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Web-based Supplementary Materials for
"Quantile Regression for Recurrent Gap Time Data"
by

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Web Appendix A: Proof of Uniform Consistency

Define the functions $F(t \mid \boldsymbol{z}) = \Pr(X_{i1} \leqslant t \mid \boldsymbol{Z}_i = \boldsymbol{z})$ and $\widetilde{F}(t \mid \boldsymbol{z}) = \Pr(X_{i1} \leqslant t, \delta_i = 1 \mid \boldsymbol{Z}_i = \boldsymbol{z})$. Let $\bar{F}(t \mid \boldsymbol{z}) = 1 - F(t \mid \boldsymbol{z})$, $\bar{f}(x \mid \boldsymbol{z}) = -f(x \mid \boldsymbol{z}) = -dF(x \mid \boldsymbol{z})/dx$, and $\widetilde{f}(x \mid \boldsymbol{z}) = d\widetilde{F}(x \mid \boldsymbol{z})/dx$. For a column vector $\boldsymbol{a}$, let $\boldsymbol{a}^{\otimes 2}$ denote $\boldsymbol{a}\boldsymbol{a}^{\intercal}$. We impose the same regularity conditions for quantile regression for univariate survival data given in Peng and Huang (2008) as follows:

C1: $\boldsymbol{Z}_i$ is uniformly bounded, that is, $\sup_i \|\boldsymbol{Z}_i\| \leqslant M$ for some $M < \infty$.

C2: (i) Each component of $\mathrm{E}(\boldsymbol{Z}_i N_{i1}[\exp\{\boldsymbol{Z}_i^{\intercal}\boldsymbol{\beta}_0(\tau)\}])$ is a Lipschitz function of τ, and (ii) $\widetilde{f}(t \mid \boldsymbol{z})$ and $f(t \mid \boldsymbol{z})$ are bounded above uniformly in t and $\boldsymbol{z}$.

C3: (i) $\widetilde{f}\{\exp(\boldsymbol{z}^{\intercal}\boldsymbol{b}) \mid \boldsymbol{z}\} > 0$ for all $\boldsymbol{b} \in \mathcal{B}(d_0)$, (ii) $\mathrm{E}(\boldsymbol{Z}_i^{\otimes 2}) > 0$, (iii) each component of $\boldsymbol{J}(\boldsymbol{b})\{\boldsymbol{B}(\boldsymbol{b})\}^{-1}$ is uniformly bounded in $\boldsymbol{b} \in \mathcal{B}(d_0)$, where $\boldsymbol{B}(\boldsymbol{b}) = \mathrm{E}[\boldsymbol{Z}_i^{\otimes 2}\widetilde{f}\{\exp(\boldsymbol{Z}_i^{\intercal}\boldsymbol{b}) \mid \boldsymbol{Z}_i\}\exp(\boldsymbol{Z}_i^{\intercal}\boldsymbol{b})]$, $\boldsymbol{J}(\boldsymbol{b}) = \mathrm{E}[\boldsymbol{Z}_i^{\otimes 2}\bar{f}\{\exp(\boldsymbol{Z}_i^{\intercal}\boldsymbol{b}) \mid \boldsymbol{Z}_i\}\exp(\boldsymbol{Z}_i^{\intercal}\boldsymbol{b})]$, and $\mathcal{B}(d_0)$ is a neighborhood containing the truth $\{\boldsymbol{\beta}_0(\tau), \tau \in (0, \tau_U]\}$, defined in the next paragraph.

C4: $\inf_{\tau \in [\nu, \tau_U]} \operatorname{eigmin} \mathrm{E}[\boldsymbol{Z}_i^{\otimes 2}\widetilde{f}[\exp\{\boldsymbol{Z}_i^{\intercal}\boldsymbol{\beta}_0(\tau)\} \mid \boldsymbol{Z}_i]\exp\{\boldsymbol{Z}_i^{\intercal}\boldsymbol{\beta}_0(\tau)\}] > 0$ for any $\nu \in (0, \tau_U]$, where $\operatorname{eigmin}(\cdot)$ denotes the minimum eigenvalue of a matrix.

First, we prove the uniform consistency of the proposed estimator $\widehat{\boldsymbol{\beta}}^*(\cdot)$ in Theorem 1. Define the functions $\boldsymbol{\mu}(\boldsymbol{b}) = \mathrm{E}[\boldsymbol{Z}_i N_{i1}\{\exp(\boldsymbol{Z}_i^{\intercal}\boldsymbol{b})\}]$ and $\widetilde{\boldsymbol{\mu}}(\boldsymbol{b}) = \mathrm{E}[\boldsymbol{Z}_i R_{i1}\{\exp(\boldsymbol{Z}_i^{\intercal}\boldsymbol{b})\}]$. By double expectation and the exchangeability of $X_{i1}, \ldots, X_{im_i^*}$ conditional on $(\gamma_i, \boldsymbol{Z}_i, m_i, C_i)$, it can be shown that $\boldsymbol{\mu}(\boldsymbol{b}) = \mathrm{E}[\boldsymbol{Z}_i N_i^*\{\exp(\boldsymbol{Z}_i^{\intercal}\boldsymbol{b})\}]$ and $\widetilde{\boldsymbol{\mu}}(\boldsymbol{b}) = \mathrm{E}[\boldsymbol{Z}_i R_i^*\{\exp(\boldsymbol{Z}_i^{\intercal}\boldsymbol{b})\}]$. For $d > 0$, define $\mathcal{B}(d) = \{\boldsymbol{b} \in \mathcal{R}^p : \inf_{\tau \in (0, \tau_U]} \| \boldsymbol{\mu}(\boldsymbol{b}) - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\} \| \leqslant d\}$ and $\mathcal{A}(d) = \{\boldsymbol{\mu}(\boldsymbol{b}) : \boldsymbol{b} \in \mathcal{B}(d)\}$. Peng and Huang (2008) showed that, under condition C3, the mapping defined by $\boldsymbol{\mu} : \mathcal{B}(d_0) \to \mathcal{A}(d_0)$ is one-to-one. Hence there exists an inverse function of $\boldsymbol{\mu}$, denoted by $\boldsymbol{\kappa}$, such that $\boldsymbol{\kappa}\{\boldsymbol{\mu}(\boldsymbol{b})\} = \boldsymbol{b}$ for any $\boldsymbol{b} \in \mathcal{B}(d_0)$.

The proposed estimator $\widehat{\boldsymbol{\beta}}^*(\tau_k)$ $(k = 1, \ldots, L)$ are generalized solutions of the monotone estimating equation (5). Because all generalized solutions of a monotone estimating equation lie in a convex set with diameter $O(n^{-1})$ (Fygenson and Ritov, 1994), we have

$$n^{-1}\sum_{i=1}^{n}\boldsymbol{Z}_iN_i^*[\exp\{\boldsymbol{Z}_i^\intercal\widehat{\boldsymbol{\beta}}^*(\tau_k)\}] = n^{-1}\sum_{i=1}^{n}\int_0^{\tau_k}\boldsymbol{Z}_iR_i^*[\exp\{\boldsymbol{Z}_i^\intercal\widehat{\boldsymbol{\beta}}^*(u)\}]dH(u) + \boldsymbol{\zeta}_{nk},$$

for $k = 1, \ldots, L$, with $\max_{k=1,\ldots,L} \parallel \boldsymbol{\zeta}_{nk} \parallel \leqslant n^{-1}\sup_{1\leqslant i\leqslant n} \parallel \boldsymbol{Z}_i \parallel$. Then simple algebra yields

$$\begin{aligned}\boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}^*(\tau_k)\} - \boldsymbol{\mu}(\boldsymbol{\beta}_0(\tau_k)\} \quad &= \quad -\boldsymbol{\nu}_n^*\{\widehat{\boldsymbol{\beta}}^*(\tau_k)\} + \int_0^{\tau_k}\widetilde{\boldsymbol{\nu}}_n^*\{\widehat{\boldsymbol{\beta}}^*(u)\}dH(u)\\ &\quad + \sum_{l=1}^{k}\int_{\tau_{l-1}}^{\tau_l}[\widetilde{\boldsymbol{\mu}}\{\widehat{\boldsymbol{\beta}}^*(u)\} - \widetilde{\boldsymbol{\mu}}\{\boldsymbol{\beta}_0(u)\}]dH(u) + \boldsymbol{\zeta}_{nk},\end{aligned}$$

where $\boldsymbol{\nu}_n^*(\boldsymbol{b}) = n^{-1}\sum_{i=1}^{n}\boldsymbol{Z}_iN_i^*\{\exp(\boldsymbol{Z}_i^\intercal\boldsymbol{b})\} - \boldsymbol{\mu}(\boldsymbol{b})$ and $\widetilde{\boldsymbol{\nu}}_n^*(\boldsymbol{b}) = n^{-1}\sum_{i=1}^{n}\boldsymbol{Z}_iR_i^*\{\exp(\boldsymbol{Z}_i^\intercal\boldsymbol{b})\} - \widetilde{\boldsymbol{\mu}}(\boldsymbol{b})$. Because the class of indicator functions of polytopes in $\mathcal{R}^p$ is Givenko-Cantelli and $\boldsymbol{Z}_i$ is bounded, it follows from the Glivenko-Cantelli theorem that both $\sup_{\boldsymbol{b}\in\mathcal{R}^p} \parallel \boldsymbol{\nu}_n^*(\boldsymbol{b}) \parallel \to 0$ and $\sup_{\boldsymbol{b}\in\mathcal{R}^p} \parallel \widetilde{\boldsymbol{\nu}}_n^*(\boldsymbol{b}) \parallel \to 0$ almost surely as $n \to \infty$. Hence, for any given $\eta_1 > 0$, $\sup_k \parallel -\boldsymbol{\nu}_n^*\{\widehat{\boldsymbol{\beta}}(\tau_k)\} + \int_0^{\tau_k}\widetilde{\boldsymbol{\nu}}_{\boldsymbol{n}}^*\{\widehat{\boldsymbol{\beta}}(u)\}dH(u) \parallel < \eta_1$ with probability 1 as $n \to \infty$. Arguing as in Peng and Huang (2008), under conditions C1, C2(i), and C3(iii), we can define a sequence $\{\epsilon_l, l = 1, \ldots, L-1\}$ such that $\epsilon_l \leqslant 2\exp\{\tau_U/(1-\tau_U)\}\eta_1 < d_0$ for sufficiently large n and $\sup_{\tau_l\leqslant\tau<\tau_{l+1}} \parallel \boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}^*(\tau)\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\} \parallel \leqslant \epsilon_l$. Hence we can show that $\sup_{\tau\in(0,\tau_U]} \parallel \boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}^*(\tau)\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\} \parallel \to 0$ in probability as $n \to \infty$.

Write $\widehat{\boldsymbol{\alpha}}(\tau) = \boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}^*(\tau)\}$ and $\boldsymbol{\alpha}_0(\tau) = \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\}$. Applying a Taylor series expansion of $\boldsymbol{\kappa}\{\widehat{\boldsymbol{\alpha}}(\tau)\}$ around $\boldsymbol{\alpha}_0(\tau)$ for $\tau \in [\nu, \tau_U]$, we have, under condition C4, $\parallel \widehat{\boldsymbol{\beta}}^*(\tau) - \boldsymbol{\beta}_0(\tau) \parallel = \parallel \boldsymbol{\kappa}\{\widehat{\boldsymbol{\alpha}}(\tau)\} - \boldsymbol{\kappa}\{\boldsymbol{\alpha}_0(\tau)\} \parallel \leqslant \parallel \boldsymbol{B}\{\boldsymbol{\beta}_0(\tau)\}^{-1}\{\widehat{\boldsymbol{\alpha}}(\tau) - \boldsymbol{\alpha}_0(\tau)\} \parallel + \parallel \boldsymbol{\epsilon}^*(\tau) \parallel \leqslant \eta^* \parallel \widehat{\boldsymbol{\alpha}}(\tau) - \boldsymbol{\alpha}_0(\tau) \parallel + \parallel \boldsymbol{\epsilon}^*(\tau) \parallel$, where η^* does not depend on τ and $\sup_{\tau\in[\nu,\tau_U]} \parallel \boldsymbol{\epsilon}^*(\tau) \parallel \to 0$ in probability as $n \to \infty$. Thus we establish the uniform consistency of the proposed estimator.

Web Appendix B: Proof of Asymptotic Normality

Next, we prove the asymptotic normality of the proposed estimator $\widehat{\boldsymbol{\beta}}^*(\cdot)$. Define the functions $\mu_1(\boldsymbol{b}) = \mathrm{E}[N_{i1}\{\exp(\boldsymbol{Z}_i^\intercal\boldsymbol{b})\}]$, $\sigma_d^2(\boldsymbol{b}) = \mathrm{var}(N_{i1}\{\exp(\boldsymbol{Z}_i^\intercal\boldsymbol{b})\} - N_{i1}[\exp\{\boldsymbol{Z}_i^\intercal\boldsymbol{\beta}_0(\tau)\}] - \mu_1(\boldsymbol{b}) + \mu_1\{\boldsymbol{\beta}_0(\tau)\})$, and $\sigma_d^{2*}(\boldsymbol{b}) = \mathrm{var}(N_i^*\{\exp(\boldsymbol{Z}_i^\intercal\boldsymbol{b})\} - N_i^*[\exp\{\boldsymbol{Z}_i^\intercal\boldsymbol{\beta}_0(\tau)\}] - \mu_1(\boldsymbol{b}) + \mu_1\{\boldsymbol{\beta}_0(\tau)\})$. We first note that $\sigma_d^{2*}(\boldsymbol{b}) \leqslant \sigma_d^2(\boldsymbol{b})$. Following the proof of Lemma B.1. in Peng and Huang (2008), we can show that $\sigma_d^{2*}\{\widetilde{\boldsymbol{\beta}}_n(\tau)\} \leqslant \sigma_d^2\{\widetilde{\boldsymbol{\beta}}_n(\tau)\} \to 0$ in probability for any sequence $\{\widetilde{\boldsymbol{\beta}}_n(\tau), \tau \in (0, \tau_U]\}_{n=1}^{\infty}$ such that $\sup_{\tau\in(0,\tau_U]} \parallel \boldsymbol{\mu}\{\widetilde{\boldsymbol{\beta}}_n(\tau)\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\} \parallel \to 0$ in probability. Given the boundedness of $\boldsymbol{Z}_i$ and arguing as in Alexander (1984) and Lai (1988), we can show that

$$\sup_{\tau\in(0,\tau_U]}\left\| n^{-1/2}\sum_{i=1}^{n}\boldsymbol{Z}_i\left(N_i^*[\exp\{\boldsymbol{Z}_i^\intercal\widetilde{\boldsymbol{\beta}}_n(\tau)\}] - N_i^*[\exp\{\boldsymbol{Z}_i^\intercal\boldsymbol{\beta}_0(\tau)\}]\right) \right.$$
$$\left. -n^{1/2}\left[\boldsymbol{\mu}\{\widetilde{\boldsymbol{\beta}}_n(\tau)\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\}\right]\right\| \to 0 \qquad \text{(A.1)}$$

in probability. Similarly, we can show that

$$\sup_{\tau\in(0,\tau_U]}\left\| n^{-1/2}\sum_{i=1}^{n}\boldsymbol{Z}_i\left(R_i^*[\exp\{\boldsymbol{Z}_i^\intercal\widetilde{\boldsymbol{\beta}}_n(\tau)\}] - R_i^*[\exp\{\boldsymbol{Z}_i^\intercal\boldsymbol{\beta}_0(\tau)\}]\right) \right.$$
$$\left. -n^{1/2}\left[\widetilde{\boldsymbol{\mu}}\{\widetilde{\boldsymbol{\beta}}_n(\tau)\} - \widetilde{\boldsymbol{\mu}}\{\boldsymbol{\beta}_0(\tau)\}\right]\right\| \to 0 \qquad \text{(A.2)}$$

in probability, provided $\sup_{\tau\in(0,\tau_u]} \parallel \boldsymbol{\mu}\{\widetilde{\boldsymbol{\beta}}_n(\tau)\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\} \parallel \to 0$ in probability.

Let $o_I(a_n)$ denote a term that converges uniformly to 0 in probability in $\tau \in I$ after being divided by a_n. Then it follws from (A.1), (A.2) and the uniform consistency of $\widehat{\boldsymbol{\beta}}^*(\tau)$ for $\boldsymbol{\beta}_0(\tau)$ that

$$\begin{aligned}
&n^{1/2}\boldsymbol{U}^*(\widehat{\boldsymbol{\beta}}^*, \tau) - n^{1/2}\boldsymbol{U}^*(\boldsymbol{\beta}_0, \tau)\\
&\quad= n^{1/2}[\boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}^*(\tau)\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\}] - \int_0^\tau n^{1/2}[\widetilde{\boldsymbol{\mu}}\{\widehat{\boldsymbol{\beta}}^*(u)\} - \widetilde{\boldsymbol{\mu}}\{\boldsymbol{\beta}_0(u)\}]dH(u) + o_{(0,\tau_U]}(1)\\
&\quad= n^{1/2}[\boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}^*(\tau)\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\}] - \int_0^\tau \left[\boldsymbol{J}\{\boldsymbol{\beta}_0(u)\}\boldsymbol{B}\{\boldsymbol{\beta}_0(u)\}^{-1} + o_{(0,\tau_U]}(1)\right] \times\\
&\qquad\qquad n^{1/2}[\boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}^*(u)\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(u)\}]dH(u) + o_{(0,\tau_U]}(1).
\end{aligned}$$

Moreover, provided $n^{1/2} \parallel \mathcal{S}_L \parallel \to 0$, the inequality $\sup_{\tau\in[\tau_k,\tau_{k+1}]} n^{1/2} \parallel \boldsymbol{U}^*(\widehat{\boldsymbol{\beta}}^*, \tau) - \boldsymbol{U}^*(\widehat{\boldsymbol{\beta}}^*, \tau_j) \parallel \leqslant n^{1/2}\sup_i \parallel \boldsymbol{Z}_i \parallel \times\{H(\tau_{k+1}) - H(\tau_k)\}$ implies that $n^{1/2}\boldsymbol{U}^*(\widehat{\boldsymbol{\beta}}^*, \tau) = o_{(0,\tau_U]}(1)$ almost surely. Hence we can establish the asymptotic representation

$$n^{1/2}[\boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}^*(\tau)\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\}] = \boldsymbol{\phi}\{-n^{1/2}\boldsymbol{U}^*(\boldsymbol{\beta}_0, \tau)\} + o_{(0,\tau_U]}(1),$$

where $\boldsymbol{\phi}$ is a linear operator defined by

$$\boldsymbol{\phi}(\boldsymbol{g})(\tau) = \int_0^\tau \left(\prod_{u \in (s,\tau]} [\boldsymbol{I}_p + \boldsymbol{J}\{\boldsymbol{\beta}_0(u)\}\boldsymbol{B}\{\boldsymbol{\beta}_0(u)\}^{-1} dH(u)] \right) d\boldsymbol{g}(s)$$

for $\boldsymbol{g} \in \mathcal{F} = \{\boldsymbol{h} : [0, \tau_U] \rightarrow \mathcal{R}^p, \boldsymbol{g}$ is left-continuous with right limit, $\boldsymbol{g}(0) = \mathbf{0}\}$.

By noting that the class of monotone process $\{\boldsymbol{Z}_i N_i^*[\exp\{\boldsymbol{Z}_i^\intercal \boldsymbol{\beta}_0(\tau)\}], \tau \in (0, \tau_U]\}$ is Donsker and that $\int_0^\tau \boldsymbol{Z}_i R_i^*[\exp\{\boldsymbol{Z}_i^\intercal \boldsymbol{\beta}_0(u)\}]dH(u)$ is Lipschitz in τ, we can show that the class of empirical process $\{\boldsymbol{Z}_i N_i^*[\exp\{\boldsymbol{Z}_i^\intercal \boldsymbol{\beta}_0(\tau)\}] - \int_0^\tau \boldsymbol{Z}_i R_i^*[\exp\{\boldsymbol{Z}_i^\intercal \boldsymbol{\beta}_0(u)\}]dH(u), \tau \in [\nu, \tau_U]\}$ is Donsker. By the central limit theorem for empirical processes, $-n^{1/2}\boldsymbol{U}^*(\boldsymbol{\beta}_0, \tau)$ converges weakly to a tight Gaussian process, $\boldsymbol{\mathcal{U}}(\tau)$, with mean 0 and covariance $\boldsymbol{\Sigma}(s,t) = E\{\boldsymbol{\xi}_i^*(s)\boldsymbol{\xi}_i^*(t)^\intercal\}$ for $\tau \in (0, \tau_U]$, where $\boldsymbol{\xi}_i^*(t) = \boldsymbol{Z}_i N_i^*[\exp\{\boldsymbol{Z}_i^\intercal \boldsymbol{\beta}_0(\tau)\}] - \int_0^\tau \boldsymbol{Z}_i R_i^*[\exp\{\boldsymbol{Z}_i^\intercal \boldsymbol{\beta}_0(u)\}]dH(u)$. By applying a Taylor series expansion to $\boldsymbol{\kappa}[\boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}^*(\tau)\}]$ around $\boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\}$ and the continuous mapping theorem, we can show that $n^{1/2}\{\widehat{\boldsymbol{\beta}}^*(\tau) - \boldsymbol{\beta}_0(\tau)\}$ converges to the Gaussian process $\boldsymbol{B}\{\boldsymbol{\beta}_0(\tau)\}^{-1}\boldsymbol{\phi}\{\boldsymbol{\mathcal{U}}(\tau)\}$.

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