

# The Endogenous Grid Method for Discrete-Continuous Dynamic Choice Models with (or without) Taste Shocks, Quantitative Economics, 2017

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## The Sequential Problem

- Consider the discrete-continuous (DC) dynamic optimization problem.

$$\max_{\{c_t, d_t\}_{t=1}^T} \sum_{t=1}^T \beta^t (\log(c_t) - \delta d_t), \quad (1)$$

involving choices of consumption  $c_t$  and whether to keep working  $d_t$ . Let  $d_t = 0$  denote retirement, let  $d_t = 1$  denote continued work, and let  $\delta > 0$  be the disutility of work. We assume retirement is absorbing.

- A sequence of period-specific borrowing constraints,  $c_t \leq M_t$ , where

$$M_t = R(M_{t-1} - c_{t-1}) + yd_{t-1}$$

is consumer's consumable resources (wealth) at the beginning of period  $t$ .

- The continuous consumption decision and discrete retirement decision are made at the start of each period, whereas interest earnings and labor income are paid at the end of the period.

## The Functional Problem

- Let  $V_t(M)$  and  $W_t(M)$  be the expected discounted lifetime utility of a worker and a retiree, respectively, in period  $t$  of their life.
- The Bellman equation for  $V_t(M)$  is

$$V_t(M) = \max\{v_t(M, 0), v_t(M, 1)\}, \quad (2)$$

where the **choice-specific value functions**  $v_t(M, d)$ ,  $d \in \{0, 1\}$  are given by

$$v_t(M, 0) = \max_{0 \leq c \leq M} \{\log(c) + \beta W_{t+1}(R(M - c))\}, \quad (3)$$

$$v_t(M, 1) = \max_{0 \leq c \leq M} \{\log(c) - \delta + \beta V_{t+1}(R(M - c) + y)\}. \quad (4)$$

- The Bellman equation for  $W_t(M)$  is

$$\max_{0 \leq c \leq M} \{\log(c) + \beta W_{t+1}(R(M - c))\}. \quad (5)$$

- RHS of Equation 3 is identical of that of Equation 5. Therefore, we have  $W_t(M) = v_t(M, 0)$ , and the consumption function of the retiree is identical to the choice-specific consumption function of the worker who decided to retire,  $c_t(M, 0)$ .

## Kinks and Discontinuities

- Even if  $v_t(M, 0)$  and  $v_t(M, 1)$  are concave functions of  $M$ , the value function as the maximum of these two concave functions will generally not be globally concave.
- Further,  $V_t(M)$  will generally have a kink point at the value  $M = \bar{M}_t$  where the two choice-specific value functions cross, that is,  $v_t(\bar{M}_t, 1) = v_t(\bar{M}_t, 0)$ . We refer to these as **primary kinks** because they constitute optimal retirement thresholds for the worker in each period  $t$ .
- The worker is indifferent between retiring and working at the primary kink  $\bar{M}_t$ , and  $V_t(M)$  is nondifferentiable at this point. However, the left and right hand side derivatives,  $V_t^-(M)$  and  $V_t^+(M)$ , exist and satisfy  $V_t^-(M) < V_t^+(M)$ .
- The discontinuity in the derivative of  $V_t(M)$  at  $\bar{M}_t$  leads to a discontinuity in the optimal consumption function in the previous period  $t - 1$  because the Bellman equation for  $V_{t-1}(M)$  depends on  $V_t(M)$ . In turn, this causes a kink in  $V_{t-1}(M)$  that we label a **secondary kink** since it is a reflection of the primary kink in  $V_t(M)$ .

## Analytical Solution to the Retirement Problem

## Theorem

*Theorem 1. Assume that income and disutility of work are time-invariant, the discount factor  $\beta$  and the disutility of work  $\delta$  are not too large, that is,*

$$\beta R \leq 1 \quad \text{and} \quad \delta < (1 + \beta) \log(1 + \beta), \quad (6)$$

and instantaneous utility is given by  $u(c) = \log(c)$ . Then for  $\tau \in \{1, \dots, T\}$  the optimal consumption rule in the worker's problem 2-4 is given by

# Analytical Solution to the Retirement Problem

## Theorem

$$c_{T-\tau}^w(M) = \begin{cases} M & \text{if } M \leq y/R\beta, \\ [M + y/R]/(1 + \beta) & \text{if } y/R\beta \leq M \leq \bar{M}_{T-\tau}^{l_1}, \\ [M + y(1/R + 1/R^2)]/(1 + \beta + \beta^2) & \text{if } \bar{M}_{T-\tau}^{l_1} \leq M \leq \bar{M}_{T-\tau}^{l_2}, \\ \dots & \dots \\ \left[ M + y \left( \sum_{i=1}^{\tau-1} R^{-i} \right) \right] \left( \sum_{i=0}^{\tau-1} \beta^i \right)^{-1} & \text{if } \bar{M}_{T-\tau}^{l_{\tau-2}} \leq M \leq \bar{M}_{T-\tau}^{l_{\tau-1}}, \\ \left[ M + y \left( \sum_{i=1}^{\tau} R^{-i} \right) \right] \left( \sum_{i=0}^{\tau} \beta^i \right)^{-1} & \text{if } \bar{M}_{T-\tau}^{l_{\tau-1}} \leq M < \bar{M}_{T-\tau}^{r_{\tau-1}}, \\ \left[ M + y \left( \sum_{i=1}^{\tau-1} R^{-i} \right) \right] \left( \sum_{i=0}^{\tau} \beta^i \right)^{-1} & \text{if } \bar{M}_{T-\tau}^{r_{\tau-1}} \leq M < \bar{M}_{T-\tau}^{r_{\tau-2}}, \\ \dots & \dots \\ [M + y(1/R + 1/R^2)] \left( \sum_{i=0}^{\tau} \beta^i \right)^{-1} & \text{if } \bar{M}_{T-\tau}^{r_2} \leq M < \bar{M}_{T-\tau}^{r_1}, \\ [M + y/R] \left( \sum_{i=0}^{\tau} \beta^i \right)^{-1} & \text{if } \bar{M}_{T-\tau}^{r_1} \leq M < \bar{M}_{T-\tau}, \\ M \left( \sum_{i=0}^{\tau} \beta^i \right)^{-1} & \text{if } M \geq \bar{M}_{T-\tau}. \end{cases} \quad (7)$$





## Analytical Optimal Consumption Function

- Theorem 1 establishes that the optimal consumption rule of the worker  $c_{T-t}(M, 1)$  is piecewise linear in  $M$ , and in period  $t$  consists of  $2(T - t) + 1$  segments.
- The first segment where  $M < \frac{y}{R\beta}$  is the credit constrained region where the agent consumes all available wealth and does not save.
- The next  $T - t - 1$  segments are demarcated by the **liquidity constraint kink points**  $\overline{M}_t^{l_j}$  that define values of  $M$  at which the consumer is liquidity constrained at age  $t + j$  but not at any earlier age.
- The remaining segments are defined by the secondary kinks,  $\overline{M}_t^{r_j}, j = 1, \dots, T - t - 1$ , and represent the largest level of saving for which it is optimal to retire at age  $t + j$  but not at any earlier age.
- Finally,  $\overline{M}_t$  is the retirement threshold, which denotes the minimum level of wealth that is required to retire.
- The optimal consumption function is discontinuous at points  $\overline{M}_t^{r_j}$  and  $\overline{M}_t$ , so in total there are  $T - t$  downward jumps in the consumption function.

## Analytical Value Function $V_t(M)$

- Theorem 1 implies that the value function  $V_t(M)$  is piecewise logarithmic with the same kink points, and can be written as  $V_t(M) = B_t \log(c_t(M, d)) + C_t$  for constants  $(B_t, C_t)$  that depend on the region that  $M$  falls into.
- The function  $V_t(M)$  has one primary kink at the optimal retirement threshold  $\bar{M}_t$  and  $T - t - 1$  secondary kinks at  $\bar{M}_t^{r_j}, j = 1, \dots, T - t - 1$ .
- In addition, there are  $T - t$  kinks related to current and future liquidity constraints at  $M = \frac{y}{R\beta}$  and  $\bar{M}_t^{l_j}, j = 1, \dots, T - t - 1$ .





## Solution in Period $T$

- Consider the terminal period  $T$ . The optimal consumption rule is to consume all available wealth and, thus, is given by  $c_T(M, d) = M$ . With positive disutility of working, all agents retire since income is paid at the end of the period. This  $T$  period solution is the base for backward induction.









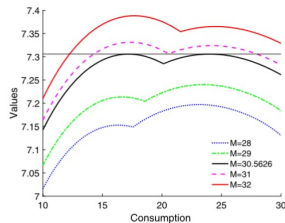




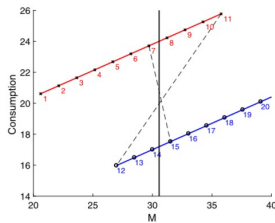


# Graphical Illustration of the Selection Process

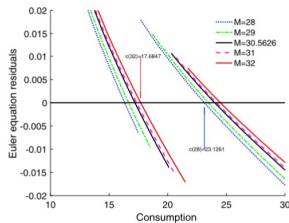
(a) Multiple local optima near secondary kink



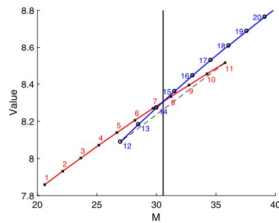
(b) Non-monotonic endogenous grid  $\vec{M}$



(c) Multiple solutions to Euler equations



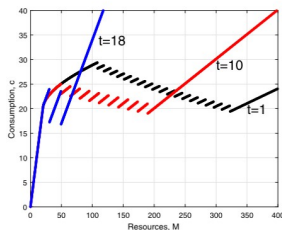
(d) Construction of the upper envelope



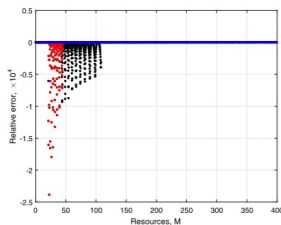
- The most important step is in panel (b) and (d).
- We need to check monotonicity of the endogenous wealth grid for each choice and construct the upper envelope for choice-specific value function in certain regions.
- Link to algorithm [◀ Upper Envelope](#)

# Graphical Illustration of the Nonmonotonicity and Selection Process

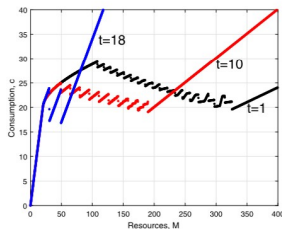
(a) Analytical Solution



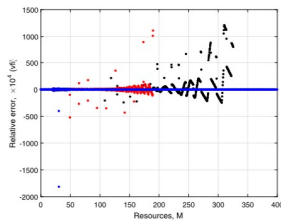
(b) Relative Error of DC-EGM solution



(c) Off the shelf VFI solution



(d) Relative Error of VFI solution



$R = 1, \beta = 0.98,$   
 $y = 20, T = 20.$

**MATLAB Implementation!!!**

## Effects of introducing income shocks and taste shocks

- First, because the discrete choice policy is expressed in probabilistic terms, the calculation of primary kink points is no longer needed.
- Second, the process of accumulation of secondary kinks is perturbed: the perturbations caused by the primary kinks that remain throughout the backward induction process in the deterministic setting “fade out” in the presence of shocks.
- Third, the calculation of expectations over random income in the problem with taste shocks can be performed with standard numerical algorithms, as opposed to the setting with random income but without taste shocks.









## Smoothed Euler Equation

- If we continue to assume that retirement is an absorbing state, the problem of the retiree remains the same, and we focus again on the worker's problem.
- The smoothed Euler equation with taste and income shocks derived from equations 12, 3, and 13 is

$$\begin{aligned}
 0 &= u'(c) - \beta R \int [u'(c_{t+1}(R(M-c) + y\eta, 1)) P_{t+1}(1 | R(M-c) + y\eta) + \\
 &\quad u'(c_{t+1}(R(M-c) + y\eta, 0)) P_{t+1}(0 | R(M-c) + y\eta)] f(d\eta) \\
 &= \frac{1}{c} - \beta R \int \left[ \frac{P_{t+1}(1 | R(M-c) + y\eta)}{c_{t+1}(R(M-c) + y\eta, 1)} + \frac{P_{t+1}(0 | R(M-c) + y\eta)}{c_{t+1}(R(M-c) + y\eta, 0)} \right] f(d\eta),
 \end{aligned} \tag{16}$$

where  $P_{t+1}(d | M)$  and conditional choice probabilities 15.

## Solution in Period $T$

- The induction starts at the terminal period  $T$  with the easily derived consumption functions

$$c_T(M, 0) = c_T(M, 1) = M,$$

choice-specific value functions

$$v_T(M, 0) = u(M) = v_T(M, 1) + \delta,$$

and the probability of remaining working

$$P_T(1 | M) = \frac{1}{1 + \exp(\delta/\sigma_\varepsilon)}.$$





## Pseudo Codes for EGM Algorithm: Set up the Iteration

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### Algorithm 1 The DC-EGM algorithm

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**Input:** Structural parameters, utility function  $u(c)$ , number of time periods  $T$ , number of grid points  $J$ , upper bound on wealth  $\bar{M}$ .

```

1: Fix the grid over savings  $\bar{A} = \{A_1, \dots, A_J\}$  such that  $A_1 = 0$  and  $A_j < A_{j+1}$ 
2: for  $t = T, \dots, 1$  do                                     ▷ Backward induction over time periods
3:   for  $st \in S$  do                                           ▷ For every state (worker, retired)
4:     for  $d = \{0, 1\}$  if  $st = \text{worker, or}$ 
                           $d = 0$  if  $st = \text{retired}$  do           ▷ For all admissible discrete choices
5:       if  $t = T$  then                                         ▷ Terminal period
6:         Set consumption function  $c_T(M, d) = M$ 
7:         Set policy function  $v_T(M, d) = u(M) + d\delta$ 
8:       else                                                   ▷ All periods  $t < T$ 
9:         Call EGM STEP (Algorithm 2)
          Input: next period consumption and value functions  $c_{t+1}(M, d')$ ,  $v_{t+1}(M, d')$ ,
           $d' \in \{0, 1\}$ 
          Output: consumption and value functions  $c_t(\bar{M}_t^d, d)$ ,  $v_t(\bar{M}_t^d, d)$  over endoge-
          nous grid  $\bar{M}_t^d$ 
10:        Call UPPER ENVELOPE (Algorithm 3)
          Input: endogenous grid  $\bar{M}_t^d$ , consumption and value functions  $c_t(\bar{M}_t^d, d)$ ,
           $v_t(\bar{M}_t^d, d)$ 
          Output: refined grid  $\bar{M}_t^{sd}$ , consumption and value functions  $c_t(\bar{M}_t^{sd}, d)$ ,
           $v_t(\bar{M}_t^{sd}, d)$ 
11:      end if
12:    end for
13:  end for
14: end for

```

**Output:** The collection of the choice-specific consumption and value functions  $c_t(M, d)$  and  $v_t(M, d)$  defined on the endogenous grids  $\bar{M}_t^d$  for both worker and retiree,  $d = \{0, 1\}$  and  $t = \{1, \dots, T\}$  constitutes the solution of the consumption-savings and retirement model



