Practical Methods for Estimation of Dynamic Discrete Choice Models, Annual Review of Economics, 2011

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Assumptions

- In each period $t \in \{1, 2, \ldots, T\}, T \leq \infty$, the individual makes a decision d_t from among a finite set of alternatives D_t.
- The flow utility at period t from choosing d_t is the sum of two components.
  - The first component is a function of state variables, x_t, that are observed by both the individual and the econometrician.
  - The second component is a choice-specific variable $\epsilon(d_t)$ that is observed by the individual but not by the econometrician and has support on the real line.
- Assume additive separability of the utility function: $u(x_t, d_t) + \epsilon(d_t)$.

Assumptions

- ϵ_t is iid over time, with the PDF $g(\epsilon_t)$.
- x_t is a Markov process. Denote the PDF of x_{t+1} conditional on x_t, d_t, and ϵ_t as $f(x_{t+1} \mid x_t, d_t, \epsilon_t)$.
- (Conditional Independence) After controlling for both the decision and observed state at t, the unobserved state at t has no effect on the observed state at $t+1$:

$$f(x_{t+1} \mid x_t, d_t, \epsilon_t) = f(x_{t+1} \mid x_t, d_t)$$

Sequential Problem

- Individuals discount the future at a rate $\beta \in (0,1)$, maximizing the present discounted value of their lifetime utilities.
- They maximize by choosing δ^*, a set of decision rules for all possible realizations of the observed and unobserved variables in each time period, whose elements are denoted by $\delta_t(x_t, \epsilon_t)$.

$$\delta^* = \arg\max_{\delta} E_{\delta}\left(\sum_{t=1}^{T} \beta^{t-1}\left[u\left(x_t, d_t\right) + \epsilon\left(d_t\right)\right] \mid x_1, \epsilon_1\right), \tag{1}$$

where the expectations are taken over the future realizations of x and ϵ induced by δ^*.

Value Function

- Define the value function as

$$V_t(x_t,\epsilon_t) \equiv \max_{\delta} E_{\delta}\left(\sum_{t'=t}^{T} \beta^{t'-t}\left[u(x_{t'},d_{t'})+\epsilon(d_{t'}) \mid x_t,\epsilon_t\right]\right) \tag{2}$$

- It can also be defined recursively as follows:

$$\begin{aligned} V_t(x_t,\epsilon_t) &= \max_{d_t}\left\{u(x_t,d_t)+\epsilon_t+\beta E\left[V_{t+1}(x_{t+1},\epsilon_{t+1} \mid x_t,d_t)\right]\right\} \\ &= \sum_{d_t} I\left[\delta_t(x_t,\epsilon_t)=d_t\right]\left\{u(x_t,d_t)+\epsilon(d_t)+\beta \iint \left[V_{t+1}(x_{t+1},\epsilon_{t+1}) g(\epsilon_{t+1}) d\epsilon_{t+1}\right] f(x_{t+1} \mid x_t,d_t) dx_{t+1}\right\}. \end{aligned} \tag{3}$$

Note how we express $E\left[V_{t+1}(x_{t+1},\epsilon_{t+1} \mid x_t,d_t)\right]$: The inner integral integrates out the unobserved state variables ϵ_{t+1} and the outer integral integrates out the observable state variables x_{t+1}.

Intergratedv (Ex Ante) Value Function

- Because ϵ_t is unobservable, we define the ex ante value function (or integrated value function), $\overline{V}_t(x_t)$, as the continuation value of being in state x_t just before ϵ_t is revealed. $\overline{V}_t(x_t)$ is then given by integrating $V_t(x_t, \epsilon_t)$ over ϵ_t:

$$\overline{V}_t(x_t) \equiv \int V_t(x_t, \epsilon_t) g(\epsilon_t) d\epsilon_t. \tag{4}$$

- Or we can express it following the recursive structure

$$\overline{V}_t(x_t) = \sum_{d_t} \int I[\delta_t(x_t, \epsilon_t) = d_t] \left[u(x_t, d_t) + \epsilon(d_t) + \beta \int \overline{V}_{t+1}(x_{t+1}) f(x_{t+1} \mid x_t, d_t) dx_{t+1} \right] g(\epsilon_t) d\epsilon_t \tag{5}$$

Conditional Value Function

- Define the conditional value function $v_t(x_t, d_t)$ as the present discounted value (net of ϵ_t) of choosing d_t and behaving optimally from period $t+1$ on

$$v_t(x_t, d_t) \equiv u(x_t, d_t) + \beta \int \overline{V}_{t+1}(x_{t+1}) f(x_{t+1} \mid x_t, d_t) dx_{t+1}. \tag{6}$$

- Therefore, the individual's optimal decision rule at t solves

$$\delta_t(x_t, \epsilon_t) = \arg\max_{d_t} [v_t(x_t, d_t) + \epsilon_t] \tag{7}$$

Conditional Choice Probabilities

- Now, the probability of observing d_t conditional on x_t, $p_t(d_t \mid x_t)$, is found by integrating out ϵ_t from the decision rule:

$$p_t(d_t \mid x_t) = \int I\{\delta_t(x_t, \epsilon_t) = d_t\} g(\epsilon_t) d\epsilon_t = \int I\left\{\underset{d_t \in D_t}{\arg\max}[v_t(x_t, d_t) + \epsilon_t(d_t)] = d_t\right\} g(\epsilon_t) d\epsilon_t. \tag{8}$$

- We call $p_t(d_t \mid x_t)$ conditional choice probability. And this is what we actually can see in the data.
- Compared to the static discrete choice model: Replace the per-period utility u_t with v_t, which takes the future consequences of current choice into consideration.

Parameterization

- $u(x_t, d_t)$ is assumed to be a known function of x_t, d_t, and a parameter vector θ_1.
- Estimation of the structural model also requires the estimation of the transition functions governing the observed states, $f(x_{t+1} \mid x_t, d_t)$, typically parameterized by a vector θ_2.
- For our current purpose, the parameters governing these transitions are then treated as known, as is the distribution of the unobservables, $g(\epsilon)$.

Finite-Horizon Conditional Value Function

- The conditional value function at T is just the flow payoff function:

$$v_T(x_T, d_T) = u(x_T, d_T). \tag{9}$$

- Knowledge of $g(\epsilon_T)$ implies that we can calculate the integrated value function at T using

$$\overline{V}_T(x_T) = \int \sum_{d_T} I[\delta_T(x_T, \epsilon_T) = d_T][v_T(x_T, d_T) + \epsilon_T(d_T)] g(\epsilon_T) d\epsilon_T \tag{10}$$

- Finally, we can obtain the conditional value function at $T-1$

$$v_{T-1}(x_{T-1}, d_{T-1}) = u(x_{T-1}, d_{T-1}) + \beta \int \overline{V}_T(x_T) f(x_T \mid x_{T-1}, d_{T-1}) dx_T. \tag{11}$$

Finite-Horizon Conditional Value Function

- In general, at time t, the conditional value function is given by

$$v_t(x_t, d_t) = u(x_t, d_t) + \beta \int \overline{V}_{t+1}(x_{t+1}) f(x_{t+1} \mid x_t, d_t) dx_{t+1} \tag{12}$$

- Moving back each period requires integrating out over both the observed and unobserved state variables, which can cause computational burden when the size of the state space grows large or the time horizon becomes long.
- In such cases, we could instead evaluate the future utility terms at a subset of the possible observed and unobserved states. At states for which the value function is not calculated directly, the value function can be interpolated using the values at the points at which the function was calculated (Keane and Wolpin (1994)).

Infinite-Horizon Conditional Value Function

- When the horizon is infinite but the environment is stationary, we can remove the t subscripts from both the conditional and ex ante value functions.
- Denoting the current value of the observed state variable by x and the next period's state variable by x'

$$
\begin{aligned}
v(x,d) &= u(x,d) + \beta \int \overline{V}\left(x'\right) f\left(x' \mid x,d\right) dx' \\
&= u(x,d) + \beta \int \left[\int \left\{ \max_{d' \in D} \left[v\left(x',d'\right) + \epsilon\left(d'\right) \right] \right\} f\left(x' \mid x,d\right) dx' \right] g(\epsilon) d\epsilon.
\end{aligned} \tag{13}
$$

- To compute the future value term, note that the ex ante value function can be expressed as

$$
\overline{V}(x) = \int \left\{ \max_{d} \left[u(x,d)\beta \int \tilde{V}\left(x'\right) f\left(x' \mid x,d\right) dx' \right] \right\} g(\epsilon) d\epsilon. \tag{14}
$$

Forming the Likelihood

- For both finite- and infinite-horizon problems, the log likelihood function is formed by calculating the probabilities of the decisions observed in the data.
- Let d_{nt}, x_{nt}, and ϵ_{nt} indicate the choice, observed state, and unobserved state at time t for individual n, respectively.
- The likelihood contribution of the choice for individual n at time t is

$$
\begin{aligned}
p_t(d_{nt} \mid x_{nt}, \theta) &= \int I(\delta(x_{nt}, \epsilon_{nt}, \theta) = d_{nt}) g(\epsilon_{nt}) d\epsilon_{nt} \\
&= \int I \left\{ \arg\max_{d_t} [v_t(x_{nt}, d_t, \theta) + \epsilon_{nt}(d_t)] = d_{nt} \right\} g(\epsilon_{nt}) d\epsilon_{nt},
\end{aligned} \tag{15}
$$

Forming the Likelihood

- With $\mathcal{N}$ individuals for $\mathcal{T}$ periods, estimates are obtained via

$$\hat{\theta} = \arg\max_{\theta} \sum_{n=1}^{\mathcal{N}} \sum_{t=1}^{\mathcal{T}} \left(\ln [p_t(d_{nt} \mid x_{nt}, \theta)] + \ln \left[f\left(x_{n,t+1} \mid x_{nt}, d_{nt}, \theta_2 \right) \right] \right). \qquad (16)$$

- Because the log likelihood function is additively separable, a consistent estimate of θ_2 can be obtained using information on the state transitions alone. Then, taking $\hat{\theta}_2$ as given, the data on the choice probabilities can be used to estimate θ_1.

Normal Errors

- There are two advantages to choosing a normal distribution.
- First, it has a more flexible correlation structure and is therefore able to capture richer patterns of substitution across choices.
- Second, it is easy to draw from a normal distribution.

Generalized Extreme Value Errors

- First, type I extreme value distribution gives us a closed-form representation:

$$p_t(d_t \mid x_t) = \frac{\exp[v_t(x_t, d_t)]}{\sum_{d'_t \in D} \exp\left[v_t\left(x_t, d'_t\right)\right]} = \frac{1}{\sum_{d'_t \in D} \exp\left[v_t\left(x_t, d'_t\right) - v_t(x_t, d_t)\right]},$$
$$\overline{V}_t(x_t) = \ln\left\{\sum_{d'_t \in D} \exp\left[v_t\left(x_t, d'_t\right)\right]\right\} + \gamma, \tag{17}$$

  where γ is Euler's constant.
- The second advantage of working with the GEV distribution concerns the mapping from choice probabilities back to ex ante value functions.

GEV Errors: Mapping between CCP and Integrated Value Function

- Given the following assumptions,
    - (a) structural errors that are additively separable from the flow payoffs,
    - (b) conditional independence of the state transitions, and
    - (c) independence of the structural errors over time,

  differences in conditional value functions can always be expressed as functions of the choice probabilities alone.

GEV Errors: Mapping between CCP and Integrated Value Function

- In the dynamic logit setting, the ex ante value function (see previous slide) can be rewritten with respect to the conditional value function associated with an arbitrarily selected choice, say d_t^*:

$$
\begin{aligned}
\overline{V}_t(x_t) &= \ln\left(\exp\left[v_t\left(x_t, d_t^*\right)\right]\left\{\frac{\sum_{d_t' \in D} \exp\left[v_t\left(x_t, d_t'\right)\right]}{\exp\left[v_t\left(x_t, d_t^*\right)\right]}\right\}\right) + \gamma \\
&= \ln\left\{\sum_{d_t' \in D} \exp\left[v_t\left(x_t, d_t'\right) - v_t\left(x_t, d_t^*\right)\right]\right\} + v_t\left(x_t, d_t^*\right) + \gamma \\
&= -\ln\left[p\left(d_t^* \mid x_t\right)\right] + v_t\left(x_t, d_t^*\right) + \gamma.
\end{aligned} \tag{18}
$$

GEV Errors: Mapping between CCP and Integrated Value Function

- To form a likelihood, we also need to calculate the conditional value function:

$$v_t(x_t,d_t)=u(x_t,d_t)+\beta\int\left(v_{t+1}\left(x_{t+1},d^*_{t+1}\right)-\ln\left[p_{t+1}\left(d^*_{t+1}\mid x_{t+1}\right)\right]\right)f(x_{t+1}\mid x_t,d_t)\,dx_{t+1}+\beta\gamma \tag{19}$$

  where d^*_{t+1} is an arbitrary choice in period $t+1$.

- The future value term now has three components:
  - the function characterizing the transitions of the state variables,
  - the CCPs for the arbitrary choice d^*_{t+1}, and
  - the conditional value function associated with d^*_{t+1}.
- The first two can often be estimated separately in a first stage. It still remains to be shown how to deal with the remaining conditional value function.
- The key to the argument is that the researcher can choose to which choice (and hence which conditional value function) to make the future value term relative!

Models Requiring Only One-Period-Ahead CCPs: Triggering

- To illustrate, suppose the choice set D_t includes an action (terminal action) that, when taken, implies that no further decisions are made.
- Once the terminal action is chosen, the agent's decision problem is no longer dynamic, allowing the future value term to be replaced with a known parametric form (or normalized to zero).
- Let $d = R$ denote the terminal action, then the conditional value function becomes

$$v_t(x_t, d_t) = u(x_t, d_t) + \beta \int (v_{t+1}(x_{t+1}, R) - \ln[p_{t+1}(R \mid x_{t+1})]) f(x_{t+1} \mid x_t, d_t) dx_{t+1} + \beta\gamma \tag{20}$$

Models Requiring Only One-Period-Ahead CCPs: Terminal Actions

- Claim: $v_{t+1}(x_{t+1}, R)$ is just a component of static utility, which is typically assumed to follow a known parametric form or normalize to zero.
- Examples of terminal actions:
  - sterilization in fertility choices
  - housing construction in land development choices

Rewriting the Conditional Value Function

- Another class of models that only require one-period-ahead CCPs are settings in which there exists a choice (renewal choice) that makes the choice in the previous period irrelevant. For example, engine replacement in Rust (1987).
- By taking the renewal action at $t+1$, the effect of the choice at t on the state at $t+2$ is removed so that

$$\int f\left(x_{t+1} \mid x_t, d_t^*\right) f\left(x_{t+2} \mid x_{t+1}, R\right) d x_{t+1} = \int f\left(x_{t+1} \mid x_t, d_t'\right) f\left(x_{t+2} \mid x_{t+1}, R\right) d x_{t+1} \tag{21}$$

Models Requiring Only One-Period-Ahead CCPs: Renewal Actions

- To see how the renewal property can be exploited in estimation, recall Equation 19. Substitute in for $v_{t+1}(x_{t+1}, R)$ with the flow payoff of replacement plus the ex ante value function at $t+2$:

$$
\begin{aligned}
v_t(x_t, d_t) &= u(x_t, d_t) + \beta \int \left(v_{t+1}(x_{t+1}, R) - \ln[p_{t+1}(R \mid x_{t+1})]\right) f(x_{t+1} \mid x_t, d_t) dx_{t+1} + \beta\gamma \\
&= u(x_t, d_t) + \beta \int \left(u(x_{t+1}, R) - \ln[p_{t+1}(R \mid x_{t+1})]\right) f(x_{t+1} \mid x_t, d_t) dx_{t+1} + \beta\gamma \\
&+ \beta^2 \iint V_{t+2}(x_{t+2}) f(x_{t+2} \mid x_{t+1}, R) f(x_{t+1} \mid x_t, d_t) dx_{t+2} dx_{t+1}.
\end{aligned} \tag{22}
$$

- The last term is constant across all choices. And therefore it will drop out for the likelihood function.

Models Requiring Multiple-Period-Ahead CCPs: Triggering

- Different trigger mechanisms to reset the system:
  - It might take more than just a single action at time t to reset the system.
  - The same renewal action might need to be repeated for a fixed number of periods.
  - It might require a particular sequence of actions.
- Examples:
  - A woman who stays out of the workforce for ρ consecutive periods effectively resets her human capital.
  - Some marketing models of state-dependent demand: consumers face a switching cost when choosing a different product than the one they consumed last period, but this cost is reset to zero whenever the consumer chooses the outside good.

Models Requiring Multiple-Period-Ahead CCPs: Notations

- Consider an individual in state x_t and a candidate sequence of decisions from t to $t+\rho$ periods: $\{d_t^*, d_{t+1}^*, \ldots, d_{t+\rho}^*\}$.
- For $\tau \in \{t+1, \ldots, t+\rho\}$, denote $\kappa_\tau^*(x_{\tau+1} \mid x_t)$ as the cumulative probability of being in state $x_{\tau+1}$ given the decision sequence and initial state, defined recursively as

$$\kappa_\tau^*(x_{\tau+1} \mid x_t) = \begin{cases} f\left(x_{\tau+1} \mid x_\tau, d_\tau^*\right) & \text{if } \tau = t \\ \int f\left(x_{\tau+1} \mid x_\tau, d_\tau^*\right) \kappa_{\tau-1}^*(x_\tau \mid x_t)\, dx_\tau & \text{otherwise.} \end{cases} \tag{23}$$

Rewrite Conditional Value Function 6 with the Renewal Action Sequence

- We can rewrite the expression for $v_t(x_t, d_t)$ given in Equation 6 such that the future utility term is expressed relative to the choices in the sequence$\{d_t^*, d_{t+1}^*, \ldots, d_{t+\rho}^*\}$.
- Intuition: Executive the same procedure as that in the terminal actions ρ times.

$$
\begin{aligned}
v_t\left(x_t, d_t^*\right) = u\left(x_t, d_t^*\right) + \sum_{\tau=t+1}^{t+\rho} \int \beta^{\tau-t}\left[u\left(x_\tau, d_\tau^*\right) - \ln\left[p_\tau\left(x_\tau, d_\tau^*\right)\right] + \gamma\right] \kappa_{\tau-1}^*\left(x_\tau \mid x_t\right) d x_\tau \\
+ \int \beta^{\rho+1} V_{t+\rho+1}\left(x_{t+\rho+1}\right) \kappa_{t+\rho}^*\left(x_{t+\rho+1} \mid x_t\right) d x_{t+\rho+1}
\end{aligned} \tag{24}
$$

Difference of the Conditional Value Function

- Consider another decision sequence $\{d'_t, d'_{t+1}, \ldots, d'_{t+\rho}\}$, with the corresponding cumulative probability $\kappa'_\tau(x_{\tau+1} \mid x_t)$.
- Suppose these sequences of decisions lead the individual to the same state in expectation. Then

$$\kappa^*_{t+\rho}\left(x_{t+\rho+1} \mid x_t\right) = \kappa'_{t+\rho}\left(x_{t+\rho+1} \mid x_t\right) \quad \text{for all } x_{t+\rho+1}. \tag{25}$$

- Hence, when calculating CCPs, since what matters is the differences between conditional value functions, the last term (the most computationally difficult part) cancels out.

Difference of the Conditional Value Function

- In particular, for these two decision sequences.

$$
\begin{aligned}
v_t\left(x_t, d_t^*\right) - v_t\left(x_t, d_t'\right) = u\left(x_t, d_t^*\right) - u\left(x_t, d_t'\right) \\
+ \sum_{\tau=t+1}^{t+\rho} \int \beta^{\tau-t}\left(u\left(x_\tau, d_\tau^*\right) - \ln\left[p_\tau\left(x_\tau, d_\tau^*\right)\right]\right) \kappa_{\tau-1}^*\left(x_\tau \mid x_t\right) d x_\tau \\
- \sum_{\tau=t+1}^{t+\rho} \int \beta^{\tau-t}\left(u\left(x_\tau, d_\tau'\right) - \ln\left[p_\tau\left(x_\tau, d_\tau'\right)\right]\right) \kappa_{\tau-1}'\left(x_\tau \mid x_t\right) d x_\tau .
\end{aligned}
\tag{26}
$$

- Note that the last line of Equation 24 disappears because of finite dependence. Namely, because the two choice sequences lead to the same states in expectation, the last line of

Case Without Finite Dependence

- When finite dependence does not hold, there are certain cases - namely stationary, infinite horizon settings - in which CCP estimation may still prove particularly advantageous.
- Detailed Calculation is on Page 18.

Stage 1: Conditional Choice Probabilities and Transition Functions

- Given unlimited data, we can do nonparametric estimation

$$p_t(d_t \mid x_t) = \frac{\sum_{n=1}^{\mathcal{N}} I(d_{nt} = j, x_{nt} = x)}{\sum_{n=1}^{\mathcal{N}} I(x_{nt} = x)} \tag{27}$$

- Similar expressions could be formed for the transition probabilities over the observed state variables.
- Data limitations, particularly when the state space is large (or continuous), will often make the above Equation infeasible. In this case, nonparametric kernels, basis functions, or flexible logits could be employed.

Stage 2: Estimating the Structural Parameters

- When finite dependence holds, the only components of the future utility term that are not estimated in the first stage are the flow payoff terms associated with the finite dependence sequences.
- In the case of renewal or terminating actions, the payoff for these actions may be normalized to zero, so estimation is as simple as a multinomial or binary logit with an offset term. For example, for Equation 22, the offset term is

$$-\beta \int \ln[p_{t+1}(R \mid x_{t+1})] f(x_{t+1} \mid x_t, d_t),$$

which is calculated outside the model.

- In other cases, the flow payoff terms that are accumulated over the relevant sequences must be multiplied by the relevant transitions of the state variables and discounted.

Improving the Precision of CCPs

- For example, suppose type 1 extreme value errors are used. The probability of choice d given observed state x and structural parameters $\hat{\theta}$ is

$$p(d \mid x, \hat{\theta}) = \frac{\exp[v(x,d)]}{\sum_{d'} \exp[v(x,d')]}$$

.

- Given estimates of the structural parameters, the above Equation can be used to update the CCPs. Then the value function can be updated.

Dealing With Large State Spaces

- To deal with large state space, we can use the CCPs to forward simulate the value function.
    - Given the individual's current state, use the estimated conditional probability functions to draw a choice and then, conditional on the choice, draw a realized state using the estimated transition function.
    - This process is continued until the time horizon is reached or, in the case of infinite-horizon models, the increment to the value function is sufficiently small due to discounting.
    - Taking many paths and averaging approximate the value function.
- Forward simulation is particularly powerful when coupled with finite dependence. Specifically, rather than taking draws out for the full time horizon or waiting until discounting makes the increment to the value function small, we instead use the CCPs associated with paths that lead to finite dependence.

Mixture Distributions and the Expectation-Maximization Algorithm

- The standard approach to account for unobserved heterogeneity in dynamic discrete models is to employ finite mixture distributions.
- In addition to x_{nt}, there is an unobserved state variable s_{nt}, which takes on one of S values, $s_{nt} \in \{1, \ldots, S\}$. To keep the exposition simple, we focus on the case in which the unobserved state does not vary over time.
- The joint likelihood of d_{nt} and $x_{n,t+1}$, conditional on x_{nt} and unobserved state s, is

$$\mathcal{L}_t(d_{nt}, x_{nt+1} \mid x_{nt}, s; \theta) = p_t(d_{nt} \mid x_{nt}, s; \theta_1) f_t(x_{nt+1} \mid d_{nt}, x_{nt}, s; \theta_2). \tag{28}$$

Forming the Log-Likelihood

- Because s_n is unobserved, we integrate it out of the likelihood. Denote $\pi(s_j, x_{n1})$ as the probability of being in unobserved state s given the data at the first observed time period, x_{n1}.
- The likelihood of the observed data for n is then given by

$$\sum_{s=1}^{S} \pi(s \mid x_{n1}) \mathscr{L}_t(d_{nt}, x_{nt+1} \mid x_{nt}, s, \theta) \tag{29}$$

- MLE now requires that one solve for both θ and π, where π refers to all possible values of $\pi(s_j, x_{n1})$

$$(\hat{\theta}, \hat{\pi}) = \underset{\theta, \pi}{\arg\max} \sum_{n=1}^{\mathscr{N}} \ln \left[\sum_{s=1}^{S} \pi(s \mid x_{n1}) \prod_{t=1}^{\mathscr{T}} \mathscr{L}_t(d_{nt}, x_{nt+1} \mid x_{nt}, s; \theta) \right]. \tag{30}$$

FOC of the Log-Likelihood

- The FOC to this problem is

$$\frac{\partial L}{\partial \theta} = 0 = \sum_{n=1}^{\mathcal{N}} \frac{\sum_s \sum_{t'} \pi(s \mid x_{n1}) \prod_{t \neq t'} \mathcal{L}_t(d_{nt}, x_{nt+1} \mid x_{nt}, s; \theta) \frac{\partial \mathcal{L}_{t'}(d_{nt'}, x_{nt'+1} \mid x_{nt'}, s; \theta)}{\partial \theta}}{\sum_{s=1}^{S} \pi(s \mid x_{n1}) \prod_{t=1}^{T} \mathcal{L}_t(d_{nt}, x_{nt+1} \mid x_{nt}, s; \theta)}$$
$$= \sum_{n=1}^{\mathcal{N}} \frac{\sum_{s=1}^{S} \sum_{t'} \pi(s \mid x_{n1}) \prod_{t=1}^{T} \mathcal{L}_t(d_{nt}, x_{nt+1} \mid x_{nt}, s; \theta) \frac{\partial \ln \mathcal{L}_{t'}(d_{nt'}, x_{nt'+1} \mid x_{nt'}, s; \theta)}{\partial \theta}}{\sum_{s=1}^{S} \pi(s \mid x_{n1}) \prod_{t=1}^{T} \mathcal{L}_t(d_{nt}, x_{nt+1} \mid x_{nt}, s; \theta)}. \quad (31)$$

  The second equality can be obtained by multiplying $\frac{\mathcal{L}_{t'}(d_{nt'}, x_{nt'+1} \mid x_{nt'}, s; \theta)}{\mathcal{L}_{t'}(d_{nt'}, x_{nt'+1} \mid x_{nt'}, s; \theta)}$ in the last part.

- Bayes' rule implies that the conditional probability of n being in unobserved state s given the data for n and the parameters $\{\theta, \pi\}$, $q(s \mid d_n, x_n; \theta, \pi)$ is

$$q(s \mid d_n, x_n; \theta, \pi) = \frac{\pi(s \mid x_{n1}) \prod_{t=1}^{T} \mathcal{L}_t(d_{nt}, x_{nt+1} \mid x_{nt}, s; \theta)}{\sum_{s'} \pi(s' \mid x_{n1}) \prod_{t=1}^{T} \mathcal{L}_t(d_{nt}, x_{nt+1} \mid x_{nt}, s'; \theta)}. \quad (32)$$

FOC of the Log-Likelihood

- The FOC can then be rewritten as

$$0=\sum_{n=1}^{\mathcal{N}}\sum_{s=1}^{S}\sum_{t=1}^{\mathcal{T}}q\left(s\,|\,d_n,x_n;\theta,\pi\right)\frac{\partial\ln\mathcal{L}_t\left(d_{nt},x_{nt+1}\,|\,x_{nt},s;\theta\right)}{\partial\theta} \tag{33}$$

- Because $q\left(s\,|\,d_n,x_n;\theta,\pi\right)$ is the probability of n being in unobserved state s conditional on the data for n, averaging across all individuals with $x_{1n}=x_1$ must correspond to $\hat{\pi}(s\,|\,x_1)$

$$\hat{\pi}\left(s\,|\,x_1\right)=\frac{\sum_n q\left(s\,|\,d_n,x_n;\hat{\theta},\hat{\pi}\right)I\left(x_{1n}=x_1\right)}{\sum_n I\left(x_{1n}=x_1\right)} \tag{34}$$

EM Algorithm

- Aim to yield a solution to the first-order conditions in Equation 31 upon convergence. Namely, given initial values $\theta^{(1)}$ and $\pi^{(1)}$, the $(m+1)$-th iteration is given by the following two-step process.
- Step 1 (Expectation): Update the conditional probabilities of being in each unobserved state according to

$$q^{(m+1)}(s \mid d_n, x_n) = \frac{\pi^{(m)}(s \mid x_{n1}) \prod_{t=1}^{\mathcal{T}} \mathcal{L}_t\left[d_{nt}, x_{nt+1} \mid x_{nt}, s; \theta^{(m)}\right]}{\sum_{s'} \pi^{(m)}(s' \mid x_{n1}) \prod_{t=1}^{\mathcal{T}} \mathcal{L}_t\left[d_{nt}, x_{nt+1} \mid x_{nt}, s'; \theta^{(m)}\right]} \qquad (35)$$

and update the population probability of being in each unobserved state, given values for the first-period state variables, using

$$\pi^{(m+1)}(s \mid x_1) = \frac{\sum_n q^{(m+1)}(s \mid d_n, x_n) I(x_{1n} = x_1)}{\sum_n I(x_{1n} = x_1)}. \qquad (36)$$

EM Algorithm

- Step 2 (Maximization): Take $q^{(m+1)}(s \mid d_n, x_n)$ as given, obtain $\theta^{(m+1)}$ from

$$\theta^{(m+1)} = \arg\max_{\theta} \sum_{n=1}^{\mathcal{N}} \sum_{t=1}^{\mathcal{T}} \sum_{s=1}^{S} q^{(m+1)}(s \mid d_n, x_n) \ln[\mathcal{L}_t(d_{nt}, x_{nt+1} \mid x_{nt}, s; \theta)] \qquad (37)$$

- These two steps are repeated until convergence, with each step increasing the log likelihood of the original problem.
- The maximization step can also be carried out in stages. For example, re-express Equation 37

$$\theta^{(m+1)} = \arg\max_{\theta} \sum_{n=1}^{\mathcal{N}} \sum_{t=1}^{\mathcal{T}} \sum_{s=1}^{S} q^{(m+1)}(s \mid d_n, x_n)(\ln[p_t(d_{nt} \mid x_{nt}, s; \theta)] + \ln[f_t(x_{nt+1} \mid d_{nt}, x_{nt}, s; \theta_2)]). \qquad (38)$$

EM Algorithm and CCP

- We can then express the CCP $p_t(j \mid x_t, s)$ as

$$p_t(j \mid x_t, s) = \frac{\Pr(j, s \mid x_t)}{\Pr(s \mid x_t)} = \frac{E\left[d_{njt} I(s_n = s) \mid x_{nt} = x\right]}{E\left[I(s_n = s) \mid x_{nt} = x_t\right]} \tag{39}$$

- Applying Law of Iterated Expectation, we obtain

$$p_t(j \mid x_t, s) = \frac{E\left[d_{njt} E\left\{I(s_n = s) \mid d_n, x_n\right\} \mid x_{nt} = x\right]}{E\left[E\left\{I(s_n = s) \mid d_n, x_n\right\} \mid x_{nt} = x_t\right]} \tag{40}$$

  The inner expectations of both the numerator and the denominator are the conditional probabilities of being in each unobserved state, $q(s \mid d_n, x_n; \theta, \pi)$.
- Finally, we link the CCP of individual's choice d to CCP of each individual's type s:

$$p_t(j \mid x_t, s) = \frac{E\left[d_{njt} q(s \mid d_n, x_n) \mid x_{nt} = x\right]}{E\left[q(s \mid d_n, x_n) \mid x_{nt} = x_t\right]} \tag{41}$$

Full EM Algorithm

- Given initial values $\theta^{(1)}$ and $\pi^{(1)}$, , and the vector of CCPs $p^{(1)}$, the $(m+1)$ iteration is given by the following two-step process.
- In the expectation step,

$$q^{(m+1)}(s \mid d_n, x_n) = \frac{\pi^{(m)}(s \mid x_{n1}) \prod_{t=1}^{\mathcal{T}} \mathcal{L}_t \left[d_{nt}, x_{nt+1} \mid x_{nt}, s, p^{(m)}; \theta^{(m)}\right]}{\sum_{s'} \pi^{(m)}(s' \mid x_{n1}) \prod_{t=1}^{\mathcal{T}} \mathcal{L}_t \left[d_{nt}, x_{nt+1} \mid x_{nt}, s', p^{(m)}; \theta^{(m)}\right]}$$

$$\pi^{(m+1)}(s \mid x_1) = \frac{\sum_n q^{(m+1)}(s \mid d_n, x_n) I(x_{1n} = x_1)}{\sum_n I(x_{1n} = x_1)} \tag{42}$$

$$p_t^{(m+1)}(j \mid x_t, s) = \frac{\sum_{n=1}^{\mathcal{N}} q^{(m+1)}(s \mid d_n, x_n) d_{njt} I(x_{nt} = x_t)}{\sum_{n=1}^{\mathcal{N}} q^{(m+1)}(s \mid d_n, x_n) I(x_{nt} = x_t)}$$

- In the maximization step,

$$\theta^{(m+1)} = \arg\max_{\theta} \sum_{n=1}^{\mathcal{N}} \sum_{t=1}^{\mathcal{T}} \sum_{s=1}^{S} q^{(m+1)}(s \mid d_n, x_n) \ln\left\{\mathcal{L}_t \left[d_{nt}, x_{nt+1} \mid x_{nt}, s, p^{(m)}; \theta\right]\right\} \tag{43}$$