Incomplete Markets and Volatility

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This paper shows that the precautionary motive, combined with asset incompleteness, is a major source of volatility and indeterminacy in financial markets. Price fluctuations originate from agents' efforts to insure themselves through time by borrowing and lending instead of shifting income across states of nature by trading risky assets. A *high* interest rate at a future date reduces the potential for future consumption smoothing via borrowing, which leads to a strong precautionary motive and a *low* interest rate in the current period. The negative feedback between future and current rates generates fluctuations. This logic is developed in SPEC, a CARA-normal exchange economy with many periods and endogenous interest rates. When there is an intermediate level of market incompleteness and sufficient investor impatience, fluctuations in the real interest rate can be large, even though the aggregate endowment is constant. SPEC has a unique equilibrium under a finite horizon; on the other hand, with a finite number of infinitely lived agents, there exists a robust continuum of equilibria that are neither bubbles nor sunspots. *Journal of Economic Literature.* Classification Numbers: C61, D52, D58, G11, G12.
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1. INTRODUCTION

Does market incompleteness cause price fluctuations in financial markets? We answer this question affirmatively by introducing SPEC,[2] a special general equilibrium model in which the dynamic path of equilibrium prices can be explicitly calculated for any sequence of asset structures. Intermediate levels of market incompleteness give rise to large temporal fluctuations in macro variables, even when exogenous aggregates are constant.

SPEC is a special case of a many period CARA-normal exchange economy, with the special feature that interest rates are endogenous. Because of its simplicity, the model allows us to analyze the interaction between asset span and intertemporal volatility. In a perfectly competitive economy, we consider finitely many agents with identical CARA utilities, symmetric information, and heterogeneous time-dependent endowments. The financial structure and aggregate endowment are deterministic in each period, but can vary with time. In equilibrium, individual consumption is random, but the macro variables are deterministic and non-stationary. For an intermediate level of market incompleteness and sufficient investor impatience, fluctuations in the real interest rate can be large, even though the aggregate endowment is constant.

Investors can insure themselves *across states of nature* by buying and selling assets, and *across time* by borrowing and lending. In SPEC, the first form of insurance does not cause market fluctuations because agents have identical CARA preferences and there is no aggregate risk. On the other hand, self-insurance by borrowing and lending can generate large endogenous movements in equilibrium prices. When an agent anticipates a *high* interest rate in the future, she realizes that it will be costly to borrow in future states where she will be poor, leading to a high precautionary motive and a *low* interest rate in the current period. The opposite movement in future and current interest rates generates fluctuations. When markets are complete, agents can fully insure themselves by transferring wealth across states of nature via risky assets; and thus the precautionary motive and market fluctuations disappear. Conversely, in the absence of risky assets, investors are exposed to large uninsurable risks and have a strong precautionary motive in every period; interest rates are then low and fluctuate in a narrow range along the equilibrium path.

The equilibrium calculation builds on a new result in consumption-portfolio theory. Given an exogenous stream of spot prices, a trader's maximization problem is solved in the CARA-normal case under both finite and infinite horizons. In contrast, previous authors have solved the infinite

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[2] SPEC is an acronym for Savings, Precaution, and Endogenous Cycles.

horizon decision problem assuming either complete markets (Hakansson, [28] Merton [47]) or a constant riskless rate (Caballero [11], Wang [64]), and various conditions on income, information and asset structure.

When the time horizon is finite, equilibrium is unique but non-stationary. On the other hand with a finite number of infinitely-lived agents, there is a robust region of economies with a continuum of perfect foresight equilibria. In each of these equilibria, asset prices are deterministic and equal to their fundamental values at each point in time. Indeterminacy is thus caused neither by sunspots[3] nor bubbles.

Fluctuations and indeterminacy arise despite the restrictive assumptions on financial structure: all assets are real, there are no cash-in-advance constraints, and Ponzi schemes are ruled out along every possible path. Because agents have identical CARA utilities, we will see that the asset prices are always independent of the cross-sectional distribution of wealth. SPEC thus profoundly differs from the one-asset economy of Scheinkman and Weiss [54], where macro fluctuations are generated by borrowing constraints and changes in the wealth distribution.

The model has both positive and normative implications. It shows that market incompleteness is a possible explanation for the large fluctuations reported by Shiller [56, 57] and others. A companion paper (Angeletos and Calvet [4]) adds production to the model, and shows that financial fluctuations are accompanied by fluctuations in aggregate investment and output. More asset markets may not only help improve welfare[4] and reduce social inequalities through better risk sharing; they could also dampen volatility in existing financial markets and the real economy.

1.1. *Review of Previous Literature*

For general GEI economies, a systematic analysis of the interaction between financial structure and price variability is currently elusive. The "curse of dimensionality" precludes the numerical calculation of equilibrium in incomplete market settings with many periods and assets. For this reason, the computational literature on infinite horizon economies has so far only examined the case of a stock and a bond under various market imperfections (Telmer [62], Lucas [39], Heaton and Lucas [32]).

The theoretical analysis of multiperiod GEI models is also quite difficult, and recent research has focused on a number of interesting special examples. Two-period models help study the variability of equilibrium quantities across states of nature (Detemple and Selden [17]). In

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[3] Following Cass and Shell, [13] an equilibrium is called a *sunspot* if there exist two states of nature with identical "fundamentals" (e.g., utilities and endowments) but distinct consumer allocations.

[4] See Cass and Citanna [12] and Elul [20] for characterizations of welfare-improving financial innovations.

particular, Geanakoplos [23] shows that market incompleteness increases
the variability of relative prices at the terminal date in economies with
several goods. Two-period models are difficult to test empirically, because
they predict price variability across states of nature while financial data
provide fluctuations across time.[5]

A related line of research uses quadratic or CARA utilities to build GEI
economies with closed-form solutions. Such models help characterize equi-
librium portfolio holdings (Magill and Quinzii [43]), and the effect of
asset span on various macro variables such as the trade balance (Willen
[66]) or the riskless rate (Elul [19]). CARA-normal economies are also
remarkably useful for understanding asymmetric information problems,
either in two periods (Grossman and Stiglitz [27], Admati [1]) or in
infinite horizon settings (Wang [63, 64]; Massa [44]). SPEC extends this
literature by introducing a multiperiod CARA-normal framework that
generates endogenous fluctuations in the riskless rate.

In infinite horizon GEI economies, agents only satisfy sequential budget
constraints and the possibility of Ponzi games must be addressed (Levine
[37], Magill and Quinzii [40], Hernández and Santos [33]). These
schemes cannot exist under complete markets because agents must obey a
unique intertemporal budget constraint. In the infinite horizon SPEC set-
ting, individual contingent plans are limits of the finite horizon solutions.
Ponzi games are thus ruled out in the strongest conceivable way, along
every possible path.

Indeterminacy is another source of complexity in infinite horizon
economies. With finitely many agents and complete markets, there generi-
cally exist a finite number of equilibria (Shannon [55]). SPEC contains
uncountably many equilibria for a robust family of incomplete market
economies. In each equilibrium, asset prices are deterministic and equal to
their fundamental value. Indeterminacy, therefore, originates neither in
bubbles nor sunspots. Uncountably many perfect foresight equilibria are
also encountered in deterministic overlapping generations (OLG) economies[6]
(Benhabib and Day [6], Grandmont [25]). In these models however,
there exist countably many agents, and fluctuations originate in the conflict
between the income and the intertemporal substitution effects. To the best
of my knowledge, endogenous fluctuations caused by the precautionary
motive are new to the literature.

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[5] Some authors use ergodicity arguments to show that variability across states of nature is
equivalent to variability along a given path. However, two period models cannot capture
purely dynamic effects, such as the negative feedback between future and current rates
exhibited in SPEC.

[6] The analogy with the OLG literature is even more striking because among the continuum
of equilibria, there exist cyclical equilibria of every order for a robust set of economies.

2. A MODEL OF EXCHANGE ECONOMY

This section develops the main features of SPEC: investor behavior, the financial structure and the equilibrium concept.

2.1. *Investors*

We examine a multiperiod exchange economy in discrete time with a finite or an infinite horizon T. The economy is stochastic, and all random variables are defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Finitely many agents $(1 \leqslant h \leqslant H)$ live and consume a single consumption good at dates $t = 0, ..., T$. During his life, each consumer h receives a stochastic endowment stream $\{\tilde{e}_t^h\}_{t=0}^T$, which could correspond for instance to a random labor income. The specification of the endowment process is extensively discussed in the macroeconomic literature (Caballero [11]). For our purposes, it is convenient to limit ourselves to a very simple structure.

Assumption 1. For any agent h, the random endowments $\{\tilde{e}_t^h\}$ are independent across t.

The aggregate wealth of the economy in a given period is characterized by the mean endowment

$$\tilde{e}_t \equiv \frac{1}{H} \sum_{h=1}^{H} \tilde{e}_t^h.$$

We impose

Assumption 2. The mean endowment is constant every period: $\tilde{e}_t = e_t$ for all t.

Distributional shocks are thus the only source of randomness in the economy, and states of nature correspond to different allocations of the deterministic aggregate endowment.[7]

Investors derive utility from their consumption streams. For a fixed $A > 0$, we define $u(c) = -\exp(-Ac)/A$ and introduce

Assumption 3. Each investor h has CARA utility

$$\sum_{t=0}^{T} \beta^t u(c_t) \equiv -\frac{1}{A} \sum_{t=0}^{T} \beta^t \exp(-Ac_t).$$

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[7] By the law of large numbers, Assumption 2 also holds when the economy contains infinitely many agents receiving independent income shocks (Bewley [7], Clarida [15], Aiyagari and Gertler [3], Huggett [34], Aiyagari [21]). The equilibrium calculation of Section 4 thus applies to the case of infinitely many agents.

Since consumption is generally stochastic, at each period t agents maximize their expected utility conditional on available information.

2.2. Financial Structure

A sequential security market opens every period, and allows agents to trade the consumption good across time and states. At each date $t < T$, traders can buy or sell a riskless asset delivering one unit of the consumption good at date $t + 1$. They also exchange a finite number of risky securities that have real payoffs in period $t + 1$. The number of risky securities can vary with time. The net supply of each asset is equal to zero, and no restrictions are assumed on short sales.

Assumption 4. In every period $t < T$, asset payoffs and individual endowments are jointly normal.

Assumptions 3 and 4 imply that the economy is CARA-normal.

The payoffs of assets traded in date t span a linear subspace $S(t)$ in the space $L^2(\Omega)$ of square-integrable random variables. Without loss of generality, consider an orthonormal basis of $S(t)$ consisting of the riskless asset $\tilde{a}_{0, t+1} \equiv 1$, and $N(t)$ risky securities $\tilde{a}_{1, t+1}, ..., \tilde{a}_{N(t), t+1}$. By construction, the risky assets have zero expected payoffs,

$$\mathbb{E}\tilde{a}_{i, t+1} = 0, \qquad i = 1, ..., N(t),$$

and are mutually uncorrelated. Each security i has a price $\pi_i(t)$ in period t. We introduce the vectors

$$\begin{cases} \widetilde{A}_{t+1} = [1, \tilde{a}_{1, t+1}, ..., \tilde{a}_{N(t), t+1}]' \\ \pi(t) = [\pi_0(t), ..., \pi_{N(t)}(t)]' \end{cases}$$

and define the (short) riskless rate as $R_t = 1/\pi_0(t)$.

We note that all assets traded in one period are *short-lived*, in the sense that they only deliver payoffs in the next period. Sections 4 and 5 will show that the interest rate sequence $\{R_t\}_{0 \leqslant t < T}$ is deterministic and that there is no risk premium in equilibrium. For this reason, allocations and prices do not change if we introduce a *long-lived* security delivering one unit of the good every period. This asset, called a *perpetuity*, is worth

$$\pi_L(t) = \sum_{s=t}^{T-1} \frac{1}{R_t \cdots R_s}$$

at date t after delivery of the period's coupon. A trader can also dynamically replicate any long-lived financial asset with independent random

payoffs $\tilde{d}_1, ..., \tilde{d}_T$. In equilibrium, the value of the long-lived asset at a given date t is thus $\sum_{s=t}^{T-1} (\mathbb{E}\tilde{d}_{s+1})/(R_t \cdots R_s)$.[8]

At the beginning of every period t, investors are informed of the realization of asset payoffs $\tilde{A}_t$, and endowment shocks $\tilde{e}_t^h$, $1 \leqslant h \leqslant H$. Information is thus symmetric across agents. Endowments and asset payoffs generate a filtration $(\mathscr{F}_t)_{0 \leqslant t \leqslant T}$, and it is convenient to denote by $\mathbb{E}_t$ the conditional expectation at a given date. In the space $L^2(\Omega)$, we project individual income on the asset span $S(t)$,

$$\tilde{e}_{t+1}^h = \mathbb{E}_t \tilde{e}_{t+1}^h + \sum_{i=1}^{N(t)} \gamma_{it}^h \tilde{a}_{i,t+1} + \tilde{\varepsilon}_{t+1}^h,$$

where $\gamma_{it}^h = Cov[\tilde{a}_{i,t+1}; \tilde{e}_{t+1}^h]$, and the residual $\tilde{\varepsilon}_{t+1}^h$ is the undiversifiable income risk. The (rescaled) average variance of residual income

$$V(t) = \frac{A^2}{H} \sum_{h=1}^{H} Var(\tilde{\varepsilon}_{t+1}^h)$$

is a useful *index of market incompleteness*. It equals zero when all individual endowments belong to the asset span, as is the case under complete markets.

2.3. *Equilibrium*

For any quotation $\{\pi(t)\}_{0 \leqslant t < T}$ of asset prices, each agent chooses a contingent plan $\{c_t^h, \theta_t^h, W_t^h\}_{t=0}^T$ adapted to the filtration $(\mathscr{F}_t)_{0 \leqslant t < T}$. An *admissible* plan also satisfies in every period the budget constraints

$$\begin{cases} c_t^h + \pi(t)^\top \theta_t^h = W_t^h, \\ W_{t+1}^h = e_{t+1}^h + A_{t+1}^\top \theta_t^h, \end{cases} \tag{2.1}$$

with the convention that $\theta_T^h = 0$ when T is finite and $W_0^h = e_0^h$ for all h. The variable W_t^h, called *wealth*, thus represents the trader's net credit or debt position at date t. Note that it is independent of future income.

Under an infinite horizon ($T = \infty$), the possibility of Ponzi games must be addressed. In our model, it is sufficient to impose

Assumption 5. When the time horizon is infinite, an admissible consumption plan $\{c_t, \theta_t, W_t\}_{t=0}^\infty$ satisfies $\beta^t \mathbb{E}_0 \exp(-AW_t) \to 0$ as t goes to infinity.

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[8] Furthermore, it is easy to show that a long-lived asset is dynamically redundant when: (1) payoffs $\{d_t\}$ follow an $AR(1)$ process, $d_{t+1} = pd_t + u_{t+1}$, where $|\rho| \leqslant 1$ and $\{u_t\}$ is a sequence of independent Gaussian random variables; (2) in every period t, investors can trade the *riskless* asset, and a short-lived *risky* asset delivering u_{t+1} in period $t+1$. These results directly extend to payoff sequences $\{d_t\}$ following invertible and stationary ARMA processes.

We will see that this rules out Ponzi games along every possible path.

DEFINITION. A *GEI equilibrium* consists of a price sequence $\{\pi(t)\}_{0 \leqslant t < T}$ and a collection of admissible plans $(\{c_t^h, \theta_t^h, W_t^h\}_{t=0}^T)_{0 \leqslant h \leqslant H}$ such that:

(i)   All markets clear every period,

$$\frac{1}{H} \sum_{h=1}^{H} c_t^h = e_t,$$

$$\frac{1}{H} \sum_{h=1}^{H} \theta_t^h = 0.$$

(ii) Each agent's plan is optimal.

In the next sections, we show that the equilibrium calculation is straightforward within this framework.

3. CONSUMPTION-PORTFOLIO DECISION

We now derive the optimal decision rule of an individual investor under a finite and an infinite horizon. Unless stated otherwise, all proofs are given in the Appendix.

3.1. *Finite Horizon*

Computing the optimal decision rule seems *a priori* like a difficult task, because forward-looking investors integrate all current and future financial opportunities into their decisions.[9] In SPEC however, the future is parsimoniously described and the decision problem can be solved by iterating Bellman's optimality principle. When the horizon T is finite, we calculate the individual decision problem at date $T-1$ and find that the indirect utility is CARA with modified parameters. Using this property, we then calculate demand in every period.

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[9] The behavior of a CARA agent in a multiperiod stochastic economy was first analyzed by Hakansson [28, 29] and Merton [47] in discrete and continuous time. In particular, Hakansson [28] considered the case of a non-stationary financial environment, in which the interest rate and asset structure are deterministically changing with time. This early work relied on the simplifying assumption that individual income is tradable. Using a constant financial structure and interest rate, later research shows how to calculate decision rules under incomplete markets (Sibley [59], Schechtman and Escudero [53], Kimball and Mankiw [35], Caballero [11], Wang [64]). This paper extends these results by considering a time-varying sequence of interest rates, financial structures and partly undiversifiable endowments.

Consider a fixed trader h facing an exogenous, *deterministic* path of asset prices $\{\pi(s)\}_{t \leq s < T}$. Let $J^h(W, t)$ denote the indirect utility along the exogenous price path, and let $J_w^h(W, t)$ denote the corresponding marginal utility. In date t, the investor maximizes

$$-\frac{1}{A}\exp(-Ac) + \beta \mathbb{E}_t J^h(\widetilde{W}_{t+1}^h, t+1), \tag{3.1}$$

subject to the budget constraints (2.1). This problem has a closed form solution under

Recursive Condition. In period $t+1$, the investor has marginal utility of wealth

$$J_w^h(W, t+1) = \exp[-A(a_{t+1} W + b_{t+1}^h)],$$

where $a_{t+1} > 0$ and b_{t+1}^h are two constant numbers.

We can then show

THEOREM 1. *At date t, the solution to the optimization problem* (3.1) *satisfies*

$$\begin{cases} c_t^h = a_t W_t^h + b_t^h, \\ \theta_{it}^h = -\gamma_{it}^h - R_t \pi_i(t)/(A a_{t+1}), \qquad 1 \leq i \leq N(t), \\ \theta_{0t}^h = (a_t/a_{t+1}) W_t^h - R_t b_t^h - R_t \sum_{i=1}^{N(t)} \pi_i(t)\, \theta_{it}^h, \end{cases} \tag{3.2}$$

where the quantities a_t and b_t^h are defined by

$$a_t = \frac{1}{1 + (a_{t+1} R_t)^{-1}}, \tag{3.3}$$

$$b_t^h = (a_t/a_{t+1} R_t)[b_{t+1}^h + a_{t+1}\mathbb{E}\tilde{\varepsilon}_{t+1}^h - A a_{t+1}^2\, Var(\tilde{\varepsilon}_{t+1}^h)/2 - \ln(\beta R_t)/A]$$
$$+ a_t \sum_{i=1}^{N(t)} \pi_i(t)[\gamma_{it}^h + R_t \pi_i(t)/(2 A a_{t+1})]. \tag{3.4}$$

The utility of wealth satisfies $J_w^h(W, t) = \exp[-A(a_t W + b_t^h)]$ and is therefore of the CARA-type.

The indirect utility $J^h(W, t)$ is thus CARA when asset prices are deterministic. In the terminology of dynamic programming, the CARA class is globally invariant under the Bellman operator. We observe that the Recursive Condition holds at the terminal date with coefficients $a_T = 1$ and

$b_T^h = 0$. Indirect utility is thus CARA in every period, and the coefficients a_t and b_t^h can be computed by a backward recursion of (3.3) and (3.4).

The decision rules of Theorem 1 contain some familiar features. The coefficient $-\gamma_{it}^h$ shows that the risk-averse investor trades securities to hedge her income shocks. Consumption c_t^h is a linear function of current wealth with intercept b_t^h and marginal propensity to consume a_t. Because the agent has a precautionary motive, the intercept b_t^h decreases with the variance of the nontradable shock ε_{t+1}^h. By Theorem 1, all households have the same marginal propensity to consume a_t in our market economy. This implies that aggregate demand and equilibrium prices will be independent of the distribution of wealth.

At the outset of period t, the investor receives a random wealth shock. A fraction a_t of this shock is absorbed by current consumption and the remaining fraction $(1-a_t)$ affects savings. A *high* a_t implies a variable consumption at date t and therefore a *low* level of consumption smoothing. By Eq. (3.3), the current a_t is an increasing function of the interest rate R_t and the future propensity a_{t+1}. We now interpret these monotonicities.

First, consider the effect of an exogenous increase in the interest rate R_t. An agent with a *low* wealth W_t^h, who borrows in the current period, is now impoverished by a higher interest on her debt. Current consumption is also more expensive relative to future consumption. The wealth and substitution effects thus both imply *lower* current consumption for a poor agent. On the other hand, a rich agent who lends in the current period earns a higher return on her savings. Because indifference curves are strongly convex, the wealth effect dominates and the rich agent enjoys a *higher* consumption in the current period. Consumption c_t^h is now a steeper function of wealth, and a high interest R_t thus discourages consumption smoothing through time via borrowing and lending.

Second, we consider an increase in future propensity a_{t+1},[10] and show by contradiction that a_t must also rise. If the slope a_t were *lower* in the current period, an investor with a large wealth W_t^h would reduce her consumption c_t^h, save more in period t, and therefore increase her future wealth W_{t+1}^h. At date $t+1$, the combined increase in wealth W_{t+1}^h and propensity a_{t+1} would then lead to a rise in consumption c_{t+1}^h and a violation of optimality: $\beta R_t u'(c_{t+1}^h) < u'(c_t^h)$.[11] The complementarity of future

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[10] The increase in a_{t+1} could be caused, for instance, by an exogenous increase of the riskless rate R_{t+1}.

[11] This explains the monotonocity of the decision rule in the absence of shocks ($\varepsilon_{t+1}^h = 0$). When income is stochastic, the expected marginal utility $\mathbb{E}_t u'(c_{t+1}^h)$ is also sensitive to the increased variability of c_{t+1}^h associated with a higher a_{t+1}. With a CARA utility, this effect is negligible as $W_t^h \to \infty$, since future consumption c_{t+1}^h has an unbounded mean but a bounded variance.

and current consumption thus implies a positive feedback between a_{t+1} and a_t.

We will see in Section 4 that these effects play an important role in understanding the properties of SPEC. In particular along the equilibrium path, future propensity a_{t+1} will also affect current a_t through the endogenous interest rate R_t, leading in some cases to a *negative* feedback between future and current propensities. Appendix 7.2 shows that the monotonicity of the decision rules is not specific to the CARA specification, but holds for a wide range of utility functions.

We can also derive a closed-form expression for marginal propensity. Direct iteration of (3.3) implies

$$\frac{1}{a_t} = 1 + \sum_{s=t}^{T-1} \frac{1}{R_t R_{t+1} \cdots R_s}, \tag{3.5}$$

or equivalently

$$a_t = 1/[1 + \pi_L(t)], \tag{3.6}$$

where $\pi_L(t)$ denotes the price of a perpetuity. A high current or future rate thus discourages consumption smoothing through time via borrowing. Relation (3.6) also facilitates the calculation of optimal demand when the time horizon is infinite.

3.2. *Infinite Horizon*

Consider an exogenous, deterministic sequence of asset prices $\{\pi(t)\}_{t=0}^{\infty}$. Since the indirect utility $J^h(W, t)$ is time-dependent, it is convenient to explicitly include all the exogenous parameters in the value function. Consider for all $t \geq 0$ the vector

$$Z_t^h = \langle \pi(t), \mathbb{E}e_{t+1}^h, Var(e_{t+1}^h), \{\gamma_{it}^h\}_{i=1}^{N(t)} \rangle,$$

and stack these vectors into an infinite array $Z^h = \{Z_t^h; t = 0, ..., \infty\}$. Let $\mathscr{S}$ denote the shift operator that maps an array $Z = \{Z_t\}_{t=0}^{\infty}$ into $\mathscr{S}Z = \{Z_1, Z_2, ...\}$. The *transformed value function* $\hat{J}(W, Z^h)$ explicitly incorporates the influence of all exogenous parameters on the decision problem.[12] It is invariant to the Bellman operator

$$\mathscr{B}\hat{F}(W, Z = \max_{\{c, \theta, W'\}} [u(c) + \beta\mathbb{E}\hat{F}(W', \mathscr{S}Z)],$$

and provides indirect utility $J^h(W, t) \equiv \hat{J}(W, \mathscr{S}^t Z^h)$ in every period.

[12] Since all investors solve the same problem (with different parameters), the function $\hat{J}(W, Z)$ is henceforth written without the subscript h.

We find a fixed point of the Bellman operator by taking the limit of the *finite* horizon value function as the number of periods goes to infinity. At any instant $t \geqslant 0$, the pointwise limit $J^h(W, t) \equiv \hat{J}(W, \mathscr{S}^t Z^h)$ is CARA with coefficients

$$a_t = 1/[1 + \pi_L(t)],$$
$$b_t^h = M_t^h + a_t \sum_{s=t+1}^{+\infty} \frac{M_s^h}{a_s R_t \cdots R_{s-1}}, \tag{3.7}$$

where $\pi_L(t)$ denotes the price of a perpetuity, and $M_t^h = a_t \sum_{i=1}^{N(t)} \pi_i(t)$ $[\gamma_{it}^h + R_t \pi_i(t)/(2Aa_{t+1})] + (a_t/a_{t+1}R_t)[a_{t+1}\mathbb{E}\tilde{e}_{t+1}^h - Aa_{t+1}^2 Var(\tilde{\varepsilon}_{t+1}^h)/2 - \ln(\beta R_t)/A]$. The convergence of the infinite series is guaranteed by placing restrictions on the exogenous array Z^h.

Assumption 6. The sequences of expected endowments $\{\mathbb{E}e_t^h\}$ and endowment risks $\{Var(e_t^h)\}$ are bounded.

Assumption 7. The sequences $\{\pi_0(t)\}$, $\{\pi_L(t)\}$, $\{\pi_i(t)/\pi_0(t)\}$ and $\{\sum_{i=1}^{N(t)} |\pi_i(t)|\}$ are bounded.

We can then prove

THEOREM 2. *Under Assumptions* $[1–7]$, *the indirect utility of wealth is CARA in every period,*

$$J^h(W, t) = -(Aa_t)^{-1} \exp[-A(a_t W + b_t^h)],$$

and consumption-portfolio decision rules satisfy (3.2).

We can now solve for the equilibrium dynamics.

4. EQUILIBRIUM PATH IN A FINITE HORIZON

We show that under a finite horizon, equilibrium is unique and can be computed by an elementary recursion. A negative feedback between future and current interest rates is found to generate endogenous price fluctuations along the unique equilibrium path.

4.1. *A Unique Equilibrium*

The equilibrium calculation is greatly simplified by Assumptions 1–4. Because investors have identical CARA utilities, aggregate demand in a given period t is independent of that period's distribution of wealth. Furthermore, since idiosyncratic shocks are independent across time and aggregate endowment is deterministic, period t's equilibrium prices are invariant across states of nature. In equilibrium, the marginal propensity

and asset prices are therefore *deterministic*, even though individual endowments and asset payoffs are random.

We compute equilibrium prices and marginal propensities by a backward recursion. At date $T-1$, SPEC is a mean-variance economy with homogenous preferences, in which the aggregate endowment (or "market portfolio") is *riskless*. As a consequence, there is no risk premium in equilibrium, and by Theorem 1, all agents have CARA indirect utilities $J^h(W, T-1)$ and identical marginal propensities a_{T-1}.

THEOREM 3. *Equilibrium is unique under a finite horizon. There is no risk premium*: $\pi_i(t) = 0$ *for all* $i \geq 1$, *and all agents have identical marginal propensities to consume. The GEI riskless rate and marginal propensity are deterministic. They can be computed by a backward recursion from*

$$\ln R_t = \ln(1/\beta) - \tfrac{1}{2} a_{t+1}^2 V(t) + A(e_{t+1} - e_t), \tag{4.1}$$

$$a_t = \frac{1}{1 + (a_{t+1} R_t)^{-1}}, \tag{4.2}$$

for any $t < T$, *and the final condition* $a_T = 1$.

Equilibrium uniqueness is a fundamental property of SPEC under a finite horizon. With complete markets ($V(t) = 0$), the interest rate R_t is independent of future propensity and can only vary with the exogenous growth rate. On the other hand when markets are incomplete, asset prices endogenously fluctuate along the unique equilibrium path, and the nonlinearity of (4.1) and (4.2) suggests that complicated dynamics can arise.

Note that in SPEC, the short rate R_t is constant when the aggregate endowment is stationary ($e_{t+1} = e_t$) and the financial structure complete. More generally, consider an exchange economy consisting of finitely many agents ($h = 1, ..., H$) with identical utilities $\sum_{t=0}^{T} \beta^t u(c_t)$, where the function u is strictly concave and twice continuously differentiable. Assume that the aggregate endowment is deterministic and constant through time. When markets are complete, this economy has, like SPEC, a zero risk premium and a constant riskless rate ($R_t = 1/\beta$) along its unique equilibrium path.[13] We note that these pricing rules also hold under incomplete markets when investors have quadratic utilities ($u''' = 0$). Thus, interest rate fluctuations can only arise in an economy with an incomplete financial structure and a nonlinear marginal utility ($u''' \neq 0$). Appendix 7.5 shows that the precautionary motive ($u''' > 0$) is also required to obtain a negative feedback between

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[13] This result is a direct consequence of the First Welfare Theorem. In equilibrium, investors fully share their income risks and have deterministic consumptions. The First Order Condition $u'(c_t^h) = \beta R_t \mathbb{E} u'(c+1)$ then implies that $R_t = 1/\beta$. See the proof of Proposition 2 for more details.

future and current rates, and thus the persistent fluctuations observed in SPEC.

Since there is no risk premium in our model, each investor h sells the tradable fraction of her income risk at no cost, and bears only the uninsurable component ε_t^h along the equilibrium path. One therefore observes the same equilibrium sequence (of interest rates and individual consumptions) in a modified economy where no risky asset is traded and individual endowments are replaced by $\mathbb{E}e_t^h + \varepsilon_t^h$. Risky assets thus play only one role—the definition of the uninsurable risks ε_t^h. Economies without risky assets are therefore important cases of SPEC,[14] which naturally arise when states, and thus individual incomes, are not publicly observed.[15]

The pricing relation (4.1) contains some familiar results. The riskless rate R_t increases with investor impatience and the growth rate of the aggregate endowment. Because a larger asset span dampens the precautionary motive of all investors, the equilibrium interest rate R_t is higher when markets are more complete. This well-known result holds in static exchange economies under more general conditions (Weil [65], Elul [19]).

Consider now the effect of a higher marginal propensity a_{t+1} on current a_t. As seen in Section 3, each agent knows that consumption is a steeper function of wealth at date $t + 1$, and would like to choose a *higher* level for a_t; we call this the *Direct Effect*. However, a_{t+1} also influences the interest rate R_t along the equilibrium path. Anticipating a riskier consumption in date $t + 1$, all agents have a stronger precautionary motive at date t, and thus reach an equilibrium with a lower interest rate R_t. As discussed in Section 3, a lower R_t incites a trader to select a *lower* a_t; we call this the *Indirect Effect*.

While the *direct effect* stems from the decision rule of a consumption smoothing investor ($u'' < 0$), the *indirect effect* originates in equilibrium

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[14] One-asset SPEC economies may seem similar to the model of Scheinkman and Weiss [54]. In this earlier framework, several workers face borrowing constraints, heterogenous shocks to their labor productivity, and seek to accumulate fiat money for precautionary reasons. Because agents have a disutility for labor, the poor are more willing to work than the rich, and aggregate production is higher when the more productive agents have little wealth. Changes in the cross-sectional distribution of wealth and productivity are therefore accompanied by fluctuations in prices and aggregate income. Fluctuations in SPEC have a profoundly different origin (a negative feedback between future and current rates due to the precautionary motive), and arise *even though macro variables are always independent of the distribution of wealth*.

[15] This seems to contradict the assumption in Subsection 2.2 that individual income shocks are common knowledge. However, by Assumptions 1 and 2, a trader does not extract relevant information from the observation of other agents' incomes. Therefore when only the riskless asset is traded, the same equilibrium path arises whether endowments are privately or publicly observed.

variations of the riskless rate, caused by the precautionary motive ($u''' > 0$) under incomplete markets. These tendencies move the current propensity a_t in opposite directions, and intuition suggests that the indirect effect dominates when the equilibrium rate R_t is sufficiently sensitive to a_{t+1}. Since marginal utility is convex, this should occur when future consumption c_{t+1}^h is very risky, i.e., when marginal propensity a_{t+1} is high.[16] To validate this idea, we rewrite equilibrium as a one-dimensional dynamical system.

THEOREM 4. *The marginal propensity satisfies the backward recursion*

$$a_t = \frac{1}{1 + \beta \exp[\frac{1}{2} a_{t+1}^2 V(t) - A(e_{t+1} - e_t)]/a_{t+1}}, \tag{4.3}$$

with the final condition $a_T = 1$.

Proof. Substitute (4.1) into (4.2). ∎

Current propensity a_t is a single-peaked function of a_{t+1}, and as anticipated, the indirect effect dominates for $a_{t+1} > [V(t)]^{-1/2}$. From (4.1), we also note that the indirect effect is second order ($dR_t/da_{t+1} = 0$) when future consumption c_{t+1}^h is riskless.

The equilibrium dynamics of the riskless rate can be inferred from Theorem 3. Substituting (3.5) into (4.1),

$$\ln R_t = \ln(1/\beta) - \frac{1}{2}\left(1 + \sum_{s=t+1}^{T-1} \frac{1}{R_{t+1}R_{t+2}\cdots R_s}\right)^{-2} V(t) + A(e_{t+1} - e_t), \tag{4.4}$$

we observe that the current rate R_t is a *decreasing* function of the rate R_{t+1}, provided that the numbers $A, \beta, V(t), e_t, e_{t+1}, R_{t+2}, ..., R_{T-1}$ are fixed. More generally, relation (4.4) shows a negative feedback between current and all future rates. However, since the rates $R_{t+1}, R_{t+2}, ..., R_{T-1}$ are *interdependent*, relation (4.4) should be used cautiously for comparative statics. Given an equilibrium path, consider a SPEC economy in which R_{t+2} is higher while $R_{t+3}, ..., R_T$ are unchanged.[17] While a higher value of R_{t+2} directly *decrease* R_t, it also leads to a lower R_{t+1} and therefore

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[16] This intuition follows from the First Order Condition, $R_t = u'(c_t^h)/[\beta \mathbb{E}_t u'(c_{t+1}^h)]$.

[17] The new equilibrium can be obtained, for instance, by lowering the index $V(t+2)$ and keeping all other exogenous parameters constant.

indirectly *increases* R_t. The ambiguous overall effect of R_{t+2} on R_t permits the existence of rich dynamic patterns.[18]

The negative feedback between future and current rates illustrates the difference between the two types of insurance available in the economy. Investors can insure themselves *across states of nature* by buying and selling risky assets, and *across time* by borrowing and lending. When the aggregate endowment is deterministic, individual efforts to hedge by trading in risky assets do not cause market fluctuations. In a future state, one agent expects to be poor and another to be rich, and their *ex ante* efforts to buy and sell income in that state will just match. On the other hand, the precautionary demand for savings can have a very strong effect on economy-wide prices. This is because anticipation of a high future interest rate reduces the potential of all consumers to smooth future consumption across time via borrowing; leading in the current period to a strong precautionary motive and a *low* interest rate.

This equilibrium effect can be obtained with a wide class of utility functions. Appendix 7.5 considers a special multiperiod exchange economy with three periods ($t = 0, 1, 2$), no aggregate risk and a separable utility $\sum_{t=0}^{2} \beta^t u(c_t)$ common to all agents. Given an equilibrium sequence, we perturb the last period's aggregate endowment to obtain a higher interest rate R_1 in equilibrium. As in Appendix 7.2, a higher R_1 implies a steeper and therefore riskier consumption function $c_1(W_1)$. With quadratic utilities ($u''' = 0$), the equilibrium interest rate R_0 is unaffected, as would be true in a more general setting. When investors have a precautionary motive ($u''' > 0$), riskier consumption at date $t = 1$ increases the marginal utility $\mathbb{E}_0 u'(c_1)$ and decreases the interest rate R_0; like in SPEC, there is a negative equilibrium feedback between future and current rates. Note in the unusual case where investors save less with increasing risk ($u''' < 0$), these results are reversed. Riskier consumption at date $t = 1$ then decreases the marginal utility $\mathbb{E}_0 u'(c_1)$ and increases the interest rate R_0; implying a *positive* feedback between future and current rates. The negative feedback observed in SPEC thus requires the precautionary motive ($u''' > 0$), and not simply a non-linear marginal utility.

When markets are incomplete, we have thus seen that the precautionary motive leads to a non-monotonic *propagation mechanism* in the backward dynamics. This mechanism controls the economy-wide fluctuations

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[18] The dynamics of the riskless rate in SPEC have interesting and subtle properties. A higher value of R_{t+1} has two effects on R_t. First, the direct negative feedback between R_{t+1} and R_t tends to *decrease* the current rate R_t. On the other hand, a higher R_{t+1} indicates that the rates $R_{t+2}, ..., R_{T-1}$ are low (on average), and thus tends to *increase* the current rate R_t. The overall effect of R_{t+1} on R_t is ambiguous, as illustrated by the recursive formula $\ln R_t = \ln(1/\beta) - (1/2) V(t)\{1 + [-2R_{t+1}^2 \ln(\beta R_{t+1})/V(t+1)]^{-1/2}\}^{-2}$ when the aggregate endowment is stationary.

generated by the anticipation of a future shock, such as an exogenous change in aggregate income or financial structure. For the rest of this paper, we concentrate on a perhaps surprising result, the existence of endogenous equilibrium fluctuations in a purely stationary economy.

4.2. *Fluctuations in a Stationary Economy*

Consider a stationary economy, in which the aggregate endowment $e_t = e$ and the index of market incompleteness $V(t) = V$ are invariant across time. We define the function

$$F(a) = \frac{1}{1 + \beta a^{-1} \exp(a^2 V/2)},$$

and rewrite the intertemporal equilibrium as the Iterated Function System (IFS)

$$\begin{cases} a_t = F(a_{t+1}) \\ a_T = 1. \end{cases} \tag{4.5}$$

The function F and the corresponding IFS only depend on two parameters, the discount factor β and the incompleteness index V. The reader can easily show.

PROPOSITION 1. *The function F is infinitely differentiable on $[0, \infty)$. It is single-peaked, reaches a maximum at $a^* = 1/\sqrt{V}$, and has a unique fixed point $\bar{a}$ on $(0, \infty)$.*

We also see that F takes values in $[0,1)$ and satisfies $F(0) = F(\infty) = 0$, and $F'(0) = 1/\beta > 1$. A graph of F is illustrated in Fig. 1.

The fixed point $\bar{a}$ is a natural candidate for an equilibrium steady state. Under a finite horizon, however, marginal propensity cannot be constant, because $a_T = 1$ at the terminal date T and $a_t < 1$ in every earlier period. Moreover when $V > 1$, it is easy to show that

$$a_{T-1} < a_t < a_T$$

for all $t < T - 1$. Extreme variations in marginal propensity thus arise around the terminal date when markets are sufficiently incomplete.

We now examine whether these fluctuations quickly disappear, so that a_t is approximately constant along most of the path, or whether they persist after many iterations. For fixed parameters V and β, consider a sequence $\mathcal{E}_{V, \beta} \equiv \{E_T\}_{T=1}^{\infty}$ of stationary SPEC economies, where each E_T has a finite horizon T, an incompleteness index V and a discount factor β. In each economy E_T, denote $a_t^T \equiv F^{T-t}(1)$ as the marginal propensity in period t along the equilibrium path.
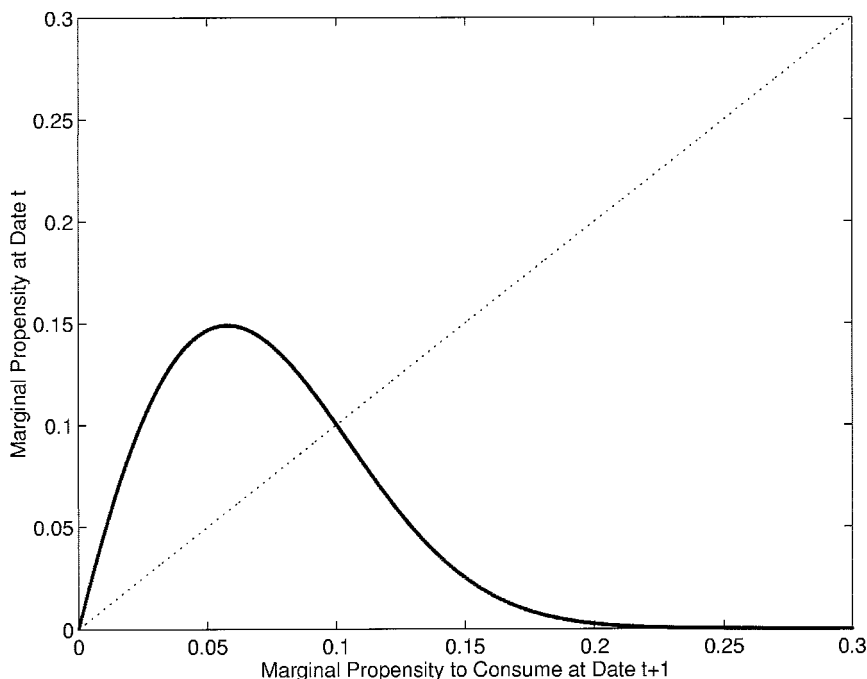
FIG. 1. Iterated function F with parameters $V = 300$, $\beta = 0.2$.

DEFINITION. The sequence of economies $\mathscr{E}_{V,\beta}$ is said to generate *persistent fluctuations* if the sequence of marginal propensities at the initial date $\{a_0^T = F^T(1)\}_{T=0}^{\infty}$ does not converge to a steady state.

Note that there is nothing specific about date 0 in this definition. We find it convenient to define the set

$$PF = \{(V, \beta) \in [0, \infty) \times (0, 1) : \mathscr{E}_{V,\beta} \text{ displays persistent fluctuations}\},$$

and now establish the existence and robustness of these fluctuations.

THEOREM 5. *There exists an open set U such that for all $(V, \beta) \in U$, the sequence of economies $\mathscr{E}_{V,\beta}$ displays persistent fluctuations. Moreover, there exists $\delta > 0$ such that*

$$\limsup_{T} |a_1^T - a_0^T| > \delta$$

for any $(V, \beta) \in U$.

The region PF of fluctuating economies contains the open set U and has therefore a non-empty interior. It easy to show that PF is bounded away from the line $\beta = 1$, or more precisely[19] that

$$\beta_{\max} = \sup_{PF} \beta < 1.$$

In a stationary SPEC economy, the existence of persistent fluctuations thus requires sufficient investor impatience.[20] This result does not, however, invalidate the empirical relevance of our approach, because SPEC is only one example in which the precautionary motive generates endogenous volatility in financial markets. The calibration of the model is addressed in Angeletos and Calvet [4], which extends SPEC to a decentralized production economy with capital accumulation and idiosyncratic technological risk. Endogenous fluctuations in prices and aggregate production can then be obtained for values of the discount factor β as high as 0.99.

In SPEC, the upper bound $\beta_{\max}$ can be interpreted in several ways. We may view the small value of $\beta_{\max}$ as an indication that the periods are relatively long.[21] Alternatively, we note that economic growth increases the

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[19] If the economy (V, β) displays persistent fluctuations, we know that $F(a^*) > a^*$, or equivalently $1/\sqrt{V} + \beta \exp(1/2) < 1$, which implies $\beta < \exp(-1/2)$. Hence $\beta_{\max} \leqslant \exp(-1/2) < 1$. In addition, we infer that $V > 1$ and therefore that the set PF is bounded away from the line $V = 0$ that corresponds to complete market economies.

[20] Levine and Zame [38] show that under incomplete markets, investors' equilibrium utilities uniformly converge to the complete markets utilities when the discount factor β goes to 1. This theorem, which holds for a large class of infinite horizon exchange economies, relies on the intuition that very patient agents can almost fully insure themselves by borrowing and lending when individual income shocks are weakly dependent across time. As good and bad shocks tend to even out, an agent can maintain a smooth consumption level without accumulating an explosive debt. These smooth consumption profiles require that richer agents be patient enough to forego current consumption and lend to poorer agents at low cost. A related intuition is that (almost) infinitely patient investors would "arbitrage away" deterministic price fluctuations, as was noted in the optimal growth literature (Boldrin and Montrucchio [9]; Deneckere and Pelikan [16]; Sorger [60, 61]; Montrucchio [49]; Mitra [48]).

[21] To give some idea of the magnitudes involved in the simulations, consider that each period represents about $L = 15$ years, and that investors are characterized by a *yearly* psychological discount factor $\beta_y = 0.9$ and a *yearly* relative risk aversion $\gamma_y = 10$ at the mean endowment point e_y. Over periods of length $L = 15$ years, the corresponding parameters are then $\beta \sim \beta_y^L \sim 0.21$ and $\gamma \sim \gamma_y \sim 10$. Assume that over each period of 15 years, yearly income $\{e_y(i)\}_{1 \leqslant i \leqslant L}$ follows a random walk (or Brownian Motion). The income $e = \sum_{1 \leqslant i \leqslant L} e_y(i)$ over the entire period then satisfies $e \sim Le_y$, $Var(\varepsilon) \sim L^3 Var(\varepsilon_y)$, and $Var(\varepsilon/e) \sim LVar(\varepsilon_y/e_y)$. When $Var(\varepsilon_y/e_y) = 0.2$, the index of market incompleteness approximately satisfies: $V \sim \gamma^2 Var(\varepsilon/e) \sim 300$. These numbers roughly correspond to the values of V and β used in the simulations.

set of parameters (V, β) giving rise to persistent fluctuations. For instance, consider a SPEC economy with fixed parameters (V, β) and a constant growth rate $g = A(e_{t+1} - e_t)$. By Theorem 4, the equilibrium sequence $\{a_t\}$ is the same in the growing economy (V, β, g) and in the stationary economy $(V, \beta e^{-g})$. The set of parameters (V, β) generating fluctuations is
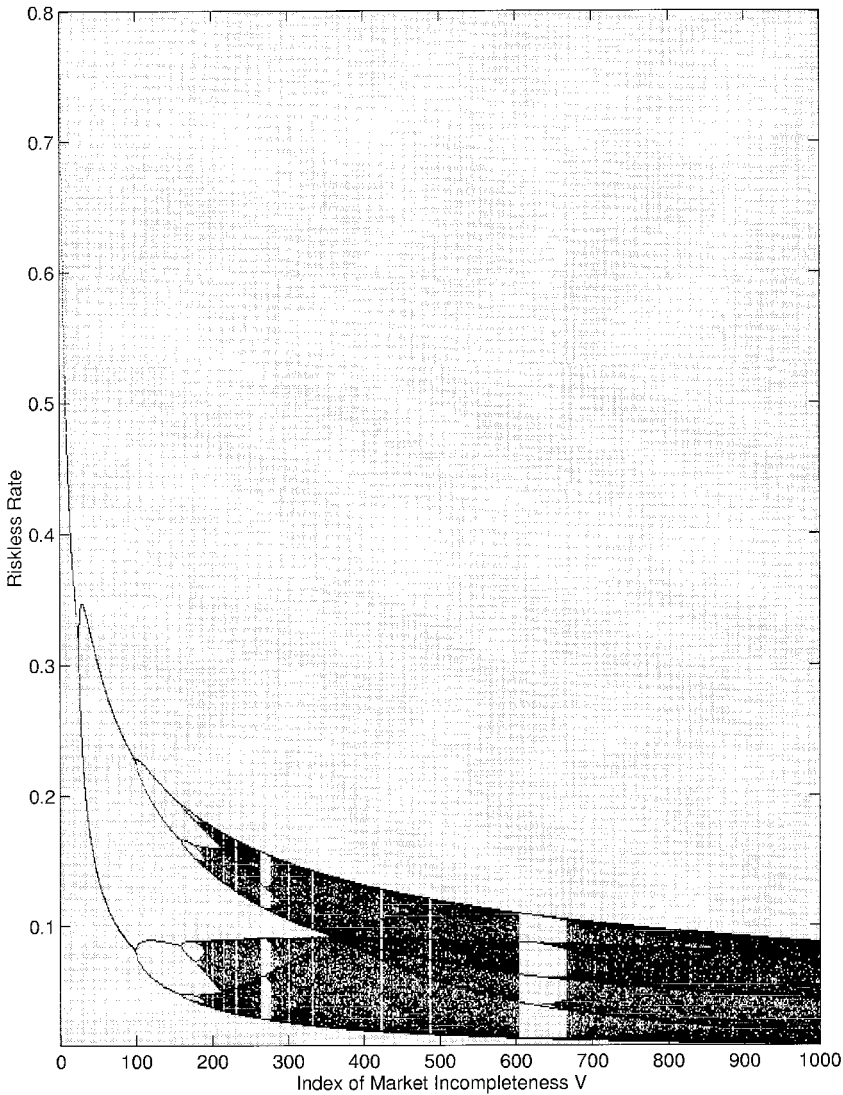

FIG. 2. Bifurcation diagram ($\beta = 0.2$).

therefore increasing with g. Thus when the growth rate g is sufficiently high, the constraint $\beta < \beta_{\max} e^g$ ceases to be binding and endogenous fluctuations are observed for discount factors β arbitrarily close to 1.

In a purely stationary economy, we perform equilibrium simulations for many values of the parameters V and β. The horizon T is set to 1,000, and the last 200 periods of each equilibrium path are neglected in order to
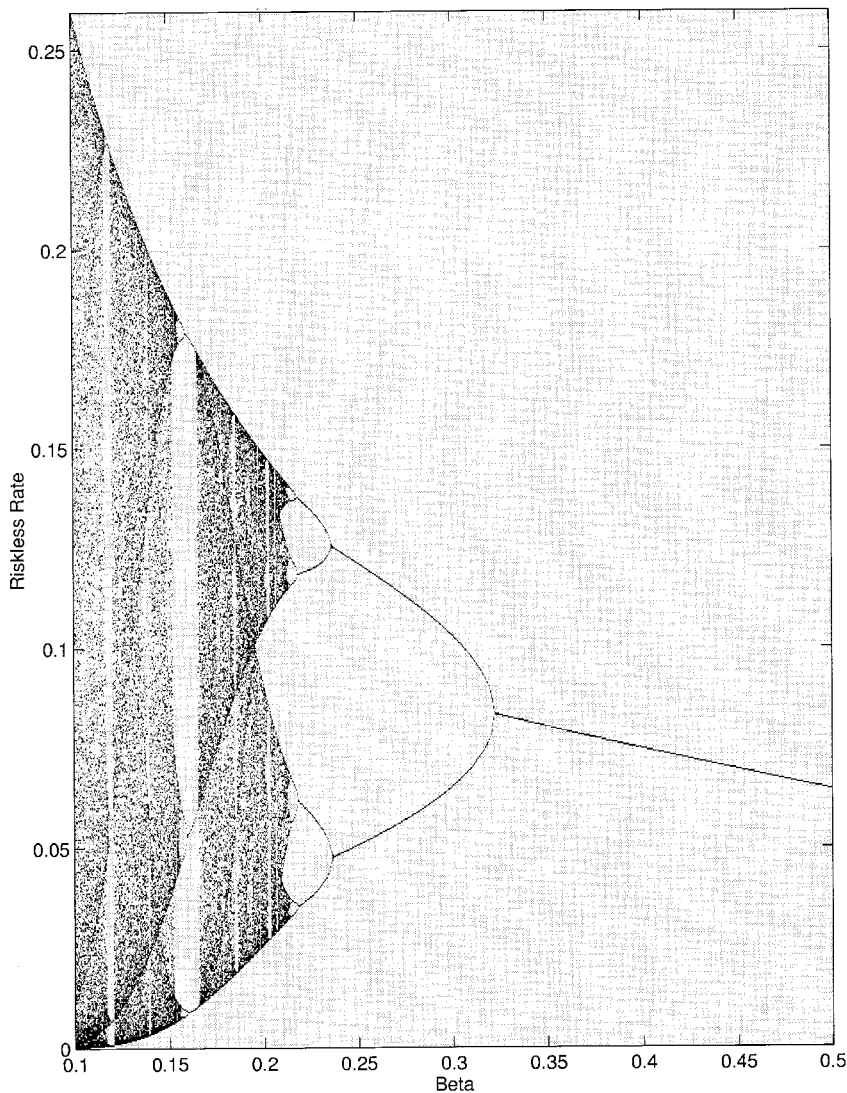

FIG. 3. Bifurcation diagram ($V = 300$).
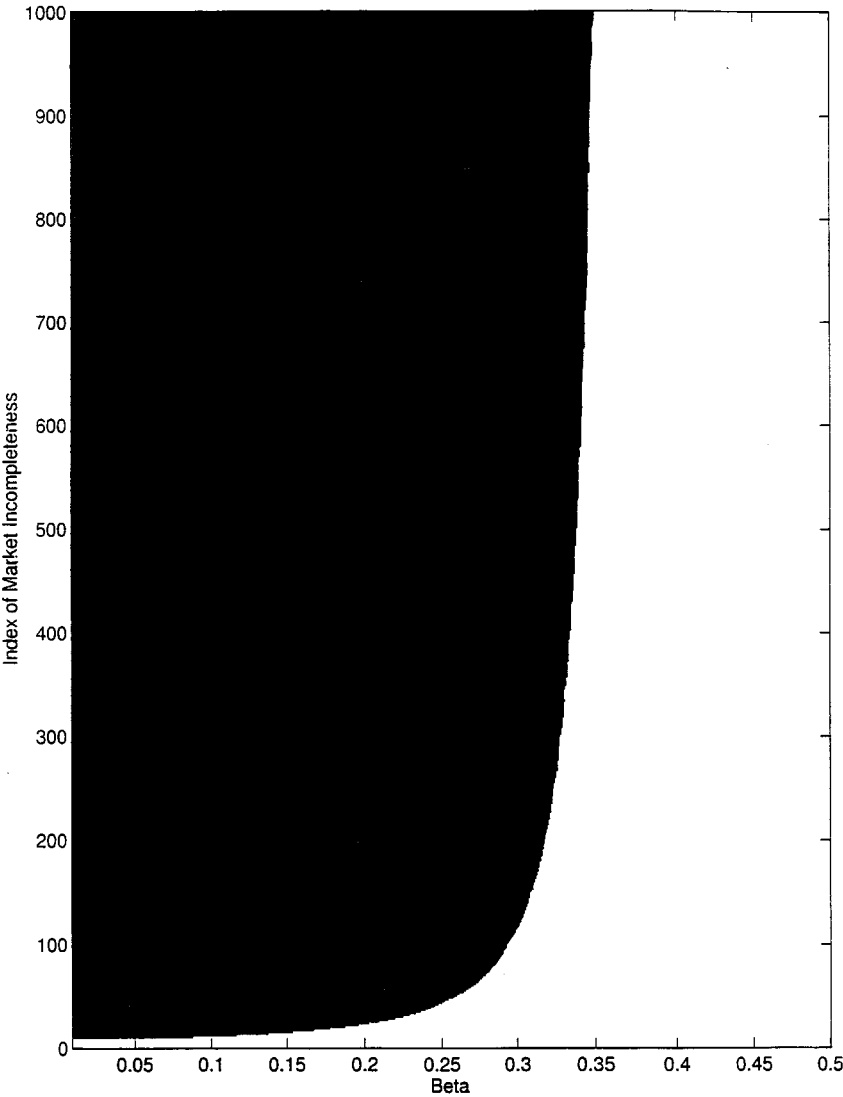
FIG. 4. Economies with persistent fluctuations.

eliminate variability around the terminal date. In Fig. 2, we take $\beta = 0.20$ and plot a bifurcation diagram[22] for various values of the incompleteness

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[22] Chaotic systems are extremely sensitive to rounding errors, and it is generally difficult to calculate a given *trajectory* with good precision. "Shadowing" theorems (Anosov [5], Bowen [10]) and numerical experiments (Hammel *et al.* [30, 31]) indicate however that simulated *orbits* stay close to a true orbit after a large number of iterations (e.g., 10^7 for the logistic map).

index. Fluctuations arise for sufficiently high values of the parameter V. Moreover, the maximal point of an orbit and the magnitude of the fluctuations both decline to zero when V is very large. Fluctuations are thus maximal for an intermediate level of market incompleteness. Figure 3 illustrates the orbits corresponding to different values of β, and a fixed index $V = 300$. When the discount factor β increases, fluctuations have decreasing
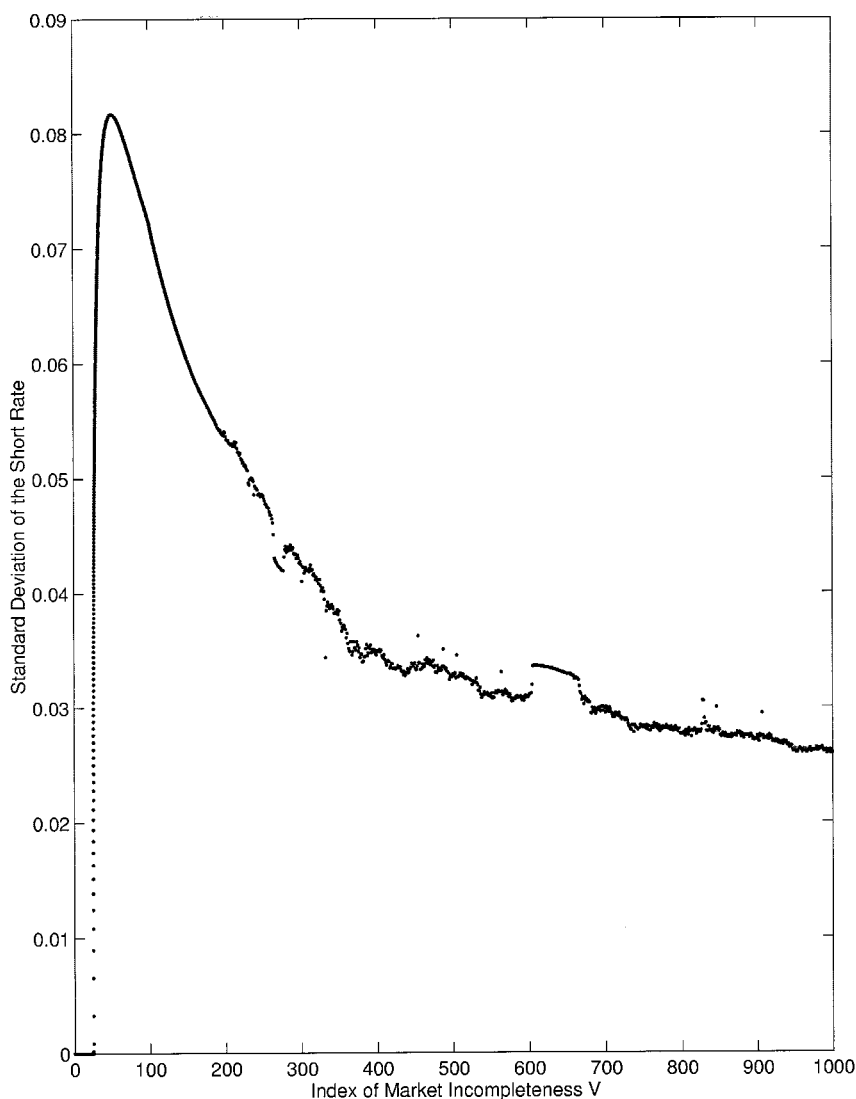

FIG. 5. Standard deviation of the short rate ($\beta = 0.2$).

amplitudes and eventually disappear. The shaded region of Fig. 4 illustrates the set PF. We find that $\beta_{\max} \approx 0.36$, and observe that fluctuations persist only under a sufficient combination of incompleteness and impatience.

Fluctuations can be quantified by the standard deviation σ of the interest rate along an equilibrium path.[23] We observe in Fig. 5 that the deviation σ is a hump-shaped function of the incompleteness index V. When markets are complete ($V = 0$), the riskless rate R_t is steady at $1/\beta$, and there are no fluctuations ($\sigma = 0$). On the other hand when unhedged risks are very large, precautionary investors have a strong demand for the riskless asset in every period, leading to small interest rates, and a small standard deviation σ. Consistent with Fig. 1, the deviation σ is maximal for intermediate values of the index V.

This section has analyzed the existence and size of fluctuations under a finite horizon. We saw that fluctuations only exist when undiversifiable risks are sufficiently large, and investors sufficiently impatient. The variability of the equilibrium path, which we quantify by the standard deviation of the riskless rate, is a single-peaked function of the index of market incompleteness. All these results are unambiguous because equilibrium is unique when the time horizon is finite.

5. EQUILIBRIUM UNDER INFINITE HORIZON

This section examines equilibrium fluctuations in an infinite horizon economy, where the aggregate endowment and the incompleteness index are constant across time. Because the economy is stationary, we might expect a unique equilibrium with steady macro variables. This intuition is valid in the complete market case. However, when markets are incomplete and investors sufficiently impatient, there also exists a continuum of equilibria in which asset prices are deterministic, bounded and varying with time.

When markets are complete ($V = 0$), investors can fully share their individual endowment risks, and the First Welfare Theorem implies

PROPOSITION 2. *Under complete markets* ($V = 0$), *there exists a unique equilibrium path, in which individual consumption is deterministic and constant across time. In every period, the marginal propensity a_t equals $1 - \beta$, risky assets are costless and the riskless rate R_t takes the value $1/\beta$.*

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[23] Ergodicity guarantees the convergence of standard errors when the number of periods T goes to infinity (Eckmann and Ruelle [18]).

The absence of price fluctuations is a familiar feature of exchange economies under complete markets.[24] In fact, the proof of Proposition 2 shows that $R_t = 1/\beta$ in an arbitrary exchange economy where (1) markets are complete, (2) the aggregate endowment is deterministic and constant through time, and (3) investors have a homogenous discount factor β. This result directly extends to incomplete market economies in which all investors have quadratic utilities, and thus no precautionary motive.

On the other hand when markets are incomplete ($V > 0$), individual consumption is necessarily random, and Subsection 4.2 suggests that deterministic price fluctuations may exist.

DEFINITION. A *perfect foresight equilibrium* (PFE) is a GEI equilibrium in which prices are deterministic and satisfy Assumption 7.

By Theorem 2, a unique collection of optimal plans corresponds to each price sequence $\{\pi(t)\}_{t=0}^{\infty}$ satisfying Assumption 7. A PFE is therefore fully characterized by its price path.

PROPOSITION 3. *In a PFE, the price of risky assets is zero, and the marginal propensity a_t and riskless rate R_t satisfy the recursions* (4.1) *and* (4.2) *in every period. Moreover, the sequence* $\{a_t\}_{t=0}^{\infty}$ *is bounded away from zero.*

In a PFE, the marginal propensity thus follows the recursion

$$a_t = F(a_{t+1}). \tag{5.1}$$

Conversely given a deterministic sequence $a = \{a_t\}_{t=0}^{\infty}$ satisfying (5.1), we can construct the *derived* price sequence $\{\pi^a(t)\}_{t=0}^{\infty}$ by

$$\begin{cases} \ln R_t^a = \ln(1/\beta) - \frac{1}{2} a_{t+1}^2 V \\ \pi_i^a(t) = 0 \qquad \text{for all } i \geqslant 1 \end{cases} \tag{5.2}$$

We now examine when the price sequence $\{\pi^a(t)\}_{t=0}^{\infty}$ defines a PFE.

PROPOSITION 4. *Let* $\{a_t\}_{t=0}^{\infty}$ *be a positive sequence satisfying the recursion* (5.1). *When* $\{a_t\}_{t=0}^{\infty}$ *is bounded away from zero, the derived price sequence* $\{\pi^a(t)\}_{t=0}^{\infty}$ *is a perfect foresight equilibrium.*

Proposition 4 shows a one-to-one correspondence between PFEs and iterated sequences $\{a_t\}_{t=0}^{T}$ bounded away from zero. In particular, relation

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[24] Note that a slightly different result holds for monetary exchange economies. Under complete markets, monetary policy can generate volatility in the price level but has no effect on the set of equilibrium allocations. See Magill and Quinzii [42] and Bhattacharya *et al.* [8] for recent discussions.

(5.1) is satisfied by deterministic equilibria in both finite and infinite horizon economies. When T is finite, equilibrium is unique and can be computed by a backward recursion from the final condition $a_T = 1$. However, for $T = \infty$, a final condition does not exist because traders do not consume all their wealth in any period. Investors can in fact coordinate on any iterated sequence $\{a_t\}_{t=0}^{\infty}$ bounded away from zero, and multiple equilibria can arise.

THEOREM 6. *In a stationary economy, the set of perfect foresight equilibria contains either the steady state as a unique element, or a continuum of equilibria including a cycle of period 2.*

We can thus distinguish two types of SPEC economies. *Neoclassical* settings contain a unique PFE, in which macro variables are time-invariant and reflect the stationarity of the fundamentals. We see that complete market economies are neoclassical. On the other hand, *Keynesian* economies allow persistent fluctuations, including a cycle of period 2, and a continuum of PFEs. The existence of a continuum of equilibria, or indeterminacy, is a specific feature of the infinite horizon economies, and is attributed to the lack of a final condition at $T = \infty$.

Neoclassical and Keynesian economies define two regions NC_{∞} and PF_{∞} of the parameter space (V, β). We now compare them to PF, the set of economies displaying persistent fluctuations under a finite horizon.

THEOREM 7. *The Keynesian region PF_{∞} contains PF, and has thus a non-empty interior. Moreover, there robustly exist cycles of every order on a subset of PF_{∞}.*

Under a finite horizon, the terminal date initiates fluctuations in an inherently unstable economy. Moving to an infinite horizon does not eliminate financial volatility, but enormously complicates the model by giving rise to a continuum of equilibria.

Indeterminacy cannot arise in SPEC under complete markets. This property is consistent with Shannon [55], who shows that market completeness generically implies the local uniqueness of competitive equilibria in a large class of economies. On the other hand when markets are incomplete, SPEC can display a robust form of indeterminacy, which to the best of my knowledge is new to GEI economies with real assets and finitely many agents. For instance with more than one commodity, Mas-Colell [45] obtains robust indeterminacy in a two-period model[25] by considering

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[25] This requires that the number of states be at least countable. See Geanakoplos and Polemarchakis [24] for an analysis of the finite state space.

endowments and utilities for which the spot market has three equilibria in every state. In SPEC however, there exists a single good, and equilibrium is unique when the horizon is finite. In infinite horizon models, indeterminacy usually originates in bubbles or sunspots (Chiappori and Guesnerie [14], Magill and Quinzii [41], Santos and Woodford [52]). Theorem 6, however, indicates that asset prices are deterministic (unlike sunspots), and equal to their fundamental values (unlike bubbles).

The model's indeterminacy is closer in spirit to the multiple, deterministic equilibria of OLG models (Gale [21], Woodford [67], Grandmont [25], Geanakoplos [22]). In both classes of economies, traders can coordinate on uncountably many deterministic equilibrium paths. The analogy with the OLG literature is all the more striking because in SPEC, there exist robust cycles of every order. In OLG models however, indeterminacy does not always imply the existence of persistent fluctuations since all equilibria can converge to steady states (Gale [21]).

Predicting which equilibrium will be reached in the Keynesian region is in general a difficult task, because of the complexities usually attached to equilibrium selection. Previous research has considered three different selection methods, which we now discuss. (The choice of a particular selection method is left to the reader). First, equilibrium fluctuations can be ruled out by assuming that agents coordinate on the unique steady state. According to this viewpoint, extending SPEC from a finite to an infinite horizon drastically simplifies the equilibrium path. Second, one can follow the mathematics literature, and select paths that are stable in the backward perfect foresight dynamics. These are the only equilibria numerically observable. The analysis is particularly simple for SPEC, because the iterated function F has a negative Schwarzian derivative. Third, we can select paths that are stable in a temporary equilibrium with learning (Grandmont [25]). In such models, each agent is endowed with, a learning rule, that predicts future prices as a function of current information.[26] Equilibrium stability then depends on the choice of the individual learning processes. Under appropriate conditions,[27] stability in the learning dynamics and in the backward perfect foresight dynamics are expected to coincide.

6. CONCLUSION

SPEC demonstrates that the precautionary motive, combined with asset incompleteness, is a leading source of volatility and indeterminacy in the

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[26] Since agents are infinitely-lived, the definition of learning rules requires particular care.
[27] See Grandmont [25].

financial markets. This is shown by calculating the equilibrium path of a CARA-normal economy with real assets and heterogeneous income streams. Under a finite horizon, the equilibrium path is unique and fluctuations are trivially robust to equilibrium selection. On the other hand with a finite number of infinitely-lived agents, there exists a robust continuum of perfect foresight equilibria, in which prices are deterministic and bounded across time.

Fluctuations originate from a *negative feedback* between future and current interest rates. In a GEI economy, agents insure themselves *through time* by borrowing and lending because they cannot freely shift income *across states of nature* by trading in risky assets. A *high* interest rate at a future date reduces the potential for future consumption smoothing across time via borrowing, leading to a strong precautionary motive and a *low* interest rate in the current period.

These results have important implications for financial markets and risk-sharing. Financial innovation may not only help improve welfare through a better allocation of risk, it should also reduce the volatility of existing assets. When markets are almost complete, the neoclassical viewpoint prevails, as volatility is entirely attributable to information flows and exogenous changes in the real sector of the economy. On the other hand when there are large undiversifiable risks and sufficient impatience, the precautionary motive generates endogenous fluctuations and large volatility; the economy thus functions in a distinctly Keynesian regime.

The full development of this theory goes well beyond the scope of the present work, and stimulates many generalizations and extensions. A companion paper (Angeletos and Calvet [4]) relates financial structure to the business cycle in an economy with decentralized production and idiosyncratic technological risk. In this setting, market incompleteness causes fluctuations not only in financial prices but also in the economy's aggregate wealth. Another possible extension would analyze how, under both symmetric and asymmetric information, exogenous changes in the financial structure affect prices and output. Empirical applications and other theoretical elaborations are also envisioned, and will be the object of further research.

7. APPENDIX

7.1. *Proof of Theorem* 1

We write the First Order Conditions to the trader's problem:

$$\pi_0(t) \exp(-Ac^h) = \beta \mathbb{E}_t J_w^h \tag{7.1}$$

$$\pi_i(t) \exp(-Ac^h) = \beta \mathbb{E}_t(\tilde{a}_{i,\,t+1} J_w^h). \tag{7.2}$$

By construction, a risky asset has zero expected payoff and therefore

$$\mathbb{E}_t(\tilde{a}_{i,\,t+1}J_w^h) = Cov_t\{\tilde{a}_{i,\,t+1}; \exp[-A(a_{t+1}\widetilde{W}_{t+1}^h + b_{t+1}^h)]\}.$$

Since $\tilde{a}_{i,\,t+1}$ and $\widetilde{W}_{t+1}^h$ are jointly normal, Stein's lemma implies

$$
\begin{aligned}
\mathbb{E}_t(\tilde{a}_{i,\,t+1}&J_w^h) \\
&= -Aa_{t+1}Cov_t(\tilde{a}_{i,\,t+1}; \widetilde{W}_{t+1}^h)\,\mathbb{E}_t\{\exp[-A(a_{t+1}\widetilde{W}_{t+1}^h + b_{t+1}^h)]\} \\
&= -Aa_{t+1}(\theta_i^h + \gamma_i^h)\,\mathbb{E}_t J_w^h,
\end{aligned}
$$

and Eq. (7.2) can be rewritten as

$$\pi_i(t)\exp(-Ac^h) = -\beta Aa_{t+1}(\theta_i^h + \gamma_i^h)\,\mathbb{E}_t J_w^h. \tag{7.3}$$

We can then divide (7.3) by (7.1) and solve for θ_i^h. The trader's wealth at date $t+1$ then simplifies to

$$
\begin{aligned}
\widetilde{W}_{t+1}^h &= \tilde{e}_{t+1}^h - \sum_{i=1}^{N(t)}\left[\gamma_{it}^h + \frac{R_t\pi_i(t)}{Aa_{t+1}}\right]\tilde{a}_{i,\,t+1} + \theta_0^h \\
&= \theta_0^h + \mathbb{E}_t\tilde{e}_{t+1}^h + \tilde{\varepsilon}_{t+1}^h - \frac{R_t}{Aa_{t+1}}\sum_{i=1}^{N(t)}\pi_i(t)\,\tilde{a}_{i,\,t+1}
\end{aligned}
$$

and thus

$$
\begin{aligned}
\mathbb{E}_t J_w^h &= \exp\left\{\frac{(Aa_{t+1})^2}{2}Var_t(\widetilde{W}_{t+1}^h) - A[a_{t+1}\mathbb{E}_t(\widetilde{W}_{t+1}^h) + b_{t+1}^h]\right\} \\
&= \exp\left\{\frac{(Aa_{t+1})^2}{2}Var_t(\tilde{\varepsilon}_{t+1}^h) + \frac{R_t^2}{2}\sum_{i=1}^{N(t)}[\pi_i(t)]^2\right. \\
&\qquad\left. - Aa_{t+1}(\mathbb{E}_t\tilde{e}_{t+1}^h + \theta_0^h) - Ab_{t+1}^h\right\}.
\end{aligned}
$$

Taking the logarithm of Eq. (7.1),

$$
\begin{aligned}
\ln\pi_0(t) &= Ac^h + \ln\beta - Ab_{t+1}^h + \frac{(Aa_{t+1})^2}{2}Var_t(\tilde{\varepsilon}_{t+1}^h) + \frac{R_t^2}{2}\sum_{i=1}^{N(t)}[\pi_i(t)]^2 \\
&\qquad - Aa_{t+1}[\mathbb{E}_t\tilde{e}_{t+1}^h + \theta_0^h], \tag{7.4}
\end{aligned}
$$

we infer

$$\theta_0^h = \frac{c^h - b_{t+1}^h}{a_{t+1}} + \frac{1}{Aa_{t+1}} \left\{ \ln(\beta R_t) + \frac{R_t^2}{2} \sum_{i=1}^{N(t)} [\pi_i(t)]^2 \right\}$$
$$+ \frac{Aa_{t+1}}{2} Var_t(\tilde{\varepsilon}_{t+1}^h) - \mathbb{E}_t \tilde{e}_{t+1}^h.$$

We now substitute θ_0^h into the budget constraint and solve for consumption c^h. By the envelope theorem $J_w^h(W_t^h, t) = \exp(-Ac^h)$, the Recursive Condition is satisfied at date t.

7.2. Properties of Optimal Consumption

The results of Subsection 3.1 can be generalized to arbitrary utility functions. Consider an agent living T periods ($t = 1, ..., T$), and receiving during her lifetime an exogenous, *deterministic* endowment stream $\{e_t\}_{t=1}^T$. In the first $T-1$ periods, the agent can trade a riskless asset with exogenous, deterministic rate of return R_t. She seeks to maximize her intertemporal utility $\sum_{t=1}^T \beta^t u(c_t)$, where the function u is strictly concave and twice continuously differentiable.

When u is either quadratic, CARA or CES, simple calculation shows that period t's consumption is a linear function of current wealth,

$$c_t \equiv a_t W_t + b_t$$

with a slope a_t given by:

CARA Utility. As in SPEC, it is easy to check that $a_t = 1/[1 + 1/(a_{t+1}R_t)]$.

Quadratic Utility. The marginal propensity $a_t = 1/[1 + (\beta a_{t+1} R_t^2)^{-1}]$ increases with the riskless rate and future propensity. By the certainty equivalence principle, this result also holds when the endowment stream $\{e_t\}_{t=1}^T$ is stochastic.

CES (or CRRA) Utility. When utility is of the form $u(c) = c^{1 - 1/\sigma}/(1 - 1/\sigma)$, marginal propensity $a_t = 1/[1 + \beta^\sigma/(a_{t+1} R_t^{1-\sigma})]$ increases with future propensity a_{t+1}, and with the riskless rate R_t if $\sigma < 1$.

This last example suggests that the desired monotonicity of a_t requires a sufficiently low elasticity of intertemporal substitution.

We now consider a general utility function u, and show that similar monotonicities still hold. Assume for simplicity that $T = 2$. The optimization problem

$$\text{Max } u(c_1) + \beta u(c_2)$$
$$\text{s.t. } c_1 + c_2/R_1 = W_1 + e_2/R_1$$

usually has a nonlinear solution $c_1(W_1)$. For this reason, it is useful to consider a positive number ε and two initial wealth levels $W_{1g} = e_1 + \varepsilon$ and $W_{1b} = e_1 - \varepsilon$. Denoting by c_{1g} and c_{1b} the corresponding first period consumptions, we can analyze the sensitivity of the slope

$$a_1(R_1; e_1, e_2, \varepsilon) = (c_{1g} - c_{1b})/2\varepsilon$$

to the riskless rate R_1. We expect that when ε is large enough, an agent endowed with $(e_1 - \varepsilon, e_2)$ will borrow at $t = 1$. An exogenous rise in R_1 increases the cost of borrowing and the relative price of first-period consumption; the wealth and substitution effects thus both imply lower consumption c_{1b}. On the other hand when endowed with $(e_1 + \varepsilon, e_2)$, the agent will save in period one, and the wealth and substitution effects are conflicting. For a sufficiently large ε, we expect the wealth effect to dominate if the indifference curves are sufficiently convex; leading to a higher consumption c_{1g} and a higher slope a_1.

We now check the validity of these conjectures. Given the wealth level W_1, we define $F(c_1; R_1) \equiv u'(c_1) - \beta R_1 u'[e_2 + R_1(W_1 - c_1)]$. Optimal consumption satisfies $F(c_1; R_1) = 0$, and by the Implicit Function Theorem,

$$\frac{\partial c_1}{\partial R_1} = -\frac{(\partial F/\partial R_1)}{(\partial F/\partial c_1)} = \frac{\beta[u'(c_2) + (c_2 - e_2)u''(c_2)]}{u''(c_1) + \beta R_1^2 u''(c_2)}.$$

The conditions

$$c_{2b} < e_2 \tag{7.5}$$

$$u'(c_{2g}) + (c_{2g} - e_2)u''(c_{2g}) < 0, \tag{7.6}$$

are therefore sufficient to guarantee that an increase in the interest rate leads to a higher c_{1g}, a lower c_{1b}, and therefore a higher slope a_1. We note that inequality (7.6) is satisfied if there exists $\sigma_0 < 1$ such that

$$-u'(c_{2g})/c_{2g}u''(c_{2g}) \leqslant \sigma_0 \qquad \text{and} \qquad e_2/c_{2g} < 1 - \sigma_0. \tag{7.7}$$

Assumption A. The utility function u is defined on the real line, $u'(c) \to +\infty$ when $c \to -\infty$, and $u'(c) \to 0$ when $c \to +\infty$. Moreover, there exists $\sigma_0 < 1$ such that elasticity $-u'(c)/cu''(c) \leqslant \sigma_0$ for large enough c.

Under this hypothesis, there exists a consumption level $\bar{c}(e_2)$ such that inequality (7.6) holds for all $c_{2g} > \bar{c}(e_2)$. We assume, without loss of generality, that $\bar{c}(e_2) > e_2$.

PROPOSITION A. *Under Assumption A and for any* (R_1, e_1, e_2), *there exists* $\underline{\varepsilon} > 0$ *such that* $\partial a_1/\partial R_1 > 0$ *for any* $\varepsilon > \underline{\varepsilon}$.

Proof. We observe that $c_{2b} < e_2$ and $c_{2g} > \bar{c}(e_2)$ are equivalent to

$$u'(e_1 - \varepsilon) > (\beta R_1)\, u'(e_2) \tag{7.8}$$

$$\beta u'[\bar{c}(e_2)] > (1/R_1)\, u'\{e_1 + \varepsilon - [\bar{c}(e_2) - e_2]/R_1\}, \tag{7.9}$$

and thus hold for large enough ε. ∎

Alternatively, we consider

Assumption B.   The utility function u is defined on $[0, \infty)$, $u'(c) \to +\infty$ when $c \to 0$, and $u'(c) \to 0$ when $c \to +\infty$, and there exists $\sigma_0 < 1$ such that elasticity $-u'(c)/cu''(c) \leqslant \sigma_0$ for all c.

We must then impose the additional restriction $0 \leqslant \varepsilon < e_1$.

PROPOSITION B.   *Under Assumption* B *and for any* (R_1, e_1), *there exist positive numbers* $\underline{e}_2, \bar{e}_2$ *and* $\underline{\varepsilon} < e_1$ *such that* $\partial a_1/\partial R_1 > 0$ *for any* $\varepsilon > \underline{\varepsilon}$ *and* $e_2 \in [\underline{e}_2, \bar{e}_2]$.

Proof.   Let $k = (1 - \sigma_0)^{-1}$, $\tilde{c}(e_2) = ke_2$, and

$$H(e_2, \varepsilon) \equiv \beta u'(ke_2) - (1/R_1)\, u'[e_1 + \varepsilon - (k-1)\, e_2/R_1].$$

We note that $H(e_2, \varepsilon) > 0$ implies that inequality (7.9) is satisfied. Since H is decreasing in e_2 and increasing in ε, choose 1) $\bar{e}_2$ such that $H(\bar{e}_2, e_1) > 0$; 2) a positive number $\underline{\varepsilon}$ such that $H(\bar{e}_2, \underline{\varepsilon}) > 0$ and $u'(e_1 - \underline{\varepsilon}) > (\beta R_1)\, u'(\bar{e}_2)$; and 3) a number $\underline{e}_2, 0 < \underline{e}_2 < \bar{e}_2$, such that $u'(e_1 - \underline{\varepsilon}) > (\beta R_1)\, u'(\underline{e}_2)$. For any $\varepsilon \in (\underline{\varepsilon}, e_1)$ and $e_2 \in [\underline{e}_2, \bar{e}_2]$, we observe that inequality (7.8) holds, that $H(e_2, \varepsilon) > H(\bar{e}_2, \underline{\varepsilon}) > 0$ and therefore that $\partial a_1/\partial R_1 > 0$. ∎

Similarly, we could use a three period setting to analyze the interaction between future and current consumption smoothing. This extension, however, is not required for the equilibrium analysis of Appendix 7.5.

7.3. *Proof of Theorem* 2

Assumption 7 guarantees that the sequence $\{a_t\}_{t=0}^{\infty}$ is contained in a compact interval of the form $[\underline{a}, 1]$, where $\underline{a} > 0$. In order to check that b_t^h is well-defined, we consider the decomposition

$$M_t^h/a_t = X_t^h + Y_t^h,$$

where

$$X_t^h = (a_{t+1} R_t)^{-1}\, [a_{t+1} \mathbb{E}\tilde{\varepsilon}_{t+1}^h - Aa_{t+1}^2\, Var(\tilde{\varepsilon}_{t+1}^h)/2 - \ln(\beta R_t)/A]$$

$$Y_t^h = \sum_{i=1}^{N(t)} \pi_i(t)[\gamma_{it}^h + R_t \pi_i(t)/(2Aa_{t+1})].$$

By Assumption 7, the interest rate sequence $\{R_t\}_{t=0}^{\infty}$ is contained in an interval of the form $[\underline{R}, \infty)$, where $\underline{R} > 0$; we thus infer that $\{\ln(\beta R_t)/a_{t+1}R_t\}_{t=0}^{\infty}$ and, by Assumption 6, the infinite sequence $\{X_t^h\}_{t=0}^{\infty}$ are bounded. By Assumption 6 and the Cauchy–Schwarz inequality $|\gamma_{it}^h| \leqslant [Var(e_t^h)]^{1/2}$, we infer the boundedness of the set $\{\gamma_{it}^h; t, t\}$ and of the sequences $\{\gamma_{it}^h + R_t\pi_i(t)/(2Aa_{t+1})\}_{t=0}^{\infty}$, $\{Y_t^h\}_{t=0}^{\infty}$ and $\{M_t^h/a_t\}_{t=0}^{\infty}$. Consequently, the deterministic sequence $\{b_t^h\}_{t=0}^{\infty}$ is well-defined and bounded.

We now check that $J(W, Z) \equiv -(Aa_0)^{-1}\exp[-A(a_0 W + b_0^h)]$ coincides with the value function. For any real number W, an admissible plan $\{c_t', \theta_t', W_t'\}_{t=0}^{\infty}$, such that $W_0' = W$, satisfies

$$J(W, Z) = \underset{\{c, \theta, W_1\}}{\text{Max}} \left[u(c) + \beta \mathbb{E} J(W_1, \mathscr{S}Z) \right]$$

$$\geqslant u(c_0') + \beta \mathbb{E}_0 J(W_1', \mathscr{S}Z)$$

$$= u(c_0') + \beta \mathbb{E}_0 \underset{\{c, \theta, W_2\}}{\text{Max}} \left[u(c) + \beta \mathbb{E}_1 J(W_2, \mathscr{S}^2 Z) \right]$$

$$\geqslant u(c_0') + \beta \mathbb{E}_0 u(c_1') + \beta^2 \mathbb{E}_0 J(W_2', \mathscr{S}^2 Z),$$

and by repetition

$$J(W, Z) \geqslant \mathbb{E}_0 \left[\sum_{t=0}^{T-1} \beta^t u(c_t') \right] + \beta^T \mathbb{E}_0 J(W_T', \mathscr{S}^T Z). \tag{7.10}$$

Since $\{\exp(-Ab_T^h)/a_T\}_{T=0}^{\infty}$ is bounded and $\exp(-Aa_T W_T') \leqslant 1 + \exp(-AW_T')$, Assumption 5 implies that

$$\beta^T \mathbb{E}_0 J(W_T', \mathscr{S}^T Z) = -(Aa_T)^{-1} \beta^T \mathbb{E}_0 \exp[-A(a_T W_T' + b_T^h)]$$

converges to zero as $T \to \infty$. Letting T go to infinity in (7.10), we obtain

$$J(W, Z) \geqslant \mathbb{E}_0 \left[\sum_{t=0}^{+\infty} \beta^t u(c_t') \right]. \tag{7.11}$$

This inequality, which holds for any admissible plan $\{c_t', \theta_t', W_t'\}$, proves that $J(W, Z)$ is an upper bound to the value function. It is then easy to check that the consumption plan defined by (3.2) is admissible and reaches $J(W, Z)$.

7.4. *Proof of Theorem* 3

It is convenient to introduce

Recursive Condition 2.   In our exchange economy at a given date $t+1$, all investors have marginal indirect utilities of the type

$$J_w^h(W, t+1) = \exp[-A(a_{t+1}W + b_{t+1}^h)],$$

where the coefficient a_{t+1} is the same for all households.

We can then compute period t's market-clearing prices.

LEMMA.   *Under Recursive Condition* 2, *the equilibrium price of each asset at date t is unique. The riskless rate satisfies*

$$\ln R_t = \ln(1/\beta) - \frac{1}{2}a_{t+1}^2 V(t) + A(a_{t+1}e_{t+1} - e_t) + \frac{A}{H}\sum_{h=1}^{H} b_{t+1}^h, \quad (7.12)$$

and the price of each risky asset is zero: $\pi_i(t) = 0$ *for all $i \geqslant 1$.*

Proof.   Since aggregate endowment is deterministic, the mean demand $\sum_{h=1}^{H} \theta_{it}^h / H$ for the ith risky asset simplifies to $-R_t\pi_i(t)/(Aa_{t+1})$ and therefore $\pi_i(t) = 0$ in equilibrium. Mean demand for the riskless asset satisfies

$$\frac{1}{H}\sum_{h=1}^{H} \theta_{0t}^h = \frac{a_t}{a_{t+1}}e_t - \frac{R_t}{H}\sum_{h=1}^{H} b_t^h.$$

Since

$$\frac{1}{H}\sum_{h=1}^{H} b_t^h$$
$$= (a_t/a_{t+1}R_t)\left[\frac{1}{H}\sum_{h=1}^{H} b_{t+1}^h + a_{t+1}e_{t+1} - a_{t+1}^2 V(t)/(2A) - \ln(\beta R_t)/A\right],$$

equilibrium can be written

$$Ae_t - \left[\frac{A}{H}\sum_{h=1}^{H} b_{t+1}^h + Aa_{t+1}e_{t+1} - a_{t+1}^2 V(t)/2 - \ln(\beta R_t)\right] = 0$$

which yields (7.12).   ∎

Since Recursive Condition 2 holds at date T, there exists a unique intertemporal equilibrium, in which agents have identical marginal propensities and there is no risk premium. Since $a_T = 1$ and $b_T^h = 0$, the lemma implies

that relation (4.1) holds in period $t = T - 1$. For any $t \leqslant T - 2$, equilibrium of the good market at date $t + 1$ implies

$$\frac{1}{H} \sum_{h=1}^{H} c_{t+1}^h = a_{t+1} e_{t+1} + \frac{1}{H} \sum_{h=1}^{H} b_{t+1}^h = e_{t+1},$$

and therefore $\sum_{h=1}^{H} b_{t+1}^h / H = (1 - a_{t+1}) e_{t+1}$. Substituting this expression in (7.12) yields (4.1).

7.5. Equilibrium Feedback with Arbitrary Utilities

We now give a simple example where the *negative* feedback between future and current rates requires that the marginal utility be *convex* ($u''' > 0$).

The Model. Consider an exchange economy $\mathscr{E}(e_0, e_1, e_2, \varepsilon)$ with H agents, three periods ($t = 0, 1, 2$) and finitely many states $\{1, ..., S\}$. Assume that exogenous uncertainty arises only in the middle period ($t = 1$). At dates $t = 0$ and 2, each agent h ($1 \leqslant h \leqslant H$) receives a deterministic endowment $e_0^h = e_0$ and $e_2^h = e_2$, and in the middle period, is endowed with random income $e_1^h = e_1 + \tilde{\varepsilon}^h$, where $\tilde{\varepsilon}^h$ takes the value ε or $-\varepsilon$ with equal probability. Moreover, assume that the mean endowment $e_1 = \sum e_1^h / H$ is deterministic. When $t \in \{0, 1\}$, investors can trade a short-lived riskless asset, which is in zero net supply and has an endogenous (gross) rate of return R_t. Preferences are identical and can be represented by a separable utility $u(c_0) + \beta \mathbb{E}u(c_1) + \beta^2 \mathbb{E}u(c_1)$, where ε is strictly concave and twice continuously differentiable.

Equilibrium. We limit our attention to *symmetric equilibria*, in which the interest rate R_1 is invariant across states. At date $t = 0$, all agents solve the same decision problem and have therefore identical demand functions, implying

$$c_0^h = e_0$$

for all h. Since the riskless asset is not traded in the initial period, each agent starts period 1 with wealth e_1^h. As in Appendix 7.2, along the equilibrium path individual consumption takes the form

$$c_1^h = e_1 + a_1(R_1, e_1, e_2) \tilde{\varepsilon}^h,$$

and the First Order Condition at the initial date is

$$u'(e_0) = \beta R_0 \mathbb{E}u'[e_1 + a_1(R_1, e_1, e_2) \tilde{\varepsilon}^h]. \tag{7.13}$$

Comparative Statics. We now consider exogenous variations in the mean endowment e_2, which leads to a new equilibrium path with a higher interest rate R_1. First examine the special case where the utility function u is either CES, CARA or quadratic. The marginal propensity $a_1(R_1, e_1, e_2)$ only depends on the riskless rate R_1, and the First Order Condition simplifies to

$$u'(e_0) = \beta R_0 \, \mathbb{E}u'[e_1 + a_1(R_1) \, \tilde{\varepsilon}^h].$$

A *higher* R_1 along the equilibrium path leads to a higher propensity a_1. With CARA or CES, marginal utility is strictly convex ($u''' > 0$), expected marginal utility $\mathbb{E}u'[e_1 + a_1(R_1) \, \tilde{\varepsilon}^h]$ is higher and the interest rate R_0 is *lower* in the new equilibrium. Like in SPEC, there is a *negative* feedback between future and current rates. On the other hand when utility is quadratic ($u''' = 0$), expected marginal utility remains unchanged, and there is no feedback between R_1 and R_0.

With an arbitrary utility function u, the marginal propensity a_1 also depends on the mean endowment e_2. An exogenous change in e_2 thus influences marginal propensity both directly and indirectly (via R_1).[28] In order to analyze the overall effect of e_2, we define

$$G(a_1, R_1; e_2) \equiv \begin{bmatrix} u'(e_1 + a_1\varepsilon) - \beta R_1 u'[e_2 + R_1(1 - a_1)\,\varepsilon] \\ u'(e_1 - a_1\varepsilon) - \beta R_1 u'[e_2 - R_1(1 - a_1)\,\varepsilon] \end{bmatrix}.$$

Equilibrium propensity a_1 and interest R_1 are simultaneously determined by the system $G(a_1, R_1; e_2) = 0$. It is convenient to define $x = (x_1, x_2) = (a_1, R_1)$, $g_{ij} = \partial G_i / \partial x_j$, and the determinant $\Delta = g_{11}g_{22} - g_{21}g_{12}$. Direct computation shows that the vector $\partial G / \partial e_2$ has positive components, and

$$g_{11} < 0, \qquad g_{12} = -\beta[u'(c_{2g}) + (c_{2g} - e_2)\,u''(c_{2g})],$$
$$g_{21} > 0, \qquad g_{22} = -\beta[u'(c_{2b}) + (c_{2b} - e_2)\,u''(c_{2b})].$$

In equilibrium, $c_{2b} < e_2$ and therefore $g_{22} < 0$.

LEMMA. *Consider a utility function satisfying Assumption A. For any* (e_0, e_1, e_2), *there exists* $\underline{\varepsilon}(e_1, e_2)$ *such that*

$$[\varepsilon > \underline{\varepsilon}(e_1, e_2)] \Rightarrow [g_{12} > 0]$$

in any symmetric equilibrium of $\mathscr{E}(e_0, e_1, e_2, \varepsilon)$.

---

[28] Note that these are not the main direct and indirect effects discussed in SPEC.

Proof.   As in Proposition A, we need to show that $c_{2g} > \bar{c}(e_2)$ for large enough ε. Since the endowment e_2 is fixed, $\bar{c}(e_2)$ is denoted $\bar{c}$ in this proof. The inequality $c_{2g} > \bar{c}$ holds if and only if the indifference curve has lower slope $|\partial c_2/\partial c_1|$ than the budget set at the point $[e_1 + \varepsilon - (\bar{c} - e_2)/R_1; \bar{c}]$ of a rich consumer's budget line. We thus define the function

$$K(\varepsilon, R_1) \equiv (\beta R_1)^{-1} u'[e_1 + \varepsilon - (\bar{c} - e_2)/R_1] - u'(\bar{c}),$$

which decreases with ε and R_1. With this notation, $c_{2g} > \bar{c}$ is equivalent to $K(\varepsilon, R_1) < 0$.

We define $\underline{R}_1 = u'(e_1)/[\beta u'(2e_2 - \bar{c})]$, and choose a number $\underline{\varepsilon}(e_1, e_2)$ satisfying the inequality: $K(\varepsilon, \underline{R}_1) < 0$. Consider now $\varepsilon > \underline{\varepsilon}(e_1, e_2)$ and an equilibrium of the economy $\mathscr{E}(e_0, e_1, e_2, \varepsilon)$. If $c_{2g} < \bar{c}$, we infer that $c_{2b} = 2e_2 - c_{2g} > 2e_2 - \bar{c}$, and thus $R_1 = u'(c_{1b})/[\beta u'(c_{2b})] > \underline{R}_1$. We conclude that $K(\varepsilon, R_1) < K[\underline{\varepsilon}(e_1, e_2), \underline{R}_1] < 0$ and thus $c_{2g} > \bar{c}$, which is a contradiction. ∎

We can now precisely analyze the comparative static properties of our model. Given an economy $\mathscr{E}(e_0^*, e_1^*, e_2^*, \varepsilon^*)$, $\varepsilon^* > \underline{\varepsilon}(e_1^*, e_2^*)$, and a utility function satisfying Assumption A, consider a symmetric equilibrium (a_1^*, R_0^*, R_1^*) where $\Delta^* \neq 0$. By the Implicit Function Theorem, the set of symmetric equilibria can be locally parameterized by a differentiable function $[a_1(e_2), R_0(e_2), R_1(e_2)]$.[29] Moreover, the components of

$$\frac{dx}{de_2}(e_2^*) = \frac{1}{\Delta^*} \begin{bmatrix} -g_{22}^* & g_{12}^* \\ g_{21}^* & -g_{11}^* \end{bmatrix} \frac{\partial G}{\partial e_2}.$$

have the same sign, implying that the equilibrium functions $a_1(e_2)$ and $R_1(e_2)$ have identical monotonicities.

We can thus consider an exogenous change in e_2 leading to *higher* R_1 and a_1. By Eq. (7.13), the equilibrium interest rate R_0 is *lower*, *unchanged* or *higher* depending on whether marginal utility is convex ($u''' > 0$), linear ($u''' = 0$) or concave ($u''' < 0$). When agents have a precautionary motive ($u''' > 0$), there is, like in SPEC, a negative feedback between future and current rates. On the other hand with a concave marginal utility ($u''' < 0$), the interest rate R_0 is *higher* in equilibrium and there is, unlike in SPEC, a *positive* feedback between future and current rates. In this example, a negative feedback can only be observed when $u''' > 0$, suggesting that the persistent fluctuations observed in SPEC require that agents have a precautionary motive ($u''' > 0$) and not simply a nonlinear marginal utility ($u''' \neq 0$).

[29] The symmetric equilibrium $[a_1(e_2), R_0(e_2), R_1(e_2)]$ implicitly depends on the fixed endowments e_0^* and e_1^*. We do not explore this dependence but focus on the effect of a future "shock," such as an exogenous change in e_2, on the current interest rate dynamics.

7.6. *Proof of Theorem* 5

This proof requires the explicit dependence of the iterated function F on the parameters V and β. We therefore define $\theta = (V, \beta)$, and denote the iterated function by $F_\theta(a)$ and its unique fixed point by $\bar{a}_\theta$. Let U_0 denote the set of parameters $\theta = (V, \beta)$ such that

$$F'_\theta(\bar{a}_\theta) < -1 \tag{7.14}$$

$$F^2_\theta(1) < \bar{a}_\theta. \tag{7.15}$$

The open set U_0 is non-empty since it contains, for instance, the point $(V = 11, \beta = 0.03)$. For any $\theta = U_0$, we infer from (7.14) that $F_\theta(1) < \bar{a}_\theta$, and that the steady state $\bar{a}_\theta$ is unstable. Therefore, if the sequence $\{a^T(\theta) = F^T_\theta(1)\}_{T=0}^{\infty}$ converges to the steady state, it must satisfy $a^T(\theta) = \bar{a}_\theta$ for some T. Condition (7.15) implies that the fixed point $\bar{a}_\theta$ is its only preimage in the interval $I_\theta = [F_\theta(1), 1]$. Since I_θ contains the entire sequence $\{a^T(\theta)\}_T$, we infer that $a^T(\theta) \neq \bar{a}_\theta$ for all T, and $a^T(\theta)$ cannot converge to the steady state.

Let θ^* be a fixed element of U_0, and let $r = \{-1 + F'_{\theta^*}(\bar{a}_{\theta^*})]\}/2$. From relation (7.14), we infer that $r < -1$. By the implicit function theorem, there exist two differentiable functions $a_+(\theta)$ and $a_-(\theta)$ defined on a compact neighborhood U_1 of θ^*, which satisfy $U_1 \subset U_0$, $F'_\theta[a_-(\theta)] = r$, $F'_\theta[a_+(\theta)] = r$ and $F'_\theta(a) \leqslant r$ for all $a \in [a_-(\theta), a_+(\theta)]$. For any $\theta \in U_1$, we define the function $m(\theta) \equiv \text{Min} \, |F_\theta(a) - a|, a \in [F_\theta(1), a_-(\theta)] \cup [a_+(\theta), 1]$.[30] Since m is positive and continuous, we know that $\delta = \text{Min}_{U_1} m > 0$.

The interior of U_1 defines an open set U. For any $\theta \in U$, consider the equilibrium sequence $\{a^t(\theta)\}_t$ and an arbitrary integer T. If $a^T(\theta) \notin [a_-(\theta), a_+(\theta)]$, then $|a^{T+1}(\theta) - a^T(\theta)| = |F[a^T(\theta)] - a^T(\theta)| > \delta$. Otherwise we know that there exists $k > 0$ such that $a^{T+k}(\theta) \notin [a_-(\theta), a_+(\theta)]$ and therefore $|a^{T+k+1}(\theta) - A^{T+k}(\theta)| > \delta$. We conclude that $\limsup_T |a^{T+1}(\theta) - a^T(\theta)| > \delta$ for all θ in U.

7.7. *Proof of Proposition* 2

Consider the allocation x defined by $x^h_t = \mathbb{E}_0 c^h_t$ for all h, t. If $Var(c^h_t) > 0$ for some h and t, the deterministic allocation x Pareto dominates the competitive equilibrium, which contradicts the First Welfare Theorem. Individual consumption is therefore deterministic along the equilibrium path, and

$$\pi_i(t) = \beta \mathbb{E}_t[\tilde{a}_{i, t+1} u'(c^h_{t+1})/u'(c^h_t)] = 0.$$

---

[30] The set $[F_\theta(1), a_-(\theta)] \cup [a_+(\theta), 1]$ cannot be empty. Otherwise $F'(a) < r$ for all $a \in I_\theta$, and therefore $|a^T(\theta) - \bar{a}_\theta| \geqslant |r|^{T-1} |F_\theta(1) - \bar{a}_\theta| \to \infty$ as $T \to \infty$, which contradicts the boundedness of the sequence $\{a^T(\theta)\}_T$.

If $\beta R_t > 1$, the individual First Order Conditions $u'(c_t^h) = \beta R_t u'(c_{t+1}^h)$ imply $c_t^h < c_{t+1}^h$ for all h, which contradicts the stationarity of the aggregate endowment. The inequality $\beta R_t > 1$ also leads to an impossibility, and we conclude that $R_t = 1/\beta$ and $c_t^h = c_{t+1}^h$ for all h, t. We then infer from the budget constraint that

$$c_t^h = (1 - \beta) \sum_{t=0}^{+\infty} \beta^t \mathbb{E}_0 e_t^h.$$

Individual wealth

$$W_t^h = \sum_{s=0}^{t-1} \beta^{-(t-s)} [\mathbb{E}_0 e_s^h - c_s^h] + \mathbb{E}_0 e_t^h$$

is therefore bounded, Assumption 5 holds, and the marginal propensity to consume satisfies $a_t = 1 - \beta$.

7.8. Proof of Proposition 3

We use Theorem 2 to compute marginal propensity and optimal demand in a PFE. Since $a_t = [1 + \pi_L(t)]^{-1}$ for all t, the sequence $\{a_t\}$ satisfies (4.2), and is bounded away from 0 by Assumption 7. The system (3.2) provides individual demand for risky assets, and implies that $\pi_i(t) = 0$ in equilibrium for all i, t. Finally, the individual First Order Condition $u'(c_t^h) = \beta R_t \mathbb{E}_t u'(c_{t+1}^h)$ simplifies to

$$A c_t^h = A \mathbb{E}_t c_{t+1}^h - (A a_{t+1})^2 \, Var_t(\varepsilon_{t+1}^h)/2 - \ln(\beta R_t), \tag{7.16}$$

and aggregation across consumers yields (4.1).

7.9. Proof of Proposition 4

By (5.2), the derived interest rate R_t^a is bounded away from zero, $R_t^a \geqslant \beta^{-1} \exp(-V/2)$, and a recursive use of Eq. (5.1) implies

$$\frac{1}{a_t} \geqslant 1 + \frac{1}{R_t^a} + \cdots + \frac{1}{R_t^a R_{t+1}^a \cdots R_s^a}$$

for all $s \geqslant t$. The sequence $\{\pi_L^a(t)\}_{t=0}^{\infty}$ is therefore well-defined and bounded by $1/\mathrm{Min}(a_t)$. Assumption 7 thus holds, and we infer from Theorem 2 that the price sequence $\{\pi^a(t)\}_{t=0}^{\infty}$ clears all markets in all periods.

7.10. *Proof of Theorem* 6

We divide the proof into several cases, depending on the relative positions of the fixed point $\bar{a}$, the maximizer $a^* = \arg\max F$, and the second iterate $F^2(a^*)$.

Case 1. $a^* \geqslant \bar{a}$. For any ε, $0 < \varepsilon < \bar{a}$, the interval $I_\varepsilon = [\varepsilon, F(a^*)]$ has a positive length, and its image $F(I_\varepsilon)$ is a compact subinterval of I_ε. The set

$$J_\varepsilon = \bigcap_{k=1}^{\infty} F^k(I_\varepsilon)$$

is the limit of a decreasing sequence of non-empty compact intervals. J_ε is therefore a compact interval $[a_\varepsilon, b_\varepsilon]$ satisfying $F(a_\varepsilon) = a_\varepsilon$, $F(b_\varepsilon) = b_\varepsilon$, and thus $J_\varepsilon = \{\bar{a}\}$. A PFE $\{a_t\}_{t=0}^{\infty}$ is bounded below by a positive number $\varepsilon < a^*$. Since $a_t = F(a_{t+1})$, we know that $a_t \leqslant F(a^*)$, and thus each a_t is contained in the interval $I_\varepsilon = [\varepsilon, F(a^*)]$. In fact, $a_t \in J_\varepsilon$ since $a_t = F^k(a_{t+k}) \in F^k(I_\varepsilon)$ for all k. We then infer that $a_t = \bar{a}$, and the steady state is the unique PFE. Note moreover that the iterated sequence $\{F^t(1)\}_{t=0}^{\infty}$ converges to $\bar{a}$.

Case 2. $a^* < \bar{a}$ and $F^2(a^*) \leqslant a^*$. Let $a = F^2(a^*)$ and $b = F(a^*)$. The function F maps the compact interval $[a, b]$ onto itself, and thus there exists a function $g : [a, b] \rightarrow [a, b]$ such that $F[g(x)] \equiv x$ for all x. For any choice of a_0 in $[a, b]$, the sequence $\{g^t(a_0)\}_{t=0}^{\infty}$ is a PFE, because it satisfies (5.1) and is bounded away from 0. Since each initial condition a_0 generates a distinct sequence $\{g^t(a_0)\}_{t=0}^{\infty}$, we can construct a continuum of equilibria. Moreover, the condition $F^2(a^*) \leqslant a^*$ implies the existence of a cycle of period 2.

Case 3. $F^2(a^*) > a^*$. For any ε, $0 < \varepsilon < a^*$, the interval $I_\varepsilon = [\varepsilon, F(a^*)]$ has positive length, and its image $F(I_\varepsilon)$ is a compact subinterval of I_ε. The set

$$J_\varepsilon = \bigcap_{k=1}^{\infty} F^k(I_\varepsilon).$$

is the limit of a decreasing sequence of non-empty compact intervals. It is therefore a compact interval $[a_\varepsilon, b_\varepsilon]$ satisfying the invariance property $F([a_\varepsilon, b_\varepsilon]) = [a_\varepsilon, b_\varepsilon]$. Since the fixed point $\bar{a}$ belongs to I_ε, we know that $\bar{a}$ is contained in J_ε, and thus $a_\varepsilon \leqslant \bar{a} \leqslant b_\varepsilon$. If $a_\varepsilon \leqslant a^*$, we successively infer that $b_\varepsilon = F(a^*)$, $F(a_\varepsilon) > a_\varepsilon$, and $F(b_\varepsilon) = F^2(a^*) > a_\varepsilon$, which contradicts the invariance of I_ε. Therefore $a^* \leqslant a_\varepsilon \leqslant \bar{a} \leqslant b_\varepsilon$, and $F(a_\varepsilon) = b_\varepsilon$, $F(b_\varepsilon) = a_\varepsilon$. We distinguish two subcases:

(3a) If the interval J_ε has positive length for some ε, $0 < \varepsilon < a^*$, the sequence $\{a_\varepsilon, b_\varepsilon, a_\varepsilon ...\}$ is a cycle of period 2. Moreover, F maps $[a_\varepsilon, b_\varepsilon]$ onto itself, and we conclude as in Case 2 that there exists a continuum of perfect foresight equilibria.

(3b) When $J_\varepsilon = \{\bar{a}\}$ for every ε, $0 < \varepsilon < a^*$, we can proceed as in Case 1. A PFE $\{a_t\}_{t=1}^\infty$ is bounded below by a positive number $\varepsilon < a^*$. Since $a_t = F(a_{t+1})$, we know that $a_t \leqslant F(a^*)$, and thus each a_t is contained in the interval $I_\varepsilon = [\varepsilon, F(a^*)]$. For a fixed $t \geqslant 0$, $a_t = F^k(a_{t+k})$ and therefore $a_t \in F^k(I_\varepsilon)$ for all k. We then infer that $a_t = \bar{a}$, and the steady state is the unique PFE. Moreover as in Case 1, the iterated sequence $\{F^t(1)\}_{t=0}^\infty$ converges to $\bar{a}$.

7.11. *Proof of Theorem* 7

When $(V, \beta) \notin PF_\infty$, we see from the proof of Theorem 6 that the sequence $F^t(1)$ converges to the steady state $\bar{a}$ and thus $(V, \beta) \notin PF$. This establishes that $PF \subseteq PF_\infty$. We also know that any IFS satisfying the condition $F^3(\bar{a}) < \bar{a}$ contains a cycle of period three and therefore, by Sarkovskii's theorem, cycles of every order. We can check numerically that this condition is satisfied for some parameters, e.g., $(V, \beta) = (500, 0.2)$, and we conclude that the set a

$$\{(V, \beta) : \mathscr{E}_{V, \beta} \text{ contains a cycle of every order}\}$$

has a non-empty interior.

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