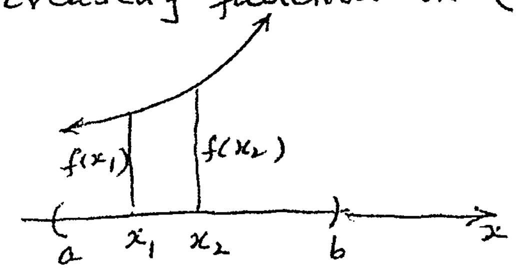
LECTURE 9

Relative Extrema & Curve Sketching

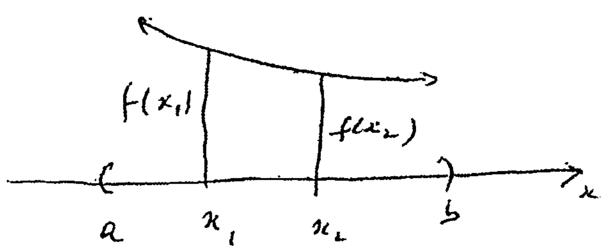
5.1 — 5.4

f is an increasing function on (a,b)

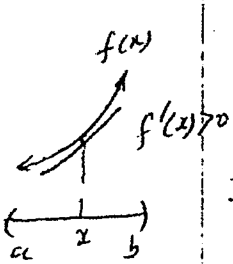


if $f(x_1) < f(x_2)$ whenever $a < x_1 < x_2 < b$.

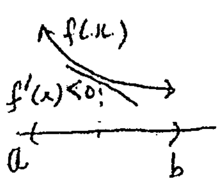
f is a decreasing function on (a,b)



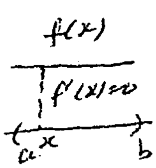
if $f(x_1) > f(x_2)$ whenever $a < x_1 < x_2 < b$.



If $f'(x) > 0$ for each x in (a,b), then f is increasing on (a,b)

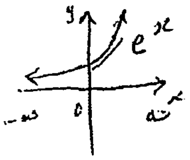


If $f'(x) < 0$ for each x in (a,b), then f is decreasing on (a,b)

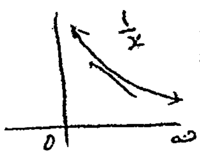


If $f'(x) = 0$ for each x in (a,b), then f is constant on (a,b).

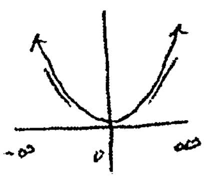
<u>Examples</u>



$f(x) = e^x$ is increasing on $(-\infty, \infty)$ since $f'(x) = e^x > 0$ for all x in $(-\infty, \infty)$.



$f(x) = \frac{1}{x}$ is decreasing on $(0, \infty)$ since $f'(x) = -\frac{1}{x^2} < 0$ for all x in $(0, \infty)$.



$f(x) = x^2$ is increasing on $(0, \infty)$ since $f'(x) = 2x > 0$ for all x in $(0, \infty)$ and $f(x) = x^2$ is decreasing on $(-\infty, 0)$ since $f'(x) = 2x < 0$ for all x in $(-\infty,$

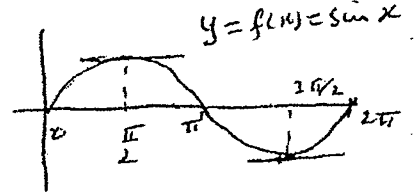
Critical numbers and critical points

c is a critical number for a function f if

(1) c is in the domain of f and

(2) $f'(c) = 0$ or $f'(c)$ does not exist.

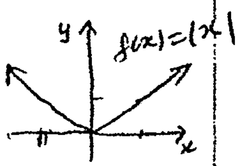
A critical point is then $(c, f(c))$.



E.g.

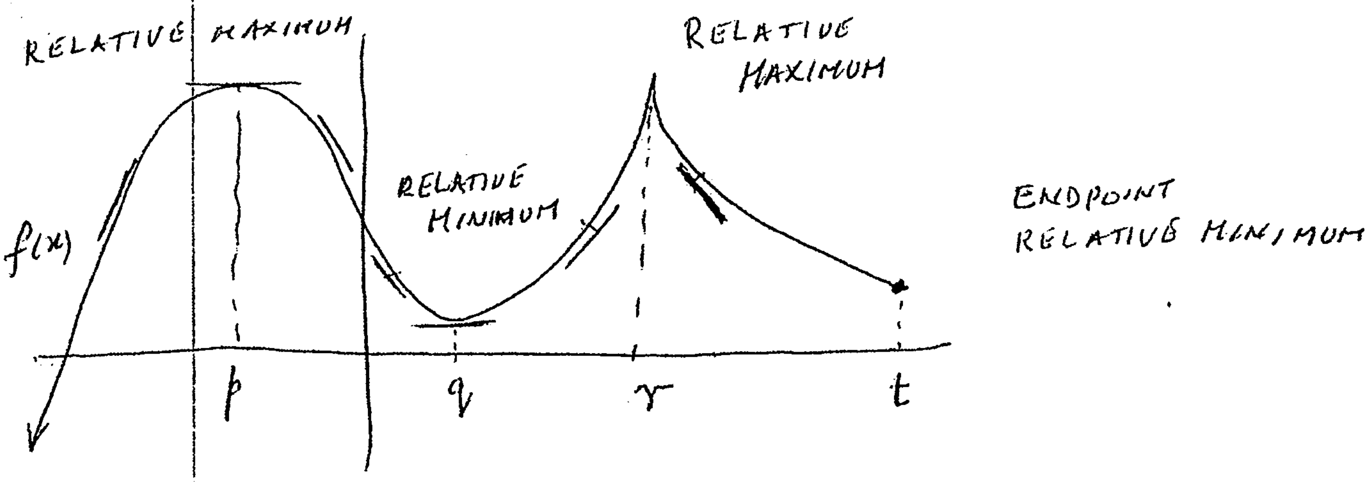
$f(x) = \sin x \qquad 0 \leq x \leq 2\pi$

has critical numbers at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, since $f'(x) = \cos x$ is zero at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.



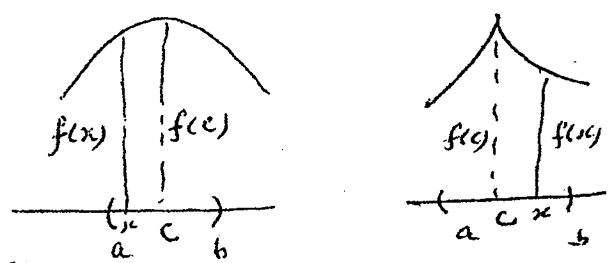
$f(x) = |x|$ has a critical number at 0 since f is not differentiable at 0 i.e. $f'(0)$ does not exist

Relative Extrema



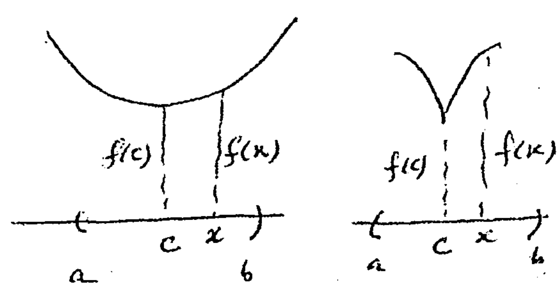
$f'(p)=0$, $f'(q)=0$ $f'(r)$ does not exist (t, $f(t)$) is the endpoint.

Relative Maximum



Let c be a critical number of function f.
Then $f(c)$ is a
RELATIVE (or LOCAL) MAXIMUM
if there exists an open interval (a,b) containing c such that $f(x) \leq f(c)$ for all x in (a,b).

Relative Minimum



Let c be a critical number of function f. Then $f(c)$ is
RELATIVE (or LOCAL) MINIMUM
if there exists an open interval (a,b) containing c such that $f(x) \geq f(c)$ for all x in (a,b).

Notes (1) A function has a relative (or local) extremum (plural: extrema) at c if it has either a relative maximum or a relative minimum there.

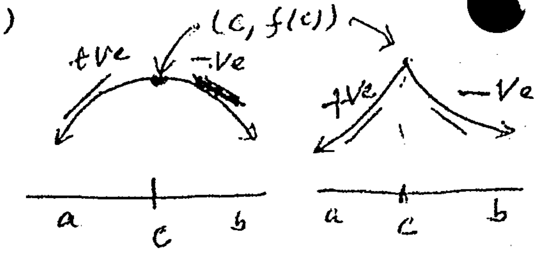
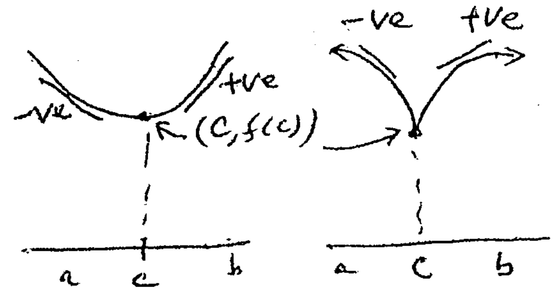
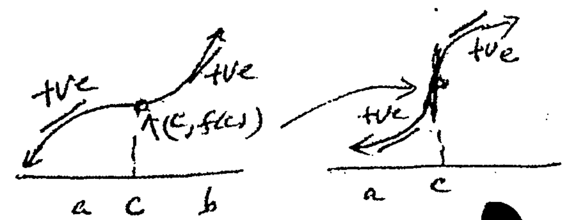
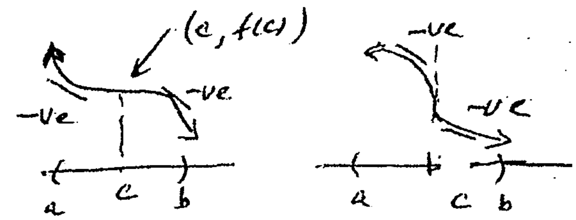
(2) If c is an endpoint of the domain of f, we only consider x in the half-open interval that is in the domain.

First Derivative Test (FDT)

Let c be a critical number for a function f and let $a < c < b$ such that c is the only critical number in (a,b). Further suppose that f is continuous on (a,b) and differentiable on (a,b) except possibly at c. Then

1. $f(c)$ is a relative maximum of f if the derivative $f'(x)$ is positive in the interval (a,c) and negative in the interval (c,b).

2. $f(c)$ is a relative minimum of f if the derivative $f'(x)$ is negative in the interval (a,c) and positive in the interval (c,b).

f(x) has:	Sign of f' in (a, c)	Sign of f' (c, b)	Sketches
Relative Maximum	+ve	−ve	
Relative Minimum	−ve	+ve	
No Relative Extremum	+ve	+ve	
No Relative Extremum	−ve	−ve	

Higher Derivatives of $y = f(x)$

First Derivative $\quad f'(x) = \frac{dy}{dx} = D_x f(x) = D_x y(x)$

Second Derivative $\quad f''(x) = [f'(x)]'$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$D_x^2[f(x)] = D_x[D_x[f(x)]]$$

Third Derivative $\quad f'''(x) = [f''(x)]'$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

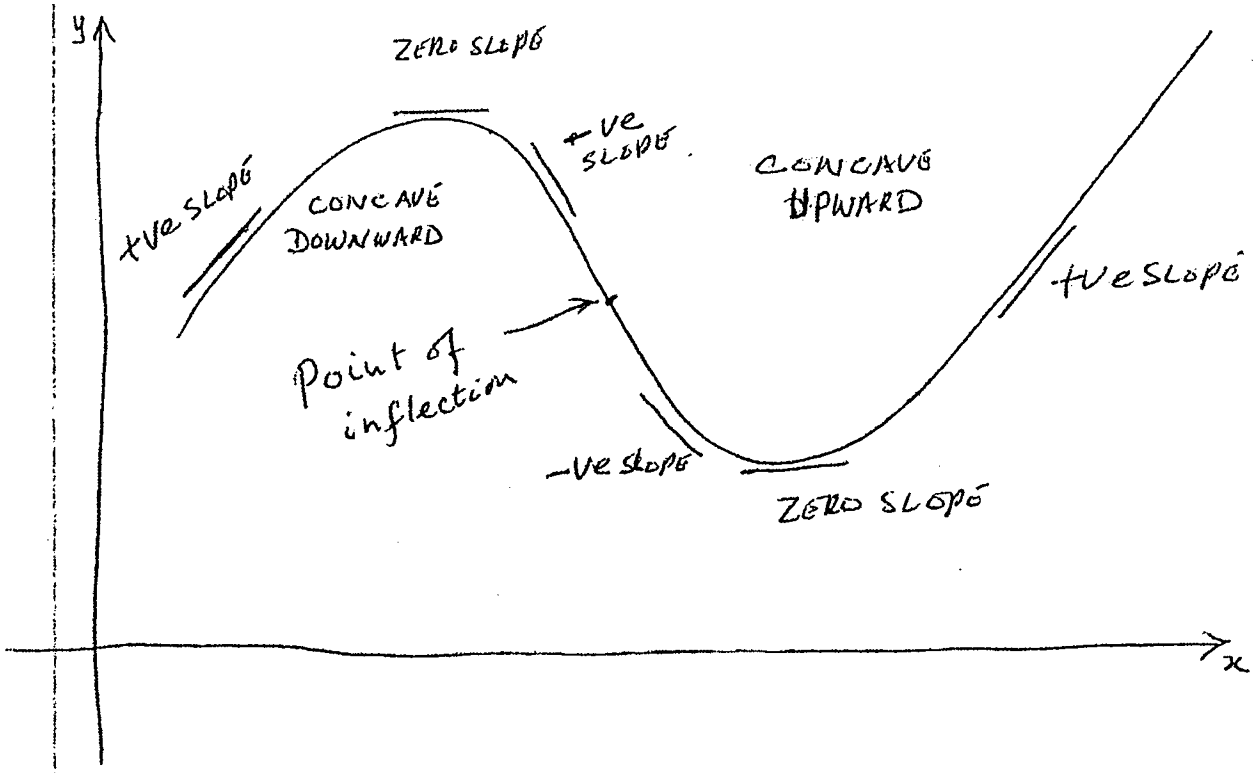
$$D_x^3[f(x)] = D_x[D_x^2[f(x)]]$$

For $n \geq 4$

the n^{th} derivative $\quad f^{(n)}(x) = [f^{(n-1)}(x)]'$

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$$

$$D_x^n[f(x)] = D_x[D_x^{n-1}[f(x)]]$$



CONCAVITY

Let f be a function with derivatives f' and f'' existing at all points in an interval (a,b). Then

1. f is concave upward on (a,b) if $f''(x) > 0$ for all x in (a,b) and

2. f is concave downward on (a,b) if $f''(x) < 0$ for all x in (a,b).

A point where a graph changes <u>concavity</u> is called a <u>point of inflection</u>.

At a point of inflection for a function f, the second derivative is 0 or does not exist.

Second Derivative Test

Let f'' exist on some open interval containing c, and $f'(c) = 0$.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.
3. If $f''(c) = 0$, then the test gives no information about extrema. We use the First Derivative Test to determine extrema.