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Constrained Efficiency in a Human Capital Model†

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This paper investigates whether capital and human capital are over-accumulated in an incomplete market economy. As in Dávila et al. (*2012*), *whether capital is over-accumulated depends on how the pecuniary externalities affect insurance and redistribution. In a human capital economy*, *however*, *not only capital but also human capital generates externalities and an additional channel arises that has implications for the overaccumulation* (*under-accumulation*) *of capital* (*human capital*). *The income sources of the poor and the correlation between wealth and human capital are crucial for the implication of pecuniary externalities. Realistically calibrated models exhibit under-accumulation* (*overaccumulation*) *of capital* (*human capital*). (*JEL* D52, D62, I26, J22, J24, J31)

Is capital in a competitive equilibrium over-accumulated in a standard incomplete market model with uninsurable idiosyncratic shocks from the perspective of the planner? The answer to this question depends on the planner's restrictions. If the planner can complete the market using transfers, then a competitive equilibrium with precautionary savings exhibits overaccumulation of capital in comparison with the first best allocation. However, if the planner faces some restrictions (e.g., limited tools of the government) and cannot complete the market, then the answer does depend on the planner's constraint. In this paper, we focus on the planner who can only internalize pecuniary externalities. That is, the only difference between the household and the constrained planner is that the planner considers how the allocation changes prices and, in turn, how the change in prices can improve welfare. Understanding the pecuniary externalities is important because they always operate as a part of the trade-off the planner faces as long as the constrained planner cannot complete the market perfectly. This paper analyzes and quantifies the pecuniary externalities in an incomplete market with *endogenous human capital* to see whether there is an under/over accumulation of capital and human capital.

Recently, Dávila et al. (2012) (hereafter DHKR) analyzed the implications of pecuniary externalities in a standard incomplete market model with exogenous, uninsured idiosyncratic labor income shocks. DHKR show that whether the capital in a competitive equilibrium is over-accumulated depends on the magnitude of

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two channels of pecuniary externalities, which have opposite implications. The first channel is the insurance effect. Since the only risk in the economy is a labor income shock, a lower wage and a higher interest rate scale down the stochastic part of the income, implying that a decrease in capital can improve the welfare of households. The second channel is the redistribution effect, which arises from the cross-sectional capital dispersion. A lower interest rate and a higher wage rate can improve the welfare of wealth-poor households, and because consumption-poor households that have a relatively higher marginal utility tend to be wealth poor, an increase in capital can improve social welfare. Thus, the over or under-accumulation of capital may arise depending on which channel of pecuniary externality dominates. DHKR consider three different calibrations and show that with a calibration that generates realistic wealth dispersion, the redistribution channel dominates, and thus, there is an under-accumulation of capital in a competitive equilibrium.

In DHKR, the key mechanism that leads to the under-accumulation of capital arises from endogenous wealth dispersion, but labor dispersion is determined by exogenous shocks. In reality, however, there is also an endogenous dispersion of labor productivity through the accumulation of human capital. More important, human capital matters for the pecuniary externalities for the following reasons. First, the planner can change prices by changing human capital as well as capital. Second, there is a new redistribution channel, which has implications opposite to those for the redistribution channel discussed in DHKR. The welfare of the human capital-poor households can be improved with a lower wage and a higher interest rate. Since the consumption poor also tend to be human capital poor, price changes through decreasing capital and increasing human capital might improve welfare, which is exactly the opposite of the redistribution channel arising from wealth dispersion. Thus, we study the implication of pecuniary externalities in an incomplete market with an endogenous distribution of both wealth and human capital.

First, using a two-period model, we analytically examine how the introduction of human capital accumulation changes the implication of the pecuniary externalities. We show that in a model where human capital is also a choice variable, the variable of interest for the implication of pecuniary externalities is not the level of capital, but the capital-labor ratio. Then, using the Euler equation of the constrained planner, we explicitly show that the introduction of human capital creates an additional channel of redistribution through human capital inequality, and this additional channel has implications for a too-high capital-labor ratio. We also show that the risk properties of human capital can change the implication of the pecuniary externalities through the endogenous response to the shock.

From the previous discussion, we can easily conjecture that wealth inequality and human capital inequality and their relative size are important for the implication of pecuniary externalities, but there can be many distributions that can have the same measure of inequality. We thus investigate further to show that the consumption poor's relative poorness in terms of human capital and the correlation between wealth and human capital is crucial for the implication of pecuniary externalities. We show that the relatively lower human capital of the consumption poor and the lower correlation between wealth and human capital tend to lower the capital-labor ratio of the constrained efficient allocation.

We quantitatively examine whether the capital-labor ratio is too high from the perspective of the constrained planner, using an infinite horizon model. We find that even though we introduce an endogenous human capital dispersion, an economy calibrated to realistic inequalities of wealth and human capital maintains the conclusion of DHKR—the capital-labor ratio is too low in a competitive equilibrium. The degree of under-accumulation, however, is smaller than that in the DHKR economy. In the baseline model, the capital-labor ratio of the constrained optimum is 2.63 times larger than that of the competitive equilibrium, while in DHKR, it is 3.65 times larger than the ratio in a competitive equilibrium. We also show that in an extended model with a progressive human capital subsidy that can generate a low correlation between wealth and human capital, the degree of under-accumulation is much smaller—the capital-labor ratio is 1.1 times larger than that of the competitive equilibrium—confirming the importance of the correlation for the redistribution channel of pecuniary externalities. The under-accumulation of capital result, however, is robust to changing key parameters of the model.

In the applied tax studies with other considerations of the government, pecuniary externalities will operate as a part of the goals of the government. We analyze the pecuniary externalities in the presence of other distortions caused by government policies. We show that the benefit of increasing the capital-labor ratio through the pecuniary externalities channel is likely to be increasing with progressive labor income distortion and is likely to be decreasing with savings distortion.

This paper is related to the literature on the constrained efficiency/inefficiency of allocation in a competitive equilibrium with uninsured shocks. After Diamond (1967) first noted the possibility of constrained inefficiency in a competitive equilibrium based on pecuniary externalities, examples of constrained inefficiency were provided in Hart (1975), Diamond (1980), Stiglitz (1982), and Loong and Zeckhauser (1982). Recently, Dávila et al. (2012) address this constrained inefficiency issue in a standard neoclassical growth model with uninsured idiosyncratic shocks, a model that is the workhorse of macroeconomics. This paper is also related to Gottardi, Kajii, and Nakajima (2015, 2016), who investigate optimal linear distortionary taxes on labor income and the return from saving in a model with labor supply or human capital. While DHKR focus on a constrained planner's problem where the planner can directly control households' labor supply and saving decisions, they consider linear tax on labor and savings, and thus, there are direct insurance and redistribution provided by the tax scheme itself as well as an indirect welfare improvement through price changes.[1]

This paper is also related to the literature on the endogenous human capital model. Huggett, Ventura, and Yaron (2006, 2011) develop a model with endogenous

[1] The old version of Gottardi, Kajii, and Nakajima (2015), which was titled "Optimal Taxation and Debt with Uninsurable Risks to Human Capital Accumulation," analyzes the constrained planner's problem who only internalizes the pecuniary externalities and analytically shows that the capital-labor ratio is too high in a competitive equilibrium. This is because their analysis abstracts away from the redistribution channel of pecuniary externalities. With a linearly additive human capital production function, the portfolio composition between human capital and other assets (capital and bond) is the same for every household, and thus, each individual's relative poorness in human capital is equal to the relative poorness in wealth even though each household can have different levels of human capital and wealth.

accumulation of human capital to match many features of the earnings distribution, and Huggett, Ventura, and Yaron's (2011) model is the most similar to the model used for this paper's quantitative investigation. Levhari and Weiss (1974) were the first to analyze the effects of risk on the human capital investment. Our analysis on the effects of risk properties builds on their framework.

Even though we focus on pecuniary externalities, since these externalities can be one of the government's considerations along with other goals, this paper is naturally connected to the optimal taxation analysis with endogenous human capital formation. da Costa and Maestri (2007), Grochulski and Piskorski (2010), Kapička and Neira (2015), and Stantcheva (2017) study optimal Mirrleesian taxation with human capital policy, and Krueger and Ludwig (2013, 2016), Peterman (2016), and Gottardi, Kajii, and Nakajima (2015) study optimal Ramsey taxation in the presence of endogenous human capital.

The rest of the paper is outlined as follows. In Section II and III, we analytically investigate the introduction of human capital and the key determinants of the pecuniary externalities. In Section IV, we quantitatively investigate the optimal capital-labor ratio. In Sections V, we discuss the policy implications, and we conclude in Section VI.

I. Two-Period Model

In this section, using a simple two-period version of an incomplete market model with endogenous human capital accumulation, we analyze the pecuniary externalities.

A. *Economy with Endogenous Human Capital Dispersion*

We start by analyzing the pecuniary externalities in an incomplete market model with human capital. To focus on the role of endogenous human capital dispersion, we consider an economy where the labor income dispersion is endogenously generated by the human capital accumulation, but the wealth dispersion is exogenously determined ex ante. Later, we endogenize both wealth and human capital.

Consider an economy populated by a continuum of households with measure one. Households live only for two periods. In period zero, a household is born with initial wealth k and human capital h. Let Γ denote the cross-sectional distribution of capital k and human capital h with density $\gamma(k,h)$. We will denote the marginal distribution of k and h by Γ_k and Γ_h, respectively. In period 0, the household with initial human capital h and wealth k will decide how much to consume c_0 and to invest in human capital x out of period 0 income $a(k,h) = w_0 h + r_0 k$, where w_0 and r_0 are the exogenously given rental rate of human capital and physical capital, respectively.[2] In period 1, human capital $h' = g(h,x)$ is produced using the human

[2] Alternatively, we can assume that production also takes place in period 0 and the rental rates are the marginal product of each factor. The following analysis does not depend on this assumption.

capital production technology g, which is increasing and concave in both arguments and differentiable. There is additive idiosyncratic risk to the human capital return:

$$e \sim N(0, \sigma_e^2),$$

and households consume both capital and labor income in period 1. The additive structure of shock is assumed for simplicity, but most of the qualitative properties of pecuniary externalities do not depend on this assumption. At the end of this section, we analyze how the risk properties of human capital affect the result.

In sum, the budget constraints of the household with initial (k, h) are

$$c_0 + x = a(k, h),$$

$$c_1(e) = w_1[g(h, x) + e] + r_1 k,$$

where w_1 is the wage rate, and r_1 is the interest rate in period 1. The household maximizes its expected lifetime utility $u(c_0) + \beta E_e[u(c_1)]$ subject to the budget constraint.

Competitive firms have the same production function $F(K, H)$, which is strictly increasing and strictly concave in both arguments and is homogeneous of degree one. We also assume that the production function exhibits complementarity between inputs $F_{KH} > 0$.

Competitive Equilibrium.—A competitive equilibrium in this economy is defined as an allocation $(x(k, h))$ and prices (r_1, w_1), such that the household chooses $x(k, h)$ to maximize its expected lifetime utility subject to the budget constraint; $K = \int k \Gamma_k(dk)$, $H = \int g(h, x(k, h)) \Gamma(dk, dh)$; and $r_1 = F_K(K, H)$, $w_1 = F_H(K, H)$.[3]

The first-order condition of the household's problem with respect to x is

$$u'(c_0) = \beta w_1 g_x E_e[u'(c_1) | k, h]. \tag{1}$$

By applying the implicit function theorem to (1), we can show that human capital investment $x(k, h)$ can increase or decrease with initial human capital h, which depends on whether the human capital h and human capital investment x are complements $(g_{xh} > 0)$ or substitutes $(g_{xh} < 0)$. However, the period 1 human capital $h'(k, h)$ is increasing in h as long as g_{xh} is not too negative, and we assume that $\partial h'/\partial h > 0$ from now on.[4]

Constrained Efficient Allocation.—The constrained planner can choose different levels of human capital investment $x(k, h)$ for each household with different (k, h),

[3] In this two-period model, we assume that the depreciation rate of capital is 100 percent for simplicity. Later, in an infinite horizon model, we use a more realistic depreciation rate of capital.

[4] By combining $\partial h'/\partial h = g_h + g_x \cdot (\partial x / \partial h)$ and $\partial x/\partial h$ obtained by applying the implicit function theorem to (1), we can easily show that $\partial h'/\partial h = \big(-(g_h + w_0)u''(c_0) + \beta w_1 E[u'(c_1)]\{g_{xh} g_x - g_h g_{xx}\}\big) / \big(-u''(c_0) - \beta w_1 g_{xx} E[u'(c_1)] - \beta w_1^2 g_x^2 E[u''(c_1)]\big)$. Thus, $\partial h'/\partial h > 0$, if $g_{xh} > g_{xx} g_h / g_x \ (< 0)$.

while respecting all the budget constraints of households and market clearing conditions and letting firms behave competitively. Because prices are determined by $r_1 = F_K(K,H)$ and $w_1 = F_H(K,H)$, the constrained planner's problem with a utilitarian social welfare function is defined as follows:

$$\max_{\{c_0(k,h),\, c_1(k,h,e),\, x(k,h)\}} \int \{u(c_0(k,h)) + \beta E_e[u(c_1(k,h,e)) \mid k,h]\} \Gamma(dk,dh)$$

subject to

$$\begin{aligned} c_0(k,h) &= a(k,h) - x(k,h), \\ c_1(k,h,e) &= F_H(K,H)(h'(k,h) + e) + F_K(K,H)k, \\ h'(k,h) &= g(h,x(k,h)), \end{aligned}$$

$$\text{where} \quad K = \int k\Gamma_k(dk) \quad \text{and} \quad H = \int h'(k,h)\Gamma(dk,dh).$$

Notice that the constrained planner cannot use transfers to complete the market. The only difference between the constrained social planner and the household is that the constrained social planner also considers pecuniary externalities—how the allocation affects prices, r_1 and w_1.

We now characterize the constrained efficient allocation focusing on its Euler equation as in DHKR. The first-order condition with respect to $x(\hat{k},\hat{h})$ is the following Euler equation $u'(c_0(\hat{k},\hat{h})) = \beta F_H g_x(\hat{h},x(\hat{k},\hat{h})) E_e[u'(c_1(\hat{k},\hat{h})) \mid \hat{k},\hat{h}] + \beta g_x(\hat{h},x(\hat{k},\hat{h}))\Delta_h$, where

$$\Delta_h = \int E_e\big[u'(c_1) \cdot \{F_{HH}(h'(k,h) + e) + F_{KH}k\} \mid k,h\big]\, d\Gamma(k,h). \tag{2}$$

By comparing (2) with (1), we can see that $\beta g_x \Delta_h$ is an additional marginal benefit of investing in human capital through the change in prices,[5] which is not internalized in the household's problem. Thus, Δ_h captures the so-called pecuniary externalities.

The sign of Δ_h is the key object of the ensuing analysis. If $\Delta_h > 0$, the constrained planner makes everyone invest more than he would do in a competitive equilibrium. Thus, positive Δ_h implies an under-accumulation of human capital in a competitive equilibrium relative to the constrained efficient allocation. However, negative Δ_h implies an overaccumulation of human capital in a competitive equilibrium.

In order to sign Δ_h, we now decompose Δ_h. For expositional purposes, it is very useful to make further assumptions on the preferences. We assume that the period utility takes the form of the constant absolute risk aversion (CARA):

$$u(c) = -\frac{1}{\psi}\exp(-\psi c),$$

[5] Note that Δ_h does not depend on the individual initial wealth k and human capital h.

where ψ is the coefficient of risk aversion. Under the CARA-normal specification, the household's expected utility is $E_e[u(c_1)] = -(1/\psi)\exp(-\psi\{E_e[c_1] - (\psi/2)\mathrm{var}_e(c_1)\})$, where $E_e[c_1] = F_H h' + F_K k$ and $\mathrm{var}_e(c_1) = F_H^2 \sigma_e^2$. Thus, Δ_h can be rewritten as follows under the CARA-normal specification:

$$\Delta_h = \int E_e[u'(c_1)]\{F_{HH}h'(k,h) + F_{KH}k - \psi F_H F_{HH}\sigma_e^2\}\, d\Gamma(k,h),$$

and we can decompose Δ_h, additively: $\Delta_h = \Delta_{h1} + \Delta_{h2}$, where

$$\Delta_{h1} = -\psi F_H F_{HH}\sigma_e^2 \int E_e[u'(c_1)\,|\,k,h]\, d\Gamma(k,h),$$

$$\begin{aligned}\Delta_{h2} &= \int E_e[u'(c_1)\,|\,k,h][F_{HH}h'(k,h) + F_{KH}k]\, d\Gamma(k,h)\\ &= \underbrace{F_{HK}K\int_{k,h} E_e[u'(c_1)\,|\,k,h]\left[\frac{k}{K} - 1\right]}_{\Delta_{h2,K}}\\ &\quad + \underbrace{F_{HK}K\int_{k,h} E_e[u'(c_1)\,|\,k,h]\left[1 - \frac{h'(k,h)}{H}\right]}_{\Delta_{h2,H}}\\ &\quad (\text{using } F_{HK}K + F_{HH}H = 0).\end{aligned}$$

This decomposition shows that the pecuniary externalities emerge from two channels. We call Δ_{h1} the "insurance effect" and Δ_{h2} the "redistribution effect." If there is no income risk ($\sigma_e^2 = 0$), then we can easily see that $\Delta_{h1} = 0$. Thus, Δ_{h1} captures the pecuniary externality due to idiosyncratic risk. However, if there are no wealth inequality ($k = K$, $\forall k$) and no human capital inequality ($h'(k,h) = H$, $\forall (k,h)$), $\Delta_{h2} = 0$. Thus, Δ_{h2} captures the pecuniary externality due to wealth and human capital inequality.[6]

We now determine the sign of each component of Δ_h. First, the sign of the insurance effect is always positive ($\Delta_{h1} > 0$), which increases optimal human capital investment. In this economy, labor income is risky due to an idiosyncratic shock, but capital income is not. If the planner decreases the wage rate and increases the interest rate, the risky part of the household income is scaled down, improving the welfare of risk-averse households. By increasing human capital, the planner can decrease the wage rate and increase the interest rate. Thus, Δ_{h1} is always positive, and the positive insurance effect gets stronger as the variance of the shock σ_e^2 increases.

[6] Under an additive shock structure, a similar additive decomposition ($\Delta_h = \Delta_{h1} + \Delta_{h2}$) is possible with more general utility functions $u(c)$:

$$\Delta_{h1} = F_{HH}\int\int eu'(c_1(k,h,e))f(e)\, de\Gamma(dk,dh), \quad \Delta_{h2} = F_{HK}K\int E_e[u'(c_1(k,h,e))]\left\{\frac{k'}{K} - \frac{h'}{H}\right\}\Gamma(dk,dh).$$

But with a more general nonadditive structure of shock, which will be analyzed in Section ID, this additive decomposition is specific to CARA normal.

However, the sign of the redistribution effect (Δ_{h2}) is ambiguous because there are two redistribution effects, and they have the opposite signs. The term $\Delta_{h2,K}$ captures the redistribution channel through wealth dispersion, and $\Delta_{h2,H}$ captures the redistribution channel through human capital dispersion. Suppose that there is no wealth dispersion $(k = K, \forall k)$, then $\Delta_{h2,K} = 0$, and we can show that the sign of the redistribution effect $(\Delta_{h2,H})$ is positive. Exploiting the CARA-normal specification, we can use $E_e[u'(c_1)] = \exp((\psi^2/2) \cdot F_H^2 \sigma_e^2) u'(E_e[c_1])$ to show that

$$\Delta_{h2} = \Delta_{h2,H} = F_{HK} K \exp\left(\frac{\psi^2}{2} F_H^2 \sigma_e^2\right) \int u'(wh'(h) + rK)\left[1 - \frac{h'(h)}{H}\right] \Gamma_h(dh),$$

and the sign of the integral factor is positive because

$$\begin{aligned}
&\int u'(F_H h'(h) + F_K K)\left[1 - \frac{h'(h)}{H}\right] \Gamma_h(dh) \\
&\quad = \int_{h:h'(h)\geq H} u'(F_H h'(h) + F_K K)\left[1 - \frac{h'(h)}{H}\right] \\
&\qquad + \int_{h:h'(h)<H} u'(F_H h'(h) + F_K K)\left[1 - \frac{h'(h)}{H}\right] \\
&\quad > u'(F_H H + F_K K)\int_{h:h'(h)\geq H} \left[1 - \frac{h'(h)}{H}\right] \\
&\qquad + u'(F_H H + F_K K)\int_{h:h'(h)<H} \left[1 - \frac{h'(h)}{H}\right] \\
&\quad = u'(F_H H + F_K K)\int \left[1 - \frac{h'(h)}{H}\right] \Gamma_h(dh) = 0,
\end{aligned}$$

where the inequality is because for h such that $h'(h) \geq H$ (and thus $1 - h'/H \leq 0$), $u'(F_H h'(h) + F_K K)(1 - h'/H) \geq u'(F_H H + F_K K)(1 - h'/H)$ and for h such that $h'(h) < H$ (and thus $1 - h'/H > 0$), $u'(F_H h'(h) + F_K K)(1 - h'/H) > u'(F_H H + F_K K)(1 - h'/H)$.

The intuition of the positive redistribution effects $\Delta_{h2,H} > 0$ is the following. The welfare of the human capital-poor household can be improved by decreasing the wage rate and increasing the interest rate. Because the marginal utility of consumption-poor households is relatively higher and human capital-poor households tend to be consumption poor, the planner wants to increase human capital to decrease the wage rate $(w = F_H)$. Thus, if there is no wealth dispersion, both Δ_{h1} and Δ_{h2} are positive, and the pecuniary externalities imply that there is underinvestment in human capital in a competitive equilibrium.

In the presence of wealth inequality, however, there is an additional term $\Delta_{h2,K}$ in the redistribution channel Δ_{h2}, which is reflecting the redistribution effects through wealth dispersion. The sign of $\Delta_{h2,K}$ is likely to be negative for exactly the opposite reason of positive $\Delta_{h2,H}$. The welfare of the wealth poor is improved when the interest rate decreases and the wage rate increases. Since the marginal utility of the consumption poor is high, if the consumption poor tend to be capital poor, the planner wants to decrease H, and thus, $\Delta_{h2,K} < 0$. Therefore, depending on which redistribution channel dominates, the redistribution channel can be either positive or negative.[7]

In sum, in an economy with *endogenous* human capital dispersion, the pecuniary externalities imply that there can be either over or underinvestment in human capital. However, in an incomplete market with *endogenous wealth dispersion* and *exogenous labor income shock*, DHKR show that the pecuniary externalities imply either over or under-accumulation of physical capital. Next, we will analyze how these pecuniary externalities from two economies are related and the role of human capital dispersion in an economy where both wealth and human capital are endogenous.

B. *Pecuniary Externalities in a Time-Investment Model*

Before getting into the comparison with DHKR's analysis, it is useful to discuss the role of time investment and money investment for the implications of pecuniary externalities in a human capital model. In the previous analysis, we assumed that households spend money (good) to invest in human capital. In reality, however, households invest both money and time, and the previous studies show that the optimal tax and education policy can be very different between the models with money investment and with time investment. In this section, we investigate whether the implications of the pecuniary externalities depend on this modeling choice. We will show that the sources of pecuniary externalities and the sign of each channel are not

[7] To explicitly show $\Delta_{h2,H} > 0$ and $\Delta_{h2,K} < 0$ in the presence of *exogenous* wealth inequality, we need one more reasonable assumption that $E_h[h'(k,h)\,|\,k]$ is weakly increasing with k (which holds if r_0 is high enough and k and h are positively correlated). Under this assumption, the integral factor in $\Delta_{h2,H}$ is positive because

$$
\begin{aligned}
&\int u'(F_H h'(h) + F_K k)\left[1 - \frac{h'(k,h)}{H}\right]\Gamma(dk,dh) \\
&\quad = \int_k \left\{ \int_{h:h'(k,h)\geq H} u'(F_H h'(k,h) + F_K k)\left[1 - \frac{h'(k,h)}{H}\right]\gamma(k,h)\,dh \right. \\
&\qquad \left. + \int_{h:h'(k,h)<H} u'(F_H h'(k,h) + F_K k)\left[1 - \frac{h'(k,h)}{H}\right]\gamma(k,h)\,dh \right\} dk \\
&\quad > \int_k u'(F_H H + F_K k)\int_h \left[1 - \frac{h'(k,h)}{H}\right]\gamma(h|k)\,dh\,\gamma_k(k)\,dk \\
&\quad = \int_{k:\omega(k)>0} u'(F_H H + F_K k)\,\omega(k)\,\gamma_k(k)\,dk + \int_{k:\omega(k)\leq 0} u'(F_H H + F_K k)\,\omega(k)\,\gamma_k(k)\,dk, \\
&\qquad \text{where} \quad \omega(k) = \int_h \left[1 - \frac{h'(k,h)}{H}\right]\gamma(h|k)\,dh = 1 - \frac{1}{H}E_h[h'(k,h)\,|\,k] \\
&\quad \geq u'(F_H H + F_K \bar{k})\int_k\int_h \left[1 - \frac{h'(k,h)}{H}\right]\gamma(k,h)\,dhdk \\
&\quad = 0, \quad \text{where } \bar{k} \text{ is such that } \omega(\bar{k}) = 0.
\end{aligned}
$$

The last inequality holds because $\omega(k)$ is weakly decreasing in k. It is easy to show $\Delta_{h2,K} < 0$ using similar arguments.

different between these two models, but the implications of pecuniary externalities for the level of human capital can be very different.

Consider the same two-period model previously mentioned in which households spend time to invest in human capital instead of money. Every household is endowed with one unit of time in each period. In period 0, each household decides how to divide the given time between human capital investment s and working $1 - s$. In period 1, human capital $h' = g(h,s)$ is produced using the human capital production technology ($g_s, g_h > 0$ and $g_{ss}, g_{hh} < 0$). Since period 1 is the last period, households spend all of their time working in period 1. Then, the budget constraints of the household are

$$c_0(k,h,e) = w_0(1-s)h + r_0 k,$$

$$c_1(k,h,e) = w_1[g(h,s) + e] + r_1 k.$$

We make one additional modification to a competitive equilibrium. We now assume that production takes place in both periods, and thus, the rental rates of period t (w_t, r_t) are determined by the marginal product of each factor in period t: $F_H(K,L_t)$, $F_K(K,L_t)$, where $K = \int k\Gamma_k(dk)$, $L_0 = \int (1 - s(k,h))h\Gamma(dk,dh)$, and $L_1 = \int g(h, s(k,h))\Gamma(dk,dh)$. Note that with money investment, the aggregate labor supply in period 0 does not depend on human capital investment x, and thus, increasing x does not change prices in period 0. With time investment, however, the aggregate labor supply in both period 0 and period 1 is affected by the choice of human capital investment s. Thus, we need to endogenize the prices for both periods to study the implication of pecuniary externalities in a time-investment model.[8]

The first-order condition of the household's problem with respect to s is $w_0 hu'(c_0) = \beta w_1 g_s E_e[u'(c_1)|k,h]$, where the marginal cost of increasing human capital investment—forgone wage earnings in period 0—is equal to the marginal benefit—additional wage earnings in period 1. We now compare the first-order condition of the constrained planner with that of the household. We assume a CARA-normal specification again for the additive decomposition of the pecuniary externalities. Then, the planner's first-order condition with respect to $s(\hat{k},\hat{h})$ is $F_{L,0}\hat{h}u'(c_0) + \hat{h}\Delta_h^0 = \beta F_{L,1} g_s E_e[u'(c_1)|\hat{k},\hat{h}] + g_s\beta\Delta_h^1$, where

$$(3) \qquad \Delta_h^0 = \int u'(c_0)[F_{LL,0}(1 - s(k,h))h + F_{KL,0}k]\Gamma(dk,dh), \quad \text{and}$$

$$\Delta_h^1 = \int u'(c_1)[F_{LL,1}h'(k,h) + F_{KL,1}k - \psi F_{L,1}F_{LL,1}\sigma_e^2]\Gamma(dk,dh).$$

By comparing the Euler equations, we can see that there are two additional terms arising from the pecuniary externalities. Note that Δ_h^t captures the welfare effects of increasing effective labor $h_t(1 - s_t)$ in period t through the change in relative prices (by decreasing the wage rate and increasing the interest rate). Increasing one unit of time for human capital investment decreases effective labor in period 0 by h and

[8] With money investment, endogenizing prices in period 0 does not change the analysis of the pecuniary externalities in a two-period model.

increases effective labor in period 1 by g_s, which leads to marginal cost $h\Delta_h^0$ and marginal benefit $g_s\beta\Delta_h^1$, respectively.

The sources of the pecuniary externalities are the same as in the money-investment model—the insurance and redistribution effects of changing factor prices, although there is no uncertainty and thus no insurance channel in period 0 in this two-period model. The sign of Δ_h^t, however, can have different implications between the two models. In the money-investment model, we could interpret positive (negative) Δ_h as underinvestment (over-investment) in human capital in a competitive equilibrium. In the time-investment model, however, the sign of Δ_h^t can imply over/undersupply of effective labor only, and it is not enough to conclude that there is over/underinvestment in human capital.

For simple exposition, consider the case where the signs of Δ_h^0 and Δ_h^1 are the same.[9] For example, suppose that Δ_h^0 and Δ_h^1 are positive. This implies that there is an undersupply of labor (in both periods) in a competitive equilibrium because increasing the labor supply in each period can improve the welfare of the economy. In the time-investment model, however, this does not mean that there should be more investment in human capital because human capital investment can be increased at the cost of decreasing working hours. In the two-period model, increasing human capital investment s decreases labor supply in period 0 and increases labor supply in period 1. Thus, even if Δ_h^0 and Δ_h^1 are positive, whether increasing human capital investment improves welfare is determined based on the comparison of the cost in period 0 and the benefit in period 1. That is, even if we know the sign of Δ_h, we need further information on the elasticity of the labor supply with respect to human capital investment to evaluate whether there is an over/underinvestment in human capital.

In the online Appendix E, we discuss the impact of the modeling choice (time versus money) on Δ_h, which can still be useful given that the sign of Δ_h is at least informative for evaluating the over/undersupply of effective labor.

We will adopt the money-investment model as the benchmark model for the analysis because the implications of pecuniary externalities are straightforward, and thus, it provides easier exposition. However, we will discuss whether the ensuing analysis is robust to the time-investment model in the following analysis.

C. *Introduction of Endogenous Wealth Dispersion*

In this section, we introduce endogenous wealth dispersion and analyze the pecuniary externalities in an economy where both wealth dispersion and human capital dispersion are endogenous. Then, we will compare our result to the previous analysis of DHKR.

Competitive Equilibrium.—The economic environment is the same as in Section IA except that households invest in both physical capital k' and human

[9] In the steady state of an infinite horizon model, Δ_h^t will be constant over time.

capital x in period 0. The household with initial (k, h) chooses k' and x subject to the budget constraints:

$$c_0(k,h) + k'(k,h) + x(k,h) = a(k,h),$$

$$c_1(k,h,e) = w[g(h,x(k,h)) + e] + rk'(k,h).$$

For simplicity, we do not impose any lower bound condition on k' and x, so the first-order conditions with respect to k' and x are $u'(c_0(k,h)) = \beta r E_e[u'(c_1)\,|\,k,h]$ and $u'(c_0(k,h)) = \beta w\, g_x(h,x,A)\, E_e[u'(c_1)\,|\,k,h]$, respectively. By combining these first-order conditions, we obtain the following no-arbitrage condition:

$$r = wg_x, \tag{4}$$

which equates the marginal return to capital investment and the marginal return to human capital investment. From (4), we can see that the human capital investment x does not depend on k, which implies $h'(h) = g(h, x(h))$, which also does not depend on k.

We assume $\partial h'/\partial h > 0$.[10] Hence, the initial inequality in h generates human capital inequality in period 1. We can also easily show that $\partial k'/\partial k > 0$,[11] and thus, the initial heterogeneity in wealth generates a dispersion of wealth in period 1.

Constrained Efficient Allocation.—The constrained planner's problem amounts to choosing the saving and human capital investment of each household with initial (k, h):

$$\max_{k'(k,h),\,x(k,h)} \int_{k,h} \Big\{ u(a(k,h) - k'(k,h) - x(k,h)) + \beta E_e\big[u(F_H(g(h,x(k,h)) + e) + F_K k'(k,h))\big]\Big\}\,\Gamma(dk,dh).$$

The first-order conditions with respect to $k'(\hat{k},\hat{h})$ and $x(\hat{k},\hat{h})$ are as follows:

$$k'(\hat{k},\hat{h}):\quad u'(c_0(\hat{k},\hat{h})) = \beta F_K E_e[u'(c_1)\,|\,\hat{k},\hat{h}] + \beta\Delta_k, \tag{5}$$

$$\text{where}\quad \Delta_k = \int E_e\big[u'(c_1)\{F_{HK}(h'(k,h) + e) + F_{KK}k'(k,h)\}\,|\,k,h\big]\,d\Gamma(k,h),$$

$$x'(\hat{k},\hat{h}):\quad u'(c_0(\hat{k},\hat{h})) = \beta F_H \cdot g_x(\hat{h},x(\hat{k},\hat{h}))\cdot E_e\big[u'(c_1)|\,\hat{k},\hat{h}\big] + \beta g_x(\hat{h},x(\hat{k},\hat{h}))\cdot\Delta_h, \tag{6}$$

$$\text{where}\quad \Delta_h = \int E_e\big[u'(c_1)\{F_{HH}(h'(k,h) + e) + F_{KH}k'(k,h)\}\,|\,k,h\big]\,d\Gamma(k,h).$$

[10] By applying the implicit function theorem to this no-arbitrage condition, we can easily show that $\partial x/\partial h = -g_{xh}/g_{xx}$, and thus, the sign of $\partial x/\partial h$ is determined by the sign of g_{xh}. As we discussed earlier, however, as long as g_{xh} is not too negative $(g_{xh} < (g_{xx}g_h)/g_x)$, we can get $\partial h'/\partial h > 0$.

[11] Using the arbitrage condition, $x(h) = g_x^{-1}(h, r/w)$ can be derived, and we can substitute $x(h)$ in the Euler equation. Then, using the implicit function theorem, $\partial k'/\partial k = ((1+r)u''(c_0))/(u''(c_0) + \beta r^2 E[u''(c_1)]) > 0$.

Note that the first-order condition with respect to physical capital investment also has an additional marginal benefit of capital investment $\beta\Delta_k$ due to a pecuniary externality. Moreover, there is a clear relationship between Δ_k and Δ_h. Using $F_{KH}K + F_{HH}H = 0$ and $F_{HK}H + F_{KK}K = 0$, we obtain the following relationship between Δ_k and Δ_h.

PROPOSITION 1:

$$\Delta_h = -\frac{K}{H}\Delta_k.$$

Proposition 1 shows that the sign of Δ_h is the opposite of the sign of Δ_k. That is, if there is an additional marginal benefit to saving from the perspective of the constrained planner, then there is an additional marginal cost to investing in human capital. Thus, we can only focus on the sign of Δ_k. We also note that the variable of interest is the capital-labor ratio (K/H), not the level of capital (K) and human capital (H) themselves, because both capital (K) and labor (H) are endogenous variables.

The sources of the pecuniary externalities are the same as in the economy with exogenous wealth dispersion, which is clearly seen from the additive decomposition of $\Delta_k = \Delta_{k1} + \Delta_{k2}$ (using CARA normal):

$$\Delta_{k1} = -\psi F_H F_{HK}\sigma_e^2 \int_{k,h} E_e[u'(c_1)|k,h]\Gamma(dk,dh), \tag{7}$$

$$\Delta_{k2} = \underbrace{F_{KK}K\int_{k,h} E_e[u'(c_1)|\,k,h]\left[\frac{k'(k,h)}{K} - 1\right]}_{\Delta_{k2,K}} + \underbrace{F_{KK}K\int_{k,h} E_e[u'(c_1)|\,k,h]\left[1 - \frac{h'(k,h)}{H}\right]}_{\Delta_{k2,H}}. \tag{8}$$

The sign of the insurance effects Δ_{k1} is always negative for the same reason as positive Δ_{h1}. The planner wants to decrease the wage rate to scale down the risky part of the income, and it can be done by decreasing saving $(\Delta_{k1} < 0)$. Similarly, the sign of each term in the redistribution effects (Δ_{k2}) is determined for the same reason as the sign of that in Δ_{h2} explained earlier, but in the opposite direction. For example, since the consumption poor are wealth poor, decreasing the interest rate and increasing the wage rate can improve welfare, which can be obtained by increasing saving $(\Delta_{k2,K} > 0)$ and decreasing human capital $(\Delta_{h2,K} < 0)$.

Since Δ_k and Δ_h are the same for all households, the sign of Δ_k (Δ_h) implies that the planner wants to increase or decrease saving (human capital investment) for every household. However, the allocation of extra saving (or human capital investment) is not the same for all households. If Δ_k (Δ_h) is positive, the constrained optimum allocates even more saving (human capital investment) to the rich because the marginal utility of consumption of the rich is small, while the extra marginal benefit of saving Δ_k (of human capital investment Δ_h) is the same for everyone.

Comparison with DHKR.—We now compare our result to that of DHKR—pecuniary externalities in a version of Aiyagari's (1994) economy with pure transitory income shock. From this comparison, we can show the role of human capital in the pecuniary externalities: introducing persistent labor income dispersion generated by human capital tends to lower the optimal capital-labor ratio.

Consider a DHKR economy. In period 0, a household with initial wealth k chooses how much to consume c_0 and save k'. In period 1, the household receives an idiosyncratic labor endowment $e \sim N(L, \sigma_e^2)$ and consumes the return on both capital investment and labor income. In sum, the budget constraints of the household with initial wealth k are $c_0 + k' = k$ and $c_1(e) = we + rk'$. To make comparison easy, we assume CARA utility.

The next lemma shows that there exists a distribution of initial wealth such that a competitive equilibrium in the DHKR economy has the same aggregate capital and labor as those in the human capital economy, and we will consider such a competitive equilibrium.

LEMMA 2: *Assume that the coefficient of risk aversion*, *the discount factor*, *and the variance of shocks are held constant across economies. Then*, *there exists a distribution of initial wealth* $\Gamma^{DHKR}(k)$ *and mean of labor endowment process* L^{DHKR} *such that*

(i) $L^{DHKR} = H^{HK}$,

(ii) *distributions of* k' *in period* 1 *are equivalent in the two economies, and thus,* $K^{DHKR} = K^{HK}$,

where HK stands for human capital. This implies that $w^{DHKR} = w^{HK}$, $r^{DHKR} = r^{HK}$.

PROOF:

(i) is automatically satisfied by setting the mean L^{DHKR} to the mean of human capital H^{HK}. We only need to show that the distributions of capital in period 1 are equivalent. Note that, using CARA-normal specification, we can easily get the closed-form solution of the k' in both economies:[12]

$$k'^{HK}(k,h) = \frac{1}{1+r}\left[\frac{1}{\psi}\log(\beta r) + \frac{\psi}{2}w^2\sigma_e^2 - wh'(h) + a(k,h) - x(h)\right], \quad \forall k, h,$$

$$k'^{DHKR}(k) = \frac{1}{1+r}\left[\frac{1}{\psi}\log(\beta r) + \frac{\psi}{2}w^2\sigma_e^2 - wL^{DHKR} + k\right], \quad \forall k.$$

Then, we can construct $\Gamma^{DHKR}(k)$ in the following way. For each initial wealth k^{DHKR} in DHKR's economy, we define the set of (k,h)'s $S(k^{DHKR})$ in the human capital economy in the following way:

$$S(k^{DHKR}) = \{(k,h) \,|\, k^{DHKR} = w(L^{DHKR} - h'(h)) + a(k,h) - x(h)\},$$

[12] It can be obtained by taking the log of the Euler equation and rearranging terms.

and set the density $\gamma^{DHKR}(k^{DHKR}) = \int_{S(k^{DHKR})} \Gamma(dk, dh)$. Note that for any $(k,h) \in S(k^{DHKR})$, $k'^{HK}(k,h) = k'^{DHKR}(k^{DHKR})$. Thus, the distributions of capital in period 1 are identical in the two economies. Since the aggregate allocations are identical in both economies, the prices in the two economies are also identical. ∎

We now ask the following question: how does the introduction of human capital accumulation change the implication of pecuniary externalities? It is obvious that there is an additional variable (human capital) that can internalize pecuniary externalities compared to the DHKR economy. More important, persistent income dispersion generated by human capital accumulation tends to lower the optimal capital-labor ratio by introducing the additional channel of pecuniary externalities.

This can be shown by comparing the optimality conditions of the constrained planner in a DHKR economy to that of a human capital economy. The additional marginal benefit of saving in the DHKR economy $\Delta_k^{DHKR} = \Delta_{k1}^{DHKR} + \Delta_{k2}^{DHKR}$ is determined by[13]

$$\Delta_{k1}^{DHKR} = -\psi F_L F_{LK} \sigma_e^2 \int_k E_e[u'(c_1)|\, k],$$

$$\Delta_{k2}^{DHKR} = F_{KK} K \int_k E_e[u'(c_1)] \left[\frac{k'(k)}{K} - 1\right].$$

In a human capital economy, there is an additional term $\Delta_{k2,H}$ in Δ_{k2} compared to Δ_{k2}^{DHKR}, and it can change the sign of the redistribution effect Δ_{k2}. That is, in the DHKR economy with a pure transitory income shock, there is only a redistribution effect through wealth heterogeneity ($\Delta_{k2,K}^{DHKR}$), and the sign of the redistribution channel is always positive. In the human capital economy, however, there is also a redistribution effect through human capital heterogeneity ($\Delta_{k2,H}$), whose sign is negative. Thus, depending on which redistribution channel dominates, even Δ_{k2} can be either positive or negative.[14]

We want to remark that if we allow persistent income shock in a DHKR economy, then a redistribution channel similar to $\Delta_{k2,H}$ can exist. Consider labor endowment $L + e_1$, where e_1 follows an AR(1) process: $e_1 = \rho e_0 + \epsilon$, with $\epsilon \sim N(0, \sigma_\epsilon^2)$ and $e_0 \sim N(0, \sigma_e^2)$. Then, the redistribution channel has additional effects $\Delta_{k2,H} = F_{KK} K \int_{k,e_0} E_{e_1}[u'(c_1)|k, e_0][1 - (L + \rho e_0)/L]$. Thus, what is crucial for the existence of the additional channel $\Delta_{k2,H}$ is persistent labor income dispersion. In a human capital economy, however, this new channel can be quantitatively more important compared to the persistent income shock. We discuss this in more detail in Section IIIA.

[13] These two sources of pecuniary externalities were already discussed in DHKR, but we show the decomposition *in additive terms* by exploiting the CARA-normal specification.

[14] By comparing Δ_{k1}^{DHKR} to (7) of the human capital economy, we can see that the introduction of human capital dispersion does not have a direct impact on the insurance channel, and the sign of the insurance effect Δ_{k1} is always negative.

The following proposition summarizes the results and provides the condition for $\Delta_{k2} < 0$ (and thus, $\Delta_k < 0$ and $\Delta_h > 0$), which can be obtained by rewriting Δ_{k2} in (8):

$$\Delta_{k2} = F_{KK} K \beta \left\{ \mathrm{cov}\left(E_e[u'(c_1)|k,h], \frac{k'(k,h)}{K} \right) - \mathrm{cov}\left(E_e[u'(c_1)|k,h], \frac{h'(k,h)}{H} \right) \right\}.$$

PROPOSITION 3: *In a DHKR economy,* $\Delta_{k1}^{DHKR} < 0$ *and* $\Delta_{k2}^{DHKR} > 0$. *However, in a human capital economy,* $\Delta_{k1} < 0$ *and* $\Delta_{k2} < 0$ *if*

$$\mathrm{cov}\left(E_e[u'(c_1)|k,h], \frac{k'(k,h)}{K} \right) > \mathrm{cov}\left(E_e[u'(c_1)|\,k,h], \frac{h'(k,h)}{H} \right).$$

We remark that this proposition still applies in the time-investment model. That is, the effects of introducing human capital on Δ_k do not depend on modeling choice.[15]

DHKR carry out three different calibrations and conclude that in a calibration that generates a realistic wealth dispersion for the United States, the redistribution effect dominates ($\Delta_k > 0$), and thus, there is an under-accumulation of capital in a competitive equilibrium, relative to the constrained efficient allocation. Proposition 3 shows that introducing human capital dispersion can change the implication of pecuniary externalities from under-accumulation to overaccumulation of capital under certain conditions. The relative inequality of capital and human capital is important for the sign of Δ_k, and the correlation between wealth and human capital is also crucial. Thus, in Section II, we analyze how these features affect the sign of Δ_k more explicitly.

D. *Risky Return to Human Capital Investment*

So far, we have analyzed the constrained efficient allocation when there is only an additive shock to human capital investment, and thus, the risk to human capital return is exogenous. However, earnings risk does depend on human capital investment in reality. The evidence regarding whether the risk increases or decreases with human capital investment is conflicting. On the one hand, education seems to increase wage variability.[16] On the other hand, job-specific human capital generated from training tends to make employment stable, and education tends to decrease the risk of being "low-paid" (Chapman 1993, Stewart and Swaffield 1999). In this section, we briefly analyze how the risk properties of human capital affect the implication of pecuniary externalities.

Suppose that the effective supply of labor in period 1 is determined by $f(h', \eta)$, and labor income is $w[f(g(h,x),\eta)]$, where $h' = g(h,x)$ is human capital produced

[15] In an infinite horizon model, period 1 is not the last period; thus, there will be a channel of labor supply dispersion in Δ_k. We discuss this in the online Appendix E.

[16] For example, Palacios-Huerta (2003) shows that human capital investment increases not only the mean but also the variance of the return to human capital in the United States.

in period 1 and η is the productivity shock to human capital. We assume that f is increasing with each argument, but $f_{h'\eta}$ can be positive or negative.[17]

By combining the first-order conditions of the households' problem, we can get the no-arbitrage condition $rE_\eta[u'(c_1)] = wg_x E[f_{h'}(h',\eta)u'(c_1)]$. Then, we can easily show that

$$wg_x E_\eta[f_{h'}] \gtreqless r \quad \Leftrightarrow \quad f_{h'\eta} \gtreqless 0.$$

That is, if $f_{h'\eta} > 0$, human capital increases risk exposure, and thus, risk-averse households invest too little, generating a risk premium for the human capital investment and vice versa.

The first-order conditions of the constrained planner's problem have the same form as in the benchmark economy but with different Δ_k (and $\Delta_h = -(K/H) \cdot \Delta_k$). We assume that $f(g(h, x(k,h)), \eta) \sim N(\mu_h(k,h), \sigma_h^2(k,h))$ for simple exposition, then using CARA utility, we get the following additive decomposition: $\Delta_k = \Delta_{k1} + \Delta_{k2}$, where

$$\Delta_{k1} = -\psi F_H F_{HK} \int_{k,h} E_\eta[u'(c_1)|\,k,h]\,\sigma_h^2(k,h)\,\Gamma(dk,dh),$$

$$\Delta_{k2} = \int_{k,h} E_\eta[u'(c_1)|\,k,h]\left[\frac{k'(k,h)}{K} - 1\right] + \int_{k,h} E_\eta[u'(c_1)|\,k,h]\left[1 - \frac{\mu_h(k,h)}{H}\right].$$

How is Δ_k affected by $f_{h'\eta}$? We can see that whether the human capital investment is a bad hedge ($f_{h'\eta} > 0$) or a good hedge ($f_{h'\eta} < 0$) has ambiguous effects on the sign of pecuniary externalities because the insurance channel (Δ_{k1}) and the redistribution channel through human capital dispersion ($\Delta_{k2,H}$) respond in opposite directions.

Note that the insurance effect is determined by the endogenous human capital risk $\sigma_h^2(k,h) = \mathrm{var}_\eta(f(h'(k,h),\eta))$, which depends on endogenous human capital investment and the distribution. If $f_{h'\eta} > 0$, then Δ_{k1} is likely to be lower than that in the exogenous risk case ($f_{h'\eta} = 0$) because the variance of the effective labor supply $\mathrm{var}_\eta(f(h',\eta))$ is increasing in human capital h'. However, the redistribution effects through human capital dispersion also depend on the endogenous risk to human capital because the dispersion of human capital is affected by the risk. Since exposure to risk is even higher for the human capital rich, they will decrease human capital investment more, and thus, the dispersion of human capital will become smaller, which will increase $\Delta_{k2,H}$ (mitigate the redistribution channel through human capital dispersion). That is, the redistribution channel can undo the effects of increasing risk on the insurance channel.

[17] The additive shock ($f(h',\eta) = h' + \eta$) we considered in the benchmark model is the special case with $f_{h'\eta} = 0$. If we consider a multiplicative shock ($f(h',\eta) = \eta h'$) as in Huggett, Ventura, and Yaron (2011), then $f_{h'\eta} = 1$. More generally, with a CES function ($f(h',\eta) = [\alpha_1 h'^{1-\rho} + \alpha_2 \eta^{1-\rho}]^{\frac{1}{1-\rho}}$), the sign of $f_{h'\eta}$ is determined by the sign of ρ.

In the study of optimal taxation with human capital policy, the sign of $f_{h'\eta}$ usually has more direct implications for the design of a human capital subsidy.[18] However, the constrained planner we consider can only change the relative factor prices to internalize pecuniary externalities, and the effects of $f_{h'\eta} \gtrless 0$ will only depend on whether the insurance effects dominate the redistribution effects.

E. *Comparison with Ex Ante Inequality and Labor Supply*

Endogenous Dispersion versus Ex Ante Inequality.—The previous analysis shows that introducing endogenous human capital dispersion can lower the optimal capital-labor ratio by introducing an additional channel of redistribution. Human capital inequality, however, can be introduced without endogenous human capital investment (e.g., ex ante initial human capital inequality). Indeed, the additional redistribution channel through human capital dispersion $\Delta_{k2,H}$ exists, regardless of whether the human capital dispersion is due to ex ante inequality or due to endogenous human capital investment. For example, if the human capital distribution is exogenously determined by the ex ante distribution of human capital, then the constrained planner's Euler equation will be the same as that of the benchmark except that $\Delta_{k2,H}$ is now determined by the ex ante distribution of human capital: $\Delta_{k2,H} = F_{KK}K\beta\int_{k,h}E_e[u'(c_1)|k,h][1-h/H]$.

However, *endogenous* human capital dispersion differs from the ex ante inequality in two aspects. First, when human capital is endogenous, not only capital but also human capital can change prices. Thus, human capital is also under or over-accumulated when human capital is endogenous, while exogenous human capital inequality only affects Δ_k. Second, and related to the first, Δ_k and the efficient level of savings in an economy with endogenous human capital will be different from those in an economy with ex ante inequality even if the two economies exhibit the same allocation (same distribution of wealth and human capital) in a competitive equilibrium. The equivalent equilibrium distributions in the two economies will have the same implications for whether increasing savings can improve welfare.[19] However, the additional optimality condition of the planner with Δ_h will lead to a different level of optimal savings in a human capital economy.[20]

[18] For example, Stantcheva (2017) shows that the sign of the net human capital subsidy (net of all wedges to undo distortions coming from other wedges) is exclusively determined by the complementarity between human capital and risk. More precisely, the net human capital subsidy is nonnegative when the Hicksian coefficient of complementarity $\left(\frac{f_{h'\eta}f}{f_{h'}f_\eta}\right)$ is equal to or smaller than one.

[19] The equivalent distribution implies the same $\Delta_k|^{CE}$ in both economies, where $\Delta_k|^{CE}$ is Δ_k evaluated with a competitive equilibrium allocation, which measures the welfare effects of increasing K at the competitive equilibrium.

[20] Although the effects of *endogenous* human capital on Δ_k and the efficient level of saving are ambiguous, we can make some predictions. For example, if Δ_k is positive and Δ_h is negative, the constrained planner wants to decrease investment in human capital for all households but more so for the consumption rich, who are relatively human capital rich. Thus, human capital inequality is likely to be lowered, then the redistribution channel through human capital dispersion $(\Delta_{k2,H})$ will be muted, which will increase Δ_k compared to that in the economy with ex ante heterogeneity. However, if Δ_k is negative and Δ_h is positive, human capital inequality is likely to be raised, which can decrease Δ_k further.

Human Capital versus Endogenous Labor Supply.—We can also endogenize labor earnings by introducing an endogenous labor supply. How does the endogenous human capital work differently from the endogenous labor supply for the implication of pecuniary externalities? To answer this question, we consider a model with a labor supply but no human capital. In period 1, the household chooses consumption (c_1) and labor (l) after the realization of productivity shock e. Then, the budget constraint of the household in period 1 is $c_1(k,e) = w(l(k,e)+e) + rk'(k)$, and for simplicity, consider an additively separable utility in period 1: $u(c) - v(l)$, where $v' > 0, v'' > 0$, and u takes the form of CARA.

The Euler equation of the constrained planner has the exact same form as in the benchmark economy, but we now have the intratemporal condition of the planner instead of the intertemporal condition with respect to human capital investment: $F_L u'(c_1(k,e)) + \Delta_L = v'(l(k,e))$, where$\Delta_L = -(K/L)\cdot\Delta_K$ as in the human capital economy. Moreover, Δ_K is determined by $\Delta_K = \int_k E_e[u'(c_1)]\{F_{LK}E_e \times [l(k,e)\,|\,k] + F_{KK}k'(k) - \psi F_L F_{LK}\sigma_e^2\}$, and thus, $\Delta_k = \Delta_{k1} + (\Delta_{k2,K} + \Delta_{k2,L})$ also has an additional term similar to $\Delta_{k2,H}$, $\Delta_{k2,L} = F_{KK}K\int_k E_e[u'(c_1)|k]\{1 - E_e[l(k,e)|\,k]/L\}$.

However, the sign of $\Delta_{k2,L}$ is likely to be different from the sign of $\Delta_{k2,H}$. Recall that the sign of the redistribution channel $\Delta_{k2,H}$ is negative because of the human capital poorness of the consumption poor. However, the consumption poor with low $k'(k)$ is likely to supply more labor because of income effects. That is, $E_e[l(k,e)\,|\,k]$ is likely to be higher for households with lower $k'(k)$ whose marginal utility of consumption is relatively high, which leads to $\Delta_{k2,L} > 0$. Thus, different from the dispersion of human capital, the additional channel through labor dispersion might increase Δ_k and optimal capital-labor ratio.[21]

II. Key Determinants of the Pecuniary Externalities

We showed that the introduction of human capital dispersion generates an additional channel of pecuniary externalities, which lowers the optimal capital-labor ratio, whose effects depend on the joint distribution of wealth and human capital. In this section, we want to further investigate the key determinants of the pecuniary externalities in terms of the shape of the distribution. We show that poor households' relative human capital poorness and the cross correlation between wealth and human capital are crucial.

A. *Human Capital Inequality and the Consumption Poor's Income Sources*

From the decomposition analysis previously mentioned, we could already conjecture that higher human capital inequality can lower Δ_k and lower the optimal capital-labor ratio because the new redistribution channel through human capital dispersion becomes stronger ($\Delta_{k2,H}$ becomes lower). But many different distributions

[21] If the productivity shock e is persistent and takes the form of a multiplicative shock with labor income $wel(k,e)$, then the sign of $\Delta_{k2,L} > 0$ becomes ambiguous because there is a high correlation between k and e, and lower e can lead to relatively lower $E_e[el(k,e)\,|\,k]$.

can have the same measure of inequality— variance, Gini, or coefficient of variance. What is crucial for the sign of Δ_{k2} is the consumption poor's relative poorness in human capital. Recall that the sign of Δ_{k2} is determined by the weighted sum of $k'/K - h'/H$ whose weight is $E_e[u'(c_1(k', h', e))]$. Thus, the consumption poor's relative poorness between k'/K and h'/H is more important than the consumption rich's relative richness. See the online Appendix B for a formal analysis.

Thus, human capital inequality and the consumption poor's relative poorness in human capital are important for the sign of Δ_{k2}. In the model, these statistics are sensitive to the complementarity (substitutability) between existing human capital and new investment $g_{hx} > (<)\ 0$. Positive g_{hx} generates higher human capital inequality and makes the consumption poor (with lower h) more human capital poor due to the relatively lower productivity of human capital production and lower investment.[22] The empirical evidence on the complementarity is conflicting. Cunha, Heckman, and Schennach (2010) find estimates for the elasticity of substitution between early stocks of human capital and investments consistent with complementarity between early human capital and investments. This is called *dynamic complementarity* in the labor literature. However, there are studies on heterogeneous interventions that show the least advantaged children benefit the most (e.g., Bitler, Hoynes, and Domina 2014; and Havnes and Mogstad 2011).

B. *Correlation between Human Capital and Wealth*

We now analyze the effect of the increasing correlation on pecuniary externalities for a given wealth inequality and earnings inequality. In this section, we focus on the local analysis—how the correlation in a competitive equilibrium affects $\Delta_k|_{CE}$ (Δ_k in (5) evaluated in a competitive equilibrium), which is the welfare effect of increasing a small amount of capital in the (status quo) competitive equilibrium. We focus on this local analysis because we are interested in how the *empirical* correlation changes the implications of the pecuniary externalities. If the constrained planner's problem is globally concave in k' and x, however, the sign of $\Delta_k|_{CE}$ will be sufficient for the sign of Δ_k.[23]

We maintain the two-period CARA-normal specification with additive risk for simplicity. Then, $\Delta_k|_{CE}$ can be rewritten with respect to period-1 capital and period-1 human capital in a competitive equilibrium:

$$(9)\quad \Delta_k|_{CE} = \int_{k',h'} E_e\big[u'(c_1(k',h',e))\big]\left\{-\psi F_H F_{HK}\sigma_e^2 + F_{KK}K\left[\frac{k'}{K} - \frac{h'}{H}\right]\right\}\Gamma_1(dk',dh'),$$

[22] The same mechanism will be generated by the inequality in learning ability that will be introduced in the full model.

[23] Under the globally concave problem, if a small increase (decrease) in capital at the competitive equilibrium improves welfare, then the efficient capital-labor ratio will be higher (lower) than that of the competitive equilibrium.

where $\Gamma_1(k', h')$ is the joint distribution of period-1 wealth (k') and human capital (h') in a competitive equilibrium.

We assume that $F(K, H) = K^{\alpha} H^{1-\alpha}$, and we also assume that $\Gamma_1(k', h')$ has a joint uniform distribution with discrete uniform marginals, and the associated density $\gamma_1(k', h')$ is given by

$$\gamma_1(k', h') = \begin{cases} \frac{1}{2}\rho & \text{if } (k', h') = (k_1, h_1) \\ \frac{1}{2}(1-\rho) & \text{if } (k', h') = (k_1, h_2) \\ \frac{1}{2}(1-\rho) & \text{if } (k', h') = (k_2, h_1) \\ \frac{1}{2}\rho & \text{if } (k', h') = (k_2, h_2) \end{cases}, \tag{10}$$

where $k_1 = (1-\theta_k)K$, $k_2 = (1+\theta_k)K$, $h_1 = (1-\theta_h)H$, $h_2 = (1+\theta_h)H$, $\theta_k, \theta_h > 0$. Notice that the marginal distributions of k' and h' are discrete uniform with $\Pr(h_i) = \Pr(k_i) = 1/2$ for $i = 1, 2$. The wealth Gini and the human capital Gini computed from this distribution are $\theta_k/2$ and $\theta_h/2$, respectively, and the correlation between k' and h' is $2\rho - 1$.

The next proposition presents the effect of the increasing correlation ρ on the insurance channel $(\Delta_{k1}|_{CE})$ and the redistribution channel $(\Delta_{k2}|_{CE})$ for fixed θ_k and θ_h. Let us emphasize that this analysis does not depend on the modeling choice of human capital. In the time-investment model, $\Delta_k|_{CE}$ will have exactly the same formula as in (8), and the ensuing analysis goes through.

PROPOSITION 4: *Suppose that* $\gamma_1(k', h')$ *is given by* (10) *with* $\rho \in (0, 1)$. *Then,*

(*i*) $\frac{\partial \Delta_{k1}|_{CE}}{\partial \rho} < 0$;

(*ii*) *If* $\theta_k > \theta_h$ *and* $\frac{\theta_k}{\theta_h} \leq \frac{1-\alpha}{\alpha}$, *then* $\frac{\partial \Delta_{k2}|_{CE}}{\partial \rho} > 0$.

PROOF:

See the online Appendix G.

The first part of Proposition 4 shows that for fixed wealth inequality and human capital inequality, an increase in the correlation between wealth and human capital strengthens the insurance channel (lowers Δ_{k1}). The second part of Proposition 4 shows that if the wealth inequality is higher than the human capital inequality and the gap is not too large—which is relevant to the data because $\theta_k/\theta_h = \text{K-Gini}/\text{H-Gini} \leq (1-\alpha)/\alpha \approx 2$ usually holds in reality,[24] then the increase in the correlation between wealth and human capital will strengthen the redistribution channel through wealth dispersion (increase Δ_{k2}).

The intuition for (ii) of Proposition 4 is the following. If the consumption poor are mostly wealth poor, then the redistribution channel through wealth dispersion

[24] The conditions are only sufficient, not necessary.

will dominate the redistribution channel through human capital dispersion, implying that increasing the K/H-ratio is welfare improving. As the correlation coefficient between wealth and human capital (ρ) increases, the wealth poor tend to also be human capital poor, which makes them more consumption poor, reinforcing the redistribution channel through wealth dispersion (increasing $\Delta_{k2,K}$).[25]

Although an increase in correlation will drive the Δ_{k1} and Δ_{k2} in opposite directions, we can expect that its impact on redistribution (Δ_{k2}) will be stronger. This is because the impact on the redistribution channel is a more direct effect through the increase in the correlation between consumption inequality and wealth inequality, while the impact on the insurance channel merely comes from the accompanying consumption dispersion when the correlation increases.

Next, we provide a numerical example to show the quantitative importance of the correlation coefficients on the implications of pecuniary externalities. We assume that the period-1 distribution of wealth and human capital is a bivariate lognormal distribution:

$$\left(\log\left(\frac{k'}{K}\right), \log\left(\frac{h'}{H}\right)\right) \sim N\left(\begin{pmatrix}\mu_{lk}\\ \mu_{lh}\end{pmatrix}, \begin{pmatrix}\Sigma_{kk} & \Sigma_{kh}\\ \Sigma_{hk} & \Sigma_{hh}\end{pmatrix}\right),$$

where $K = E[k']$ $(\mu_{lk} = -0.5\,\Sigma_{kk})$ and $H = E[h']$ $(\mu_{lh} = -0.5\,\Sigma_{hh})$.

Note that the correlation coefficient of wealth and earnings $(\mathrm{corr}(k', h' + e))$ is $(e^{\Sigma_{kh}} - 1)/\left(\sqrt{e^{\Sigma_{kk}} - 1} \cdot \sqrt{e^{\Sigma_{hh}} - 1}\right)$. Thus, for fixed Σ_{kk} and Σ_{hh}, the correlation coefficient is determined by Σ_{kh}, and by varying Σ_{kk}, Σ_{hh}, and Σ_{kh}, we can match various combinations of the wealth Gini, the human capital Gini, and the correlation coefficient between wealth and earnings.[26]

We set the capital income share α $(F(K,H) = K^{\alpha}H^{1-\alpha})$ to $1/3$ and choose $\beta = 0.96$ and $\delta_k = 0.08$.[27] The mean of human capital is normalized to 1 and the mean of capital is chosen so that the annual interest rate is 4 percent. We fix $\sigma_e^2 = 0.1$ and choose Σ_{kk} and Σ_{hh} to match the wealth Gini (0.8) and the earnings Gini (0.6). Then, we investigate the effect of increasing the correlation coefficient between wealth and earnings for a fixed wealth Gini and earnings Gini by changing Σ_{kh}.

In panel A of Figure 1, we display $\Delta_k|_{CE}$ by the correlation coefficient for three different values of the risk-aversion parameter, ψ (1, 1.5, 2). We can see that as the correlation coefficient increases, $\Delta_k|_{CE}$ increases. More important, panel A shows that $\Delta_k|_{CE}$ changes the sign from negative to positive as the correlation coefficient increases, which implies that two economies with the same wealth Gini and human capital Gini can have opposite evaluations as to whether the capital-labor ratio is too high, depending on the size of the correlation. Panel B of Figure 1 displays the

[25] DHKR also briefly mention the possibility of a stronger redistribution channel with higher correlation when discussing the effect of a serially correlated earnings shock (on page 2440).

[26] The wealth Gini of this lognormal distribution is $2\Phi(\sqrt{\Sigma_{kk}}/\sqrt{2}) - 1$, where Φ is the cumulative distribution function of the standard normal. Thus, the wealth Gini is determined by Σ_{kk}. On the other hand, the earnings Gini of this economy depends on both the variance of the e-shock (σ_e^2) and the variance of relative human capital $(\mathrm{var}(h'/H) = \exp(\Sigma_{hh}) - 1)$, because earnings in this economy are defined by $w(h' + e)$.

[27] We think of period 1 in this numerical example as the last period of a long-horizon model. That is, the model period of this numerical example is one year, and we set the parameters accordingly.

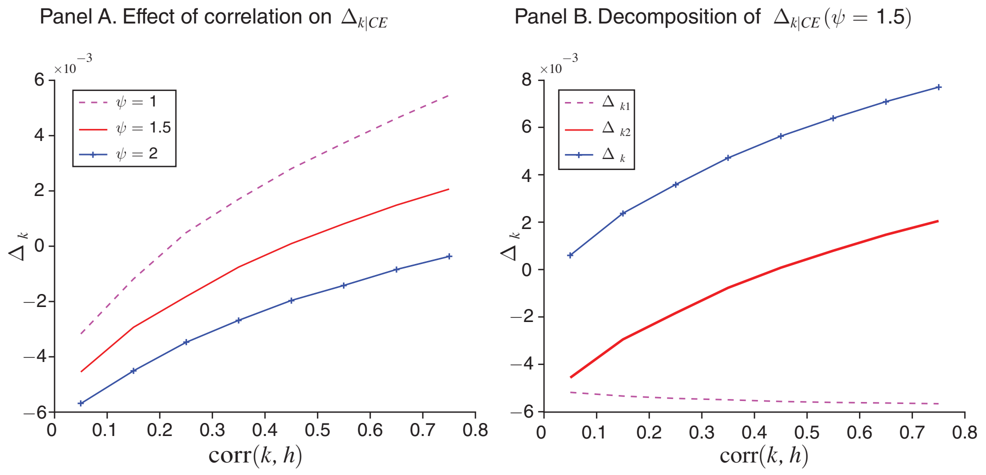

FIGURE 1. EFFECTS OF CORRELATION ON $\Delta_k|_{CE}$

decomposition of $\Delta_k|_{CE}$ into an insurance channel ($\Delta_{k1}|_{CE}$) and a redistribution channel ($\Delta_k|_{CE}$) for $\psi = 1.5$ and shows that the effect of the correlation on $\Delta_k|_{CE}$ is driven by a stronger redistribution channel effect, as we showed in Proposition 4 with the simple uniform distribution.

In the United States, the correlation coefficient between wealth and earnings has been significantly increasing. Díaz-Giménez, Quadrini, and Ríos-Rull (1997) report that the correlation coefficient in 1992 measured by the Survey of Consumer Finances (SCF) was 0.23, and Díaz-Giménez, Glover, and Ríos-Rull (2011) report that the correlation coefficient in 2007 measured by the SCF was 0.48. If we take the previously mentioned numerical example, the sign of $\Delta_k|_{CE}$ based on the 1992 statistics is negative (implying the overaccumulation of capital), while the sign of $\Delta_k|_{CE}$ based on 2007 statistics is positive (implying the under-accumulation of capital), implying that the evaluation of the overaccumulation of capital in a competitive equilibrium has changed in the United States. This numerical example also sheds some light on the cross-country analysis. Countries that have very low correlation coefficients, such as Sweden and Spain, are likely to exhibit an overaccumulation of capital in a competitive equilibrium.

III. Infinite Horizon Model

In this section, we quantitatively examine the role of human capital in the implication of pecuniary externalities, using the infinite horizon model calibrated to actual data (wealth Gini and earnings Gini).

In this economy, the preferences of infinitely lived households are represented by $E_0\{\sum_t \beta^t u(c_t)\}$, where the period utility u has the constant relative risk aversion (CRRA) form, $c^{1-\sigma}/(1-\sigma)$. A household with invariant learning ability A that has human capital h_t in period t and invests money x_t will have human capital h_{t+1} at period $t+1$:

$$h_{t+1} = \eta_{t+1} g(h_t, x_t, A),$$

where η is an idiosyncratic shock and the human capital production technology is given by (see, e.g., Ben-Porath 1967)

$$g(h,x,A) = (1-\delta_h)h + A(xh)^{\phi}.$$

The only source of risk the household faces, η, is i.i.d. across agents and time and is drawn from the lognormal distributions every period:

$$\log\eta \sim N(\mu_\eta, \sigma_\eta^2),$$

where $E[\eta] = 1$. There is initial heterogeneity in learning ability. In period 0, A is drawn from a lognormal distribution:

$$\log A \sim N(\mu_A, \sigma_A^2).$$

The period budget constraint of the household is

$$c_t + k_{t+1} + x_t = k_t(1+r) + wh_t, \quad \forall t.$$

As the budget constraint shows, households can insure against idiosyncratic shocks by accumulating capital or human capital. The households supply one unit of labor, and thus, the effective labor supply of an individual with h_t human capital is h_t, and earnings in period t are wh_t.

Competitive firms have a production function $f(K_t, L_t)$, where K_t is aggregate capital and L_t is aggregate effective labor. We assume that f is a Cobb-Douglas function $f(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$. The depreciation rate for capital is denoted by δ_k. Due to competitive firms, wages and interest rates are determined by the marginal productivity of labor and the marginal productivity of capital, respectively: $w = f_L(K,L)$, $r = f_K(K,L) - \delta_k$.

We now state the household's problem recursively. The state variables of the individual household are $(k,h,A) \in S = \mathcal{K} \times \mathcal{H} \times \mathcal{A}$. The probability measure Φ is defined over the Borel sets of S. The law of motion of the distribution is written as $\Phi' = F(\Phi)$.

Then, the household's problem is

$$v(\Phi,k,h;A) = \max_{c,k',x} u(c) + \beta E_{\eta'} v(\Phi', k', \eta' g(h,x,A);A)$$

subject to

$$c + k' + x = k[1 + r(\Phi)] + hw(\Phi) \quad \text{and} \quad \Phi' = F(\Phi).$$

The solution of the household's problem is $c(\Phi,k,h;A)$, $k'(\Phi,k,h;A)$, $x(\Phi,k,h;A)$. The intertemporal conditions the solution should satisfy are

$$u'(c) \;\geq\; \beta(1+r(\Phi'))\,E_{\eta'}\big[u'(c')\big],$$

$$u'(c) \;\geq\; g_x(h,x,A)\,\beta E_{\eta'}\left[\eta'\left\{w(\Phi') + \frac{g_h(h',x',A)}{g_x(h',x',A)}\right\}u'(c')\right],$$

where the equality holds when $k' \geq 0$ and $x \geq 0$, respectively.[28]

Using the decision rules $k'(\Phi,k,h;A)$, $h'(\Phi,k,h,\eta';A) = \eta' g(h,x(\Phi,k,h;A),A)$, a transition function is computed as follows:

$$Q(\Phi,k,h,A,B;k',x) \;=\; \int_{\eta'}\chi_{k'(\Phi,k,h;A)\in B_k}\cdot\chi_{h'(\Phi,k,h,\eta';A)\in B_h}\cdot\chi_{A\in B_A}\cdot f(\eta')\,d\eta',$$

where $f(\eta')$ is the lognormal density of η', and χ is an indicator function. Given this transition function, we define the updating operator $T(\Phi,Q)$ as $\Phi'(B) = T(\Phi,Q)(B) = \int_S Q(\Phi,k,h,A,B;k',x)\,d\Phi$. Then, a recursive competitive equilibrium is defined in a standard manner. See the online Appendix C for the formal definition.

We now characterize the allocation of the constrained planner's problem:

$$\Omega(\Phi) = \max_{\substack{k'(k,h;A),\\ x(k,h;A)}} \int_S u\big(k(1+f_K(\Phi)-\delta_k) + hf_L(\Phi) - k'(k,h;A) - x(k,h;A)\big)\,d\Phi + \beta\Omega(\Phi')$$

subject to

$$\Phi' \;=\; T(\Phi,Q(\,\cdot\,;k',s_h)).$$

The first-order conditions of the constrained planner's problem are as follows. See the online Appendix C for derivation and formal representation:

$$u'(c(k,h;A)) \;\geq\; \beta(1+r(\Phi'))\,E_{\eta'}\big[u'\big(c'(k,h,\eta';A)\big)\big] + \beta\Delta_k,$$

where the inequality becomes an equality if $k'^*(k,h;A) \;>\; 0$, and

$$u'(c(k,h;A)) = g_x(h,x^*,A)\beta E_{\eta'}\left[\eta' u'(c'(k,h,\eta';A))\left\{w(\Phi') + \frac{g_h(h',x',A)}{g_x(h',x',A)}\right\}\right]$$

$$+\, g_x\beta\Delta_h,$$

[28] $x \geq 0$ does not bind, since $\lim_{x\to 0} g_x(h,x,a) = \infty$.

where

$$\Delta_k = \int_S u'\big(c'(k',h';A)\big)\big[k' f_{KK}(K',L') - h' f_{LK}(K',L')\big]\, d\Phi',$$

$$\Delta_h = -\frac{K(\Phi')}{H(\Phi')}\Delta_k, \quad \text{and} \quad K(\Phi') = \int k' d\Phi', \quad H(\Phi') = \int h' d\Phi'.$$

We can see that the optimality conditions of the constrained efficient allocation in an infinite horizon model are the dynamic generalization of those in the two-period model, (5) and (6). The term Δ_k is still determined by the magnitude of the insurance channel and the two redistribution channels of the pecuniary externalities. In an infinite horizon model, the redistribution channel in period t refers to the effects of redistributing according to the states in period t before realization of the η-shock in period t, while the insurance channel refers to the effects of redistributing according to the shock in period t.

In the online Appendix E, we briefly analyze the time-investment model in an infinite horizon economy to show that the sources of pecuniary externalities and the relationship between Δ_k and Δ_h are the same as in the money-investment model with slight modification. However, the implications of the sign of Δ_k on the level of human capital investment require more investigation as discussed in the online Appendix E.

A. *Quantitative Analysis*

We now quantitatively investigate the optimal capital-labor ratio compared to the ratio in a competitive equilibrium. We focus on the steady-state capital-labor ratio. It is important, however, to notice that this steady-state analysis is not the result of maximizing steady-state welfare, but the result of maximizing ex ante lifetime welfare, including the transition to the steady state.

Using the model calibrated to generate a realistic dispersion of wealth and human capital, we quantitatively show that despite the introduction of risky human capital that generates a persistent labor income dispersion, the capital-labor ratio in a competitive equilibrium is lower than that of the constrained optimum, as in DHKR. But the magnitude of the capital-labor ratio in a constrained optimum is smaller than that of DHKR.

The calibrated parameters are summarized in Table 1. The parameters for preference and technology are standard. The relative risk aversion σ is 2, and the discount factor β is calibrated to place the equilibrium interest rate at 4 percent, as the calibration of DHKR (originally Díaz, Pijoan-Mas, and Ríos-Rull (2003)) does. The capital share α of the Cobb-Douglas production function $(K^\alpha L^{1-\alpha})$ is set to 0.36, and the depreciation rate of capital is set to 0.08, which are standard and equal to that of DHKR.

In the baseline model, the return to scale, ϕ, of the human capital production function is set to 0.4,[29], and the depreciation rate of human capital δ_h is set to 0.04. Next, we do the sensitivity analysis for δ_h and ϕ.

[29] In a money-investment model, to guarantee the existence of a stationary state, ϕ should be smaller than 0.5. However, in a time-investment model with the fixed amount of time in each period, the stationary allocation can be

Table 1—Calibrated Parameter Values

Category	Symbol	Parameter value
Preference	$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$	$\sigma = 2$
	β	$\beta = 0.9406$
Technology	$Y = K^{\alpha} L^{1-\alpha}$	$\alpha = 0.36$
	δ_k	$\delta_k = 0.08$
Human capital	$h' = \eta g(h, x, A)$	$\phi = 0.4$
	$g(h, x, A) = (1-\delta_h)h + A(xh)^{\phi}$	$\delta_h = 0.04$
Human capital shock	$\log \eta \sim N(\mu_\eta, \sigma_\eta^2)$	$\sigma_\eta = 0.111$
Initial heterogeneity	$\log A \sim N(\mu_A, \sigma_A^2)$	$(\mu_A, \sigma_A) = (0.069, 0.148)$

The parameters of the human capital shock process are (μ_η, σ_η), and the parameters of the distribution of the initial learning ability are (μ_A, σ_A). The standard deviation of the shock to human capital σ_η and the standard deviation of the initial learning ability σ_A are important parameters for the earnings dispersion. Parameter σ_η is set to 0.111, which is the estimate of Huggett, Ventura, and Yaron (2011), and σ_A is calibrated to 0.148, which generates an earnings Gini index of 0.60—the target of calibration of DHKR (originally Díaz, Pijoan-Mas, and Ríos-Rull (2003)). Further, μ_η is set to guarantee $E[\eta] = 1$, and μ_A is normalized to set the steady-state output without a shock (first best output) to 1.

Before presenting the constrained efficient allocation, we discuss some features of the competitive equilibrium. The competitive equilibrium of this economy has large precautionary savings. As a result, the aggregate capital-labor ratio is 1.32 times as large as that of an economy without shocks. Additionally, this economy can generate a Gini index of wealth of 0.804, which is close to the US data. This high wealth Gini can be obtained even without assuming an extreme income process, which is an interesting by-product of calibrating the human capital model. This is mainly because of the concave human capital production function. The rich can smooth out consumption against shocks by investing either in capital or in human capital. The concavity of human capital production function, however, shows that the return of the marginal human capital investment is decreasing as investment increases, thus the rich save their income by investing in physical capital once they reach the amount of human capital investment that achieves the no-arbitrage condition. As well summarized in Castañeda, Díaz-Giménez, and Ríos-Rull (2003), the literature has made a lot of attempts to account for wealth inequality. Our quantitative analysis shows that having a human capital structure can be helpful for generating wealth inequality.

The steady-state capital-labor ratio of the constrained efficient allocation is higher than that of the competitive equilibrium, maintaining DHKR's conclusion—the under-accumulation of capital. The capital-labor ratio of the constrained optimum

achieved even with $\phi > 0.5$, and the estimate of ϕ ranges from 0.5 to over 0.9 in the literature (Browning, Hansen, and Heckman 1999).

Table 2—Steady State for the Infinite Horizon Economy

	First best	CE	Constrained optimum
Capital-labor ratio	4.223	5.564	15.036
Capital-output ratio	2.514	3.000	5.667
Interest rate (percent)	6.32	4.00	−1.65
Wealth Gini	—	0.804	0.777
Earnings Gini	—	0.600	0.348
Bottom 20% $\frac{k}{K}/\frac{h}{H}$	—	0.014[a]	0.012[b]
corr(k, h)	—	0.746	0.607

[a] $k/K = 0.001, h/H = 0.059$.
[b] $k/K = 0.004, h/H = 0.309$.

is 3.56 times as large as that of the first best[30] and 2.70 times larger than that of the competitive equilibrium. Thus, the capital-labor ratio in the competitive equilibrium is too low relative to the capital-labor ratio in the constrained efficient allocation, implying an under-accumulation of capital. In DHKR's (Section VIB) economy with an exogenous labor income shock, the capital-labor ratio of a constrained efficient allocation is 8.5 times larger than that of the deterministic economy and 3.65 times larger than that of the competitive equilibrium. Thus, the degree to which the capital-labor ratio in a competitive equilibrium is lower than the optimal ratio in an economy with human capital is smaller than that in an economy with an exogenous labor income shock.

The steady-state aggregate capital (K) of the constrained efficient allocation is 1.53 times larger than that of the competitive equilibrium, while the aggregate human capital (L) is 1.77 times smaller than that of the competitive equilibrium. Thus, this human capital economy exhibits the under-accumulation of capital and the overaccumulation of human capital in the competitive equilibrium.

Based on our analysis in Section III, we can see why the pecuniary externalities imply a too low capital-labor ratio at a competitive equilibrium. Despite human capital dispersion, the consumption poor are relatively wealth poor. That is, the relative wealth poorness (k/K) of the bottom 20 percent of consumption-poor households is 0.001, which is much smaller than their human capital poorness $(h/H = 0.059)$.[31] Moreover, the correlation between wealth and human capital is 0.746. These features are crucial features for the under-accumulation of capital as we discussed in the theoretical analysis.

The efficient capital-labor ratio is lower in a human capital economy than that in DHKR's quantitative analysis mainly because of the redistribution channel through human capital dispersion $(\Delta_{k2,H})$, which lowers Δ_k. As we already discussed in Section IC, a persistent shock in a DHKR economy can also generate a channel similar to $\Delta_{k2,H}$, but its effect is much smaller because it is merely determined by

[30] We consider the first best from the perspective of the planner who decides allocation before the realization of both ability and shocks.

[31] In the US data (Díaz-Giménez, Glover, and Ríos-Rull (2011)), the relative wealth poorness (k/K) of the bottom 20 percent is about 0.013 (which is higher than that of the model), but the relative earnings poorness (h/H) of the twentieth percentile of the earnings distribution is zero (which is lower than that of the model). The zero earnings of the large number of households are due to the retirees and the disabled households.

the dispersion of the persistent part of labor income driven by a three-state Markov process. The dispersion of labor income implied by the three states is smaller than that of the human capital economy,[32] and the dispersion is scaled down by the persistence coefficient.[33] Moreover, in a human capital economy, the complementarity between existing human capital and investment and the dispersion in learning ability makes the consumption poor invest in human capital less and makes them much poorer in human capital compared to the exogenous income shock case. Thus, the $\Delta_{k2,H}$ channel is much more strong in a human capital economy.

Notice that the correlation between wealth and human capital of the efficient allocation (0.607) is much lower than that in the competitive equilibrium. This is because when the planner distributes additional saving, he or she mandates even more additional saving to the wealth rich compared to the addition to the wealth poor, while a decrease in human capital investment for the human capital rich is more than the decrease for the human capital poor. This lower correlation can mitigate the redistribution effects of pecuniary externalities through less consumption dispersion and thus can improve welfare. Another finding is that inequalities in both wealth and human capital of the constrained efficient allocation are lower than those of a competitive equilibrium.[34]

B. *Robustness*

To see how robust our result is, we change some key parameters for the human capital distribution to compare the constrained efficient capital-labor ratio. Table 3 shows the results for various depreciation rates of human capital δ_h and the return to scale of the human capital production function ϕ. We recalibrated the model for each set of δ_h and ϕ so that the generated earnings Gini is 0.60 and the interest rate is 4 percent. Thus, the capital-labor ratio in a competitive equilibrium is identical to that in the baseline model.

First, the implication of pecuniary externalities—a too low capital-labor ratio in a competitive equilibrium—is robust to changing these parameters. We can see that the optimal capital-labor ratio is not very sensitive to these parameters. This is because for these calibrations of the model, the consumption-poor households are relatively wealth poor and the correlation between wealth and human capital is high, which generates severe redistribution effects of pecuniary externalities.

Another notable observation is that for all these calibrations, the correlation of the constrained efficient allocation is much lower than the correlation in a competitive equilibrium, and a lower correlation at the optimum tends to lead to a relatively lower capital-labor ratio at the optimal allocation.

[32] In a DHKR economy, the poorness in labor income e/L of the most consumption-poor household is 0.18, while that in a human capital economy (h/H) is nearly zero, despite the same earnings Gini.

[33] To generate high wealth dispersion, DHKR assumes that there is a significant risk of a large fall in labor earning, which lowers persistence.

[34] The earnings Gini of the constrained efficient allocation becomes lower because a negative Δ_h has a stronger negative impact on the human capital accumulation of households with lower marginal utility (higher consumption) who tend to have higher earnings. The wealth Gini becomes lower because a decrease in the interest rate lowers wealth inequality, and this interest rate effect dominates the stronger positive Δ_k effects on the wealth rich.

Table 3—Sensitivity Analysis

		δ_h			ϕ	
		0.02	0.04	0.06	0.4	0.45
Optimal	Capital-labor ratio	16.69	15.04	14.70	15.04	12.19
	Interest rate (percent)	−2.06	−1.65	−1.56	−1.65	−0.74
Optimal	Bottom 20% $\frac{k}{K}/\frac{h}{H}$	0.006	0.012	0.013	0.012	0.006
	corr(k, h)	0.72	0.61	0.60	0.61	0.46
CE	Bottom 20% $\frac{k}{K}/\frac{h}{H}$	0.002	0.014	0.027	0.014	0
	corr(k, h)	0.80	0.75	0.70	0.75	0.73

In Section IIA, we discussed that human capital inequality and the consumption poor's relative poorness in human capital are sensitive to the complementarity between existing human capital and new investment. However, in order to match the realistic earnings Gini in the model, if we assume substitutability $(g_{hx} < 0)$, then we should set σ_A much higher. Then, the inequality mechanism generated by the complementarity is replaced by a very similar inequality-generating mechanism through the heterogeneity in learning ability, whose impact on Δ_k is not very significant. We also did a sensitivity analysis by changing the combination of (σ_η, σ_A) to match the same earnings Gini, but the result is not sensitive to the parameters as long as the model generates the same Gini. See the online Appendix D for details.

In our baseline and sensitivity analysis, however, the models calibrated to realistic earnings and wealth Ginis tend to generate very high correlations between wealth and human capital in a competitive equilibrium (higher than the US data). Our theoretical analysis implies that this high correlation might bias the evaluation toward a too low capital ratio in a competitive equilibrium. In the next section, we examine a model that can generate a lower correlation to see whether the implication of pecuniary externalities is changed.

C. *Model with Low Correlation between Wealth and Earnings*

We consider an economy where there is a subsidy to human capital investment, but the subsidy is decreasing in wealth. That is, we assume that the cost of investing in human capital x after getting a subsidy is $q(x,k) = (1 - \tau^{-k-\zeta})x$, which generates a higher cost of investing in human capital for wealth-rich households, and thus, the correlation between wealth and human capital can be lower than in the baseline economy.

Calibrating the model to all of the targets of the wealth Gini, the earnings Gini, and the correlation is a very difficult task, and we do not attempt to match them. As a numerical example, we set (τ, ζ) to $(1.05, 21)$ so that the subsidy rate $\tau^{-k-\zeta}$ is about 0.36 for the zero-wealth households, while it is 0.01 for the top 1 percent wealth-rich households. Then, the correlation between wealth and human capital generated from this model is 0.54, much lower than that of the baseline model (0.75). As shown in Table 4, however, this model generates a lower earnings Gini and a higher wealth Gini than those of the baseline model because a progressive human capital subsidy reduces earning inequality, and wealth inequality increases due to substitution between capital and human capital.

Table 4—Model with Low Correlation

	Baseline		Subsidy	
	CE	Optimal	CE	Optimal
Capital-labor ratio	5.56	15.04	5.56	6.21
Interest rate (percent)	4.00	−1.65	4.00	3.19
Wealth Gini	0.804	0.777	0.835	0.931
Earnings Gini	0.600	0.348	0.575	0.583
Bottom 20% $\frac{k}{K}/\frac{h}{H}$	0.014	0.012	0.033	0.049
Correlation	0.75	0.61	0.54	0.21

Table 4 compares the results from the human capital subsidy model to the results from the baseline model. Since the subsidy model is also calibrated to match an interest rate of 4 percent, the capital-labor ratio in a competitive equilibrium is not changed. We can see that the optimal capital-labor ratio is 6.21, which is 1.1 times larger than that in a competitive equilibrium. That is, the constrained efficient capital-labor ratio is still higher than that of competitive equilibrium, but the degree of under-accumulation is much smaller. The consumption poor are still relatively wealth poor, but since these households receive a lot of subsidies for human capital investment, their human capital poorness is improved. Moreover, these households can save more due to positive income effects (as can be seen from the relatively higher $(k/K)/(h/H)$), and thus, the consumption poor's relative consumption poorness is improved, which mitigates the pecuniary externalities through redistribution. In sum, lower correlation improves the consumption inequality, which weakens the redistribution channel of pecuniary externalities.

IV. Policy Implications

In this section, we discuss the policy implications of the pecuniary externalities in an economy with human capital. First, we briefly talk about the implementation of the constrained efficient allocation. Second, and more important, we relate the pecuniary externalities to existing policy analysis.

A. *Implementation of the Constrained Efficient Allocation*

If the government only cares about the pecuniary externalities as the constrained planner in our analysis does, then the government can implement the constrained efficient allocation by taxing (subsidizing) the capital income and the return to human capital whenever there is overaccmulation (under-accumulation) and giving transfers so that individual budgets are maintained, which is explicitly shown in online Appendix A.

In practice, however, a tax-transfer system is set up taking into consideration many other things, such as the equity-efficiency trade-off, insurance provision, and crowding in/out of the private market. We now discuss the policy implications of the pecuniary externalities when the government has other considerations.

B. *Relation to Existing Policy Analysis and Effects of Other Distortions*

How can we relate our analysis to the existing analysis on optimal tax and human capital policy? In all studies in the optimal taxation literature, an interesting trade-off of the policy depends on the precise restrictions imposed on the government policy,[35] and these restrictions are crucial for the optimal tax results. The goal of the constrained planner in this paper is restricted to internalizing pecuniary externalities only, but this channel of pecuniary externalities exists as a part of the government's goals in any optimal tax studies regardless of the tool of the government as long as there is a missing market against idiosyncratic uncertainty in a general equilibrium.

This paper is, in particular, focused on the pecuniary externalities in the one-sector neoclassical growth model, where changes in the relative prices of physical capital and labor are the key. This exact channel of pecuniary externalities operates as part of constituting the tax-transfer system in many previous studies on optimal tax and human capital policy. In Ramsey tax literature, Krueger and Ludwig (2013, 2016), Peterman (2016), and Gottardi, Kajii, and Nakajima (2015) analyze the optimal tax using a neoclassical growth model with human capital accumulation, and part of the taxes in their studies are set to internalize pecuniary externalities. However, in the Mirrleesian tax literature, many studies that analyze the optimal tax with human capital assume an exogenous interest rate (small open economy).[36] Thus, in their wedges, this pecuniary externality channel is missing. But the pecuniary externalities channel can be added if these analyses are extended to a general equilibrium.

In the applied tax studies with other goals of the government (arising from policy tool restrictions), there are other distortions—a labor wedge, savings wedge, and human capital wedge. These distortions can change the implication of the pecuniary externalities because these distortions change the magnitude of the insurance and redistribution channels. We now investigate the effects of other distortions on pecuniary externalities.

Effects of Progressive Income Tax and Saving Distortions.—The first example is the effect of progressive income taxation. Let's consider a simple two-period model with only an additive shock, but we keep the ex ante heterogeneity in learning ability A, thus $h' = g(h,x,A)$ with $g_{xA} > 0$. We consider a very simple version of a progressive labor income tax in period 1: $T(wh') = \tau wh' - d$, where $d = \int \tau wh' d\Phi = \tau wH$. This is a progressive tax system because the average tax net of transfers $t(wh') = (\tau w(h'-H))/(wh')$ is increasing in labor income wh', and the progressivity of the tax is increasing with τ.[37]

We can show that human capital inequality is affected by the progressivity of the income tax. By applying an implicit function theorem to the no-arbitrage condition

[35] For example, in the Mirrleesian tax literature, the main restriction is that the government cannot condition a tax-transfer system on the unobservable income-generating abilities of workers. In the Ramsey literature, a more explicit restriction—a functional form restriction—is imposed on the tool of the government.

[36] For example, da Costa and Maestri (2007), Kapička and Neira (2015), and Stantcheva (2017) study optimal policy with human capital in an economy with exogenous prices. In the Mirrleesian tax literature, Grochulski and Piskorski (2010) is the exception.

[37] With this tax system, the household's budget constraints are $c_0 = a(k,h) - x - k'$, $c_1 = (1-\tau)wg(h,x,A) + \tau wH + rk'$.

$r = w(1-\tau)\, g_x(h,x,A)$, we can easily get $\partial x/\partial\tau = g_x/((1-\tau)\, g_{xx}) < 0$, which leads to $\partial h'/\partial\tau = g_x(\partial x/\partial\tau) < 0$.[38] Moreover, increasing τ will decrease h' inequality because the distortions created by the progressive income tax τ differ across learning ability levels. For a household with low A, human capital investment is very low even without taxes (because of a low marginal product $g_x(h,x,A)$), and thus, distortions from the tax are low, while households with high A face relatively high distortions.

The planner's first-order conditions that internalize the pecuniary externalities still have additional terms Δ_k and $\Delta_h = -(K/H)\cdot\Delta_k$, but Δ_k is affected by the progressivity of the tax. With CARA-normal specification, $\Delta_k = \Delta_{k1} + \Delta_{k2}$ is represented as follows:

$$\Delta_{k1} = -(1-\tau)^2 \psi F_H F_{HK} \sigma_e^2 \beta \int_{k,h} E[u'(c_1)|k,h]\, d\Gamma,$$

$$\Delta_{k2} = \underbrace{F_{KK} K\beta \int_{k,h} E[u'(c_1)]\left\{\frac{k'(k,h)}{K} - 1\right\}}_{=\Delta_{k2,K}} + \underbrace{(1-\tau)\, F_{KK} K\beta \int_{k,h} E[u'(c_1)]\left\{1 - \frac{h'(k,h)}{H}\right\}}_{=\Delta_{k2,H}}.$$

When the progressivity (τ) increases, there is a direct effect of decreasing $(1-\tau)^2$ in the insurance channel. A progressive tax reduces income uncertainty, implying a weaker insurance channel, which increases Δ_{k1}. Higher progressivity is also likely to lead to a weaker redistribution channel through human capital dispersion (higher $\Delta_{k2,H}$) because of two effects: there is a direct effect of decreasing $1-\tau$ in $\Delta_{k2,H}$,[39] and higher τ decreases h' inequality.

In sum, the distortions generated by the progressive income taxes tend to increase Δ_k and decrease Δ_h, meaning that the part of capital income taxes due to the pecuniary externalities will be lowered and the part of human capital subsidy due to the same externalities will be lowered.

We can also discuss the effects of savings distortion on pecuniary externalities. For example, consider a linear capital income tax rate τ_k. The term Δ_k (and Δ_h) can be decomposed as usual, and we can see that there is a direct effect of mitigating the redistribution channel through wealth dispersion: $\Delta_{k2,K} = (1-\tau_k)\, F_{KK} K \int E_e[u'(c_1)]\left[k'/K - 1\right]$, which reflects that increasing K can decrease the interest rate only by $(1-\tau_k)\, F_{KK}$. Thus, capital income taxes imposed by any other goals of the government lower Δ_k, and thus, part of the capital income tax due to pecuniary externalities should be increased (and the part of the human capital subsidy due to pecuniary externalities should be raised) if the direct effect is dominating other indirect effects through the change in allocation.

[38] In the time-investment model, we need to model progressivity as an increasing marginal tax rate with income to get the same result—human capital decreases when progressivity increases.

[39] Increasing K can raise the wage rate only by $(1-\tau)\, F_{HK}$.

We want to remark that these effects of other distortions on pecuniary externalities are different from the compensating channel in the optimal taxation literature—part of the human capital subsidy is just to compensate for other distortions. Pecuniary externalities in an incomplete market exist regardless of other distortions, but the magnitude of each insurance and redistribution channel depends on other distortions. The directions are actually the opposite to those of the compensating channels. For example, progressive labor taxation discourages human capital investment, and thus, there should be a compensating subsidy to undo these distortions, while the part of the subsidies due to pecuniary externalities is lowered due to progressive taxation. In the case of introducing capital income taxation, the human capital investment could be distorted upward,[40] and the compensating channel wants to decrease the human capital subsidy to undo this distortion, while the part of the subsidy due to pecuniary externalities is raised.

Effects of Means-Tested Human Capital Subsidy and Joint Tax System.—The last example is the effects of a means-tested human capital subsidy, which was already employed in the quantitative analysis. As we discussed earlier, the means-tested human capital subsidy lowers the correlation between cross-sectional wealth and human capital, which tends to decrease Δ_k and decrease the optimal capital-labor ratio.

We can think of other policies that can decrease the correlation between wealth and human capital. For example, consider a joint tax system that is taxing capital income and labor income jointly. If the labor income tax rate is higher for people with higher capital income, then this joint tax system can also decrease the correlation between wealth and human capital, which leads to a lower Δ_k and a lower optimal capital-labor ratio.

V. Conclusion

We have analyzed the implication of pecuniary externalities in an economy with endogenous human capital dispersion. Analytically, we show that introducing human capital dispersion could lead to the evaluation that the capital-labor ratio in a competitive equilibrium is too high when an economy with a pure transitory labor shock has an opposite evaluation—a too low capital-labor ratio. The consumption poor's relative poorness in terms of human capital and the lower correlation of wealth and human capital are crucial for this implication—too high capital-labor ratio. A quantitative investigation calibrated to the realistic wealth inequality and human capital inequality, however, shows that even if we introduce human capital accumulation, the capital-labor ratio in a competitive equilibrium is lower than that of a constrained efficient allocation in the long run.

We suggest several directions for future research. This paper shows the importance of the joint distribution of wealth and earnings for the implication of pecuniary externalities. Developing a model with more structure to seriously match all

[40] This is because human capital investment allows agents to transfer resources to the future without being subject to saving taxes.

important statistics and quantifying pecuniary externalities might be an interesting topic for future research. Our analysis also shows that pecuniary externalities in a time-investment model require further investigation on the elasticity of the effective labor supply with respect to education and training. Finally, analyzing pecuniary externalities with different market structures and investigating the impact of human capital are interesting research questions.

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