3\pi}{2}$, r decreases from 1 to 0.
- As θ increases from $\dfrac{3\pi}{2}$ to 2π, r increases from 0 to 1.

Since $r = 1 + \sin\theta$ is a periodic function, letting θ be greater than 2π or smaller than 0 results in retracing our path we've already drawn. So the complete curve is given below.



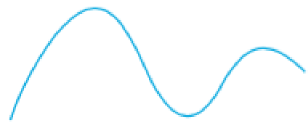
This curve is called a **cardioid** because it is shaped like a heart.

Homework. §10.3 (optional): 29, 31, 33, 39, 41, 47

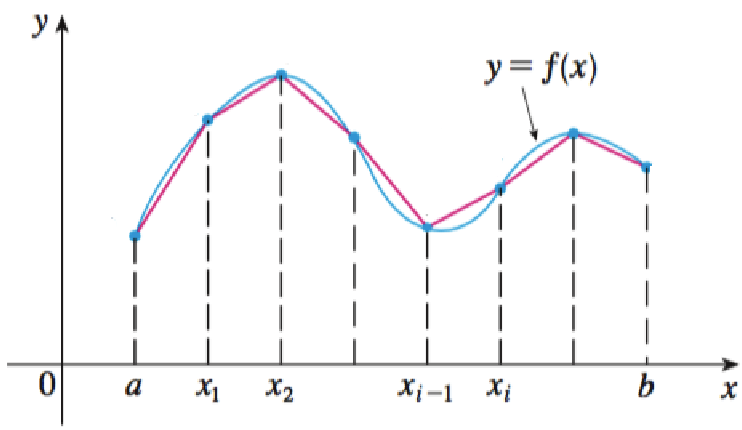
Chapter 8. Further Applications of Integration

§8.1 Arc Length

Suppose that $y = f(x)$ is a smooth curve on $[a, b]$ (i.e. no jumps, corners, or gaps on $[a, b]$).[12]

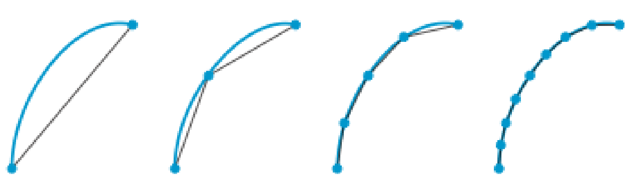


In this section we give a precise definition for the length of a curve (also known as ________________).



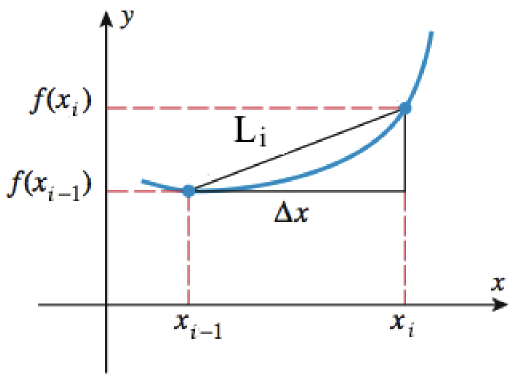
We can get the length of a curve C defined by $y = f(x)$ on $[a, b]$ as follows:

- Divide the interval $[a, b]$ into ____ subintervals of equal width ________.
- Join the endpoints of the curve in each subinterval to form a series of ________________________.
- Approximate the length of the curve C by ____________ all the lengths of the line segments.
- As $n \to$ ____, we get the ________ length of the curve, C.



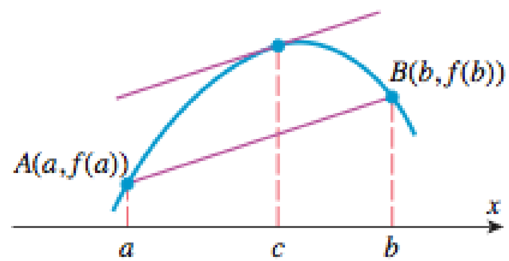
[12]Formally, $y = f(x)$ is a smooth curve on $[a, b]$ if f' exists on $[a, b]$ and f' is continuous on $[a, b]$

How to compute the length of the line segment L_i in the i-th subinterval $[x_{i-1}, x_i]$:



By the Pythagorean theorem,

$L_i =$ (1)



Recall: **The Mean Value Theorem (§4.2):** Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is at least one point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

By applying the Mean Value Theorem to f on $[x_{i-1}, x_i]$, we find that there is a number $x_i^* \in (x_{i-1}, x_i)$ such that

Hence Eq. (1) becomes:

$L_i =$

Thus the sum of the lengths of all the line segments is:

$$\sum_{i=1}^{n} L_i = \qquad \sqrt{1 + (f'(x_i^*))^2}\ \Delta x$$

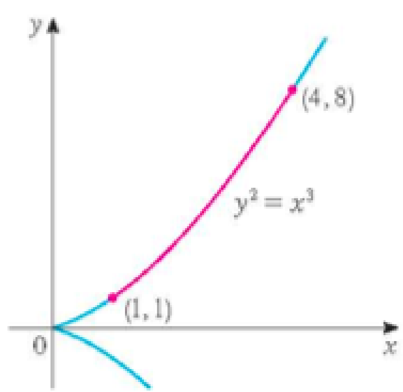
So we define the **exact** arc length L of the curve C to be:

$$L = \qquad \sum_{i=1}^{n} L_i = \qquad \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\ \Delta x =$$

In summary:

If $y = f(x)$ is a smooth curve on $[a, b]$ then the arc length of this curve over $[a, b]$ is $L =$

Example. Set up, but do not evaluate, the integral for the arc length of the curve $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.



Hence

Answer: $\boxed{\dfrac{8}{27}(10)^{3/2} - \dfrac{8}{27}\left(\dfrac{13}{4}\right)^{3/2}}$

If a smooth curve is expressed in the form $\boxed{x = g(y)}$ where g' is continuous on $[c, d]$, the arc length L from $y = c$ to $y = d$ can be expressed as

Example. Find the arc length of the curve $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.

(This is the same example as before but we find the arc length by expressing $y^2 = x^3$ as a function of y this time).

$$y^2 = x^3 \Rightarrow x =$$

Thus $L =$

(Hint for solving the integral: Factor out $y^{-2/3}$ from the terms inside the square root (i.e. write integrand as $\sqrt{y^{-2/3} \cdot \boxed{\cdots + \cdots}}$) , then take $y^{-2/3}$ out of the square root, then use substitution...)

Answer: $\boxed{\dfrac{8}{27}(10)^{3/2} - \dfrac{8}{27}\left(\dfrac{13}{4}\right)^{3/2}}$

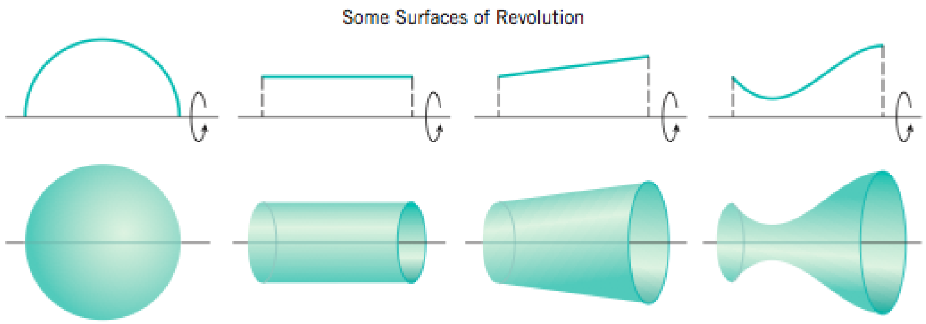
Example. Find the arc length of the curve $y = x^2 - \left(\frac{1}{8}\ln x\right)$ from $(1, 1)$ to $(3, 71/8)$. *(Hint for solving the integral: expand then factor the expression inside the square root)*

$$\boxed{\text{Answer: } 8 + \frac{\ln 3}{8}}$$

Homework. 8.1: # 3, 5, 7 ,13, 17 (odd) (hint for #17: to evaluate the integral, multiply the numerator and the denominator by $(\sec x + \tan x)$ then use substitution $u = \sec x + \tan x$...)

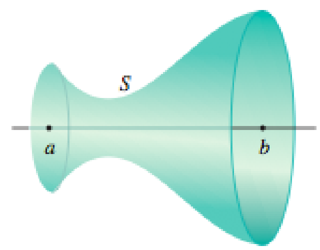
§8.2 Area of a Surface of Revolution

A **surface of revolution** is formed when a ____________ is rotated about a line.

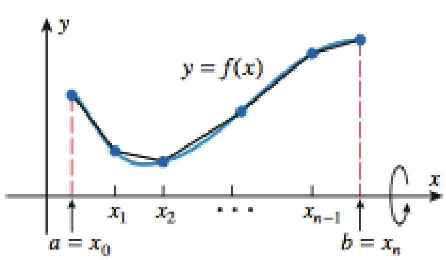


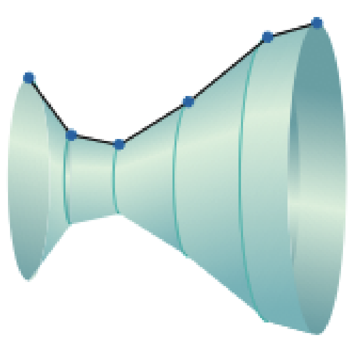
We want to define the area of a surface of revolution in such a way that it corresponds to our intuition.

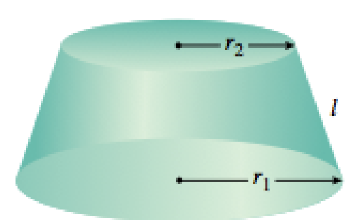
How can we calculate the surface area of the following surface?



- Approximate the curve by a series of ________________________ [as in §8.1].
- Rotating each line segment about the given axis produces a **band** whose surface area is easy to compute (see next page).
- Approximate the surface area in each subinterval by the surface area of the ________.
- Let the number of line segments go to ____ to get the ________ surface area.

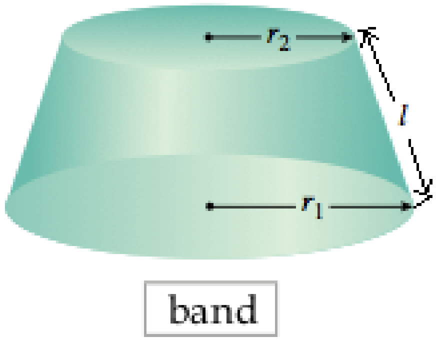






band

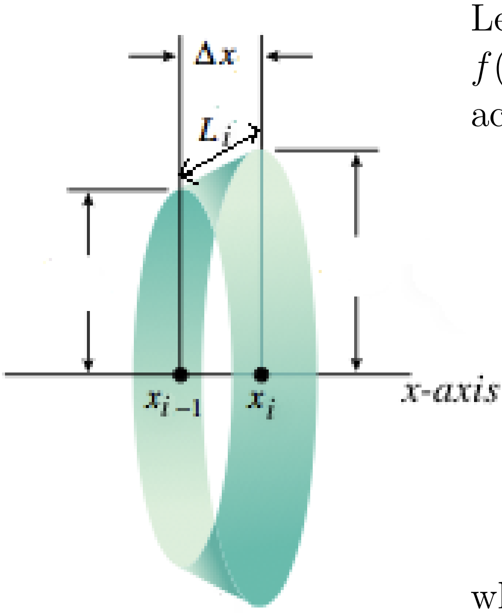
One can show that the surface area S of a band shown is:



$$\boxed{S = 2\pi r l} \tag{1}$$

where $r = \dfrac{r_1 + r_2}{2}$ (i.e. the average radius of the band).

Now consider the band obtained by rotating the line segment on the ith subinterval $[x_{i-1}, x_i]$ about the <u>x-axis</u>:



Let L_i be the length of the line segment joining $f(x_{i-1})$ and $f(x_i)$. Then the area S_i of this band, according to Formula (1), is:

$$S_i = \tag{2}$$

From §8.1 we know that

$$L_i = \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x$$

where $x_i^* \in [x_{i-1}, x_i]$.

Also, if Δx is small, then $f(x_{i-1}) \approx$ ____ and $f(x_i) \approx$ ____ (since f is continuous). Hence Eq. (2) becomes:

$$S_i \approx$$

Thus the approximation to the area of the complete surface of revolution, S, is:

$$S \approx \sum_{i=1}^{n} S_i \approx \qquad 2\pi f(x_i^*) \cdot \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x$$

This approximation becomes better as $n \to$ ____. Using the definition for the definite integral, we get the exact surface area S is:

$$S = \qquad \sum_{i=1}^{n} 2\pi f(x_i^*) \cdot \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x =$$

Recall: (§8.1): The length L of the curve $y = f(x)$ from (a, c) to (b, d) is given by:

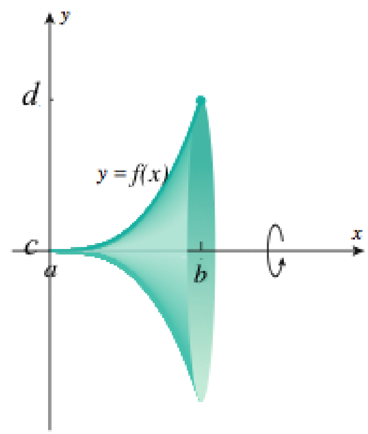
$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy}$$

In summary, we have the following definition:

Let f be a smooth, non-negative function on $[a, b]$. The **surface area** S of the surface of revolution generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the ______________ is defined as:

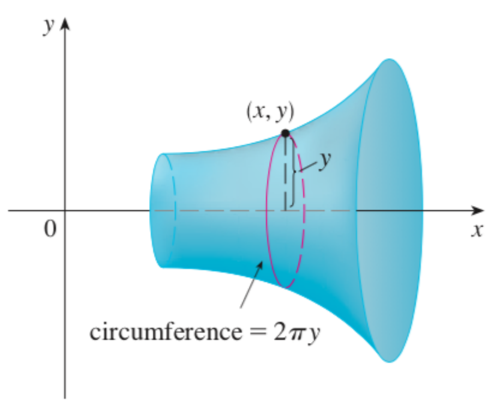
$S \ =$

$\quad =$



If a curve is described as $x = g(y)$, $\ c \le y \le d$ then

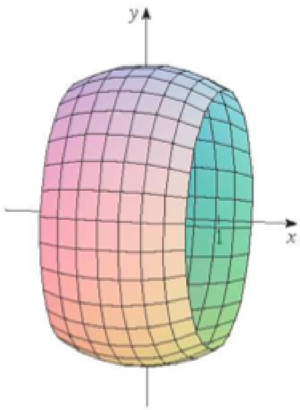
$S =$



This formula can be remembered by thinking of $2\pi y$ as the circumference of a circle traced out by the point (x, y) on the curve as it is rotated about the $x-$axis.

$$\boxed{S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}$$

Example. Find the area of the surface obtained by rotating the curve $y = \sqrt{4-x^2}$, $-1 \le x \le 1$ about the x-axis.



$$\frac{dy}{dx} =$$

(Hint for solving the integral: combine the two square roots into one then simplify...)

The surface area is:

$$S =$$

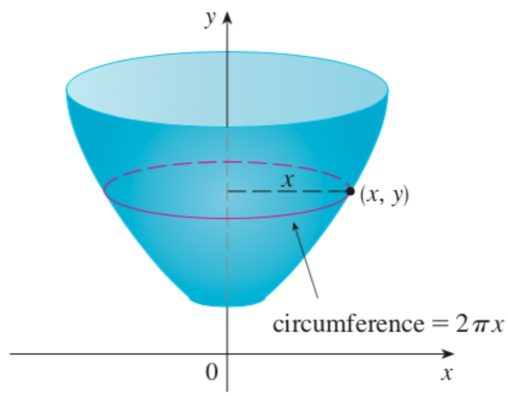
$\boxed{Answer: 8\pi}$

Similar definition exists for rotation about the y**-axis:**

Let $x = g(y)$ be smooth and non-negative curve with $y \in [c, d]$. The **surface area** S of the surface of revolution generated by revolving the portion of the curve $x = g(y)$ between $y = c$ and $y = d$ about the ____________ is defined as:

$$S = \int_{y=c}^{d} 2\pi g(y)\sqrt{1 + [g'(y)]^2}\ dy$$

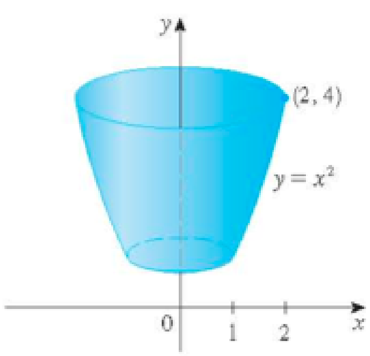
$$= \int_{y=c}^{d} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$



If a curve is described as $y = f(x)$, $a \leq x \leq b$, then

$$S = \int_{x=a}^{b} 2\pi x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Example. The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Set up, but do not evaluate, the integral for the area of the resulting surface using two methods:



a) by integrating with respect to x:

b) by integrating with respect to y:

Answer: $\frac{\pi}{6}(17^{3/2} - 5^{3/2})$

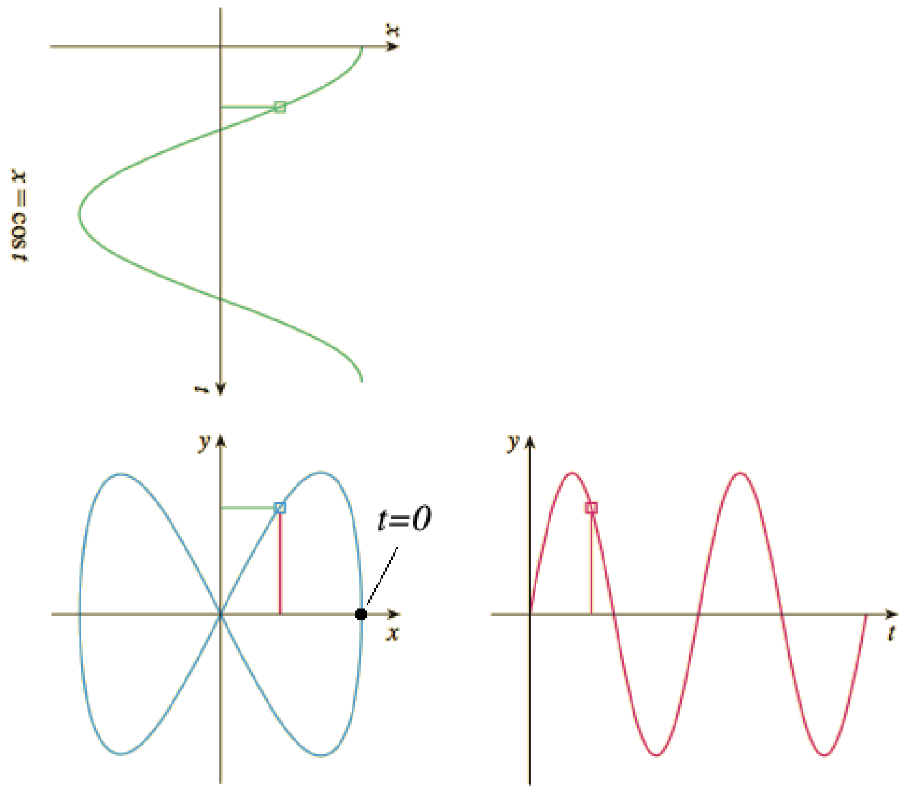
Homework. 8.2: # 1 - 11 (odd), 15, 17, 19

§10.2 Calculus with Parametric Curves

Recall: **Parametric equations** are equations of the form

$$\boxed{x = f(t), \quad y = g(t)}$$

That is, x and y are described *separately* as functions of a new variable, t (called the **parameter**). Each value of t determines a point (x, y) on the xy-plane. As t varies, $(x, y) = (f(t), g(t))$ varies and traces out a curve, which we call a **parametric curve**.



$x = \cos t \quad y = \sin 2t$ $\qquad$ $y = \sin 2t$

Arc Length

In this section, we look at a formula for arc length of a curve given by parametric equations.

Recall: (§8.1) The arc length L of a curve given in the form $y = f(x)$, $a \leq x \leq b$ (where f' is continuous in this interval) is

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx} \tag{1}$$

Suppose we want to find the arc length of the curve C as above for $\alpha \leq t \leq \beta$ where C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$ and $f(\beta) = b$ (i.e. $\dfrac{dx}{dt} = f'(t) > 0$, which means x is getting larger as t is getting larger)

Suppose that $f(\alpha) = a$ and $f(\beta) = b$. Now Eq. (1) becomes:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{t=\alpha}^{\beta} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\frac{dx}{dt}\, dt \tag{2}$$

If $\dfrac{dx}{dt} > 0$, we have $\dfrac{dx}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2}$. Hence (2) becomes

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If $\dfrac{dx}{dt} < 0$, we have $\dfrac{dx}{dt} = -\sqrt{\left(\dfrac{dx}{dt}\right)^2}$. Furthermore, since $\dfrac{dx}{dt} < 0$ implies that $a = f(\alpha) > f(\beta) = b$. Hence (1) becomes

$$L = \int_b^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{\beta}^{\alpha} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\left(-\frac{dx}{dt}\right) dt = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If $\dfrac{dx}{dt} = 0$, then clearly

$$L = |y(\beta) - y(\alpha)| = |y(t)|\Big|_{\alpha}^{\beta} = \int_{t=\alpha}^{\beta} \left|\frac{dy}{dt}\right|\, dt$$

Thus:

If a curve represented by the parametric equations

$$x = f(t), \;\; y = g(t)$$

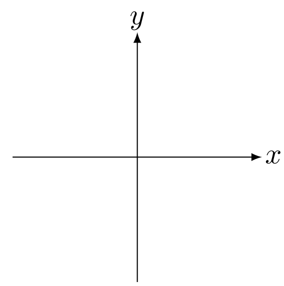
is traced **once** as t increases from α to β, and if $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are continuous on $\alpha \leq t \leq \beta$, then the arc length L of the curve is given by

$$\boxed{L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

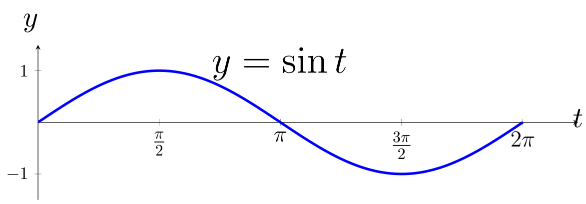
Example. Find the arc length of the curve given by the parametric equations

$$x = a\cos t, \;\; y = a\sin t \text{ (assume } a > 0)$$

Let's see what the curve looks like by eliminating the parameter t:



Let's see which interval for t traces the circle once.
If we choose $t = 0$ to be the starting value, then by observing the graph of $y = a\sin t$ we can see that the circle is traced starting from $(x, y) =$ ________, then moves counterclockwise until returning to the starting point when $t =$ ____.



(Alternatively, one may solve the equation $x = a\cos t = 1$, and you can use the interval given by two consecutive values of t)

So the arc length L of this curve is:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$$

This result makes sense intuitively; the circle of radius a has arclength (i.e. circumference in this case) of $2\pi r = 2\pi a$.

Example. Find the length of the curve given by the parametric equations:

$$x = e^t + e^{-t}, \;\; y = 5 - 2t, \;\; 0 \le t \le 3$$

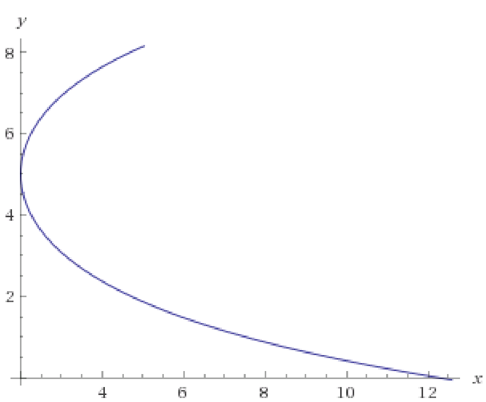
It is easy to see that the curve is traced **once** as t moves from 0 to 3; y keeps ____________ as t increases (i.e. no y-value will be repeated, so the curve will not be traced more than once). so the interval of integration will be the same as the domain, i.e. $0 \le t \le 3$.

$$\frac{dx}{dt} = \qquad\qquad\qquad\qquad \frac{dy}{dt} =$$

Therefore, the arc length is:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$=$$



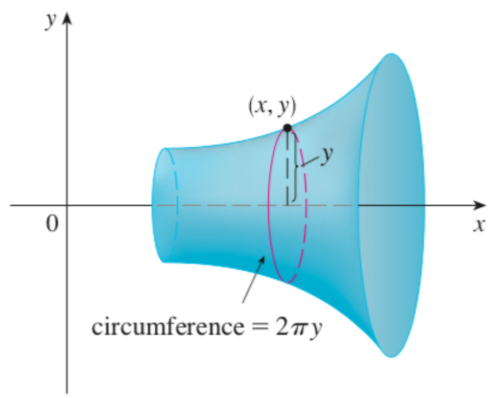
(t from −1.573 to 2.526)

$\boxed{Answer : e^3 - e^{-3}}$

Surface Area

As in arc length, we adapt the formula for surface area (§8.2) to find a formula for surface area given parametric equations.
Recall: When we rotate about the x-axis, the formula for the surface area is

$$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$



We replace the arc length part of the formula ($\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$) by the arc length formula for parametric curves to get the following result:

Suppose the curve given by the parametric equations

$$x = f(t), y = g(t), \alpha \le t \le \beta$$

where f', g' are continuous and $g(t) \ge 0$ is rotated about the $x-$**axis**. If the curve is traversed exactly once as t increases from α to β, then the area of the resulting surface is given by:

$$\boxed{S = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}$$

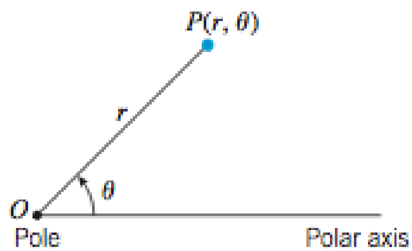
Homework. 10.2: # 47, 49, 71

§10.4 Areas and Lengths in Polar Coordinates

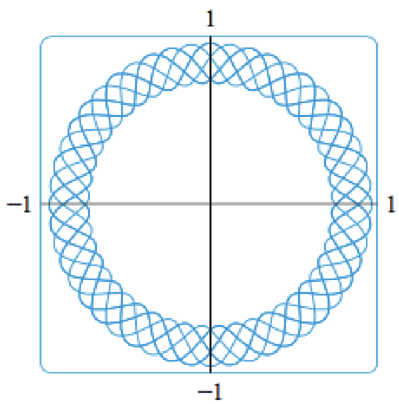
In this section, we develop the formula for the area of a region whose boundary is given by a polar equation.

Recall: (§10.3) **Polar Coordinate System**: We associate with each point in the plane a pair of **polar coordinates** (r, θ). where:

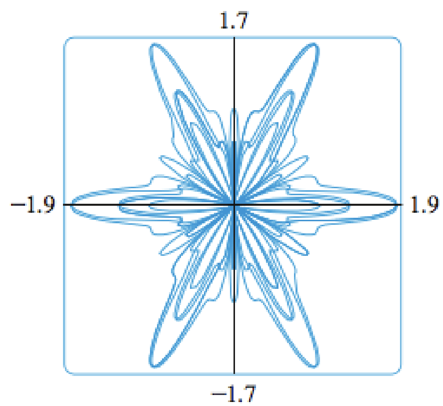
- r is the distance from P to the pole O, and
- θ is the angle (usually measured in radians) between the ray (OP) and the polar axis.



Examples of Polar Curves $r = f(\theta)$



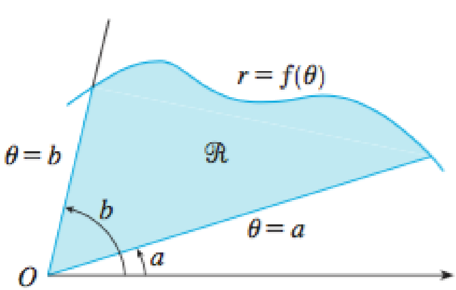
$r = \sin^2(2.4\theta) + \cos^4(2.4\theta)$



$r = \sin^2(1.2\theta) + \cos^3(6\theta)$

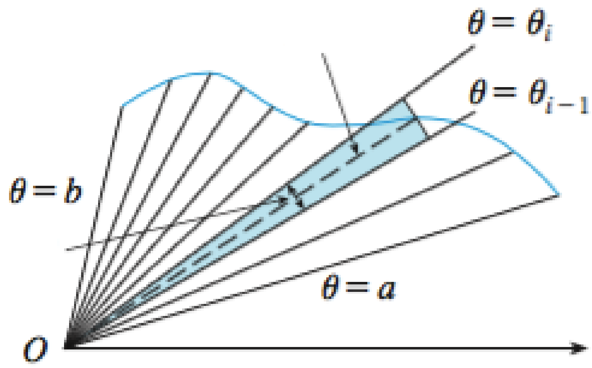
Area in Polar Coordinates

Motivation: How can we find the area, A, of a region $\mathscr{R}$ below?



Note: $r = f(\theta)$ here is a polar curve. Assume that f is a positive continuous function.

Idea:



- Divide up the interval $\theta \in [a, b]$ into n equal subintervals with endpoints

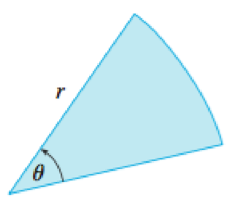
$$\theta_0, \theta_1, \cdots, \theta_n$$

and we get n areas of equal angles ________________.

- Let $\theta_i^* \in [\theta_{i-1}, \theta_i]$. Approximate the area of the i-th region (A_i) by the area of a **sector** with angle ____ and radius ________:

$$A_i \approx$$

Recall: The area A of a *sector* of a circle (i.e. a portion of a circle – see figure below) of radius r and angle θ is:



$$\boxed{A = \frac{1}{2}r^2\theta, \quad (\theta \text{ is in radians})}$$

- Adding up all the areas we get an approximation to the total area A of $\mathscr{R}$:

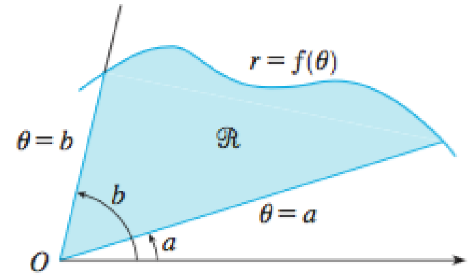
$$A = \qquad A_i \approx \qquad \frac{1}{2}(f(\theta_i^*))^2 \cdot \Delta\theta$$

- The approximation improves as n increases, if $n \to \infty$, we get the exact area of $\mathscr{R}$. That is,

$$A = \qquad \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i^*))^2 \cdot \Delta\theta =$$

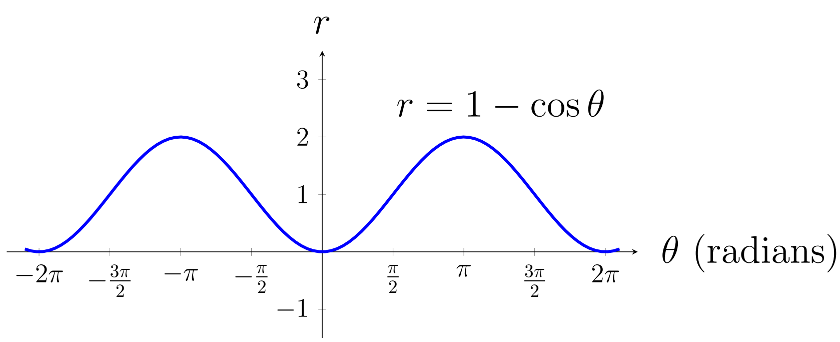
In summary, the area A of the region $\mathscr{R}$ is:

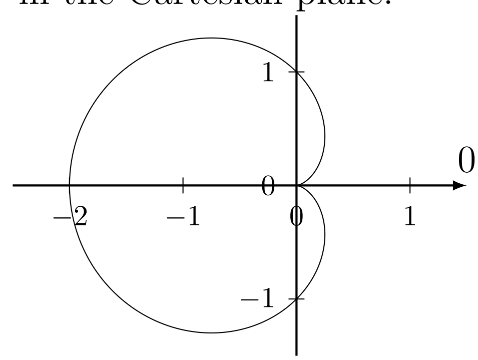
$$\boxed{A = \int_{\theta=a}^{b} \frac{1}{2}(f(\theta))^2 d\theta = \int_{a}^{b} \frac{1}{2} r^2 d\theta}$$



Example. Find the area of the region in the **first quadrant** that is within the cardioid $r = 1 - \cos\theta$.

We first need to know in which interval of θ the first quadrant is traced. To draw this curve, one can first plot the graph of $r = 1 - \cos\theta$ in the Cartesian plane:





Let's choose our lower limit of integration to be $\theta = 0$. Now, as θ increases from 0 to $\frac{\pi}{2}$, r traces the curve in the first quadrant. Since the first quadrant stops at the angle of ________, it is the upper limit of integration.
*(**shortcut: when $r \geq 0$ for all θ, we can simply think of θ as the angle, so the angle between 0 and $\pi/2$ would describe the region in the first quadrant)*

So the area, A, of the region in the first quadrant is:

$$A = \int_{a}^{b} \frac{1}{2} r^2 d\theta =$$

Arc Length of a Polar Curve

Recall:

If a curve represented by the parametric equations

$$x = f(t), \;\; y = g(t), \;\; \alpha \le t \le \beta$$

is traced once in $\alpha \le t \le \beta$, the arc length L of the curve is given by

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{1}$$

To find the length of a polar curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, we regard θ as a parameter and write the parametric equations of the curve as

$$\boxed{x = r\cos\theta, \;\; y = r\sin\theta} \;\; \Rightarrow \;\; \boxed{x = f(\theta)\cos\theta, \;\; y = f(\theta)\sin\theta}$$

Now we can rewrite (1) using the parameter θ as

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \tag{2}$$

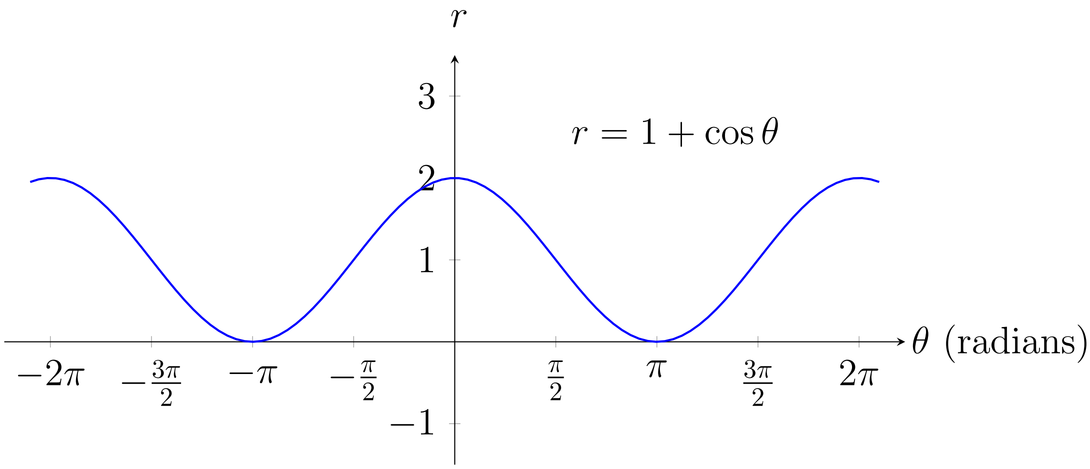
Let's find an expression for $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$ in terms of r and θ only.

$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{d}{d\theta}(r\cos\theta)\right)^2 + \left(\frac{d}{d\theta}(r\sin\theta)\right)^2 \\
&= \left(\frac{dr}{d\theta}\cos\theta + r(-\sin\theta)\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2 \\
&= \left(\left(\frac{dr}{d\theta}\right)^2\cos^2\theta - \cancel{2\left(\frac{dr}{d\theta}\cos\theta\right)(r\sin\theta)} + r^2\sin^2\theta\right) \\
&+ \left(\left(\frac{dr}{d\theta}\right)^2\sin^2\theta + \cancel{2\left(\frac{dr}{d\theta}\sin\theta\right)(r\cos\theta)} + r^2\cos^2\theta\right) \\
&= \left(\frac{dr}{d\theta}\right)^2 + r^2 \;\; (\text{ since } \sin^2\theta + \cos^2\theta = 1)
\end{aligned}$$

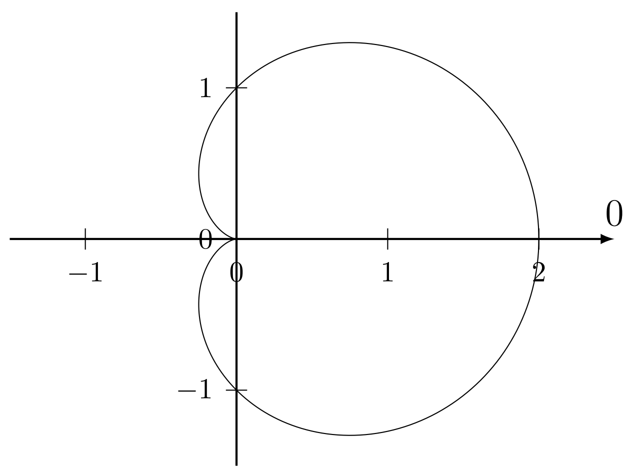
So (2) becomes $\boxed{\phantom{L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}}$

Example. Find the total arc length of the cardioid $r = 1 + \cos\theta$.

We will use the graph of $r = 1 + \cos\theta$ on the Cartesian plane to figure out which values θ should take on to trace the curve exactly once:



In the polar coordinate system:



Hence the cardioid is traced once as θ varies from $\theta =$ ____ to $\theta =$ ____.

*(**shortcut: when $r \geq 0$ for all θ, we can simply think of θ as the angle, so the angle between 0 and 2π would describe the entire region)*

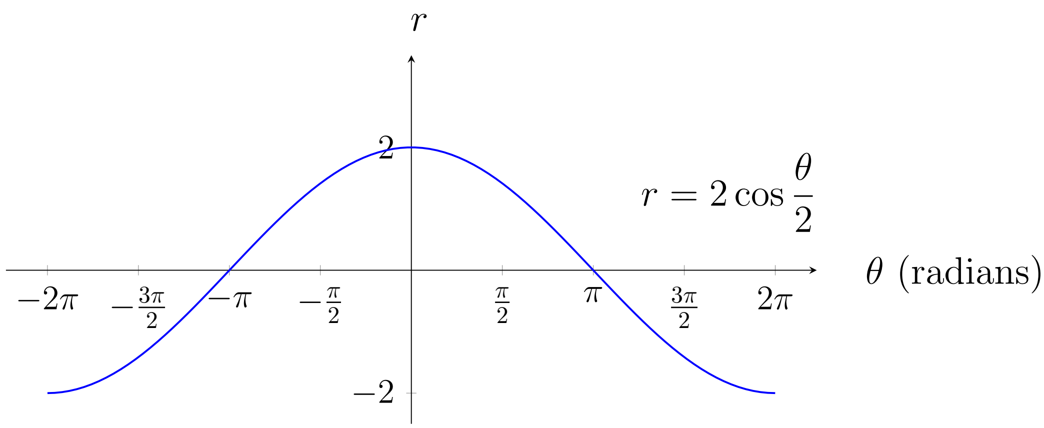
Hence

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta =$$

Hint for solving the integral: After simplifying the integral as much as possible, use the half-angle formula $\boxed{\frac{1+\cos\theta}{2} = \cos^2\left(\frac{\theta}{2}\right)}$

(Continued on the next page...)

The following graph of $r = 2\cos\dfrac{\theta}{2}$ helps us get rid of the absolute value sign:



In the interval $[0, 2\pi]$, $r = 2\cos(\theta/2)$ changes signs at $\theta =$ ____. So we must split into the sum of two integrals to get rid of the absolute value signs:

$\boxed{\text{Answer: } 8}$

Homework. §10.4 # 1,3, 5, 17 (use $\theta \in [-\pi/6, \pi/6]$), 19 (use $\theta \in [0, \pi/4]$), 49, 51

Chapter 9. Differential Equations

§9.1 Modeling with Differential Equations

Definition. ***Differential equation*** *is an equation that involves an unknown function and its* ________________.

Example. $\dfrac{dy}{dx} + 2y^2 = x^2$, $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 - xy - 7 = 0$, etc.

Definition. *A* ***solution to a differential equation*** *is any function* ____________ *that satisfies the differential equation.*

Remark. Unless otherwise specified, assume that y is a function of x, i.e. $y = f(x)$.

Example 1. *Show that any function of the form $y = ce^{-x} + x - 1$, where c is a constant, is a solution of the differential equation*

$$y' + y = x$$

It can be shown that <u>every</u> solution of the differential equation

$$y' + y = x$$

must have the form $y = ce^{-x} + x - 1$ where c is a constant; therefore, this is the **general solution of the differential equation** $y' + y = x$.

In practice, a **particular solution of a differential equation** is obtained from the general solution of the differential equation by requiring that the solution and/or its derivative(s) satisfy a certain condition at value of x, usually a condition of the form $y(0) = y_0$ (called the **initial condition**). The problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.

Example. Use the results of Example 1 to find the particular solution of the equation $y' + y = x$ that satisfies the condition $y(0) = 0$.

From Example 1, we know that the general solution to this differential equation is

$$y = ce^{-x} + x - 1$$

Now, $y(0) = 0 \Rightarrow$

Hence the *particular solution* to this initial-value problem is:

Homework. 9.1: # 7, 13, 15

§9.2 Direction Fields

Unfortunately, it is impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution. In this section, we show that we can still learn a lot about the solution through a graphical approach, called *direction fields*.

Suppose we are asked to sketch the graph of the solution of the initial-value problem

$$y' = x + y, \quad y(0) = 1$$

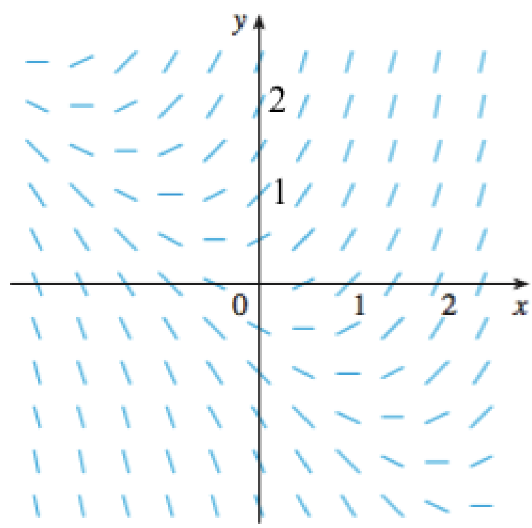
We do not know how to find the formula for the solution, but we can still figure out the general shape of the curve (called the **solution curve**) as follows.

Step 1. Compute a table with rows for x, y and y': (use random and independent choices for x and y)

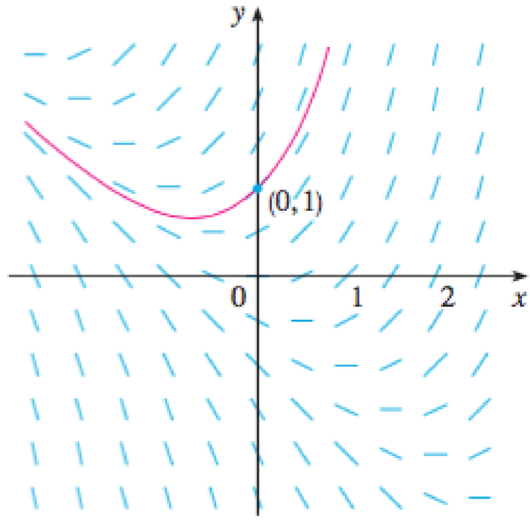
x							
y							
$y' = x + y$							

$\cdots$

Step 2. Draw short line segments with these slopes (last row of the table) at the corresponding points (x, y). The result is a **direction field**.

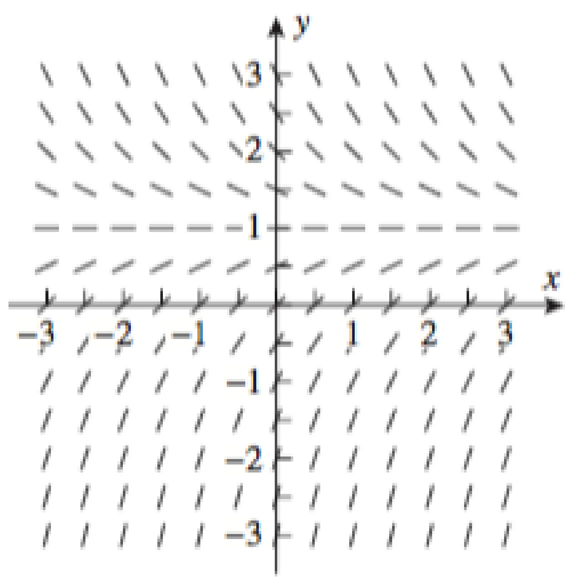


Step 3. Sketch the solution curve through the point stated in the initial-value problem (in this example, a curve through the point $(0, 1)$). Draw the curve so that it is ____________ to the nearby line segments in the direction field.



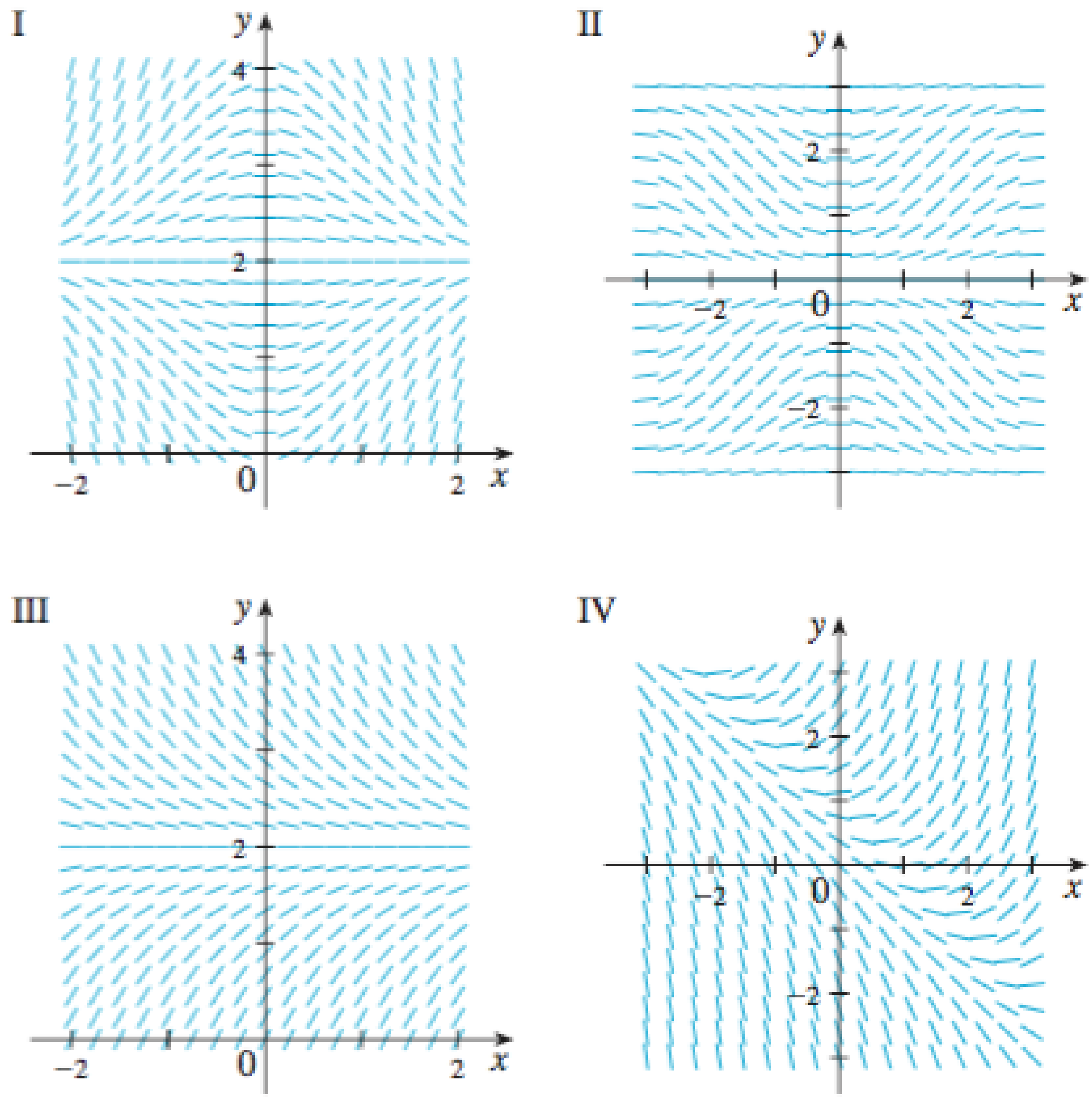
Example. A direction field for the differential equation $y' = 1 - y$ is shown below. Sketch the graph of the solutions that satisfy the initial conditions

$$y(0) = 1, \ y(0) = -1 \ \text{ and } \ y(0) = 2.$$



Remark. The general solution to the above differential equation is: $y = Ce^{-x} + 1$.

Example. Match the differential equation $y' = x(2-y)$ with its direction field from the direction fields given below.



Homework. 9.2:# 1a, 3, 5, 7

§9.3 Separable Equations

Differential equations are classified according to their basic form because different methods are used to solve different types of equations.

In this section, we describe a method for solving an important class of *first-order differential equations* (i.e. the order of the highest derivative that occurs in the differential equation is 1) that can be written in the form

$$\boxed{\phantom{\frac{dy}{dx} = f(x)g(y)}}$$

where $f(x)$ is a function of ____ only and $g(y)$ is a function of ____ only.

Such differential equations are said to be **separable** because the variables can be separated.

Separable equations can be solved using the **method of separation of variables:**

Suppose we are given a first-order separable differential equation in the form

$$\frac{dy}{dx} = f(x)g(y)$$

- **Step 1.** Write the above equation in the form

$$\tag{1}$$

  When written in this form, the variables are said to be **separated**.

- **Step 2.** ________________ each side of Eq. (1) with respect to the appropriate variable.

Example. Find the general solution of the differential equation

$$\frac{dy}{dx} = -x^2 y$$

- **Step 1.** Separate the variables:

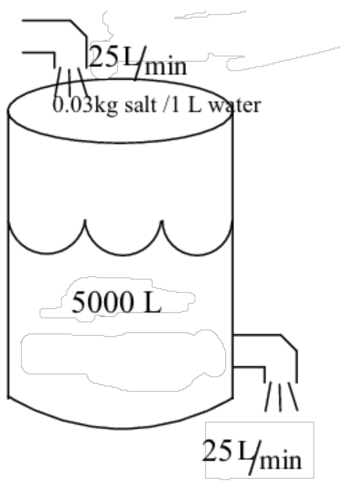
  Notice that we could only have done the above operation if $y \neq 0$. We will consider the case when $y = 0$ separately (at the end). So suppose that $y \neq 0$.

- **Step 2.** Integrate both sides:

  Let $A =$ ________. Then $y =$

  Now, let's consider the case $y = 0$. It is obvious that when $y = 0$, $\dfrac{dy}{dx} =$ ____, so the differential equation is satisfied when $y = 0$. That is, $y = 0$ is a solution to the differential equation too. So we can write the general solution as:

Example. A tank contains 20 kg of salt dissolved in 5000 L of water. Salt water that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L / min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after 30 minutes?



Let $y(t)$ be the amount of salt (kg) in the tank after t minutes. Then $\frac{dy}{dt}$ represents the

We need to find ________.

$$\boxed{} \qquad (2)$$

rate at which salt enters tank (in kg/min) =

rate at which salt exits tank (in kg/min) =

Hence (2) becomes

This is a separable equation, which we know how to solve.

Homework. 9.3 # 1 - 21 (odd) [hint for #11: to integrate $\sec x$, multiply it by $\dfrac{\sec x + \tan x}{\sec x + \tan x}$ first...], (hint: let $y(t)$ be the amount of alcohol in the vat in **gallons**).

§9.4 Models for Population Growth

In this section, we investigate differential equations that are used to model population growth.

1. Unrestricted Growth (Natural Growth)
The differential equation describing an unrestricted growth model is given by

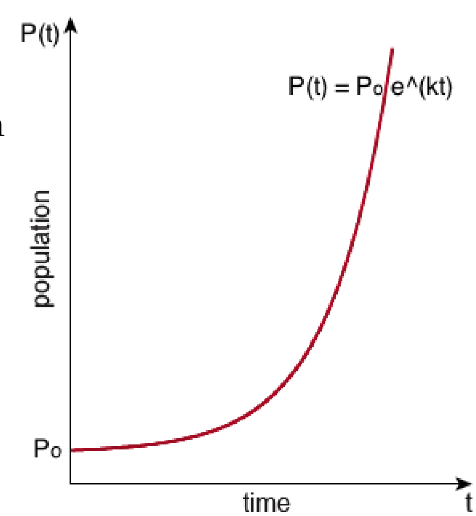
where $P(t)$ represents the size of a certain population at any time t and k is a positive constant (we saw this equation in Math 151).

Let's solve this differential equation by separating the variables to get

Suppose that the initial condition is given by $P(0) = P_0$. Then

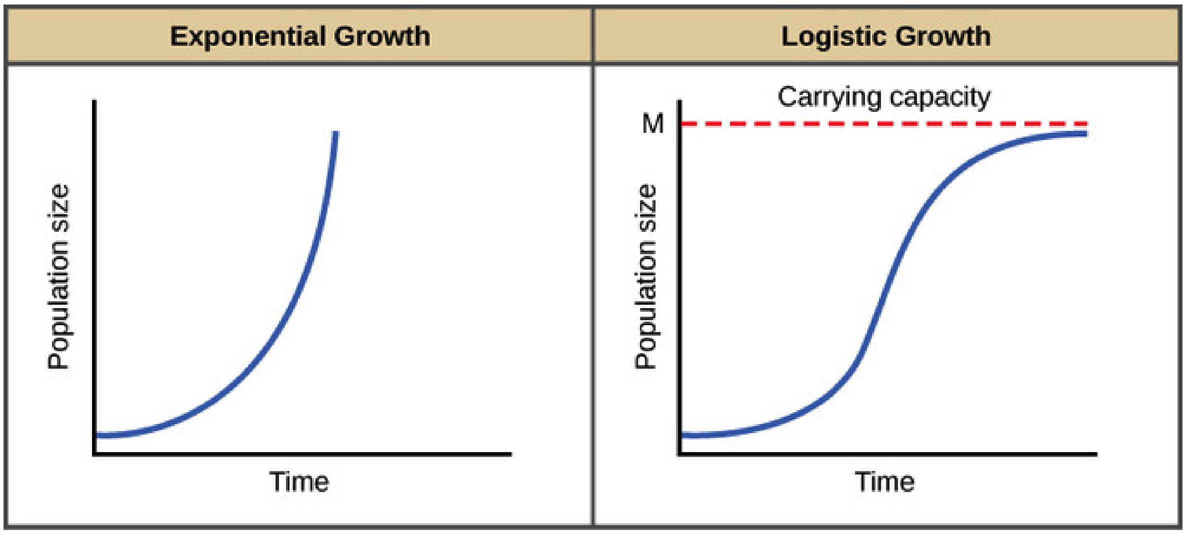
Therefore the solution to the initial-value problem

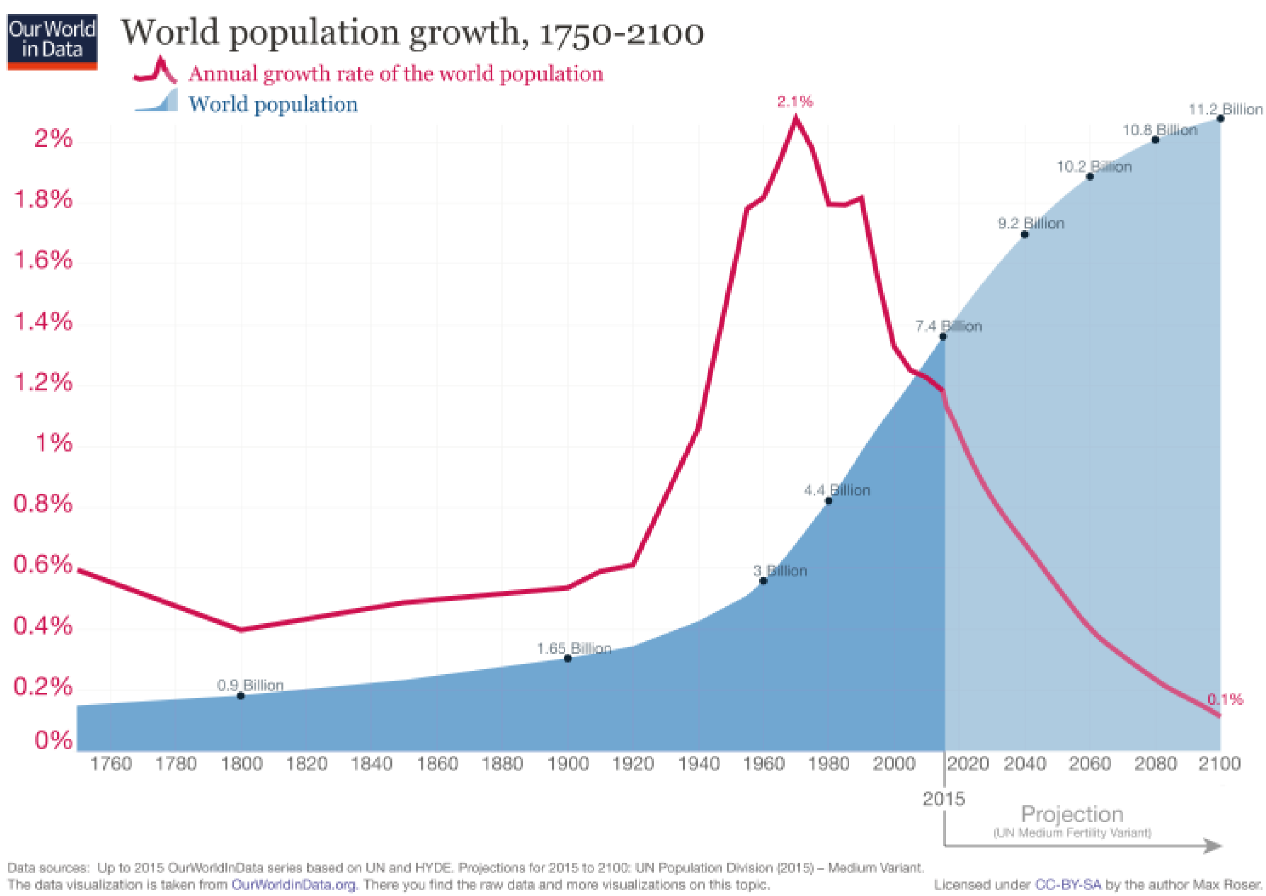
$$\boxed{\frac{dP}{dt} = kP, \quad P(0) = P_0}$$ is



2. Restricted Growth (Logistic Model)

A population often increases exponentially in its early stages but levels off eventually and approaches its "carrying capacity", M, because of limited resources.





A differential equation describing a restricted growth model (or, also known as **logistic model**) is given by $\boxed{}$ where:

- M is the ______________________________(i.e. the maximum population that the environment is capable of sustaining in the long run),
- k is a positive constant, and
- $P(t)$ is the population at time t.

The logistic equation is separable so we can solve it:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Note that the general solution to this logistic equation can be simplified to:

$$\boxed{P(t) = \frac{M}{Ae^{-kt} + 1} \qquad \text{where } A = e^{-c}}$$

Suppose that we are given the initial condition $P(0) = P_0$. Then we can solve for the constant A:

Hence the solution to the logistic model is:

$$\boxed{P(t) = \frac{M}{Ae^{-kt} + 1} \text{ where } A = \qquad\qquad}$$

Homework. 9.4 # 5 (derive the solution by using the Separation of Variables method), 21a + Chapter 9 Review on Page 657 #15

Space for Notes