Lecture 11
Curve Sketching 5.4

After the study of sections 5.1 – 5.3, we are now in a position to sketch the graph of a function f given by $y = f(x)$.

Curve sketching may be done with some or all of the following steps.

1. Find the domain
2. Find all x- and y- intercepts. ($y=0$) ($x=0$.)
3. Find all vertical and horizontal asymptotes.
4. Find if f is an even or an odd function to discuss symmetry.
5. Find all critical numbers and intervals of increase or decrease and relative extrema.
6. Find all intervals where f is concave upward or concave downward and inflection points.
7. Plot the domain, the intercepts, the asymptotes, the critical points, the inflection points, some additional points as needed, and using the symmetry (even/odd) of f, connect the points with a smooth curve.

This completes the sketching of the graph of a function f.

Examples 5.4

Use the curve sketching techniques (steps 1–7) to sketch the graph of the following functions.

A. $y = f(x) = 4x^3 - 9x^2 - 30x + 6$

1. Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$

2. y-intercept $= f(0) = 6$

To find x-intercepts solve $f(x) = 0$. f is a cubic function and therefore has at most <u>three</u> real roots.

-3	-2	-1	x	0	1	2	3	4	5
-93	-2	23	$f(x)$	6	-29	-58	-57	-2	131

Using the Intermediate Value Theorem, we observe that there is a root in the following three intervals:

$(-2, -1)$, $(0, 1)$, $(4, 5)$

With the help of a graphical calculator or otherwise the three x-intercepts are -1.96, 0.19, 4.02 approximately.

3. Since $f(x)$ is a polynomial, it has no vertical or horizontal asymptotes. However its end behavior is like $4x^3$.

4.

$f(x)$	$f(-x)$	$-f(-x)$
$4x^3-9x^2-30x+6$	$-4x^3-9x^2+30x+6$	$4x^3+9x^2-30x-6$
	$f(x)\neq f(-x)$ f is not even	$f(x)\neq -f(-x)$ f is not odd

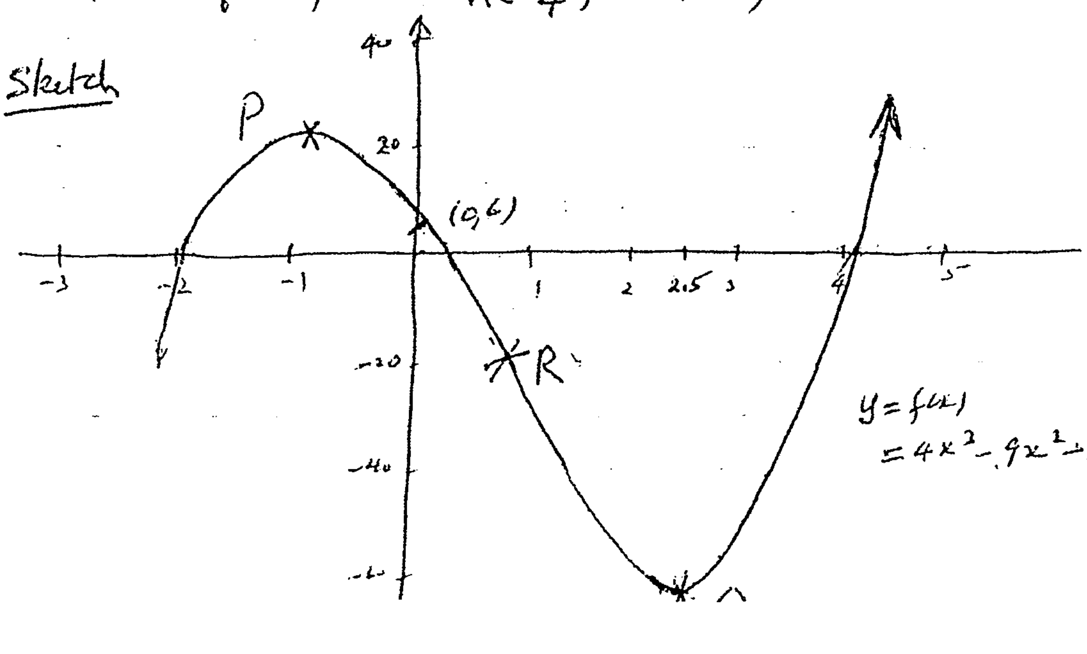
5. critical points : $(-1, 23)$ and $(\frac{5}{2}, -62.75)$ see Lecture 10
Intervals of increase: $(-\infty,-1)$ and $(\frac{5}{2},\infty)$
Interval of decrease : $(-1,\frac{5}{2})$

$$\nearrow \quad -1 \quad \searrow \quad 5/2 \quad \nearrow$$

Relative maximum at $P(-1, 23)$.
Relative minimum at $Q(5/2, -62.75)$

6. Concave downward on $(-\infty,\frac{3}{4})$. see Lecture 10
Concave upward on $(\frac{3}{4},\infty)$
point of inflection $R(\frac{3}{4}, -19.88)$

7. Sketch



$y=f(x)$
$=4x^3-9x^2-30x+6$

B. $y = f(x) = x\sqrt{4-x^2}$

1. Domain $[-3, 3]$

2. y-intercept $= f(0) = 0$
   x-intercepts: $-3, 0, 3$

3. No vertical or horizontal asymptotes. (Why?)

4.

$f(x)$	$f(-x)$	$-f(-x)$
$x\sqrt{9-x^2}$	$-x\sqrt{9-x^2}$	$x\sqrt{9-x^2}$
	$f(-x) \neq f(x)$	$f(x) = -f(-x)$
	f is not even	f is an odd function

Since f is an odd function, the graph is symmetric about the origin.

5. From Lecture 10:
   critical points: $P(-3,0)$, $Q(-\frac{3\sqrt{2}}{2}, -\frac{9}{2})$, $R(\frac{3\sqrt{2}}{2}, \frac{9}{2})$, $S(3,0)$

   Relative maxima points: P and R

   Relative minima points: Q and S.

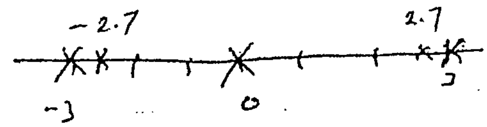
6. $f'(x) = \dfrac{9-2x^2}{\sqrt{9-x^2}}$

$$f''(x) = \frac{(-4x)\sqrt{9-x^2} - (9-2x^2)\cdot\frac{1}{2}\frac{1}{\sqrt{9-x^2}}\cdot -2x}{9-x^2}$$

$$= \frac{(-4x)(9-x^2) - x(9-2x^2)}{(9-x^2)^{3/2}} = \frac{6x^3-45x}{(9-x^2)^{3/2}}$$

$$f''(x) = \frac{3x(2x^2-15)}{(9-x^2)^{3/2}}$$

$f''(x) = 0$ at $x = 0, \pm\sqrt{\frac{15}{2}} = \pm 2.7$
$f''(x)$ doesn't exist at $x = \pm 3$

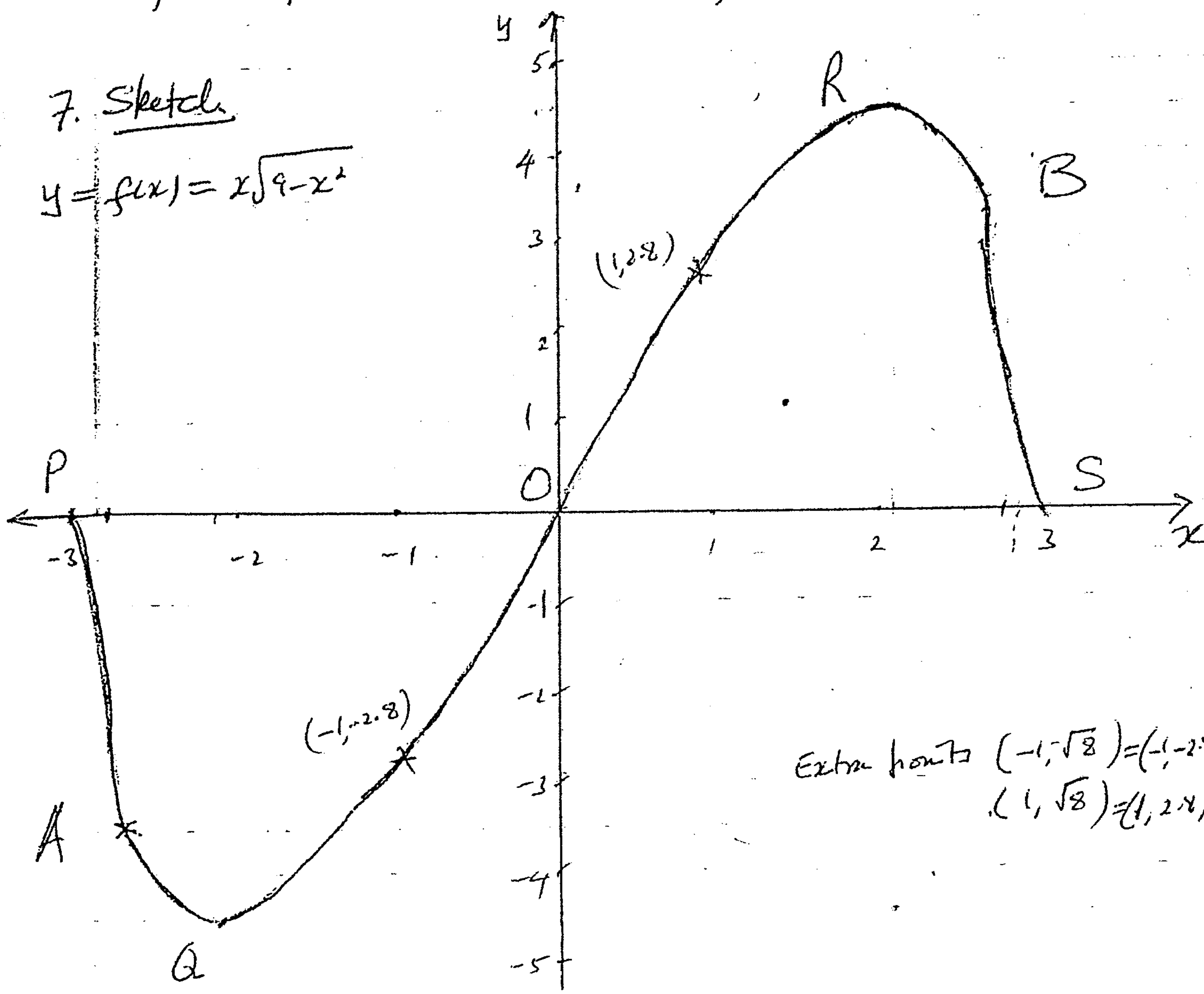


Interval	$(-3, -2.7)$	$(-2.7, 0)$	$(0, 2.7)$	$(2.7, 3)$
Sign of f''	$f''(-2.9) < 0$	$f''(-1) > 0$	$f''(1) < 0$	$f''(2.9)$
concavity	downward	upward	downward	upward

Concavity changes at $x \approx -2.7$, $x = 0$, and $x \approx 2.7$
Inflection points are $A(-2.7, -3.5)$, $O(0,0)$ and $B(2.7, 3.5)$

7. <u>Sketch</u>

$y = f(x) = x\sqrt{9-x^2}$



Extra points $(-1, -\sqrt{8}) = (-1, -2.8)$
$(1, \sqrt{8}) = (1, 2.8)$

C $y = f(x) = x - 4\ln(3x-9)$

1. Domain : $x > 3$

2. No y-intercept. x-intercept $\simeq 3.87912\ldots$ and $\simeq 13.96$

3. Vertical asymptote $x = 3$,

4. Since domain is $x > 3$, there is no possibility of even/odd symmetry (why?)

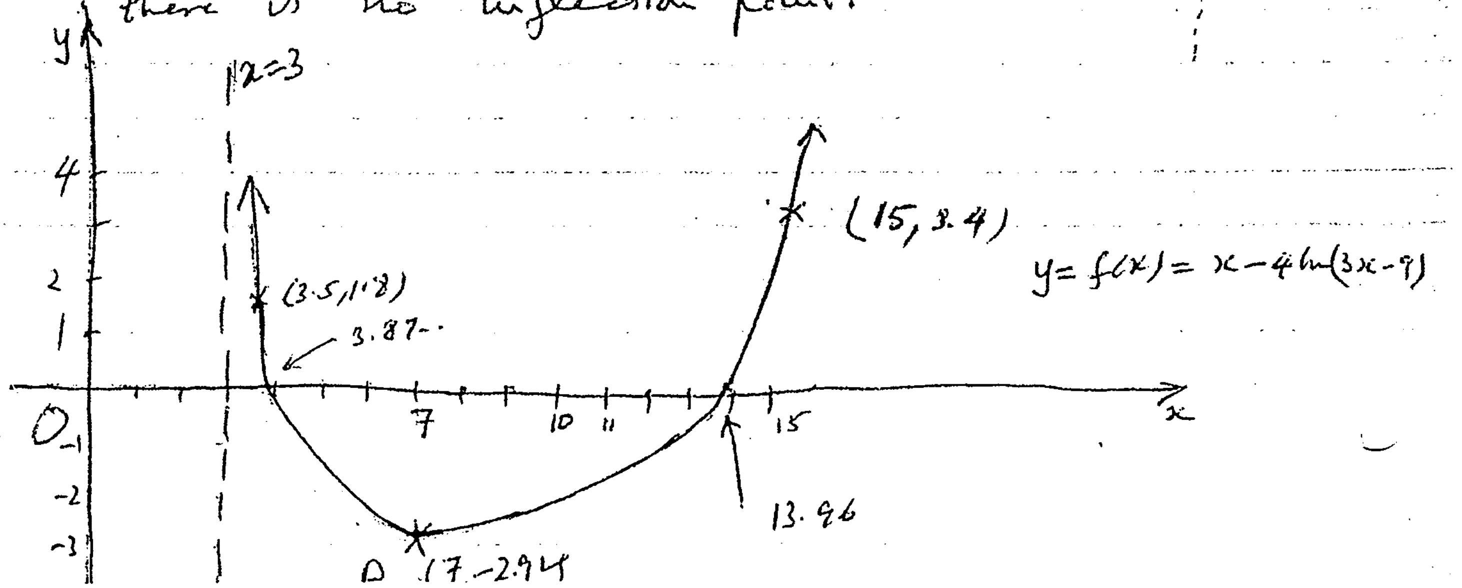
5. From Lecture 10
one critical point $= P(7, -2.94)$. This critical point is also the relative minimum point.

6. From Lecture 10 $f'(x) = \dfrac{x-7}{x-3}$

$$f''(x) = \frac{(x-7)'(x-3) - (x-7)(x-3)'}{(x-3)^2} = \frac{(x-3)-(x-7)}{(x-3)^2} = \frac{4}{(x-3)^2}$$

Since $f''(x) = \dfrac{4}{(x-3)^2} > 0$ for all $x > 3$, f is concave upward for $x > 3$. Since there is no change in concavity, there is no inflection point.

Sketch



D. $y = f(x) = x e^{x^2-3x}$

1. Domain: $(-\infty, \infty)$ 2. x-intercept: 0, y-intercept 0
3. There are no Vertical or horizontal asymptotes.
4.

$f(x)$	$f(-x)$	$-f(-x)$
$x e^{x^2-3x}$	$-x e^{x^2+3x}$	$x e^{x^2+3x}$
	$f(-x) \neq f(x)$ f is not even	$f(x) \neq -f(-x)$ f is not odd

5. From Lecture 10
critical points: $P\left(\frac{1}{2}, \frac{1}{2} e^{-5/4}\right)$ and $Q(1, e^{-2})$

$\frac{1}{2} e^{-5/4} = 0.143\ldots$, $e^{-2} = 0.135\ldots$

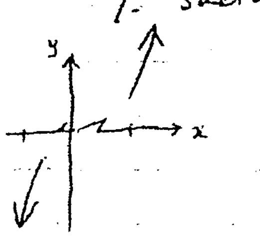
Relative maximum at P & Relative Minimum at Q using the First Derivative Test.

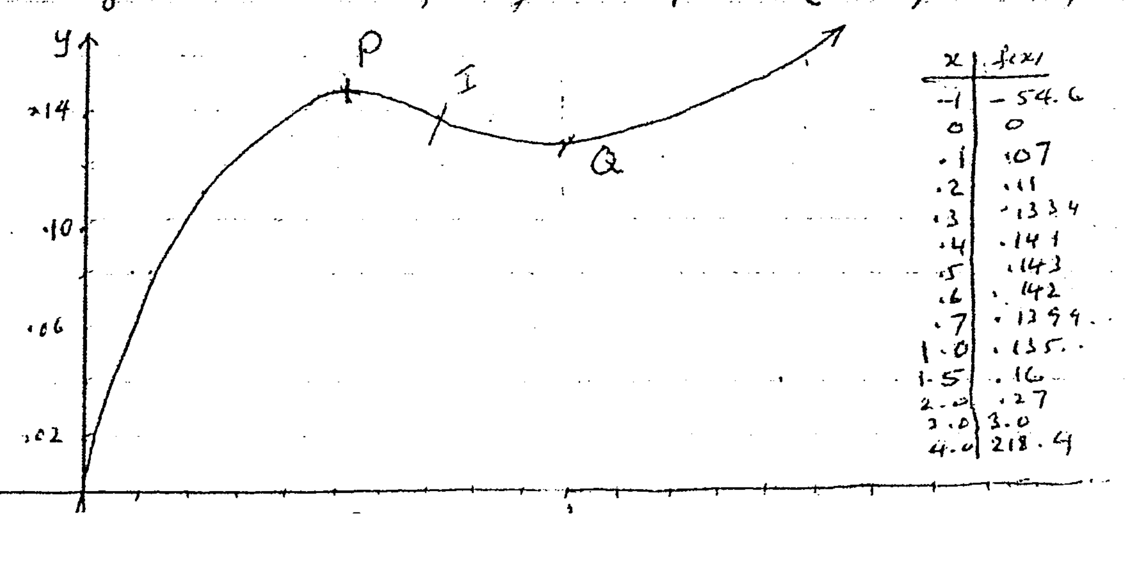
6. $f'(x) = (2x^2-3x+1) e^{x^2-3x}$, $f''(x) = (4x^3-12x^2+15x-6) e^{x^2-3x} = p(x) e^{x^2-3x}$
$f''(x) = 0$ has only one root at $x \approx 0.702$.

Interval	$(-\infty, .702)$	$(0.702, \infty)$
Sign of $f''(x)$ = Sign of the cubic p	$p(0) < 0$	$p(1) > 0$
Concavity	Downward	upward

Concavity changes at $x \approx 0.702$, Inflection point $= I(0.702, 0.1399)$

7. Sketch





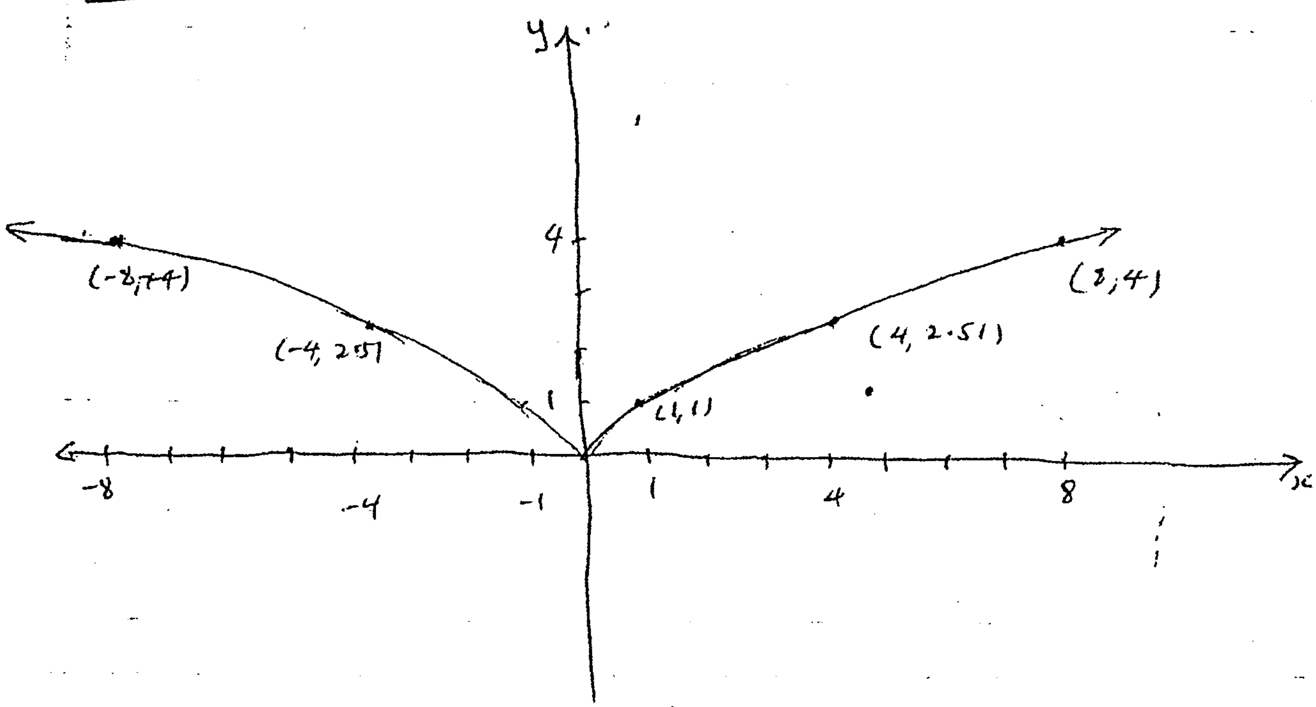
x	$f(x)$
-1	-54.6
0	0
.1	.07
.2	.11
.3	.1334
.4	.141
.5	.143
.6	.142
.7	.1399
1.0	.135..
1.5	.16
2.0	.27
3.0	3.0
4.0	218.4

E. $y = f(x) = x^{2/3}$

1. Domain $(-\infty, \infty)$ 2. x-intercept = y-intercept = 0
3. No vertical or horizontal asymptotes.
4. Since $f(-x) = f(x)$, it is an even function and its

5. from Lecture 10
critical point $(0,0)$ which is also the relative minimum point.
$f'(0)$ does not exist. There is a vertical tangent at $x=0$.
6. f is concave downward on $(-\infty, 0)$ and $(0, \infty)$
There is no inflection point. $f''(0)$ does not exist.

7. Sketch



F. $y = f(x) = x^{7/3} + 56x^{4/3} = x^{4/3}(x+56)$

1. Domain $(-\infty, \infty)$
2. y-intercept: 0 $\quad x$-intercepts: $-56, 0$
3. There are no horizontal or vertical asymptotes.

4.

$f(x)$	$f(-x)$	$-f(-x)$
$x^{4/3}(x+56)$	$x^{4/3}(-x+56)$	$-x^{4/3}(-x+56)$
	$\neq f(x)$	$\neq f(x)$
	f is not an even function	f is not an odd function

5. From Lecture 10

$f'(x) = \frac{7}{3}x^{1/3}(x+32)$ $\qquad f'(x)=0$ when $x=-32, 0$

$f\uparrow$		$f\downarrow$		$f\uparrow$
$f'(-64)>0$	-32	$f'(-8)<0$	0	$f'(1)>0$

By the first derivative test there is a relative maximum at $x=-32$ and a relative minimum at $x=0$. The critical point $P(-32, 2438.248..)$ is a relative maximum and the critical point $Q(0,0)$ is a relative minimum.

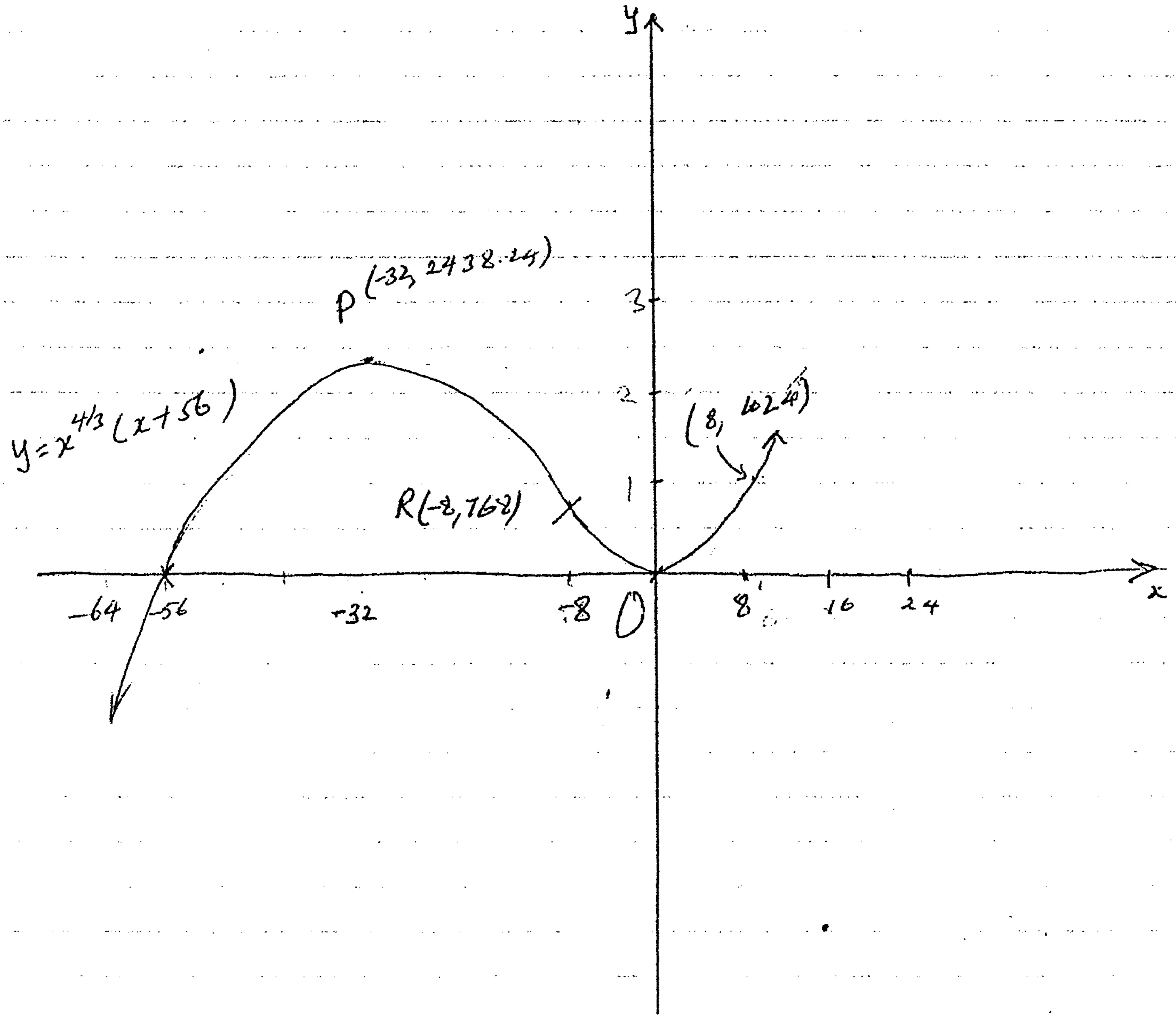
6. Concavity

$f''(x) = \frac{28}{9}\ \frac{x+8}{x^{2/3}}$ $\qquad f''(x)=0$ when $x=-8$ and $f''(x)$ fails to exist when $x=0$.

f concave downward		f concave upward		f concave upward
$f''(-27)<0$	-8	$f''(-1)>0$	0	$f''(1)>0$

f is concave downward on $(-\infty,-8)$ and f is concave upward on $(-8,0)$ and $(0,\infty)$. Since concavity changes only at $x=-8$, there is a point of inflection at $R(-8, 768)$.

Sketch



G. $y = f(x) = \dfrac{4x}{x^2+1}$

1. Domain : $(-\infty, \infty)$ 2. y-intercept : 0 , x-intercept : 0.
3. Horizontal asymptote : $y=0$
4. $f(x)$ $\quad$ $f(-x)$ $\quad$ $-f(-x)$

$\dfrac{4x}{x^2+1}$ $\quad$ $-\dfrac{4x}{x^2+1}$ $\quad$ $\dfrac{4x}{x^2+1}$

$f(x) \neq f(-x)$ $\quad$ $f(x) = -f(-x)$

f is not an even function. $\quad$ f is an odd function graph of f is symmetric about the origin.

5. $f'(x) = 4\left[\dfrac{(x)'(x^2+1) - x(x^2+1)'}{(x^2+1)^2}\right] = 4\left[\dfrac{x^2+1-2x^2}{(x^2+1)^2}\right] = 4\,\dfrac{(1-x^2)}{(x^2+1)^2}$

$f'(x)=0$ when $1-x^2=0$ or $x=-1, 1$

$f \searrow$		$f \nearrow$		$f \searrow$
$f'(-2)<0$	-1	$f'(0)>0$	1	$f'(2)<0$

By the first derivative test there is a relative minimum at $x=-1$ and a relative maximum at $x=1$.
The critical point $P(-1,-2)$ is a relative minimum and the critical point $Q(1,2)$ is a relative maximum.

6. Concavity

$f''(x) = 4\left[(1-x^2)'(x^2+1)^{-2} + (1-x^2).\{(x^2+1)^{-2}\}'\right]$

$= 4\left[-2x(x^2+1)^{-2} + (1-x^2).-2(x^2+1)^{-3}.2x\right]$

$= -8x(x^2+1)^{-3}\left[+(x^2+1) + 2(1-x^2)\right] = -8x(x^2+1)^{-3}.(3-x^2)$

$= \dfrac{-8x(3-x^2)}{(x^2+1)^3} = \dfrac{8x(x^2-3)}{(x^2+1)^3}$

$f''(x)=0$ when $x = \pm\sqrt{3}$ and $x=0$.

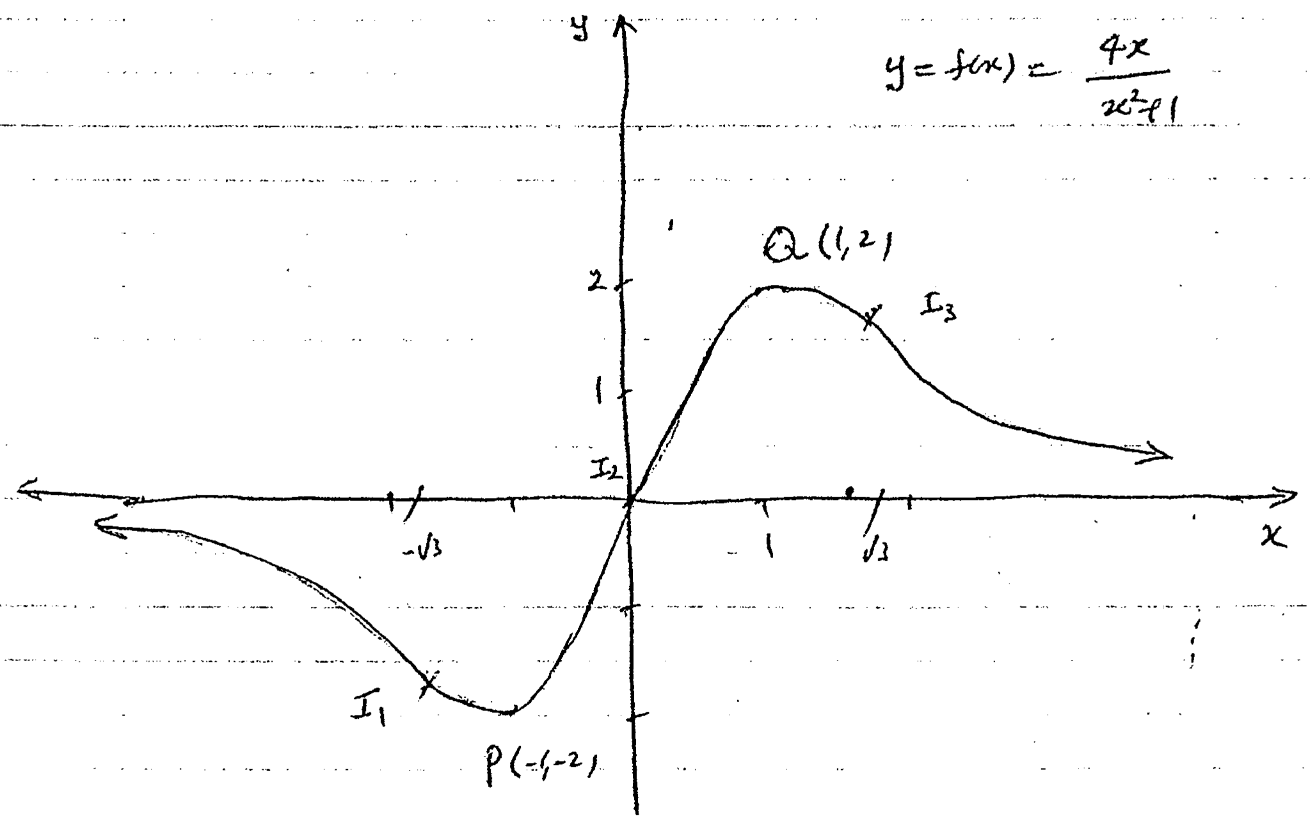
$$f''(x) = \frac{8x(x^2-3)}{(x^2+1)^3}$$

CD		CU		CD		CU
$f''(-2)<0$	$-\sqrt{3}$	$f''(-1)>0$	0	$f''(1)<0$	$\sqrt{3}$	$f''(2)>0$

f is concave downward on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ and
f, is concave upward on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$.

Concavity changes at $-\sqrt{3}$, 0, and $\sqrt{3}$. Therefore there are three inflection points $I_1(-\sqrt{3}, -\sqrt{3})$, $I_2\,(0,0)$ and $I_3(\sqrt{3}, \sqrt{3})$

7. Sketch



II. $y = f(x) = 2x + \frac{8}{x}$

1. Domain : $(-\infty, 0) \cup (0, \infty)$
2. There are no x - or y - intercepts.
3. $x = 0$ is a vertical asymptote. $y = x$ is an oblique asymptote.
4. $f(x) = 2x + \frac{8}{x}$, $f(-x) = -2x - \frac{8}{x}$, $-f(-x) = 2x + \frac{8}{x}$

   $f(x) \neq f(-x)$ — f is not an even function

   $f(x) = -f(-x)$ — f is an odd function, graph of f is symmetric about the origin

5. $f'(x) = 2 - \frac{8}{x^2}$ f' exists throughout the domain of f and $f'(x) = 0$ when $x = -2$ and $x = 2$.

$f \uparrow$	-2	$f \downarrow$	0	$f \downarrow$	2	$f \uparrow$
$f'(-3) > 0$		$f'(-1) < 0$		$f'(1) < 0$		$f'(3) > 0$

By the first derivative test the critical point $P(-2, -8)$ is a relative maximum point and the critical point $Q(2, 8)$ is a relative minimum point.

6. Concavity $f''(x) = \frac{16}{x^3}$

CD	0	CU
$f''(-1) < 0$		$f''(1) > 0$

since 0 is not in the domain of f, there is no point of inflection.

7. Sketch $y = f(x) = 2x + \frac{8}{x}$