MATH 157.

5.2. Related Rates (相关变率)

When defining the derivative $f'(x)$, we define it to be exactly the rate of change of $f(x)$ with respect (遵守) to x. Consequently, any question about rates of change can be rephrased (改述) as a question about derivatives. When we calculate derivates, we are calculating rates of change. Results and answers we obtain (得到) for derivatives translate (相当于) directly into results and answers about rates of change (变率). Let us look at some examples where more than one variable is involved, and where our job is to analyze and exploit (运用) relations between the rate of change of there variables. As an aside (除…以外), this class of problems is know as (被称为) related rates problems. The mathematical step of relating to rate of change turns out to be (原来是) largely an exercise in differentiation (微分) using the chain rule or implicit (隐式的) differentiation. This explains why some textbooks place this sections shortly after chain rule and implicit differentiation.

Let's say we are interested in the relationship between the rate of change of a mortgage (抵押贷款) rate and the rate of change of the number of houses sold over time. If x represents the mortgage rate and y the number of houses sold at anytime t, then x and y are each functions of this third variable t. Suppose furthermore (此外) that the mortgage rate x is related to the number of houses sold y, we also have an equation relating x to y:

$$f(x) = g(y)$$

Then we can differentiate both sides of the equation (等式) implicitly (暗中地) with respect to t, and get

$$f'(x)\frac{dx}{dt} = g'(y)\frac{dy}{dt}$$

In other words, we now have an equation that relates $\frac{dx}{dt}$ to $\frac{dy}{dt}$. In term of (依据) our problem this means that the rate of change of the mortgage rate and the rate of change of the number of houses sold are related as a function of time. And so, as $\frac{dx}{dt}$ changes determines (决定) how $\frac{dy}{dt}$ changes, i.e. the rate of change of mortgage w.r.t time controls the rate of change of houses at the instant (当时) of time.

Example 5.6. Speed at which a Coordinate (坐标) is changing.

Suppose an object is moving along a path (小径) described by $y=x^2$, that is, it is moving on a parabolic (抛物线的) path. At a particular time, say $t=5$, the x-coordinate is 6 and we measure the speed at which the x-coordinate of the object is changing and find that $\frac{dx}{dt}=3$

At the same time, how fast is the y-coordinate changing?

$$y=x^2.$$
$$1.\ \frac{dy}{dt}=2x\frac{dx}{dt}$$
$$\frac{dy}{dt}=2\cdot 6\cdot 3$$
$$\frac{dy}{dt}=36$$

$\therefore$ the speed is 36.

In many cases, particularly interesting ones, x and y will be related in some other way, for example $x=f(y)$ or $F(x,y)=k$ or perhaps (可能) $F(x,y)=G(x,y)$, where $F(x,y)$ and $G(x,y)$ are expressions (表达) involving both variables. In all cases, you can solve the related rates problem by taking the derivative of both sides, plugging in (插入) all the known values (namely (也就是) x, y and $\frac{dx}{dt}$) and the solving for $\frac{dy}{dt}$.

To summarize, here are the steps in doing a related rates problem

Step for solving Related Rates Problems.

① Read the problem at least twice

② Sketch and label (贴标签) a diagram (几何图形)