6月16日 midterm

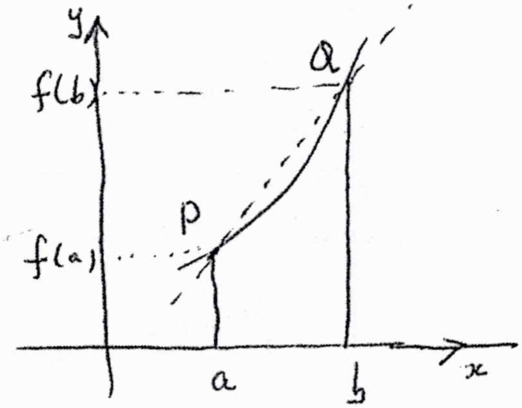
Lecture 7

$\lim_{x\to 0^-} \sqrt{x}$

Graphical Diff.

3.3 — 3.5

Rates of Change



Average Rate of change

The average rate of change of $f(x)$ with respect to x for a function f as x changes from a to b is

$$\frac{\Delta y}{\Delta x} \quad \frac{f(b) - f(a)}{b - a} = \text{slope of line } PQ.$$

$f'(x)$ = slow of the tangent

De

$f'(a)$ 斜率

Notes.

1. We refer to $\frac{f(b)-f(a)}{b-a}$ as the difference quotient.

2. We note that $\frac{f(b)-f(a)}{b-a}$ is the slope of the line joining points $(a, f(a))$ and $(b, f(b))$ on the graph of $y = f(x)$. The line PQ is called the secant line.

Instantaneous Rate of Change

The instantaneous rate of change for a function f when $x = a$ is

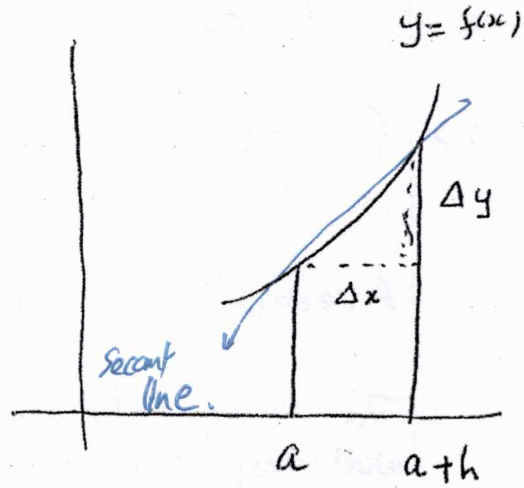
$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists. This is written as $f'(a)$. We then say that f is differentiable at $x = a$ and its derivative is $f'(a)$.

Notes.

(1) $f(a+h) - f(a) = \text{change in } y$
$= \Delta y$

$h = (a+h) - a = \text{change in } x$
$= \Delta x$



$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \frac{dy}{dx} \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=a}$$

$$= \left.\frac{df(x)}{dx}\right|_{x=a}$$

(2) $f'(a)$ may be also written as

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

or

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(3) $f'(a)$ is the slope of the tangent line to the curve $y = f(x)$ at $x = a$.

Thus the slope of the tangent line gives us the instantaneous rate of change while the slopes of the secant lines give us average rates of change.

(4)
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \qquad = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \frac{df}{dx} \qquad = \frac{dy}{dx}$$

$$= f'(x) \qquad = y'(x)$$

$$= D_x f = D_x f(x) = D_x y = D_x y(x)$$

These are various alternate notations of the derivative.

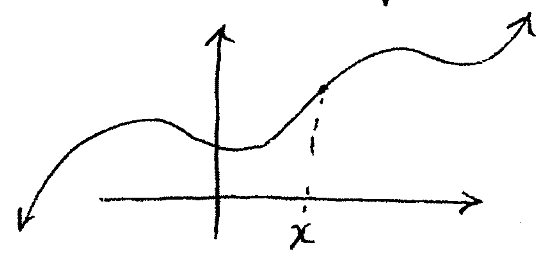
$f'(x)$ — read: "f - prime of x"

As x varies we have derivative function $f'(x)$.

For a fixed number $x = a$, we also write

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = y'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

(5) The derivative exists for a function f at a point x if all of the following conditions are satisfied.

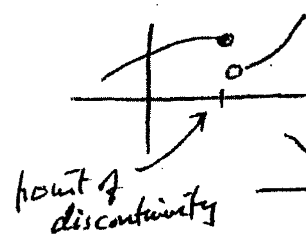


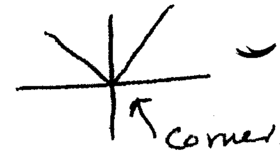
(i) f is continuous

(ii) f is smooth.

(iii) f does not have a vertical tangent line.

(6) The derivative does not exist at a point x when any of the following conditions hold.

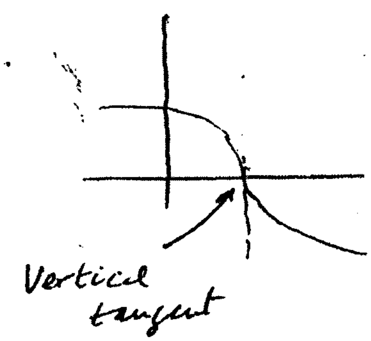




(i) f is discontinuous

(ii) f has a sharp corner

(iii) f has a vertical tangent line.



Examples

1. Find the average rate of change for $y = f(x) = x^2 + x + 1$ between $x = 3$ and $x = 5$.

$$\text{Average rate of change on } [3,5] = \frac{f(5) - f(3)}{5-3} = \frac{31 - 13}{2} = 9$$

2. Suppose the position of an object moving in a straight line is given by

$$s(t) = 2t^3 - 5t^2 + t - 1.$$

Find the instantaneous velocity at time $t = 4$.

$$s'(4) = \lim_{h \to 0} \frac{s(4+h) - s(4)}{h}$$

$$= \lim_{h \to 0} \frac{\left[2(4+h)^3 - 5(4+h)^2 + (4+h) - 1\right] - \left[2.4^3 - 5.4^2 + 4 - 1\right]}{h}$$

$$= \lim_{h \to 0} \frac{\left[2(\cancel{4^3} + 3.4^2.h + 3.4.h^2 + h^3) - 5(\cancel{4^2} + 8h + h^2) + \cancel{(4+h)} - \cancel{1}\right] - \left[\cancel{2.4^3} - \cancel{5.4^2} + \cancel{4} - \cancel{1}\right]}{h}$$

$$= \lim_{h \to 0} \frac{h\left[96 + 12h + h^2 - 40 - 5h + 1\right]}{h}$$

$$= \lim_{h \to 0} 57 + 7h + h^2 = 57$$

3. Marginals

$$\frac{dC}{dx} = C'(x) = \text{marginal cost}$$

$$\frac{dR}{dx} = R'(x) = \text{marginal revenue}$$

$$\frac{dP}{dx} = P'(x) = \text{marginal profit}$$

① $f'(x) \approx \frac{f(x+h) - f(x)}{h}$

② $hf'(x) \approx f(x+h) - f(x)$

③ $f(x+h) \approx f(x) + hf'(x)$

suppose $h = 1$ is small

$f(x+1) - f(x) \approx f'(x)$

$C(x+1) - C(x) \approx C'(x)$

The cost in dollars of producing x tacos is

$$C(x) = 1000 + .24x^2, \qquad 0 \le x \le 30{,}000$$

(a) Find the marginal cost

$$C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h} = \lim_{h \to 0} \frac{\left[1000 + 0.24(x+h)^2\right] - \left[1000 + 0.24x^2\right]}{h}$$

$$= \lim_{h \to 0} \frac{0.24(x^2 + 2hx + h^2) - 0.24x^2}{h} = \lim_{h \to 0} \frac{h(0.48x + 0.24h)}{h}$$

$$= \lim_{h \to 0} 0.48x + 0.24h$$

$$= 0.48x \qquad 0 < x < 30{,}000$$

(b) Find and interpret the marginal cost at a production level of 100 tacos.

$C'(100) = 0.48x\Big|_{x=100} = 48$. $48 is approximately the cost of producing the 101st taco. The exact cost of the 101st taco is $= C(101) - C(100) = [1000 + 0.24 \cdot 101^2] - [1000 + 0.24 \cdot 100^2] = \$48.24$.

$48.24 > $48 but quite close

4. Find the equation of the tangent line to the graph of $y = f(x) = \sqrt{x}$ at $x = 25$.

$$m = f'(25) = \lim_{h \to 0} \frac{f(25+h) - f(25)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{25+h} - \sqrt{25}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{25+h} - 5}{h} \cdot \frac{\sqrt{25+h} + 5}{\sqrt{25+h} + 5}$$

$$= \lim_{h \to 0} \frac{(25+h) - (25)}{h(\sqrt{25+h} + 5)} = \lim_{h \to 0} \frac{\cancel{h}\, 1}{\cancel{h}(\sqrt{25+h} + 5)}$$

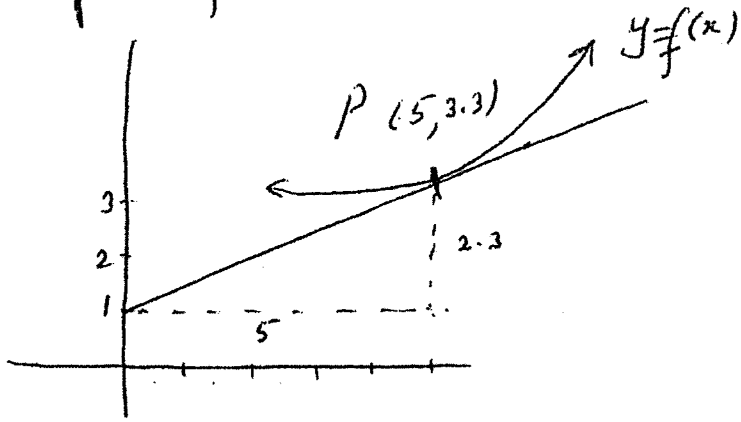
$$= \lim_{h \to 0} \frac{1}{\sqrt{25+h} + 5} = \frac{1}{10}$$

The tangent line is:

$$y - \sqrt{25} = \frac{1}{10}(x - 25)$$

$$y = \frac{1}{10}x + \frac{5}{2}$$

5. We can estimate the slope of the tangent from the graph of a function:



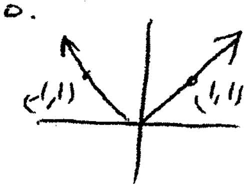
Slope of the tangent to the curve $y = f(x)$ at $P(5, 3.3)$

$$= \frac{2.3}{5} = 0.46$$

Note. Slope of the curve at P = slope of the tangent line at P.

6. Show that f is not differentiable at $x = 0$.

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Since $\lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{f(h) - 0}{h} = \lim\limits_{h \to 0} \dfrac{|h|}{h}$

does not exist, because $\lim\limits_{h \to 0^+} \dfrac{|h|}{h} = \lim\limits_{h \to 0} \dfrac{h}{h} = 1$

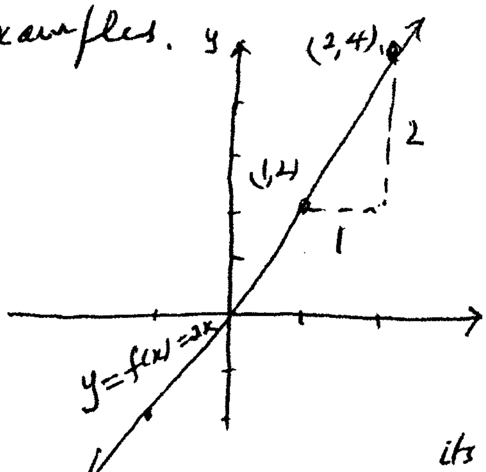
and $\lim\limits_{h \to 0^-} \dfrac{|h|}{h} = \lim\limits_{h \to 0^-} \dfrac{-h}{h} = -1$.

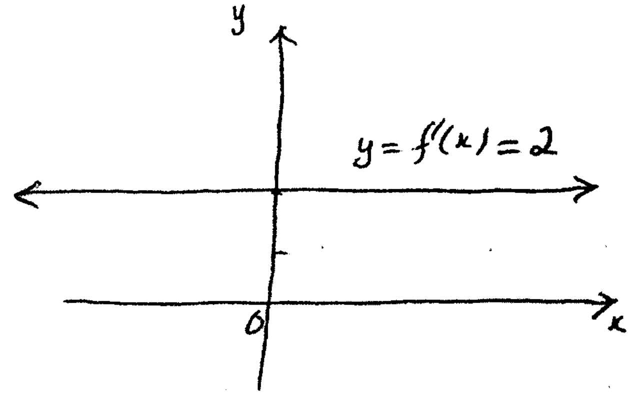
7. Graphical Differentiation

Given a graph of a function f and using the fact that $f'(x)$ equals the slope of the tangent to the graph of $y=f(x)$ at x, we are able to estimate $f'(x)$ and hence sketch the graph of the derivative of $f(x)$ i.e. $f'(x)$.

We illustrate this with the help of following examples.

(i)



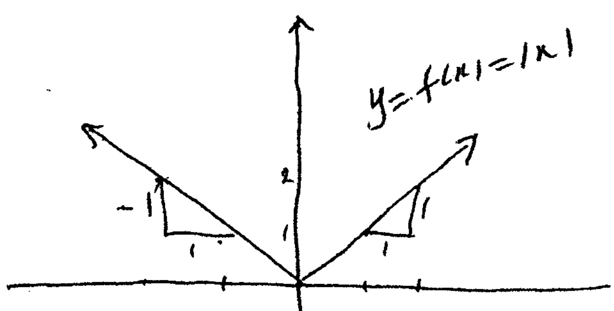


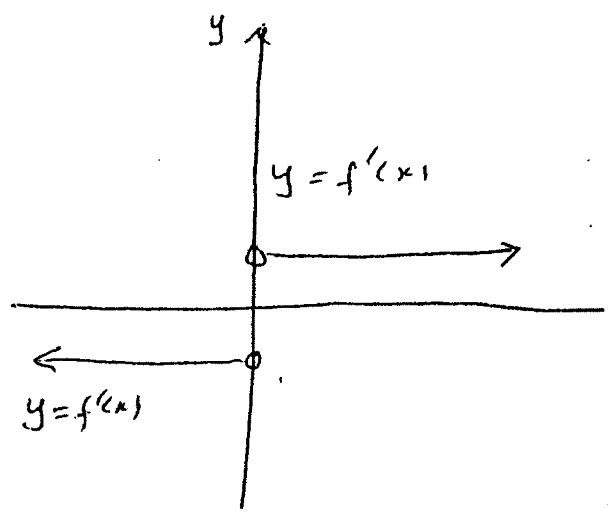
Here the curve itself is its tangent,

for all x: slope of the tangent $= \frac{2}{1} = 2$

$\therefore \quad f'(x) = 2$ for all x.

(ii)

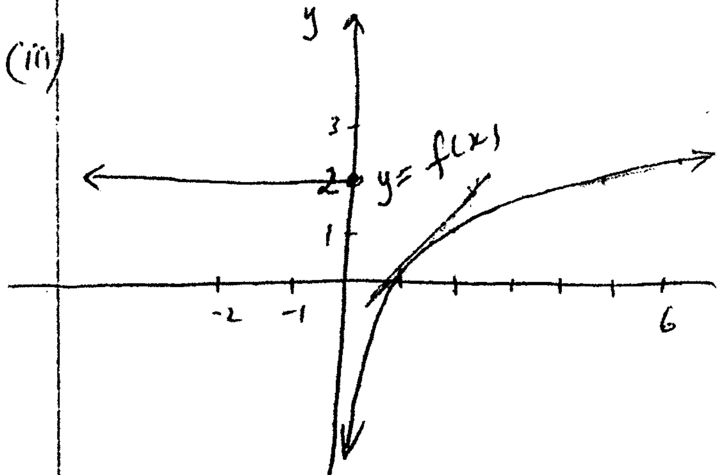


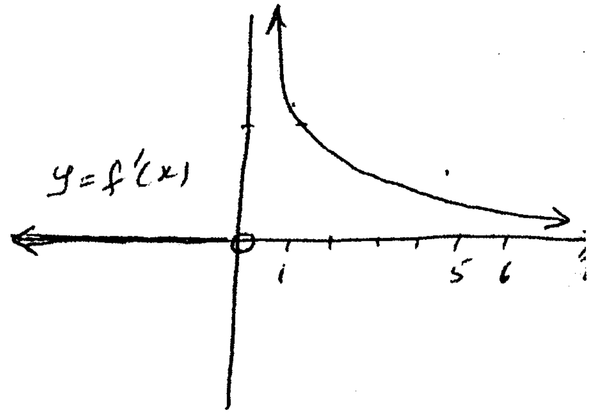


Here $f'(0)$ does not exist.

$$f'(x)=\begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

(iii)





$(-\infty, 0)$ the graph of $y = f(x)$ is a horizontal line and so its slope is zero. Therefore $f'(x) = 0$ on $(0, \infty)$

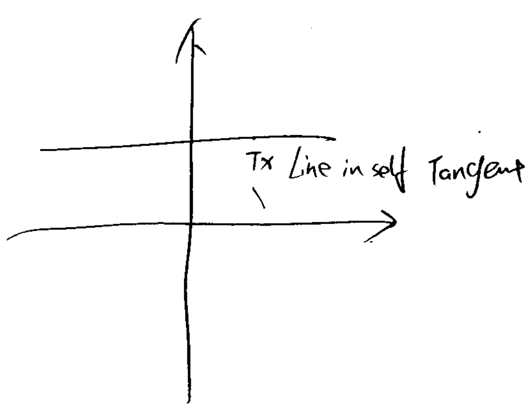
f is not differentiable at 0, since f is discontinuous at $x = 0$. Thus the graph of f' has an open circle at $x = 0$.

on $(0, \infty)$, the slopes of the tangents are positive

(at $x = 1$ it is about 2)

but decreasing. However as $x \to \infty$, $f' \to 0^+$

at slower and slower rate.



$f(x) = 100$

$f'(x) = 0$

$(100)' = 0.$

(IV)