Two-way Heteroscedastic ANOVA when the Number of Levels is Large

Short Running Title: Two-way ANOVA

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Abstract

We consider testing the main treatment effects and interaction effects in crossed two-way layout when one factor or both factors have large number of levels. Random errors are allowed to be nonnormal and heteroscedastic. In heteroscedastic case, we propose new test statistics. The asymptotic distributions of our test statistics are derived under both the null hypothesis and local alternatives. The sample size per treatment combination can either be fixed or tend to infinity. Numerical simulations indicate that the proposed procedures have good power properties and maintain approximately the nominal α-level with small sample sizes. A real data set from a study evaluating forty varieties of winter wheat in a large-scale agricultural trial is analyzed.

Key words: heteroscedasticity, large number of factor levels, unbalanced designs, quadratic forms, projection method, local alternatives

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1 Introduction

In many experiments, data are collected in the form of a crossed two-way layout. If we let X_{ijk} denote the k-th response associated with the i-th level of factor A and j-th level of factor B, then the classical two-way ANOVA model specifies that:

$$X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk} \tag{1.1}$$

where $i = 1, \ldots, a,\ j = 1, \ldots, b,\ k = 1, 2, \ldots, n_{ij}$, and to be identifiable the parameters are restricted by conditions such as $\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = \sum_{i=1}^{a} \gamma_{ij} = \sum_{j=1}^{b} \gamma_{ij} = 0$. The classical ANOVA model assumes that the error terms ϵ_{ijk} are iid normal with mean 0, in which case the F statistics for testing the null hypotheses of no treatment effects or no interaction effects have certain optimality properties (cf. Arnold, 1981, Chapter 7).

The study of properties of F-tests under violation of the classical assumptions of normality and homoscedasticity has a long history. See for example Box (1954), Box and Andersen (1955), Scheffé (1959, Chapter 10), Miller (1986, Chapter 4). However, these studies pertain only to the case when the number of treatment levels is small. In this case, Arnold (1980) showed that the classical F-test is robust to the normality assumption if in addition the sample size per treatment level tends to infinity. Portnoy (1984, 1986) investigated the asymptotic behaviors of M-estimators in general linear model when the number of treatment levels goes to infinity with the sample size. Li, Lindsay and Waterman (2003) discussed adjusted maximun likelihood estimator for multistratum data, in which both the number of strata and the number of within-stratum replications go to ∞.

Recently, there has been some interest in investigating the behavior of testing procedures when the number of treatment levels is large, see Boos and Brownie (1995), Akritas and Arnold (2000), Bathke (2002), Akritas and Papadatos (2004). The results in this present paper can be applied in many disciplines. For example, in agricultural trials it is not uncommon to see large number of treatments (cultivars, pesticides, fertilizers, etc) but limited replications per treatment combination. In a recent statewide agricultural study performed by Washington State University, 40 different varieties/lines of winter wheat are investigated. This data set will be analyzed in Section 5 below in detail. Our tests can also be applied to certain type of microarray data, where one factor corresponds to the large number of genes. Dudoit et al. (2000) described a replicated cDNA microarray experiment to compare the expression of genes in the livers of SR-BI transgenic mice with that of the corresponding wild-type mice. The data were summarized in a two-way layout, our method can be applied to test for the main and interaction effects.

The asymptotic theory of ANOVA test statistics when the number of levels tends to infinity is more complex than that when the levels are fixed and the sample sizes tend to infinity. For example, the test statistic for no main factor A effects in the balanced case (so $n_{ij} = n$) is MST_A/MSE, where $MST_A = b(a-1)^{-1}\sum_i n(\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot\cdot\cdot})^2$ and $MSE = [ab(n-1)]^{-1}\sum_i\sum_j\sum_k(\overline{X}_{ijk} - \overline{X}_{ij\cdot})^2$. Thus it is easily guessed that its limiting distribution, as $n \to \infty$ and a, b are fixed, will be a constant multiple of a χ^2 distribution. When, on the other hand, $a \to \infty$ the degrees of freedom of both MST_A and MSE tend to infinity and finding its limiting distribution requires that we study $a^{1/2}(MST_A/MSE - 1)$. Certainly this result cannot be guessed. Except for the one-way design, however, the aforementioned papers have considered only balanced homoscedastic models and showed that the usual F-procedure (i.e. using the F-test statistic with critical points from the F distribution) is asymptotically correct.

Hypothesis testing in unbalanced design is much more complex than in balanced case. For the general situation, a close-form expression for the test statistic may not be available (Arnold, 1981). Besides, one often needs to specify appropriate weights for the hypothesis. Different methods have been proposed (mostly for homoscedastic case) and there still remain many controversies. Arnold (1981) reviewed five different methods and also discussed the problem of choosing appropriate weights for the hypotheses, see also discussions in Ananda and Weerahandi (1997), Rencher (2000), Sahai and Ageel (2000). In this paper, we are interested in unbalanced heteroscedastic two-way ANOVA design when at least one of the factor levels tends to infinity. The cell sizes can be either fixed or also tend to infinity. For homoscedastic model, we consider test statistics based on the method of unweighted means, which was originally proposed by Yates (1934), see also Searle (1971), Hocking (1996). Since the method of unweighted means is valid only in the homoscedastic case, we propose and study a new statistic when the errors are heteroscedastic. One interesting aspect of the new statistic we propose for the heteroscedastic case is that its limiting distribution does not depend on the fourth moment. Since higher moments are typically more difficult to estimate accurately, the new statistic is computationally competitive even in cases where homoscedasticity can be ascertained.

We will focus on the hypotheses of no main factor A effects and no interaction effects:

$$H_0(A) : \alpha_i = 0,\ i = 1, \ldots, a, \text{ and } H_0(C) : \gamma_{ij} = 0, \quad i = 1, \ldots, a,\ j = 1, \ldots, b.$$

When both a and b tend to infinity, the problem of testing $H_0(B) : \beta_j = 0,\ j = 1, \ldots, b$, is symmetric to that of testing $H_0(A)$. When only a tends to infinity but b stays fixed, it can also be shown that the test statistic for $H_0(B)$ has asymptotically a chi-square distribution. The nature of this result is different because the numerator has fixed degrees of freedom, and will not be presented here. Finally, similar techniques apply for testing for no simple effects. For the sake of brevity, these results are not presented here but can be found in Wang (2003).

The rest of the paper is organized as the following. In Section 2, we present the main results for the homoscedastic and heteroscedastic case, including a limiting result under local alternatives. The projection method is discussed in Section 3. Numerical results and a data analysis are given in Sections 4, 5, respectively. Section 6 discusses conclusions and further work, while a sketch of the technical proofs is given in the Appendix.

2 Main Results

Throughout the article, X_{ijk} are assumed to be iid in each cell (i, j). Thus, the errors ϵ_{ijk} in (1.1) are not assumed to be iid. We write $\mathbf{X} = (X_{111}, \ldots, X_{11n_{11}}, \ldots, X_{1b1}, \ldots, X_{1bn_{1b}}, \ldots, X_{ab1}, \ldots, X_{abn_{11}})'$, $\mathbf{X}_i = (X_{i11}, \ldots, X_{i1n_{i1}}, \ldots, X_{ib1}, \ldots, X_{ibn_{ib}})'$, $\mathbf{X}_{ij} = (X_{ij1}, \ldots, X_{ijn_{ij}})'$, $\overline{X}_{ij.} = n_{ij}^{-1} \sum_{k=1}^{n_{ij}} X_{ijk}$, $\widetilde{X}_{i..} = b^{-1} \sum_{j=1}^{b} \overline{X}_{ij.}$, $\widetilde{X}_{...} = (ab)^{-1} \sum_{i=1}^{a} \sum_{j=1}^{b} \overline{X}_{ij}$ and $\widetilde{X}_{.j.} = a^{-1} \sum_{i=1}^{a} \overline{X}_{ij.}$, $N = \sum_i \sum_j n_{ij}$. If the cell sizes also tend to infinity with a or b, we can also write $n_{ij}(a, b)$ instead of n_{ij}, we set $n(a, b) = \min\{n_{ij}; i = 1, \ldots, a, j = 1, \ldots, b\}$, $\kappa(a, b) = \max\{n_{ij}; i = 1, \ldots, a, j = 1, \ldots, b\}$, and will assume that $n(a, b) \to \infty$ and $\kappa(a, b)/n(a, b) \leq C < \infty$, for all a, b, for some $C \geq 1$. In the case when only $a \to \infty$, we also use the notations: $n_{ij}(a)$, $n(a)$ and $\kappa(a)$ without confusion. Finally, $\mathbf{1}_d$ and $\mathbf{I}_d$ denote the $d \times 1$ column vector of 1's and the d-dimensional identity matrix, respectively, $\mathbf{J}_d = \mathbf{1}_d \mathbf{1}_d'$ and $\mathbf{P}_d = \mathbf{I}_d - \frac{1}{d}\mathbf{J}_d$.

2.1 Homoscedastic Model

The test statistics for testing hypotheses $H_0(A)$ and $H_0(C)$ are

$$Q_A = MST_A/MSE \quad \text{and} \quad Q_C = MST_C/MSE, \tag{2.1}$$

respectively, where

$$\begin{aligned}
MST_A &= \frac{b}{\frac{a-1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}}}\sum_{i=1}^{a}\left(\widetilde{X}_{i..} - \widetilde{X}_{...}\right)^2, \\
MST_C &= \frac{1}{\frac{(a-1)(b-1)}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}}}\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\overline{X}_{ij.} - \widetilde{X}_{i..} - \widetilde{X}_{.j.} + \widetilde{X}_{...}\right)^2, \\
MSE &= \frac{1}{N-ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}\left(X_{ijk} - \overline{X}_{ij.}\right)^2,
\end{aligned}$$

In the balanced case, Q_A and Q_C are the same as the classical F test statistics. In unbalanced case, they correspond to the test statistics of Yates, also known as the method of unweighed, or harmonic means.

Theorem 2.1 (Balanced case) *Let $Var(X_{ijk}) = \sigma^2 > 0$ and assume $E(X_{ijk}^4)$ are uniformly bounded.*
(a) If for all i, j, $n_{ij} = n \geq 2$ remains fixed,

1. *under $H_0(A)$,*

$$\begin{aligned}
a^{1/2}(Q_A - 1) &\to N\left(0, 2 + \frac{2}{b(n-1)}\right), \quad \text{as } a \to \infty \text{ and } b \text{ is fixed,} \\
a^{1/2}(Q_A - 1) &\to N(0, 2), \quad \text{as } a \to \infty \text{ and } b \to \infty.
\end{aligned}$$

2. *under $H_0(C)$,*

$$\begin{aligned}
a^{1/2}(Q_C - 1) &\to N\left(0, \frac{2}{b-1} + \frac{2}{b(n-1)}\right), \quad \text{as } a \to \infty \text{ and } b \text{ is fixed,} \\
(ab)^{1/2}(Q_C - 1) &\to N\left(0, 2 + \frac{2}{n-1}\right), \quad \text{as } a \to \infty \text{ and } b \to \infty.
\end{aligned}$$

(b) If for all i, j, $n_{ij} = n = n(a) \to \infty$, as $a \to \infty$, then

1. under $H_0(A)$,

$$
\begin{aligned}
a^{1/2}(Q_A - 1) &\rightarrow N(0,2), \quad \text{as } a \to \infty \text{ and } b \text{ is fixed} \\
a^{1/2}(Q_A - 1) &\rightarrow N(0,2), \quad \text{as } a \to \infty \text{ and } b \to \infty,
\end{aligned}
$$

2. under $H_0(C)$,

$$
\begin{aligned}
a^{1/2}(Q_C - 1) &\rightarrow N\left(0, \frac{2}{b-1}\right), \quad \text{as } a \to \infty \text{ and } b \text{ is fixed,} \\
(ab)^{1/2}(Q_C - 1) &\rightarrow N(0,2), \quad \text{as } a \to \infty \text{ and } b \to \infty.
\end{aligned}
$$

Remark 2.1. a) If we assume all the errors ϵ_{ijk} are iid then the result of Theorem 2.1 requires only finite second moments instead of fourth.
b) Results for the balanced case with fixed sample sizes when $a \to \infty$ and b is fixed overlap with those of Akritas and Arnold (2000), but our assumptions are weaker.

Theorem 2.2 (Unbalanced case) *Let $Var(X_{ijk}) = \sigma^2 > 0$, assume that for some $\delta > 0$, $E\left(|X_{ijk}|^{4+\delta}\right) < \infty$ are uniformly bounded, and set $\mu_4 = E(X_{ijk} - EX_{ijk})^4/\sigma^4$.*
(a) If $\limsup\,(ab)^{-1}\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{2+\delta} < \infty$, for some $\delta > 0$, then with

$$
\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij} \to b_1 \in (1,\infty), \quad \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}} \to b_2, \quad \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}^2} \to b_3,
$$

$$
\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}^3} \to b_4, \quad \frac{1}{ab^2}\sum_{i=1}^{a}\left(\sum_{j=1}^{b}\frac{1}{n_{ij}}\right)^2 \to b_5, \text{ and}
$$

$$
\tau_1^2 = \tau^* + \frac{2bb_5}{b_2^2}, \quad \tau_2^2 = \tau^* + \frac{2(b_3 b^2 - 2bb_3 + bb_5)}{b_2^2(b-1)^2}, \quad \tau_3^2 = \tau^* + \frac{2b_3}{b_2^2},
$$

where $\tau^* = \left[(b_2 - b_1)(b_1 - 1)^{-2} + b_4 b_2^{-2} + 2b_3 b_2^{-1}(b_1 - 1)^{-1}\right](\mu_4 - 3) + 2/(b_1 - 1)$,

1. under $H_0(A)$,

$$
\begin{aligned}
a^{1/2}(Q_A - 1) &\rightarrow N\left(0, \tau_1^2/b\right), \quad \text{as } a \to \infty \text{ and } b \text{ is fixed,} \\
a^{1/2}(Q_A - 1) &\rightarrow N\left(0, \frac{2b_5}{b_2^2}\right), \quad \text{as } a \to \infty \text{ and } b \to \infty.
\end{aligned}
$$

2. under $H_0(C)$,

$$\begin{aligned} a^{1/2}(Q_C - 1) &\rightarrow N\left(0, \tau_2^2/b\right), \quad \textit{as } a \to \infty \textit{ and } b \textit{ is fixed,} \\ (ab)^{1/2}(Q_C - 1) &\rightarrow N\left(0, \tau_3^2\right), \quad \textit{as } a \to \infty \textit{ and } b \to \infty. \end{aligned}$$

(b) If n_{ij} *also tend to infinity with* a *or* b*, then with*

$$\frac{a}{\left(\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}\right)^2} \sum_{i=1}^{a}\left(\sum_{j=1}^{b} n_{ij}^{-1}\right)^2 \to \tau_4^2, \quad \frac{ab}{\left(\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}\right)^2} \sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-2} \to \tau_5^2,$$

1. under $H_0(A)$,

$$\begin{aligned} a^{1/2}(Q_A - 1) &\rightarrow N\left(0, 2\tau_4^2\right), \quad \textit{as } a \to \infty \textit{ and } b \textit{ is fixed,} \\ a^{1/2}(Q_A - 1) &\rightarrow N\left(0, 2\tau_4^2\right), \quad \textit{as } a \to \infty \textit{ and } b \to \infty, \end{aligned}$$

2. under $H_0(C)$,

$$\begin{aligned} a^{1/2}(Q_C - 1) &\rightarrow N\left(0, \frac{2\tau_4^2}{(b-1)^2} + \frac{2\tau_5^2}{b}\left(1 - \frac{1}{(b-1)^2}\right)\right), \quad \textit{as } a \to \infty,\ b \textit{ fixed,} \\ (ab)^{1/2}(Q_C - 1) &\rightarrow N\left(0, 2\tau_5^2\right), \quad \textit{as } a \to \infty \textit{ and } b \to \infty. \end{aligned}$$

Remark 2.2. The exact distribution of the unweighted means statistic in the unbalanced case is not known even under normality and homoscedasticity. Neter, Wasserman and Kutner (1985, p.753) indicate that the usual F critical points give satisfactory approximation provided the ratios of sample sizes do not exceed 2, with most n_{ij} agreeing more closely. Contrary to the balanced case, the presence of β_2 in the limiting distribution given in Theorem 2.2 for fixed sample sizes implies that the unweighted means procedure (i.e. using the unweighted means test statistic with critical points from the F distribution) is not robust to departures from the normality assumption even under homoscedasticity. If the sample sizes also tend to infinity, the contribution of μ_4 in the limiting distribution becomes negligible, but even then Theorem 2.2 shows that the procedure is not valid asymptotically. To see how this statement is implied, let $N = \sum_i \sum_j n_{ij}$, suppose that only $a \to \infty$ while b remains fixed, and set $\overline{n} = \lim_{a\to\infty}(ab)^{-1}N$. Let U_{ab} have an $F_{a-1,N-ab}$ distribution, which is the approximate distribution of the unweighted means statistic for no main factor A effects, under homoscedasticity and normality. It is easily verified that if $\overline{n} < \infty$, $a^{1/2}(U_{ab} - 1) \to N(0, 2 + 2/[b(\overline{n} - 1)])$, as $a \to \infty$, while if N/a also tends to infinity with a then $a^{1/2}(U_{ab} - 1) \to N(0, 2)$. It follows, that the unweighted means procedure for testing for no main factor A effects will be asymptotically valid if the limiting distribution of $a^{1/2}(MST_A/MSE - 1)$ is as above. Similarly the usual F-procedure for testing for no interaction effects will be asymptotically valid if the limiting, as $a \to \infty$, distribution of $a^{1/2}(MST_C/MSE - 1)$ is $N(0, 2/(b - 1) + 2/[b(\overline{n} - 1)])$, if $\overline{n} < \infty$, and $N(0, 2/(b - 1))$ if N/a also tends to infinity with a. According to Theorem 2.2 the limiting distributions of $a^{1/2}(MST_A/MSE - 1)$ and $a^{1/2}(MST_C/MSE - 1)$ are different. To appreciate the difference, and to see if Neter, Wasserman and Kutner's recommendation applies also to the case where one of the factors has many levels, we considered the test for no factor A effects using normal observations (i.e. $\beta_2 = 3$) in a design with $a = 100$, $b = 4$, and $n_{ij} = 4$ for $i = 1, \ldots, 50$ and all j, and $n_{ij} = 8$ for $i = 51, \ldots, 100$ and all j. Then the 95th percentiles of the limiting distribution of $a^{1/2}(MST_A/MSE - 1)$ and the limit of its approximate distribution are $W_{1,0.05} = 2.51$ and $W_{2,0.05} = 2.38$, respectively. Use of $W_{1,0.05}$ results in approximate rejection rate of 0.059 under the null hypothesis. Changing the sample size 8 to 12, 16 and 20, the rejection rates become 0.070, 0.079, and 0.085, respectively. Thus, Neter, Wasserman and Kutner's recommendation seems valid even with many levels, provided the normality assumption holds.

Remark 2.3. a) It can be shown that if $n_{ij} = n$, then in Theorem 2.2, $\tau_1^2 = 2(n-1)^{-1} + 2b$ and $\tau_2^2 = 2(n - 1)^{-1} + 2b(b - 1)^{-1}$, $2b_5 b_2^{-1} = 2$ and $\tau_3^2 = 2 + 2(n - 1)^{-1}$, so the results are consistent with those in Theorem 2.1 if the design is balanced.
b) We need not assume, as we do in Theorem 2.2, that the kurtosis μ_{4ij} is the same in all cells (i, j). In this more general case, the test statistics will have more complex variance expressions, but the asymptotic normality still holds under similar assumptions.

2.2 Heteroscedastic Model

In the balanced heteroscedastic case, we still have $E(MSE) = E(MST_A)$, $E(MSE) = E(MST_C)$, under $H_0(A)$, $H_0(C)$, respectively. Thus, we may still use Q_A, Q_C as test statistics. This, however, is not true in the unbalanced case. A simple remedy is to replace MSE by a different linear combination of the cell sample variances. This yields

$$T_A = (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[\left(\widetilde{X}_{i\cdot\cdot} - \widetilde{X}_{\cdot\cdot\cdot} \right)^2 - \frac{1}{b}\left(1 - \frac{1}{a}\right) \frac{S_{ij}^2}{n_{ij}} \right], \tag{2.2}$$

$$T_C = (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[\left(\overline{X}_{ij\cdot} - \widetilde{X}_{i\cdot\cdot} - \widetilde{X}_{\cdot j\cdot} + \widetilde{X}_{\cdot\cdot\cdot} \right)^2 - \frac{(a-1)(b-1)}{ab} \frac{S_{ij}^2}{n_{ij}} \right], \tag{2.3}$$

where $S_{ij}^2 = (n_{ij}-1)^{-1} \sum_{k=1}^{n_{ij}} (X_{ijk} - \overline{X}_{ij\cdot})^2$, as test statistics for $H_0(A)$, $H_0(C)$, respectively. In balanced case, T_A and T_C are related to Q_A and Q_C by

$$T_A = (ab)^{-1/2} \frac{a-1}{n} (Q_A - 1) MSE, \quad T_C = (ab)^{-1/2} \frac{(a-1)(b-1)}{n} (Q_C - 1) MSE.$$

Theorems 2.3 deals with both the balanced and unbalanced case.

Theorem 2.3 *Let $0 < Var(X_{ijk}) = \sigma_{ij}^2 < \infty$. Then,*
(a) For $n_{ij} \geq 2$ fixed, if for some $\delta > 0$, $\limsup \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} E|X_{ij1} - E(X_{ij1})|^{4+\delta} < \infty$, and also

$$\frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{\sigma_{ij}^4}{n_{ij}(n_{ij}-1)} \to \phi^4, \quad \frac{1}{ab^2} \sum_{i=1}^{a} \sum_{j_1 \neq j_2}^{b} \frac{\sigma_{ij_1}^2}{n_{ij_1}} \frac{\sigma_{ij_2}^2}{n_{ij_2}} \to \eta^4,$$

1. *under $H_0(A)$,*

$$T_A \to N\left(0, \frac{2(\phi^4 + b\eta^4)}{b^2}\right), \text{ as } a \to \infty \text{ and } b \text{ is fixed,}$$
$$b^{1/2} T_A \to N\left(0, 2\eta^4\right), \quad \text{as } a \to \infty \text{ and } b \to \infty,$$

2. *under $H_0(C)$,*

$$T_C \to N\left(0, \frac{2(b-1)^2\phi^4}{b^2} + \frac{2\eta^4}{b}\right), \quad \text{as } a \to \infty \text{ and } b \text{ is fixed,}$$
$$T_C \to N\left(0, 2\phi^4\right), \quad \text{as } a \to \infty \text{ and } b \to \infty.$$

(b) If n_{ij} also tend to infinity with a or b, and for some $\delta > 0$, $\limsup \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} E|X_{ij1} - E(X_{ij1})|^{4+\delta} < \infty$, then with

$$\frac{n^2(a,b)}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{\sigma_{ij}^4}{n_{ij}(a,b)(n_{ij}(a,b)-1)} \to \phi_1^4, \quad \frac{n^2(a,b)}{ab^2} \sum_{i=1}^{a} \sum_{j_1 \neq j_2}^{b} \frac{\sigma_{ij_1}^2}{n_{ij_1}(a,b)} \frac{\sigma_{ij_2}^2}{n_{ij_2}(a,b)} \to \eta_1^4,$$

1. under $H_0(A)$,

$$
\begin{aligned}
n(a,b)T_A &\to N\left(0, \frac{2(\phi_1^4 + b\eta_1^4)}{b^2}\right), \quad \textit{as } a \to \infty \textit{ and } b \textit{ is fixed,} \\
n(a,b)b^{1/2}T_A &\to N\left(0, 2\eta_1^4\right), \quad \textit{as } a \to \infty \textit{ and } b \to \infty,
\end{aligned}
$$

2. under $H_0(C)$,

$$
\begin{aligned}
n(a,b)T_C &\to N\left(0, \frac{2(b-1)^2\phi_1^4}{b^2} + \frac{2\eta_1^4}{b}\right), \quad \textit{as } a \to \infty \textit{ and } b \textit{ is fixed,} \\
n(a,b)T_C &\to N\left(0, 2\phi_1^4\right), \quad \textit{as } a \to \infty \textit{ and } b \to \infty.
\end{aligned}
$$

Remark 2.4. To be able to apply the above theorem, we need to estimate ϕ^4, η^4, ϕ_1^4 and η_1^4. We estimate η^4 by $(ab^2)^{-1}\sum_{i=1}^{a}\sum_{j_1 \neq j_2}^{b} n_{ij_1}^{-1}n_{ij_2}^{-1}s_{ij_1}^2 s_{ij_2}^2$, where s_{ij}^2 denote sample variance for cell (i,j). This estimator is consistent. To obtain consistent estimation of η^4, we need to be able to estimate σ_{ij}^4 unbiasedly. We used a U-statistic to estimate σ_{ij}^4:

$$
\begin{aligned}
\binom{n_{ij}}{4}^{-1}\frac{1}{12}\sum &\left[(X_{ijk_1} - X_{ijk_2})^2(X_{ijk_3} - X_{ijk_4})^2 + (X_{ijk_1} - X_{ijk_3})^2(X_{ijk_2} - X_{ijk_4})^2\right. \\
&\left. + (X_{ijk_1} - X_{ijk_4})^2(X_{ijk_2} - X_{ijk_3})^2\right],
\end{aligned}
$$

where the sum is over all subsects of distinct values of (k_1, k_2, k_3, k_4), $k_l \in [1, n_{ij}]$, $l = 1, \ldots, 4$. ϕ_1^4 and η_1^4 are estimated similarly. For the consistency of U-statistics, we refer to Lee (1990).

We next investigate the asymptotic distributions of T_A and T_C under local alternative sequences. We consider only the case that $a \to \infty$ but b remains fixed, and thus we denote the smallest cell size by $n(a)$. It is convenient to represent the random variables under local alternatives as simple translations of random variables that satisfy the null hypotheses. Thus, for local alternatives to the hypothesis of no main row effect we let

$$
Y_{ijk} = X_{ijk} + \alpha_i(a), \quad i = 1, ..., a, \ j = 1, ..., b, \ k = 1, ..., n_{ij}, \tag{2.4}
$$

where X_{ijk} is a sequence of random variables satisfying $H_0(A)$, and

$$
\alpha_i(a) = a^{3/4}n(a)^{-1/2}\int_{(i-1)/a}^{i/a} g(t)dt, \tag{2.5}
$$

where g is a continuous function on [0,1] such that $\int_0^1 g(t)dt = 0$.

For local alternatives to the hypothesis of no interaction effect we will let

$$Y_{ijk} = X_{ijk} + \gamma_{ij}(a), \quad i = 1, ..., a, \; j = 1, ..., b, k = 1, ..., n_{ij}, \tag{2.6}$$

where X_{ijk} is a sequence of random variables satisfying $H_0(C)$, and

$$\gamma_{ij}(a) = a^{3/4} n(a)^{-1/2} \int_{(i-1)/a}^{i/a} g_j(t)dt, \tag{2.7}$$

where $g_j(t)$ are continuous on [0,1] such that $\sum_{j=1}^b g_j(t) = 0, \forall\, t$, and $\int_0^1 g_j(t)dt = 0, \forall\, j$.

Let $T_A(\mathbf{Y})$, $T_C(\mathbf{Y})$ be the statistics T_A, T_C evaluated on the Y_{ijk}.

Theorem 2.4 *Assume* $\max\{\sigma_{ij}^2; 1 \le i \le a, \; 1 \le j \le b\} = o(a^{-1/2})$ *and denote*

$$\theta_A^2 = \int_0^1 g^2(t)dt \quad \text{and} \quad \theta_C^2 = \frac{1}{b}\sum_{j=1}^b \int_0^1 g_j^2(t)dt.$$

(a) If $n_{ij} \ge 2$ remains fixed, and under the conditions of Theorem 2.3(a) imposed on X_{ijk},

1. *if Y_{ijk} are given in (2.4) with $\alpha_i(a)$ given by (2.5), then*

$$T_A(\mathbf{Y}) \quad \to \quad N\left(\frac{b^{1/2}\theta_A^2}{n(\infty)}, \; \frac{2(\phi^4 + b\eta^4)}{b^2}\right), \quad \text{as } a \to \infty \text{ and } b \text{ is fixed,}$$

2. *if Y_{ijk} are given in (2.6) with $\gamma_{ij}(a)$ given by (2.7), then, with $n(\infty) = \lim n(a)$,*

$$T_C(\mathbf{Y}) \quad \to \quad N\left(\frac{b^{1/2}\theta_C^2}{n(\infty)}, \; \frac{2(b-1)^2\phi^4}{b^2} + \frac{2\eta^4}{b^2}\right), \quad \text{as } a \to \infty \text{ and } b \text{ is fixed.}$$

(b) If $n = n(a) \to \infty$, as $a \to \infty$, and under the conditions of Theorem 2.3(b) imposed on X_{ijk},

1. *if Y_{ijk} are given in (2.4) with $\alpha_i(a)$ given by (2.5), then*

$$n(a)T_A(\mathbf{Y}) \quad \to \quad N\left(b^{1/2}\theta_A^2, \; \frac{2(\phi_1^4 + b\eta_1^4)}{b^2}\right), \quad \text{as } a \to \infty \text{ and } b \text{ is fixed,}$$

2. *if Y_{ijk} are given in (2.6) with $\gamma_{ij}(a)$ given by (2.7), then*

$$n(a)T_C(\mathbf{Y}) \quad \to \quad N\left(b^{1/2}\theta_C^2, \; \frac{2(b-1)^2\phi_1^4}{b^2} + \frac{2\eta_1^4}{b^2}\right), \quad \text{as } a \to \infty \text{ and } b \text{ is fixed.}$$

3 Projection Method

In this section we apply the Hájek's projection method to linearize our test statistics. If the statistic S is based on independent random vectors $\mathbf{U}_1, ..., \mathbf{U}_n$ and has finite second moment, then its projection onto the class of random variables of the form $\sum_{i=1}^{n} g_i(\mathbf{U}_i)$ where g_i are measurable with $Eg_i^2(\mathbf{U}_i) < \infty$ is given by $\hat{S} = \sum_{i=1}^{n} E(S|\mathbf{U}_i) - (n-1)ES$. See, for example, van der Vaart (1998, Chapter 11). If $\hat{S}$ is asymptotically equivalent to S, then it can be used for finding the asymptotic distribution of S. To achieve asymptotic equivalence, the space onto which we project has to be chosen appropriately. Akritas and Papadatos (2004) demonstrated that an appropriate space can be chosen for projecting quadratic forms that arise in one-way ANOVA designs. In this paper we show that the appropriate spaces to project the statistics Q_A and T_A is that of random variables of the form $\sum_{i=1}^{a} g_i(\mathbf{X}_i)$. The same space works also for Q_C and T_C if b is fixed, but if $b \to \infty$ the appropriate space for projecting Q_C and T_C is that of $\sum_{i=1}^{a} \sum_{j=1}^{b} g_{ij}(\mathbf{X}_{ij})$.

The first lemma and proposition below consider the projection of $MST_A - MSE$ and $MST_C - MSE$. The projection of T_A and T_C is considered in the following lemma and proposition. Note that by Slutsky's theorem the asymptotic distribution of $Q_A - 1$ and $Q_C - 1$ follows from that of $MST_A - MSE$, $MST_C - MSE$.

Lemma 3.1 *a) Under $H_0(A)$, the projection $\widehat{S}_A = \sum_{i=1}^{a} E\left(MST_A - MSE|\mathbf{X}_i\right)$ of $MST_A - MSE$ is given by*

$$\sum_{i=1}^{a} \sum_{j=1}^{b} \left(\frac{1}{\sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij}^{-1}} \frac{1}{n_{ij}^2} \mathbf{X}_{ij}' \mathbf{J}_{n_{ij}} \mathbf{X}_{ij} + \frac{1}{N-ab} \frac{1}{n_{ij}} \mathbf{X}_{ij}' \mathbf{J}_{n_{ij}} \mathbf{X}_{ij} - \frac{1}{N-ab} \mathbf{X}_{ij}' \mathbf{X}_{ij} \right)$$

$$+ \frac{1}{\sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij}^{-1}} \sum_{i=1}^{a} \sum_{j_1 \neq j_2}^{b} \frac{1}{n_{ij_1} n_{ij_2}} \mathbf{X}_{ij_1}' \mathbf{1}_{n_{ij_1}} \mathbf{1}_{n_{ij_2}}' \mathbf{X}_{ij_2}. \tag{3.1}$$

b) Under $H_0(C)$, the projection $\widehat{S}_{1C} = \sum_{i=1}^{a} E\left(MST_C - MSE|\mathbf{X}_i\right)$ of $MST_C - MSE$ is given by

$$\sum_{i=1}^{a} \sum_{j=1}^{b} \left(\frac{1}{\sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij}^{-1}} \frac{1}{n_{ij}^2} \mathbf{X}_{ij}' \mathbf{J}_{n_{ij}} \mathbf{X}_{ij} + \frac{1}{N-ab} \frac{1}{n_{ij}} \mathbf{X}_{ij}' \mathbf{J}_{n_{ij}} \mathbf{X}_{ij} - \frac{1}{N-ab} \mathbf{X}_{ij}' \mathbf{X}_{ij} \right)$$

$$- \frac{1}{(b-1) \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij}^{-1}} \sum_{i=1}^{a} \sum_{j_1 \neq j_2}^{b} \frac{1}{n_{ij_1} n_{ij_2}} \mathbf{X}_{ij_1}' \mathbf{1}_{n_{ij_1}} \mathbf{1}_{n_{ij_2}}' \mathbf{X}_{ij_2},$$

while $\widehat{S}_{2C} = \sum_{i=1}^{a}\sum_{j=1}^{b} E\left(MST_C - MSE|\mathbf{X}_{ij}\right)$ *is given by*

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\frac{1}{\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}}\frac{1}{n_{ij}^{2}}\mathbf{X}_{ij}'\mathbf{J}_{n_{ij}}\mathbf{X}_{ij} + \frac{1}{N-ab}\frac{1}{n_{ij}}\mathbf{X}_{ij}'\mathbf{J}_{n_{ij}}\mathbf{X}_{ij} - \frac{1}{N-ab}\mathbf{X}_{ij}'\mathbf{X}_{ij}\right).$$

Proposition 3.2 *Under homoscedasticity, and with* n_{ij} *either fixed or going to* ∞,
a) when $a \to \infty$, b *is fixed or both* a *and* b *go to* ∞, $a^{1/2}\left(MST_A - MSE - \widehat{S}_A\right) \xrightarrow{p} 0$, *under* $H_0(A)$.
b) under $H_0(C)$, *when* $a \to \infty$ *and* b *is fixed,* $a^{1/2}\left(MST_C - MSE - \widehat{S}_{1C}\right) \xrightarrow{p} 0$, *and when both* a *and* b *tend to infinity,* $(ab)^{1/2}\left(MST_C - MSE - \widehat{S}_{2C}\right) \xrightarrow{p} 0$.

Lemma 3.3 *a) Under* $H_0(A)$, *the projection* $\widetilde{T}_A = \sum_{i=1}^{a} E(T_A|\mathbf{X}_i)$ *of* T_A *is*

$$(ab)^{-1/2}\sum_{i=1}^{a}\frac{a-1}{ab}\left[\left(\sum_{j=1}^{b}\overline{X}_{ij.}\right)^{2} - \sum_{j=1}^{b}\frac{S_{ij}^{2}}{n_{ij}}\right]. \tag{3.2}$$

b) Under $H_0(C)$, *the projection* $\widetilde{T}_{1C} = \sum_{i=1}^{a} E(T_C|\mathbf{X}_i)$ *of* T_C *is*

$$\frac{(a-1)(b-1)}{(ab)^{3/2}}\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\overline{X}_{ij.}^{2} - \frac{S_{ij}^{2}}{n_{ij}}\right) - \frac{a-1}{(ab)^{3/2}}\sum_{i=1}^{a}\sum_{j_1\neq j_2}^{b}\overline{X}_{ij_1.}\overline{X}_{ij_2.}\ ,$$

while $\widetilde{T}_{2C} = \sum_{i=1}^{a}\sum_{j=1}^{b} E(T_C|\mathbf{X}_{ij})$ *is given by*

$$\frac{(a-1)(b-1)}{(ab)^{3/2}}\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\overline{X}_{ij.}^{2} - \frac{S_{ij}^{2}}{n_{ij}}\right).$$

Proposition 3.4 *Assume* $\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\sigma_{ij}^{2} \to \sigma^2 \in (0,\infty)$, *for* $\widetilde{T}_A$ *and* $\widetilde{T}_{1C}$ *defined in Lemma 3.3,*
a) Under $H_0(A)$, $n(a,b)b^{1/2}(T_A - \widetilde{T}_A) \xrightarrow{p} 0$, *in all cases considered.*
b) Under $H_0(C)$, $n(a,b)(T_C - \widetilde{T}_{1C}) \xrightarrow{p} 0$, *and* $n(a,b)(T_C - \widetilde{T}_{2C}) \xrightarrow{p} 0$, *when* b *stays fixed and tends to infinity, respectively, in all cases.*

Propositions 3.2 and 3.4 are used in the appendix for proving the main results.

4 Numerical Results

All simulations pertain to main row effects when only a is large; similar results are obtained for interaction effects and also for both a and b large but are not reported here.

They are based on 5,000 runs, using software Matlab 6.1 to generate random data. The results reported in the tables pertain to balanced designs, with $n = 4$ and $b = 2$. In the following, "CF" denotes the classical F test, "BHOM", "BHET", "UBHOM", "UB-HET" denote tests for the balanced homoscedastic, balanced heteroscedastic, unbalanced homoscedastic, unbalanced heteroscedastic situations, respectively.

Table 1 investigates the achieved levels using iid normal(0,1) and lognormal(0,1) variables. In this case the statistics used in the "CF", "BHOM" and "BHET" procedures are equivalent (see also the comment preceding Theorem 2.3). Thus, the three procedures differ only because they use different cut-off points (which are asymptotically equivalent), and that explains the difference in the achieved α-levels. Surprisingly, the level of the "BHET" procedure is quite accurate, though a bit on the conservative side, in this homoscedastic setting. In practice, of course, it is difficult to justify homoscedasticity with samples of size 4. Thus, given the results of Table 2 which used heteroscedastic errors, we can recommend the "BHET" procedure. In Table 2 the observations in cell (i, j) are generated from normal(0, 1) if $i \leq a/2$ and from normal(0, 25) otherwise.

Put Tables 1 and 2 about here

The effects of unequal variances can be more serious if the design is unbalanced. Some of our simulation results in heteroscedastic unbalanced case indicates that "CF" can be very liberal while the test based on our asymptotic results can be very accurate. In one such simulation with a=20, b=2, we used sample sizes 7, 6, 7, 5, 5, 14, 14, 4, 7, 5, 5, 7, 4, 7, 6, 7, 5, 7, 5, 6), for $j = 1$, and (14, 16, 5, 16, 16, 7, 4, 14, 6, 7, 4, 7, 14, 6, 4, 6, 5, 7, 4, 6), for $j = 2$. The observations in cell (i, j) are generated from normal(0, $(1 + i/4 + j/4)^2$) distribution. In this example, the level of "CF" is 0.1652, the level of "UBHOM" is 0.2528, while the level of test "UBHET" is 0.0806.

Tables 3 and 4 examine the power of the procedures in a balanced setting with a=20. In Table 3, the observations in the (i, j)-th cell are generated from $i * \tau/a + \text{normal}(0, 1)$, for $\tau = 0,\ 0.3,\ 0.6,\ 0.9,\ 1.2$. In Table 4, the observations in the (i, j)-th cell are generated from $i * \tau/a + \text{lognormal}(0, 1)$, for $\tau = 0,\ 1,\ 2,\ 3$. In both tables, $\tau = 0$ corresponds to the null hypothesis. In the normal setting of Table 3, "CF" is optimal so the increased power of our tests is due to their liberality. Table 4, however, indicates clearly that the power of "BHET" is higher than that of the others..

Put Tables 3 and 4 about here

5 Data Analysis

In an agricultural trial performed in 2002 by the Washington State University Statewide Extension Uniform Cereal Variety Testing Program (http://variety.wsu.edu/), 40 different varieties/lines of winter wheat are studied in terms of yield. The growing conditions are classified into four classes by the amount of rainfall and soil condition. The number of observations is 4 for each variety/line in the first class, 5 in the second class, 6 in the third class, and 4 in the fourth class. Of interest is to test whether there is any difference among varieties/lines and whether there is interaction effect between varieties/lines and growing conditions. This is an unbalanced two-way layout with one factor having large number of levels. The usual F test (CF) based on type III sum of squares gives p-value=0.0690 for the variety/line main effect and p-value=1 for interaction effect. "UBHOM" gives p-value=0.0535 for variety/line main effect and p-value=1 for interaction effect, while "UBHET" gives p-value=0.0886 for variety/line main effect and p-value=1 for interaction. Therefore, all approaches suggest marginally strong variety/line effect, but there is no evidence suggesting interaction effect.

To demonstrate the value of our nonparametric tests, we disturbed the data in both the mean values and the variances. We first disturbed the mean values by adding 5 (respectively 7) to the observations corresponding to varieties 15-26 for the treatment corresponding to the lowest (respectively second lowest) profile. As a result of this disturbance, the p-values for the variety/line main effect are 0.0083, 0.0022, 0.0067, for the CF, UBHOM and UBHET tests, respectively. Our heteroscedastic test distinguished itself when we also introduced a disturbance in the variances. To describe this disturbance, let X_{ijk}, $i = 1, \ldots, 40$, $j = 1, \ldots, 4$, denote the yield data after their mean was disturbed as described above. Then the variance-disturbed data are $\widetilde{X}_{ijk} = \overline{X}_{ij.} + (X_{ijk} - \overline{X}_{ij.})\sigma_j$, where $\sigma_j = (1.5)^{-1}$, for $j = 1, 4$, and $\sigma_j = 1.5$, for $j = 2, 3$. As a result of this additional disturbance, the p-values for the variety/line main effect are 0.2020, 0.2038, 0.0116, for the CF, UBHOM and UBHET tests, respectively. Thus, even though the design is nearly balanced, this modest disturbance in the variances is enough to affect the classical F-test as well as the present test which is based on the assumption of homoscedasticity.

6 Some Extensions

Viewing the covariate in a one-way ANCOVA design as a factor with many levels, the present methodology (with $a \to \infty$ and b) fixed, can be applied to the yet unresolved problem of testing for covariate effects and for interaction effects between the factor and covariate in the nonlinear fully nonparametric ANCOVA model proposed by Akritas, Arnold and Du (2000). The extension is not trivial since the present methodology requires some replication in the cells while in typical ANCOVA designs there is only one observation per covariate value. As demonstrated in the regression setting of Wang, Akritas and Van Keilegom (2002), this difficulty can be overcome using smoothness assumptions and considering windows around each covariate value. This generates cells with replicated observations, but cells will have common observations which destroys the present independence assumption. The methodology with both a and b tending to infinity, can be applied to extend the lack-of-fit test of Wang, Akritas and Van Keilegom (2002) to regression designs with two covariates.

Appendix: Proofs

Due to space limitation, we only give the proofs for the results on testing for main row effects and only when a goes to ∞. Proofs for other cases are similar and can be found in Wang (2003). Under $H_0(A)$, we may assume $E(X_{ijk}) = 0$ without loss of generality. We may also assume $Var(X_{ijk}) = 1$ in the homoscedastic setting.

Proof of Theorem 2.1. It is straightforward to show $MSE \xrightarrow{p} 1$ as $a \to \infty$ under current moment assumptions, both when n is fixed and $n = n(a) \to \infty$, so by Slutsky's theorem we may consider the numerator $a^{1/2}(MST_A - MSE)$ in both cases. By Proposition 3.2, it suffices to find the asymptotic distribution of $a^{1/2}\widehat{S}_A$. In the present balanced case, the expression in Lemma 3.1 is simplified to $a^{1/2}\widehat{S}_A = a^{-1/2}\sum_{i=1}^{a} Y_i$, where

$$Y_i = \frac{1}{b(n-1)} \sum_{j=1}^{b} \sum_{k_1 \neq k_2}^{n} X_{ijk_1} X_{ijk_2} + \frac{1}{bn} \sum_{j_1 \neq j_2}^{b} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} X_{ij_1k_1} X_{ij_2k_2} \tag{A.1}$$

are independent random variables. Clearly, $E(Y_i) = 0$, $E(Y_i^2) = 2+2(b(n-1))^{-1}$. Thus we have $Var(a^{1/2}\widehat{S}_A) = 2 + \frac{2}{b(n-1)}$. Under both conditions (n fixed or $n \to \infty$), Liapounov's CLT can be applied to prove asymptotic normality. More specifically, we want to check that for some $\delta > 0$, $\sum_{i=1}^{a} E|a^{-1/2}Y_i|^{2+\delta} \to 0$ when $a \to \infty$. This is easily seen to be true when taking $\delta = 2$. □

Proof of Theorem 2.2. (a) Using Proposition 3.2 and the fact that $MSE \xrightarrow{p} 1$, it suffices to show $a^{1/2}\widehat{S}_A \to N(0, \tau_1^2/b)$. Let $c_1 = \left(\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}}\right)^{-1}$, $c_2 = (N-ab)^{-1}$, it follows that: $a^{1/2}\widehat{S}_A = \sum_{i=1}^{a} Y_{a,i}$, where

$$\begin{aligned} Y_{a,i} &= a^{1/2}\left[\sum_{j=1}^{b}\sum_{k_1=1}^{n_{ij}}\sum_{k_2=1}^{n_{ij}}\left(\frac{c_1}{n_{ij}^2}+\frac{c_2}{n_{ij}}\right)X_{ijk_1}X_{ijk_2} - \sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} c_2 X_{ijk}^2\right. \\ &\quad \left. + c_1\sum_{j_1\neq j_2}^{b}\sum_{k_1=1}^{n_{ij_1}}\sum_{k_2=1}^{n_{ij_2}}\frac{1}{n_{ij_1}n_{ij_2}}X_{ij_1k_1}X_{ij_2k_2}\right] \end{aligned} \tag{A.2}$$

are independent random variables. $\sum_{i=1}^{a} E(Y_{a,i}) = 0$. Straightforward but tedious calculations show that $Var\left(a^{1/2}\widehat{S}_A\right) \to \tau_1^2/b$. It remains to check Lyapounov's condition, i.e., that for some $\delta > 0$,

$$\sum_{i=1}^{a} E\left[|Y_{a,i} - E(Y_{a,i})|^{2+\delta}\right] = (abc_1)^{2+\delta}\frac{1}{a^{1+\delta/2}b^{2+\delta}}\sum_{i=1}^{a} R_{a,i} \to 0, \text{ as } a \to \infty, \tag{A.3}$$

where

$$\begin{aligned} R_{a,i} &= E\left|\sum_{j=1}^{b}\sum_{k_1=1}^{n_{ij_1}}\sum_{k_2=1}^{n_{ij_2}}\left(\frac{1}{n_{ij}^2}+\frac{c_2}{c_1}\frac{1}{n_{ij}}\right)X_{ijk_1}X_{ijk_2} - \sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}\frac{c_2}{c_1}X_{ijk}^2\right. \\ &\quad \left. + \sum_{j_1\neq j_2}^{b}\sum_{k_1=1}^{n_{ij_1}}\sum_{k_2=1}^{n_{ij_2}}\frac{1}{n_{ij_1}n_{ij_2}}X_{ij_1k_1}X_{ij_2k_2} - \sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}\left(\frac{1}{n_{ij}^2}+\frac{c_2}{c_1 n_{ij}}-\frac{c_2}{c_1}\right)\right|^{2+\delta}. \end{aligned}$$

Note that $abc_1 \to 1/b_2$. $(A.3)$ is proved by applying the following inequality repeatedly,

$$\left|\sum_{i=1}^{m} z_i\right|^p \leq m^{p-1}\sum_{i=1}^{m}|z_i|^p, \quad m \geq 1, \quad p \geq 1. \tag{A.4}$$

More specifically, by (A.4),

$$
\begin{aligned}
R_{a,i} \;\leq\; & 3^{1+\delta}\Bigg\{E\left|\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}\left(\frac{1}{n_{ij}^2}+\frac{c_2}{c_1}\frac{1}{n_{ij}}-\frac{c_2}{c_1}\right)(X_{ijk}^2-1)\right|^{2+\delta} \\
& +E\left|\sum_{j=1}^{b}\sum_{k_1\neq k_2}^{n_{ij}}\left(\frac{1}{n_{ij}^2}+\frac{c_2}{c_1}\frac{1}{n_{ij}}\right)X_{ijk_1}X_{ijk_2}\right|^{2+\delta}+E\left|\sum_{j_1\neq j_2}^{b}\sum_{k_1=1}^{n_{ij_1}}\sum_{k_2=1}^{n_{ij_2}}\frac{1}{n_{ij_1}n_{ij_2}}X_{ij_1k_1}X_{ij_2k_2}\right|^{2+\delta}\Bigg\} \\
& = 3^{1+\delta}(D_1+D_2+D_3).,
\end{aligned}
$$

where the definition of D_i, i=1, 2, 3, is clear from the context. First, notice that $\frac{1}{n_{ij}^2}\leq 1$, $\left|\frac{1}{n_{ij}}\frac{c_2}{c_1}-\frac{c_2}{c_1}\right| = \frac{n_{ij}-1}{n_{ij}}\cdot\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}}}{N-ab}\leq\frac{n_{ij}-1}{n_{ij}}\cdot\frac{ab}{N-ab}\leq 1$ and apply (A.4), we have

$$
\begin{aligned}
D_1 \;\leq\; & b^{1+\delta}\sum_{j=1}^{b}E\left|\left(\frac{1}{n_{ij}^2}+\frac{c_2}{c_1}\frac{1}{n_{ij}}-\frac{c_2}{c_1}\right)\sum_{k=1}^{n_{ij}}(X_{ijk}^2-1)\right|^{2+\delta}\leq b^{1+\delta}2^{2+\delta}\sum_{j=1}^{b}E\left|\sum_{k=1}^{n_{ij}}(X_{ijk}^2-1)\right|^{2+\delta} \\
\leq\; & b^{1+\delta}2^{3+2\delta}\sum_{j=1}^{b}n_{ij}^{1+\delta}\sum_{k=1}^{n_{ij}}(E|X_{ijk}|^{4+2\delta}+1)=b^{1+\delta}2^{3+2\delta}\sum_{j=1}^{b}n_{ij}^{2+\delta}(E|X_{ij1}|^{4+2\delta}+1).
\end{aligned}
$$

Similarly,

$$
D_2 \;\leq\; b^{1+\delta}\sum_{j=1}^{b}E\left|\left(\frac{1}{n_{ij}^2}+\frac{c_2}{c_1}\frac{1}{n_{ij}}\right)\sum_{k_1\neq k_2}^{n_{ij}}X_{ijk_1}X_{ijk_2}\right|^{2+\delta}\leq b^{1+\delta}2^{2+\delta}\sum_{j=1}^{b}E\left|\sum_{k_1\neq k_2}^{n_{ij}}X_{ijk_1}X_{ijk_2}\right|^{2+\delta}.
$$

We now apply a new inequality: for any $p\geq 2$ there exists a finite positive constant A_p (depending only on p) such that for any i.i.d random variables $Z_1,\ldots,Z_n$ with $E(Z_i)=0$,

$$
E|Z_1+\cdots+Z_n|^p\leq A_p\; n^{p/2}E|Z_1|^p. \tag{A.5}
$$

This inequality can be proved by first using Marcinkiewicz-Zygmund inequality (see Chow and Teicher, 1997, pp. 386-387): For a sequence of independent random variables $V_1,\ldots,V_n$ with mean 0, there exists a finite positive constant A_p depending only on p such that

$$
E\left|\sum_{i=1}^{n}V_i\right|^p\leq A_pE\left|\left(\sum_{i=1}^{n}V_i^2\right)\right|^{p/2} \tag{A.6}
$$

and then applying inequality $(A.4)$. Thus, writing

$$
\left|\sum_{k_1\neq k_2}^{n_{ij}}X_{ijk_1}X_{ijk_2}\right|^{2+\delta}=\left|\left(\sum_{k=1}^{n_{ij}}X_{ijk}\right)^2-\sum_{k=1}^{n_{ij}}\left(X_{ijk}^2-1\right)-n_{ij}\right|^{2+\delta},
$$

and using inequality $(A.4)$, (A.5) and Hölder's inequality, we have $E\left|\sum_{k_1\neq k_2}^{n_{ij}} X_{ijk_1}X_{ijk_2}\right|^{2+\delta} \leq 3^{1+\delta}D_\delta n_{ij}^{2+\delta}E|X_{ij1}|^{4+2\delta}$, where D_δ are finite positive constants depending only on δ. We thus have

$$D_2 \;\leq\; b^{1+\delta}2^{2+\delta}\sum_{j=1}^{b}3^{1+\delta}D_\delta n_{ij}^{2+\delta}E|X_{ij1}|^{4+2\delta}.$$

Writing $D_3 = E\left|\left(\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}\frac{X_{ijk}}{n_{ij}}\right)^2 - \sum_{j=1}^{b}\sum_{k_1=1}^{n_{ij}}\sum_{k_2=1}^{n_{ij}}\frac{X_{ijk_1}X_{ijk_2}}{n_{ij}^2}\right|^{2+\delta}$ and apply inequality $(A.4)$, (A.5) repeatedly, we have

$$D_3 \;\leq\; 2^{1+\delta}b^{1+\delta}G_\delta\sum_{j=1}^{b}\frac{E\left|X_{ij1}\right|^{4+2\delta}}{n_{ij}^{2+\delta}},$$

where G_δ are finite positive constants depending only on δ. Combining the upper bounds we obtained above on D_1, D_2 and D_3, we have

$$R_{a,i} \leq H_\delta b^{1+\delta}\sum_{j=1}^{b}n_{ij}^{2+\delta}E|X_{ij1}|^{4+2\delta},$$

for some positive constant H_δ depending only on δ. (A.3) thus holds.

(b) Under the new set of conditions, we have $\mathrm{Var}\left((ab)^{1/2}\widehat{S}_A\right) \to 2b\tau_4^2$. The asymptotic normality is proved by checking Lyapounov's condition in (A.3). The proof follows the same lines as in (a) except that now since we have

$$\left|\frac{1}{n_{ij}^2(a)} + \frac{c_2}{c_1}\frac{1}{n_{ij}(a)} - \frac{c_2}{c_1}\right| \leq \frac{2}{n^2(a)}, \quad \left|\frac{1}{n_{ij}^2(a)} + \frac{c_2}{c_1}\frac{1}{n_{ij}(a)}\right| \leq \frac{2}{n^2(a)},$$

The upper bounds on D_1, D_2 and D_3 become:

$$D_1 \leq b^{1+\delta}n(a)^{-4-2\delta}\kappa(a)^{2+\delta}2^{3+2\delta}\sum_{j=1}^{b}(E|X_{ij1}|^{4+2\delta}+1),$$

$$D_2 \leq b^{1+\delta}n(a)^{-4-2\delta}\kappa(a)^{2+\delta}2^{2+\delta}\sum_{j=1}^{b}3^{1+\delta}D_\delta E|X_{ij1}|^{4+2\delta},$$

$$D_3 \leq 2^{1+\delta}n(a)^{-2-\delta}b^{1+\delta}G_\delta\sum_{j=1}^{b}E\left|X_{ij1}\right|^{4+2\delta}.$$

Notice that $abc_1(a) \leq \kappa(a)$, (A.3) still holds. $\square$

Proof of Theorem 2.3. (a) For the case n_{ij} are fixed, Proposition 3.4 indicates that the distributions of T_A and $\widetilde{T}_A$ are asymptotically equivalent. $\widetilde{T}_A$ can be expressed as $\widetilde{T}_A = \sum_{i=1}^{a} \widetilde{T}_{Ai}$, where $\widetilde{T}_{Ai} = (ab)^{-1/2}\frac{a-1}{ab}\left[\sum_{j=1}^{b}\sum_{k_1\neq k_2}^{n_{ij}} \frac{X_{ijk_1}X_{ijk_2}}{n_{ij}(n_{ij}-1)} + \sum_{j_1\neq j_2}^{b} \overline{X}_{ij_1.}\overline{X}_{ij_2.}\right]$ are independent with $E(\widetilde{T}_{Ai})$=0. The two terms in $\widetilde{T}_{Ai}$ are uncorrelated,

$$E(\widetilde{T}_A^2) \;=\; \frac{2(a-1)^2}{(ab)^3}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{\sigma_{ij}^4}{n_{ij}(n_{ij}-1)} + \sum_{i=1}^{a}\sum_{j_1\neq j_2}^{b}\frac{\sigma_{ij_1}^2}{n_{ij_1}}\frac{\sigma_{ij_2}^2}{n_{ij_2}}\right] \to \frac{2}{b^2}(\phi^4 + b\eta^4).$$

It remains to verify the Lyapounov's condition. By inequality $(A.4)$,

$$\begin{aligned}
E\left|\widetilde{T}_{Ai}\right|^{2+\delta} &\leq (ab)^{-1-\delta/2}\frac{1}{b^{2+\delta}}2^{1+\delta}\left[E\left|\sum_{j=1}^{b}\frac{\sum_{k_1\neq k_2}^{n_{ij}} X_{ijk_1}X_{ijk_2}}{n_{ij}(n_{ij}-1)}\right|^{2+\delta} + E\left|\sum_{j_1\neq j_2}^{b}\overline{X}_{ij_1.}\overline{X}_{ij_2.}\right|^{2+\delta}\right] \\
&\leq (ab)^{-1-\delta/2}\frac{1}{b^{2+\delta}}2^{1+\delta}\left[b^{1+\delta}\sum_{j=1}^{b}\frac{3^{1+\delta}D_\delta E|X_{ij1}|^{4+2\delta}}{(n_{ij}-1)^{2+\delta}} + 2^{1+\delta}b^{1+\delta}G_\delta\sum_{j=1}^{b}\frac{E\left|X_{ij1}\right|^{4+2\delta}}{n_{ij}^{2+\delta}}\right] \\
&\leq a^{-1-\delta/2}b^{-2-\delta/2}H_\delta\sum_{j=1}^{b}\frac{E|X_{ij1}|^{4+2\delta}}{(n_{ij}-1)^{2+\delta}},
\end{aligned}$$

where D_δ, G_δ and H_δ are finite positive constants depending only on δ, The second inequality uses (A.6) and $(A.4)$, similar as in the proof of Theorem 2.2. Thus, $\sum_{i=1}^{a} E\left|\widetilde{T}_{Ai}\right|^{2+\delta} \to 0$ and Lyapounov's condition is satisfied.

(b) For the case that $n_{ij}\to\infty$, a similar calculation yields

$$Var\left(n(a)\widetilde{T}_A\right) \to \frac{2(\phi_1^4 + b\eta_1^4)}{b^2}.$$

Lyapounov's condition will be satisfied if : $\exists\ \delta > 0$, such that

$$L(a) = \sum_{i=1}^{a} E\left|n(a)\widetilde{T}_{Ai}\right|^{2+\delta} \to 0. \tag{A.7}$$

Similarly as in (a), we have

$$L(a) \leq \frac{n(a)^{2+\delta}}{(n(a)-1)^{2+\delta}}a^{-1-\delta/2}b^{-2-\delta/2}H_\delta\sum_{i=1}^{a}\sum_{j=1}^{b}E|X_{ij1}|^{4+2\delta} \to 0.$$

□

Proof of Theorem 2.4. Write

$$T_A(\mathbf{Y}) = T_A(\mathbf{X}) + (ab)^{-1/2}\sum_{i=1}^{a}\sum_{j=1}^{b}\alpha_i^2(a) + 2(ab)^{-1/2}\sum_{i=1}^{a}\sum_{j=1}^{b}\alpha_i(a)\widetilde{X}_{i..},$$

Denote $h(a) = a^{-1/2}\sum_{i=1}^{a}\alpha_i^2(a)$, $\quad H(a) = 2a^{-1/2}\sum_{i=1}^{a}\alpha_i(a)\widetilde{X}_{i\cdot\cdot}$. In both cases, $n(a)h(a) = a\sum_{i=1}^{a}(\int_{(i-1)/a}^{i/a} g(t)dt)^2 = a^{-1}\sum_{i=1}^{a} g^2(\xi_{ia}) \to \int_0^1 g^2(t)dt = \theta_A^2$, where $\xi_{ia} \in [\frac{i-1}{a}, \frac{i}{a}]$. $E(H(a)) = 0$.

$$Var(H(a)) = 4a^{-1}\sum_{i=1}^{a}\alpha_i^2(a)\frac{1}{b^2}\sum_{j=1}^{b}\frac{\sigma_{ij}^2}{n_{ij}(a)} \le \frac{4a^{-1/2}max_{\substack{1\le i\le a\\ 1\le j\le b}}\sigma_{ij}^2}{bn(a)}h(a) \to 0$$

as $a \to \infty$. Thus $H(a) \xrightarrow{p} 0$. The proof is finished by combining the results of Theorem 2.3 and Slutsky's theorem. $\square$

Proof of Lemma 3.1.

$$\begin{aligned} & MST_A - MSE \\ = & \sum_{i=1}^{a}\sum_{j=1}^{b}\Bigg(\frac{1}{\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}}\frac{1}{n_{ij}^2}\mathbf{X}_{ij}'\mathbf{J}_{n_{ij}}\mathbf{X}_{ij} + \frac{1}{N-ab}\frac{1}{n_{ij}}\mathbf{X}_{ij}'\mathbf{J}_{n_{ij}}\mathbf{X}_{ij} \\ & -\frac{1}{N-ab}\mathbf{X}_{ij}'\mathbf{X}_{ij}\Bigg) + \frac{a}{(a-1)\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}}\sum_{i=1}^{a}\sum_{j_1\neq j_2}^{b}\frac{1}{n_{ij_1}n_{ij_2}}\mathbf{X}_{ij_1}'\mathbf{1}_{n_{ij_1}}\mathbf{1}_{n_{ij_2}}'\mathbf{X}_{ij_2} \\ & -\frac{1}{(a-1)\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}}\sum_{(i_1,j_1)}\sum_{\neq(i_2,j_2)}\frac{1}{n_{i_1j_1}n_{i_2j_2}}\mathbf{X}_{i_1j_1}'\mathbf{1}_{n_{i_1j_1}}\mathbf{1}_{n_{i_2j_2}}'\mathbf{X}_{i_2j_2}. \end{aligned}$$

Using this expression and the fact that

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}\left(\frac{1}{\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}}\frac{1}{n_{ij}^2} + \frac{1}{N-ab}\frac{1}{n_{ij}} - \frac{1}{N-ab}\right) = 0,$$

We easily obtain the expression of projection $\sum_{i=1}^{a} E\left(MST_A - MSE|\mathbf{X}_i\right)$. $\square$

Proof of Proposition 3.2. From the proof of Lemma 3.1,

$$MST_A - MSE - \widehat{S}_A = -\frac{1}{(a-1)\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}}\sum_{i_1\neq i_2}^{a}\sum_{j_1=1}^{b}\sum_{j_2=1}^{b}\sum_{k_1=1}^{n_{i_1j_1}}\sum_{k_2=1}^{n_{i_2j_2}}\frac{1}{n_{i_1j_1}n_{i_2j_2}}X_{i_1j_1k_1}X_{i_2j_2k_2}.$$

We have

$$\begin{aligned} aE\left(MST_A - MSE - \widehat{S}_A\right)^2 &= \frac{2a}{(a-1)^2(\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}})^2}\sum_{i_1\neq i_2}^{a}\sum_{j_1=1}^{b}\sum_{j_2=1}^{b}\sum_{k_1=1}^{n_{i_1j_1}}\sum_{k_2=1}^{n_{i_2j_2}}\frac{1}{n_{i_1j_1}^2n_{i_2j_2}^2} \\ &\le \frac{2a}{(a-1)^2(\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}})^2}\left\{\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}}\right\}^2 = \frac{2a}{(a-1)^2} \to 0, \end{aligned}$$

as $a \to \infty$. $\square$

Proof of Lemma 3.3. The expression for the projection of the statistic T_A is easily derived by rewriting T_A as

$$T_A = (ab)^{-1/2} \sum_{i=1}^{a} \frac{a-1}{ab} \left[\left(\sum_{j=1}^{b} \overline{X}_{ij.} \right)^2 - \sum_{j=1}^{b} \frac{S_{ij}^2}{n_{ij}} \right] - (ab)^{-3/2} \sum_{i_1 \neq i_2}^{a} \left(\sum_{j=1}^{b} \overline{X}_{i_1 j.} \right) \left(\sum_{j=1}^{b} \overline{X}_{i_2 j.} \right).$$

$\square$

Proof of Proposition 3.4. Write

$$T_A - \widetilde{T}_A = -(ab)^{-3/2} \sum_{i_1 \neq i_2}^{a} \left(\sum_{j=1}^{b} \overline{X}_{i_1 j.} \right) \left(\sum_{j=1}^{b} \overline{X}_{i_2 j.} \right).$$

If $n_{ij}(a) \to \infty$ as $a \to \infty$, then

$$\begin{aligned} E\left(b^{1/2} n(a)(T_A - \widetilde{T}_A) \right)^2 &= \frac{2n^2(a)}{a^3 b^2} \sum_{i_1 \neq i_2}^{a} E \left(\sum_{j=1}^{b} \overline{X}_{i_1 j.} \right)^2 E \left(\sum_{j=1}^{b} \overline{X}_{i_2 j.} \right)^2 \\ &\leq \frac{2M}{a} \left(\frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \sigma_{ij}^2 \right)^2 \to 0, \end{aligned}$$

where M is a positive constant. For the case that $n_{ij} \geq 2$ is fixed, simply treat $n(a)$ as fixed in the above derivation. $\square$

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Table 1: Estimated levels for nominal 0.05 level tests, homoscedastic errors

a	error	CF	BHOM	BHET
20	normal(0,1)	0.055	0.084	0.088
	lognormal(0,1)	0.044	0.064	0.050
30	normal(0,1)	0.050	0.075	0.077
	lognormal(0,1)	0.042	0.060	0.040
40	normal(0,1)	0.049	0.072	0.073
	lognormal(0,1)	0.049	0.070	0.044

Table 2: Estimated levels for nominal 0.05 level tests, heteroscedastic errors.

	CF	BHOM	BHET
a=20	0.095	0.125	0.076
a=30	0.114	0.139	0.084
a=40	0.107	0.131	0.075
a=50	0.100	0.120	0.065

Table 3: Estimated power for nominal 0.05 level tests, normal(0,1) errors

τ	CF	BHOM	BHET
0	0.048	0.076	0.078
0.3	0.070	0.110	0.114
0.6	0.163	0.224	0.226
0.9	0.403	0.485	0.489
1.2	0.694	0.761	0.763

Table 4: Estimated power for nominal 0.05 level tests, lognormal(0,1) errors

tao	CF	BHOM	BHET
0	0.046	0.065	0.047
1	0.130	0.177	0.172
2	0.509	0.583	0.637
3	0.858	0.889	0.922