Homework 2 - Berkeley STAT 157

Handout 1/29/2019, due 2/5/2019 by 4pm in Git by committing to your repository.

In [1]:
```
from mxnet import nd, autograd, gluon
from matplotlib import pyplot as plt
import numpy as np
```

1. Multinomial Sampling

Implement a sampler from a discrete distribution from scratch, mimicking the function `mxnet.ndarray.random.multinomial`. Its arguments should be a vector of probabilities p. You can assume that the probabilities are normalized, i.e. tha they sum up to 1. Make the call signature as follows:

```
samples = sampler(probs, shape)

probs   : An ndarray vector of size n of nonnegative numbers summing up to 1
shape   : A list of dimensions for the output
samples : Samples from probs with shape matching shape
```

Hints:

1. Use `mxnet.ndarray.random.uniform` to get a sample from $U[0, 1]$.
2. You can simplify things for `probs` by computing the cumulative sum over `probs`.

In [2]:
```
def sampler(probs, shape):
    ## Add your codes here
    samples = nd.zeros(shape).reshape(-1,)
    num_items = shape[0] * shape[1]
    for i in range(num_items):
        ran_gen = nd.random.uniform()
        counter = 0
        while ran_gen > 0:
            ran_gen -= probs[counter]
            if (ran_gen) < 0:
                break
            counter += 1
        samples[i] = probs[counter]
    samples = samples.reshape(shape)
    return samples

# a simple test
sampler(nd.array([0.2, 0.3, 0.5]), (2,3))
```

Out[2]:
```
[[0.5 0.5 0.5]
 [0.5 0.5 0.5]]
<NDArray 2x3 @cpu(0)>
```

2. Central Limit Theorem

Let's explore the Central Limit Theorem when applied to text processing.

- Download https://www.gutenberg.org/ebooks/84 (https://www.gutenberg.org/files/84/84-0.txt) from Project Gutenberg
- Remove punctuation, uppercase / lowercase, and split the text up into individual tokens (words).
- For the words `a`, `and`, `the`, `i`, `is` compute their respective counts as the book progresses, i.e.

$$n_{\text{the}}[i] = \sum_{j=1}^{i} \{w_j = \text{the}\}$$

- Plot the proportions $n_{\text{word}}[i]/i$ over the document in one plot.
- Find an envelope of the shape $O(1/\sqrt{i})$ for each of these five words. (Hint, check the last page of the sampling notebook (http://courses.d2l.ai/berkeley-stat-157/slides/1_24/sampling.pdf))
- Why can we **not** apply the Central Limit Theorem directly?
- How would we have to change the text for it to apply?

- Why does it still work quite well?

2.1 Answer) Download [https://www.gutenberg.org/ebooks/84 (https://www.gutenberg.org/files/84/84-0.txt)](https://www.gutenberg.org/files/84/84-0.txt) from Project Gutenberg

- Remove punctuation, uppercase / lowercase, and split the text up into individual tokens (words).
- For the words `a`, `and`, `the`, `i`, `is` compute their respective counts as the book progresses, i.e.

$$n_{\text{the}}[i] = \sum_{j=1}^{i} \{w_j = \text{the}\}$$

- Plot the proportions $n_{\text{word}}[i]/i$ over the document in one plot.

In [20]:

```
filename = gluon.utils.download('https://www.gutenberg.org/files/84/84-0.txt')
with open(filename) as f:
    book = f.read()

def preprocess(text):
    punctuation = set([',', '.', '-', '(', ')', ';', '/', '*', "'", ' ', ':', '"', '['])
    nums = set('0123456789$')

    return [word.lower() for word in text.split() if not any((x in word) for x in punctuation) and not a

lst = preprocess(book)

a_counts = np.zeros(len(lst))
and_counts = np.zeros(len(lst))
the_counts = np.zeros(len(lst))
i_counts = np.zeros(len(lst))
is_counts = np.zeros(len(lst))

for i in range(0,len(lst)):
    if (i != 0):
        a_counts[i] = a_counts[i-1]
        and_counts[i] = and_counts[i-1]
        the_counts[i] = the_counts[i-1]
        i_counts[i] = i_counts[i-1]
        is_counts[i] = is_counts[i-1]
        if (lst[i] == 'a'):
            a_counts[i] += 1
        elif (lst[i] == 'and'):
            and_counts[i] += 1
        elif (lst[i] == 'the'):
            the_counts[i] += 1
        elif (lst[i] == 'i'):
            i_counts[i] += 1
        elif (lst[i] == 'is'):
            is_counts[i] += 1
    else:
        if (lst[0] == 'a'):
            a_counts[0] = 1
        elif (lst[0] == 'and'):
            and_counts[0] = 1
        elif (lst[0] == 'the'):
            the_counts[0] = 1
```

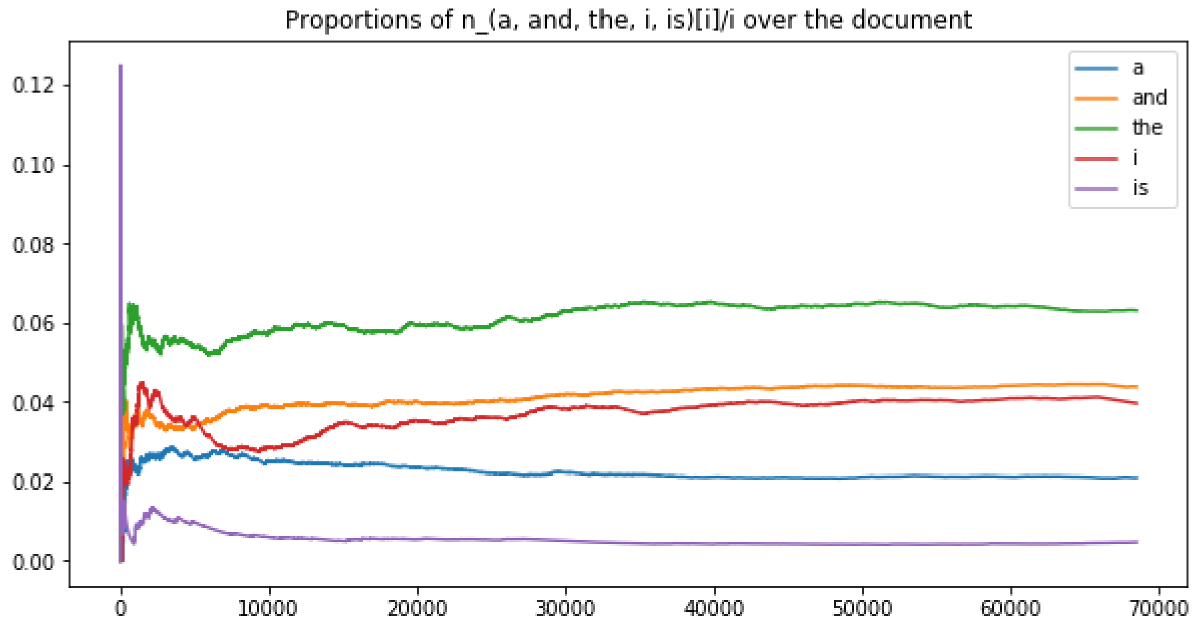
```python
        elif (lst[0] == 'i'):
            i_counts[0] = 1
        elif (lst[0] == 'is'):
            is_counts[0] = 1

a_prop = np.zeros(len(lst))
and_prop = np.zeros(len(lst))
the_prop = np.zeros(len(lst))
i_prop = np.zeros(len(lst))
is_prop = np.zeros(len(lst))

for i in range(1,len(lst)):
    a_prop[i] = a_counts[i]/i
    and_prop[i] = and_counts[i]/i
    the_prop[i] = the_counts[i]/i
    i_prop[i] = i_counts[i]/i
    is_prop[i] = is_counts[i]/i
```

2.2 Answer) Plot the proportions $n_{\text{word}}[i]/i$ over the document in one plot.

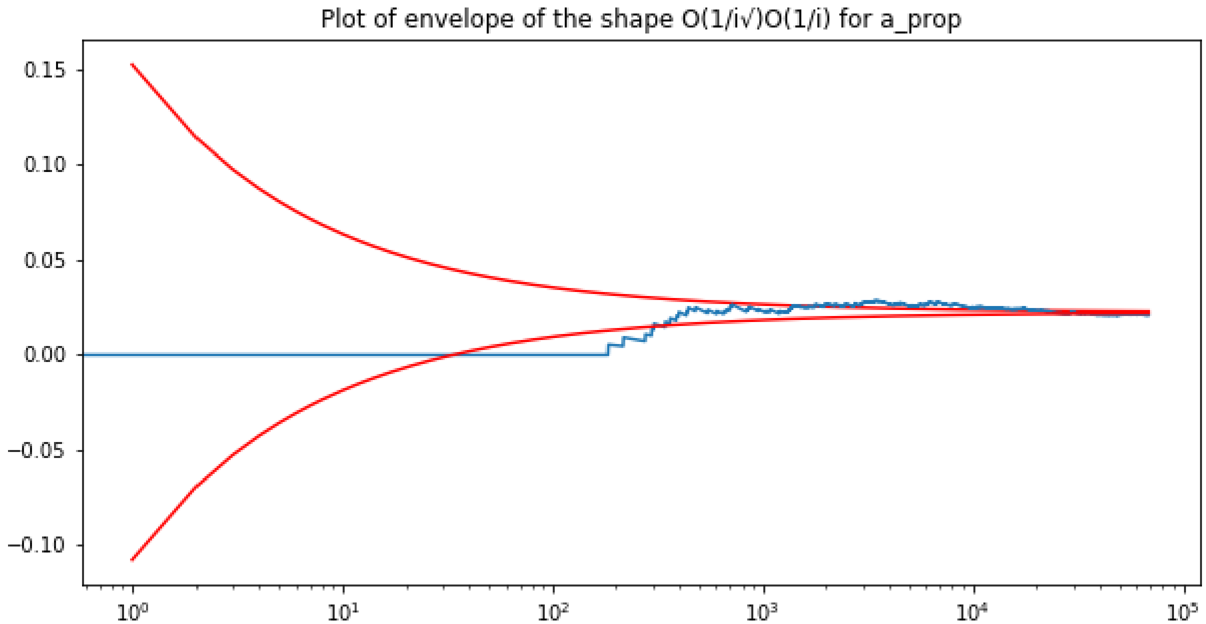
In [147]:
```
plt.figure(figsize=(10,5))
plt.plot(a_prop, label ='a')
plt.plot(and_prop, label ='and')
plt.plot(the_prop, label ='the')
plt.plot(i_prop, label ='i')
plt.plot(is_prop, label ='is')
plt.title('Proportions of n_(a, and, the, i, is)[i]/i over the document')
plt.legend()
plt.show()
```



2.3. Answer) Plot of envelope of the shape O(1/i√)O(1/i) for a_prop.

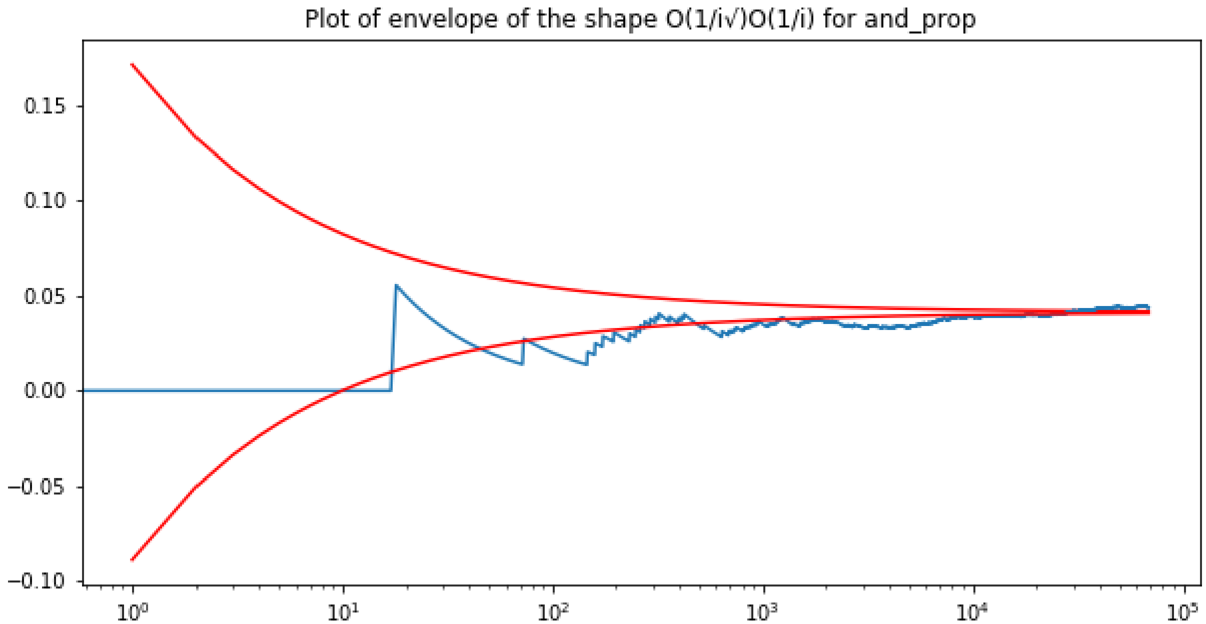
In [148]:

```
y = np.arange(1,len(lst)+1).reshape(len(lst),1)
plt.figure(figsize=(10,5))
plt.semilogx(y, 0.13*np.power(y,-0.5) + np.mean(a_prop), 'r')
plt.plot(a_prop)
plt.semilogx(y, -0.13*np.power(y,-0.5) + np.mean(a_prop), 'r')
plt.title('Plot of envelope of the shape O(1/i√)O(1/i) for a_prop')
plt.show()
```



2.3. Answer) Plot of envelope of the shape O(1/i√)O(1/i) for and_prop.

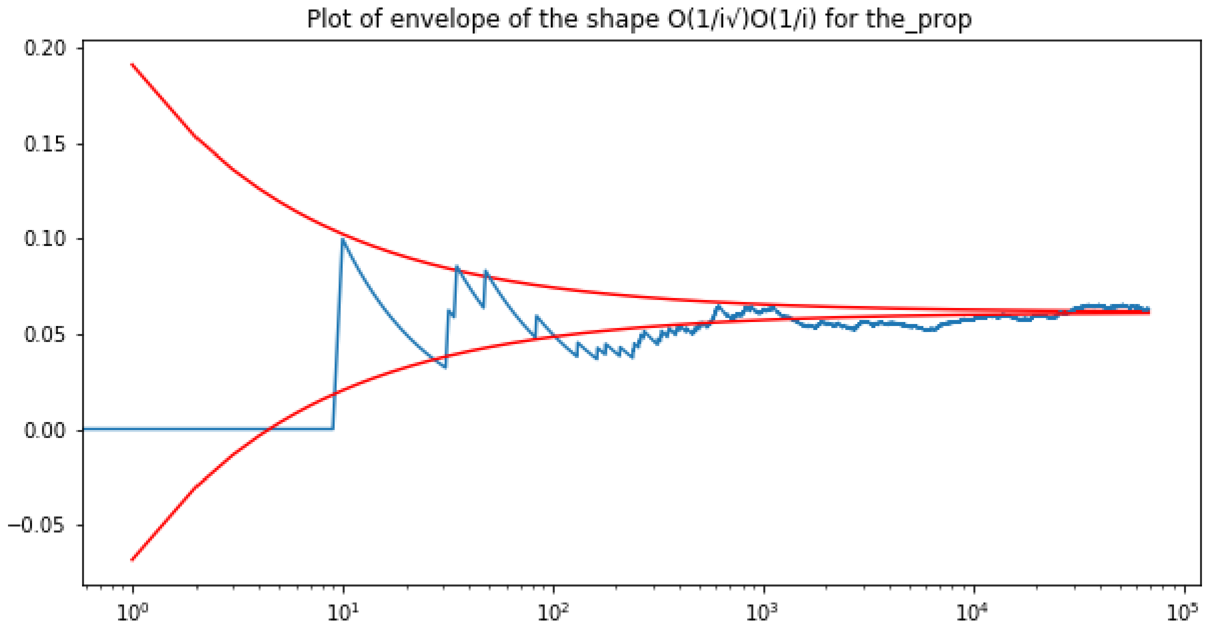
In [149]:
```
y = np.arange(1,len(lst)+1).reshape(len(lst),1)
plt.figure(figsize=(10,5))
plt.semilogx(y, 0.13*np.power(y,-0.5) + np.mean(and_prop), 'r')
plt.plot(and_prop)
plt.semilogx(y, -0.13*np.power(y,-0.5) + np.mean(and_prop), 'r')
plt.title('Plot of envelope of the shape O(1/i√)O(1/i) for and_prop')
plt.show()
```



2.3. Answer) Plot of envelope of the shape O(1/i√)O(1/i) for the_prop.

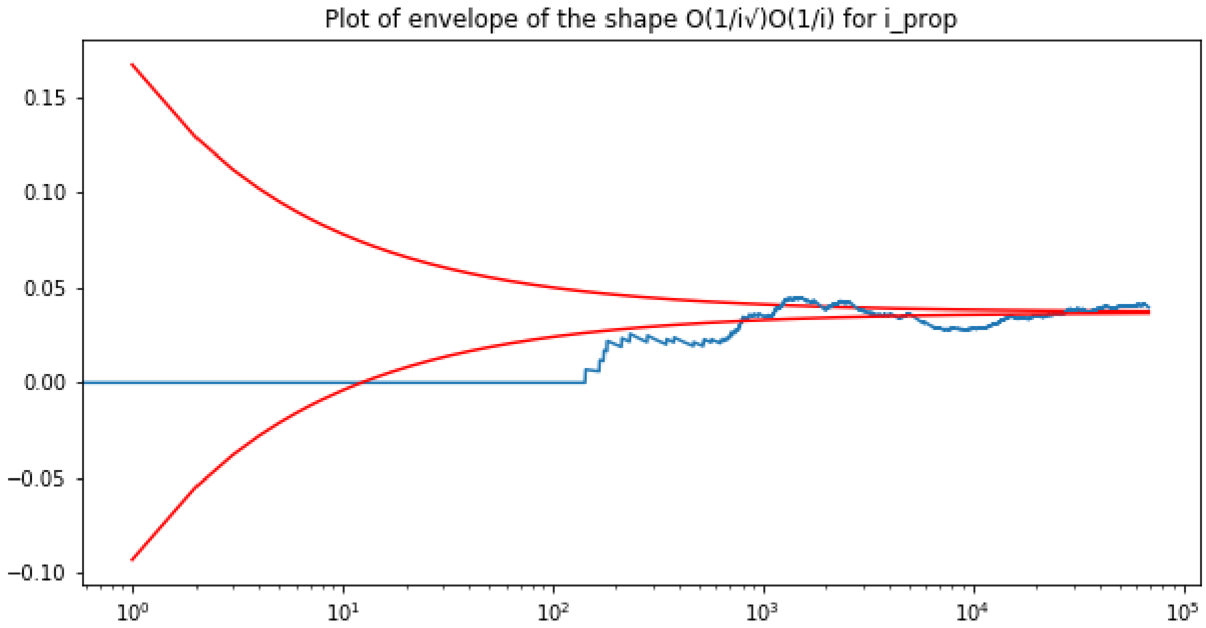
In [150]:

```
y = np.arange(1,len(lst)+1).reshape(len(lst),1)
plt.figure(figsize=(10,5))
plt.semilogx(y, 0.13*np.power(y,-0.5) + np.mean(the_prop), 'r')
plt.plot(the_prop)
plt.semilogx(y, -0.13*np.power(y,-0.5) + np.mean(the_prop), 'r')
plt.title('Plot of envelope of the shape O(1/i√)O(1/i) for the_prop')
plt.show()
```



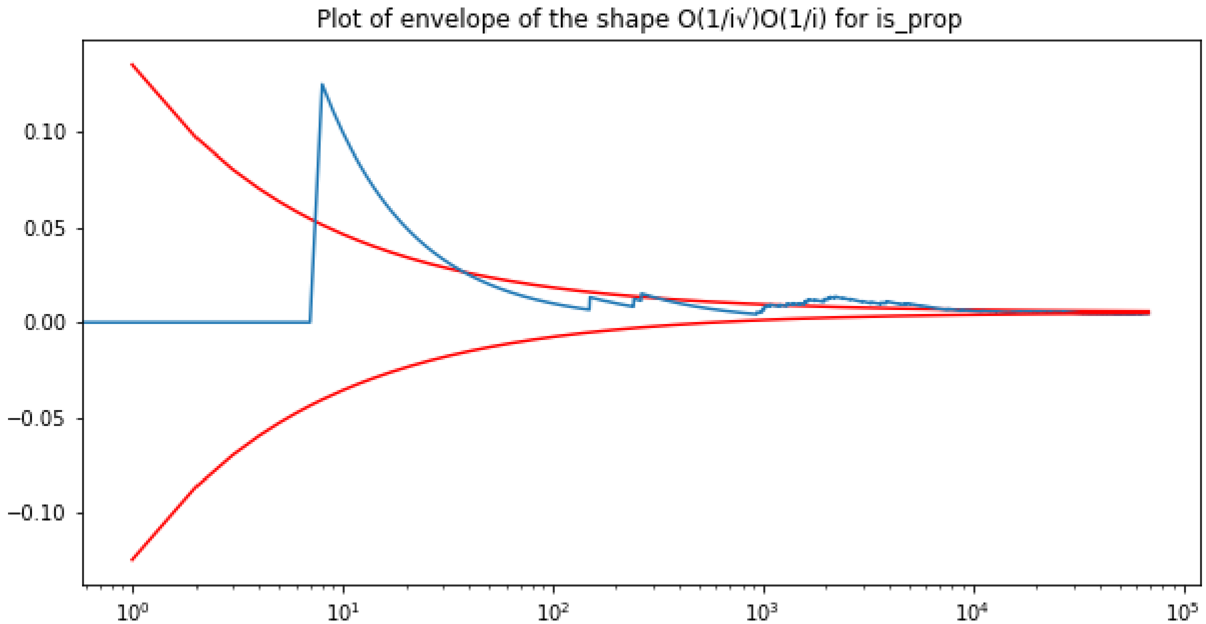
2.3. Answer) Plot of envelope of the shape O(1/i√)O(1/i) for i_prop.

In [151]:
```
y = np.arange(1,len(lst)+1).reshape(len(lst),1)
plt.figure(figsize=(10,5))
plt.semilogx(y, 0.13*np.power(y,-0.5) + np.mean(i_prop), 'r')
plt.plot(i_prop)
plt.semilogx(y, -0.13*np.power(y,-0.5) + np.mean(i_prop), 'r')
plt.title('Plot of envelope of the shape O(1/i√)O(1/i) for i_prop')
plt.show()
```



2.3. Answer) Plot of envelope of the shape O(1/i√)O(1/i) for is_prop.

In [152]:
```
y = np.arange(1,len(lst)+1).reshape(len(lst),1)
plt.figure(figsize=(10,5))
plt.semilogx(y, 0.13*np.power(y,-0.5) + np.mean(is_prop), 'r')
plt.plot(is_prop)
plt.semilogx(y, -0.13*np.power(y,-0.5) + np.mean(is_prop), 'r')
plt.title('Plot of envelope of the shape O(1/i√)O(1/i) for is_prop')
plt.show()
```



Answer) Why can we not apply the Central Limit Theorem directly?

We can't apply the Central Limit Theorem directly because the sequence in which words appear in a document is not independent of one another.

Answer) How would we have to change the text for it to apply?

If we shuffle the texts, then we can apply the CLT.

Answer) Why does it still work quite well?

It still works well because sample size (# of words) is large and our bag of words are relatively small so dependency isn't magnified.

3. Denominator-layout notation

We used the numerator-layout notation for matrix calculus in class, now let's examine the denominator-layout notation.

Given $x, y \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, we have

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}, \quad \frac{\partial \mathbf{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x}, \frac{\partial y_2}{\partial x}, \ldots, \frac{\partial y_m}{\partial x} \end{bmatrix}$$

and

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{y}}{\partial x_1} \\ \frac{\partial \mathbf{y}}{\partial x_2} \\ \vdots \\ \frac{\partial \mathbf{y}}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_1}, \ldots, \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2}, \frac{\partial y_2}{\partial x_2}, \ldots, \frac{\partial y_m}{\partial x_2} \\ \vdots \\ \frac{\partial y_1}{\partial x_n}, \frac{\partial y_2}{\partial x_n}, \ldots, \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

Questions:

1. Assume $\mathbf{y} = f(\mathbf{u})$ and $\mathbf{u} = g(\mathbf{x})$, write down the chain rule for $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$
2. Given $\mathbf{X} \in \mathbb{R}^{m \times n},\ \mathbf{w} \in \mathbb{R}^n,\ \mathbf{y} \in \mathbb{R}^m$, assume $z = ||\mathbf{X}\mathbf{w} - \mathbf{y}||^2$, compute $\frac{\partial z}{\partial \mathbf{w}}$.

Answers:

$1. \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \frac{\partial \mathbf{y}}{\partial \mathbf{u}}$ (performed in this order for matrix dimensions to match denominator layout notation.)

2. First expand z. $z = \|\mathbf{Xw} - \mathbf{y}\|^2, = (Xw-y)^T(Xw-y)$

$$
\begin{aligned}
&= (w^T X^T - y^T)(Xw-y) \\
&= w^T XTXw - w^T XTy - y^T Xw + y^T y \\
&= w^T XTXw - 2y^T Xw + y^T y \\
Now\ &\frac{\partial z}{\partial \mathbf{w}}(w^T X^T Xw - 2y^T Xw + y^T y) = 2w^T(X^T X) - 2y^T X
\end{aligned}
$$

4. Numerical Precision

Given scalars `x` and `y`, implement the following `log_exp` function such that it returns

$$-\log\left(\frac{e^x}{e^x + e^y}\right)$$

.

In [145]:
```
def log_exp(x, y):
    ## add your solution here
    return -nd.log(nd.exp(x) / (nd.exp(x) + nd.exp(y)))
```

Test your codes with normal inputs:

In [146]:
```
x, y = nd.array([2]), nd.array([3])
z = log_exp(x, y)
z
```

Out[146]:
```
[1.3132617]
<NDArray 1 @cpu(0)>
```

Now implement a function to compute $\partial z / \partial x$ and $\partial z / \partial y$ with `autograd`

In [138]:
```
def grad(forward_func, x, y):
    ## Add your codes here
    x.attach_grad()
    y.attach_grad()
    with autograd.record():
        z = forward_func(x,y)
    z.backward()
```

Test your codes, it should print the results nicely.

In [139]:
```
grad(log_exp, x, y)
print('x.grad =', x.grad)
print('y.grad =', y.grad)
```

```
x.grad =
[-0.7310586]
<NDArray 1 @cpu(0)>
y.grad =
[0.7310586]
<NDArray 1 @cpu(0)>
```

But now let's try some "hard" inputs

In [140]:
```
x, y = nd.array([50]), nd.array([100])
grad(log_exp, x, y)
print('x.grad =', x.grad)
print('y.grad =', y.grad)
```

```
x.grad =
[nan]
<NDArray 1 @cpu(0)>
y.grad =
[nan]
<NDArray 1 @cpu(0)>
```

Does your code return correct results? If not, try to understand the reason. (Hint, evaluate `exp(100)`). Now develop a new function `stable_log_exp` that is identical to `log_exp` in math, but returns a more numerical stable result.

In [143]:

```
def stable_log_exp(x, y):
    if x > y:
        a = x
        b = y
    else:
        a = y
        b = x
    return a + nd.log(1 + nd.exp(b-a)) - x
grad(stable_log_exp, x, y)
print('x.grad =', x.grad)
print('y.grad =', y.grad)
```

```
x.grad =
[-1.]
<NDArray 1 @cpu(0)>
y.grad =
[1.]
<NDArray 1 @cpu(0)>
```

In [ ]: