# Lecture 8

## Differentiation Rules (4.1–4.5, 13.2, 6.4)

Here, we assume that all functions below are differentiable.

### 1. Constant Function

Let $y = f(x) = k$, where $k$ is a constant real number, then

$$\frac{dy}{dx} = D_x y = f'(x) = 0.$$

| $f(x)$ | $f'(x)$ |
|---|---|
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\sec^2 x$ |

| $f(x)$ | $f'(x)$ |
|---|---|
| ~~cos~~ $\csc x$ | $-\csc x \cdot \cot x$ |
| $\sec x$ | $\sec x \cdot \tan x$ |
| $\cot x$ | $-\csc^2 x$ |

e.g.

$$f(x) = 25.$$

$$f'(x) = 0$$

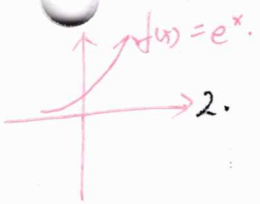
| $x$ | $f(x)=\sin x$ | $f'(x)$ |
|---|---|---|
| $\pi$ | 0 | $-1$ |
| $\frac{\pi}{2}$ | 1 | 0 |
| 0 | 0 | 1 |
| $\frac{3\pi}{2}$ | $-1$ | 0 |

$f'(x)\sin x = \cos x$

$y = x^3$

$y' = 3x^2$

| $x$ | $f'(x)$ |
|---|---|
| 2 | 12 |
| 1 | 3 |
| 0 | 0 |
| $-1$ | 3 |
| $-2$ | 12 |




### 2. Power Rule

Let $y = f(x) = x^n$, where $n$ is a constant real number, then

$$\frac{dy}{dx} = D_x y = f'(x) = n x^{n-1}.$$

1. $f(x) = \sqrt{x}$. $f'(x) = \frac{1}{2\sqrt{x}}$
2. $f(x) = \frac{1}{\sqrt{x}}$ $f'(x) = -\frac{1}{2\sqrt{x^3}} = \frac{1}{-2x\sqrt{x}}$
3. $f(x) = \frac{1}{x}$ $f'(x) = -\frac{1}{x^2}$
4. $f(x) = x^{\frac{3}{2}}$ $= \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

| $f(x)$ | $f'(x)$ |
|---|---|
| $e^x$ | $e^x$ |
| $a^x$ | $(\ln a)\cdot a^x$ |
| $\ln x$ | $\frac{1}{x}$ |
| $\log_a x$ | $\frac{1}{\ln a}\cdot\frac{1}{x}$ |

e.g.

$$y = x^5, \quad \frac{dy}{dx} = 5x^4$$

$$y = \sqrt{x} = x^{1/2}, \quad \frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2}\frac{1}{\sqrt{x}}$$

$$y = x^{-6.6}, \quad \frac{dy}{dx} = -6.6\, x^{-7.6}$$

$$y = \frac{1}{x} = x^{-1}, \quad \frac{dy}{dx} = (-1)x^{-2} = -\frac{1}{x^2}.$$

| $x$ | $f(x)=e^x$ | $f'(x)=$ |
|---|---|---|
| 2 | $e^2$ | $e^2$ |
| 1 | $e$ | $e$ |
| 0 | 1 | 1 |
| $-1$ | $e^{-1}$ | $e^{-1}$ |

$\therefore f'(x) = f(x) = e^x$.

3. Constant Times a Function

Let $y = k\,f(x)$, where $k$ is a constant real number,
then $\dfrac{dy}{dx} = D_x[k\,f(x)] = k\,f'(x)$.

e.g.

$$y = 5x^3, \quad \frac{dy}{dx} = 5(D_x\,x^3) = 5 \cdot 3x^2$$
$$= 15x^2$$

$$y = \sqrt{2}\,p^{3/2}, \quad \frac{dy}{dp} = \sqrt{2}(D_p\,p^{3/2})$$
$$= \sqrt{2} \cdot \frac{3}{2}\,p^{1/2}$$
$$= \frac{3\sqrt{2}}{2}\,p^{1/2}$$

4. Sum or Difference of Functions

Let $y = f(x) \pm g(x)$, then

$$\frac{dy}{dx} = \frac{df}{dx} \pm \frac{dg}{dx}$$

e.g.

$$y = x + \frac{1}{x}$$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2}$$

5. Product Rule

If $y = f(x) = u(x)\,v(x)$, then

$$\frac{dy}{dx} = f'(x) = u'(x) \cdot v(x) + u(x)\,v'(x)$$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot v + u\,\frac{dv}{dx}$$

e.g.

$$y = f(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}} + 7\right)\left(x^2 + x + 1\right)$$

$$\frac{dy}{dx} = \left(\sqrt{x} + \frac{1}{\sqrt{x}} + 7\right)' \cdot (x^2 + x + 1) + \left(\sqrt{x} + \frac{1}{\sqrt{x}} + 7\right)\left(x^2 + x + 1\right)'$$

$$= \left(\frac{1}{2}\frac{1}{\sqrt{x}} - \frac{1}{2}\frac{1}{x\sqrt{x}}\right)(x^2 + x + 1) + \left(\sqrt{x} + \frac{1}{\sqrt{x}} + 7\right)(2x + 1)$$

| $f(x)$ | $f'(x)$ |
|---|---|
| $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\cos^{-1} x$ | $\frac{-1}{\sqrt{1-x^2}}$ |
| $\tan^{-1}$ | $\frac{1}{1+x^2}$ |

| $f(x)$ | $f'(x)$ |
|---|---|
| $\sec^{-1} x$ | $\frac{1}{\lvert x \rvert \sqrt{x^2-1}}$ |
| $\csc^{-1} x$ | $-\frac{1}{\lvert x \rvert \sqrt{x^2-1}}$ |
| $\cot^{-1} x$ | $-\frac{1}{1+x^2}$ |

## 6. Quotient Rule

If $y = f(x) = \dfrac{u(x)}{v(x)}$, then

$$\frac{dy}{dx} = f'(x) = \frac{u' \cdot v - u v'}{v^2}$$

e.g. $y = f(x) = \dfrac{2x-1}{x^2+9}$

$$\frac{dy}{dx} = f'(x) = \frac{(2x-1)' \cdot (x^2+9) - (2x-1)(x^2+9)'}{(x^2+9)^2}$$

$$= \frac{2(x^2+9) - (2x-1)(2x)}{(x^2+9)^2}$$

$$= \frac{2x^2+18 - 4x^2+2x}{(x^2+9)^2}$$

$$= \frac{-2x^2+2x+18}{(x^2+9)^2}$$

## 7. Chain Rule : Derivative of a Composite Function

Suppose we have two functions $f$ and $g$ defined by $y = f(u)$ and $u = g(x)$, then a composite function $f \circ g$ is defined by $f(g(x))$.

$D_{f \circ g} = \{$ all $x$ in the $D_g$ such that $g(x)$ is in the $D_f \}$

Example $f(x) = \sqrt{1-x}$, $g(x) = x^2$. The $D_{f \circ g}$ is the set of all $x$ in the domain of $g$ such that $g(x) = x^2$ is in the domain of $f$. That is $x^2 \leq 1$ or $-1 \leq x \leq 1$.

$\therefore \quad D_{f \circ g} = [-1, 1]$

$f \circ g$ (read '$g$ composed with $f$') is given by

$$y = f(g(x)) = f(x^2) = \sqrt{1-x^2}.$$

The composite function $g \circ f$ (read $f$ composed with $g$') is given by $y = g(f(x)) = g(\sqrt{1-x}) = 1-x$.

$D_{g \circ f} =$ The set of all $x$ in the $D_f$ such that $f(x) = \sqrt{1-x}$ is in the domain of $g$. That $x \leq 1$.

$$= (-\infty, 1]$$

Note

1. $f \circ g$ and $g \circ f$ are in general not equal.

2. If $f$ and $g$ are inverse functions of each other, then $f \circ g = g \circ f =$ identity function.

Chain Rule

If $y$ is a function of $u$, say $y = f(u)$, and if $u$ is a function of $x$, say $u = g(x)$, then $y$ is a function of $x$ and $y = f(u) = f[g(x)]$, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy(u)}{du} \cdot \frac{du(x)}{dx}$$

$$= \frac{df}{du} \cdot \frac{du}{dx} = \frac{df(u)}{du} \cdot \frac{du(x)}{dx}$$

$$= \frac{df}{dg} \cdot \frac{dg}{dx} = \frac{df(g)}{dg} \cdot \frac{dg(x)}{dx}$$

$$= f'[g(x)] \cdot g'(x)$$

e.g.

$$y = u^2, \qquad u = x^2 + x + 1$$

$$\frac{dy}{dx} = \frac{d(u^2)}{du} \cdot \frac{du}{dx}$$

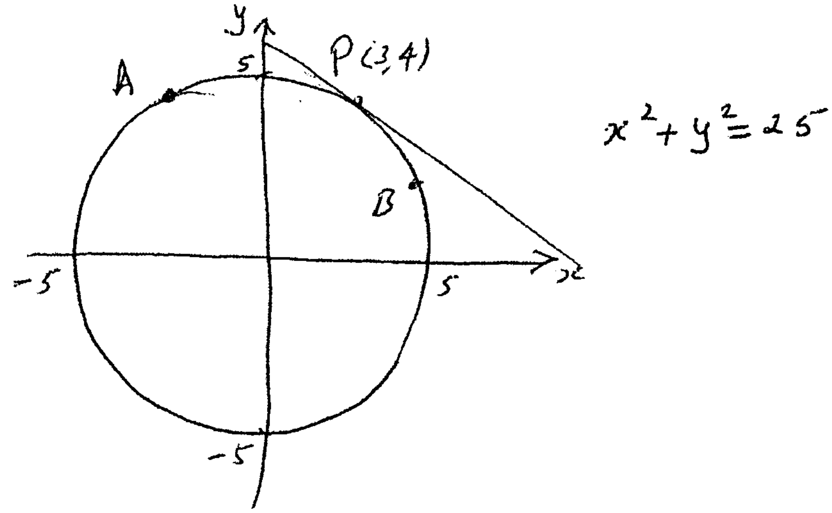
$$= 2u \cdot (2x+1)$$

$$= 2(x^2+x+1) \cdot (2x+1)$$

$$\left.\frac{dy}{dx}\right|_{x=3} = 2(3^2+3+1)((2)(3)+1)$$

$$= 2(13)(7) = 182$$

## 8. Implicit Differentiation




Problem. Find an equation of the tangent to the circle $x^2+y^2=25$ at the point $P(3,4)$. Note here that $AB$ arc of the circle is not given explicitly like $y=f(x)$. Near $P$, *if* we can write $y$ as function of $x$ explicitly.

Then we have

$$x^2+[y(x)]^2=25$$

Differentiating this equation with respect to $x$, we have:

$$D_x(x^2)+D_x[y(x)]^2=D_x(25)$$

$$2x+2y(x)\cdot y'(x)=0$$

In order to find $y'(3)$, the slope of the tangent at $P$, we substitute $x=3$, $y=4$ into this equation.

$$2(3)+2\cdot 4\cdot y'(3)=0 \quad \text{or } y'(3)=-\frac{3}{4}$$

Hence an equation of the tangent at $P$ is

$$y-4=-\frac{3}{4}(x-3)$$

$$y=\frac{-3}{4}x+\frac{25}{4}$$

In this particular example, it is not difficult to write an explicit function for the arc AB:

$$y = f(x) = \sqrt{25 - x^2}$$

$$\left.\frac{dy}{dx}\right|_{x=3} = \frac{1}{2} \cdot \frac{1}{\sqrt{25-x^2}} \cdot -2x \Bigg|_{x=3}$$

$$= \left.\frac{-x}{\sqrt{25-x^2}}\right|_{x=3}$$

$$= \frac{-3}{\sqrt{25-9}} = \frac{-3}{4}$$

9. Derivative of the exponential function $y = f(x) = e^x$

$$y = f(x) = e^x$$

$$\frac{dy}{dx} = f'(x) = e^x$$

We can prove that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$ but it is beyond the scope of this course and therefore we will just use it to obtain the above result.

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} e^x \cdot \frac{e^h - 1}{h}$$

$$= e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x \cdot 1$$

$$= e^x$$

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$$y = e^{u(x)}$$

$$\frac{dy}{dx} = \frac{d\,e^u}{du} \cdot \frac{du}{dx} = e^u \cdot u'(x)$$

e.g.

1. $y = e^{-x}$,

$$\frac{dy}{dx} = e^{-x} \cdot \frac{d}{dx}(-x) = -e^{-x}$$

2. $y = e^{x^2}$

$$\frac{dy}{dx} = e^{x^2} \cdot \frac{d}{dx}(x^2) = 2x e^{x^2}$$

3. $y = 2^x$ $\qquad 2 = e^{\ln 2}$

$$= \left(e^{\ln 2}\right)^x$$

$$= e^{(\ln 2)x}$$

$$\frac{dy}{dx} = e^{(\ln 2)x} \cdot \frac{d}{dx}((\ln 2)x)$$

$$= 2^x \cdot \ln 2$$

$$\therefore \frac{dy}{dx} = (\ln 2)\, 2^x$$

4. $y = a^x = e^{(\ln a)x}$

$$\frac{dy}{dx} = e^{(\ln a)x} \cdot \frac{d}{dx}((\ln a)x) = a^x \cdot \ln a.$$

$$\frac{dy}{dx} = \ln a \cdot a^x$$

10. Derivative of logarithmic function

$$y = \ln x = \log_e x \iff e^y = x$$

Implicitly differentiating $e^{y(x)} = x$ with respect to $x$:

$$e^{y(x)} \cdot y'(x) = 1$$

$$y'(x) = \frac{1}{e^y} = \frac{1}{x}$$

Hence, if $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$

e.g.

$$y = \log_a x \iff a^{y(x)} = x$$

Implicitly differentiating $a^{y(x)} = x$ with respect to $x$, we have

$$a^{y(x)} \cdot \ln a \cdot y'(x) = 1$$

$$\therefore \quad y'(x) = \frac{1}{a^{y(x)}} \cdot \frac{1}{\ln a} = \frac{1}{\ln a} \cdot \frac{1}{x}$$

Summary:

| $y(x)$ | $\frac{dy}{dx}$ |
|---|---|
| $e^x$ | $e^x$ |
| $a^x$ | $\ln a \ a^x$ |
| $\ln x$ | $\frac{1}{x}$ |
| $\log_a x$ | $\frac{1}{\ln a} \cdot \frac{1}{x}$ |

e.g. $y = \ln|x|$

if $x>0$, $y = \ln|x| = \ln x$ and $\frac{dy}{dx} = \frac{1}{x}$

if $x<0$ $y = \ln|x| = \ln(-x)$ and $\frac{dy}{dx} = \frac{1}{(-x)} \cdot \frac{d}{dx}(-x)$

$= \frac{1}{-x} \cdot -1 = \frac{1}{x}$

$$\therefore \quad \frac{d}{dx} \ln|x| = \frac{1}{x}.$$

## 11. Differentiation of Trigonometric functions

$$y = f(x) = \sin x$$

It can be proved that $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$ & $\lim\limits_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cdot \cos h + \sin h \cdot \cos x - \sin x}{h}$$

$$= \lim_{h \to 0} \left[ \sin x \; \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right]$$

$$= \sin x \left[ \lim_{h \to 0} \frac{\cos h - 1}{h} \right] + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \sin x \cdot [0] + \cos x \cdot [1]$$

$$= \cos x$$

Hence, $\dfrac{d}{dx}(\sin x) = \cos x$

The derivatives of the remaining five trigonometric functions can be obtained easily using trigonometric identities and the chain rule.

$y = f(x) = \cos x$ , then $\frac{dy}{dx} = -\sin x$

[[ Since $\cos x = \sin(\frac{\pi}{2} - x)$, we have

$y = f(x) = \cos x = \sin(\frac{\pi}{2} - x)$

$$\frac{d(\cos x)}{dx} = \frac{d}{dx} \sin(\frac{\pi}{2} - x)$$

$$= \cos(\frac{\pi}{2} - x) \cdot \frac{d}{dx}(\frac{\pi}{2} - x)$$

$$= \sin x \cdot (-1)$$

$$= -\sin x$$ ]]

$y = f(x) = \tan x$ , then $\frac{dy}{dx} = \sec^2 x$

[[ $y = f(x) = \tan x = \frac{\sin x}{\cos x}$

$$\frac{d(\tan x)}{dx} = \frac{(\sin x)' \cos x - \sin x \cdot (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$ ]]

$y = f(x) = \sec x$, then $\frac{dy}{dx} = \sec x \tan x$

$$\left[\left[ \frac{d(\sec x)}{dx} = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d}{dx}(\cos x)^{-1} \right.\right.$$

$$= (-1)(\cos x)^{-2} \cdot (\cos x)' = (-1)\frac{1}{\cos^2 x} \cdot (-\sin x)$$

$$\left.\left. = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \sec x \cdot \tan x \right]\right]$$

$y = f(x) = \csc x$, then $\frac{dy}{dx} = -\csc x \cdot \cot x$

$$\left[\left[ \frac{d}{dx}(\csc x) = \frac{d}{dx}(\sin x)^{-1} = (-1)(\sin x)^{-2} \cdot (\sin x)' \right.\right.$$

$$= -\frac{1}{\sin^2 x} \cdot \cos x = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$\left.\left. = -\csc x \cdot \cot x \right]\right]$$

$y = f(x) = \cot x$, then $\frac{dy}{dx} = -\csc^2 x$

$$\left[\left[ \frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{(\cos x)' \sin x - (\cos x)(\sin x)'}{\sin^2 x} \right.\right.$$

$$= \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x}$$

$$\left.\left. = -\frac{(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x \right]\right]$$

## Examples

1. $$g(x) = \frac{x^2 - x + 1}{\sqrt{x}} = x^{3/2} - x^{1/2} + x^{-1/2}$$

$$\frac{dg}{dx} = g'(x) = \frac{3}{2} x^{\frac{3}{2}-1} - \frac{1}{2} x^{\frac{1}{2}-1} + (-\tfrac{1}{2}) x^{-\frac{1}{2}-1}$$

$$= \frac{3}{2} x^{1/2} - \frac{1}{2} x^{-1/2} - \frac{1}{2} x^{-3/2}$$

$$= \frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}.$$

2. $$f(t) = t^{-2} - 2t^{-3} + \sqrt{11}$$

$$\frac{df}{dt} = f'(t) = -2t^{-3} - 2 \cdot -3 \cdot t^{-4} + 0$$

$$= -\frac{2}{t^3} + \frac{6}{t^4}$$

3. An analyst has found that a company's costs and revenues in dollars for one product are given by

$$C(x) = 2x \quad \text{and} \quad R(x) = 6x - \frac{x^2}{1000}.$$

respectively, where $x$ is the number of items produced.

a. Find the marginal cost function.

$$C'(x) = \frac{d}{dx}(2x) = 2\frac{d(x)}{dx} = 2$$

b. Find the marginal revenue function.

$$R'(x) = \frac{dR}{dx} = 6 - \frac{2x}{1000} = 6 - \frac{x}{500}$$

c. Find the marginal profit function.

$$P(x) = R(x) - C(x) = 6x - \frac{x^2}{1000} - 2x = 4x - \frac{x^2}{1000}$$

$$P'(x) = \frac{dP}{dx} = 4 - \frac{2x}{1000} = 4 - \frac{x}{500}.$$

d. What value of $x$ makes marginal profit equal to 0?

$P'(x) = 0 \Rightarrow 4 - \frac{x}{500} = 0$ or $x = 2000$

e. Find the profit when the marginal profit is 0.

$$P(2000) = 4x - \frac{x^2}{1000}\Big|_{x=2000} = 8000 - \frac{(2000)^2}{1000} = \$4000$$

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3. Marginal Average Cost $= \frac{d}{dx}\left(\frac{C(x)}{x}\right) = \frac{d\,\bar{C}(x)}{dx}$

The total cost (in hundreds of dollars) to produce $x$ units of perfume is

$$C(x) = \frac{3x+2}{x+4}$$

a. Find the average cost function $\bar{C}(x)$. $\bar{C}(x) = \frac{3x+2}{x(x+4)}$

b. Find the marginal average cost function

$$\bar{C}'(x) = \frac{d\bar{C}}{dx} = \frac{d}{dx}\left(\frac{3x+2}{x^2+4x}\right)$$

$$= \frac{(3x+2)'(x^2+4x) - (3x+2)(x^2+4x)'}{(x^2+4x)^2}$$

$$= \frac{3(x^2+4x) - (3x+2)(2x+4)}{x^2(x+4)^2} = \frac{3x^2+12x - 6x^2 - 14x - 8}{x^2(x+4)^2} = -\frac{(3x^2+2x+8)}{x^2(x+4)^2}$$

4. $f(t) = \dfrac{\sqrt{t}}{t-1}$

$$\frac{df}{dt} = f'(t) = \frac{(\sqrt{t})'(t-1) - \sqrt{t}\,(t-1)'}{(t-1)^2} = \frac{\frac{1}{2}\frac{1}{\sqrt{t}}(t-1) - \sqrt{t}}{(t-1)^2}$$

$$= \frac{(t-1)-2t}{2\sqrt{t}\,(t-1)} = -\frac{t+1}{2\sqrt{t}\,(t-1)}$$

5. $R(x) = \dfrac{x^2+7x-2}{x-2}$. $R'(x) = \dfrac{(x^2+7x-2)'(x-2) - (x^2+7x-2)(x-2)'}{(x-2)^2}$

$$= \frac{(2x+7)(x-2) - (x^2+7x-2)}{(x-2)^2}$$

$$= \frac{2x^2+3x-14 - x^2-7x+2}{(x-2)^2}$$

$$= \frac{x^2-4x-12}{(x-2)^2}$$

6. $y = \dfrac{1}{(3x^2-4)^6} = (3x^2-4)^{-6}$

$$\frac{dy}{dx} = -6\,(3x^2-4)^{-6-1} \cdot \frac{d}{dx}(3x^2-4)$$

$$= -\frac{6}{(3x^2-4)^7} \cdot 6x = \frac{-36x}{(3x^2-4)^7}$$

7. Find the equation of the tangent line to the graph of $y = f(x) = \sqrt{x^2+16}$ at $x = 3$.

$$f'(3) = \frac{d}{dx}(x^2+16)^{1/2}\Big|_{x=3}$$

$$= \frac{1}{2}\cdot(x^2+16)^{-1/2}\cdot\frac{d}{dx}(x^2+16)\Big|_{x=3}$$

$$= \frac{1}{\cancel{2}}\cdot\frac{1}{\sqrt{3^2+16}}\cdot\cancel{(2)}(3) = \frac{3}{5}$$

The required equation is:

$$y - y(3) = \frac{3}{5}(x-3)$$

$$y - 5 = \frac{3}{5}x - \frac{9}{5}$$

or $$\underline{y = \frac{3}{5}x + \frac{16}{5}}$$

8. $$y = e^{-x^2}$$
$$\frac{dy}{dx} = e^{-x^2} \cdot \frac{d}{dx}(-x^2) = -2x\, e^{-x^2}$$

9. $$y = x e^{-x}$$
$$\frac{dy}{dx} = (x)' e^{-x} + x \cdot (e^{-x})'$$
$$= e^{-x} + x(-e^{-x})$$
$$= (1-x) e^{-x}$$

10. $$y = x \ln x$$
$$\frac{dy}{dx} = (x)' \ln x + x (\ln x)'$$
$$= \ln x + x \cdot \frac{1}{x}$$
$$= 1 + \ln x$$

11. $$y = \frac{x^2 - 1}{x \ln x}$$
$$\frac{dy}{dx} = \frac{(x^2-1)' x \ln x - (x^2-1)(x \ln x)'}{(x \ln x)^2}$$
$$= \frac{2x^2 \ln x - (x^2-1)(1+\ln x)}{(x \ln x)^2}$$
$$= \frac{x^2 \ln x - x^2 + \ln x + 1}{x^2 (\ln x)^2} = \frac{(x^2+1)\ln x - x^2 + 1}{x^2 (\ln x)^2}$$

12. $y = 10^{x^2} + \log_2 |\sin x|$

$$\frac{dy}{dx} = 10^{x^2} . \ln 10 . 2x + \frac{1}{\ln 2 \sin x} . \cos x$$

$$\left[\left[ \frac{d(a^x)}{dx} = (\ln a)\, a^x . \quad \frac{d}{dx} \log_a |x| = \frac{1}{\ln a} . \frac{1}{x} \right]\right]$$

13. $y = \tan^5 3x$

$$\frac{dy}{dx} = 5 \tan^4 3x . \sec^2 3x . 3$$

$$= 15 \tan^4 3x . \sec^2 3x$$

14. $y = e^{\cos(x^2+x+1)}$

$$\frac{dy}{dx} = e^{\cos(x^2+x+1)} . -\sin(x^2+x+1) . (2x+1)$$

15. $y = \dfrac{\sin x}{1+\sin x}$

$$\frac{dy}{dx} = \frac{(\sin x)'(1+\sin x) - \sin x . (1+\sin x)'}{(1+\sin x)^2}$$

$$= \frac{\cos x\,(1+\sin x) - \sin x . \cos x}{(1+\sin x)^2}$$

$$= \frac{\cos x}{(1+\sin x)^2} .$$