The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems - Economics Letter - 2006 (Part 3 Macro Specialization)

1. Model:

1. The goal is to $\max \sum_{t=1}^{T} u(C_t)$, where $u(C_t) = \frac{C_t^{1-\rho}}{1-\rho}$.
2. The laws of motion for whole resources M_t, for labor L_t and for capital K_t are:

$$\begin{gathered} M_t = W_t L_t + R_t K_t + K_t, \\ L_{t+1} = L_t G_{t+1} \phi_{t+1}, \\ K_{t+1} = (M_t - C_t) * \tau. \end{gathered}$$

3. Denote period t saving by $A_t = M_t - C_t$.
   Denote the only stochastic terms by $\Delta_t = G_t \phi_t$, that is, stochasticity of this model only comes from labor supply.
   Denote the per capita terms by $m_t = \frac{M_t}{L_t}, k_t = \frac{K_t}{L_t}, a_t = \frac{A_t}{L_t}, c_t = \frac{C_t}{L_t}$.
4. The returns W_t and R_t are pinned down by the production function $Y_t = K_t^{\alpha} L_t^{1-\alpha}$ and competitive market assumption, i.e.,

$$\begin{gathered} W_t = (1-\alpha) k_t^{\alpha}, \\ R_t = \alpha k_t^{\alpha-1}. \end{gathered}$$

5. Given all the above assumptions, we can easily get the laws of motion for per capita terms:

$$\begin{gathered} a_t = m_t - c_t, \\ k_{t+1} = (m_t - c_t) \tau \Delta_{t+1}^{-1} = a_t \tau \Delta_{t+1}^{-1}, \\ m_{t+1} = W_{t+1} + R_{t+1} k_{t+1} + k_{t+1} = k_{t+1} + k_{t+1}^{\alpha}. \end{gathered}$$

6. Now, we can re-formulate the problem into a functional equation with m_t as the only state variable, and c_t as the choice variable:

$$v_t(m_t) = \max_{c_t} \left\{ u(c_t) + \beta E \left[\Delta_{t+1}^{1-\rho} v_{t+1}(m_{t+1}) \right] \right\} = \max_{a_t} \left\{ u\left(m_t - a_t\right) + \beta E \left[\Delta_{t+1}^{1-\rho} v_{t+1}\left(\mathbf{m_{t+1}}(\mathbf{k_{t+1}}(\mathbf{a_t}))\right) \right] \right\},$$

where the bold part in the last term means the function from our choice variable c_t to next period state variable m_{t+1}.

7. Give specific values:

$$\begin{gathered} \beta = 0.96, \rho = 2, \\ \alpha = 0.36, \tau = 0.9, \\ \Delta_t \in \{0.99, 1.00, 1.21\}, \text{with probabilities } 0.25,\ 0.5,\ 0.25, \text{ respectively.} \end{gathered}$$

2. EGM: Euler equation

1. The first-order condition with respect to k_{t+1} is

$$u'(c_t) = \beta E \left[\Delta_{t+1}^{1-\rho} v'_{t+1}\left(m_{t+1}\right) * \frac{\partial m_{t+1}}{\partial k_{t+1}} * \frac{\partial k_{t+1}}{\partial a_t} \right] = \beta E \left[\Delta_{t+1}^{1-\rho} v'_{t+1}\left(m_{t+1}\right) * \left(1 + \alpha k_{t+1}^{\alpha-1}\right) * \tau \Delta^{-1} \right] = \tau \beta E \left[\Delta_{t+1}^{-\rho} v'_{t+1}$$

2. The envelopment theorem is

$$\begin{aligned} v'_t(m_t) &= \beta E \left[\Delta_{t+1}^{1-\rho} v'_{t+1}\left(m_{t+1}\right) * \frac{\partial m_{t+1}}{\partial m_t} \right] = \beta E \left[\Delta_{t+1}^{1-\rho} v'_{t+1}\left(m_{t+1}\right) * \frac{\partial m_{t+1}}{\partial k_{t+1}} * \frac{\partial k_{t+1}}{\partial m_t} \right] \\ &= \tau \beta E \left[\Delta_{t+1}^{-\rho} v'_{t+1}\left(m_{t+1}\right) * \left(1 + \alpha k_{t+1}^{\alpha-1}\right) \right] = u'(c_t), \end{aligned}$$

where the last equality comes from the FOC. So, we have

$$u'(c_{t+1}) = v'_{t+1}(m_{t+1}).$$

3. Finally, the Euler equation is

$$u'(c_t) = \tau\beta E\left[\Delta^{-\rho}u'(c_{t+1}) * \left(1+\alpha k_{t+1}^{\alpha-1}\right)\right].$$

4. For our purpose,
   (1) we first generate a exogenous grid of end-of-period savings $\mathcal{A} = \{a^1, \ldots, a^J\}$;
   (2) for the last period T, the optimal consumption level is $c_T^*(m_T) = m_T$;
   (3) having obtained period T consumption level, we step backward to period $T-1$, again, for a given grid point $a_{T-1}^i = a^i$, $k_T^i = a_{T-1}^i\tau\Delta^{-1}, m_T^i = k_T^i + (k_T^i)^\alpha$, and $c_T^i = c_T^*(m_T)$ (note that k_t^i, m_T^i, c_T^i are all random variables);
   (4) therefore, RHS of the Euler function can be calculated for this specific grid point, $\text{RHS} = \tau\beta E\left[\Delta^{-\rho}u'(c_T^i) * (1+\alpha(k_T^i)^\alpha)\right]$, combining with the Euler function, we can get $\overline{c}_{T-1}^{i*}$ (a function of a^i);
   (5) finally, the endogenous grid of resource is $m_{T-1}^i = \overline{c}_{T-1}^{i*} + a^i$.
   (6) Now we have obtained the optimal consumption function $c_{T-1}^*(m_{T-1})$, i.e., the mapping from m_{T-1} to c_{T-1}^*, given that we can interpolate it according to all the corresponding points $(m_{T-1}^1, \overline{c}_{T-1}^{1*}), \ldots, (m_{T-1}^J, \overline{c}_{T-1}^{J*})$.
   (7) Step backward one more period and continue this process, since we have already set up

3. MATLAB Implementation

See https://github.com/wangwz-econ/EGM-2006EL for the details.