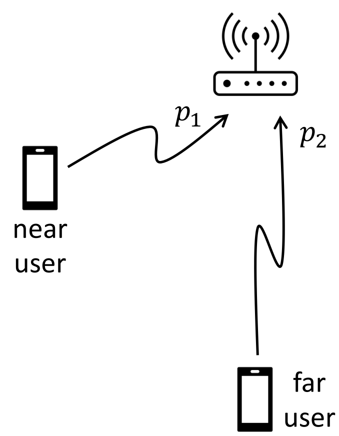
Multi-Packet Reception

In our presentation of WiFi CSMA/CD, we assumed a standard collision channel, when only one packet can be transmitted at a time. While this is a common assumption for wireline networks, it does not reflect the capability of a wireless channel.

In fact, typical wireless systems allow multiple packets to be transmitted simultaneously. Indeed, this is the whole point of multiple-access: FDMA, TDMA, CDMA, etc. To understand this, we need models for **multi-packet reception** (MPR).

MPR Scenarios

There are a number of settings in which multiple simultaneous transmissions do not result in the destruction of all packets involved. Here are four important ones:



(1) Capture from near-far effect: Even if all users employ the same transmission power, due to their distance, the received powers will vary. We studied this extensively in the cellular context. The result is that users with stronger received powers are easier to decode correctly.

Specifically, consider two users with transmit powers p_1 and p_2. If $p_1 \gg p_2$, then user 1's packet may still be received correctly, i.e., **captured**, even when user 2 is transmitting at the same time. We can quantify this effect through the received SINR for user 1:

$$\frac{p_1}{p_2 + N_0} \geq \gamma, \tag{1}$$

where γ is the required threshold for successful reception. However, we can see how this may be bad for user 2. In fact, with **successive interference cancellation** (SIC) techniques, it may be possible to even recover user 2's information. Note that if power control is used, i.e., resulting in $p_1 = p_2$, then any such capability will be lost.

(2) Multiple orthogonal channels: We have studied multiple access techniques with channels corresponding to frequency, time, or code dimensions. In the wireless LAN case, one possibility is to assume each user picks one of the available channels randomly (like randomly picking timeslots in the CSMA/CD contention window). Then, collisions only occur when two users pick the same channel. Some of the 5G proposals have referred to this as "coded multiple access."

(3) Alternate CDMA model: With CDMA, suppose each user has a unique spreading code, and that received power control is employed. Then, due to the spreading gain, K users can have their packets received simultaneously as long as

$$\frac{P}{(K-1)P} \geq \gamma \tag{2}$$

for some γ.

(4) MIMO: MIMO can accomplish MPR across multiple users through having multiple receive antennas. With two users and two receive antennas,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{FH} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \tag{3}$$

where s_i is the transmit symbol for user i and y_i is the corresponding received symbol after applying a decoding vector $\mathbf{F}$. When the channel matrix $\mathbf{H}$ is random, e.g., Raleigh, the strength of y_1 and y_2 vary with the channel H, leading to a similar capture effect as in near-far situations.

It should be obvious that MPR will improve random access performance. The question is, by how much?

General MPR Model

Let us now consider a fairly general MPR model that can be applied to the settings above. If there are k transmissions, we define $\Sigma_{kj} \in [0, 1]$ to be the probability that j of them will succeed, independently of the other timeslots.

The specific subset of $j \leq k$ packets that succeed is determined randomly and uniformly across the transmissions. This is natural for the multi-channel and CDMA models discussed above, where differences in received power do not come into play. For the MIMO and near-far models, though, this also requires that we assume the channel/location changes are independent and identically distributed across timeslots.

We can construct the matrix $\Sigma = [\Sigma_{kj}]$ for different settings:

(1) Collision. For the collision channel model, whenever more than one packet is transmitted simultaneously, all of them are lost. Thus,

$$\Sigma \quad = \quad \begin{array}{c c} & \begin{array}{cccc} (j=0) & (j=1) & (j=2) & \cdots \end{array} \\ \begin{array}{c} (k=1) \\ (k=2) \\ (k=3) \\ \vdots \end{array} & \left[\begin{array}{cccc} 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}\right] \end{array} \tag{4}$$

(2) Capture. In the capture model, when more than one packet is transmitted, we consider that the strongest packet may survive, with probability p_k. This leads to the following matrix:

$$\Sigma \quad = \quad \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 1-x_2 & x_2 & 0 & \cdots \\ 1-x_3 & x_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5}$$

(3) Multi-channel. Assume that there are q channels available, and that the k users choose one of them uniformly randomly. For channels $i = 1, ..., q$, we define

$$S_i = \begin{cases} 1 & \text{if channel } i \text{ sees a successful transmission} \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

The probability of a given user selecting channel i is $1/q$. Across the k transmissions, then,

$$P[S_i = 1] = k\frac{1}{q}\left(1 - \frac{1}{q}\right)^{k-1}. \tag{7}$$

The explicit expression for Σ_{kj} in this case is complicated, since the S_i's are not independent: success on one channel impacts success on another. However, we can define a random variable for the number of successful transmissions across k packets as

$$J = \sum_{i=1}^{q} S_i. \tag{8}$$

The mean of J for k transmissions can in turn be calculated as

$$\mathbb{E}[J|k] = \sum_{j=1}^{k} j \cdot \Sigma_{jk} = \sum_{i=1}^{q} \mathbb{E}[S_i] = k\left(1 - \frac{1}{q}\right)^{k-1}. \tag{9}$$

We will see shortly that it is indeed typically $\sum_j j \cdot \Sigma_{jk}$ that we are after.

(4) CDMA. In the CDMA case, let $\bar{K}$ be the maximum number of concurrent code transmissions that can be supported from (2). All transmissions will be successful as long as $k \leq \bar{K}$, and otherwise, they will all fail:

$$\Sigma = \begin{array}{cc} & \begin{array}{cccccc} (j=0) & & & & (j=\bar{K}) & \end{array} \\ \begin{array}{c} (k=1) \\ \\ \\ (k=\bar{K}) \\ \\ \\ \\ \end{array} & \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & 1 & \cdots \\ 1 & 0 & 0 & \cdots & 0 & \cdots \\ 1 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix} \end{array} \tag{10}$$

Fixed Number of Users

Suppose the total number of users in the system, n, is fixed over time. As in the slotted-Aloha model we looked at before, the main question we are after is the throughput of the system.

Let $p \in [0, 1]$ be the probability that a user transmits in a timeslot, independently of the other users. The expected number of successful transmissions can then be expressed as

$$\sum_{k=1}^{n} \binom{n}{k} p^k (1-p)^{n-k} C_k, \tag{11}$$

where $\binom{n}{k}$ is the Binomial coefficient, and

$$C_k = \sum_{j=1}^{k} j \cdot \Sigma_{kj} \tag{12}$$

is the expected number of successful transmissions if k users are transmitting. The expression (11) can be compared to the throughput expression we saw previously for slotted-Aloha with no MPR:

$$np(1-p)^{n-1}. \tag{13}$$

We see that (13) is (11) when $C_1 = 1$ and $C_k = 0$ for $k > 1$, i.e., for the collision channel (4).

The idea is that we then solve for p to maximize the throughput in (11). Recall that when we did this for Aloha, we found that $p^\star = 1/n$.

Dynamic Arrivals

We now consider the case where new packets arrive to the system for transmission at a rate λ in each time period.

We first suppose that each node transmits with a *fixed* probability p in each time period. With Aloha, the system is unstable for any $\lambda > 0$. To see this, note that if the number of backlogged packets is n, then the probability of success $np(1-p)^{n-1} \to 0$ as $n \to \infty$ for any fixed p. The number of backlogged packets will keep growing and eventually there will be no successful transmissions.

Let us now look at the case of MPR for a fixed p:

Proposition 1: Suppose the number of backlogged packets at time t is $n(t) = n$, in which case the number of successful MPR transmissions is given by (11). If $\lim_{k\to\infty} C_k = C$, then (11) also approaches C as $n \to \infty$.

To see this, since $\lim_{k\to\infty} C_k = C$ by assumption, then for any $\epsilon > 0$, there exists an M such that $|C_k - C| < \epsilon$ for all $k \geq M$. In other words, $-\epsilon < C_k - C < \epsilon$ for all $k \geq M$. Further, when n is sufficiently large,

$$\sum_{k=m}^{n} \binom{n}{k} p^k (1-p)^{n-k} \geq 1 - \epsilon. \tag{14}$$

Together, we have

$$\sum_{k=1}^{n}\binom{n}{k}p^k(1-p)^{n-k}\cdot C_k \geq \sum_{k=m}^{n}\binom{n}{k}p^k(1-p)^{n-k}\cdot C_k$$
$$\geq (1-\epsilon)\cdot(C-\epsilon), \tag{15}$$

since $C_k < C-\epsilon$ by assumption. Therefore, $\lim_{n\to\infty}(11) \geq (1-\epsilon)(C-\epsilon)$. Since ϵ is arbitrary, we can consider a vanishingly small ϵ and conclude $\lim_{n\to\infty}(11) \geq C$. Using similar techniques, we can show that $\lim_{n\to\infty}(11) \leq C$, which proves the proposition.

The value of C in Proposition 1 directly determines which arrival rates λ an MPR system can stabilize. In particular, we have the following theorem:

Theorem 1: If $\lim_{k\to\infty} C_k = C < \infty$, then the system is stable for $\lambda < C$, and unstable for $\lambda > C$. If $\lim_{k\to\infty} C_k = \infty$, then the system is always stable, and if $\lim_{k\to\infty} C_k = 0$, it is never stable.

We consider a few examples illustrating Theorem 1:

(1) Collision or CDMA. In both of these cases, $\lim_{k\to\infty} C_k = 0$. Thus, the system is not stable for any $\lambda > 0$.

(2) Multi-channel. With q channels, we had shown in (9) that

$$C_k = k\left(1-\frac{1}{q}\right)^{k-1}. \tag{16}$$

If q is fixed, then we have the same result as for Collision/CDMA: $\lim_{k\to\infty} C_k = 0$. On the other hand, suppose we have $q = \alpha k$, i.e., the number of channels increasing linearly with the number of transmissions k. Then,

$$C_k = k\Big(1-\frac{1}{\alpha k}\Big)^{k-1} = k\Big(1-\frac{1}{\alpha k}\Big)^{-1}\left[\Big(1-\frac{1}{\alpha k}\Big)^{\alpha k}\right]^{\frac{1}{\alpha}} \tag{17}$$

As $k\to\infty$, $\Big(1-\frac{1}{\alpha k}\Big)^{\alpha k} \to e^{-1}$. Thus,

$$\lim_{k\to\infty} C_k = \lim_{k\to\infty} k\Big(1-\frac{1}{\alpha k}\Big)^{-1}\cdot e^{-\frac{1}{\alpha}} = \infty \tag{18}$$

Hence, the system is always stable. However, it seems unlikely to have the number of channels growing with k. Instead, we should investigate what happens if we allow the system to adjust p based on the number of backlogged packets.

Backoff Control

With Aloha, if we can properly control $p_n \propto \frac{1}{n}$, then the system can be stable when $\lambda < \frac{1}{e}$. More generally, for MPR, proper control of p_n allows the system to support even higher λ than that provided by Theorem 1, i.e., $C = \lim_{k\to\infty} C_k$. Formally, we have the following theorem:

Theorem 2: Let $p_n^\star$ be the retransmission probability which minimizes the expected increase of the backlog when the backlog is equal to n packets. If we employ this as the MPR retransmission probability at each timeslot, then the system is stable if

$$\lambda < \eta_c = \sup_{x>0}\ e^{-x} \sum_{k=1}^{\infty} \frac{x^k}{k!} C_k, \tag{19}$$

and unstable if $\lambda > \eta_c$, where sup denotes the supremum.

To understand the proof of Theorem 2, we write the expected number of successes when the backlog is n as

$$t_n(p) = \sum_{k=1}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot C_k, \quad C_k = \sum_{j=1}^{k} j \cdot \Sigma_{kj} \tag{20}$$

where from (11) we have just made p an explicit variable. We should choose p that maximizes (20). At $p_n^\star$, we have $t_n(p_n^\star)$, which varies with n. However, we only care about the behavior when $n \to \infty$. In particular, we need

$$\lambda < \lim_{n\to\infty} t_n(p_n^\star) = \lim_{n\to\infty} \sup_{p\in[0,1]} t_n(p), \tag{21}$$

i.e., for stability, the arrival rate must be less than the expected number of successes. We then need to show that the right hand side of (21) is η_c from (19). Let

$$t(x) = e^{-x} \sum_{k=1}^{\infty} \frac{x^k}{k!} C_k. \tag{22}$$

The key is that when n is large, $t_n(p)$ will be close to $t(x)$, for a suitable change of variable. To see this, let $p = \frac{x}{n}$. Then,

$$t_n\left(\frac{x}{n}\right) = \sum_{k=1}^{n} \binom{n}{k} \left(\frac{x}{n}\right)^k \left(1 - \frac{x}{n}\right)^{n-k} \cdot C_k \tag{23}$$

$$= \sum_{k=1}^{n} \underbrace{\frac{n(n-1)\cdot(n-k+1)}{k!}\cdot\frac{x^k}{n^k}}_{\text{(a)}} \cdot \underbrace{\left(1-\frac{x}{n}\right)^n}_{\text{(b)}} \cdot \underbrace{\left(1-\frac{x}{n}\right)^{-k}}_{\text{(c)}} \cdot C_k \quad (24)$$

For each fixed k, as $n \to \infty$, term (a) approaches $\frac{x^k}{k!}$, term (b) approaches e^{-x}, and term (c) approaches 1. Further, we can show that this convergence is uniform in k. Thus,

$$\lim_{n\to\infty} t_n\left(\frac{x}{n}\right) = e^{-x}\sum_{k=1}^{\infty}\frac{x^k}{k!}C_k. \quad (25)$$

Using this relationship,

$$\lim_{n\to\infty}\sup_{p\in[0,1]} t_n(p) = \lim_{n\to\infty}\sup_{x\in(0,n)} t_n\left(\frac{x}{n}\right) = \sup_{x>0} t(x),$$

which concludes the proof.

The insight here is that there is an optimal value of x such that $p_n^\star = \frac{x}{n}$. As $n \to \infty$, the number of retransmissions becomes Poisson distributed, with $P[k] = e^{-x}\frac{x^k}{k!}$. With C_k being the expected number of transmissions that are successful, we get the maximum rate η_c in (19).

We consider a few MPR settings as examples:

(1) Collision. For the collision channel model, we have

$$t(x) = e^{-x}\sum_{k=1}^{\infty}\frac{x^k}{k!}C_k = e^{-x}\cdot x. \quad (26)$$

Setting $t'(x) = 0$, we find $x = 1$, giving $\eta_c \sup_x t(x) = \frac{1}{e}$ and $p_n = \frac{1}{n}$. This is consistent with what we expect for slotted Aloha.

(2) q-channels. With $C_k = k\left(1-\frac{1}{q}\right)^{k-1}$ defined in (16),

$$\begin{aligned} t(x) &= e^{-x}\sum_{k=1}^{\infty}\frac{x^k}{k!}\cdot k\left(1-\frac{1}{q}\right)^{k-1} = e^{-x}\sum_{k=1}^{\infty} x\cdot\frac{\left(x\left(1-\frac{1}{q}\right)\right)^{k-1}}{(k-1)!} \\ &= e^{-x}\cdot x\cdot e^{x\left(1-\frac{1}{q}\right)} = xe^{-x/q}. \end{aligned} \quad (27)$$

Setting $t'(x) = 0$, we need $e^{-x/q} - \frac{x}{q}e^{-x/q} = 0$. This gives $x = q$, and $\eta_c = qe^{-1}$. The rate is exactly q-times that of a single channel (or the collision model), as we would expect.

(3) CDMA. For the CDMA model, $C_k = k$ if $k \leq \bar{K}$, and $C_k = 0$ if $k > \bar{K}$. In a homework problem, you are asked to derive the expression for $t(x)$ and η_c. An interesting point for comparison is when $\bar{K} = q$, i.e., a multi-channel model with q channels vs. a CDMA system that can allow q simultaneous transmissions. Which is better? In fact, in CDMA, η_c will be close to $\bar{K}$, while in a q-channel system, it is q/e.

Finally, note that in practice, we need to introduce a control policy based on *estimates* for the current backlog. In a distributed setting, it is impractical to assume that each user has perfect knowledge of the backlog across all other users. Binary exponential backoff in CSMA/CD is an example of such a policy, for the collision channel case: the contention window size responds to estimates of system demand based on lack of acknowledgment packets.