3.1: 19 c

$$(AB)_{11} = \vec{r_1}(A) \cdot \vec{c_1}(B) = 6 + 49 = 55$$
$$(AB)_{22} = \vec{r_2}(A) \cdot \vec{c_2}(B) = -12+5+28 = 21$$
$$(AB)_{33} = \vec{r_3}(A) \cdot \vec{c_3}(B) = 0+12+45 = 57$$

$$\left.\right\} tr(AB) = 55+21+57 = 112+21 = 133$$

$$tr(B) = 6+1+5 = 12$$
$$tr(A) = 3+5+9 = 17$$

$$\therefore\ tr(BA) - tr(B) \cdot tr(A) = 133 - 12 \cdot 17 = -71$$

3.1: 21 (a,b)

a) matrix inner product of $\vec{u}, \vec{v} = \vec{u}^T \vec{v}$ $\qquad \vec{u} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix}$

b) " outer " " " $= \vec{u}\vec{v}^T$

a) $\begin{bmatrix} 3 & -4 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 - 28 \end{bmatrix} = \begin{bmatrix} -22 \end{bmatrix}$ (or simply -22)

b) $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} \begin{bmatrix} 2 & 7 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 21 & 0 \\ -8 & -28 & 0 \\ 10 & 35 & 0 \end{bmatrix}$

3.1: D9

a) False: $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$

AB is defined b/c $(2 \times 3) \cdot (3 \times 2) = 2 \times 2$

BA " " b/c $(3 \times 2)(2 \times 3) = 3 \times 3$.

b) True: For AB & BA to both be defined, we need:

$A \quad m \times n$

$B \quad n \times m$

But: $AB = m \times m$ & $BA = n \times n$.

to add these $m = n$.
$\therefore$ A & B are square matrices of same size.

c) True: if A $m \times n$ & B is $n \times k$. Then AB is defined.
assume $\vec{c_j}(B) = \vec{0}$ (for some $1 \leq j \leq k$)
then
$\vec{r_i}(A) \cdot \vec{c_j}(B) = 0$ for all $i = 1, \ldots, m$.
But these are precisely entries in j^{th} column of AB.
(see def. of matrix Mult.)

d) False: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$ a column of zeros.

$BA = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 9 & 12 \end{bmatrix}$ No zero column.

e) True: A $m \times n$, then $A^T = n \times m$ so $A^TA = n \times n$
$AA^T = m \times m$.
trace is defined for any square matrix.

f) False: $\vec{u} = [1 \ -1]$ $\vec{v} = [3 \ 4]$ then $\vec{u}^T v$ is a 2×2 matrix which not the same as $\vec{u} \cdot \vec{v}$.

3.2: 21

Since A is 2×2, A^{-1} exists if and only if $\det(A) \neq 0$

$A = \begin{bmatrix} c & 1 \\ c & c \end{bmatrix}$ $\therefore$ need $c^2 - c \neq 0$
$c \neq 0$ or 1.

3.2: D1

Note: $(A+B)(A-B) = A^2 - AB + BA - B^2$
* in order for this to simplify to $A^2 - B^2$, we need $AB = BA$ so the middle terms cancel. However, since matrix mult. is, in general, not commutative, most examples you pick should fail this property

ex: $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

then $AB = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \neq BA = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ . . ∴ $(A+B)(A-B) \neq A^2 - B^2$.

3.2: D3

a useful strategy for attempting T/F problems:

when attempting such questions, it is best to first play around w/ some examples / especial cases in order to get a better 'feel' (small).

for the problem. While doing so, aim to look for counterexamples; i.e., begin w/ the assumption that the given statement is false, If after a while you cannot find a counterexample, there is a decent chance that the statement is true.

3.3: 7a

Should be clear that rows 1 & 3 of A are swapped to get B.

$$\therefore \quad E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

3.3: 11a

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 2 & -5 & 1 & -3/2 & 0 \\ 0 & 1 & -2 & 0 & -2/5 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -11/10 & -6 \\ 0 & 1 & 0 & -1 & 3/10 & 3 \\ 0 & 0 & 1 & -1 & 7/10 & 2 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3 & -1.1 & -6 \\ -1 & 0.3 & 3 \\ -1 & 0.7 & 2 \end{bmatrix}.$$

3.3: 13

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R.$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \underbrace{E_4 E_3 \cdot E_2 \cdot E_1}_{B} A = R.$$

3.3: 29

$$\left[\begin{array}{cc|c} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -2 & b_2 \\ 0 & 0 & b_1 - 2b_2 \end{array}\right]$$

need $b_1 - 2b_2 = 0$ or else the system is inconsistent.

$\therefore \quad b_1 = 2b_2.$ for any $b_2 \in \mathbb{R}$.

3.3: P7

The idea is illustrated in (3.3 # 13) see above; use this to try and give a formal argument.

Extra 2

A $m \times n$. $\vec{u}, \vec{v} \in \mathbb{R}^n$. (so $n \times 1$ matrices).

a) let $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$. $A = \begin{bmatrix} | & & | \\ \vec{c_1} & \cdots & \vec{c_n} \\ | & & | \end{bmatrix}$, $k \in \mathbb{R}$.

$$\begin{aligned} A(\vec{u} + \vec{v}) &= (u_1 + v_1)\vec{c_1} + \cdots + (u_n + v_n)\vec{c_n} \\ &= \underbrace{\left(u_1 \vec{c_1} + \cdots + u_n \vec{c_n}\right)}_{A\vec{u}} + \underbrace{\left(v_1 \vec{c_1} + \cdots + v_n \vec{c_n}\right)}_{A\vec{v}} \end{aligned}$$

b) $$A(k\vec{u}) = (ku_1)\vec{c_1} + \cdots + (ku_n)\vec{c_n}$$
$$= k\underbrace{(u_1\vec{c_1} + \cdots + u_n\vec{c_n})}_{A\vec{u}}.$$

Extra 3:

the idea is given in part c of D9 from 3.1 (see above).

Extra 5:

This should follow directly from IMT:

Performing EROs on A is equivalent to multiplying A on the left by the corresponding elementary matrices. $E_1, \ldots, E_k$.

$$\underbrace{E_k \cdots E_1}_{} A = R$$

product of elementary matrices is invertible b/c each elementary matrix is invertible so $P = E_k \cdots E_1$.

Extra 8:

$$\begin{bmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix} \xrightarrow{r_2 \to r_1 + r_2} \begin{bmatrix} 2 & 5 & -1 \\ 6 & 4 & 1 \\ 6 & 4 & 1 \end{bmatrix} \xrightarrow{r_3 \to -r_2 + r_3} \begin{bmatrix} 2 & 5 & -1 \\ 6 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

→ Cannot be row reduced to I_3

→ NOT invertible by IMT.