Jiaxuan Xu (vivy)

PRACTICE TEST 2

$\perp$

$\sqrt[3]{x}$ ~~min~~

$\sqrt{-2}$ ≥ 0

8/10

9/10

$\sqrt{x}$

$x \geq 0$

1

1. Differentiate the following functions as indicated. [12 marks]

a) $f(x) = \sin(2x) + \tan(2x) + 16^x + ln|1-x|$, *find $f'(0)$.*

$f'(x) = 2\cos(2x) + 2\sec^2(2x) + \ln 16 \cdot 16^x + \frac{-1}{1-x}$

$f'(0) = 2\cos(0) + 2\sec^2(0) + \ln 16 \cdot 1 + \frac{-1}{1}$

$= 2 + 2 + \ln 16 - 1$

$= 3 + \ln 16$

$= 5.7726$

b) $f(x) = \sqrt{3x+4}^{\sqrt{(3x+4)}}$, find $f'(4)$.

$\ln y = \sqrt{(3x+4)} \ln \sqrt{3x+4}$

$\frac{y'}{y} = \frac{1}{2}(3x+4)^{-\frac{1}{2}} \cdot 3 \cdot \frac{1}{\sqrt{3x+4}} \cdot \frac{1}{2}(3x+4)^{-\frac{1}{2}} \cdot 3$

$\frac{y'}{y} = \frac{3}{2\sqrt{3x+4}} \cdot \frac{1}{\sqrt{3x+4}} \cdot \frac{1}{2\sqrt{(3x+4)}} \cdot 3$

$y' = \frac{9}{\cancel{2}\sqrt{3x+4} \cdot (3x+4)} \cdot \sqrt{3x+4}^{\sqrt{(3x+4)}}$

$f'(4) = \frac{9}{16 \cdot 16} \cdot \sqrt{16}^{4}$

$= \frac{9}{\cancel{256}} \cdot \cancel{256}$

$= 9$

2

1 c) **Use implicit differentiation to find $\frac{dy}{dx}$ at P(3,2) for the function $y = y(x)$, given by the equation:**

$$2x^2 + ye^{(x-3)} + 5y^2 = 40.$$

$ye^{(x-3)} + 5y^2 = 40 - 2x^2$

$ye^{(x-3)} \cdot dy + 10y \cdot dy = -4x \cdot dx.$

$dy\,(ye^{(x-3)} + 10y) = -4x \cdot dx$

$\frac{dy}{dx} = \frac{-4x}{ye^{(x-3)} + 10y}.$

$\frac{dy}{dx} = \frac{-4(3)}{2+20}.$

$\frac{dy}{dx} = \frac{\cancel{-12}\ 6}{\cancel{22}\ 11}$

$\frac{dy}{dx} = -\frac{6}{11}$

3

$5x - 8x^{-1}$

2. (a) **Find the linearization $L(x)$ of $f(x) = 5x - \frac{8}{x}$ at $a = 2$. [3 marks]**

$L(x) = f'(a)(x-a) + f(a)$ $\qquad f'(x) = 5 + \frac{8}{x^2}$

$= (5 + \frac{8}{2^2})(x-2) + [5(2) - \frac{8}{2}]$

$= 7(x-2) + 6$

$= 7x - 14 + 6$

$= 7x - 8$.

$\therefore L(x) = 7x - 8$.

(b) **Use $L(x)$ to approximate $f(1.95)$. [2 marks]**

$L(x) = 7(1.95) - 8$

$= 5.65$.

Horizontal Asymptote.
=> End Behavior.

4.

dy dx

3.Find the horizontal and vertical asymptotes of the graphs of the given functions. Do not sketch the graphs. [4 marks]

a) $f(x) = \frac{x^2+x-2}{x(x-1)}$

$= \frac{\cancel{(x-1)}(x+2)}{x\cancel{(x-1)}}$

$= \frac{x+2}{x}$. ∴ vertical asymptotes is $x=0$ ~~and~~. $\lim_{x\to\infty} \frac{x+2}{x}$.

$\lim_{x\to\infty} \frac{x+2}{x}$

$= \lim_{x\to\infty} \frac{1}{1}$

$=1$ ∴ horizontal asymptotes is $y=1$

b) $f(x) = \frac{6e^{2x}}{e^{2x}-1}$.

$e^0=1$

$e^{2x}-1 \neq 0$

$e^{2x} \neq 1$

$2x \neq 0$

$x \neq 0$

∴ vertical asymptotes is $x=0$.

$\lim_{x\to\infty} \frac{6e^{2x}}{e^{2x}-1}$

$= \cancel{\lim 12x \ln e}$

$\cancel{\ln(e^{2x}-1)}$

$= \lim_{x\to\infty} \frac{6\cancel{e^{2x}}\cdot\cancel{x}}{\cancel{e^{2x}}\cdot\cancel{x}}$

$= \lim_{x\to\infty} 6$

$=6$

$\lim_{x\to-\infty} \frac{6e^{2x}}{e^{2x}-1} = \frac{0}{0-1} = 0.$

$y=0$ H.A.

5

4. Consider the equation $f(x) = x + \ln x - 7 = 0$. [6 marks]

(a) Explain that there is a root r in the interval (5,6).

$f(5) = \ln 5 - 2 < 0$ there is a root in (5,6).

$f(6) = \ln 6 - 1 > 0$

By the Intermediate Value [IVT]

(b) Explain that there are no other real roots.

$f' = 1 + \frac{1}{x} > 0$

$\therefore f$ increasing $(0, \infty)$

it only intercepts

the x-axis cannot be others.

(c) Use Newton's method to approximate r to the nearest hundredth. Explain the accuracy.

$N = x - \frac{x + \ln x - 7}{1 - \frac{1}{x}}$

$= \frac{8 - \ln x}{1 - x}$

$x_1 = 5.1$

$\frac{23.7 - 6.5.1}{f(5.1)} = N(5.1)$

$= 5.327178.$

+ ↓ -

6

5. A rock is thrown into a still pond. The circular ripples move outward from the point of impact of the rock so that the radius of the circle formed by a ripple increases at the rate of 1.5m ~~per minute.~~ 1.5m/min Find the rate at which the area is changing at the instant the radius is 20m.[6 marks]

$S = \pi r^2$

$\frac{ds}{dt} = 2r\pi \cdot \frac{dr}{dt}$

$\frac{ds}{dt} = 2 \cdot 20 \cdot \pi \cdot 1.5$

$\frac{ds}{dt} = 60\pi \ m^2/min$

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6. Consider the function $f(x) = \frac{2x^2}{x^2+3}$; $f'(x) = \frac{12x}{(x^2+3)^2}$; $f''(x) = \frac{36(1-x^2)}{(x^2+3)^3}$.
[12 marks]

(a) Find the domain , intercepts, symmetry and asymptotes of f.

$(-\infty, \infty)$

$0 = \frac{2x^2}{x^2+3}$ $\quad f(0) = \frac{20^2}{0^2+3}$

$2x^2 = 0$ $\quad = 0$

$x = 0$

$f(-x) = \frac{2(-x)^2}{(-x)^2+3}$

$= \frac{2x^2}{x^2+3} = f(x)$

Have symmetry

$\therefore$ intercepts is $x=0$
$y=0$

2

asymptotes :

$\lim_{x\to+\infty} \frac{2x^2}{x^2+3}$

$= \lim_{x\to+\infty} \frac{4x}{2x+3}$

$= \lim_{x\to+\infty} \frac{4}{5}$

$= \frac{4}{5}$

$\therefore$ Horizontal asymptotes is $\frac{4}{5}$

$\lim_{x\to0} = \frac{0}{3}$

$= 0$

$\therefore$ NO Vertical asymptotes.

(b) Find the intervals where the function is increasing or decreasing and the relative extrema.

$f'(x) = \frac{12x}{(x^2+3)^2}$

$0 = \frac{12x}{(x^2+3)^2}$

$x = 0$

$f(0) = \frac{20^2}{0^2+3}$

$= 0$

$\therefore$ the relative extrema is $(0,0)$

$f'(-1) < 0$, $f'(1) > 0$

$\searrow$ 0 $\nearrow$

$\therefore$ When $x < 0$ is decreasing.
when $x > 0$ is increasing.

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(c) Find the intervals where the function is concave upward or concave downward, and the inflection points.

$$f''(x) = \frac{36(1-x^2)}{(x^2+3)^3}$$

$$0 = \frac{36(1-x^2)}{(x^2+3)^3}$$

$0 = 1 - x^2$

$x^2 = 1$

$x = 1$

$\therefore$ concave upward when $x < 1$.

concave downward when $x > 1$.

$$f(x) = \frac{2x^2}{x^2+3}$$

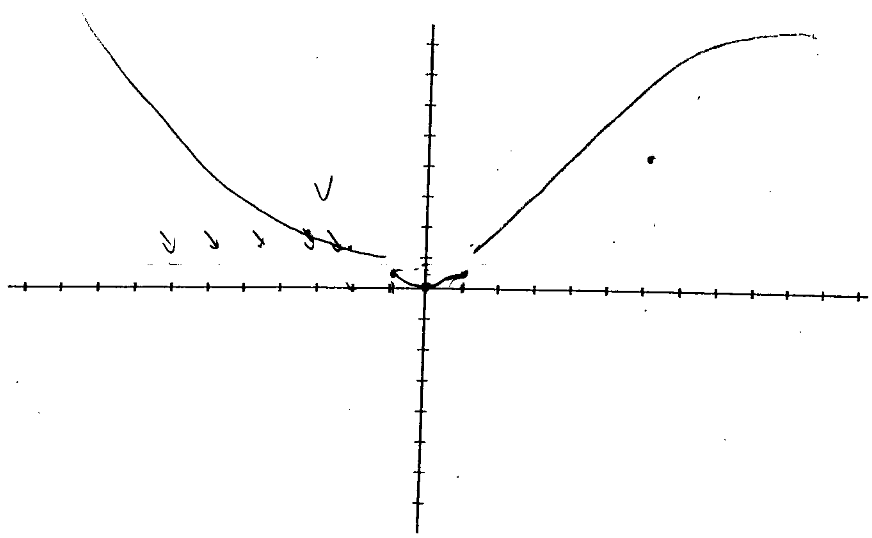
$$f(1) = \frac{2}{4} = \frac{1}{2}$$

$\therefore$ inflection point is $(1, \frac{1}{2})$.

$f''(0) > 0$, $f''(2) < 0$.

$\cup \quad 1 \quad \cap$

d) Graph the function.



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In planning a restaurant, it is estimated that a profit of \$6 per seat will be made if the number of seats is less than or equal to 50. On the other hand, the profit will decrease by 10 cents for each seat above 50. Find the number of seats that will produce the maximum profit. What is the maximum profit? [7 marks]

0.1

dp

quantity $= x$

when $x \le 50$

$6x = P$

when $x > 50$

$[6 - 0.1(x-50)]x = P$

$[6 - 0.1x + 5]x = P$

$[11 - 0.1x]x = P$

$11x - 0.1x^2 = P$

$-0.1x^2 + 11x = P$

$a = -0.1 \quad b = 11$

$x = -\frac{b}{2a} = -\frac{11}{-0.2}$

$x = \frac{11}{0.2}$

$x = 55$

$P = -0.1(55)^2 + 11(55)$

$= 302.5$

$\therefore$ the maximum profit is 302.5.

$\therefore x = 50$

$6 \cdot 50 = 300$

300 is the maximum profit

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4. ~~3.~~ Consider the equation $f(x) = x^3 + 6x - 29 = 0$. [5 marks]

(a) Explain that there is a root r in the interval (2,3).

$f(2) = 2^3 + 6 \cdot 2 - 29$

$= -9 < 0$

$f(3) = 3^3 + 6 \cdot 3 - 29$

$= 16 > 0$ ∴ There is the root in the (2,3)

By the [IVT]

(b) Use Newton's method to approximate r to the nearest hundredth. Explain the accuracy.

$x_1 = x - \dfrac{x^3 + 6x - 29}{3x^2 + 6}$

$x_1 = 3 - \dfrac{3^3 + 6 \cdot 3 - 29}{3 \cdot 3^2 + 6}$

$x_1 = 2.52$

$x_2 = 2.52 - \dfrac{2.52^3 + 6 \cdot 2.52 - 29}{3 \cdot 2.52^2 + 6}$

$= 2.4352$

$x_3 = 2.4352 - \dfrac{2.4352^3 + 6 \cdot 2.4352 - 29}{3 \times 2.4352^2 + 6}$

$= 2.432997$

∴ $r = 2.43$

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4. Consider the function $f(x) = x^4 - 6x^2 + 18$ [12 marks]

a) On which intervals is $f(x)$ increasing or decreasing?

$f'(x) = 4x^3 - 12x$

$0 = 4x^3 - 12x$

$0 = 4x(x^2 - 3)$

$x^2 = 3$ or $x = 0$

$x = \pm\sqrt{3}$

$f'(-2) < 0$, $f'(-1) > 0$, $f'(1) < 0$, $f'(2) > 0$

$\searrow\ -\sqrt{3}\ \nearrow\ 0\ \searrow\ \sqrt{3}\ \nearrow$

$\therefore$ when $x < -\sqrt{3}$ is decreasing.
when $-\sqrt{3} < x < 0$ is increasing
when $0 < x < \sqrt{3}$ is decreasing.
when $\sqrt{3} < x$ is increasing.

b) On which intervals is $f(x)$ concave up or down?

$f''(x) = 12x^2 - 12$

$0 = 12x^2 - 12$

$0 = 12(x^2 - 1)$

$0 = (x+1)(x-1)$

$x = -1$ or $x = 1$

$f''(-2) > 0$, $f''(0) < 0$, $f''(2) > 0$

$\cup\ -1\ \cap\ 1\ \cup$

$\therefore$ when $x < -1$ is concave up.
when $-1 < x < 1$ is concave down.
when $x > 1$ is concave up

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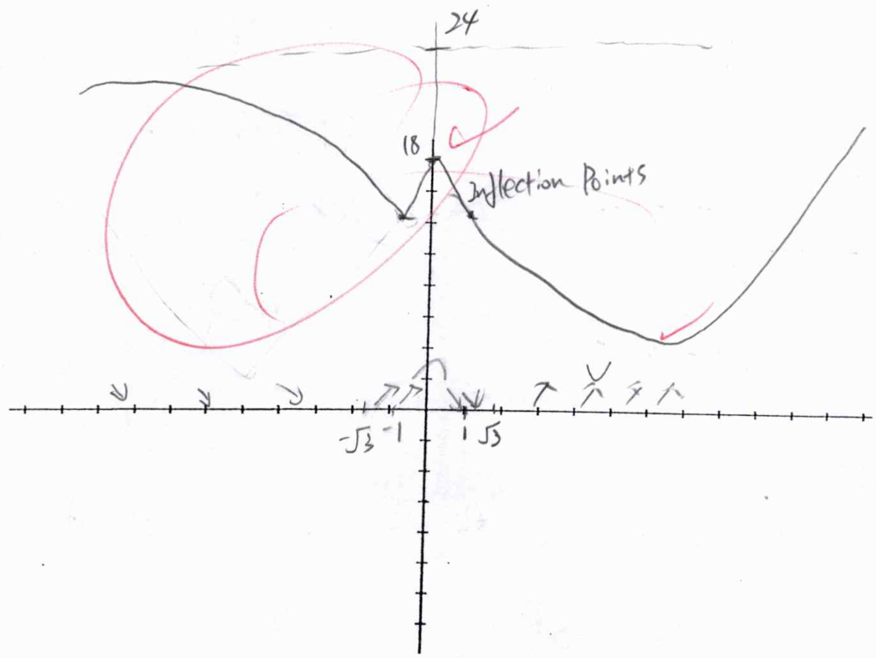
c) Sketch the graph of $f(x)$ below. Label any intercepts, relative minima, relative maxima and inflection points.



Intercepts :

$f(x) = x^4 - 6x^2 + 18$

$0 = x^4 - 6x^2 + 18$ No x intercepts

$f(0) = 0 - 0 + 18$

$= 18$ ∴ y intercepts 18.

$\lim_{x\to\infty} x^4 - 6x^2 + 18$

$= \lim_{x\to\infty} 4x^3 - 12x$

$= \lim_{x\to\infty} 12x^2 - 12$

$= \lim_{x\to\infty} 24x$

$= \lim_{x\to\infty} 24$

$= 24$

∴ H.A asymtoptote is 24.

No V.A.

$f(-1) = (-1)^4 - 6(-1)^2 + 18$

$= 1 - 6 + 18$

$= 13$

∴ inflection point is $(-1, 13)$ and $(1, 13)$

(13)

10. A company needs to design cylindrical metal containers with a volume of $32\pi\ m^3$. The top and bottom will made of a sturdy material that costs \$2 per m^2, while the material for the side costs \$1 per m^2. Find the radius, height, and cost of the least expensive ~~container.~~ [5 marks]

$V = 32\pi m^3$ $\quad V = r^2\pi \cdot h$

side $= 2\pi r \cdot h$

The top and botton $= 2r^2\pi = 2\frac{V}{h}$

$32\pi = r^2\pi h$

$32 = r^2 h$

$h = \frac{32}{r^2}$

the cost of top and bottom $= 4\pi r^2$

The cost of side $= 2\pi r h = 2\pi r \cdot \frac{32}{r^2} r$

all the cost $= 4\pi r^2 + \frac{64\pi}{r}$

$S' = 8\pi r - \frac{64\pi}{r^2}$

when $S' = 0$

$8\pi r - \frac{64\pi}{r^2} = 0$

$8\pi r = \frac{64\pi}{r^2}$

$r = \frac{8}{r^2}$

$r^3 = 64$

$r = 2$

$\therefore h = \frac{32}{2^2}$

$= 8$

$S = 4\pi \cdot 2^2 + \frac{64\pi}{2}$

$= 16\pi + 32\pi$

$= 48\pi$

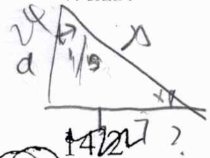
$\therefore$ radius is 2

height is 8

cost is 48π

More questions - (14)

(11) The top of a 25-foot ladder, leaning against a vertical wall is slipping down the wall at the rate of 1 foot per second. How fast is the bottom of the ladder slipping along the ground when the bottom of the ladder is 7 feet away from the base of the wall?



$25^2 = a^2 + b^2$

$25^2 = a^2 + 7^2$

$a = 24$

$0 = 2a \cdot \frac{da}{dt} + 2b\frac{db}{dt}$

$0 = 2 \cdot 24 \cdot 1 + 2 \cdot 7 \cdot \frac{db}{dt}$

$-48 = 14\frac{db}{dt}$

$\frac{db}{dt} = -3.4287$ ✓

(12) 圆柱体 A cylindrical tank of radius 10 feet is being filled with wheat at the rate of 314 $= \frac{dv}{dt}$ = V/s cubic feet per minute. How fast is the depth (深度) of the wheat increasing?

$r = 10$ $\quad V = r^2\pi \cdot h$ $\quad$ h.

$V = 100\pi h$

$314 \div 10^2 \div \pi = h$

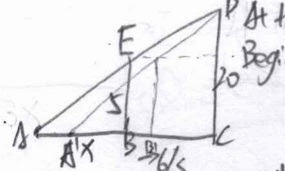
$3.14 \div \pi = h$

$h = 1$ feet.

(13) ~~14.3 and 14.4~~

A 5-foot girl is walking toward a 20-foot lamppost at the rate of 6 feet per second. How fast is the tip of her shadow (cast by the lamp) moving?

How fast is the length of the girl's shadow changing?



At the Begin: $AB = x$ $\quad BC = 3x$

$\vec{BC} = 3x - 6s$

$\vec{AB} = \frac{3x - 6s}{3} = x - 2s$

$\vec{B'C} = 3x - 6s + (x - 2s) = 4x - 8s$

$AC = 4x$

$AA' = 4x - 4x + 8s = 8s$

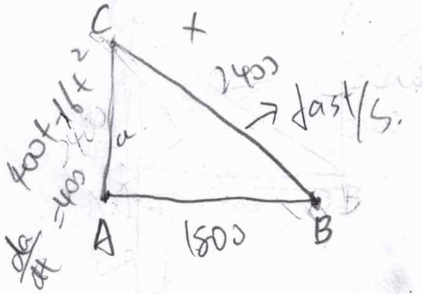
∴ the fast is $\frac{8s}{s} = 8$ foot/s.

$V_s = 400$

x is the distance, and s is time

(14) ~~14.5~~

A rocket is shot vertically upward with an initial velocity of 400 feet per second. Its height s after t seconds is $s = 400t - 16t^2$. How fast is the distance changing from the rocket to an observer on the ground 1800 feet away from the launching site, when the rocket is still rising and is 2400 feet above the ground?



$AC = \sqrt{2400^2 - 1800^2}$

$= \sqrt{2520000} = 600\sqrt{7}$

$600\sqrt{7} = 400t - 16t^2$

$x_1 = -3.4833$ (x)

$x_2 = 28.4833$

$(400t - 16t^2)^2 + 1800^2 = x^2$

$2(400t - 16t^2) \cdot (400 - 32t) + 1800^2 = 2x \cdot \frac{dx}{dt}$

$(800t - 16t^2)(400 - 32t) + 1800^2 = 2x \cdot \frac{dx}{dt}$

$(800t - 16t^2)(400 - 32t) + 3240000 = 2x \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = -369.8673$