Bewley-Huggett-Aiyagari Models: Computation, Simulation, and Uniqueness of General Equilibrium - Macroeconomic Dynamics - 2019

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1 Bewley-Huggett-Aiyagari Models

Let $X \subseteq \mathbb{R}^l$ be the endogenous state variable, $Y = Y_1 \times Y_2 \subseteq X \times \mathbb{R}^c$ the choice variable, and $Z \subseteq \mathbb{R}^k$ the exogenous state variable. Let $\Theta \subseteq \mathbb{R}^q$ be a parameter space. The state of an agent is then a pair (x, z). A value function maps $V_p : X \times Z \to \mathbb{R}$. A policy function maps $g_p : X \times Z \to Y$. Let $S = X \times Z$, and let $\mathscr{S}$ be its Borel σ-field. The measure of agents μ_p is a probability distribution over $(S, \mathscr{S})$. The return function maps $R_p : X \times Y \times Z \to \mathbb{R}$, and the discount factor is $0 < \beta < 1$.

The aggregate variable sare $A \in \mathbb{A} \subseteq \mathbb{R}^a$. A price vector is $p \in \mathbb{P} \subseteq \mathbb{R}^p$. The exogenous shock follows a Markov chain with transition function Q mapping from Z to Z. The aggregate function maps $\mathscr{M}(S, \mathscr{S}) \to \mathbb{R}^a$, where $\mathscr{M}(S, \mathscr{S})$ is the space of probability measures on $(S, \mathscr{S})$. The market clearance function maps $\lambda : \mathbb{R}^a \times \mathbb{R}^p \to \mathbb{R}$.

Definition 1. A competitive equilibrium is an agent's value function V_p, agent's policy function g, vector of prices p, and measure of agent's μ_p, such that

1. given prices p, the agent's value function V_p, and policy function g_p solve the agents problem:

$$V_p(x, z) = \max_{y=(y_1, y_2) \in Y} \left\{ R_p(x, y, z) + \beta \int V_p\left(y_1', z'\right) Q\left(z, dz'\right) \right\}; \tag{1}$$

2. aggregates are determined by individual actions: $A_p = \mathscr{A}(\mu_p)$;

3. markets clear (in terms of prices): $\lambda(A_p, p) = 0$;

4. the measure of agents is invariant:

$$\mu_p(x, z) = \iint \left[\int \mathbb{1}\left\{ x = g_p^{y_1}(\hat{x}, z) \right\} \mu_p(\hat{x}, z) Q\left(z, dz'\right) \right] d\hat{x} dz. \tag{2}$$

*This note is written down during my M.phil. period at the University of Oxford.

2 The Algorithm

The solution of BHA models involves three steps: (i) computation of the optimal policy function; (ii) simulation of the stationary distribution of the agents (or at least some of its moments); and (iii) finding the general equilibrium.

The following briefly describes how pure discretization of the state space works in each of these three steps.

2.1 Discretized value function iteration and the optimal policy function

The algorithm for discretized VFI is largely standard:

Declare initial value V_0.
Declare iteration count $n = 0$.
while $\| V_{n+1} - V_n \| >$ "tolerance constant" do
Increment n. Let $V_{\text{old}} = V_{n-1}$.
for $x = 1, \dots, n_x$ do
    for $z = 1, \dots, n_z$ do
        Calculate $E[V_{\text{old}} \mid z]$
        Calculate $V_n(x,z) = \max_{y=1,\dots,n_y} R(x,y,z) + \beta E\left[V_{\text{old}}(y,z') \mid z\right]$
        Calculate $g(x,z) = \arg\max_{y=1,\dots,n_y} R(x,y,z) + \beta E\left[V_{\text{old}}(y,z') \mid z\right]$
    end for
end for
for $i = 1, \dots, H$ do (Howards Improvement Algorithm (do H updates))
    $V_{\text{old}} = V_n$
    for $x = 1, \dots, n_x$ do
        for $z = 1, \dots, n_z$ do
            Calculate $V_n(x,z) = R(x, g(x,z), z) + \beta E\left[V_{\text{old}}(g(x,z), z') \mid z\right]$
        end for
    end for
end for
end while
for $x = 1, \dots, n_x$ do
    for $z = 1, \dots, n_z$ do
        Calculate $g_n(x,z) = \arg\max_{y=1,\dots,n_y} R(x,y,z) + \beta E\left[V_{\text{old}}(y,z') \mid z\right]$
    end for
end for

This algorithm has several alternatives for solving VFI, involving various degrees of discretization. For example, here z is fully discretized [using e.g., quadrature methods such as Tauchen (1986) method]; an alternative would be to use Monte Carlo integration methods [e.g., Pal and Stachurski (2013)]. The choice of the next period's state is also discretized, but partial discretization would be an alternative [Santos and Vigo-Aguiar (1998), Aruoba et al. (2006)]. Instead of discretizing the value function itself, one can use fitted value function methods [Benitez-Silva et al. (2005), Cai and

Judd (2014)]. Other related algorithms for VFI are the endogenous grid method [Carroll (2006), Barillas and Fernandez-Villaverde (2007), Fella (2014)] and the envelope condition method [Maliar and Maliar (2013), Tsyrennikov et al. (2016)].

2.2 Iteration on discretized agents distribution

The algorithm for iterating on the discretized agent distribution is now given:

Declare initial distribution μ_0.(A matrix the elements of which sum to one)
Declare iteration count $n = 0$.
while $||\mu_{n+1} - \mu_n|| >$ "tolerance constant" do
Increment n. $\mu_n = 0$ (A matrix of zeros on x-by-z)
for $x = 1, \ldots, n_x$ do
    for $z = 1, \ldots, n_z$ do
        for $z' = 1, \ldots, n_z$ do
            $\mu_n\left(x(g(x,z)), z'\right) = \mu_n\left(x(g(x,z)), z'\right) + \left(\mu_{n-1}(x,z) * Q(z,z')\right)$ [1]
        end for
    end for
end for
end while

The algorithm consists of a while-loop for convergence of the agents distributions, together with iterative updating of the agents distribution based on the following formula:

$$\mu_p(x,z) = \iint \left[\int \mathbb{1}\left\{x = g_p^{y_1}(\hat{x}, z)\right\} \mu_p(\hat{x}, z) Q\left(z, dz'\right) \right] d\hat{x} dz.$$

The main alternative to iterating on the agents' distribution is to use simulated paths to generate an approximation of the distribution.

2.3 Calculate aggregate variables and evaluate market clearance

Evaluate $A = \sum_{x=1,\ldots,n_x, z=1,\ldots,n_z} A(x,z)\mu_n(x,z)$ and $\lambda(A,p)$.

2.4 Find a/the general equilibrium using a grid on prices

The algorithm is:

declare grid for prices $p = 1, \ldots, n_p$
for $p = 1, \ldots, n_p$ do
    Given p, solve Step (2.1) to obtain the value function and optimal policy function.
    Given p, solve Step (2.2) to obtain agents distribution.
    Given p, solve Step (2.3) to obtain the aggregate variables and market clearance.

[1] This equation in the original paper seems wrong.

end for

Evaluate every pair of two consecutive grid points on the price grid; if the market clearance conditions switch their sign (e.g., from negative to positive), then whichever of the two prices has the lowest absolute value of the market clearance condition is the equilibrium price. (For simplicity, this has been written assuming price is one-dimensional; for an n-dimensional price, the consecutive grid points would be a consecutive 2n neighborhood of grid points.)

Algorithmically, this step will be accomplished by placing a grid on prices and simply evaluating the model at each point on the grid. While this is slower than the more commonly used algorithms (function minimization algorithms, such as Newton-Raphson or simplex-search methods), it allows us to prove that the algorithm will converge for any and all equilibria.

3 Example Models

3.1 Aiyagari-1994

In this model, the exogenous shock (z) is the labor supply h, the endogenous state (x) is the capital holdings k, and the decision variable (y) is the next period's capital k'. The state of a household is their current capital holding and their exogenous labor supply shock (k,h). Individual household capital holdings, given an interest rate, aggregate to give aggregate capital holdings. The market clearance condition is that the interest rate will be determined by perfect competition in the goods market together with a representative firm with Cobb-Douglas production function. In short,

Definition 2. A competitive equilibrium is an agent's value function $V(k,h)$, agent's policy function $k' = g(k,h)$, an interest rate r and wage w, aggregate capital K and labor H, and a measure of agents $\mu(k,h)$, such that

1. given prices r and w, the agent's value function $V(k,h)$, and policy function $k' = g(k,h)$ solve the agents' problem:

$$V(k,h) = \max_{k'} \left\{ u(c) + \beta \int V\left(k',h'\right) Q\left(h,dh'\right) \right\},$$
$$\text{s.t.} \quad c + k' = wh + (1+r)k;$$
$$c \geq 0, k' \geq \underline{k}.$$

2. the aggregates are determined by individual actions: $K = \int k d\mu(k,h)$ and $H = \int h d\mu(k,h)$;

3. markets clear (in terms of prices): $r - \left(\alpha K^{\alpha-1} H^{1-\alpha} - \delta\right) = 0$, and $w = (1-\alpha)\left((r+\delta)/\alpha\right)^{\alpha/(\alpha-1)}$;

4. the measure of agents is invariant:

$$\mu(k,h) = \iint \left[\int \mathbb{1}\{k = g(\hat{k},h)\} \mu(\hat{k},h) Q\left(h,dh'\right) \right] d\hat{k} dh,$$

where h is the labor supply shock, which takes values in $Z = \{h_1,\ldots,h_{n_h}\}$ and evolves according to the Markov transition function $Q\left(h,h'\right)$. Since $H = E(h) = 1$, the Cobb-Douglas production function is really only based on the aggregate capital (in the sense that H is a fixed constant).