# Conditional Choice Probabilities and the Estimation of Dynamic Models, RES, 1993

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## 1. Introduction

This paper presents a new estimator for dynamic discrete choice models, called the Conditional Choice Probability (CCP) estimator. Our approach does not require econometricians to explicitly solve the valuation functions used to characterize optimal decision rules via backwards recursion methods. It is based on a new representation of the valuation function which is expressed in terms of the utility payoffs, choice probabilities, and probability transitions of choices and outcomes that remain feasible in future periods. Under conditions presented below, this representation can be exploited to estimate the model's structural parameters by employing non-parametric estimates of these future choice probabilities and probability transitions in place of their true values.

## 2. The Framework

Consider a typical agent making choices over time in an uncertain environment. She is assumed to choose one action from a set, $\mathscr{J}$, which contains up to $J$ alternatives at each period, $t$, over a finite life of length $T$. Her objective is to maximize the expected value of a sum of period-specific payoffs or utilities. Let $d_{tj} = 1$ indicate the agent chooses action $j$ in $t$, and setting $d_{tj} = 0$ means she does something else. Then $\boldsymbol{d}_t = \left(d_{t1}, \dots, d_{t,J-1}\right)'$ describes her action in period $t$. Alternatively expressed:

$$
\begin{aligned}
& d_{tj} \in \{0,1\} \text{ for all } (t,j) \in T \times \mathscr{J}, \\
& \sum_{j=1}^{J} d_{tj} = 1 \text{ for all } t \in T.
\end{aligned}
\tag{2.1}
$$

The action taken at period $t$ typically affects the outcome, $b_t \in \mathscr{B}$, which arrives at the end of the period. Let $H_t = \left(\boldsymbol{b}_0', b_1, \dots, b_{t-1}\right)'$ represent the agent's history as of the beginning of period $t$; it

[*]This note is written during my M.Phil. study at the University of Oxford.

includes a $L \times 1$ vector of the agent's initial endowment of characteristics, $\boldsymbol{b}_0 \in \mathscr{B}_0$, and the agent's entire history of outcomes from period 1 through $t-1$. We assume that $\mathscr{B}$ is a finite set, but only that $\mathscr{B}_0$ is compact. The outcome, $b_t$, is either fully determined by action $\boldsymbol{d}_t$, or generated according to the transition probabilities [1]:

$$F_j \left(H_{t+1} \mid H_t\right), \tag{2.2}$$

where $H_{t+1} = (H_t, b_t)'$. In the context of econometric analysis, such distribution functions, as well as the agent's payoff functions described below, may be specified in terms of a vector of structural parameters.

In each period $t$, there is a current utility or payoff, $u_{tj}$, associated with each choice $j$. Let $u_j^* \left(H_t\right) \equiv \mathbb{E}\left(u_{tj} \mid H_t\right)$ denote the conditional expectation of $u_{tj}$, given $H_t$. It follows that:

$$u_{tj} = u_j^* \left(H_t\right) + \varepsilon_{tj}, \tag{2.3}$$

where the stochastic utility component, $\varepsilon_{tj}$ is, by construction, conditionally independent of $H_t$. Let $\boldsymbol{u}^* \left(H_t\right) = \left(u_1^* \left(H_t\right), \ldots, u_J^* \left(H_t\right)\right)'$ and $\boldsymbol{\varepsilon}_t = \left(\varepsilon_{t1}, \ldots, \varepsilon_{tJ}\right)'$, respectively, denote the $J \times 1$ vectors of deterministic and stochastic utility components. We write the distribution function of $\boldsymbol{\varepsilon}_t$ as:

$$G \left(\boldsymbol{\varepsilon}_t \mid H_t\right) \tag{2.4}$$

and assume it has a well-defined, joint probability density function, $dG \left(\boldsymbol{\varepsilon}_t \mid H_t\right)$.

The agent sequentially chooses $\{\boldsymbol{d}_t\}_{t \in T}$ to maximize the objective function:

$$E_0 \left( \sum_{t=0}^{T} \sum_{j=1}^{J} d_{tj} \left[ u_j^* \left(H_t\right) + \varepsilon_{tj} \right] \right). \tag{2.5}$$

Let $\boldsymbol{d}_s^0 = \left(d_{s1}^0, \ldots, d_{s,J-1}^0\right)'$ denote the agent's optimal choice in period $s$. We define the conditional valuation function associated with choosing $j$ in period $t$ as:

$$v_j \left(H_t\right) = E \left[ \sum_{s=t+1}^{T} \sum_{j=1}^{J} d_{sj}^0 \left[ u_{sj}^* + \varepsilon_{sj} \right] \mid H_t, d_{tj} = 1 \right], \tag{2.6}$$

where $u_{tj}^* \equiv u_j^* \left(H_t\right)$ is adopted for notational simplicity. Optimal decision making implies that $d_{tk}^0 = 1$ if and only if:

$$k = \text{argmax}_{j \in \mathscr{J}} \left[ u_{tj}^* + \varepsilon_{tj} + v_{tj} \right], \tag{2.7}$$

where $v_{tj} \equiv v_j(H_t)$. Conditional on history $H_t$, the probability the agent chooses action $k$ is therefore:

$$p_k \left(H_t\right) = \Pr \left\{ k = \text{argmax}_{j \in \mathscr{J}} \left[ u_{tj}^* + \varepsilon_{tj} + v_{tj} \right] \mid H_t \right\}. \tag{2.8}$$

Let $\boldsymbol{p} \left(H_t\right) \equiv \left(p_1(H_t), \ldots, p_{J-1}(H_t)\right)'$ denote the $(J-1)$ dimensional vector of conditional choice prob-

[1] Since $H_t$ is a vector of length $t$, there is an argument subscripting $F$ (as well as many other mappings in this paper) by $t$.

abilities associated with the first $J-1$ actions in period $t$.

# 3. An Alternative Representation of Conditional Valuation Functions

In general, the conditional valuation function, $v_j(H_t)$, does not have a closed form solution. The standard practice is to exploit Bellman equation and use backward recursion methods to obtain one. This section provides an alternative representation of $v_j(H_t)$, which will prove convenient when estimating a parametric representation of such models.

Note that (2.3) and (2.8) imply that the conditional probability of making choice 1, say, can be written as:

$$\begin{aligned} p_1(H_t) &= \mathbb{E}\left(d_{t1}^0 = 1 \mid H_t\right) \\ &= \int_{\varepsilon_1=-\infty}^{\infty} \int_{\varepsilon_2=-\infty}^{\varepsilon_1+u_{t1}^*+v_{t1}-u_{t2}^*-v_{t2}} \cdots \int_{\varepsilon_J-=\infty}^{\varepsilon_1+u_{t1}^*+v_{t1}-u_{tJ}^*-v_{tJ}} dG\left(\varepsilon_1,\ldots,\varepsilon_J \mid H_t\right) \\ &= \int_{-\infty}^{\infty} G_1\left(\varepsilon_1,\left[\varepsilon_1+u_{t1}^*-u_{t2}^*+v_{t1}-v_{t2}\right],\ldots,\left[\varepsilon_1+u_{t1}^*-u_{tJ}^*+v_{t1}-v_{tJ}\right] \mid H_t\right) d\varepsilon_1, \end{aligned} \tag{3.1}$$

where

$$G_j(\boldsymbol{\varepsilon}_t \mid H_t) = \frac{\partial G(\boldsymbol{\varepsilon}_t \mid H_t)}{\partial \varepsilon_{tj}}.$$

Equation (3.1) holds true because up to normalization, $p_1(H_t)$, the integrand of the last line is the probability density function for $\varepsilon_{t1}$, given history $H_t$ and $\left[u_{t1}^*+v_{t1}+\varepsilon_{t1} \geq u_{tk}^*+v_{tk}+\varepsilon_{tk}\right]$ for all $k \in \mathscr{J}$ (or in words, when the first choice is optimal).

For each $k \in \mathscr{J}$, the expression corresponding to (3.1) is a positive, real-valued, mapping from the differences in conditional valuation functions associated with the optimal choice and the alternative actions. We now show that these differences can be expressed as functions of conditional choice probabilities.

Let $\boldsymbol{s} = \left(s_1,\ldots,s_{J-1}\right)'$ be a $(J-1)$-dimensional vector. For each $t \in T$ and $j \in \{1,\ldots,J-1\}$, define the real-valued function, $Q_j(\boldsymbol{s}, H_t)$, as :

$$\begin{aligned} &Q_j(\boldsymbol{s}, H_t) \\ &= \int G_j\left(\left[\varepsilon+u_{tj}^*-u_{t1}^*+s_j-s_1\right],\ldots,\left[\varepsilon+u_{tj}^*-u_{t,j-1}^*+s_j-s_{j-1}\right],\varepsilon,\left[\varepsilon+u_{tj}^*-u_{t,j+1}^*+s_j-s_{j+1}\right],\ldots,\left[\varepsilon+u_{tj}^*-u_{tJ}^*+s_j\right] \mid H_t\right) d\varepsilon \end{aligned} \tag{3.2}$$

and $\boldsymbol{Q}(\boldsymbol{s}, H_t)$, a $(J-1)$ dimensional vector function, as:

$$\boldsymbol{Q}(\boldsymbol{s}, H_t) = \left(Q_1(\boldsymbol{s}, H_t),\ldots,Q_{J-1}(\boldsymbol{s}, H_t)\right)'. \tag{3.3}$$

If $\underline{\boldsymbol{v}}$ comprises the differences in conditional valuation functions, namely,

$$\underline{\boldsymbol{v}} = \left(v_{t1}-v_{tJ},\ldots,v_{t,J-1}-v_{tJ}\right)' \equiv \underline{\boldsymbol{v}}(H_t), \tag{3.4}$$

then $\boldsymbol{p}(H_t) = \boldsymbol{Q}\left(\underline{\boldsymbol{v}}(H_t), H_t\right)$ [2]. The cornerstone of our estimation strategy is to express $\underline{\boldsymbol{v}}(H_t)$ as a function of $\boldsymbol{p}(H_t)$. This requires $\boldsymbol{Q}\left(\underline{\boldsymbol{v}}(H_t), H_t\right)$ to be invertible in $\underline{\boldsymbol{v}}(H_t)$. By the following proposition, its inverse exists.

**Proposition 1.** For each $H_t$, the mapping $\boldsymbol{Q}\left(\underline{\boldsymbol{v}}, H_t\right)$ is invertible in $\underline{\boldsymbol{v}}$.

Proposition 1 enables one to express $v_j(H_t)$ in terms of the choice probabilities, transition probabilities and expected (per period) payoffs associated with future histories. To demonstrate this, we proceed in several steps. First, the expected optimal per-period payoff in period $t$, condition on history $H_t$, is:

$$\mathbb{E}\left(\sum_{j=1}^{J} d_{tj}^{0}\left[u_j^*(H_t) + \varepsilon_{tj}\right] \mid H_t\right) = \sum_{j=1}^{J} p_{tj}\left[u_j^*(H_t) + \mathbb{E}\left(\varepsilon_{tj} \mid H_t, d_{tj}^0 = 1\right)\right], \tag{3.5}$$

where $p_{tj} \equiv p_j(H_t)$. To express $\mathbb{E}\left(\varepsilon_{tj} \mid H_t, d_{tj}^0 = 1\right)$ as a function of conditional choice probabilities, Proposition 1 implies that the (unnormalized) conditional density function, appearing in (3.1), can be written as:

$$\begin{aligned}
&G_j\left(\left[\varepsilon + u_{tj}^* - u_{i1}^* + v_j - v_1\right], \ldots, \left[\varepsilon + u_{tj}^* - u_{i,j-1}^* + v_j - v_{j-1}\right], \varepsilon, \left[\varepsilon + u_{tj}^* - u_{t,j+1}^* + v_j - v_{j+1}\right], \ldots, \left[\varepsilon + u_{ij}^* - u_{ij}^* + v_j\right] \mid H_t\right) \\
&= G_j\left(\left[\varepsilon + u_{tj}^* - u_{t1}^* + Q_{tj}^{-1} - Q_{t1}^{-1}\right], \ldots, \left[\varepsilon + u_{tj}^* - u_{t,j-1}^* + Q_{tj}^{-1} - Q_{t,j-1}^{-1}\right], \varepsilon, \left[\varepsilon + u_{ij}^* - u_{t,j+1}^* + Q_{tj}^{-1} - Q_{t,j+1}^{-1}\right], \ldots, \left[\varepsilon + u_{tj}^* - u_{tj}^* + Q_{tj}^{-1}\right] \mid H_t\right)
\end{aligned} \tag{3.6}$$

where $v_j \equiv v_{tj}(H_t)$, $Q_{tj}^{-1} \equiv Q_j^{-1}\left(\boldsymbol{p}_t, H_t\right)$, and $\boldsymbol{p}_t \equiv \left(p_{t1}(H_t), \ldots, p_{t,J-1}(H_t)\right)'$. Therefore, the expectation of $\varepsilon_{tj}$, when $j$ is the optimal action for history $H_t$, is[3]:

$$\begin{aligned}
&W_j(\boldsymbol{p}_t, H_t) \\
&\equiv \mathbb{E}\left(\varepsilon_{tj} \mid H_t, d_{tj}^0 = 1\right) \\
&= \int \frac{\varepsilon G_j\left(\left[\varepsilon + u_{tj}^* - u_{t1}^* + Q_{tj}^{-1} - Q_{t1}^{-1}\right], \ldots, \left[\varepsilon + u_{tj}^* - u_{t,j-1}^* + Q_{tj}^{-1} - Q_{t,j-1}^{-1}\right], \varepsilon, \left[\varepsilon + u_{ij}^* - u_{t,j+1}^* + Q_{tj}^{-1} - Q_{t,j+1}^{-1}\right], \ldots, \left[\varepsilon + u_{tj}^* - u_{tj}^* + Q_{tj}^{-1}\right] \mid H_t\right)}{p_j(H_t)} d\varepsilon.
\end{aligned} \tag{3.7}$$

Using (3.5) and (3.7), it follows that the agent's expected utility in period $t$, conditional on $H_t$, is:

$$U\left(\boldsymbol{p}_t, H_t\right) = \sum_{k=1}^{J} p_{tk}\left[u_{tk}^* + W_k\left(\boldsymbol{p}_t, H_t\right)\right]. \tag{3.8}$$

To complete the representation of $v_j(H_t)$, we characterize the sequences of choices and state variables which would be feasible for the agent in future periods if she was to choose action $j$ in period $t$. Let $\mathscr{A}_s(H_t)$ denote the set of histories which remain feasible for some age $s$ following $t$, given history $H_t$ :

$$\mathscr{A}_s(H_t) = \left\{H_s : H_s = (H_t, b_t, \ldots, b_{s-1})'\right\}. \tag{3.9}$$

[2]This is actually quite a large gap in the derivation. To understand this, we can go back to the interpretation of equation (3.1) and note that $v_{t1} - v_{t2} = (v_{t1} - v_{tJ}) - (v_{t2} - v_{t2})$. And when we define $Q_1$, we actually use notations $s_1, \ldots, s_{J-1}$, where $s_j$ actually means $v_{tj} - v_{tJ}$.

[3]The most difficult derivation is the last line of equation (3.7). This is actually a problem of computing conditional expectation. We need to have a better understanding of the interpretation of the integrand in the equation (3.1).

Denote the conditional choice probabilities associated with this finite set of possible histories by the vector set, $p(H_t)$:

$$\overline{p}\,(H_t) = \left\{\boldsymbol{p}\,(H_s) : H_s \in \mathscr{A}_s\,(H_t) \text{ for } s = t+1,\ldots,T\right\}. \tag{3.10}$$

Then the agent's conditional value function for $d_{tj} = 1$ is given by the sum (over the periods $s \in \{t+1,\ldots,T\}$ and over histories which might eventuate, $H_s \in \mathscr{A}_s\,(H_t)$) of the associated expected payoffs, $U_s \equiv U(\boldsymbol{p}_s, H_s)$, times the probability of each $H_s$ occurring. The probability of $H_s$ occurrring, conditional $H_t$ and $d_{tj} = 1$, is, in turn, given by the product of the relevant conditional choice and state transition probabilities:

$$F_j\,(H_{t+1} \mid H_t) \prod_{r=t+1}^{s-1} \left[\sum_{k=1}^{j} p_{rk} F_k\,(H_{r+1} \mid H_r)\right], \tag{3.11}$$

where $H_s \in \mathscr{A}_s\,(H_r)$ for each $r \in \{t+1,\ldots,s-1\}$. Adopting the abbreviated notation $F_{rk} \equiv F_k(H_{r+1|H_r})$, we have thus established that one can represent any conditional valuation function as a real valued mapping of the current history and future conditional choice probabilities; denoting this new representation by $V_j\,(H_t, p(H_t))$, it is defined as:

$$V_j\left(H_t, \overline{p}\,(H_t)\right) = \sum_{s=t+1}^{T} \sum_{H_s \in \mathscr{A}_s(H_r)} \left\{U_s \left[F_{tj} \prod_{r=t+1}^{s-1} \left(\sum_{k=1}^{J} p_{rk} F_{rk}\right)\right]\right\}. \tag{3.12}$$

It immediately follows that $V_j\left(H_t, \overline{p}\,(H_t)\right) = v_j\,(H_t)$.

Representation (3.12) can be further simplified in some circumstances. Consider histories where there exists at least one action, which, if taken next period, eliminates the differential impact of any subsequent choices on outcomes. Such histories are said to be terminal histories. More precisely, a history $H_t$ is said to be terminal if and only if there exists at least one action, say $J \in \mathscr{J}$, (called a terminating action) which, if chosen in $t+1$, must be picked for all periods $s \in \{t+2,\ldots,T\}$. A search model where agents cannot change jobs exemplifies dynamic discrete choice models with this terminal history property. Suppose the $j$-th action is a terminating action associated with some history $H_t$. Then the indirect utility associated with this action at all $H_{t+1} \in \mathscr{A}_{t+1}(H_t)$ simplifies to:

$$V_J(H_{t+1}) = \sum_{s=t+2}^{T} \sum_{H_s \in \mathscr{A}_s(H_{t+1})} \mathbb{E}\left(u_{sJ} \mid H_s\right) \left[\prod_{r=t+1}^{s-1} F_J(H_{r+1} \mid H_r) \mid H_s \in \mathscr{A}_s(H_r)\right] \tag{3.13}$$

and it follows from Proposition 1 that the conditional valuation functions at time $t$ associated with the remaining choices may be expressed as:

$$\begin{aligned} &V_j\left(H_t, \overline{p}\,(H_t)\right) \\ &\sum_{H_{t+1} \in \mathscr{A}_{t+1}(H_t\}} \left\{U_{t+1} + V_J\,(H_{t+1}) + \sum_{k=1}^{J-1} p_{t+1,k} v_{t+1,k}\right\} F_j\,(H_{t+1} \mid H_t) \\ &\sum_{H_{t+1} \in \mathscr{A}_{t+1}} (H_t) \left\{U_{t+1} + V_J\,(H_{t+1}) + \left(\sum_{k=1}^{J-1} p_{t+1,k} Q_k^{-1}\left(\boldsymbol{p}_{t+1}, H_{t+1}\right)\right)\right\} F_j\,(H_{t+1} \mid H_t). \end{aligned} \tag{3.14}$$

Equation (3.14) shows that if $H_t$ is a terminal history, then $V_j\big(H_t, \overline{p}(H_t)\big)$ is a function of the values taken on by $\boldsymbol{p}(H_{t+1})$ and $V_J(H_{t+1})$ as $H_{t+1}$ ranges over the elements in the set $\mathscr{A}_{t+1}(H_t)$. Note that (3.13) implies that the conditional probabilities, associated with future choices beyong $t+1$, do not ener the expressions for valuations of period $t$ choices. Consequently, the existence of terminal states greatly reduces the number of future choice probabilities required to calculate conditional valuation functtions.

The new representation of conditional valuation functions has two uses: one in forming orthogonality conditions for estimation purposes and the other for interpreting the comparative dynamics associated with changes in the state variables. We conclude this section with a brief discussion of the latter. From the definition of $\boldsymbol{Q}(\boldsymbol{v}, H_t)$ and (3.12), it follows that:

$$\boldsymbol{p}(H_t) = \boldsymbol{Q}\big(\boldsymbol{v}\big(H_t, p(H_t), H_t\big)\big). \tag{3.15}$$

Differentiating with respect a continuous component in $\boldsymbol{b}_0$, say $b_{01}$, yields a taxonomy of the various contribution pieces:

$$\frac{\partial \boldsymbol{p}(H_t)}{\partial b_{01}} = \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{v}_t}\left(\left.\frac{\partial \boldsymbol{v}_t}{\partial b_{01}}\right|_{d\boldsymbol{p}_t=0} + \frac{\partial \boldsymbol{v}_t}{\partial p(H_t)}\frac{\partial p(H_t)}{\partial b_{01}}\right) + \left.\frac{\partial \boldsymbol{Q}_t}{\partial b_{01}}\right|_{d\boldsymbol{v}=0} \tag{3.16}$$

The last term in (3.16) captures changes in $\boldsymbol{p}(H_t)$ due to the impact of $b_{01}$ on current utility, holding constant the conditional valuation function. The effect is through two channels: one through the effects of $b_{01}$ on $\boldsymbol{u}^*(H_t)$, the deterministic components of current utility and the other through on $FG(\boldsymbol{\varepsilon}_t \mid H_t)$, the probability distribution of $\boldsymbol{\varepsilon}_t$. The other way in which $b_{01}$ affects $\boldsymbol{p}(H_t)$ is through its impact on expected future utility, namely, $\boldsymbol{v}_t$. Again, this overall effect (the first term in (3.16)) consists of the influence of $b_{01}$ through several channels: one through its effect on $\boldsymbol{v}_t$, holding $\boldsymbol{p}(H_t)$ constant and, the other, through its effect on future choice probabilities $\boldsymbol{p}(H_t)$, characterizing $U\big(\boldsymbol{p}_t, H_t\big)$ and, ultimately, $V_j\big(H_t, \overline{p}(H_t)\big)$. The latter effects reflect the changes in the probabilities ofreaching future nodes and adjustments to the dynamic selection correction terms in future periods.

# 4. An Example: Optimal Stopping

Before discussing the CCP estimator in detail, we consider the form of $V_j\big(H_t, \overline{p}(H_t)\big)$ for a simple optimal stopping model to illustrate the content of Proposition 1.

Suppose a couple must decide when, over the course of their lifetime, to permanently sterilize and no longer be at risk to bear children. Prior to sterilizing, births occur according to (an exogenous) stochastic process. Each period, the couple receives a level of utility which depends on the number of their offspring; this payoff reflects the balance between the satisfaction derived from and the costs associated with rearing these children. The example ignores other forms of heterogeneity across couples; hence, $\mathscr{B}_0$ can be ignored in this illustration.

In terms of the notation developed above, let $d_{t1} = 1$ if the couple does not sterilie in period $t$ and $d_{t2} = 1$ if it sterilizes. Because $J = 2$, $d_{t2} = 1 - d_{t1}$. Since sterilization is assumed irreversible and available at any $t$, every history $H_t$ is terminal; thus, $d_{t1} = 0$ implies $d_{s1} = 0$ for all $s \in \{t, \ldots, T\}$. The period of the couple's lifetime in which sterilization takes place, $\tau$, is called the stopping time (for childbearing).

The outcomes in this model are births. We let $b_s = 1$ if a child is born in period $s$, and let $b_s = 0$ otherwise. For simplicity, we assume a birth occurs at the end of period $t$ to unsterilized women with probability $\alpha \in (0,1]$. That is,

$$F_1\left((H_t, 1) \mid H_t\right) = \alpha. \tag{4.1}$$

Sterilized women cannot bear children, so:

$$F_2\left((H_t, 1) \mid H_t\right) = \alpha. \tag{4.2}$$

We assume that the nonstochastic component of the woman's utility in any period $t$ depends only on the number of existing children. Consequently, this problem has a finite state space; the couple only has to keep track of periods since their marriage and current family size in making contraception decisions. Let $\widetilde{H}_t$ denote family size at $t$. In terms of the couple's birth history, $\widetilde{H}_t = H_t' \iota$, where $\iota$ denotes a $t \times 1$ vector of ones, or more simply:

$$\widetilde{H}_t \equiv \sum_{s=1}^{t} b_{t-s} \tag{4.3}$$

For the sake of illustration, let the couple's current utility be quadratic in $\widetilde{H}_t$:

$$u_1^*\left(H_t\right) = u_2^*\left(H_t\right) = \beta^t \left(\delta_1 \widetilde{H}_1 + \delta_2 \widetilde{H}_t^2\right) \tag{4.4}$$

where $\beta \in (0,1)$ is the discount factor. The choice specific idiosyncratic component associated with each action, $\varepsilon_{tj}$ is assumzed to be iid across $(t, j)$ as a Type I Extreme Value random variable with location parameter 0. The couple's decision problem, then, is to sequentially choose $\{d_s\}_{s=0}^{T}$ (or, equivalently the optimal stopping time $\tau$) which maximizes:

$$\mathbb{E}_0 \left[ \sum_{t=0}^{T} \beta^s \left(\delta_1 \widetilde{H}_s + \delta_2 \widetilde{H}_s^2 + d_s \varepsilon_{s1} + (1 - d_s)\, \varepsilon_{s2}\right) \right] \tag{4.5}$$

To characterize the optimal decision rule for this model, we need the conditional valuation functions of each action $j \in [1,2]$. Because sterilization is a terminating action, the value of setting $d_t = 0$ is just the expected dicounted utility derived from the stock of existing children, $\widetilde{H}_t$. The above assumptions imply this value is:

$$v_2\left(H_t\right) = \beta' \left(\gamma + \delta_1 \widetilde{H}_t + \delta_2 \widetilde{H}_t^2\right) / \left(1 - \beta^{T-1}\right) \tag{4.6}$$

where $\gamma$ is Euler's constant. The discounted value of not sterilizing in period $t$, and remaining fertile at least one more period is:

$$v_1(H_t) = \max_{\{d_s\}_{s=t+1}^{T}} E\left[\sum_{s=t+1}^{T} \beta^s \left(\delta_1 \widetilde{H}_s + \delta_2 \widetilde{H}_s^2 + d_s \varepsilon_{s1} + (1-d_s)\varepsilon_{s2}\right) \mid H_1\right]. \tag{4.7}$$

Aside from $\varepsilon_{t1}$ and $\varepsilon_{t2}$, the only difference in expected utility from the two actions the parents can take is due to the value of birth. Therefore, the optimal deccision rule is:

$$d_t = \begin{cases} 0, & \text{if } \varepsilon_{t1} - \varepsilon_{t2} \geqq \beta^{-t} v_1(H_t) - \beta^{-t} v_2(H_t) \\ 1, & \text{if } \varepsilon_{t1} - \varepsilon_{t2} < \beta^{-t} v_1(H_t) - \beta^{-t} v_2(H_t) \end{cases} \tag{4.8}$$

where $\beta^{-t} v_j(H_t)$ is the current (undiscounted) conditional valuation function for history $H_t$ and action $j$. This implies the conditional probability of choosing not to sterilize in period $t$ is:

$$p_1(H_t) = \left\{1 + \exp\left[\beta^{-t} v_1(H_t) - \beta^{-1} v_2(H_t)\right]\right\}^{-1}. \tag{4.9}$$

Since the right-hand side of (4.6) already corresponds to the form of $V_{t2}$ in (3.13), we only need to derive the expression for $V_1\big(H_t, \overline{p}(H_t)\big)$. To proceed, note that $\mathscr{A}_{t+1}(H_t) = \{(H_t, 1), (H_t, 0)\}$ and the associated set of conditional choice probabilities is:

$$\overline{p}(H_t) = \left\{p_1(()H_t, 0), p_1(()H_t, 1)\right\}. \tag{4.10}$$

Using (4.9), it follows from Proposition 1 that $Q^{-1}\big(p_1(H_{t+1}), H_{t+1}\big)$ is:

$$Q^{-1}\big(p_1(H_{t+1}), H_{t+1}\big) = \beta^{t+1} \ln\left(\frac{p_1(H_{t+1})}{p_2(H_{t+1})}\right). \tag{4.11}$$

To complete the expression for $V_1\big(H_1, \overline{p}(H_t)\big)$, we need to characterize the form of the $W_{t+1,j}$ functions associated with $U_{t+1}$ in (3.8). Given the assumed distributions for $\varepsilon_{t1}$ and $\varepsilon_{t2}$, these functions take the form:

$$\begin{aligned} W_j\left(\boldsymbol{p}_{t+1}, H_{t+1}\right) &\equiv E\left(\varepsilon_{t+1,j} \mid H_{t+1}, d_{t+1,j}^0 = 1\right) \\ &= E\left(\varepsilon_{t+1,j} \mid H_{t+1}, \beta^{t+1}\varepsilon_{t+1,j} - \beta^{t+1}\varepsilon_{t+1,k} > v_j(H_{t+1}) - v_k(H_{t+1})\right) \\ &= E\left(\varepsilon_{t+1,j} \mid H_{t+1}, \varepsilon_{t+1,j} > \varepsilon_{t+1,k} + \ln\left\{p_j(H_{t+1}) / \left[1 - p_j(H_{t+1})\right]\right\}\right) \\ &= \gamma - \ln\left[p_j(H_{t+1})\right], \end{aligned} \tag{4.12}$$

for $j = 1, 2$, where $\gamma$ is Euler's constant. Then, we get

$$\begin{aligned} &V_t\left(\overline{p}(H_t), H_t\right) = \\ &\beta^t \alpha \left\{\delta_1\left(\widetilde{H}_t + 1\right) + \delta_2\left(\widetilde{H}_t + 1\right)^2 + \sum_{j=1}^{2} p_j(H_t, 1)\left(\gamma - \ln\left[p_j(H_t, 1)\right]\right) + \frac{\beta\left[\gamma + \delta_1\left(\widetilde{H}_t + 1\right) + \delta_2\left(\widetilde{H}_t + 1\right)^2\right]}{1 - \beta^{T-t-1}} + p_1(H_t, 1)\left(\ln\left[\frac{p_1(H_{t+1})}{p_2(H_{t+1})}\right]\right)\right\} \\ &+ \beta^t(1-\alpha)\left\{\delta_1 \widetilde{H}_t + \delta_2 \widetilde{H}_t^2 + \sum_{j=1}^{2} p_j(H_t, 0)\left(\gamma - \ln\left[p_j(H_t, 0)\right]\right) + \frac{\beta\left[\gamma + \delta_1 \widetilde{H}_t + \delta_2 \widetilde{H}_t^2\right]}{1 - \beta^{T-t-1}} + p_2(H_t, 0)\left(\ln\left[\frac{p_1(H_{t+1})}{p_2(H_{t+1})}\right]\right)\right\}. \end{aligned} \tag{4.13}$$

This function consists of two expressions in braces: the one in the first line gives the expected lifetime utility from period $t+1$ on if a child is born in period $t$ and $H_{t+1} = \{H_t, 1\}$, weighted by the probability that such abirth will occur; the second gives the probability-weighted utility associated with the birth not occurring and $H_{t+1} = \{H_t, 0\}$. Each of these expressions, in turn, consists of the sum of: (a) the expected payoff in period $t+1$, $U_{t+1}$; (b) the value of sterilizing at $t+1$, $V_2$; and (c) a term which adjusts for the fact that sterilization may not be optimal in $t+1$.

Using $V_1\big(\overline{p}(H_t), H_t\big)$ in place of $V_1(H_t)$ and the expression in (4.6) for $V_2(H_t)$, one can represent the conditional probability of choosing either action as a function of the couple's history, $H_t$, and the one period ahead choice probabilities $p(H_t)$. Consequently $p(H_t)$ and $v_2(H_t)$ are sufficient to summarize the expected future value of an action in period $t$. Provided we can obtain consistent estimates of the future choice probabilities cheaply, the representation developed here can be used to formulate estimators for $\alpha, \beta, \delta_1, \delta_2$, the structural parameters of interest.