Math157 Solution Midterm 1

1. For x units sold, the total revenue function is $R(x) = 30x + 100$.
The total cost function is $C(x) = 500 + 8x + \frac{1}{8}x^2$. [6 marks]

a) Find the profit function $P(x)$.

$$P(x) = R(x) - C(x) = 22x - \frac{1}{8}x^2 - 400$$

b) Find the marginal profit when 100 units are sold.

$$\frac{dP}{dx} = P'(x) = 22 - \frac{1}{4}x$$

$$\left.\frac{dP}{dx}\right|_{x=100} = P'(100) = -3$$

c) If $P(100) = 550$, use your part b answer to estimate the total profit if 101 units sold.

$$P(100+1) - P(100) \approx P'(100) = -3$$

$$\therefore \quad P(101) \approx P(100) - 3 = 550 - 3 = 497$$

d) Should the company sell the 101^{st} unit? Explain using answers above.

No.

Since the profit falls when 101^{st} unit is sold!

2. Find the instantaneous rate of change for $f(x) = 3x^2 - 5x + 1$ at $x = 4$ using the limit definition of the derivative. [4 marks]

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} \quad \text{if the limit exists}$$

$$= \lim_{h \to 0} \frac{3\left[(4+h)^2 - 4^2\right] - 5\left[4+h-4\right]}{h}$$

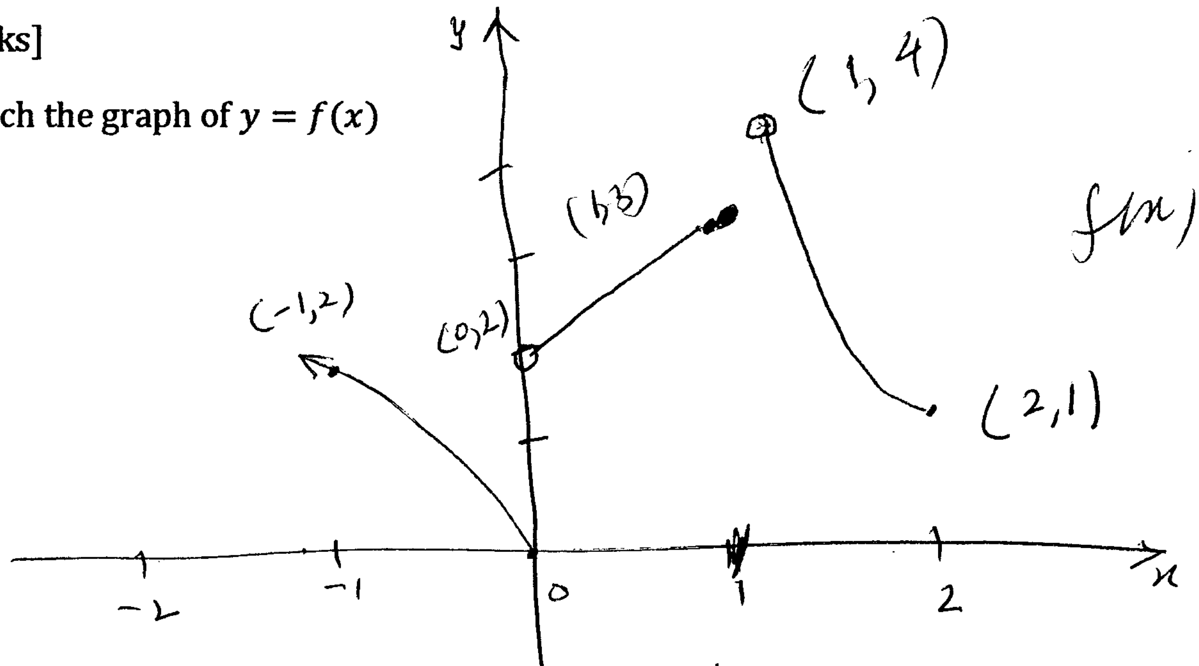
$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}}\left[3(8+h) - 5\right]$$

$$= 19$$

3. Let $f(x) = \begin{cases} \sqrt{-4x} & if -1 \le x \le 0 \\ 2 + x, & if\ 0 < x \le 1 \\ (3x-5)^2 & if\ 1 < x \le 2 \end{cases}$

[6 marks]

a) Sketch the graph of $y = f(x)$



b) Is f continuous at $x = 0$? Justify your answer.

NO

Condition ② fails

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sqrt{-4x} = 0$

$\neq$ $\therefore \lim_{x\to 0} f(x)$ DNE

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2+x = 2$

f is continuous at a if

1. $f(a)$ is defined
2. $\lim_{x\to a} f(x)$ exists
3. $f(a) = \lim_{x\to a} f(x)$

c) Is f continuous at $x = 1$? Justify your answer.

NO

Condition ② fails

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2+x = 3 \neq \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3x-5)^2$

$= 4$

$\therefore \lim_{x\to 1} f(x)$ DNE

4. Find an equation of the tangent line to the curve $x^5 - x^2y - y^4 = 27$ at the point $P(2,1)$. [4 marks]

Apply Dx:

$$5x^4 - [2x \cdot y + x^2 \cdot y'] - 4y^3 y' = 0$$

Put $x=2, y=1, m = y'(P)$

$$(5)(16) - [4 + 4m] - 4m = 0$$

$$8m = 76$$

$$m = \frac{19}{2}$$

Tp: $y - 1 = \frac{19}{2}(x - 2)$

$$y = \frac{19}{2}x - 18$$

5. Find the following limits, if they exist. [9 marks]

a) $\lim_{x\to 5}\frac{x^2-(10-x)^2}{10-2x}$ $= \lim_{x\to 5}\frac{[x-(10-x)][x+(10-x)]}{10-2x}$

$= \lim_{x\to 5} -\frac{(10-2x)\cdot 10}{(10-2x)} = \lim_{x\to 5} -10 = \underline{-10}$

b) $\lim_{x\to 4}\frac{-4+\sqrt{4x}}{x-4}$ $\cdot \frac{-4-\sqrt{4x}}{-4-\sqrt{4x}}$

$= \lim_{x\to 4}\frac{16-4x}{x-4}\cdot\frac{1}{-(4+\sqrt{4x})} = \lim_{x\to 4}\frac{-4\cancel{(x-4)}}{\cancel{(x-4)}}\cdot\frac{1}{-(4+\sqrt{4x})}$

$= \frac{1}{2}$

c) $\lim_{x\to -\infty}\frac{\sqrt{9x^2+7x-1}}{3x-5}$ $\begin{matrix}\div x\\ \div x\end{matrix}$

Note:- if $x<0$, $x = -\sqrt{x^2}$

$= \lim_{x\to -\infty} -\frac{\sqrt{\frac{9x^2+7x-1}{x^2}}}{3-\frac{5}{x}} = -\lim_{x\to -\infty}\frac{\sqrt{9+\frac{7}{x}-\frac{1}{x^2}}}{3-\frac{5}{x}} = -1$

6. Differentiate the following functions as indicated: [12 marks]

a) $y = f(x) = x^5 + \frac{1}{x^2} - \frac{1}{\sqrt{x}} + 5^\pi, find\ f'(1).$

b) $y = g(x) = \frac{2x^2-1}{x^2+1}, find\ g'(1).$ $\rightarrow \left(\frac{u}{v}\right)' = [u'v] - [uv']$

c) $y = f(x) = \log_4[tan^{-1}(x+1)], find\ f'(0).$

a) $f'(x) = 5x^4 - \frac{2}{x^3} + \frac{1}{2} \cdot \frac{1}{x^{3/2}}$

$f'(1) = 5 - 2 + \frac{1}{2} = 7/2$

b) $g'(x) = \frac{(4x)(x^2+1) - [(2x^2-1) \cdot 2x]}{(x^2+1)^2}$

$g'(1) = \frac{8-2}{4} = \frac{3}{2}$

c) $f'(x) = \frac{1}{\ln 4} \cdot \frac{1}{\tan^{-1}(x+1)} \cdot \frac{1}{1+(x+1)^2} \cdot 1$

$\tan^{-1} 1 = \pi/4$

$f'(0) = \frac{1}{\ln 4} \cdot \frac{1}{\tan^{-1}(1)} \cdot \frac{1}{2} = \frac{1}{\ln 4} \cdot \frac{4}{\pi} \cdot \frac{1}{2}$

$= \frac{2}{\pi \ln 4}$

7. Use logarithmic differentiation to find the derivative of $f(x) = (\sin x + \cos x)^{(2x+1)}$. Calculate $f'(0)$. [4 marks]

$$\ln f(x) = (2x+1)\ln(\sin x + \cos x)$$

D_x:

$$\frac{1}{f(x)} \cdot f'(x) = 2\ln(\sin x + \cos x) + (2x+1) \cdot \frac{1}{\sin x + \cos x}(\cos x - \sin x)$$

Put $x=0$

$$\frac{f'(0)}{f(0)} = 1$$

$$f'(0) = f(0) = 1$$