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Complete solution of the integrability problem for homothetic demand functions

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For any Walrasian demand function, the Strong Axiom implies (and is implied by) rationalizability by a complete preorder. However, these equivalent conditions do not ensure the existence of a *continuous* utility function or complete preorder giving raise to the primitive demand. We here propose a self-contained proof of a related fact: if the demand is homothetic and continuous, the Strong Axiom characterizes the existence of a continuous and homogeneous of degree one subjective utility function (or a continuous and homothetic complete preorder) representing the demand. Our contruction depends upon standard tools and overturns the need for ad-hoc axioms that were used in prior published literature on the topic.

Key words revealed preference, continuous homogeneous of degree one utility, integrability of demand

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1 Introduction

Since its introduction in the works by Ville (1946) and Houtakker (1950), the Strong Axiom (of revealed preference, sometimes called of consumer behavior) has played a central role in the pure theory of consumer behavior. If one wonders whether a given demand function is "sufficiently reasonable," it could be agreed that the meaning of such term is that the agent has a definite preference (complete and transitive) on the set of possible commodities, and that choices are the maxima of that preference subject to budget constraints. Remarkable contributions by Ville and Houthakker proved that the Strong Axiom characterizes demand functions that are rational in this particular sense. Richter (1966, 1971) first used a

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set-theoretic approach to obtain that result (see also Mas-Colell *et al.* (1995), proposition 3.J.1).

Nonetheless, this clear picture blurs when one intends to recognize whether a more stringent behavior can explain the primitive demand or not, and then some kind of continuity is an unavoidable request. Therefore, to answer the questions that arise one must identify the precise conditions that assure that a consumer's demand derives, for example, from (semi)continuous preferences or utilities. The majority of the works that investigate this topic are highly involved and provide partial answers: the only contributions that follow elementary approaches are Sondermann (1982) and Alcantud and Rodríguez-Palmero (2002), but neither of them provide necessary and sufficient conditions. Actually, these works do not aim at ensuring that a continuous utility can explain the demand, but rather only semicontinuity is achieved. As of now one is forced to conclude that in this field, striking results are forcefully linked to severely complex technicalities.

In this connection, consider now the homothetic case. After some previous approaches that appealed to integrability techniques (see e.g. Chipman 1974, theorem 2), Liu and Wong (2000) proved that a so-called Strong Axiom of Homothetic Revelation characterizes homothetic demand functions that are derived from continuous utilities of homogeneous type. Their technique has not been an exception in terms of the need for mathematically complex tools: the keystone in their argument is the appeal to the Compactness theorem of first-order predicate logic (Liu and Wong 2000, lemma 1). In the present work, we prove that neither the use of non-standard techniques nor the introduction of an ad-hoc axiom are needed to elicit the real situation.[1] We uncover the utterly transparent structure of the problem that was tackled in Liu and Wong (2000): the Strong Axiom is necessary and sufficient for a continuous homothetic demand to be represented by a continuous and homogeneous of degree one utility function. By contrast with the Liu–Wong approach, this full solution to a severe integrability problem is self-contained and depends upon standard tools only. Besides, it is free of superfluous assumptions like differentiability or restrictions on the Slutsky matrix. We convert the integrability question into one of representability of the revealed preference by *weak utilities*, and then we study that relation in check that it admits adequate utilities. The fact that only the Strong Axiom has to be requested (aside from the obvious continuity and homotheticity of the demand) is much in the direction of the original Ville–Houthakker achievement, thus a variation of their characterization follows from our main result without effort.[2]

The structure of this paper is as follows. We introduce some notation in Section 2, where some auxiliary results are also proven. Section 3 presents our main result. We summarize our contribution in the concluding Section 4.

[1] In a related line of inquiry, Alcantud *et al.* (2006) follows the approach initiated by Alcantud and Rodríguez-Palmero (2002) by yielding a proof of its main result that uses standard tools only.

[2] Namely, the Strong Axiom together with continuity and homotheticity characterize demand functions that are rationalized by continuous and homothetic complete preorders. Necessity is clear, and now sufficiency stems from our result.

2 Notation and auxiliary results

A binary relation P on a set X is said to be *acyclic* if $x_1\ P\ x_2\ P\ \dots\ P\ x_n$ implies $x_1 \neq x_n$ for all $x_1, \dots, x_n \in X$. Given an acyclic binary relation P on a set X, we denote by P^∞ the *transitive closure* of P (i.e. $x\ P^\infty\ y$ if and only if there exist $x_1, \dots, x_n \in X$ such that $x = x_1\ P\ x_2\ P\ \dots\ P\ x_n = y$), which is a *partial order* (i.e. it is irreflexive and transitive). An acyclic binary relation P on a topological space X is said to be *upper semicontinuous* if $P(x) = \{y \in X:\ x\ P\ y\}$ is an open set for every $x \in X$ and it is *tc-upper semicontinuous* if its transitive closure is upper semicontinuos (i.e. if $P^\infty(x) = \{y \in X:\ x\ P^\infty\ y\}$ is an open set for every $x \in X$).

A *weak utility* for an acyclic binary relation P on a set X is a real-valued function $u: X \to \mathbb{R}$ such that $u(x) > u(y)$ for all $x, y \in X$ with $x\ P\ y$.

If A is any subset of a real cone X, then define, for every $t \in \mathbb{R}$, $tA = \{tx : x \in A\}$. We recall that a real cone X in a subset of a real vector space E such that $tx \in X$ for every $x \in X$ and $t \in \mathbb{R}_{++}$. An acyclic binary relation P on a real cone X is said to be *homothetic* if $x\ P\ y$ implies $tx\ P\ ty$ for every $x, y \in X$ and $t \in \mathbb{R}_{++}$. A real-valued function $\varphi: X \longrightarrow \mathbb{R}$ on a real cone X is *homogeneous of degree one* if $\varphi(\lambda x) = \lambda \varphi(x)$ for every $x \in X$ and $\lambda \in \mathbb{R}_{++}$.

The following context is assumed in the remaining of this section. X denotes a fixed consumption set (i.e. a non-empty subset of the non-negative orthant of $\mathbb{R}^n$ for some n). By $\|\mathbf{x}\|$ we mean the Euclidean norm of $\mathbf{x}$ in $\mathbb{R}^n$. We adopt the usual notation for vector prices $\mathbf{p} = (p^1, \dots, p^n)$, income w and budget sets $B(\mathbf{p}, w) = \{x \in X:\ \mathbf{p} \cdot x \leqslant w\}$. $\mathcal{B}$ will be a collection of non-empty budget sets of X associated with some subset of price–income pairs of $P \times M = \mathbb{R}^n_{++} \times (0, +\infty)$. Let h be a *demand function* on $\mathcal{B}$; that is, a function that selects exactly one element (denoted by $h(\mathbf{p}, w)$ or $h(B)$) for each $B = B(\mathbf{p}, w) \in \mathcal{B}$. The subset $R(h) = \{h(B) : B \in \mathcal{B}\} \subseteq X$ is the *range* of h. Throughout the present paper we assume that the null vector does not belong to $R(h)$.

Let us fix a demand function h on X. We say that h is *rationalizable* if there exists a *preference* (i.e. complete, transitive binary relation) R such that $h(B) = \{x \in B : x\ R\ y \text{ for all } y \in B\}$, for all $B \in \mathcal{B}$. We also say that R is a *rationalization* of h. Similarly, we say that h on X is *representable* if there exists a function $u: X \longrightarrow \mathbb{R}$ such that $h(B) = \{x \in B:\ u(x) \geqslant u(y) \text{ for all } y \in B\}$, for all $B \in \mathcal{B}$, and u is called a *representation* of h. Obviously, representability implies rationalizability.

If $x, y \in X$, $x \neq y$, and there is a $B \in \mathcal{B}$ such that $y \in B$ and $x = h(B)$ then we say that x is *directly revealed preferred to* y, and we write $x\ S\ y$ (see Samuelson 1938, 1950). The binary relation S (usually called *revealed preference*) thus defined on X is obviously irreflexive and depends on $\mathcal{B}$. Strong requirements for S typically define rationality assumptions on h as follows: h satisfies the *Weak Axiom of Revealed Preference* (WARP, also Weak Axiom) if S is asymmetric, and h satisfies the *Strong Axiom of Revealed Preference* (SARP, also Strong Axiom) if S is acyclic or equivalently if S^∞ is irreflexive.

It is obvious that SARP implies WARP. Moreover, if a demand function h satisfies WARP, then (i) h must be univalued; (ii) SARP characterizes rationability of h by complete preorders (see Richter 1966; also Richter 1971, corollary 1); and (iii) a function $u: X \to \mathbb{R}$ is a representation of h if and only if u is a weak utility for its revealed preference S (see Alcantud and Rodríguez-Palmero 2002, lemma 1).

A demand function h on a real cone X is said to be *homothetic* if

$$\lambda \cdot h(p, w) = h(p, \lambda w), \quad \text{for all } B(p, w) \in \mathcal{B} \quad \text{and all } \lambda > 0.$$

In order for this property of h to hold true, its range $R(h)$ must be a real cone.

Among the many types of restrictions that the axioms of revealed preference impose on demand behavior, we list some that we use afterwards. First, we put forward a useful translation of Walras' law into another simple property of the demand, whenever it agrees with the Weak Axiom.

Proposition 1 *For any demand function h satisfying WARP, the following conditions are equivalent:*

(a) $[h(p, w) = h(p, w') \Rightarrow w = w']\ \forall p \in P,\ \forall w, w' \in M$.
(b) $[p \cdot h(p, w) = w]\ \forall B(p, w) \in \mathcal{B}$ *(Walras's Law).*

PROOF:

(a) $\Rightarrow$ *(b)* Let $B(p, w) \in \mathcal{B}$. If $h(p, w) = h(p, p \cdot h(p, w))$, then $p \cdot h(p, w) = w$ and the result is true. If $h(p, w) \neq h(p, p \cdot h(p, w))$, then $h(p, w)\ S\ h(p, p \cdot h(p, w))$, because $h(p, p \cdot h(p, w)) \in B(p, w)$; and $h(p, p \cdot h(p, w))\ S\ h(p, w)$, because $h(p, w) \in B(p, p \cdot h(p, w))$. Therefore, it follows $h(p, w)\ S\ h(p, p \cdot h(p, w))\ S\ h(p, w)$, which contradicts WARP.

(b) $\Rightarrow$ *(a)* If $h(p, w) = h(p, w')$, then by Walras's Law:

$$w = p \cdot h(p, w) = p \cdot h(p, w') = w'. \qquad \square$$

We next list two further consequences of the Weak Axiom.

Proposition 2 *For any demand function h satisfying WARP, the following statements are equivalent:*

(a) S is homothetic.
(b) h is homothetic.
If h agrees with SARP, these two properties are equivalent to
(c) S^{∞} is homothetic.

PROOF:

$(a) \Rightarrow (b)$ Suppose that there are $(p, w) \in P \times M$ and $\lambda > 0$ such that $h(p, \lambda w) \neq \lambda h(p, w)$. Then $h(p, \lambda w)\ S\ \lambda h(p, w)$, because $\lambda h(p, w) \in B(p, \lambda w)$, and $h(p, w)\ S\ \frac{1}{\lambda} h(p, \lambda w)$, because $\frac{1}{\lambda} h(p, \lambda w) \in B(p, w)$. Therefore, from homotheticity of S, $\lambda h(p, w)\ S\ h(p, \lambda w)\ S\ \lambda h(p, w)$, which contradicts WARP.

$(b) \Rightarrow (a)$ Let $x, y \in X$ such that $x\ S\ y$. There is a pair $(p, w) \in P \times M$ such that $x = h(p, w)$. From WARP, $x \neq y$ and, therefore, $\lambda x \neq \lambda y$. In addition, $p \cdot x \geqslant p \cdot y$, and $p \cdot \lambda x \geqslant p \cdot \lambda y$ for any $\lambda > 0$. However, because h is homothetic, $\lambda x = h(p, \lambda w)$, which means $\lambda x\ S\ \lambda y$.

Suppose now that the Strong Axiom holds. Trivially *(a)* $\Rightarrow$ *(c)*; hence, it suffices to prove *(c)* $\Rightarrow$ *(b)*. Assume that S^{∞} is homothetic but there are $(p, w) \in P \times M$ and $\lambda > 0$ such that $h(p, \lambda w) \neq \lambda h(p, w)$. Then, as proven above in this proof, $h(p, \lambda w)\ S\ \lambda h(p, w)$ and $h(p, w)\ S\ \frac{1}{\lambda} h(p, \lambda w)$; therefore, from homotheticity of S^{∞}, one gets $h(p, \lambda w)\ S\ \lambda h(p, w)\ S^{\infty}\ h(p, \lambda w)$, which contradicts SARP. $\square$

Corollary 1 *Let h be any homothetic demand function satisfying WARP. Then h satisfies Walras's Law.*

PROOF:

From Proposition 1, it is enough to check $[\, h(p, w) = h(p, w') \Rightarrow w = w' \,]$. Suppose there are $p \in P$ and $w, w' \in M$ such that $h(p, w) = h(p, w')$, and $w \neq w'$. Then there is a $\lambda > 0$ satisfying $w' = \lambda\, w$, and, consequently, because h is homothetic, $h(p, w) = h(p, w') = h(p, \lambda\, w) = \lambda\, h(p, w)$. This implies $h(p, w) = \bar{0}$, contradicting $\bar{0} \notin R(h)$. □

The technical result that puts an end to this section shows that in order to deduce semicontinuity properties of the revealed preference the full force of the Strong Axiom is not needed, but rather WARP may well suffice (see also Alcantud and Rodríguez-Palmero 2002, proposition 2 and theorem 1):

Theorem 1 *Let $P \subseteq \mathbb{R}^n$, $M \subseteq \mathbb{R}$ be sets of prices and incomes such that $P \times M$ is open in $\mathbb{R}^{n+1}$, and let h be any demand function on X. Assume that h is continuous and satisfies Walras' Law. Then WARP implies that S is tc-upper semicontinuous (i.e. $S^{\infty}(x)$ is an open set of X for every $x \in X$).*

PROOF:

Let $x, y \in X$ such that $x\, S^{\infty}\, y$. Then either $x = z\, S\, y$ or there is $z = h(p_z, w_z) \in X$ with $x\, S^{\infty}\, z\, S\, y$ and $p_z \cdot y \leqslant z$. Clearly, we are done if we justify the existence of a neighborhood of y relative to X, $N(y)$, such that $z\, S^{\infty}\, N(y)$, irrespective of the case.

If $p_z \cdot y < p_z \cdot z = w_z$, then the statement is immediate because there will be a neighborhood of y relative to X, $N(y)$, satisfying $p_z \cdot N(y) < p_z \cdot z = w_z$, entailing $z\, S\, N(y)$.

Suppose now $p_z \cdot y = p_z \cdot z = w_z$. In this case, let $t = \frac{z+y}{2}$, and, for each $i \in \mathbb{N}$, define the sequence of prices and income pairs $\{(p_i, w_i)\}_{i \in \mathbb{N}}$ on $\mathbb{R}^{n+1}$ by $p_i = p_z + \frac{1}{i}(z - y)$, and $w_i = p_i \cdot t$, respectively.[3] Then, because

$$w_i = p_i \cdot t = \left(p_z + \frac{1}{i}(z-y) \right) \cdot t = w_z + \frac{1}{i}\, \frac{(z-y)\cdot(z+y)}{2}$$
$$= w_z + \frac{1}{i}\, \frac{\|z\|^2 - \|y\|^2}{2}, \tag{1}$$

it follows $\lim_{i\to\infty}(p_i, w_i) = (p_z, w_z)$ and, therefore, as $P \times M$ is open, there is $i_0 \in \mathbb{N}$ such that $(p_i, w_i) \in P \times M$ for all $i \geqslant i_0$. Let $x_i = h(p_i, w_i)$ for all $i \geqslant i_0$. Now,

$$\begin{aligned} p_z \cdot x_i &= \left(p_i - \frac{1}{i}\,(z-y) \right) \cdot x_i \\ &= w_i - \frac{1}{i}\,(z-y)\cdot x_i \\ &= w_z + \frac{1}{i}\, \frac{\|z\|^2 - \|y\|^2}{2} - \frac{1}{i}\,(z-y)\cdot x_i \qquad \text{(from [1])} \\ &= w_z + \frac{1}{i} \left(\frac{\|z\|^2 - \|y\|^2 - 2\,(z-y)\cdot x_i}{2} \right) \\ &< w_z, \end{aligned} \tag{2}$$

[3] Observe that $z \neq y$ from WARP and, consequently, $\|z - y\| > 0$.

where the last inequality holds for i large enough because, from continuity of h, it follows $\lim_{i\to\infty} \frac{\|z\|^2 - \|y\|^2 - 2(z-y)\cdot x_i}{2} = \frac{\|z\|^2 - \|y\|^2 - 2(z-y)\cdot z}{2} = -\frac{\|z-y\|^2}{2} < 0$. In contrast,

$$\begin{aligned} p_i \cdot y &= (p_z + \frac{1}{i}(z-y)) \cdot y \\ &= w_z + \frac{1}{i}(z-y)\cdot y \\ &< w_z + \frac{1}{i}\frac{\|z\|^2 - \|y\|^2}{2} \\ &= w_i, \end{aligned} \tag{3}$$

where the last inequality holds because $(z-y)\cdot y = \frac{\|z\|^2 - \|y\|^2 - \|z-y\|^2}{2} < \frac{\|z\|^2 - \|y\|^2}{2}$. Hence, from (2) and (3) there must be a sufficiently large index $i \in \mathbb{N}$ such that $z\, S\, x_i\, S\, y$ and $p_i \cdot y < p_i \cdot x_i = w_i$, which implies the existence of a neighborhood of y relative to X, $N(y)$, satisfying $z\, S\, x_i\, S\, N(y)$. This completes the proof. □

3 Main result

Our final result solves the integrability question we had posed ourselves.

Theorem 2 *Let $P \subseteq \mathbb{R}^n$, $M \subseteq \mathbb{R}$ be sets of prices and incomes such that $P \times M$ is open in $\mathbb{R}^{n+1}$, and let h be any demand function on the real cone X. The following conditions are equivalent:*

(a) h is continuous, homothetic, and satisfies SARP.
(b) h can be represented by a continuous and homogeneous of degree one utility function u on X.

Proof: $(a) \Rightarrow (b)$ A first step consists of studying the behavior of some countable collection of open sets with a particular form. We keep $p_0 \gg 0$ and $w_0 > 0$ arbitrary but fixed and, for simplicity, x_{w_0} denotes the only element in $h(p_0, w_0)$. Now, for each rational number q define the countable collection $\{G_q\}_{q\in\mathbb{Q}}$ as follows:

$$G_q = \begin{cases} S^{\infty}(x_{w_0}), & \text{if } q = 1; \\ qG_1, & \text{if } q > 0; \\ \varnothing, & \text{otherwise.} \end{cases} \tag{4}$$

Next, we claim that the following conditions[4] hold for the countable collection $\{G_q\}_{q\in\mathbb{Q}}$:

(α) $\bigcup_{q\in\mathbb{Q}} G_q = X$.
(β) G_q is an open set of X $\forall q \in \mathbb{Q}$, due to Theorem 1.
(γ) $[q_1 < q_2 \Rightarrow \overline{G_{q_1}} \subseteq G_{q_2}]\ \forall q_1, q_2 \in \mathbb{Q}$.
(δ) $[x\, S^{\infty}\, y \Rightarrow \exists q_1 < q_2 :\ y \in G_{q_1}, x \notin G_{q_2}]\ \forall x, y \in X$.

[4] $\overline{C}$ denotes the topological closure of C relative to X.

To check that (α) holds, fix $x \in X$. Because there is $q > 0$ rational such that $q\, x_{w_0} > p_0 \cdot x$, it follows $x_{w_0} S \frac{x}{q}$, which means $\frac{x}{q} \in G_1 = S^\infty(x_{w_0})$ or, equivalently, $x \in qG_1 = G_q$. Hence, we have verified that $\cup_{r\in\mathbb{Q}} G_r = X$.

Let us now check condition (γ). To do so, as $\overline{\alpha C} = \alpha \overline{C}$, we only need to check that $\overline{G_1} \subseteq qG_1$ for each $q > 1$ rational because then $q_2 > q_1$ will entail $\overline{G_{q_1}} = \overline{q_1 G_1} = q_1 \overline{G_1} \subseteq q_1 \frac{q_2}{q_1} G_1 = G_{q_2}$. We use the following straightforward fact: if $z \in G_1$ and $z \geqslant a$[5] (with $a \in X$) then $a \in G_1$. Fix $q > 1$ rational, and take $x \in \overline{G_1}$. If x is the null vector then $x_{w_0}\ S\ x$ because we assumed that $x \notin R(h)$, and, therefore, $\frac{1}{q}x = x \in G_1$. Otherwise, there is $\epsilon > 0$ such that every $z \in B(x,\epsilon) \cap X$ satisfies $z \geqslant \frac{1}{q}x,\ z \neq \frac{1}{q}x$. Since there must be $z \in B(x,\epsilon) \cap X \cap G_1$ because $x \in \overline{G_1}$, one obtains that $\frac{1}{q}x \in G_1$ and, therefore, $x \in qG_1$.

Finally, let us show that condition (δ) holds. Consider any two elements $x, y \in X$ such that $x\, S^\infty\, y$. Then either $x = z\, S\, y$ or there is $z \in X$ with $x\, S^\infty\, z\, S\, y$. Clearly, we are done if we justify the existence of $q_1, q_2 \in \mathbb{Q}$ such that $q_1 < q_2,\ z \notin G_{q_2}$, and $y \in G_{q_1}$, irrespective of the case. Let us define the set $A(z) = \{t > 0 : tz \in G_1\}$. This set is non-empty because $(0, \frac{w_0}{p_0 \cdot z}) \subseteq A(z)$. Furthermore, it is bounded above: as $z \in R(h)$, there is a pair $(p_z, w_z) \in P \times M$ such that $z = h(p_z, w_z)$. Then, for all $t > \frac{p_z \cdot x_{w_0}}{w_z}$ it follows $tz\, S\, x_{w_0}$ because h is homothetic, and now acyclicity of S^∞ assures $A(z) \subseteq (0, \frac{p_z \cdot x}{w_z}]$. Denote $\lambda_0 = \sup A(z)$. Because S^∞ is upper semicontinuous, we have that $\lambda_0 \notin A(z)$ due to the continuity of scalar multiplication. We have thus shown $\lambda_0 z \notin G_1$. Now, note that $\lambda_0 z\, S\, \lambda_0 y$ by homotheticity. Then, upper semicontinuity of S^∞ grants the existence of $\epsilon_0 > 1$ with $\lambda_0 z\, S^\infty\, (\lambda_0 \epsilon_0) y$, and, therefore, $\frac{\lambda_0}{\sqrt{\epsilon_0}} z\, S^\infty\, (\lambda_0 \sqrt{\epsilon_0}) y$. By definition of λ_0, $x_{w_0}\, S^\infty\, \frac{\lambda_0}{\sqrt{\epsilon_0}} z\, S\, (\lambda_0 \sqrt{\epsilon_0}) y$ and then $(\lambda_0 \sqrt{\epsilon_0}) y \in G_1$. Now fix any rational numbers q_1, q_2 satisfying $1/(\lambda_0\sqrt{\epsilon_0}) < q_1 < q_2 < 1/\lambda_0$. Then, since the collection $\{G_q\}_{q\in\mathbb{Q}}$ is increasing we deduce $y \in G_{q_1}$ because $(\lambda_0\sqrt{\epsilon_0}) y \in G_1$, and $z \notin G_{q_2}$ because $\lambda_0 z \notin G_1$, and, consequently, condition (δ) obtains.

A second step consists of defining a suitable solution function $u : X \longrightarrow \mathbb{R}$ in the following way: let u be defined on X by

$$u(x) = \inf_{q\in\mathbb{Q}} \{q : x \in G_q\} \geqslant 0. \tag{5}$$

Observe that u is well defined from conditions (α). We proceed to show that u is a homogeneous of degree one and continuous weak utility for S as claimed.

To prove that u is homogeneous of degree one, assume that there exists some $t > 0$ such that $u(tx) < tu(x)$. Select $r \in \mathbb{Q}$ such that $u(tx) < r < tu(x)$. As scalar multiplication and u are continuous, there exists $q \in \mathbb{Q}^{++}$ such that $u(qx) < r < qu(x)$, which in particular implies that $qx \in G_r$ because $u(qx) < r$, and $x \notin G_{\frac{r}{q}}$ because $u(x) > \frac{r}{q} \in \mathbb{Q}$. From homotheticity of h one obtains $x \in G_{\frac{r}{q}}$, a contradiction. Likewise, whenever $t \in \mathbb{R}^{++}$ and $x \in X$ the inequality $tu(x) < u(tx)$ cannot hold.

We now justify that u is upper semicontinuous. Let $0 < \alpha,\ y \in \{x \in X : u(x) < \alpha\}$. Pick $q \in \mathbb{Q}$ such that $u(y) < q < \alpha$, hence $y \in G_q$. Thus, from condition (β), G_q is an open set containing y such that $u(z) \leqslant q < \alpha$ for every $z \in G_q$, which means that the set $\{x \in X :$

[5] By this we mean $z_i \geqslant a_i$ for each possible component.

$u(x) < \alpha\}$ is open in X. To show that u is lower semicontinuous, let $y \in \{x \in X : u(x) > \alpha\}$ and let q_1 and q_2 be rational numbers satisfying $u(y) > q_2 > q_1 > \alpha$. Then $y \notin G_{q_2}$ and from condition (γ), $y \in (\overline{G_{q_1}})^c$, the complement of $\overline{G_{q_1}}$ in X. Because $u(z) \geqslant q_1 > \alpha$ for every $z \in (\overline{G_{q_1}})^c$ and $(\overline{G_{q_1}})^c$ is open in X, the set $\{x \in X : u(x) > \alpha\}$ is open in X. Therefore, u is continuous.

Furthermore, it is straightforward to check that u is a weak utility for S^∞ by condition (δ).

$(b) \Rightarrow (a)$ Let u be a continuous and homogeneous of degree one utility representing h. We first check the homotheticity of the demand function. Suppose, by means of contradiction, that there exists a pair $(p, w) \in P \times M$ and a real number $\lambda > 0$ such that $h(p, \lambda w) \neq \lambda h(p, w)$. Then $h(p, \lambda w)\, S\, \lambda h(p, w)$ and $h(p, w)\, S\, \frac{1}{\lambda} h(p, \lambda w)$. Because u is a weak utility for S^∞ (or, equivalently, for S) we deduce

$$u(h(p, \lambda w)) > u(\lambda h(p, w)) \quad \text{and} \quad u(h(p, w)) > u\left(\frac{1}{\lambda} h(p, \lambda w)\right),$$

and then, as u is homogeneous of degree one, we obtain the absurd conclusion

$$\begin{aligned} u(h(p, \lambda w)) > u(\lambda h(p, w)) = \lambda u(h(p, w)) &> \lambda u\left(\frac{1}{\lambda} h(p, \lambda w)\right) \\ &= u(h(p, \lambda w)). \end{aligned}$$

To show the continuity of h we only need to argue that because homogeneous of degree one utilities induce locally nonsatiated preferences on X, proposition 3.AA.1 in Mas-Colell *et al.* (1995) applies and, therefore, the demand function is continuous. The fact that h fulfils SARP is obvious and well known, and the proof is completed. □

4 Conclusion

The realization that without further restrictions the solutions of the integrability problem associated with a demand function coincide with the weak utilities for its revealed preference S (see Alcantud and Rodríguez-Palmero 2002) permits us to tackle that problem from an entirely different perspective. The intuition under the process is simple to state, but not quite to develop: one can produce conditions for the binary relation S to admit weak utilities, a problem with a large tradition behind it, and this solves the initial problem. In the "general" case (i.e. no functional form of the demand is presumed) this view of the problem has produced *sufficient* conditions for *semicontinuous* representations only (see Sondermann 1982, Alcantud and Rodríguez-Palmero 2002). Here, we fully exploit the possibilities of the technique by taking advantage of the very tractable functional form of homotheticity. Not only does this mean that *necessary and sufficient* conditions are reached, but also *continuity* is characterized. Even though our analysis requires intermediate technical assertions about the behavior of the revealed preference, such technicalities do not interfere with a very clear and comprehensible final statement.

There are other advantages of our proposal that should be emphasized. For one thing, the argument is self-contained and can be followed by the average analyst equipped with the standard mathematical toolbox. In this we separate from the main trend in the field and, particularly, from previous approaches to this specification of the integrability problem like Liu and Wong (2000). Our characterization avoids the appeal to ad-hoc axioms made in previous approaches: the Strong Axiom is necessary and sufficient for a continuous homothetic demand to be represented by a continuous and homogeneous of degree one utility function, an indisputable statement that favors comparisons.

References

Alcantud, J. C. R., G. Bosi, C. Rodríguez-Palmero, and M. E. Zuanon (2006), "Mathematical utility theory and the representability of demand by continous homogeneous functions," *Portuguese Economic Journal* **5**, 195–205.

Alcantud, J. C. R., and C. Rodríguez-Palmero (2002), "A note on representability of consumer's behavior," *Economic Theory* **20**, 207–16.

Chipman, J. S. (1974), "Homothetic preferences and aggregation," *Journal of Economic Theory* **8**, 26–38.

Houtakker, H. S. (1950), "Revealed preference and the utility function," *Economica N. S.* **7**, 159–74.

Liu, P.-K., and K.-C. Wong (2000), "Revealed homothetic preference and technology," *Journal of Mathematical Economics* **34**, 287–314.

Mas-Colell, A., M. D. Whinston, and J. D. Green (1995), *Microeconomic Theory*, Oxford: Oxford University Press.

Richter, M. K. (1966), "Revealed preference theory," *Econometrica* **34**, 635–45.

Richter, M. K. (1971), "Rational choice," J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein, eds, *Preferences, Utility and Demand*, 29–58, New York: Harcourt, Brace and Jovanovich.

Samuelson, P. A. (1938), "A note on the pure theory of consumer's behavior," *Economica* **5**, 61–71.

Samuelson, P. A. (1950), "The problem of integrability in utility theory," *Economica* **17**, 355–85.

Sondermann, D. (1982), "Revealed preference: An elementary treatment," *Econometrica* **50**, 777–9.

Ville, J. (1946), "Sur les conditions d'existence d'une ophélimité totale et d'un indice du niveau des prix," *Annales de l'Université de Lyon* Sec. A(3) n. 9, 32–39. Translated by P. K. Newman (1951), *Review of Economic Studies* **19**, 123–28.