1. Differentiate the following functions as indicated:

a) $h(x) = 3^{-x^2} + \ln|-5+4x| - \sin(\pi x)$, find $h'(0)$

$h'(x) = \ln 3 \cdot 3^{-x^2} \cdot (-x^2)' + \frac{1}{-5+4x} \cdot (-5+4x)' - \cos(\pi x) \cdot (\pi x)'$

$= \ln 3 \cdot 3^{-x^2} \cdot (-2x) + \frac{4}{-5+4x} - \pi \cdot \cos(\pi x)$

$h'(0) = -\frac{4}{5} - \pi$ ✓

b) $y = (3x+1)^{\cos x}$, find $\frac{dy}{dx}$ when $x = 0$.

$\ln y = \cos x \cdot \ln(3x+1)$

D_x: $\frac{y'}{y} = -\sin x \ \ln(3x+1) + \cos x \cdot \frac{1}{3x+1} \cdot 3$

$\frac{y'(0)}{y(0)} = 3$

$y'(0) = 3\, y(0) = \underline{\underline{3}}$

c) $f(x) = (x^5 - 4x^{-1.5})(\sqrt{3x^2+1})$, find $f'(1)$

$f'(x) = (5x^4 + 6x^{-2.5})\sqrt{3x^2+1} + (x^2 - 4x^{-1.5}) \cdot \frac{1}{2}(3x^2+1)^{-1/2} \cdot 6x$

$f'(1) = (11)(2) + (-3) \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 6$

$= 22 - 9/2 = 35/2 = \underline{\underline{17.5}}$

d) $g(x) = \frac{6}{\sqrt[3]{7-2x}}$ find $g'(-\frac{1}{2})$ $\rightarrow$ $6(7-2x)^{-1/3}$

$g'(x) = 6 \cdot \frac{-1}{3}(7-2x)^{-4/3} \cdot (-2)$

$g'(-1/2) = -2 \cdot 8^{-4/3} \cdot -2 = +4 \cdot \frac{1}{8^{4/3}} = \frac{4}{(8)(2)}$

$= \underline{\underline{+1/4}}$

6 2. Given the equation $x.\cos y - \sin y = 5x$ find $\frac{dy}{dx}$ by implicit differentiation.

D_x :

$$1.\cos y - x\sin y . y' - \cos y . y' = 5$$

$$y' = \frac{-5 + \cos y}{x\sin y + \cos y} = \frac{\cos y - 5}{\cos y + x\sin y}$$

3. Find the points on the graph of the function $f(x) = -2xe^{-x}$ where the tangent line in horizontal. $\times$ 0.4.

$\overline{\wedge}$ $\underline{\vee}$

$$f'(x) = -2\left[1.\,e^{-x} + x.\,e^{-x}.(-1)\right]$$

$$= -2e^{-x}\left[1 - x\right]$$

$$f'(x) = 0 \Rightarrow x = 1$$

6

Required point $= (1, f(1))$

$$= (1, \frac{-2}{e})$$

$$\approx (1, -0.7358)$$

1/1 ~~~~

4. Find all the points on the graph of $f(x)=\frac{1}{3}x^3-3x^2+5x-4$ where the tangent line is parallel to the line $3x+y=5$.

$y=-3x+5 \quad \therefore m=-3$

平行的

$f'(x)= x^2-6x+5$

$f'(x)=-3$

$x^2-6x+5=-3 \longrightarrow x^2-6x+8=0$

$(x-4)(x-2)=0$

$x=2,4$

Required points are $\left(2, \frac{-10}{3}\right)$
$\left(4, \frac{-32}{3}\right)$

5. Use the limit definition of the derivative to find $f'(-3)$ when $f(x)=1-\sqrt{4-7x}$.

$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ if the limit exist.

$f'(-3)=\lim_{h\to 0}\frac{f(-3+h)-f(-3)}{h}$

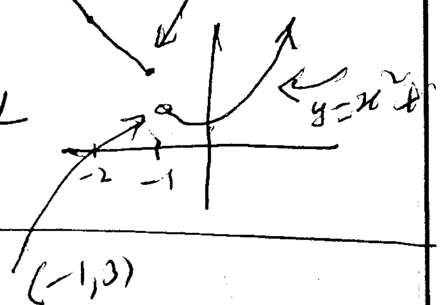
$=\lim_{h\to 0}\frac{\left[1-\sqrt{4-7(-3+h)}\right]-\left[1-5\right]}{h}$

$=\lim_{h\to 0}\frac{5-\sqrt{25-7h}}{h}\cdot\frac{5+\sqrt{25-7h}}{5+\sqrt{25-7h}}=\lim_{h\to 0}\frac{25-(25-7h)}{h\left[5+\sqrt{25-7h}\right]}$

$=\lim_{h\to 0}\frac{7\cancel{h}}{\cancel{h}\left[5+\sqrt{25-7h}\right]}=\frac{7}{10}$

$y=5-2x$

6. Graph the function $f(x)=\begin{cases}5-2x & if\ \ x\le -1\\ x^2+2 & if\ \ x>-1\end{cases}$. Use the definition of continuity to find if the function is continuous at x=-1.



f is continuous at a if
1. $f(a)$ exists
2. $\lim_{x\to a} f(x) = L$
3. $L = f(a)$

1. $f(-1) = 5-2x\big|_{x=-1} = 7$

2. $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 5-2x = 7$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x^2+2 = 3$

$\therefore \lim_{x\to -1} f(x)$ does not exist

$\therefore$ conditions (2) & (2) fail.

$\therefore f$ discontinuous at $x=-1$

7. Find the value(s) of c and d such that the function $f(x)=\begin{cases}3+cx & if\ \ x\le 2\\ dx^2+c & if\ \ x>2\end{cases}$ is continuous and differentiable at x=2.

f is continuous at $x=2$; 1. $f(2) = 3+cx\big|_{x=2}$
$= 2c+3$

2. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 3+cx = 2c+3$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} dx^2+c = c+4d$

$\therefore \lim_{x\to 2} f(x)$ exists if $2c+3 = c+4d$

$\boxed{c = 4d-3} \to (1)$

3. ✓

$f'(x) = \begin{cases} c & if\ x<2 \\ 2dx & if\ x>2\end{cases}$

$f'(2) = \lim_{x\to 2^-} f'(x) = c = \lim_{x\to 2^+} 2dx = 4d$

$\therefore f$ is differentiable at $x=2$ if $c = 4d \to (2)$

Since solving (1) & (2) we find no solution.
This function cannot be continuous and differentiable.

$\frac{dx}{dt} = 80$ units/month

8. The production of a certain commodity is increasing at a rate of 80 units per month. The demand and cost functions are respectively: $p = 240 - 0.15x$ and $C(x) = 9600 + 100x + 0.05x^2$.

a) Find the marginal cost of producing 600 units. Interpret your result.

b) Find the rate of change of the profit with respect to time if the production level is 600 units.

Profit $= P(x) = R(x) - C(x) = px - C = (240x - 0.15x^2) - C(x)$

a) $M.C = \frac{dC}{dx} = 100 + 0.1x$

$C'(600) = \$160/\text{unit}$

$C(600+1) - C(600) \approx C'(600)$

Cost of producing 601^{st} item is approximately $160.

$P(x) = 240x - 0.15x^2 - 9600 - 100x - 0.05x^2$

$= -9600 + 140x - 0.2x^2$

$\left.\frac{dP}{dt}\right|_{x=600} = 140\frac{dx}{dt} - 0.4x\frac{dx}{dt}\Big|_{x=600} = [140 - (0.4)(600)](80)$

$= -8000$

9. Find the first partial derivatives of the function $f(x, y) = -3xy + 6x + \ln(x + y) + 2y^3$ at the point (0, 1).

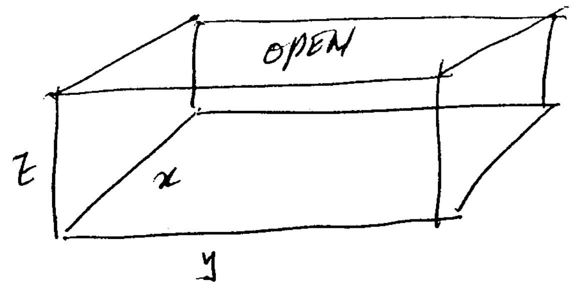
$f_x = \frac{\partial f}{\partial x} = -3y + 6 + \frac{1}{x+y} \cdot 1$

$f_x(0,1) = -3 + 6 + 1 = 4$

$f_y = \frac{\partial f}{\partial y} = -3x + \frac{1}{x+y} \cdot 1 + 6y^2$

$f_y(0,1) = 7$

10. An open rectangular box having a surface area of 300 cm^2 is to be constructed from a tin sheet. Find the dimensions of the box if the volume of the box is to be as large as possible. What is the maximum volume?



Surface area $= 300 = xy + 2xz + 2yz$

$$xy + 2(x+y)z = 300$$

$$\therefore\ z = \frac{300 - xy}{2(x+y)}$$

$$\text{Volume} = V(x,y) = xyz = \frac{300xy - x^2y^2}{2(x+y)}$$

$$V_x = \frac{(300y - 2xy^2)(2x+2y) - \left((300xy - x^2y^2)\cdot 2\right)}{4(x+y)^2}$$

$$V_y = \frac{(300x - 2x^2y)(2x+2y) - (300xy - x^2y^2)\cdot(2)}{4(x+y)^2}$$

$V_x = 0 \Rightarrow (300y - 2xy^2)(x+y) = (300xy - x^2y^2) \rightarrow (1)$

$V_y = 0 \Rightarrow (300x - 2x^2y)(x+y) = (300xy - x^2y^2) \rightarrow (2)$

(1) ÷ (2) $\quad 300y - 2xy^2 = 300x - 2x^2y$

$300(x-y) = 2xy(-y+x) \Rightarrow x = y$ or $\boxed{2xy = 300}$

↳ leads to a solution [illegible] Not possible

Put $x = y$ in (1) to get $x = y = 10,\ z = 5$

Since we have only one critical number

$$\therefore\ V_{max} = 10\cdot 10\cdot 5 = 500\ cm^3$$

11. Find the relative extrema of the function $f(x, y) = xy + \frac{4}{x} + \frac{2}{y}$.

$f_x = y - \frac{4}{x^2}$; $f_y = x - \frac{2}{y^2}$

$f_{xy} = 1$ $\qquad f_{yx} = 1$

$f_{xx} = \frac{8}{x^3}$ $\qquad f_{yy} = \frac{4}{y^3}$

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2$

$D_{xy} = \frac{32}{x^3 y^3} - 1$

For critical points

$f_x = 0 \longrightarrow y - \frac{4}{x^2} = 0 \Rightarrow x^2 y = 4$ (1)

$f_y = 0 \longrightarrow x - \frac{2}{y^2} = 0 \Rightarrow x y^2 = 2$ (2)

(1) $\div$ (2) $\Rightarrow \frac{x}{y} = 2$ or $x = 2y \to$ (3)

put (3) into (1) we get

$4y^3 = 4 \therefore y = 1$

$x = 2$

critical point $A = (2, 1)$

Since $D(A) = \frac{32}{2^3} - 1 = 3 > 0$ & $f_{xx}(A) = \frac{8}{8} = 1 > 0$

$\therefore$ Relative min at A & $f(2,1) = 6$

12. Evaluate the following limits if they exist. If they do not exist, indicate why not.

a) $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x^2 - 6x + 8}$

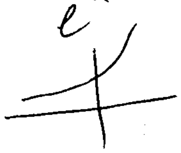
$\lim\limits_{x \to 4} \dfrac{\cancel{(x-4)}(x+4)}{(x-2)\cancel{(x-4)}}$

$\lim\limits_{x \to 4} \dfrac{x+4}{x-2} = 4$

b) $\lim\limits_{x \to 1} \dfrac{\sqrt{5+11x} - 4}{2x - 2} \cdot \dfrac{\sqrt{5+11x}+4}{\sqrt{5+11x}+4}$

$= \lim\limits_{x \to 1} \dfrac{5 + 11x - 16}{2(x-1)\left[\sqrt{5+11x}+4\right]}$

$= \lim\limits_{x \to 1} \dfrac{11\cancel{(x-1)}}{2\cancel{(x-1)}} \cdot \dfrac{1}{\left[\sqrt{5+11x}+4\right]} = \dfrac{11}{16}$



$\lim\limits_{x \to -\infty} e^x = 0$

c) $\lim\limits_{x \to -\infty} (x^3 - e^x)$

$= \lim\limits_{x \to -\infty} x^3 - \lim\limits_{x \to -\infty} e^x$

$= -\infty - 0$

$= -\infty$ / DNE

d) $\lim\limits_{x \to 5^+} \dfrac{15x - 3x^2}{|10 - 2x|}$

$= \lim\limits_{x \to 5^+} \dfrac{15x - 3x^2}{2x - 10}$

$= \lim\limits_{x \to 5^+} \dfrac{3x(5-x)}{2(x-5)} = \lim\limits_{x \to 5^+} \dfrac{-3x}{2}$

$= -15/2$

13. Use the linear approximation to approximate $63^{2/3}$. Use the second derivative test to check if your approximation is too big or too small compared to the real value.

$f(x) = x^{2/3}$; $f'(x) = \frac{2}{3} x^{-1/3}$; $f''(x) = -\frac{2}{9} x^{-4/3}$

$L(x) = f(a) + f'(a) \cdot (x - a)$

$a = 64$,

$L(x) = f(64) + f'(64)\,(x - 64)$

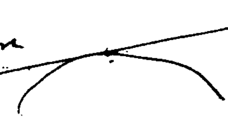
$L(x) = 16 + \frac{1}{6}(x - 64) \approx f(x)$

$f(63) = 63^{2/3} \approx 16 + \frac{1}{6}(63 - 64) = 16 - \frac{1}{6}$

$= 95/6$

$f''(64) < 0$

$\therefore$ f concave down



$\therefore$ approximation is too big.

$2x^2-12x+15=0$

$x=\frac{12\pm\sqrt{144-120}}{4}=3\pm\frac{\sqrt{24}}{4}=\begin{matrix}4.22\\1.78\end{matrix}$

14. Given $f(x)=x^3(x-5)^2$, $f'(x)=5x^2(x-5)(x-3)$, and $f''(x)=10x(2x^2-12x+15)$, sketch a complete graph of f. Be sure to clearly indicate all intercepts, relative extrema, concavity, and inflection points.

x-intercept 0, 5   y-intercept 0 ; $f'(x)=0 \Rightarrow x=0,3,5$

critical pts. $A\equiv(0,0)$, $B\equiv(3,108)$, $C\equiv(5,0)$

I	$(-\infty,0)$	$(0,3)$	$(3,5)$	$(5,\infty)$
Z	-1	1	4	10
$f'(x)$	+ve	+ve	−ve	+ve
f	↗	↗	↘	↗
FDT	Neither		Rel. max at B	Local min at C

$f''(x)=0 \Rightarrow x=0, 1.78, 4.22$

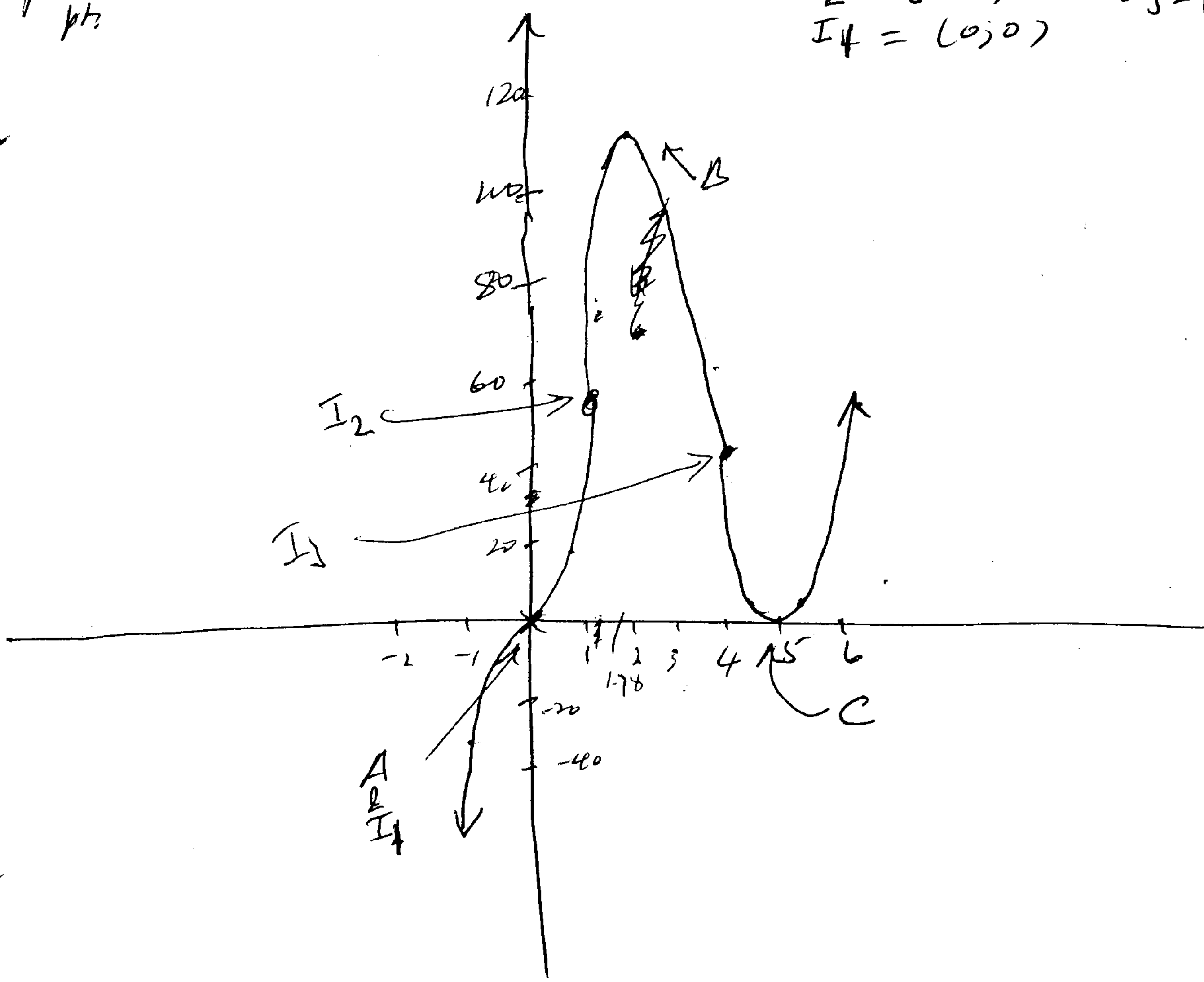
I	$(-\infty,0)$	$(0,1.78)$	$(1.78,4.22)$	$(4.22,\infty)$
Z	-1	1	2	10
$f''(x)$	−ve	+ve	−ve	+ve
f	∩	∪	∩	∪

Inflection pts

Possible inflection pts

$I_2=(1.78, 58)$ $I_3=(4.22, 46)$

$I_1=(0,0)$



15. Consider the function $f(x)=\dfrac{4(4+3x)}{(x+2)^2}$ and its derivatives $f'(x)=\dfrac{-4(3x+2)}{(x+2)^3}$ and $f''(x)=\dfrac{24x}{(x+2)^4}$. Sketch the graph of the function.

$D_f = (-\infty,-2) \cup (-2,\infty)$

x-intercept $= -4/3$

y-intercept $= 4$

V.A $x=-2$

H.A $y=0$

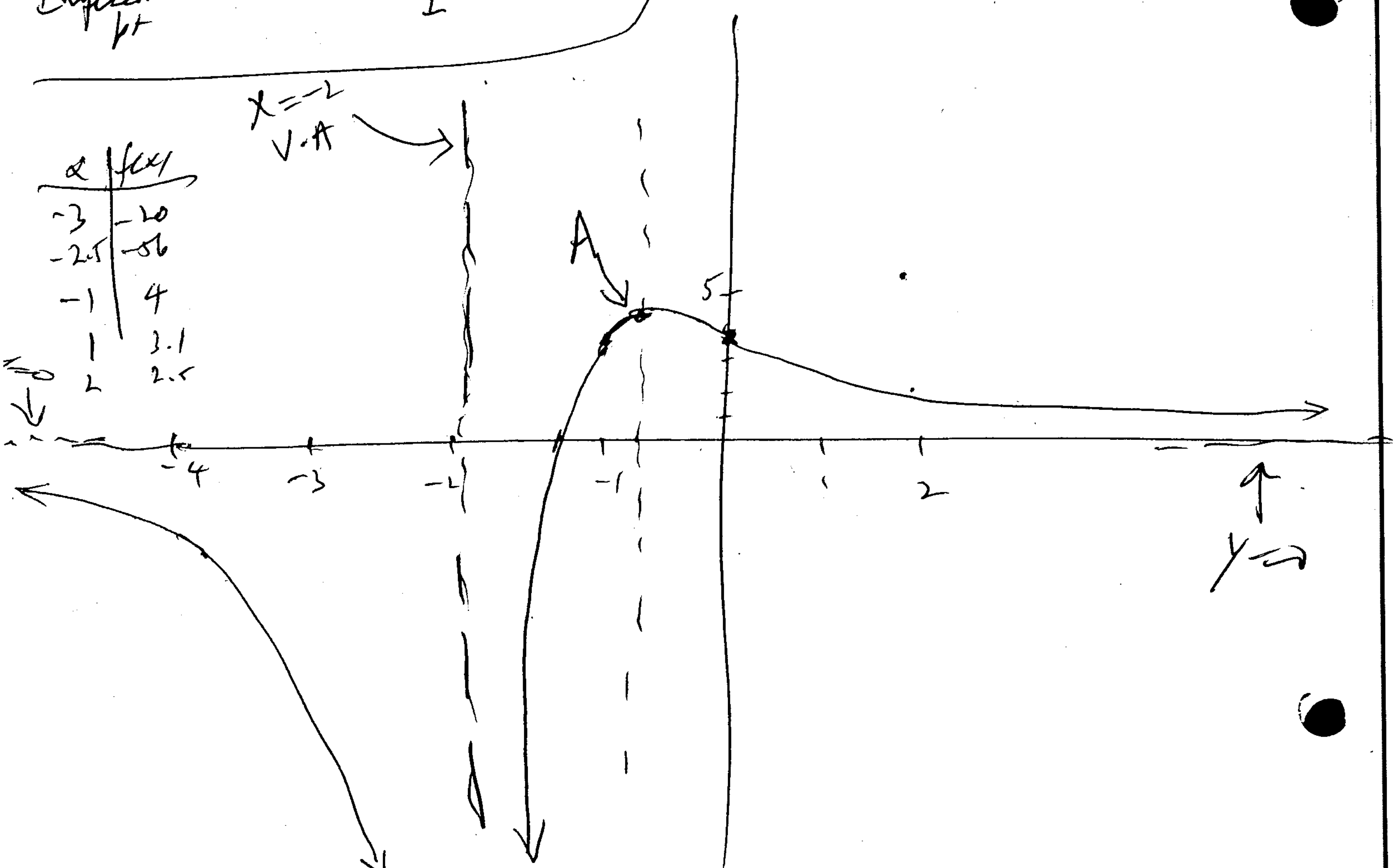
$f'=0 \Rightarrow x=-2/3$ critical pt. $A=(-2/3, 4.5)$

I	$(-\infty,-2)$	$(-2,-2/3)$	$(-2/3,\infty)$
z	-10	-1	0
$f'(z)$	$-ve$	$+ve$	$-ve$
f	↘	↗	↘
FDT		Rel. max at A	

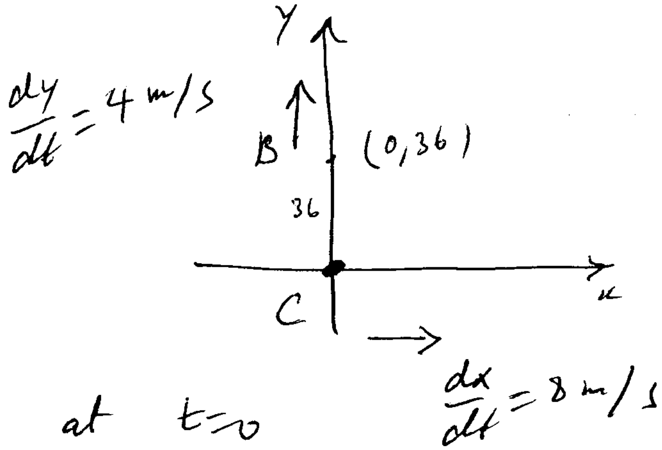
$f''=0 \Rightarrow x=0$ possible inflection pt. $I=(0,4)$

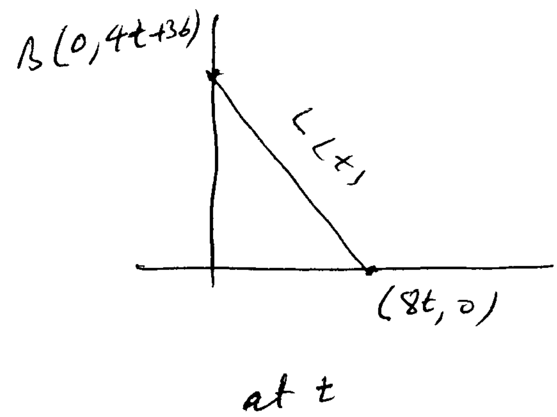
I	$(-\infty,-2)$	$(-2,0)$	$(0,\infty)$
z	-10	-1	1
$f''(z)$	$-ve$	$-ve$	$+ve$
f	$\cap$	$\cap$	$\cup$
Inflection pt			I

x	$f(x)$
-3	-20
-2.5	-56
-1	4
1	3.1
2	2.5



16. A balloon is rising at a constant speed 4m/sec. A boy is cycling along a straight road at a speed of 8m/sec. When he passes under the balloon, it is 36 metres above him. How fast is the distance between the boy and balloon increasing 3 seconds later?





$$[L(t)]^2 = (4t+36)^2 + (8t)^2$$

$\frac{d}{dt}$:

$$2L(t)\frac{dL}{dt} = 2\cdot(4t+36)\cdot 4 + 128t$$

$$L(t)\frac{dL}{dt} = 4(4t+36) + 64t$$

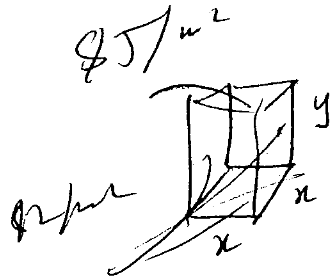
$$= 80t + 144.$$

$$\therefore \left.\frac{dL}{dt}\right|_{t=3} = \left.\frac{80t+144}{\sqrt{(4t+36)^2+(8t)^2}}\right|_{t=3}$$

$$= \frac{240+144}{\sqrt{(48)^2+(24)^2}} = \frac{\cancel{284}\ \cancel{142}\ 71}{\cancel{(24)}\ \cancel{12}\ 6\sqrt{5}}$$

$$= \frac{71}{6\sqrt{5}} \approx 5.3 \text{ m/s}$$

17. A closed rectangular box of volume 20 m^3 is to be constructed with a square base of width x. The material for the top costs \$5 per square metre whereas the material for the remaining sides costs \$2 per square metre. Find the cost of the box as a function of the width of the base. Find the dimensions of the cheapest box.



$$\text{Volume} = x^2 y = 20$$

$$\therefore y = \frac{20}{x^2}$$

$$\text{Total Cost} = 5x^2 + 2x^2 + 2\cdot 4xy$$

$$= 7x^2 + 8xy$$

$$C(x) = 7x^2 + \frac{160}{x}$$

$$C'(x) = 14x - \frac{160}{x^2} \qquad C' = 0 \Rightarrow x^3 = \frac{160}{14} = \frac{80}{7}$$

$$C''(x) = 14 + \frac{320}{x^3} \qquad x = \sqrt[3]{80/7} \approx 2.25 \text{ m}$$

$$y = \frac{20}{x^2} = 3.94 \text{ m}$$

Since $C''\left(\sqrt[3]{\frac{80}{7}}\right) > 0$, by SDT

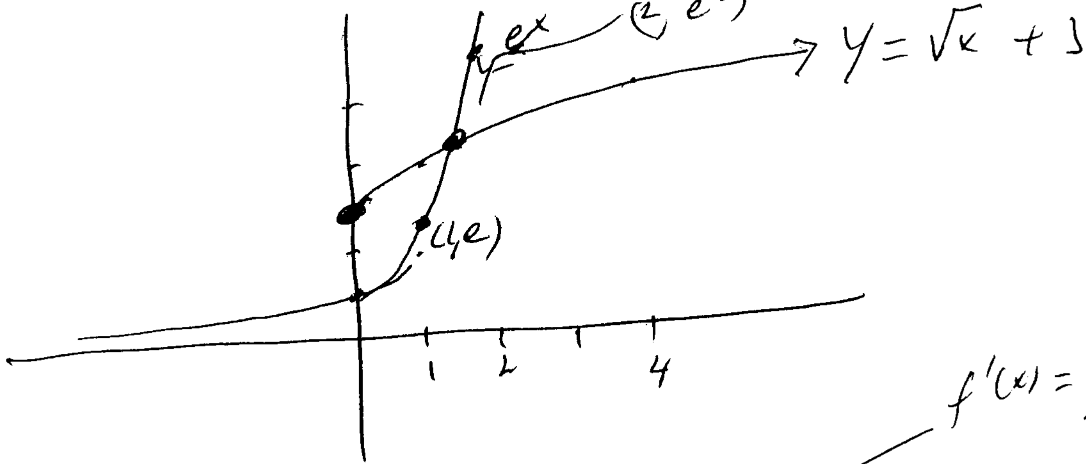
Rel. min & by CPT abs. min

when $x = \sqrt[3]{\frac{80}{7}}$.

Dimensions of the cheapest box are

$2.25 \times 2.25 \times 3.94$

18. Graph the functions $y = e^x$ and $y = \sqrt{x} + 3$. Use Newton's Method to approximate the x-value of intersection of the two graphs.



$f'(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} - e^x$

intersection $1 < x < 2$

Solve $f(x) = \sqrt{x} + 3 - e^x = 0$

$1 < \text{Root} < 2$

x	$f(x)$
1	> 0 (1.28)
2	< 0 (-2.9)

choose $x_1 = 1.5$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = N(x_1)$

$= 1.4369221$

~~$= 1.5 + 0.02540569$~~

~~$= 1.52540569$~~

$$\boxed{x \approx 1.43}$$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 1.4345457$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$= 1.4345424$

$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.4345424$

$N(x) = \frac{x f'(x) - f(x)}{f'(x)}$

$= \frac{x\left(\frac{1}{2} \cdot \frac{1}{\sqrt{x}} - e^x\right) - \sqrt{x} - 3 + e^x}{\frac{1}{2} \cdot \frac{1}{\sqrt{x}} - e^x}$

$= \frac{\frac{\sqrt{x}}{2} - x e^x - \sqrt{x} + 3 - e^x}{\frac{1}{2} \cdot \frac{1}{\sqrt{x}} \cdot e^x}$

$= x - \frac{(\sqrt{x} + 3 - e^x)}{\left(\frac{1}{2} \cdot \frac{1}{\sqrt{x}} - e^x\right)}$

$=$

$=$

$\left(\frac{dx}{dp} = 60(p-45) \right.$

19. A company's demand equation is given by $x = 30(p-45)^2$, where $0 \le p < 45$, and p is price in dollars.
a) Find the prices for which demand is *elastic*, and for which demand is *inelastic*.
b) Use *point elasticity of demand* to help you determine whether revenue will increase or decrease if the unit price is increased by 4% from an original price of \$10.
c) Find the price for the maximum revenue.

$$E(p) = -\frac{p}{x} \cdot \frac{dx}{dp} = -\frac{p}{30(p-45)^2} \cdot 60(p-45)$$

$$= \frac{2p}{45-p}$$

a) $E = 1 \Rightarrow 2p = 45 - p \Rightarrow p = 15$

$\therefore$ $E < 1$ if $0 \le p < 15$

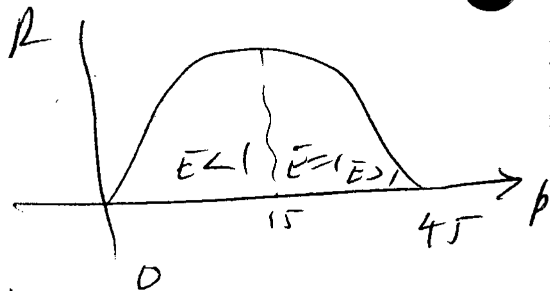
$E > 1$ if $45 > p > 15$

$0 \; (\; E<1 \; 15 \; , \; E>1 \;) \; 45$

inelastic elastic

b) $E(10) = \frac{20}{35} < 1$

$\therefore$ Revenue increases if price is increased slightly (4%)



c) Revenue is maximum when $E = 1$

i.e. $p = \$15$