DC-EGM, QE, 2017

1. Model with Only One Continuous Choice Variable

1.1 Model Setup

$$V_t(M_t) = \max_{c_t \in [0, M_t]} [u(c_t) + \beta E V_{t+1} (R(M_t - c_t) + \tilde{y})]$$

1. M_t is the cash-in-hand, all resources available at period t.
   $A_t = M_t - c_t$ is the assets at the end of period t (savings)
   R is the deterministic return on savings
   $\tilde{y}$ is the stochastic income

1.2 Traditional Value Function Iteration (VFI)

1. Fix a grid over $M_t, t = 1, \dots, T$, $\mathcal{M} = \{M^1, \dots, M^J\}$.
2. In the terminal period T, we calculate the value function $V_T(M_T) = \max_{c_T} \{u(c_T)\} = u(M_T)$ and $c_T^*(M_T) = M_T$, where $M_T \in \mathcal{M}$.
3. In period $T-1$, we similarly do the same procedure,

$$V_{T-1}(M) = \max_{c_{T-1}} \{u(c_{T-1}) + \beta E [V_T (R(M - c_{T-1}) + \tilde{y})]\},$$

where the expectation is over all possible states of income $\tilde{y}$. And for each possible state of $\tilde{y}$, we can get $V_T (R(M - c_{T-1}) + \tilde{y})$ for each grid point $M \in \mathcal{M}$. And $c_{T-1}^*(M) = \arg\max \{u(c_{T-1}) + E V_T (R(M - c_{T-1}) + \tilde{y})\}$.

4. Note that we need interpolate the function $V_T(M)$ since we only know its values at grid points $M^j \in \mathcal{M}$, while to step forward, we need to calculate its values at points $R(M^j - c_{T-1}) + \tilde{y}$.

1.3 Standard Endogenous Grid Method

1. We can first obtain the Euler function

$$u'(c_t) = \beta E [u'(c_{t+1}) R]$$

2. The traditional approach to solve the Euler equation:
   1. Fix a grid over $M_t, t = 1, \dots, T$, $\mathcal{M} = \{M^1, \dots, M^J\}$. For every point M in the grid:
   2. first calculate the policy function in period T, $c_T^*(M) = M$;
   3. then proceed backward and get $c_{T-1}^*(M) = (u')^{-1} (E [\beta u'(c_T(M)) R])$.
   4. But when M_t is small enough, the credit constraint is binding, so the Euler equation is not a necessary condition for optimization; thus, special treatment is needed.

3. EGM Algorithm proposed by Carroll (2006, Economics Letters):
   1) Fix a grid over $A_t, t = 1, \dots, T$, $\mathcal{A} = \{A^1, \dots, A^J\}$. For every point A^j in the grid:
   2) first calculate the policy function in period T, $c_T^*(M) = M, \overline{c}_T^*(A) = 0$ (note that the function $\overline{c}$ maps end-of-period savings to optimal consumption in that period);
   3) then proceed back and first get $M_T^j = RA^j + \tilde{y}$, then get $\overline{c}_{T-1}^*(A^j) = (u')^{-1} \left(E \left[\beta u'(c_T^*(M_T^j)) R\right]\right)$, finally according to the mapping from A^j to M_{T-1}^j: $M_{T-1}^j = A^j + \overline{c}_{T-1}^*(A^j)$, we have set up the mapping from M_{T-1}^j to $c_{T-1}^*(M_{T-1}^j)$.
   4) Note that importantly, the grid points of $\mathcal{M}_{T-1}$ are endogenously determined by the Euler equation and linking between end-of-period saving, optimal consumption as a function of end-of-period savings, and period state variable (cash in hand, total resources).

4. In terms of implementation, we need to interpolate the function $c_{T-1}^*(M_{T-1})$ from all the endogenous grid points we know $M_{T-1}^j, j = 1, \dots, J$.

5. Thanks to this, we can easily incorporate credit constraint $M_t \leq M^{cc}, \forall t$ into this algorithm, as long as we set up two points $(0, 0)$ and (M^{cc}, M^{cc}) and use linear interpolation, then naturally it connects these two points with a 45 degree line and the constraint is satisfied.

1.4 Parametrization and MATLAB Implementation

$$\beta = 0.95, \quad \log(\tilde{y}) \sim \text{Normal}(1, \sigma^2), \quad R = 1.05,$$
$$T = 25, \quad M \in [0, 10], \quad J = 100.$$

2. A Deterministic Model with One Continuous and One Discrete Choice Variable

2.1 Model Setup

1. The sequential problem:

$$\max_{\{c_t, d_t\}_{t=1}^T} \sum_{t=1}^{T} \beta^t \left(\log\left(c_t\right) - \delta d_t\right),$$

where $d_t = 1$ denotes remaining working, while $d_t = 0$ denotes retirement. And the flow budget constraint is

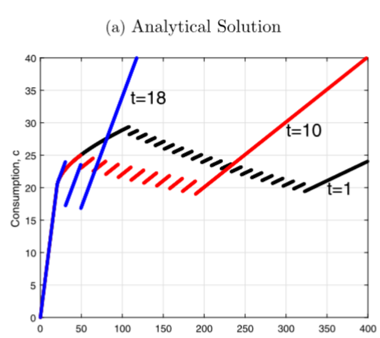
$$M_t = R(M_{t-1} - c_{t-1}) + y d_{t-1}.$$

2. Problem in the functional equation form:

$$V_t(M, 1) = \max\left\{v_t(M, 0), v_t(M, 1)\right\},$$
$$V_t(M, 0) = \max_{0 \leq c \leq M} \left\{\log(c) + \beta V_{t+1}\left(R(M - c), 0\right)\right\},$$
$$v_t(M, 0) = \max_{0 \leq c \leq M} \left\{\log(c) + \beta V_{t+1}(R(M - c), 0)\right\},$$
$$v_t(M, 1) = \max_{0 \leq c \leq M} \left\{\log(c) - \delta + \beta V_{t+1}(R(M - c) + y, 1)\right\}.$$

where $V_t(M, d)$ denotes the value function as a function of start-of-period wealth (state variable 1) is M and working status (state variable 2) is d; and $v_t(M, d)$ denotes the condition value function when start-of-period wealth (state variable 1) is M conditional on agent's choice of working status in period t is d. Note the difference between state variable 2, the working status (remaining working or retirement) and choice in period t (in this period, does the agent retire or not).

3. Analytical Solution (optimal consumption as a function of state variable M in time t): primary kinks and secondary kinks



Note: This is the consequence of combining a continuous state variable and a discrete state variable!

4. Euler Equations:

$$u'(c_t) = \beta R u'\left(c_{t+1}^{w*}\left(R(M - c) + y\right)\right),$$
$$u'(c_t) = \beta R u'\left(c_{t+1}^{r*}\left(R(M - c)\right)\right),$$

where c^{w*}_{t+1} denotes the *state-specific* optimal policy function if the agent remains working in period $t+1$, c^{r*}_t denotes the *state-specific* optimal policy function if the agent is retiring in period $t+1$, while $c_t = c_t(M, d)$ denotes the *choice-specific* consumption function in period t if he chooses d.

5. Theoretical Foundation of Improving the Standard EGM Algorithm:
   Theorem 2 (Monotonicity of the Saving Function). Let $A_t(M, d) = M - c_t(M, d)$ denote the savings function implied by the optimal consumption function $c_t(M, d)$. If $u(c)$ is a concave function, then for each $t \in \{1, \ldots, T\}$ and each discrete choice $d \in \{0, 1\}$ the optimal saving function $A_t(M, d) = M - c_t(M, d)$ is monotone nondecreasing in M.

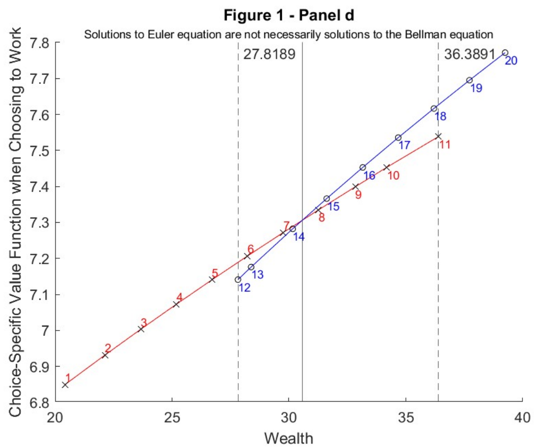
2.2 DC-EGM for the Deterministic Model

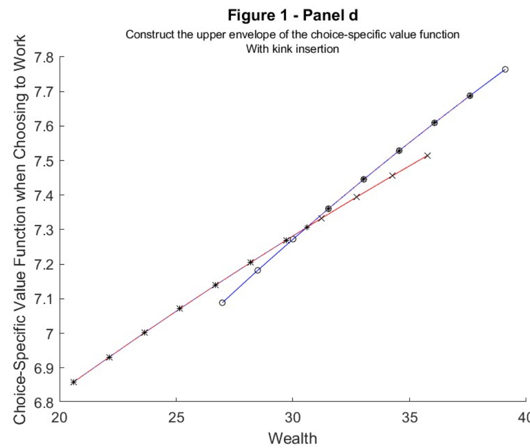
1. Fix a grid over $A_t, t = 1, \ldots, T$, $\mathcal{A} = \{A^1, \ldots, A^J\}$.
2. In the terminal period T, optimal consumption rule is to consume all available wealth and, thus, is given by $c^{d*}_T(M) = M$, for each state variable 2, $d \in \{0, 1\}$.
3. In the period $T-1$, for the retirees, consider a specific grid point A^j, we first get $M^j_T = RA^j$ and then we can use the corresponding Euler equation $u'(\overline{c}^*_{T-1}(A^j, 0)) = \beta R u'\left(c^{r*}_T\left(M^j_T\right)\right)$ to get the mapping between A^j to $\overline{c}^*_{T-1}(A^j, 0)$. Finally, we know that $M^j_{T-1} = A^j + \overline{c}^*_{T-1}(A^j, 0)$, thus, we can build up the mapping from M^j_{T-1} to $\overline{c}^*_{T-1}(A^j, 0)$, i.e., $c^*_{T-1}(M_{T-1}, 0)$.
4. In the period $T-1$, we conduct the same process for workers, except this time, we use the corresponding Euler equation, $u'(\overline{c}^*_{T-1}(A^j, 1)) = \beta R u'\left(c^{w*}_T\left(RA^j + y\right)\right)$ and build up the policy function $c^*_{T-1}(M_{T-1}, 1)$.
5. Given our calculation of the choice-specific consumption function, we can obtain the choice-specific value function $v_{T-1}(M, d)$, which is calculated by $v_{T-1}(M, 0) = \log(c^*_{T-1}(M, 0)) + \beta V_T\left(R(M - c^*_{T-1}(M, 0)), 0\right)$ and $v_{T-1}(M, 1) = \log(c^*_{T-1}(M, 1)) - \delta + \beta V_T\left(R(M - c^*_{T-1}(M, 1)) + y, 1\right)$. Note that interpolation of the next-period value function is needed.
6. Now, we construct the upper envelope of these two choice-specific value function $v_{T-1}(M, 1)$ and $v_{T-1}(M, 0)$ to get the unconditional value function $V_{T-1}(M, 1)$. While the other unconditional value function $V_{T-1}(M, 0)$ is obtained by $V_{T-1}(M, 0) = \log(c^*_{T-1}(M, 0)) + \beta V_T\left(R(M - c^*_{T-1}(M, 0)), 0\right)$.
7. Finally, we can assert that the optimal consumption function in period $T-1$ is

$$c^{w*}_{T-1} = c^*_{T-1}(M, 1) \cdot \mathbf{I}\left(v_{T-1}(M, 1) > v_{T-1}(M, 0)\right) + c^*_{T-1}(M, 0) \cdot \mathbf{I}\left(v_{T-1}(M, 1) < v_{T-1}(M, 0)\right),$$
$$c^{r*}_{T-1} = c^*_{T-1}(M, 0).$$

8. To proceed into period $T-2$, difficulties occur because of the kinks and multiple local maxima, implying that the Euler equations are not necessary for global maximization.
9. To see the difficulties, see the next two graphs which show that the solutions to the Euler equations are not necessarily strictly increasing!

These two panels are constructed by constructing exactly the same exogenous savings grid as in the paper. In the left panel, the number index shows the exogenous savings levels. The x and y axis are endogenous wealth grid points and choice specific value function, respectively. The endogenous wealth grid clearly has a non-monotonic region. In the right panel, the y axis shows the conditional value function. Clearly, the solution to the Bellman equation is the upper envelope of these two segments. But direct solution to the Euler equation fails to account for this!






```
% To construct these two graphs, run test_fun_UpperEnvlp_Graphical_Illustration.m.
% The loaded test_data_forFig1.mat is constructed by setting gridN = 399 in model2_deterministic_retirement.m file
%   and then collect T=18 choice-specific
gridN = 399;
wealth = choice_w{18}(1:21, 2);
consum = choice_w{18}(1:21, 1);
valuec = choice_w{18}(1:21, 3);
clearvars -except wealth consum valuec
wealth=wealth(2:end); consum=consum(2:end); valuec=valuec(2:end);
a1 = wealth(1:11); a2 = wealth(12:20);
b1 = valuec(1:11); b2 = valuec(12:20);
save test_data_forFig1
```

10. To tackle this problem, we need the further steps to refine the endogenous wealth grid to delete the suboptimal points in the sense that they are not solutions to the Bellman equation.
    a. Step 1: Detect the non-increasing wealth region, that is, find the index of which there is a sudden decrease in endogenous wealth.
    b. Step 2: Find the next increasing wealth region, that is, find another index of which endogenous wealth grid points are increasing.
    c. Step 3: Now, these two points cut the original wealth grid into three segments. Construct the upper envelope of these three segments and store which indices are deleted.
    d. Step 4: Delete the corresponding consumption and conditional value points.

Specifically, in MATLAB, we use the `cls_line` class with two important `cls_upper_envlp()` and `cls_outer_refinement()` methods to implement this step!

11. Having obtained the correctly approximated choice-specific value and consumption function, we then construct the upper envelope of the two conditional value functions to determine one's optimal choice in any period, and construct one's unconditional value function and optimal consumption function. (Same as step 6 and 7 above.)

Remark:
The key difficulty in this problem is that due to nonconvexity of the value function, solutions to the Euler equation do not necessarily solve the Bellman equation.
And the only additional step is to delete the suboptimal solutions by first detecting non-increasing region then using upper envelope of conditional value function to determine which points in that region are to be deleted.

2.3 MATLAB Implementation

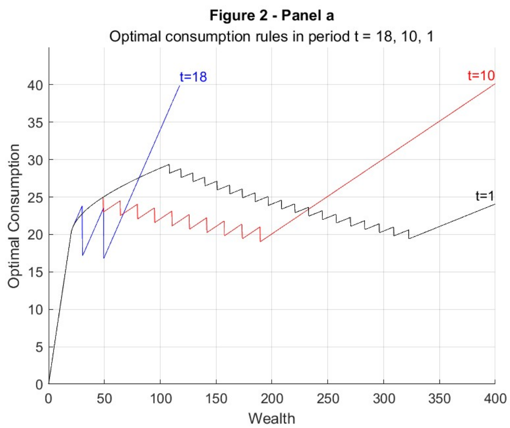
1. The retiree's problem is the same as a standard EGM algorithm.
2. For the worker's problem, the additional step to delete the suboptimal endogenous wealth grid points are integrated in the class `cls_line`. In this class, the most important method is `cls_upper_envlp()`, which deletes original points based on upper envelope calculation and adds intersection points.

Inside this method, the most confusing part is to determine the upper envelope when the interpolated values reach maximum on another line. Details are the red part in the following code file.

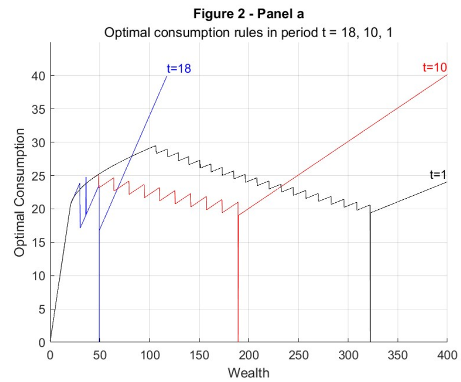
cls_upper_envlp() method for DC-EGM Algorithm

3. The other parts are the same as the standard EGM algorithm, except here we need to compare the two choice-specific value functions to determine one's choice in any period. And this is realized through the function fun_Choice2.

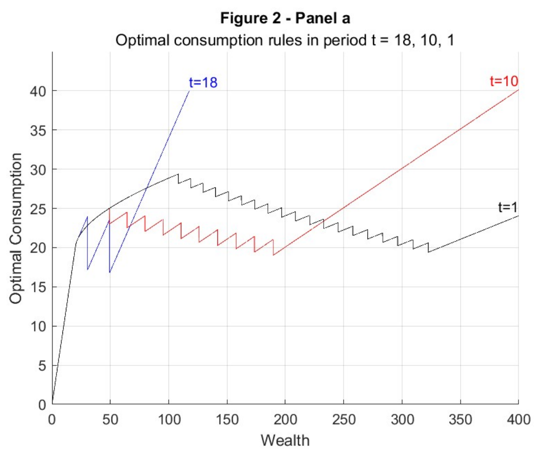
4. The results are



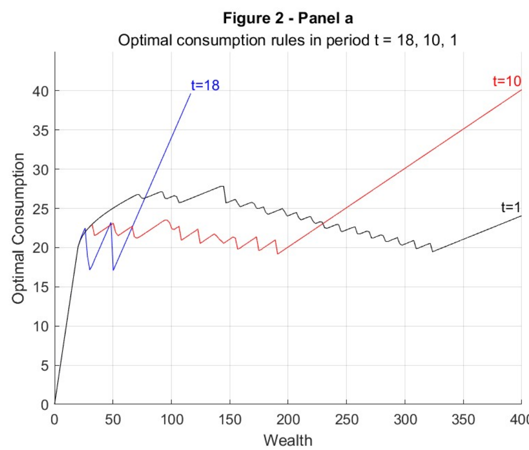
gridN=2e+3; InterpMethod = 'spline';



gridN=2e+3; InterpMethod = 'linear';



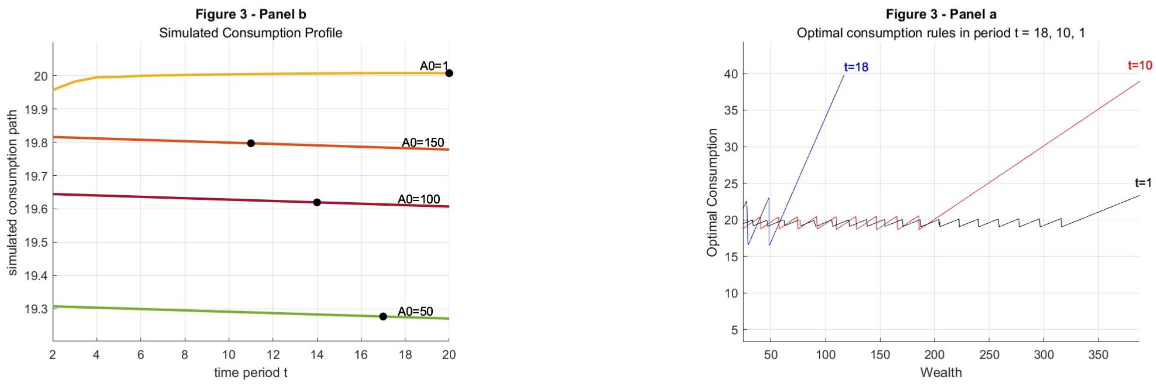
gridN=2e+4; InterpMethod = 'spline';



gridN=2e+2; InterpMethod = 'spline';

Notes: when interpolating next-period value function, since it is non-linear, InterpMethod = 'linear' can sometimes generate some weird results. The number of grid points used in exogenous savings grid also have considerable effects on the final results. In this case, when gridN>1e+3, the results are indistinguishable but the computational time can increase greatly.

5. Another special case occurs when $R = 1/\beta = 1.02$. The last two graphs show the optimal consumption function and simulated consumption profile given different initial asset levels. Under this parameterization, individuals fully smooth their consumption path.



Even in the original graph, the simulated consumption path is quite weird. Why a higher initial wealth level does not translate into higher lifecycle consumption?

3. A Model of Consumption-Savings and Retirement with Income and Taste Shocks

3.1 Model Setup

1. The income process is $y_t = y\eta_t$, where $\log \eta_t \sim \mathcal{N}(-\frac{\sigma_\eta^2}{2}, \sigma_\eta^2)$.
2. The taste shock is additive separable and has the form of $\sigma_\varepsilon \varepsilon(d_t)$, where $\varepsilon(d_t)$ are iid standard extreme value distribution.
3. For the worker's problem, the Bellman equation is

$$V_t(M, \varepsilon) = \max \left\{ v_t(M, 0) + \sigma_\varepsilon \varepsilon(0), v_t(M, 1) + \sigma_\varepsilon \varepsilon(1) \right\}.$$

where the conditional value function $v_t(M, 0)$ is the same as the baseline model, while

$$v_t(M, 1) = \max_{0 \le c \le M} \left\{ \log(c) - \delta + \beta \int EV_{t+1}^{\sigma_\varepsilon}(R(M-c) + y\eta) f(d\eta) \right\}.$$

The integrated value function (integrated in the sense that it takes expectation over future taste shocks) is

$$\begin{aligned} EV_{t+1}^{\sigma_\varepsilon}(M, 1) &= E_\varepsilon \left[V_{t+1}(M, \varepsilon) \right] \\ &= E_\varepsilon \left[\max \left\{ v_{t+1}(M, 0) + \sigma_\varepsilon \varepsilon(0), v_{t+1}(M, 1) + \sigma_\varepsilon \varepsilon(1) \right\} \right] \\ &= \sigma_\varepsilon \log \left(\exp \left\{ \frac{v_{t+1}(M, 0)}{\sigma_\varepsilon} \right\} + \exp \left\{ \frac{v_{t+1}(M, 1)}{\sigma_\varepsilon} \right\} \right). \end{aligned}$$

4. Therefore, the conditional choice in period t is

$$P_t(d \mid M) = \frac{\exp \{ v_t(M, d) / \sigma_\varepsilon \}}{\exp \{ v_t(M, 1) / \sigma_\varepsilon \} + \exp \{ v_t(M, 0) / \sigma_\varepsilon \}}, \quad d \in \{0, 1\}.$$

5. Finally, the smoothed Euler equation is

$$\begin{aligned} u'(c) =& \beta R \int \left[u'\left(c_{t+1}(R(M-c) + y\eta, 1) \right) P_{t+1}(1 \mid R(M-c) + y\eta) + u'\left(c_{t+1}(R(M-c) + y\eta, 0) \right) P_{t+1}(0 \mid R(M-c) \right. \\ \frac{1}{c} =& \beta R \int \left[\frac{P_{t+1}(1 \mid R(M-c) + y\eta)}{c_{t+1}(R(M-c) + y\eta, 1)} + \frac{P_{t+1}(0 \mid R(M-c) + y\eta)}{c_{t+1}(R(M-c) + y\eta, 0)} \right] f(d\eta). \end{aligned}$$

3.2 MATLAB Implementation

1. Compared to the baseline deterministic model,
    a. We do not need to construct the upper envelope of the two choice-specific value functions. Instead, we calculate the probability of choosing to work and retire in period t when having wealth M.
    b. But, we need to do some integration to calculate the expectation over a stochastic income.
    c. More importantly, the refinement process of endogenous wealth grid is still necessary!
2. See file `model3_AddShocks.m`.