Limit and Continuity

Graphical Diffi

Limit of a Function

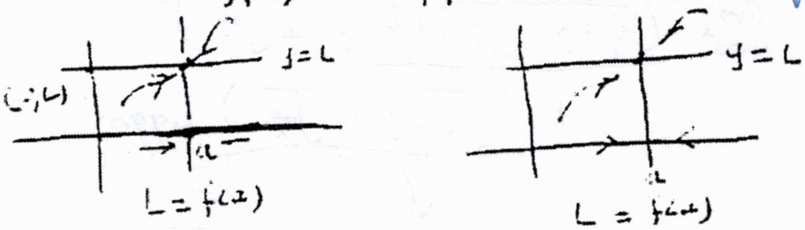
Let f be a function and let a and L be real numbers. If, as x approaches (逼近) a from left or right, the values of $f(x)$ approach or equal L, we say that the "limit of $f(x)$ as x approaches a is L" = y. and we write this as:

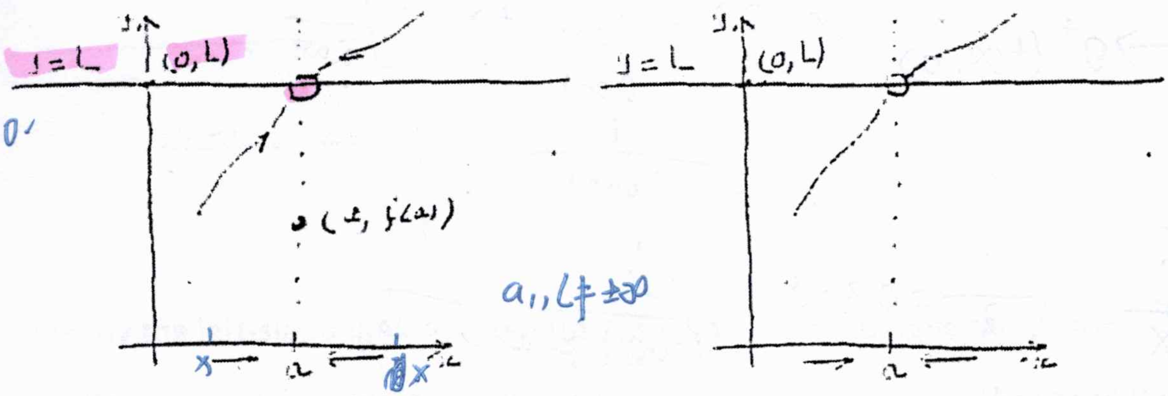
$$\lim_{x \to a} f(x) = L = y.$$

$f(a) \neq L$

任 a

$x - a \neq 0$





$a, L \neq \pm\infty$

f(a) exists, $L \neq f(a)$ f(a) does not exist, $L \neq f(a)$

1. It is not necessary that $f(a)$ be defined or that $f(a)$ = L.

e.g.

Consider the function $f(x) = x / x$. $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x}{x} = 1$

but f(0) is not defined.

Consider the function $f(x) = \frac{1}{x^2+1}$. $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2+1} = 1$

and here $f(0) = 1$.

$\lim_{x \to 0^+} \sqrt{\frac{1}{x} + 2} - \sqrt{\frac{1}{x}}$

$\lim_{x \to 0^+} \frac{\left(\sqrt{\frac{1}{x}+2} - \sqrt{\frac{1}{x}}\right)\left(\sqrt{\frac{1}{x}+2} + \sqrt{\frac{1}{x}}\right)}{1 \left(\sqrt{\frac{1}{x}+2} + \sqrt{\frac{1}{x}}\right)}$ bigger

$\downarrow$

$x \to 0^+ \; (+\infty, 0) \quad (\infty, 0) \quad \frac{1}{0.001} + 2 + \sqrt{\frac{1}{0.001}}$

$\frac{1}{0.0001}$

$\frac{1}{x} \quad \frac{1}{3} = 0.333.$

$\Downarrow$

$\sqrt{1000 + 2} + \sqrt{1000}$

$\frac{1}{0.3} =$

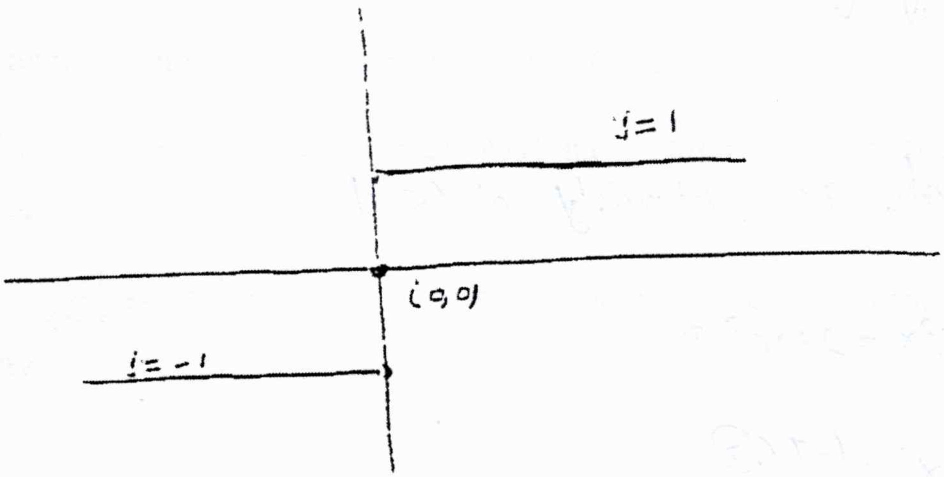
$\frac{2}{\text{bigger}} =$ ~~sm~~small.

$\frac{2}{1} = 2 \quad \frac{2}{4} = \underset{\text{near}}{0.2} \to 0.$

2. For $\lim_{x \to a} f(x)$ to exist we must have the two one-sided limits to exist and to be equal. Otherwise, the two-sided limit $\lim_{x \to a} f(x)$ does not exist.

e.g. Consider the function $f(x)$ defined below:

$$f(x) = \begin{cases} 1 \; if \; x > 0 \\ 0 \; if \; x = 0 \\ -1 \; if \; x < 0 \end{cases}$$



Clearly the left-sided limit ,written $\lim_{x \to 0-} f(x)$ is -1, and the right-sided limit written $\lim_{x \to 0+} f(x)$ is 1 and since they are not equal, the two-sided limit $\lim_{x \to 0} f(x)$ does not exist even though $f(0) = 0$.

In the summary for $\lim_{x \to a} f(x)$ to exist we must have:

1. $\lim_{x \to a-} f(x) = L_1$
2. $\lim_{x \to a+} f(x) = L_2$
3. $L_1 = L_2$

How to find limits?

1. Table/graph.

compute $\lim_{x \to 0} = \frac{\cancel{\sin x} \sin x}{x}$

X must be Radians

$\lim_{x \to 0} \frac{\cos x - 1}{x} = -1.$

$\frac{\cos 1 - 1}{1} = -1.$

$\frac{\cos 2 - 1}{2} = 1.$

$\frac{\cos 3 - 1}{3} = 1$

$\lim_{x \to 0} \frac{\sin x}{x} = 1.$

The Intermediate Value theorem (IVT)

Re : $f(x)=0$

If f continuines m $[a,b]$ and N between $f(a)$ & $f(b)$,

$a<c<b$

such that $f(c)=N$.

futhermore if $f(a)\ f(b)<0$.

$+ \quad -$

$- \quad +$

$\therefore N=0$

3.7.5.

We need only check g's continuisly at $x=-1$

1. $g(-1)=6-7c$.

2. $\lim_{x\to-1^-} = 2-2c^2x = 2+2c^2$ ①

$\lim_{x\to-1^+} 6-7cx^2 = 6-7c$ ②.

$\lim_{x\to-1} g(x)$ exists if ① = ②

$2+2c^2=6-7c$

$2c^2+7c-4=0 \quad \vee \quad (2c-1)(c+4)=0$.

$c=\dfrac{-7\pm\sqrt{49+32}}{4}$

$=\dfrac{-7\pm9}{4}$

$=-4, \frac{1}{2}$

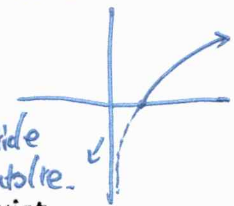
$c=-4, \frac{1}{2}$ make g cont everywhere $(-\infty,\infty)$

Limits of infinity

$\lim_{x \to 0^+} \frac{1}{x} = +\infty$ / No exist

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$ / No exist.

$\lim_{x \to 0^+} \ln x = -\infty$



Vertide asymtolte.

If

1. $\lim_{x \to a} f(x) = \infty$
2. $\lim_{x \to a} f(x) = -\infty$
3. $\lim_{x \to a-} f(x) = \infty$ or $-\infty$
4. $\lim_{x \to a+} f(x) = \infty$ or $-\infty$

We say that limits of $f(x)$ at a, a, a^-, a^+ <u>do not exist.</u>

E.g. $\lim_{x \to 1} \frac{1}{(x-1)^2} = +\infty \Rightarrow f(x) = 1/(x-1)^2$ has no limit at $x = 1$ or the limit is infinity.

$\lim_{x \to 1} \frac{-1}{(x-1)^2} = -\infty \Rightarrow f(x) = -1/(x-1)^2$ has no limit at $x = 1$ 1 or the limit is minus infinity.

$\lim_{x \to 0^+} \frac{1}{\sqrt{x}} = \infty \Rightarrow f(x) = 1/\sqrt{x}$ has no left -side limit 1 or the limit is infinity.

Similarly,

$\lim_{x \to 0+} \frac{-1}{\sqrt{x}} = -\infty$ i.e. has no right – sided limit.

$\frac{2(t+2)}{7(4t+1)}$

$\frac{4t+8\ 4t=1}{t+2}$ $\frac{1}{t+2} \times 2$ $\frac{4}{4t+1}$

$\frac{2\cancel{4}}{4t+1} \cdot \frac{t+2}{\cancel{t+7}}$ $\frac{14}{t+2}$

Limits at Infinity

These reflect the behavior of f(x) as $|x|$ increases without bound i.e. as $|x| \to \infty$ (that is, as $x \to \infty$ or $x \to -\infty$). Thus as $x \to \infty$ or $x \to -\infty$ f(x) → a finite number L, we then write:

$\lim_{x \to \infty} f(x) = \mathrm{L}$ or $\lim_{x \to \infty} f(x) = \mathrm{M}$

Where L and M are finite numbers.

For any positive number P.

$\lim_{x\to\infty} \frac{1}{x^p} = 0$ and $\lim_{x \to -\infty} \frac{1}{x^p} = 0$

$\lim\limits_{x \to \infty} e^{-x} = 0$, $\lim\limits_{x \to -\infty} e^{x} = 0$

$$\lim_{x \to \infty} \frac{2x}{x+1} = 2$$

Finding limits

These three are general methods for finding limits.

1. $\lim_{x\to a} f(x)$ may be found by substitution.

e.g. $\lim_{x \to 2} \frac{(x^2+1)}{(x-1)} = 5$ $4+1 = \frac{5}{1} = 5$.

2. $\lim_{x\to a} f(x)$ may be found by cancellation.

e.g. $\lim_{x\to 7} \frac{(x+7)(x-7)}{(x-7)} = \lim_{x\to a} x + 7 = 14$

3. $\lim_{x\to a} f(x)$ may be found by some algebraic manipulation.

e.g. by rationalization

$\lim_{h\to 0} \frac{h}{2-\sqrt{h+4}} = \lim_{h\to 0} \frac{h}{2-\sqrt{h+4}} \cdot \frac{2+\sqrt{h+4}}{2+\sqrt{h+4}} = \lim_{h\to 0} \frac{h(2+\sqrt{h+4})}{4-h-4} = \lim_{h\to 0} \frac{h(2+\sqrt{h+4})}{-h}$
$= \lim_{h\to o} -(2+\sqrt{h+4}) = -4$

Continuity 连续性

A function f is continuous at x = a if the following three conditions are satisfied

1. $f(a)$ is defined.
2. $\lim_{x\to a} f(x)$ exists, say L.
3. L = $f(a) = L$.

If f is not continuous at a, it is said to be discontinuous at a.

Continuity on an open interval (a, b) = a< x <b
间隔

A function is said to be continuous on (a, b) if it is continuous at every x –value in the interval.

Continuity on a closed interval [a, b] = a ≤ x ≤b

A function is continuous on a closed interval [a, b] if

1. It is continuous on the open interval (a, b).
2. It is continuous from the right at x =a, that is, $\lim_{x\to a+} f(x) = f(a)$
3. It is continuous from the left at x =b, that is, $\lim_{x\to b-} f(x) = f(b)$

Examples

1. Removable discontinuity

$$f(x) = \frac{x}{x} = \begin{cases} 1 \; if \; x \neq 0 \\ 15 \; if \; x = 0 \end{cases}$$

f is continuous everywhere except at x = 0

Since $\lim_{x\to 0-} \frac{x}{x} = 1$, $\lim_{x\to 0+} \frac{x}{x} = 1$ therefore, $\lim_{x\to 0} \frac{x}{x} = 1$

But $f(0)$ ≠1 it is 15

However, by redefining $f(0)$ = 1, we may remove this discontinuity. Such a discontinuity is called a removable discontinuity.

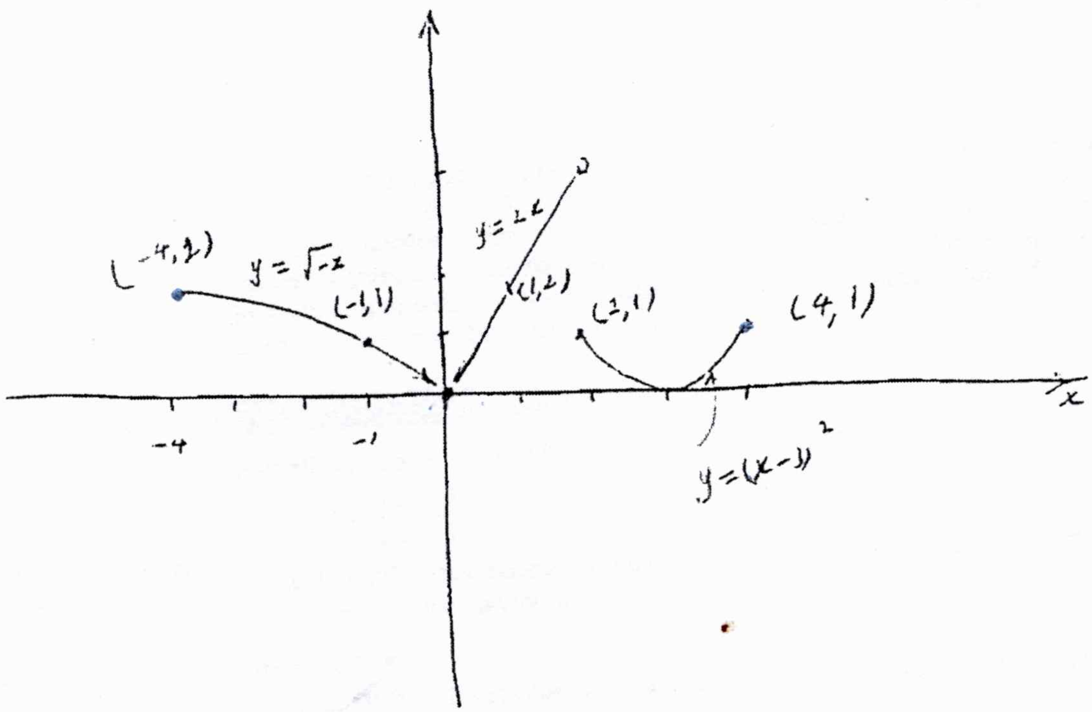
3.1
3.3
3.4
3.5

2. Let $f(x) = \begin{cases} \sqrt{-x}, -4 \le x < 0 \\ 2x, \quad 0 \le x < 2 \\ (x-3)^2, 2 \le x \le 4 \end{cases}$

a. Sketch the graph of f.
b. What are the domain and range of f?
c. Is f continuous at x =0, x =2 ?

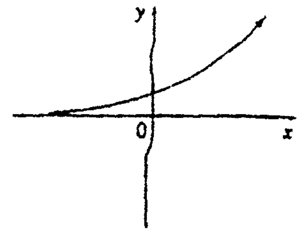
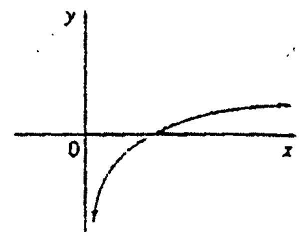
Solution

a.



b. D_f = [-4, 4]
R_f = [0, 4)
c. $\lim_{x\to 0^-} f(x) = 0 = \lim_{x\to 0} f(x)$ = 0 and since f(0)
Hence f is continuous at x =0
$\lim_{x\to 2^-} f(x) = 4 \neq \lim_{x\to 2^+} f(x) = 1$, therefore f is not continuous at x=2

Continuous Functions (cont.)

Type of Function	Where It Is Continuous	Graphic Example
Exponential Function $y = a^x$ where $a > 0$	For all x	
Logarithmic Function $y = \log_a x$ where $a > 0$, $a \neq 1$	For all $x > 0$	

Continuous functions are nice to work with because finding $\lim_{x \to c} f(x)$ is simple if f is continuous: just evaluate $f(c)$.

When a function is given by a graph, any discontinuities are clearly visible. When a function is given by a formula, it is usually continuous at all x-values except those where the function is undefined or possibly where there is a change in the defining formula for the function, as shown in the following examples.

EXAMPLE 2 Continuity

Find all values $x = a$ where the function is discontinuous.

(a) $f(x) = \dfrac{4x - 3}{2x - 7}$

SOLUTION This rational function is discontinuous wherever the denominator is zero. There is a discontinuity when $a = 7/2$.

(b) $g(x) = e^{2x-3}$

SOLUTION This exponential function is continuous for all x. TRY YOUR TURN 1

YOUR TURN 1 Find all values $x = a$ where the function is discontinuous.

$f(x) = \sqrt{5x + 3}$

EXAMPLE 3 Continuity

Find all values of x where the following piecewise function is discontinuous.

$$f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ x^2 - 3x + 4 & \text{if } 1 \leq x \leq 3. \\ 5 - x & \text{if } x > 3 \end{cases}$$

SOLUTION Since each piece of this function is a polynomial, the only x-values where f might be discontinuous here are 1 and 3. We investigate at $x = 1$ first. From the left, where x-values are less than 1,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 1) = 1 + 1 = 2.$$

From the right, where x-values are greater than 1,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 - 3x + 4) = 1^2 - 3 + 4 = 2.$$

we must decide what to do with the endpoints. We will say that a function f is **continuous from the right** at $x = c$ if $\lim_{x\to c^+} f(x) = f(c)$. A function f is **continuous from the left** at $x = c$ if $\lim_{x\to c^-} f(x) = f(c)$. With these ideas, we can now define continuity on a closed interval.

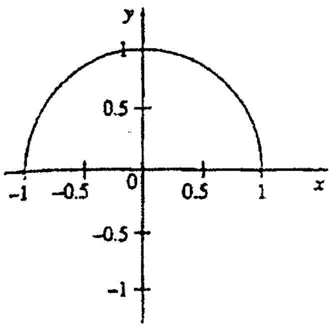


FIGURE 19

Continuity on a Closed Interval

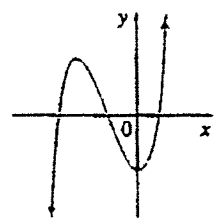
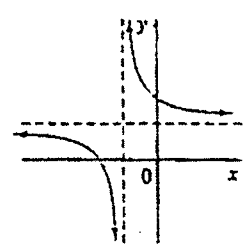
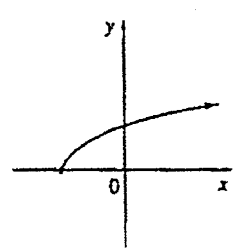
A function is **continuous on a closed interval** $[a, b]$ if

1. it is continuous on the open interval (a, b),
2. it is continuous from the right at $x = a$, and
3. it is continuous from the left at $x = b$.

For example, the function $f(x) = \sqrt{1 - x^2}$, shown in Figure 19, is continuous on the closed interval $[-1, 1]$. By defining continuity on a closed interval in this way, we need not worry about the fact that $\sqrt{1 - x^2}$ does not exist to the left of $x = -1$ or to the right of $x = 1$.

The table below lists some key functions and tells where each is continuous.

Continuous Functions

Type of Function	Where It Is Continuous	Graphic Example
Polynomial Function $y = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers, not all 0	For all x	
Rational Function $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, with $q(x) \neq 0$	For all x where $q(x) \neq 0$	
Root Function $y = \sqrt{ax + b}$, where a and b are real numbers, with $a \neq 0$ and $ax + b \geq 0$	For all x where $ax + b \geq 0$	

(continued)

Figure 8 illustrates these three facts.

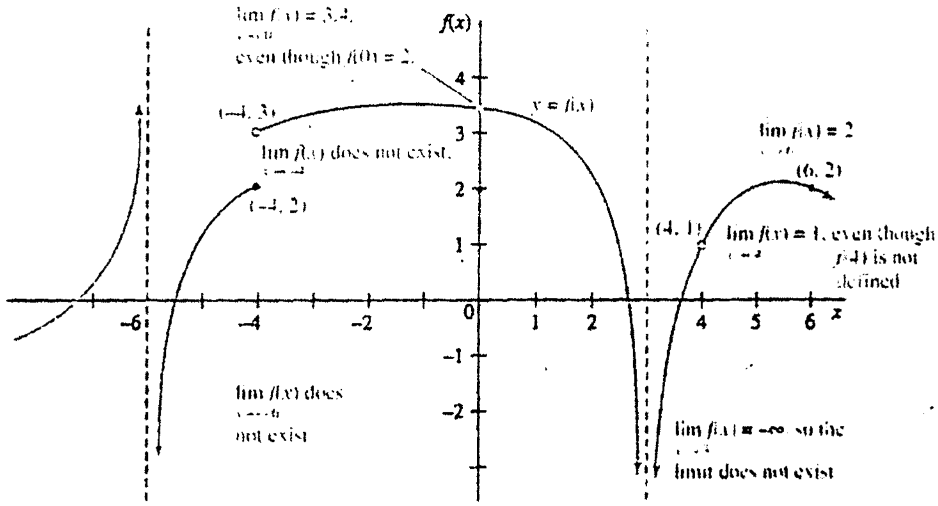


FIGURE 8

Rules for Limits As shown by the preceding examples, tables and graphs can be used to find limits. However, it is usually more efficient to use the rules for limits given below. (Proofs of these rules require a formal definition of limit, which we have not given.)

Rules for Limits

Let a, A, and B be real numbers, and let f and g be functions such that

$$\lim_{x\to a} f(x) = A \qquad \text{and} \qquad \lim_{x\to a} g(x) = B.$$

1. If k is a constant, then $\lim_{x\to a} k = k$ and $\lim_{x\to a} [k \cdot f(x)] = k \cdot \lim_{x\to a} f(x) = k \cdot A$.
2. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A \pm B$

   (The limit of a sum or difference is the sum or difference of the limits.)
3. $\lim_{x\to a} [f(x) \cdot g(x)] = [\lim_{x\to a} f(x)] \cdot [\lim_{x\to a} g(x)] = A \cdot B$

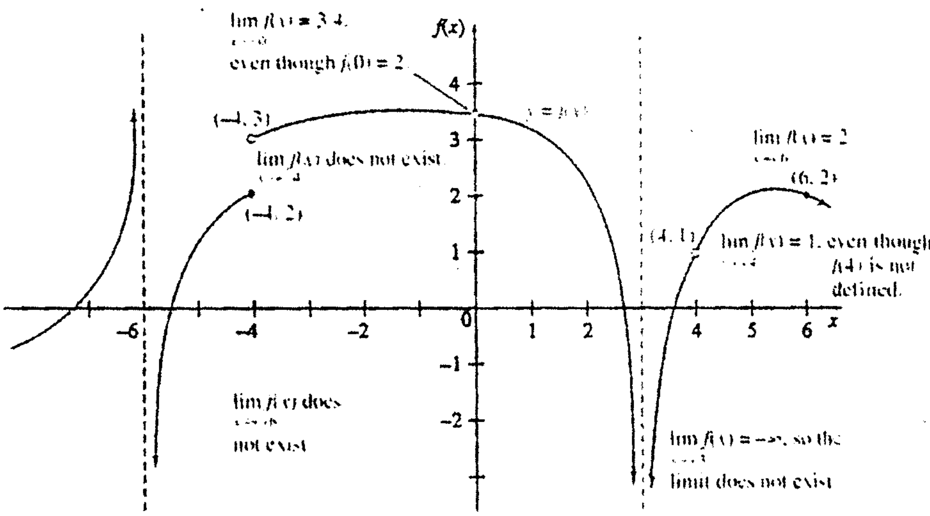
   (The limit of a product is the product of the limits.)
4. $\displaystyle \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}$ if $B \neq 0$

   (The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero.)
5. If $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.
6. For any real number k, $\lim_{x\to a} [f(x)]^k = [\lim_{x\to a} f(x)]^k = A^k$, provided this limit exists.*
7. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if $f(x) = g(x)$ for all $x \neq a$.
8. For any real number $b > 0$, $\lim_{x\to a} b^{f(x)} = b^{[\lim_{x\to a} f(x)]} = b^A$.
9. For any real number b such that $0 < b < 1$ or $1 < b$,

   $\lim_{x\to a} [\log_b f(x)] = \log_b[\lim_{x\to a} f(x)] = \log_b A$ if $A > 0$.

*This limit does not exist, for example, when $A < 0$ and $k = 1/2$, or when $A = 0$ and $k \leq 0$.

Figure 8 illustrates these three facts.



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