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Idiosyncratic production risk, growth and the business cycle ☆

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Abstract

We introduce a neoclassical growth economy with idiosyncratic production risk and incomplete markets. Each agent is an entrepreneur operating her own technology with her own capital stock. The general equilibrium is characterized by a closed-form recursion in the CARA-normal case. Incomplete markets introduce a risk premium on private equity, which reduces the demand for investment. As compared to complete markets, the steady state can thus have both a lower capital stock due to investment risk, and a lower interest rate due to precautionary savings. Furthermore, the anticipation of high real interest rates in the future feeds back into high risk premia and low investment in the present, thus slowing down convergence to the steady state.

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Our results highlight the importance of private risk premia for capital accumulation and business cycles.

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1. Introduction

Undiversifiable entrepreneurial and investment risks pervade economic activity in the developing world. Even in the United States, privately-held companies account for about half of production, employment, and corporate equity, and represent more than half the financial wealth of rich households (Carroll, 2001). Moskowitz and Vissing-Jørgensen (2002) further document that entrepreneurs and private investors face a dramatic lack of diversification and an extreme dispersion in returns.[1] The survival rate of private firms is only 39% over the first 5 years, and returns on investment vary widely among surviving firms. These large undiversifiable risks are potentially important for macroeconomic performance because rich households and entrepreneurs control a large fraction of savings and investment in the economy.[2]

Following Bewley (1977), an extensive literature has investigated the macroeconomic impact of labor-income risk in the neoclassical growth model.[3] Partly because of its focus on precautionary savings and the wealth distribution, this earlier research has generally neglected idiosyncratic uncertainty in private production, investment, and entrepreneurial activity. In such settings, incomplete markets lead to a lower interest rate and a higher capital stock in the steady state (Huggett, 1993; Aiyagari, 1994) and tend to have small effects on business-cycle dynamics (Krusell and Smith, 1998). Consequently, financial innovations such as improvements in borrowing limits or risk-sharing opportunities are predicted to reduce capital accumulation and medium-run growth. Traditional Bewley models thus do not help explain the empirical evidence on the positive impact of financial sophistication on productivity and growth (e.g., King and Levine, 1993; Levine, 1997).

The development literature, on the other hand, has proposed that financial innovation promotes productivity and growth by helping the reallocation of savings to more productive activities.[4] In one-sector linear-growth economies, a complementary argument

[1]Moskowitz and Vissing-Jørgensen (2002) also observe: "About 75% of all private equity is owned by households for whom it constitutes at least half of their total net worth. Furthermore, households with entrepreneurial equity invest on average more than 70% of their private holdings in a single private company in which they have an active management interest."

[2]Idiosyncratic production risks affect more generally a wide class of decision-makers. For instance in publicly traded corporations, the tenure and compensation of executives are closely tied to the outcome of the investment decisions they make on behalf of shareholders. Similarly, labor income often includes returns to education, learning-by-doing, or some form of human or intangible capital.

[3]Examples of Bewley models include Aiyagari (1994), Calvet (2001), Huggett (1993, 1997), Krusell and Smith (1998), Ríos-Rull (1996), and Weil (1992). See Ljungqvist and Sargent (2000, Chapter 14) for a review.

[4]See, for example, Greenwood and Jovanovic (1990), Bencivenga and Smith (1991), King and Levine (1993), Obstfeld (1994), and Acemoglu and Zilibotti (1997). More recently, Krebs (2003) considers a linear growth economy in which labor-income risk distorts the allocation of savings between physical and human capital.

originates in the distinction between endowment and rate-of-return shocks. Devereux and Smith (1994) thus consider a simple AK economy under either financial autarchy or complete markets. With idiosyncratic endowment risk, the growth rate unambiguously falls as the economy moves to complete markets; with rate-of-return risk, on the other hand, financial innovation stimulates growth for sufficiently high elasticities of intertemporal substitution. It remains an open question, however, whether these intuitions extend to the Bewley class of models, as well as how entrepreneurial risks might affect the transitional dynamics and the business cycle.

The paper builds a bridge between the two literatures by investigating how idiosyncratic production risk affects the steady state and the transitional dynamics of a neoclassical growth economy. We follow the Bewley paradigm by assuming diminishing returns to capital accumulation and allowing a credit market, but extend the standard framework by introducing idiosyncratic risk in private production and investment. The economy is populated by a large number of infinitely-lived agents, or entrepreneurs, each of whom operates her own neoclassical technology with her own labor and capital. Production is subject to firm-specific uncertainty, which generates idiosyncratic risk in entrepreneurial income and investment returns.

Incomplete markets generally imply that aggregate dynamics depend on the wealth distribution. We overcome this "curse of dimensionality" by adopting a CARA-normal specification for preferences and risks, which ensures that risk-taking and therefore investment are independent of wealth.[5] We are thus able to characterize the general equilibrium by a tractable closed-form recursion, which, to the best of our knowledge, is a methodological innovation.

Idiosyncratic production risks introduce a risk premium on private capital, which reduces the demand for investment at any given interest rate. Uninsurable income risks also encourage the precautionary supply of savings, implying a lower interest rate as compared to complete markets. The overall effect of incomplete markets on capital accumulation is therefore ambiguous in general. Nevertheless, the reduction in investment demand dominates the reduction in the interest rate unless the interest elasticity of savings is sufficiently low, thus resulting in underaccumulation of capital in the steady state as compared to complete markets. Hence, improvements in entrepreneurial risk sharing—induced, for instance, by financial liberalization or the introduction of new hedging instruments—are likely to have a positive effect on savings and medium run growth. This result holds even though there is no margin of reallocation towards more productive activities. Our findings thus extend and reinforce the findings of the development literature in the context of the neoclassical growth model.

Our framework also allows us to derive novel implications for the transitional dynamics. The expectation of high real interest rates in the near future implies a low willingness to take risk in the present and therefore discourages current investment. This feedback slows down convergence to the steady state and generates a dynamic macroeconomic complementarity that, as Cooper (1999) and others have argued, may increase the persistence and magnitude of the business cycle.

[5] Similar dynamic CARA-normal specifications have been used to investigate the impact of precautionary savings by Caballero (1990) in a decision-theoretic context and by Calvet (2001) in a general-equilibrium endowment economy.

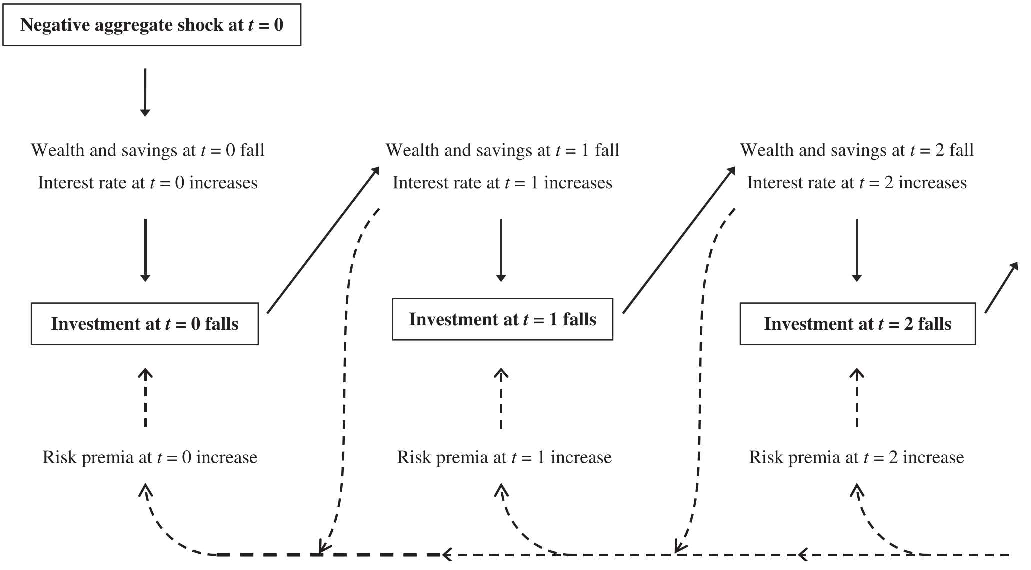

Fig. 1. The figure illustrates how countercyclicality in private risk premia introduces amplification and persistence in the business cycle. The solid arrows represent propagation of the shock under complete markets, whereas the dashed arrows represent the additional feedback through risk premia.

Fig. 1 illustrates the transitional dynamics in an economy hit by an unanticipated negative shock to aggregate wealth. The solid lines represent transmission under complete markets. Agents smooth consumption by reducing current investment, which results in low wealth, low savings and high real interest rates in later periods. This is the fundamental propagation mechanism in the neoclassical growth paradigm. In the presence of idiosyncratic production uncertainty, the traditional channel is complemented by the endogenous countercyclicality of private risk premia, as illustrated by the dashed arrows in the figure. Anticipating higher real interest rates and thus larger self-insurance costs in the near future, agents become less willing to take risk in the present, which further reduces investment and hinders the recovery. This mechanism is shown to increase persistence in the transitional dynamics. More generally, we expect that macroeconomic fluctuations are further amplified by additional sources of countercyclicality in private premia, such as business-cycle variation in firm-specific volatility or risk-sharing opportunities.

The effects exhibited in this paper originate in the non-marketability of idiosyncratic risks. Our approach thus complements, but also differs from, the literature examining the effects of credit imperfections on entrepreneurial activity (e.g., Bernanke and Gertler, 1989, 1990; Banerjee and Newman, 1993; Galor and Zeira, 1993; Kiyotaki and Moore, 1997; Aghion et al., 1999). This earlier research has focused on the effect of wealth and borrowing constraints on the individual *ability* to invest. We show that incomplete markets also affect the *willingness* to invest, which has novel implications for capital accumulation and the business cycle. We illustrate these effects in their sharpest version by using a model where agents face no borrowing constraints, individual investment does not depend on own wealth, and wealth heterogeneity has no impact on aggregate dynamics.

This paper is organized as follows. Section 2 introduces the economy and Section 3 analyzes the individual decision problem. Section 4 characterizes the general equilibrium in closed form and analyzes the steady state and the propagation mechanism. Illustrative numerical examples are presented in Section 5. Unless stated otherwise, all proofs are given in Appendix.

2. A Ramsey economy with idiosyncratic production risk

Time is discrete and infinite, indexed by $t \in \{0, 1, \ldots\}$. The economy is populated by a continuum of agents, indexed by $j \in [0, 1]$, who are born at $t = 0$ and live forever. Each individual is an "entrepreneur" who operates her own production scheme using her own labor and capital.

2.1. *Technology and idiosyncratic risks*

The gross output of entrepreneur j at date t is given by $A_t^j f(k_t^j)$, where k_t^j is her capital stock at the beginning of the period, A_t^j is her random total factor productivity (TFP), and f is a neoclassical production function. The function f is common across households and satisfies $f' > 0$, $f'' < 0$, $\lim_{k\to 0} f'(k) = +\infty$, and $\lim_{k\to +\infty} f'(k) = 0$. The individual controls k_t^j through her investment choice at date $t-1$, but only observes the idiosyncratic productivity A_t^j at date t. The return on investment is thus subject to idiosyncratic uncertainty.

For comparison with production risk, we find it useful to also introduce endowment risk. The individual receives an exogenous idiosyncratic income e_t^j, which does no affect investment or production opportunities.[6] The overall non-financial income of household j in period t is

$$y_t^j = A_t^j f(k_t^j) + e_t^j. \tag{1}$$

Variation in A_t^j captures idiosyncratic *entrepreneurial* or *production risk*, whereas variation in e_t^j captures *endowment risk*.

We assume that idiosyncratic production and endowment risks are Gaussian, mutually independent, and i.i.d. across time and individuals:

$$A_t^j \sim \mathcal{N}(1, \sigma_A^2) \quad \text{and} \quad e_t^j \sim \mathcal{N}(0, \sigma_e^2).$$

The averages $\mathbb{E}A_t^j = 1$ and $\mathbb{E}e_t^j = 0$ are simple normalizations. The standard deviations σ_A and σ_e parsimoniously parameterize the magnitude of the uninsurable production and endowment shocks. Under complete markets, σ_A and σ_e are both equal to zero. Traditional Bewley economies only consider idiosyncratic labor-income risk, which corresponds to $\sigma_e > 0$ but $\sigma_A = 0$. This paper focuses instead on the case $\sigma_A > 0$.

2.2. *Financial markets*

Agents can buy and sell a riskless short-term bond. One unit of the bond purchased at date t yields $1 + r_t$ units of the good with certainty at date $t + 1$. In equilibrium, the interest rate r_t clears the period-t bond market. We rule out default, borrowing constraints, and

[6] For instance, e_t^j may be interpreted as a taste shock or labor income from an unmodeled outside firm.

any other credit-market imperfections. Without loss of generality, the riskless bond is in zero net supply.[7]

Let c_t^j, i_t^j and θ_t^j denote the consumption, capital investment, and bond purchases of agent j in period t. The budget constraint in period t is given by

$$c_t^j + i_t^j + \theta_t^j = y_t^j + (1 + r_{t-1})\theta_{t-1}^j, \tag{2}$$

where y_t^j is the non-financial income defined in (1). The law of capital accumulation, on the other hand, is

$$k_{t+1}^j = (1 - \delta)k_t^j + i_t^j. \tag{3}$$

where $\delta \in [0, 1]$ is the depreciation rate. To simplify notation, we combine (2) and (3) and conveniently rewrite the budget set in terms of stock variables:

$$c_t^j + k_{t+1}^j + \theta_t^j = w_t^j, \tag{4}$$

where

$$w_t^j \equiv A_t^j f(k_t) + (1 - \delta)k_t^j + e_t^j + (1 + r_{t-1})\theta_{t-1}^j \tag{5}$$

represents the agent's total *wealth* at date t.

2.3. Preferences

The model is most tractable when agents have exponential expected utility $\mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t u(c_t)$, where $u(c) = -\Psi \exp(-c/\Psi)$. It is useful for the analysis, however, to distinguish between intertemporal substitution and risk aversion. We thus assume more generally that agents have preferences of the Kreps–Porteus/Epstein–Zin type. For every stochastic consumption stream $\{c_t\}_{t=0}^{\infty}$, the utility stream $\{u_t\}_{t=0}^{\infty}$ is recursively defined by

$$u_t = u(c_t) + \beta u(\mathbb{CE}_t[u^{-1}(u_{t+1})]), \tag{6}$$

where $u(c) = -\Psi \exp(-c/\Psi)$, $v(c) = -\exp(-\Gamma c)/\Gamma$, and $\mathbb{CE}_t x_{t+1} \equiv v^{-1}[\mathbb{E}_t v(x_{t+1})]$ is the certainty equivalent of x_{t+1} conditional on period t information. A high Ψ corresponds to a strong willingness to substitute consumption through time, while a high Γ implies a high degree of risk aversion. When $\Gamma = 1/\Psi$, the functions u and v coincide and the preference structure (6) reduces to standard expected utility, $u_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$.

The CARA specification entails both costs and benefits. It implies, for instance, that private productive investment is independent of individual wealth; our framework thus cannot be used to analyze the interaction between wealth inequality and aggregate capital formation. The CARA-normal specification, however, is probably not essential for the main insights of the paper and brings the benefit of tractability. As will be shown in Section 3, the cross-sectional distribution of wealth—an infinite dimensional object—will not be a relevant state variable for aggregate dynamics, and general equilibrium will be characterized in closed form.

[7] Ricardian equivalence holds in our model because agents have infinite horizons and can freely trade the riskless bond. Therefore, as long as public debt is financed by lump-sum taxation and the economy is closed, there is no loss of generality in assuming zero net supply for the riskless bond.

2.4. Equilibrium

Idiosyncratic risks are independent in the population and cancel out in the aggregate. We thus consider equilibria in which the aggregate dynamics are deterministic.

Definition. An incomplete-market equilibrium is a deterministic interest-rate sequence $\{r_t\}_{t=0}^{\infty}$ and a collection of contingent plans $(\{c_t^j, k_{t+1}^j, \theta_t^j\}_{t=0}^{\infty})_{j\in[0,1]}$ such that: (i) the plan $\{c_t^j, k_{t+1}^j, \theta_t^j\}_{t=0}^{\infty}$ maximizes the utility of each agent j; and (ii) the bond market clears in every date and event.

The next section characterizes optimal individual behavior for given interest rates. In Section 4, we aggregate across individuals and characterize the general equilibrium dynamics.

3. Decision theory

In this section we characterize the solution of the individual decision problem. To simplify the exposition, we focus on the special case of expected utility ($\Psi = 1/\Gamma$) and refer the reader to the Appendix for the general proof. We also drop the agent-specific index j from all decision variables.

3.1. Optimal savings and investment

Given a deterministic interest rate sequence $\{r_t\}_{t=0}^{\infty}$, the household chooses a contingent plan $\{c_t, k_{t+1}, \theta_t\}_{t=0}^{\infty}$ that maximizes expected lifetime utility subject to (4). Since idiosyncratic risks are uncorrelated over time, individual wealth w_t fully characterizes the state of the household in period t. The value function $V_t(w)$ satisfies the Bellman equation:

$$V_t(w_t) = \max_{(c_t,k_{t+1},\theta_t)} u(c_t) + \beta \mathbb{E}_t V_{t+1}(w_{t+1}),$$

where the maximization is subject to the budget constraint (4). Given the CARA-normal specification, an educated guess is to consider an exponential value function and a linear consumption rule:

$$V_t(w) = u(a_t w + b_t) \quad \text{and} \quad c_t = \widehat{a}_t w + \widehat{b}_t, \tag{7}$$

where $a_t, \widehat{a}_t \in \mathbb{R}_+$ and $b_t, \widehat{b}_t \in \mathbb{R}$ are non-random coefficients to be determined.

By (5), individual wealth is Gaussian with conditional mean $\mathbb{E}_t w_{t+1} = f(k_{t+1}) + (1-\delta)k_{t+1} + (1+r_t)\theta_t$, and variance $\mathrm{Var}_t(w_{t+1}) = \sigma_e^2 + f(k_{t+1})^2\sigma_A^2$. The value function thus satisfies

$$\mathbb{E}_t V_{t+1}(w_{t+1}) = V_{t+1}\left(\mathbb{E}_t w_{t+1} - \frac{\Gamma_t}{2}\mathrm{Var}_t w_{t+1}\right),$$

where $\Gamma_t \equiv \Gamma a_{t+1}$ measures absolute risk aversion in period t with respect to wealth variation in period $t+1$. We henceforth call Γ_t the *effective* risk aversion at date t. We will later see that endogenous variations in Γ_t can generate persistence in the transitional dynamics.

Taking the first-order conditions (FOCs) with respect to k_{t+1} and θ_t, we infer that optimal investment satisfies

$$r_t + \delta = f'(k_{t+1})[1 - \Gamma_t f(k_{t+1})\sigma_A^2]. \tag{8}$$

In the absence of idiosyncratic production uncertainty ($\sigma_A = 0$), the agent equates the net marginal product of capital with the interest rate: $r_t + \delta = f'(k_{t+1})$. This is a familiar result in complete-market or Bewley-type economies. When instead $\sigma_A > 0$, the mean return on investment must be compensated for risk. The quantity $\Gamma_t f(k_{t+1}) f'(k_{t+1})\sigma_A^2$ measures the *risk premium on private equity*. This premium increases with idiosyncratic production risk σ_A and effective risk aversion Γ_t.

The optimal level of savings is determined by the Euler equation $u'(c_{t+1}) = \beta(1 + r_t)\mathbb{E}_t u'(c_{t+1})$, or equivalently

$$\mathbb{E}_t c_{t+1} - c_t = \Psi \ln[\beta(1 + r_t)] + \frac{\Gamma}{2}\,\mathrm{Var}_t(c_{t+1}), \tag{9}$$

where $\mathrm{Var}_t(c_{t+1}) = (\widehat{a}_{t+1})^2[\sigma_e^2 + f(k_{t+1})^2\sigma_A^2]$. The precautionary motive has the familiar implication that expected consumption growth increases with the variance of future consumption (Leland, 1968; Sandmo, 1970; Caballero, 1990; Kimball, 1990).

The envelope and Euler conditions imply after simple manipulation that $\widehat{a}_t = a_t$ and $1/a_t = 1 + 1/[a_{t+1}(1 + r_t)]$. Iterating forward, and using $\Gamma_t \equiv \Gamma a_{t+1}$, we infer that effective risk aversion in period t is inversely proportional to the price of a perpetuity delivering one unit of the consumption good in every period $s \geqslant t + 1$:

$$\Gamma_t = \frac{\Gamma}{1 + \sum_{s=1}^{+\infty} \frac{1}{(1 + r_{t+1}) \cdots (1 + r_{t+s})}}. \tag{10}$$

Thus, in our model, the willingness to take risk decreases with future real interest rates. More generally, we expect that the demand for risky investment decreases with future borrowing costs or the ability to smooth intertemporally uninsurable shocks.

We show in Appendix that these results easily extend to non-expected utility ($\Psi \neq 1/\Gamma$):

Proposition 1 (*Individual choice*). *Given any interest rate path* $\{r_t\}_{t=0}^{\infty}$, *the demand for investment is given by*

$$r_t + \delta = f'(k_{t+1})[1 - \Gamma_t f(k_{t+1})\sigma_A^2], \tag{11}$$

and consumption and savings are characterized by the Euler equation

$$\mathbb{E}_t c_{t+1} - c_t = \Psi \ln[\beta(1 + r_t)] + \Gamma_t^2[\sigma_e^2 + f(k_{t+1})^2\sigma_A^2]/(2\Gamma). \tag{12}$$

Effective risk aversion Γ_t *is given by* (10) *and increases with future interest rates.*

Condition (11) characterizes the demand for investment. An increase in the contemporaneous interest rate raises the cost of capital and reduces the demand for investment under any market structure. Under incomplete markets, investment demand is independent of endowment risk σ_e but is negatively affected by production risk σ_A. Moreover, if and only if $\sigma_A > 0$, an increase in future interest rates raises Γ_t and therefore reduces investment.

Condition (12), on the other hand, characterizes the supply of savings. Endowment risk unambiguously increases the conditional variance of individual wealth, $\mathrm{Var}_t(w_{t+1}) =$

$\sigma_e^2 + f(k_{t+1})^2\sigma_A^2$, and therefore expected consumption growth. The effect of production shocks, however, is generally ambiguous because the optimal investment k_{t+1} and thus risk exposure fall with σ_A. In the Appendix, we check this intuition by constructing a simple example in which $\mathrm{Var}_t(w_{t+1})$ varies non-monotonically with σ_A.

3.2. Extension to risky financial assets

The analysis is now generalized to an economy with risky financial securities, which agents can use to partially hedge their idiosyncratic shocks. We show below that under reasonable assumptions, the consumption-savings results of Proposition 1 remain valid when σ_A and σ_e are reinterpreted as the standard deviations of the residual risks faced by the agents after hedging.

Idiosyncratic shocks are now correlated across agents, but aggregate risk is again ruled out.[8] The risky securities are indexed by $m \in \{1, \dots, M\}$ and are traded in all periods. One unit of security m bought in period t yields a random amount of consumption $d_{m,t+1}$ next period. The idiosyncratic shocks (A^j_{t+1}, e^j_{t+1}) and asset payoffs $\mathbf{d}_{t+1} = (d_{m,t+1})_{m=1}^M$ are jointly normal and independent through time.[9] Without loss of generality, the risky assets are normalized to have zero expected payoffs: $\mathbb{E}_t\mathbf{d}_{t+1} = 0$. We rule out default and short-sales constraints, and assume for simplicity that all assets are in zero net supply.

Consider now the decision problem of a fixed agent and, as previously, drop the index j. Let $\boldsymbol{\varphi}_t = (\varphi_{m,t})_{m=1}^M$ denote the portfolio of risky financial assets purchased in period t, and $\boldsymbol{\pi}_t = (\pi_{m,t})_{m=1}^M$ the vector of prices. The budget constraint is

$$c_t + k_{t+1} + \theta_t + \boldsymbol{\pi}_t \cdot \boldsymbol{\varphi}_t = w_t,$$

where total wealth is defined as

$$w_t = A_t f(k_t) + (1-\delta)k_t + e_t + (1 + r_{t-1})\theta_{t-1} + \mathbf{d}_t \cdot \boldsymbol{\varphi}_{t-1}.$$

Since there is no aggregate risk, we focus on price systems with zero risk premia on financial securities. This implies $\pi_{m,t} = \mathbb{E}_t d_{m,t+1}/(1 + r_t) = 0$. Risky assets thus permit agents to partially hedge their idiosyncratic shocks at no cost. It follows that the optimal portfolio minimizes the conditional variance of wealth and is therefore given by $\varphi_{m,t} = -\mathrm{Cov}_t[A_{t+1}f(k_{t+1}) + e_{t+1};\, d_{m,t+1}]$.

A convenient geometric representation is obtained by projecting A_{t+1} and e_{t+1} on the span of risky assets:

$$A_{t+1} = \boldsymbol{\kappa}_A \cdot \mathbf{d}_{t+1} + \tilde{A}_{t+1}, \quad e_{t+1} = \boldsymbol{\kappa}_e \cdot \mathbf{d}_{t+1} + \tilde{e}_{t+1}.$$

The quantities $\boldsymbol{\kappa}_A \cdot \mathbf{d}_{t+1}$ and $\boldsymbol{\kappa}_e \cdot \mathbf{d}_{t+1}$ represent the diversifiable components of production and endowment shocks, while undiversifiable risks are captured by the residuals $\tilde{A}_{t+1}$ and

[8] The economy may for instance be generated by a linear factor model of the form $A^j_{t+1} = \sum \lambda_{j,m} f_m + \tilde{A}^j_{t+1}$ and $e^j_{t+1} = \sum \mu_{j,n} g_n + \tilde{e}^j_{t+1}$, where the factors $(f_1, \dots, f_p, g_1, \dots, g_q)$ are mutually independent and individual loadings aggregate up to zero in the population.

[9] Our results are also robust to the introduction of long-lived securities. This is easily shown by considering an economy with short-lived assets and deterministic equilibrium interest rates. A trader can then dynamically replicate any long-lived asset $\{d_t\}$ if: (i) the payoff sequence $\{d_t\}$ follows an $AR(1)$ process, $d_{t+1} = \rho d_t + u_{t+1}$, where $|\rho| \leqslant 1$ and $\{u_t\}$ is a sequence of independent Gaussians; (ii) in every period t, agents can trade the bond and a short-lived asset delivering u_{t+1} in period $t+1$.

$\tilde{e}_{t+1}$.[10] The optimal portfolio fully hedges the diversifiable component of the idiosyncratic income shocks, and individual wealth reduces to $w_{t+1} = \tilde{A}_{t+1} f(k_{t+1}) + (1-\delta)k_{t+1} + \tilde{e}_{t+1} + \theta_t$.

Individual symmetry is preserved by assuming that $\tilde{A}_{t+1}$ and $\tilde{e}_{t+1}$ are identically distributed in the population. The degree of financial incompleteness is then quantified by the variances

$$\sigma_A^2 \equiv \mathrm{Var}(\tilde{A}_{t+1}) \quad \text{and} \quad \sigma_e^2 \equiv \mathrm{Var}(\tilde{e}_{t+1}).$$

The rest of the decision analysis can be carried out as previously. Risky assets thus only play one role in this economy: the definition of σ_A and σ_e. For this reason, we henceforth take these indexes as given and focus on the equilibrium dynamics of production, consumption and the interest rate.

Finally, it is easy to relax the assumption that the incompleteness indexes are invariant through time or with the state of the economy. For instance, individual decision and equilibrium are easily derived in closed-form when σ_A and σ_e are exogenous and deterministic functions of time. Another tractable extension specifies σ_A and σ_e as functions of aggregate wealth, for instance to capture that uninsurable risk tends to worsen during recessions.[11] Variations in σ_A and σ_e then influence the transitional dynamics and may strengthen the results of the paper. For expositional convenience, however, we do not pursue these possibilities and instead develop economic intuition in the simplest version of the model.

4. General equilibrium

In this section we characterize the general equilibrium, and then examine the steady state and transitional dynamics.

4.1. Closed-form equilibrium dynamics

Let C_t and K_t denote, respectively the population averages of consumption and capital in period t. Because agents have CARA preferences and face no borrowing constraints, private investment is independent of wealth ($K^j_{t+1} = K_{t+1}$ for all j) and consumption is linear in wealth. As a result, the wealth distribution does not affect the aggregate dynamics and the equilibrium path is easily characterized by a closed-form recursion.[12]

Proposition 2 (*General equilibrium*). *In an incomplete-market equilibrium, the macro path* $\{C_t, K_{t+1}, r_t\}_{t=0}^{\infty}$ *is deterministic and satisfies the recursion*

$$C_t + K_{t+1} = f(K_t) + (1-\delta)K_t, \tag{13}$$

[10]Since the projection space does not include the bond, we also know that $\mathbb{E}_t\tilde{A}_{t+1} = \mathbb{E}_t A_{t+1} = 1$ and $\mathbb{E}_t\tilde{e}_{t+1} = \mathbb{E}_t e_{t+1} = 0$.

[11]See Mankiw (1986) and Constantinides and Duffie (1996) for countercyclicality in labor-income risk and Campbell et al. (2001) for countercyclicality in idiosyncratic stock-return volatility.

[12]Aggregation is further simplified by the following assumptions: (i) idiosyncratic shocks are serially uncorrelated; (ii) preferences, technology and the structure of risks are identical across agents. The first assumption implies that individual investment and precautionary savings are independent of contemporaneous idiosyncratic shocks; the second makes investment and precautionary savings identical across agents.

$$r_t + \delta = f'(K_{t+1})[1 - \Gamma_t f(K_{t+1})\sigma_A^2], \tag{14}$$

$$C_{t+1} - C_t = \Psi \ln[\beta(1 + r_t)] + \Gamma_t^2[\sigma_e^2 + f(K_{t+1})^2\sigma_A^2]/(2\Gamma), \tag{15}$$

for all $t \geqslant 0$, *where effective risk aversion* Γ_t *is given by* (10).[13]

Condition (13) is the resource constraint of the economy; it is equivalent to the sum of individual budget constraints when the bond market clears. Conditions (14) and (15) characterize the aggregate demand for investment and the aggregate supply of savings. Under complete markets, Eqs. (14) and (15) reduce to the familiar conditions $r_t + \delta = f'(K_{t+1})$ and $u'(C_t)/u'(C_{t+1}) = \beta(1 + r_t)$.

Idiosyncratic production shocks introduce a risk premium on private equity and *discourage* aggregate investment for any given interest rate. The precautionary motive, on the other hand, tends to *stimulate* savings and reduce the interest rate, as is well known in Bewley models. We will see that incomplete markets can therefore lead to both a lower capital stock and a lower interest rate as compared to complete markets. Moreover, as an increase in σ_A or Γ_t raises private premia, cyclical variation in idiosyncratic production risk or risk tolerance may induce persistence and amplification in the business cycle.

4.2. Steady state

We now analyze how undiversifiable idiosyncratic production risks affect the capital stock in the steady state. A *steady state* is a fixed point $(C_\infty, K_\infty, r_\infty)$ of the dynamic system (13)–(15). We easily show:

Proposition 3 (*Steady state*). *A steady state always exists. In any steady state, the interest rate and the aggregate capital stock satisfy*

$$r_\infty + \delta = f'(K_\infty)\left[1 - \Gamma \frac{r_\infty}{1 + r_\infty} f(K_\infty)\sigma_A^2\right], \tag{16}$$

$$\ln[\beta(1 + r_\infty)] = -\frac{\Gamma}{2\Psi}\left(\frac{r_\infty}{1 + r_\infty}\right)^2[\sigma_e^2 + f(K_\infty)^2\sigma_A^2], \tag{17}$$

while aggregate consumption is $C_\infty = f(K_\infty) - \delta K_\infty$. *When the steady state is unique, the interest rate* r_∞ *decreases with either* σ_e *or* σ_A; *in contrast, the capital stock* K_∞ *increases with* σ_e *but is ambiguously affected by* σ_A. *Near complete markets, the steady state is unique and* K_∞ *decreases with* σ_A *if and only if* $\Psi/C_\infty > \underline{\psi}$, *where* $\underline{\psi} \equiv (1 - \beta)/[2(1 - \beta + (1 - \alpha)\beta\delta)]$ *and* $\alpha \equiv f'(K_\infty)K_\infty/f(K_\infty)$.

The first equation corresponds to the aggregate demand for productive investment, and the second to the aggregate supply of savings. We note that $1 + r_\infty = 1/\beta$ under complete markets, but $1 + r_\infty < 1/\beta$ in the presence of undiversifiable idiosyncratic risks ($\sigma_A > 0$ or $\sigma_e > 0$). The property that the risk-free rate is below the discount rate under incomplete markets has been proposed as a possible solution to the low risk-free rate puzzle (e.g., Weil, 1992; Huggett, 1993).

Endowment and production risks have very different effects on the steady state. Consider first the case $\sigma_e > 0$ and $\sigma_A = 0$, as in a traditional Bewley economy (e.g.,

[13]Existence and determinacy of equilibrium are examined in Angeletos and Calvet (2000, 2005).

Aiyagari, 1994). A higher σ_e increases consumption risk, stimulates the precautionary supply of savings and reduces the interest rate. Since investment is still determined by the equation $r_\infty + \delta = f'(K_\infty)$, it follows that the capital stock necessarily increases with the level of endowment risk.

Consider next the case $\sigma_A > 0$. Production risk affects both the savings supply and investment demand. On one hand, a higher σ_A tends to encourage precautionary savings and reduce the interest rate. On the other hand, a higher σ_A increases the private risk premium and reduces the demand for investment at any level of the interest rate. There is thus a conflict between the savings and the investment effect. Intuition suggests that the investment effect dominates when the supply of savings is sufficiently elastic with respect to real returns: an increase in precautionary savings then only has a small effect on the equilibrium interest rate. The proposition confirms this argument by showing that K_∞ decreases with σ_A when the elasticity of intertemporal substitution at the steady state, Ψ/C_∞, is sufficiently high.

4.3. Transitional dynamics and propagation

Idiosyncratic production risks have novel implications for the business cycle and medium-run growth. We illustrate these effects in Fig. 1 for an economy hit at date $t = 0$ by an unanticipated negative shock to aggregate wealth. The solid lines represent transmission under complete markets. Agents smooth consumption by reducing current investment, which results in low wealth, low savings and high real interest rates in later periods. This is the fundamental propagation mechanism in the neoclassical growth paradigm.

When production is subject to uninsurable idiosyncratic shocks, the traditional channel is complemented by the endogenous countercyclicality of risk premia, as illustrated by the dashed arrows in the figure. The anticipation of high real interest rates at $t = 1$ leads to an increase in the risk premium at $t = 0$ and hence a further reduction of investment at $t = 0$. Similarly, the anticipation of higher interest rates in any period $t > 1$ feeds back to even higher risk premia and even lower investment in earlier periods. As a result, the impact of the exogenous shock on investment and output is amplified, and the recovery of the economy is slowed down.

We note that the propagation mechanism originates in a pecuniary externality in risk-taking. When private agents decide how much to save and invest in a future period, they do not internalize the impact of their choices on future real interest rates and thereby on current risk-taking and investment. This externality generates a novel dynamic macroeconomic complementarity: the anticipation of low aggregate investment in the future feeds back into low aggregate investment in the present through the effect of real interest rates on risk-taking. Note that this mechanism arises even though σ_A is assumed to be constant through time. The model suggests that countercyclicality in idiosyncratic volatility σ_A could reduce the willingness to invest during a recession even further and thus amplify the business cycle.[14]

The reader may be familiar with a standard example of macroeconomic complementarity—the production externalities considered by Bryant (1983), Cooper and John (1988),

[14] We could easily incorporate a countercyclical σ_A in our model by letting $\mathrm{Var}_{t-1}(A_t) = v(Y_t/Y_\infty)$ for some decreasing function $v(.)$.

Benhabib and Farmer (1994) and others.[15] In this literature, individual marginal productivity is assumed to increase in the aggregate stock of capital, which generates a complementarity in investment. Such production externalities have been shown to generate amplification and persistence in business-cycle dynamics. This type of explanation seems unsatisfactory since it relies on an ad hoc exogenous effect. In contrast, the complementarity in our model is a genuine general-equilibrium implication of a market imperfection.

This complementarity can generate poverty traps, endogenous fluctuations and multiple self-fulfilling equilibria (Angeletos and Calvet, 2000, 2005). Our work thus also contributes to the literature on equilibrium indeterminacy (e.g., Benhabib and Farmer, 1994). Such phenomena, however, only occur when idiosyncratic risks are very large or agents are very impatient, and do not arise for the plausible parameter values considered in the simulations of the next section. For this reason, this paper focuses on the effects of incomplete markets in economies with a unique equilibrium and a unique stable steady state.

Finally, the propagation mechanism hinges on the existence of uninsurable idiosyncratic risk in production and investment. It is thus not present in Bewley-type economies that only consider endowment or labor-income risk (e.g., Aiyagari, 1994; Krusell and Smith, 1998). We conjecture more generally that a similar propagation mechanism is likely to arise in other frameworks where the risk premium on private investment is countercyclical. Consider for instance an economy with both credit-market imperfections and uninsurable investment risks. We expect that an entrepreneur will invest less in the present when he anticipates a higher borrowing rate or a higher probability to face a binding borrowing constraint in the near future. The anticipation of a downturn then raises risk premia and discourages capital accumulation in the present, thus making the recession partially self-fulfilling. We observe that these effects could occur whether or not *current* investment is financially constrained and are likely to be reinforced by countercyclicality in σ_A.

5. Numerical examples

The CARA-normal specification enables us to solve the general equilibrium in closed form, and thereby to clearly identify the impact of incomplete markets on the steady state and the transitional dynamics. While our framework does not permit a precise quantitative assessment of these mechanisms, it is nevertheless useful to illustrate the effect of market incompleteness in plausible numerical simulations.

The calibration is explained in detail in the Appendix. To map our exponential preferences to more standard isoelastic preferences, we choose parameters Γ and Ψ that match a given relative risk aversion γ and a given elasticity of intertemporal substitution ψ at the complete-markets steady state. We set $\gamma = 2$ and $\psi = 1$. To capture persistence in idiosyncratic productivity and investment returns, we interpret the length of a time period as the horizon of an investment project or the average life of an idiosyncratic shock. We choose a length of 5 years and let the discount and depreciation rates be 5% per year. Finally, we assume a Cobb–Douglas production function, $f(K) = K^{\alpha}$, and consider two values for the income share of capital: $\alpha = 0.4$, which corresponds to the standard

[15] Cooper (1999) provides an excellent overview of macroeconomic complementarities.

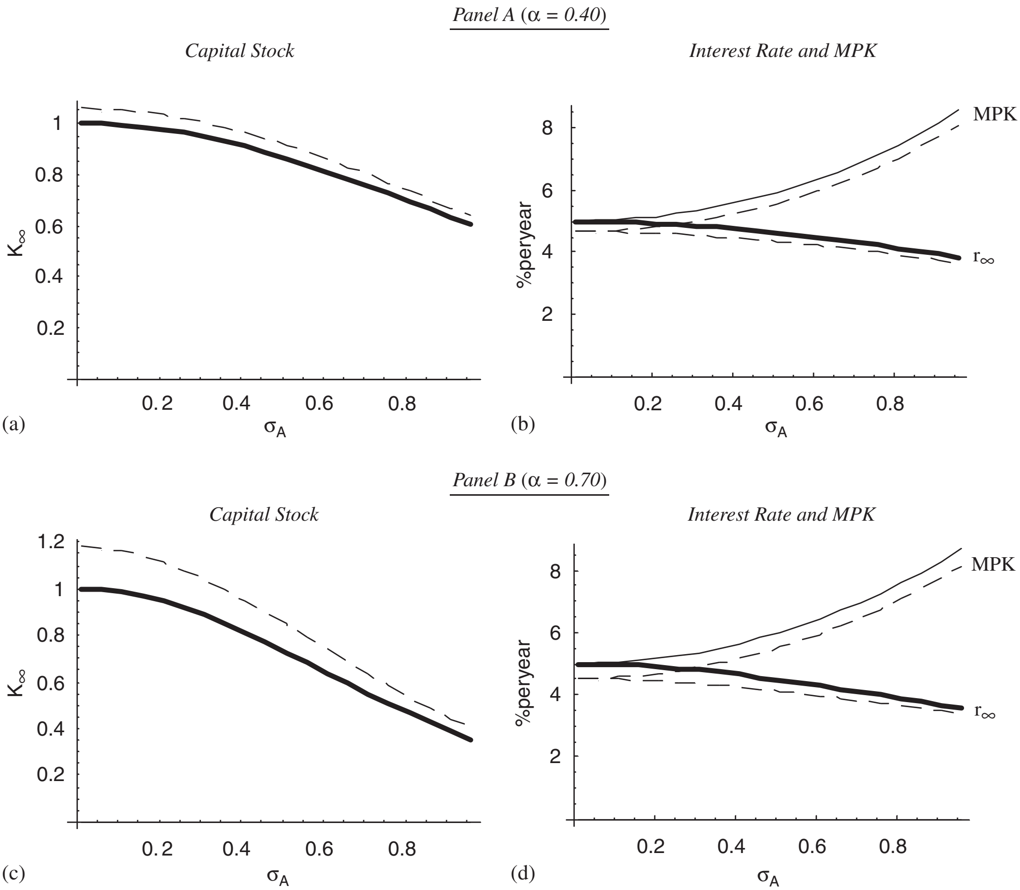

Fig. 2. The plots illustrate the impact of incomplete markets on the capital stock, the marginal product of capital and the interest rate. The length of a period is 5 years, the discount rate is 5% per year, the depreciation rate is 5% per year, the income share of capital is 40% (Panel A) or 70% (Panel B), the degree of relative risk aversion is 2, and the elasticity of intertemporal substitution is 1. The solid lines correspond to $\sigma_e = 0$ and the dashed ones to $\sigma_e = 0.50$. The capital stock is normalized by its complete-market value.

definition of capital; and $\alpha = 0.7$, which corresponds to a broad definition that includes both physical and human (or other intangible) capital.

Under the normalization $\mathbb{E}A = 1$, the parameter σ_A corresponds to the coefficient of variation in private production and investment returns.[16] Although accurate estimates of σ_A are not readily available, idiosyncratic risks in production, investment, and entrepreneurial returns are known to be substantial. Carroll (2001) and Moskowitz and Vissing-Jørgensen (2002) show that households with private equity suffer from a dramatic lack of diversification. They hold more than half of their financial wealth in private firms,

[16]To see this, note that $\mathrm{Var}_{t-1}[A_t^j f(k_t^j)]^{1/2}/\mathbb{E}_{t-1}[A_t^j f(k_t^j)] = \mathrm{Var}_{t-1}[A_t^j]^{1/2}/\mathbb{E}_{t-1}[A_t^j] = \sigma_A$. We also normalize σ_e so that $\sigma_e = 0.50$, for example, means that the standard deviation of the endowment equals 50% of the steady-state level of mean gross income.

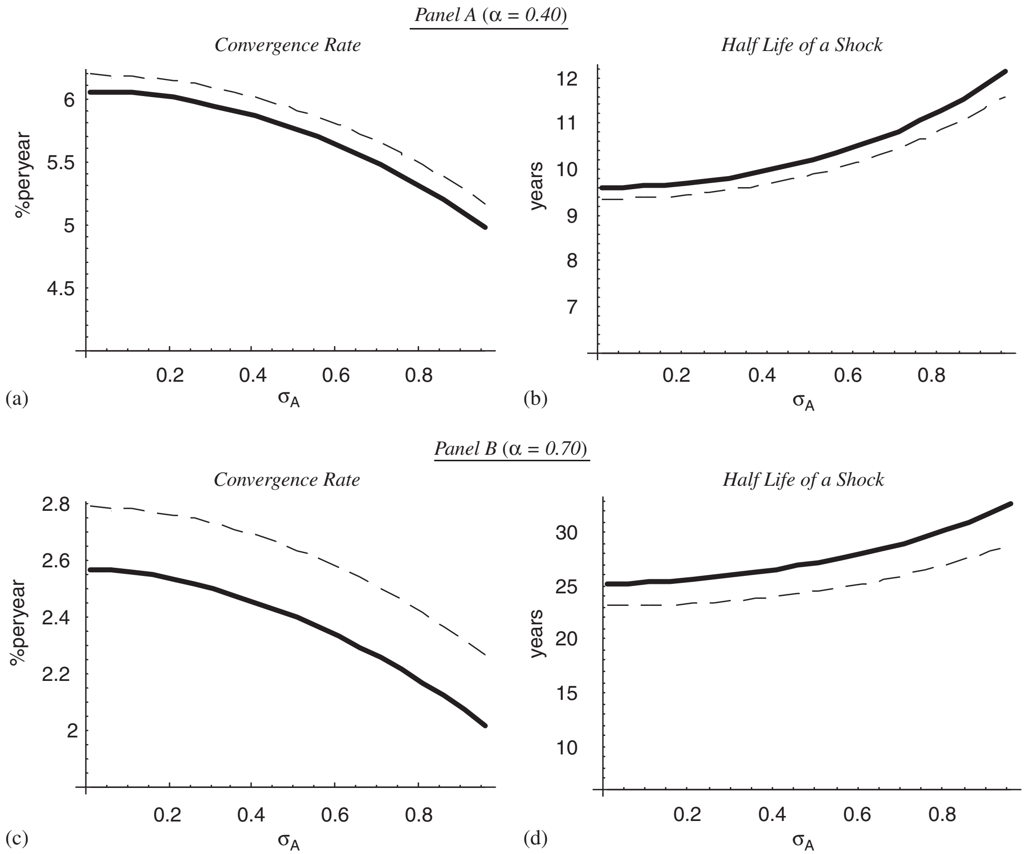

Fig. 3. The plots illustrate the impact of incomplete markets on the transitional dynamics for the same calibration parameters as in Fig. 2.

70% of which is invested in a single company. The failure rate of a new private firm is as high as 61% over the first 5 years. The distribution of returns to entrepreneurial activity is extremely wide even conditionally on survival. The average and median returns of a single private company differ by as much as 121 percentage points. The returns to a value-weighted *index* of private equity funds have a standard deviation of 17% as compared to a mean of 14%, which implies a coefficient of variation larger than 100%. Since the index diversifies away much of the firm-specific risk, this is probably a lower bound for σ_A. To be conservative, we consider values for σ_A between 0% and 100%.

In Fig. 2, we illustrate the impact of production risk on the steady-state capital stock and interest rate. The effect of σ_A on K_∞ is very strong. For example, with a narrow definition of capital ($\alpha = 0.4$, Panel A), the capital stock is about 40% lower than its complete-market value when $\sigma_A = 100\%$; equivalently, the saving rate falls from 17% when markets are complete to 13% when $\sigma_A = 100\%$. With a broad definition of capital ($\alpha = 0.7$, Panel B), the corresponding reduction in the capital stock is 60%; and the saving rate falls from 30% to 23%. Hence, in contrast to endowment risks, production risks may cause a significant reduction in aggregate savings. Furthermore, the risk-free rate can be a very poor proxy for the marginal productivity of capital. When $\alpha = 0.4$ and $\sigma_A = 100\%$, the mean return to capital is 8.5% per year as compared to a yearly interest rate of 4%.

The simulation also provides useful insights on the interaction between endowment and production risks. In Fig. 2, the dashed lines correspond to $\sigma_e = 50\%$, whereas the solid ones to $\sigma_e = 0$. The steady-state level of capital increases with σ_e, but becomes less sensitive to σ_e as σ_A also increases. These findings suggest that precautionary savings for one type of risk are substitutes for precautionary savings for the other risk.

Fig. 3 illustrates the impact of incomplete markets on the rate of convergence to the steady state. The local dynamics around the steady state are approximated by $(K_{t+1} - K_\infty) = \lambda(K_t - K_\infty)$, where λ is the stable eigenvalue of the Jacobian of the dynamic system (13)–(15) evaluated at the steady state. The convergence rate is then given by $1 - \lambda$. We also report the half-life τ of an aggregate wealth shock, which by definition satisfies $\lambda^\tau = 1/2$ or, equivalently, $\tau = -\log_2 \lambda$. As σ_A increases, convergence is slowed down. When $\alpha = 0.4$ (Panel A), the half-life of a shock is almost doubled at $\sigma_A = 100\%$; and the effect is even stronger when $\alpha = 0.7$ (Panel B).

These results suggest that the magnitude of uninsurable production and investment risks appears as a potential determinant of both the steady state and the rate of convergence. Cross-country variation in the degree of risk sharing may thus help explain the large diversity and persistence of productivity levels and growth rates around the world (e.g. Barro, 1997; Jones, 1997).

6. Concluding remarks

This paper examines a neoclassical growth economy with uninsurable idiosyncratic production and endowment risks. Under a CARA-normal specification of preferences and incomes, we obtain closed-form solutions for individual choices and aggregate dynamics. Uninsurable production shocks introduce a risk premium on private equity and reduce the aggregate demand for investment. As a result, the steady-state capital stock tends to be lower under incomplete markets despite the low risk-free rate induced by the precautionary motive.[17] Countercyclicality in private risk premia may also increase the amplitude and persistence of the business cycle.

The tractability of our setup easily generalizes to multiple sectors (Angeletos and Calvet, 2005). For instance, agents can have access to two private technologies—one with high-risk and high-mean return, and another with low risk and low return. We can also consider several forms of investment, such as physical, human or intangible capital. In such an environment, incomplete risk sharing distorts not only the aggregate levels of savings and investment but also the cross-sectoral allocation of capital and labor.

Wealth heterogeneity and credit-market imperfections have been viewed by many authors as a source of amplification and persistence. Although these departures from the neoclassical growth model are not considered here, we find that incomplete risk-sharing in production and investment generates a novel propagation mechanism, which originates in the feedback from anticipated future economic conditions to current risk premia and investment demand. Introducing borrowing constraints may increase the sensitivity of the

[17] Angeletos (2005) finds similar steady-state results in a model where agents have constant *relative* risk aversion and can invest part of their wealth in public-traded capital.

private equity premium to future credit conditions. Cyclical variation in borrowing capacity or firm-specific risks may also strengthen the mechanism.

The next steps would be to extend these insights to more general frameworks and conduct a careful quantitative evaluation of the interaction between private risk premia and the business cycle. Idiosyncratic entrepreneurial risks can also affect the market price of tradeable risks and thus have important implications for asset pricing.[18] We leave these questions open for future research.

Appendix A

Proof of Proposition 1 (*Individual choice*). In the main text we solve the decision problem of an agent with expected utility ($\Gamma = 1/\Psi$). Here, we derive the optimal choice of an Epstein–Zin agent. An educated guess is that the agent has linear value function $J_t(w) = a_t w + b_t$, and consumption policy $c_t = \hat{a}_t w_t + \hat{b}_t$. Since w_t is normal, the certainty equivalent of $J_{t+1}(w_{t+1})$ is $J_{t+1}(\mathbb{E}_t w_{t+1} - (\Gamma_t/2)\,\mathrm{Var}_t\, w_{t+1})$. The agent solves in period t the optimization problem:

$$u[J_t(w_t)] = \max_{(c_t^j, k_{t+1}^j, \theta_t^j)} u(c_t) + \beta u\left[J_{t+1}\left(\mathbb{E}_t w_{t+1} - \frac{\Gamma_t}{2}\,\mathrm{Var}_t\, w_{t+1}\right)\right]. \tag{18}$$

We know that $\mathbb{E}_t w_{t+1} = f(k_{t+1}) + (1-\delta)k_{t+1} + (1+r_t)\theta_t$ and $\mathrm{Var}_t\, w_{t+1} = \sigma_e^2 + f(k_{t+1})^2\sigma_A^2$. It is convenient to consider the function $G(k_{t+1}, \Gamma_t) \equiv f(k_{t+1}) + (1-\delta)k_{t+1} - \Gamma_t[\sigma_e^2 + f(k_{t+1})^2\sigma_A^2]/2$. The agent thus maximizes

$$u(c_t) + \beta u\{J_{t+1}[(1+r_t)\theta_t + G(k_{t+1}, \Gamma_t)]\}, \tag{19}$$

subject to $c_t + k_{t+1} + \theta_t = w_t$. The FOCs with respect to k_{t+1} and θ_t give

$$u'(c_t) = \beta u'(J_{t+1})a_{t+1}\{1 - \delta + f'(k_{t+1})[1 - \Gamma_t f(k_{t+1})\sigma_A^2]\},$$

$$u'(c_t) = \beta u'(J_{t+1})a_{t+1}(1+r_t).$$

Dividing these equalities yields $1 + r_t = 1 - \delta + f'(k_{t+1})[1 - \Gamma_t f(k_{t+1})\sigma_A^2]$.

We now write the envelope condition: $u'[J_t(w_t)]a_t = u'(c_t)$, or equivalently $c_t = a_t w_t + b_t - \Psi \ln a_t$. We infer that $\widehat{a}_t = a_t$ and $\widehat{b}_t = b_t - \Psi \ln a_t$. We then rewrite the FOC with respect to θ_t as

$$\begin{aligned} u'(c_t) &= \beta a_{t+1}(1+r_t)u'\left[J_{t+1}\left(\mathbb{E}_t w_{t+1} - \frac{\Gamma_t}{2}\,\mathrm{Var}_t\, w_{t+1}\right)\right] \\ &= \beta(1+r_t)u'\left(a_{t+1}\mathbb{E}_t w_{t+1} - \frac{\Gamma}{2}\, a_{t+1}^2 \mathrm{Var}_t\, w_{t+1} + b_{t+1} - \Psi \ln a_{t+1}\right). \end{aligned}$$

Since $\widehat{a}_{t+1} = a_{t+1}$ and $\widehat{b}_{t+1} = b_{t+1} - \Psi \ln a_{t+1}$, the FOC reduces to

$$u'(c_t) = \beta(1+r_t)u'[\mathbb{E}_t c_{t+1} - \Gamma\,\mathrm{Var}_t(c_{t+1})/2],$$

which implies Euler condition (12).

[18]This intuition is consistent with the empirical evidence in Heaton and Lucas (2000) that entrepreneurship has a strong impact on portfolio holdings.

The budget constraint and the consumption rule imply that $\theta_t = (1-\widehat{a}_t)w_t - k_{t+1} - \widehat{b}_t$. Since $\widehat{a}_t = a_t$, we infer from Euler condition (12) that

$$a_t w_t + b_t = a_{t+1}(1-a_t)(1+r_t)w_t + a_{t+1}[G(k_{t+1},\Gamma_t) - (1+r_t)(k_{t+1}+\widehat{b}_t)] \\ + b_{t+1} - \Psi \ln[\beta(1+r_t)a_{t+1}/a_t].$$

Since this linear relation holds for every w_t, we conclude that $a_t = a_{t+1}(1+r_t)(1-a_t)$ or equivalently $1/a_t = 1 + 1/[a_{t+1}(1+r_t)]$. Iterating forward yields (10).

We finally turn to the effect of risk on savings. Since $\mathrm{Var}_t(w_{t+1}) = \sigma_e^2 + \sigma_A^2 f(k_{t+1})^2$, we infer that $\partial \mathrm{Var}_t(w_{t+1})/\partial\sigma_e^2 > 0$. On the other hand, $\partial \mathrm{Var}_t(w_{t+1})/\partial\sigma_A^2 = f(k_{t+1})^2 + [2\sigma_A^2 f(k_{t+1})f'(k_{t+1})](\partial k_{t+1}/\partial\sigma_A^2)$ has an ambiguous sign. Consider the case $f(k)=\sqrt{k}$. The FOC $r_t + \delta = (2\sqrt{k_{t+1}})^{-1}(1-\Gamma_t\sigma_A^2\sqrt{k_{t+1}})$ implies $k_{t+1} = [\Gamma_t\sigma_A^2 + 2(\delta+r_t)]^{-2}$. We conclude that $\mathrm{Var}_t(w_{t+1}^j) = \sigma_e^2 + \sigma_A^2/[\Gamma_t\sigma_A^2 + 2(\delta+r_t)]^2$ is a single-peaked function of σ_A. □

Proof of Proposition 2 (*General equilibrium*). We now derive the equations characterizing general equilibrium. First, note that (10) implies $a_t^j = a_t$ and $\Gamma_t^j = \Gamma_t$ for all j, t. We infer from the optimality condition (11) that $k_{t+1}^j = K_{t+1}$ for all j. Eq. (11) is then equivalent to (14). We aggregate Euler equation (12) across agents and infer (15). Finally, the aggregation of budget constraints yields (13). Existence and local determinacy are examined in Angeletos and Calvet (2000, 2005). □

Proof of Proposition 3 (*Steady state*). We first prove the existence of the steady state and then examine its comparative statics.

(A) *Existence*: The steady state is defined by system (16)–(17). The second equation implies that the interest rate r_∞ belongs to the interval $(0, \beta^{-1}-1]$. The first equation implies $f'(K_\infty) > \delta$, or equivalently $K_\infty < \widehat{K} \equiv (f')^{-1}(\delta)$. The capital stock K_∞ is thus contained in the interval $[0, \widehat{K})$.

Each steady-state equation implicitly defines the interest rate as a function of the capital stock. Consider for instance Eq. (17). It is useful to define the functions $X : (0, \beta^{-1}-1] \to [0, +\infty)$, $X(r) \equiv 2\Psi/\Gamma(1+(1/r))^2 \ln[1/\beta(1+r)]$, and $V : [0,\widehat{K}) \to [\sigma_e^2, \sigma_e^2 + f(\widehat{K})^2\sigma_A^2)$, $V(K) \equiv \sigma_e^2 + f(K)^2\sigma_A^2$. We observe that X is decreasing in r and V is increasing in K. The steady-state equation (17) is equivalent to $X(r) = V(K)$. For each $K \in [0,\widehat{K})$, the equation $X(r) = V(K)$ has a unique solution $r_2(K) \equiv X^{-1}[V(K)]$, which maps $[0,+\infty)$ onto $(0, X^{-1}(\sigma_e^2)] \subseteq (0, \beta^{-1}-1]$. Similarly, the steady-state equation (16) implicitly defines a decreasing function $r_1(K)$, which maps $(0,\widehat{K}]$ onto $[0,+\infty)$. The steady state K_∞ is given by the intersection of r_1 and r_2.

Consider the function $\Delta(K) \equiv r_2(K) - r_1(K)$. When $K \to 0$, we know that $r_1(K) \to +\infty$ and $r_2(K)$ is bounded, implying $\Delta(K) \to -\infty$. Since $\Delta(\widehat{K}) = r_2(\widehat{K}) > 0$, there exists at least one steady state for any (σ_A, σ_e). Under complete markets, the steady state is unique since the function r_2 is constant and r_1 is decreasing. By continuity, the steady state is also unique when σ_A and σ_e are sufficiently small.

(B) *Comparative statics*: The functions $r_1(K)$ and $r_2(K)$ are both decreasing. We know that $|r_1'(K_\infty)| > |r_2'(K_\infty)|$ when the steady state is unique. An increase in σ_e leaves the function $r_1(K)$ unchanged and pushes down the function $r_2(K)$. The steady state is therefore characterized by a lower interest rate and a higher capital stock. An increase in σ_A reduces both $r_1(K)$ and $r_2(K)$, reflecting the fact that σ_A enters in both the investment demand and the savings supply. It follows that an increase in σ_A unambiguously reduces r_∞, but can have an ambiguous effect on K_∞. Let $\Gamma_\infty = \Gamma r_\infty/(1+r_\infty)$. In the neighborhood of $\sigma_A = \sigma_e = 0$, Eqs. (16) and (17) imply

$$\frac{\mathrm{d}r_\infty}{1+r_\infty} = -\frac{\Gamma_\infty^2}{2\Psi\Gamma} f(K_\infty)^2\,\mathrm{d}(\sigma_A^2) < 0,$$

and

$$\begin{aligned} f''(K_\infty)\,\mathrm{d}K_\infty &= \mathrm{d}r_\infty + \Gamma_\infty f'(K_\infty) f(K_\infty)\,\mathrm{d}(\sigma_A^2) \\ &= \Gamma_\infty f'(K_\infty) f(K_\infty)\left[1 - \frac{(1+r_\infty)\Gamma_\infty}{2\Psi\Gamma}\frac{f(K_\infty)}{f'(K_\infty)}\right]\mathrm{d}(\sigma_A^2) \\ &= \Gamma_\infty f'(K_\infty) f(K_\infty)\left(1 - \frac{r_\infty}{2\Psi}\frac{K_\infty}{\alpha}\right)\mathrm{d}(\sigma_A^2), \end{aligned}$$

where $\alpha \equiv f'(K_\infty)K_\infty/f(K_\infty)$. Therefore, $\mathrm{d}r_\infty/\mathrm{d}(\sigma_A^2) < 0$. We note that $\mathrm{d}K_\infty/\mathrm{d}(\sigma_A^2) < 0$ if and only if $\Psi/C_\infty > \underline{\psi} \equiv (r_\infty K_\infty)/(2\alpha C_\infty)$. Since $r_\infty = \beta^{-1} - 1 = f'(K_\infty) - \delta$ and $C_\infty = f(K_\infty) - \delta K_\infty = [(\beta^{-1} - 1 + \delta)/\alpha - \delta]K_\infty$, we infer $\underline{\psi} = (1-\beta)/2[1-\beta+(1-\alpha)\beta\delta]$. □

Calibrated economies: We now examine in detail the calibration of Γ and Ψ. Relative risk aversion at the steady-state consumption level is ΓC_∞. We restrict the incomplete-market economy so that ΓC_∞ remains invariant at a fixed level γ. The elasticity of intertemporal substitution (EIS) is equal to Ψ/C_∞ at the steady-state consumption level. Similar to the calibration of risk aversion, we could restrict Ψ/C_∞ to remain constant at a fixed level ψ. We adopted this method in an earlier version (Angeletos and Calvet, 2000) along with the additional restriction $\Psi = 1/\Gamma$ (expected utility). We found that idiosyncratic production risks slow down convergence to the steady state, as predicted in Section 4.

This paper proposes a slightly more elaborate calibration method for Ψ that stems from the following observations. Consider a *complete-market* Ramsey economy with intertemporal utility $\sum_{t=0}^{+\infty}\beta^t u(c_t)$, where u is a smooth strictly concave function. Gross output is $\Phi(K) = f(K) + (1-\delta)K$. The local dynamics around the steady state are approximated by $(K_{t+1} - K_\infty) \approx \lambda(K_t - K_\infty)$, where λ is the stable eigenvalue of the linearized system. It is easy to show that[19]

$$\lambda = \frac{1}{2}\left\{1 + \beta(\beta^{-1} - 1 + \delta)M_\infty + \frac{1}{\beta} - \sqrt{[1 + \beta(\beta^{-1} - 1 + \delta)M_\infty]^2 - \frac{4}{\beta}}\right\},$$

where M_∞ quantifies the relative curvatures of the production and utility functions:

$$M_\infty = \frac{f''(K_\infty)/f'(K_\infty)}{u''(C_\infty)/u'(C_\infty)}.$$

The eigenvalue λ is thus fully determined by (β, δ) and M_∞. With a Cobb–Douglas production $f(K) = K^\alpha$ and a CARA utility $u(C) = \Psi\exp(-C/\Psi)$, the ratio M_∞ reduces to $(1-\alpha)\Psi/K_\infty$. Under complete markets, the convergence rate $g = 1 - \lambda$ is thus fully determined by the parameters (α, β, δ) and the ratio Ψ/K_∞.

When we move from complete to incomplete markets, two phenomena affect the eigenvalue λ. First, the transitional dynamics are affected by new terms in (14) and (15): the

[19] Cass (1965) derives a similar result for continuous time economies.

risk premium in the investment–demand equation and the consumption variance in the Euler equation. Second, changes in the steady-state affect the relative curvature Ψ/K_∞ and thereby the eigenvalue λ. This second effect reflects the shift of the steady state to different points on the production and utility functions. It is thus purely mechanical and sheds little light on the impact of incomplete risk sharing on the transitional dynamics. For this reason, we prefer to neutralize this effect by keeping Ψ/K_∞ (or equivalently M_∞) invariant at a specified level as we vary σ_A and σ_e.[20]

This in turn requires an appropriate calibration of Ψ/K_∞. When markets are complete, we impose that the intertemporal elasticity Ψ/C_∞ be equal to a given coefficient ψ. This allows us to choose a value of ψ that matches empirical estimates of the EIS. A simple calculation also implies that, when $\sigma_A = \sigma_e = 0$, $C_\infty/K_\infty = (\beta^{-1} - 1 + \delta)/\alpha - \delta$. Therefore, when markets are complete, $\Psi/K_\infty = \psi C_\infty/K_\infty = \psi[(\beta^{-1} - 1 + \delta)/\alpha - \delta]$. When markets are incomplete, we keep Ψ/K_∞ invariant at this level. Our calibration thus disentangles the dynamic effect of financial incompleteness from purely mechanical changes in the relative curvatures of the production and utility functions.[21]

To summarize, a *calibrated economy* $\mathscr{E}^{\text{cal}} = (T, \beta, \gamma, \psi, \alpha, \delta, \sigma_A, \sigma_e)$ is an exponential Epstein–Zin economy $\mathscr{E} = (\beta', \Gamma, \Psi, f, \delta', \sigma'_A, \sigma'_e)$ such that $\beta' = \beta^T$, $1 - \delta' = (1 - \delta)^T$, $\Gamma C_\infty = \gamma$, $\Psi/K_\infty = \psi[(\beta'^{-1} - 1 + \delta')/\alpha - \delta']$, $f(k) = k^\alpha$, $\sigma'_A = \sigma_A$, and $\sigma'_e = \sigma_e f(K_\infty)$.

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[20] The alternative calibration method, which keeps constant the EIS Ψ/C_∞ at ψ but lets Ψ/K_∞ vary, also implies an increase in persistence when σ_A increases from zero. But, because K_∞ typically decreases with σ_A, the change in Ψ/K_∞ tends to reduce persistence. For large production risks, the convergence rate $g = 1 - \lambda$ is then slightly non-monotonic in σ_A in some simulations, but remains below the complete market value. See Angeletos and Calvet (2000) for further details.

[21] The calibration method used here also has the following alternative interpretation. Instead of adjusting the incomplete-markets steady-state EIS, we can set it at a predetermined level, but assume that the production function is exponential: $f(K) = 1 - \exp(-\phi K)$. We then calibrate the coefficient ϕ by setting the income share of capital equal to α in the complete-markets steady state. This specification implies exactly the same calibrated steady state and convergence rate.

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