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Computational Finance
Lecture 6 – Robust Portfolio Selection and Factor Models
MFIN 706

Lecture outline

Robust portfolio selection

- Risk parity, equal risk contributions
- Robust mean-variance optimization

Factor models

- Capital Asset Pricing Model (CAPM)
- Arbitrage Pricing Theory (APT)

📖 Robust Portfolio Selection

Problems with mean-variance model

- **Mean-variance model:**
  - Significantly over-estimates return and under-estimates risk in optimized portfolios
  - Produces instable optimal solutions as small changes in input estimation often generate large changes in the optimal portfolio
  - Optimal mean-variance portfolio are not necessarily well diversified
  - Assumes a single-period framework, but many investors, e.g., in pension funds and insurance, have long-term investment objectives

- **Enhancements to mean-variance model:**
  - Practitioners and investors are more interested in downside risk and usually ignore upside risk: use downside risk measures (semi-variance)
  - Distribution of returns is not Normal: use tail-based (downside) risk measures, e.g., Value-at-Risk, Conditional-Value-at-Risk
  - Mean-variance model is sensitive to estimation errors: better estimation techniques, re-sampling, risk parity, robust optimization, factor models
  - Multi-period models

Robust portfolio selection

- Mean-variance model overuses statistically estimated information and, as a result, maximizes the effect of estimation error
  - Estimation error is the difference between the true parameters (mean, variance and covariance) and the estimated value of these parameters
- Chopra and Ziemba (1993) showed that errors in the expected returns are about 10 times more important than errors in covariances:
  - Avoid using return estimates: use covariances only
  - Estimate returns robustly: robust parameter estimation (Black-Litterman model), portfolio resampling 稳健的
  - Assume an uncertainty set around expected returns: robust optimization
- Fixing errors in return estimates: avoid using return estimates
  - Select "1/*n*" portfolio
  - Select minimum variance portfolio
  - Select portfolio with equal risk contributions (ERC portfolio) from each asset (also known as *risk parity* or *risk budgeting*)
- Fixing errors in return estimates: robust mean-variance optimization
- Fixing errors in covariance estimates: factor models weather

Risk Parity, Equal Risk Contributions

Decomposition of portfolio risk measures

f(w) is the variance of a function of weight=wTQw

- $f(w)$ denotes a risk measure for a portfolio with asset weights w
- As portfolio standard deviation is homogeneous function of degree one, from Euler's homogeneous function theorem:

$$f(w) = \sum_{i=1}^{n} w_i \frac{\partial f(w)}{\partial w_i}$$

- From this decomposition of risk measure $f(w)$, a contribution of asset i to the risk measure $f(w)$ denoted as RC_i or $C_i\, f(w)$ is

$$\mathrm{RC}_i = C_i f(w) = w_i \frac{\partial f(w)}{\partial w_i}$$

- Percentage contribution of risk measure $f(w)$, denoted as $\%C_i\, f(w)$, is equal

$$\%C_i f(w) = \frac{C_i f(w)}{f(w)} = \frac{w_i}{f(w)} \frac{\partial f(w)}{\partial w_i}$$

- Marginal contribution to risk

$$\mathrm{MCR}_i = \frac{\partial f(w)}{\partial w_i}$$

Homogenous functions and Euler's theorem

- **Definition:** homogeneous function of degree one

  Let $f(\boldsymbol{x}) = f(x_1, \ldots, x_n)$ be a continuous and differentiable function of the variables $\boldsymbol{x} = (x_1, \ldots, x_n)^T$. f is homogeneous of degree one if for any constant c, $f(c \cdot x_1, \ldots, c \cdot x_n) = c \cdot f(x_1, \ldots, x_n)$.

- Examples: Define $\boldsymbol{x} = (x_1, x_2)^T$ and $\boldsymbol{e} = (1, 1)^T$.

  Let $f(x_1, x_2) = x_1 + x_2 = \boldsymbol{e}^T \boldsymbol{x} = f(\boldsymbol{x})$. Then
  $$f(c \cdot \boldsymbol{x}) = \boldsymbol{e}^T (c \cdot \boldsymbol{x}) = c \cdot (\boldsymbol{e}^T \boldsymbol{x}) = c \cdot f(\boldsymbol{x})$$

  Let $f(x_1, x_2) = x_1^2 + x_2^2 = \boldsymbol{x}^T \boldsymbol{x} = f(\boldsymbol{x})$. Then
  $$f(c \cdot \boldsymbol{x}) = (c \cdot \boldsymbol{x})^T (c \cdot \boldsymbol{x}) = c^2 \cdot (\boldsymbol{x}^T \boldsymbol{x}) \neq c \cdot f(\boldsymbol{x})$$
  homogeneous of degree two

  Let $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2} = (\boldsymbol{x}^T \boldsymbol{x})^{1/2} = f(\boldsymbol{x})$. Then
  $$f(c \cdot \boldsymbol{x}) = ((c \cdot \boldsymbol{x})^T (c \cdot \boldsymbol{x}))^{1/2} = c \cdot (\boldsymbol{x}^T \boldsymbol{x})^{1/2} = c \cdot f(\boldsymbol{x})$$

- **Theorem**: Euler's theorem

  Let $f(\boldsymbol{x}) = f(x_1, \ldots, x_n)$ be a continuous, differentiable and homogenous of degree one function of the variables $\boldsymbol{x} = (x_1, \ldots, x_n)^T$. Then
  $$f(\boldsymbol{x}) = x_1 \cdot \frac{\partial f(\boldsymbol{x})}{\partial x_1} + \ldots + x_n \cdot \frac{\partial f(\boldsymbol{x})}{\partial x_n} = \boldsymbol{x}^T \cdot \nabla f(\boldsymbol{x})$$

Decomposition of portfolio risk measures

- Risk measure is a portfolio variance $f(w) = \sigma_p^2(w) = w^T Q w$:

$$\mathrm{RC}_i = C_i \,\mathrm{var}(w) = w_i \frac{\partial f(w)}{\partial w_i} = \underline{2 w_i (Qw)_i}$$

- Risk measure is a portfolio standard deviation $f(w) = \sigma_p(w) = \sqrt{w^T Q w}$:

$$\mathrm{RC}_i = C_i \,\mathrm{std}(w) = w_i \frac{\partial f(w)}{\partial w_i} = w_i \frac{(Qw)_i}{\sqrt{w^T Q w}} \qquad \frac{w_i\ (Qw)}{\text{sqrt}(w' Q w)}$$

- Deviation of a portfolio *w* from the equal risk contribution (ERC) portfolio:

$$d(w) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathrm{RC}_i - \mathrm{RC}_j)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (C_i f(w) - C_j f(w))^2$$

- Our goal is to find a portfolio, where

$$\mathrm{RC}_i = \mathrm{RC}_j \quad \forall\ i, j$$

Equal risk contribution (ERC) portfolios with standard deviation risk measure

- Risk measure is portfolio standard deviation $f(w) = \sigma_p(w) = \sqrt{w^T Q w}$:

$$\mathrm{RC}_i = C_i \,\mathrm{std}(w) = w_i \boxed{\frac{\partial f(w)}{\partial w_i}} = \boxed{\frac{w_i (Qw)_i}{\sqrt{w^T Q w}}}$$

- Minimize deviation of a portfolio *w* from the equal risk contribution portfolio:

$$d(w) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathrm{RC}_i - \mathrm{RC}_j)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (C_i f(w) - C_j f(w))^2$$

- Solve non-linear optimization problem (non-linear objective):

$$\begin{aligned} \min_{w} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \left(w_i (\underbrace{Qw}_{=f(x)})_i - w_j (Qw)_j\right)^2 \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{n} w_i = 1 \\ & w \geq 0 \end{aligned}$$

Computing equal risk contributions portfolio in Matlab

```
% Random data for 10 stocks
n = 10;
Q = randn(n); Q = Q*Q'/1000; % covariance matrix
mu  = rand(1,n)'/100;        % expected return

% Equality constraints
A_eq = ones(1,n); b_eq = 1;

% Inequality constraints
A_ineq = []; b_ineql = []; b_inequ = [];

% Define initial portfolio ("1/n portfolio")
w0 = repmat(1.0/n, n, 1);

options.lb = zeros(1,n);        % lower bounds on variables
options.lu = ones (1,n);        % upper bounds on variables
options.cl = [b_eq' b_ineql']; % lower bounds on constraints
options.cu = [b_eq' b_inequ']; % upper bounds on constraints

% Set the IPOPT options
options.ipopt.jac_c_constant        = 'yes';
options.ipopt.hessian_approximation = 'limited-memory';
options.ipopt.mu_strategy           = 'adaptive';
options.ipopt.tol                   = 1e-10;

% The callback functions
funcs.objective         = @computeObjERC;
funcs.constraints       = @computeConstraints;
funcs.gradient          = @computeGradERC;
funcs.jacobian          = @computeJacobian;
funcs.jacobianstructure = @computeJacobian;

%% Run IPOPT
[w_erc info] = ipopt(w0',funcs,options);

% Compute variance and asset risk contributions for the ERC portfolio
std_ERC = sqrt(w_erc*Q*w_erc');
 RC_ERC = (w_erc' .* ( Q*w_erc' )) / sqrt(w_erc*Q*w_erc');
```

Equal risk contributions portfolio

$$\min_{w} \sum_{i=1}^{n}\sum_{j=1}^{n}\left(w_i(Qw)_i - w_j(Qw)_j\right)^2$$
$$\text{s.t. } \sum_{i=1}^{n} w_i = 1$$
$$w \geq 0$$

compute gradient in the function called compute constraints

Sum of risk contributions should be equal to portfolio standard deviation

`norm(sum(RC_ERC)-std_ERC)`

Computing equal risk contributions portfolio in Matlab

```
     Portfolio ERC return = 0.00548
  Portfolio minVar return = 0.00605
     Portfolio 1/n return = 0.00477
   Portfolio ERC st.dev. = 0.02455
Portfolio minVar st.dev. = 0.02071
   Portfolio 1/n st.dev. = 0.03380
```

$$\begin{aligned} \min_{w} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\left(w_i(Qw)_i - w_j(Qw)_j\right)^2 \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{n} w_i = 1 \\ & w \geq 0 \end{aligned}$$

```
Portfolio weights for ERC, minVar and 1/n portfolios:
   0.180981953239785   0.218673166680919   0.10000
   0.079302394525101   0.061325366015795   0.10000
   0.070617222967983                   0   0.10000
   0.135809605682682   0.199507146162841   0.10000
   0.049097575088658                   0   0.10000
   0.084084332175664   0.061738170598153   0.10000
   0.038147423686914                   0   0.10000
   0.180240217294946   0.196791100523478   0.10000
   0.079497762703981   0.080742768882074   0.10000
   0.102221512634286   0.181222281136741   0.10000
```

$\frac{1}{N}$

non of them use expected returns

the middle one is not fully diversified, some are zero

```
Asset risk contributions for ERC, minVar and 1/n portfolios:
   0.002455426192429   0.004527838374640   0.000495142771429
   0.002455434272175   0.001269800724980   0.005041476137304
   0.002455526996161                   0   0.003015840420291
   0.002455413513190   0.004130987473781   0.002045922904311
   0.002455485309035                   0   0.006070144152725
   0.002455436210822   0.001278348241155   0.003312375325107
   0.002455550799736                   0   0.007991408240698
   0.002455423088444   0.004074749134804  -0.001376023614983
   0.002455418932625   0.001671856739297   0.004466899192400
   0.002455411858867   0.003752381745439   0.002736187000100
```

sum up

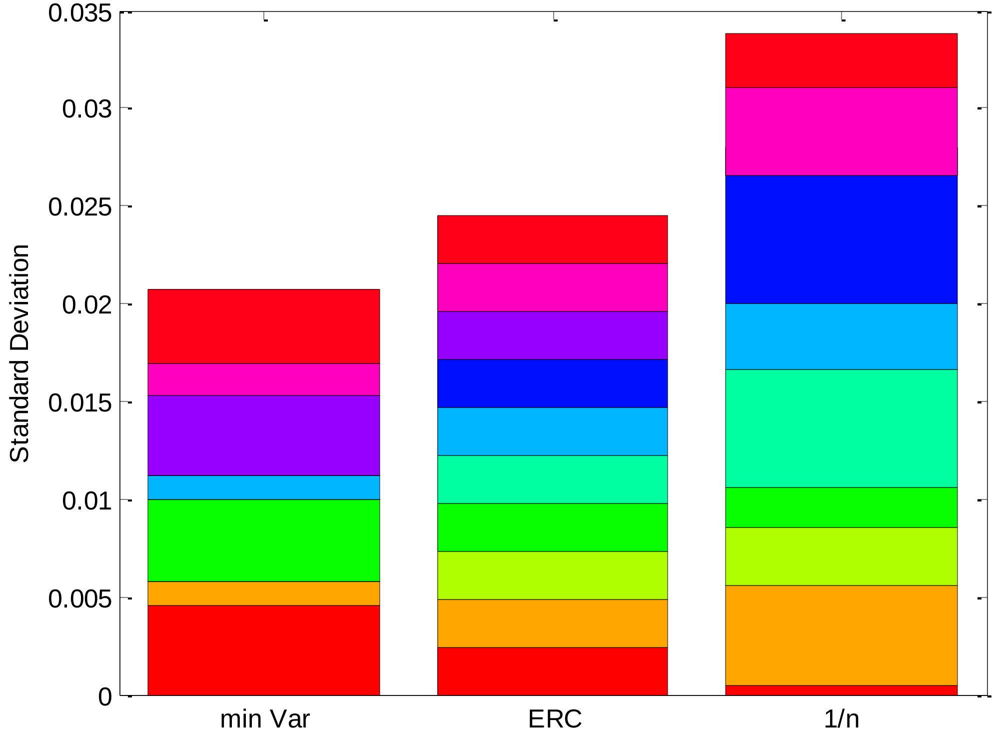
```
Sum of asset risk contributions for ERC, minVar and 1/n portfolios:
   0.024554527173484   0.020705962434096   0.033799372529381

Standard deviation for ERC, minVar and 1/n portfolios:
   0.024554527173484   0.020705962434096   0.033799372529381
```

Computing equal risk contributions portfolio in Matlab

```
Asset risk contributions for ERC, minVar, 1/n portfolios:
   0.00245542   0.00452783   0.00049514
   0.00245543   0.00126980   0.00504147
   0.00245552            0   0.00301584
   0.00245541   0.00413098   0.00204592
   0.00245548            0   0.00607014
   0.00245543   0.00127834   0.00331237
   0.00245555            0   0.00799140
   0.00245542   0.00407475  -0.00137602
   0.00245541   0.00167185   0.00446689
   0.00245541   0.00375238   0.00273618

St. dev. for ERC, minVar, 1/n portfolios:
   0.02455453   0.02070596   0.03379937
```

all weather funds

$$\sigma_{\text{min Var}} \leq \sigma_{\text{ERC}} \leq \sigma_{1/n}$$



📖 Robust Mean-Variance Optimization

Improving stability – example

Three asset example:

shorting allowed

$$
\begin{aligned}
\max_{w} \quad & \mu^T w \\
\text{s.t.} \quad & (w - w_b)^T Q (w - w_b) \leq 0.1^2 \\
& \textstyle\sum_{i=1}^{n} w_i = 1 \\
& w \text{ unconstrained}
\end{aligned}
$$

the squared tracking error

- Expected returns and standard deviations (correlations = 20%)

	μ^1	μ^2	σ
Asset 1	7.15%	7.16%	20%
Asset 2	7.16%	7.15%	24%
Asset 3	7.00%	7.00%	28%

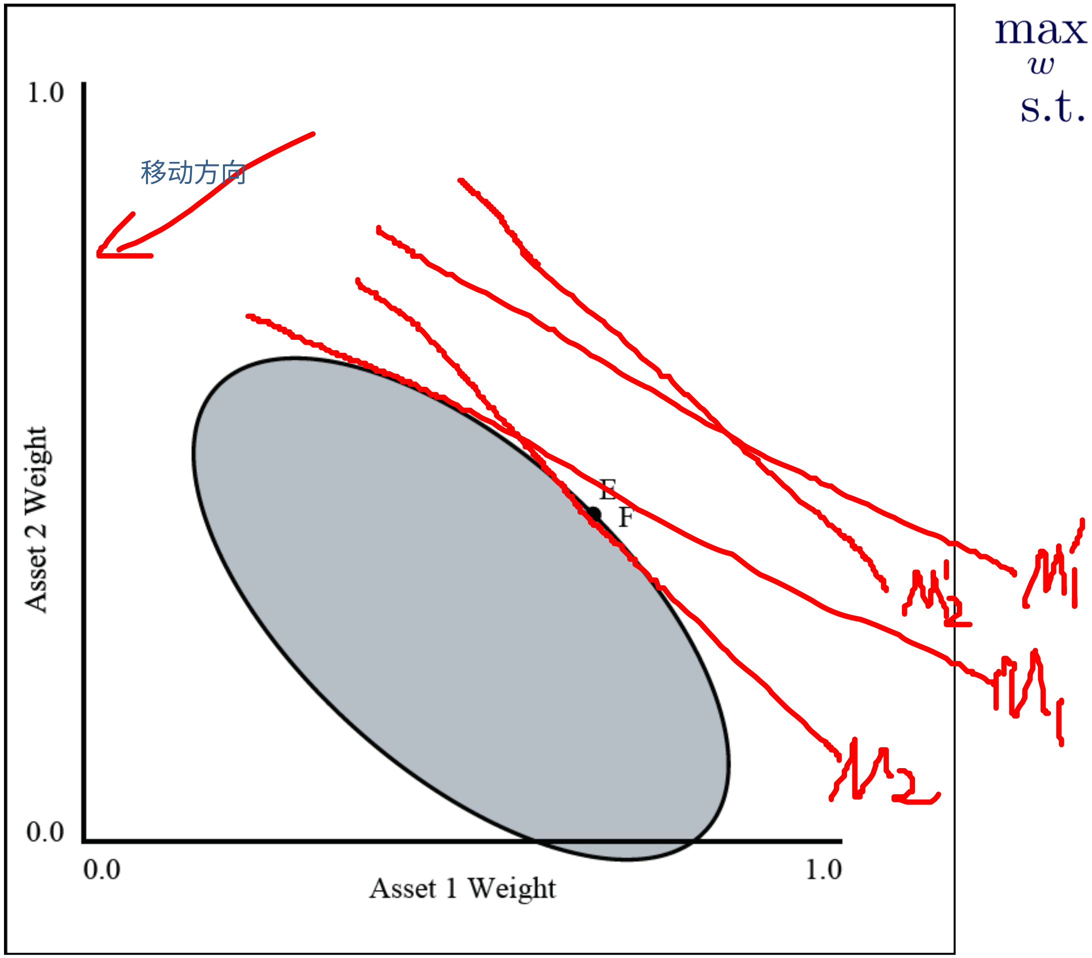
- Optimal weights

	Portfolio E	**Portfolio F**
Asset 1	67.18%	67.26%
Asset 2	43.10%	43.01%
Asset 3	-10.28%	-10.28%

Source: S. Ceria. Robust portfolio construction, 2006

Improving stability – example

Three asset example: graphical representation



$$
\begin{aligned}
\max_{w} \quad & \mu^T w \\
\text{s.t.} \quad & (w - w_b)^T Q (w - w_b) \le 0.1^2 \\
& \textstyle\sum_{i=1}^{n} w_i = 1 \\
& w \text{ unconstrained}
\end{aligned}
$$

Source: S. Ceria. Robust portfolio construction, 2006

Improving stability – example

Three asset example:

no shorting allowed

$$\begin{array}{ll} \max_{w} & \mu^T w \\ \text{s.t.} & (w - w_b)^T Q (w - w_b) \le 0.1^2 \\ & \sum_{i=1}^{n} w_i = 1 \\ & w \ge 0 \end{array}$$

- Expected returns and standard deviations (correlations = 20%)

	μ^1	μ^2	σ
Asset 1	7.15%	7.16%	20%
Asset 2	7.16%	7.15%	24%
Asset 3	7.00%	7.00%	28%

- Optimal weights

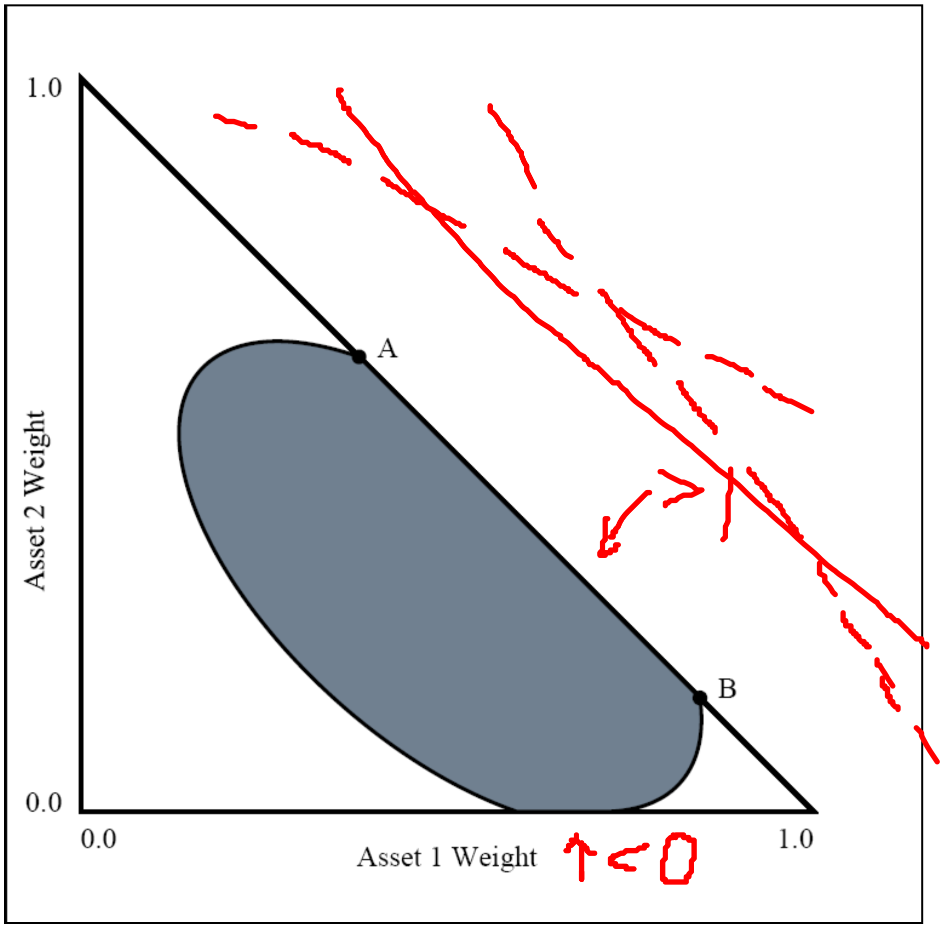
	Portfolio A	Portfolio B
Asset 1	38.1%	84.3%
Asset 2	61.9%	15.7%
Asset 3	0.0%	0.0%

big jump because boundary

Source: S. Ceria. Robust portfolio construction, 2006

Improving stability – example

Three asset example: constraints create instability

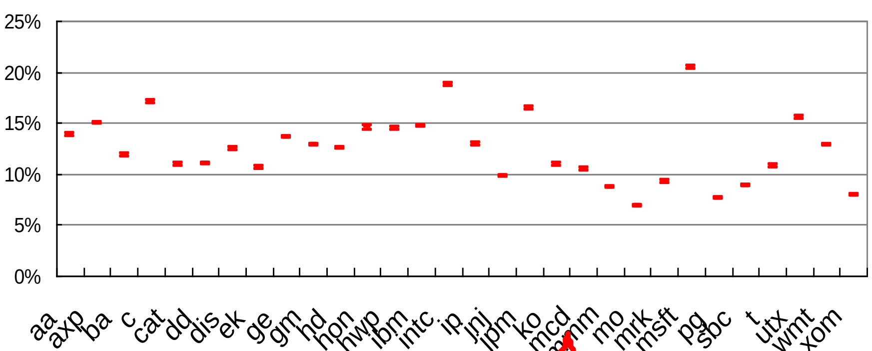


$$
\begin{array}{ll}
\max\limits_{w} & \mu^T w \\
\text{s.t.} & (w - w_b)^T Q (w - w_b) \leq 0.1^2 \\
 & \sum_{i=1}^{n} w_i = 1 \\
 & w \geq 0
\end{array}
$$

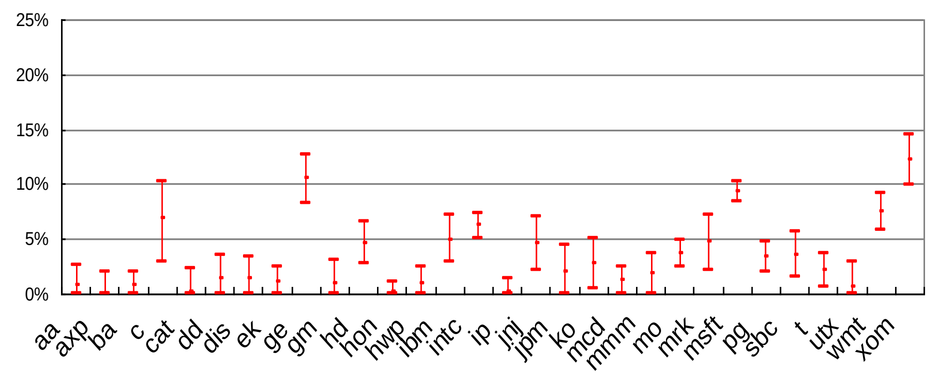
Source: S. Ceria. Robust portfolio construction, 2006

Stability experiment on the Dow 30

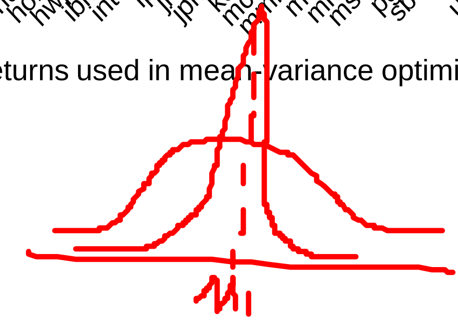
- Instability due to changes in expected returns:
  - Use expected returns and covariance from Idzorek (2002) for Dow 30
  - Randomly generate 10,000 expected return estimate vectors from a Normal distribution with mean equal to the expected return and std equal to 0.1% of the std of return of the corresponding asset
  - Run 10,000 traditional mean-variance optimizations and record the weights of the resulting portfolios
  - Use a fixed risk aversion coefficient



Range of expected returns used in mean-variance optimization



Range of asset weights obtained from mean-variance optimization



Source: S. Ceria. Robust portfolio construction, 2006

Estimation error generates inefficiencies

Estimated Frontier

Efficient Frontier computed using the estimated expected returns

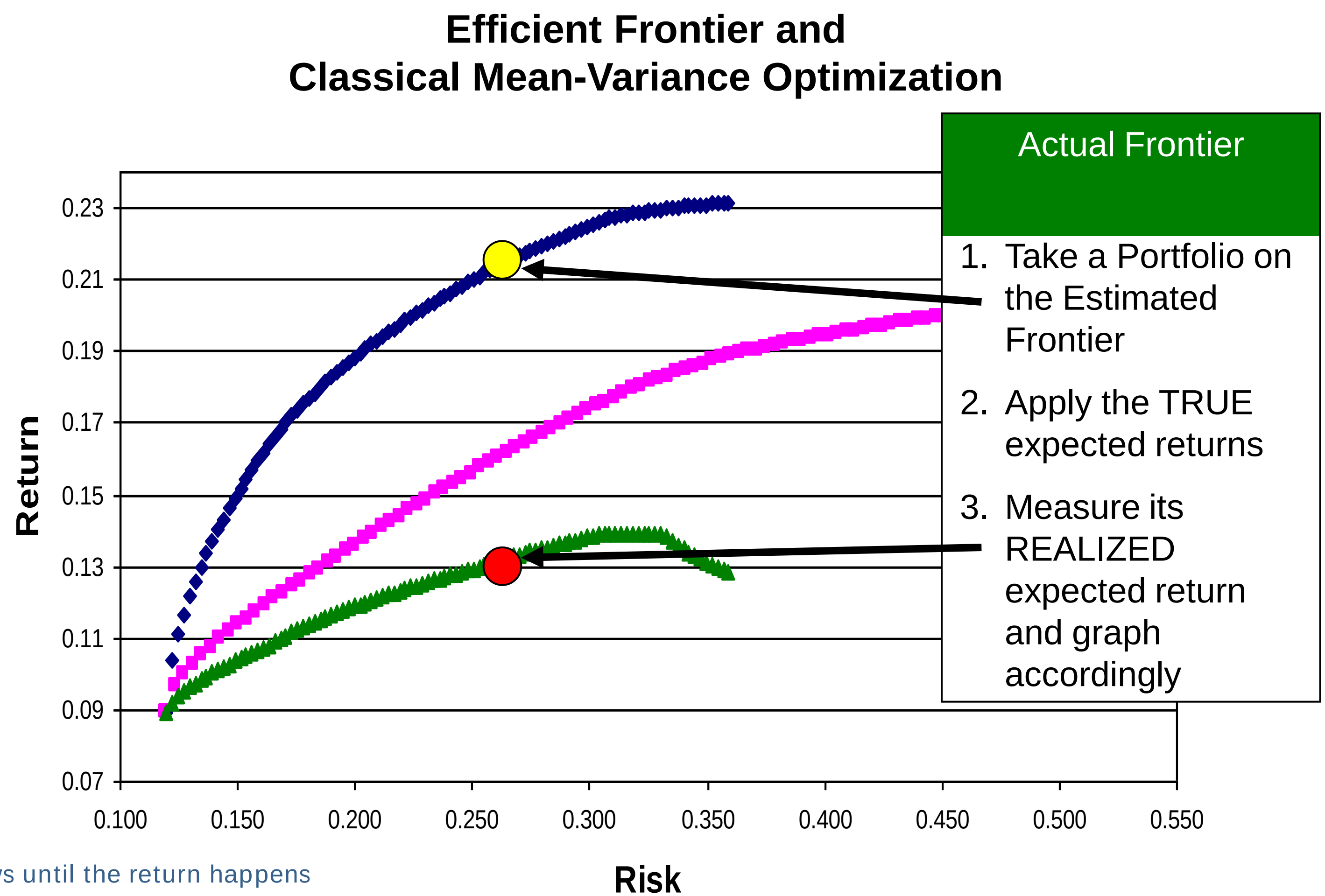
True Frontier

Efficient Frontier computed using the true expected returns

Actual Frontier

Return for the portfolios in the Estimated Frontier using the true expected returns

knows until the return happens



- How does the True Efficient Frontier differ from the Actual Frontier?

Source: S. Ceria. Robust portfolio construction, 2006

Robust optimization

- What is robust optimization?
  - An optimization process that incorporates uncertainties of the inputs into an optimization problem
  - It explicitly considers estimation error within the optimization process
  - It was developed independently by Ben-Tal and Nemirovski, initial applications were in the area of engineering
- What are the advantages of using robust mean-variance optimization?
  - Recognize that there are errors in the estimation process and directly “exploit” that knowledge
  - Address portfolio construction practice
  - We can solve the robust mean-variance optimization problem “efficiently” in “roughly” the same time as the classical mean variance optimization problem
- How do we solve robust mean-variance optimization problems?
  - Robust mean-variance optimization problem can be formulated as a Quadratically Constrained Quadratic Optimization Problem (Second Order Cone Optimization Problem)
  - Interior Point Methods (IPMs) are used to solve such problems, e.g., with CPLEX

Robust optimization – ellipsoidal uncertainty set

- The vector of true expected returns *r* lies in the ellipsoidal uncertainty set:

$$r \in \mathcal{U}(\mu)_\delta = \{r \,:\, (r-\mu)^T \Theta^{-1} (r-\mu) \leq \delta^2\}$$

define the shape of the set

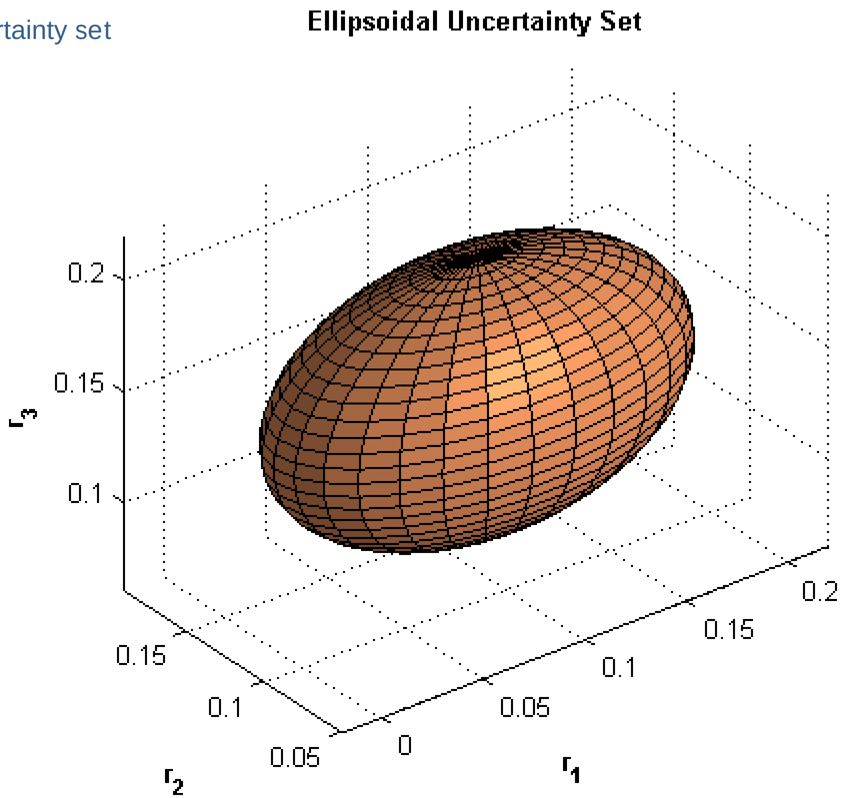
- Robust portfolio optimization:

$$\begin{aligned} \min \quad & -r^T w + \lambda w^T Q w \\ \text{s.t.} \quad & \textstyle\sum_i w_i = 1 \\ & w \geq 0 \\ & \forall\, r \in \mathcal{U}(\mu)_\delta \end{aligned}$$

r is a random number from the uncertainty set

- Ellipsoidal uncertainty set:

u1, u2, u3 ….ui
will be the center of uncertainty set



Robust optimization – box uncertainty set

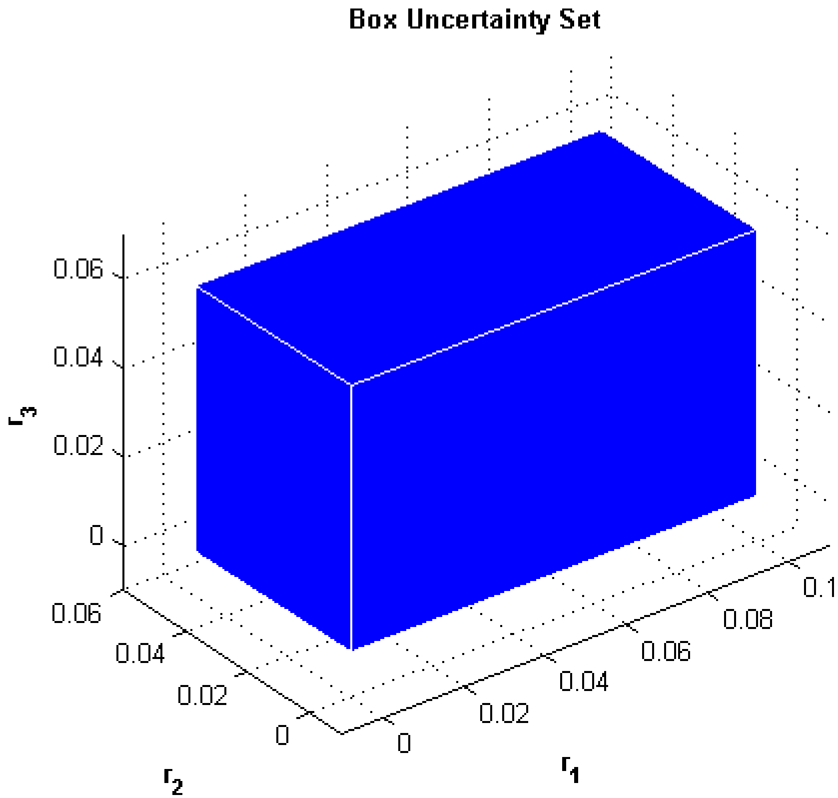
- The vector of true expected returns *r* lies in the box uncertainty set:

$$r \in \mathcal{U}(\mu)_\delta = \{r : |r_i - \mu_i| \le \delta_i,\ i = 1, \dots, n\}$$

- Robust portfolio optimization:

$$\begin{array}{ll} \min & -r^T w + \lambda w^T Q w \\ \text{s.t.} & \sum_i w_i = 1 \\ & \quad w \ge 0 \\ & \quad \forall\, r \in \mathcal{U}(\mu)_\delta \end{array}$$

- Box uncertainty set:



Robust optimization

- The vector of true expected returns *r* lies in the box uncertainty set or ellipsoidal uncertainty set :

$$r \in \mathcal{U}(\mu)_\delta = \{r \,:\, |r_i - \mu_i| \leq \delta_i,\ i = 1, \ldots, n\}$$

$$r \in \mathcal{U}(\mu)_\delta = \{r \,:\, (r - \mu)^T \Theta^{-1} (r - \mu) \leq \delta^2\}$$

- Objectives:
  - minimize variance of portfolio return $w^T Q w$
  - maximize portfolio expected return $\mu^T w$
  - minimize portfolio return estimation error
    - box uncertainty set $\delta^T |w| = \|Dw\|_1,\ D = \mathrm{diag}(\delta_j)$
    - ellipsoidal uncertainty set $\|\Theta^{1/2} w\| = \sqrt{w^T \Theta w}$

- Constraints (as before):

$$\begin{aligned} \textstyle\sum_i w_i &= 1 \\ w &\geq 0 \end{aligned}$$

Mathematical formulation of robust optimization

then add constraint $w^T\Theta w \le t^2$

let t

ε-constrained formulation

$$\begin{aligned} \max \quad & \mu^T w \\ \text{s.t.} \quad & w^T Q w \le \varepsilon_1 \\ & \|\Theta^{1/2} w\| \le \varepsilon_2 \\ & \textstyle\sum_{i=1}^n w_i = 1 \\ & w \ge 0 \end{aligned}$$

ensure in the constraint is a convex set

weighted sum formulation

$$\begin{aligned} \min \quad & \boxed{-\mu^T w} + \boxed{\lambda w^T Q w} + \boxed{\lambda_{\text{er}} \|\Theta^{1/2} w\|} \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^n w_i = 1 \\ & w \ge 0 \end{aligned}$$

the bigger the lamda, the more important this portion is

Maximize expected return

Additional term that “corrects” for risk

Additional term that “corrects” for estimation error

- μ vector of expected returns
- Q covariance matrix of returns
- Θ covariance matrix of estimated returns
- λ risk aversion
- λ_{er} estimation error aversion
- ε_1 bound on variance (risk)
- ε_2 size of uncertainty set

$$\|\Theta^{1/2} w\| = \sqrt{w^T \Theta w}$$

$$\|x\| = \sqrt{x^T x}$$

Typical choices for Θ:

- $\Theta = I$ (identity matrix)
  shape will be like a ball
- $\Theta = \mathrm{diag}(Q)$ (variances)

Robust mean-variance portfolio optimization in Matlab

```
% Random data for 10 stocks
n = 10;
Q = randn(n); Q = Q*Q'/1000; % covariance matrix
mu  = rand(1,n)'/100;         % expected return

%% Initial portfolio ("equally weighted" or "1/n")
w0 = ones(n,1) ./ n;

ret_init = dot(mu, w0);   % 1/n portfolio return
var_init = w0' * Q * w0; % 1/n portfolio variance

% Bounds on variables
lb_rMV = zeros(n,1); ub_rMV = inf*ones(n,1);

% Target portfolio return estimation error
var_matr = diag(diag(Q));
rob_init = w0' * var_matr * w0; % r.est.err. of 1/n portf
rob_bnd = rob_init;          % target return estimation error

% Compute minimum variance portfolio (MVP)
% Target portfolio return = return of MVP
Portf_Retn = ret_minVar;

%% Formulate and solve robust mean-variance problem
f_rMV  = zeros(n,1); % objective function
% Constraints
A_rMV  = sparse([  mu'; ones(1,n)]);the first constraint
lhs_rMV = [Portf_Retn; 1]; rhs_rMV = [inf; 1];
% Create CPLEX model
cplex_rMV = Cplex('Robust_MV');
cplex_rMV.addCols(f_rMV, [], lb_rMV, ub_rMV);
cplex_rMV.addRows(lhs_rMV, A_rMV, rhs_rMV);the first constraint
% Add quadratic objective
cplex_rMV.Model.Q = 2*Q;
% Add quadratic constraint on return estimation error (robustness constraint)
cplex_rMV.addQCs(zeros(size(f_rMV)), var_matr, 'L', rob_bnd, {'qc_robust'});
% Solve
cplex_rMV.solve();
```

sparse save the space by omitting storage of value zero， instead, it saves the non-zero value's column and row number

L means less or equal to, G means greater or equal to

Robust mean-variance optimization

$$\begin{aligned} \min \quad & w^T Q w \\ \text{s.t.} \quad & \mu^T w \geq \varepsilon_{\text{ret}} \\ & \|\Theta^{1/2} w\| \leq \varepsilon_{\text{rob}} \\ & \textstyle\sum_{i=1}^n w_i = 1 \\ & w \geq 0 \end{aligned}$$

yita

$$w^T \Theta w \leq \tilde{\varepsilon}_{\text{rob}} = \varepsilon_{\text{rob}}^2$$

$$\varepsilon_{\text{ret}} = \mu^T w_{\text{mvp}}$$

$$\tilde{\varepsilon}_{\text{rob}} = w_0^T \Theta w_0$$

Add quadratic constraint in CPLEX

$$c^T w + w^T \Theta w \leq \tilde{\varepsilon}_{\text{rob}}$$

Robust mean-variance portfolio optimization in Matlab

```
      Solution status = optimal
        Solution time = 0.062 seconds
   Solution objective = 0.00051938

     Portfolio rMV return = 0.00626
  Portfolio minVar return = 0.00605
    Portfolio init return = 0.00477

    Portfolio rMV st.dev. = 0.02279
 Portfolio minVar st.dev. = 0.02071
   Portfolio init st.dev. = 0.03380

   Portfolio rMV r.est.err. = 0.03015
 Portfolio r.est.err. bound = 0.03015
Portfolio minVar r.est.err. = 0.03725
  Portfolio init r.est.err. = 0.03015


Portfolio weights before and after rounding
of near-zero elements:
   0.179615611138064   0.179615611138467
   0.038042817087437   0.038042817087522
   0.043800866287312   0.043800866287410
   0.138742661246450   0.138742661246761
   0.006269906626988   0.006269906627002
   0.088177664144388   0.088177664144586
   0.000000000002309                   0
   0.127195720474382   0.127195720474668
   0.075087726943317   0.075087726943486
   0.303067026049420   0.303067026050099
```

Robust mean-variance optimization

$$
\begin{aligned}
\min \quad & w^T Q w \\
\text{s.t.} \quad & \mu^T w \geq \varepsilon_{\text{ret}} \\
& \|\Theta^{1/2} w\| \leq \varepsilon_{\text{rob}} \\
& \textstyle\sum_{i=1}^{n} w_i = 1 \\
& w \geq 0
\end{aligned}
$$

$$w^T \Theta w \leq \tilde{\varepsilon}_{\text{rob}}$$

$$\varepsilon_{\text{ret}} = \mu^T w_{\text{mvp}}$$

$$\tilde{\varepsilon}_{\text{rob}} = w_0^T \Theta w_0$$

Intuition behind robust mean-variance optimization

Estimated Frontier

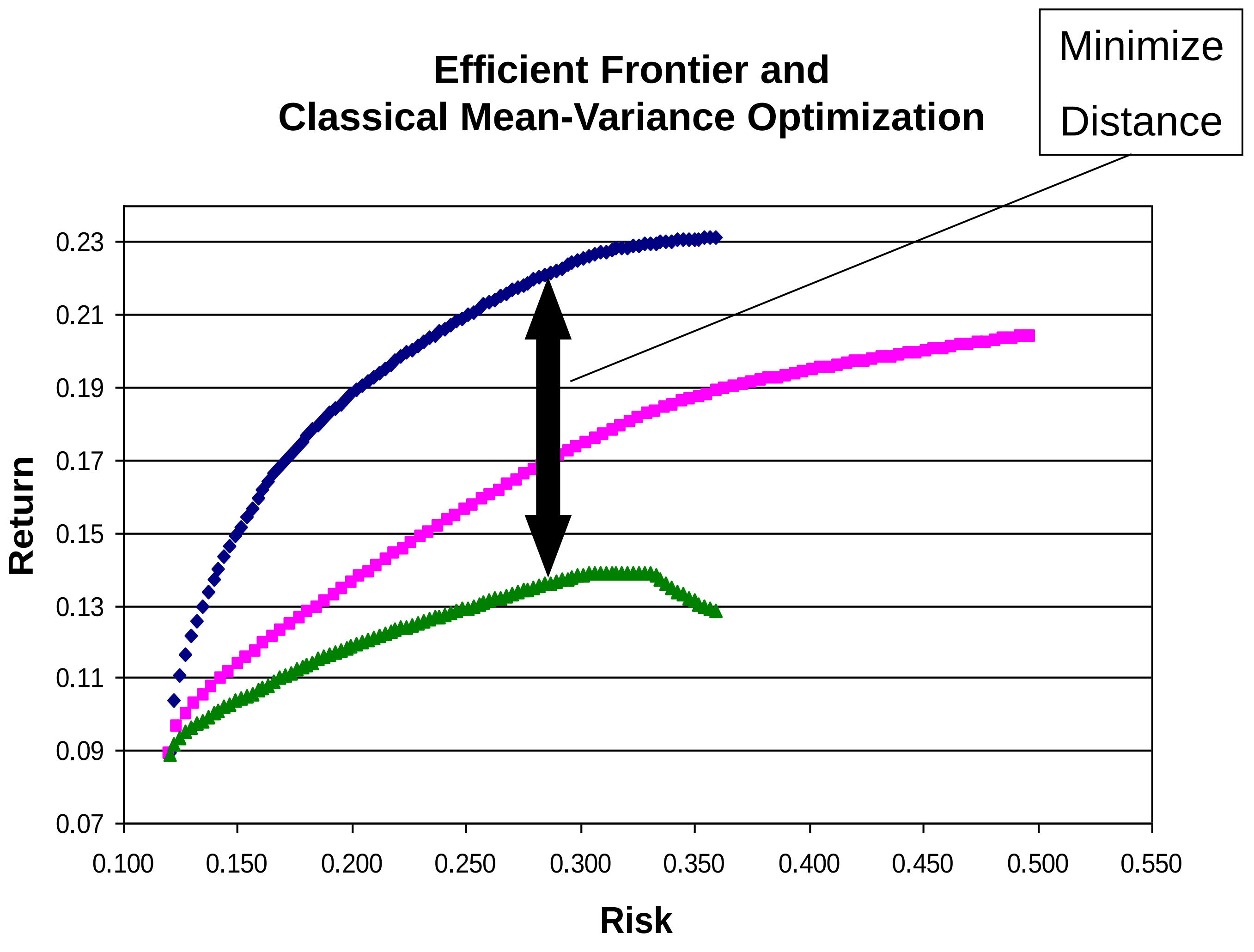
Efficient Frontier computed using the estimated expected returns

True Frontier

Efficient Frontier computed using the true expected returns

Actual Frontier

Return for the portfolios in the Estimated Frontier using the true expected returns



Source: S. Ceria. Robust portfolio construction, 2006

Intuition behind robust mean-variance optimization – example

Three asset example:

no shorting allowed

$$\begin{aligned} \max_{w} \quad & \mu^T w \\ \text{s.t.} \quad & (w - w_b)^T Q (w - w_b) \leq 0.1^2 \\ & \textstyle\sum_{i=1}^{n} w_i = 1 \\ & w \geq 0 \end{aligned}$$

- Expected returns and standard deviations (correlations = 20%)

	μ^1	μ^2	σ
Asset 1	7.15%	7.16%	20%
Asset 2	7.16%	7.15%	24%
Asset 3	7.00%	7.00%	28%

- Optimal weights

two μ and 3 lamda represent high, medium and low aversion condition

	High Aversion		Medium Aversion		Low Aversion	less robust
	A μ_1	more robust B μ_2	A	B	A	B
Asset 1	35.26%	35.68%	43.38%	45.54%	47.36%	52.65%
Asset 2	35.69%	35.27%	45.55%	43.39%	52.64%	47.35%
Asset 3	29.05%	29.05%	11.07%	11.07%	0.0%	0.0%

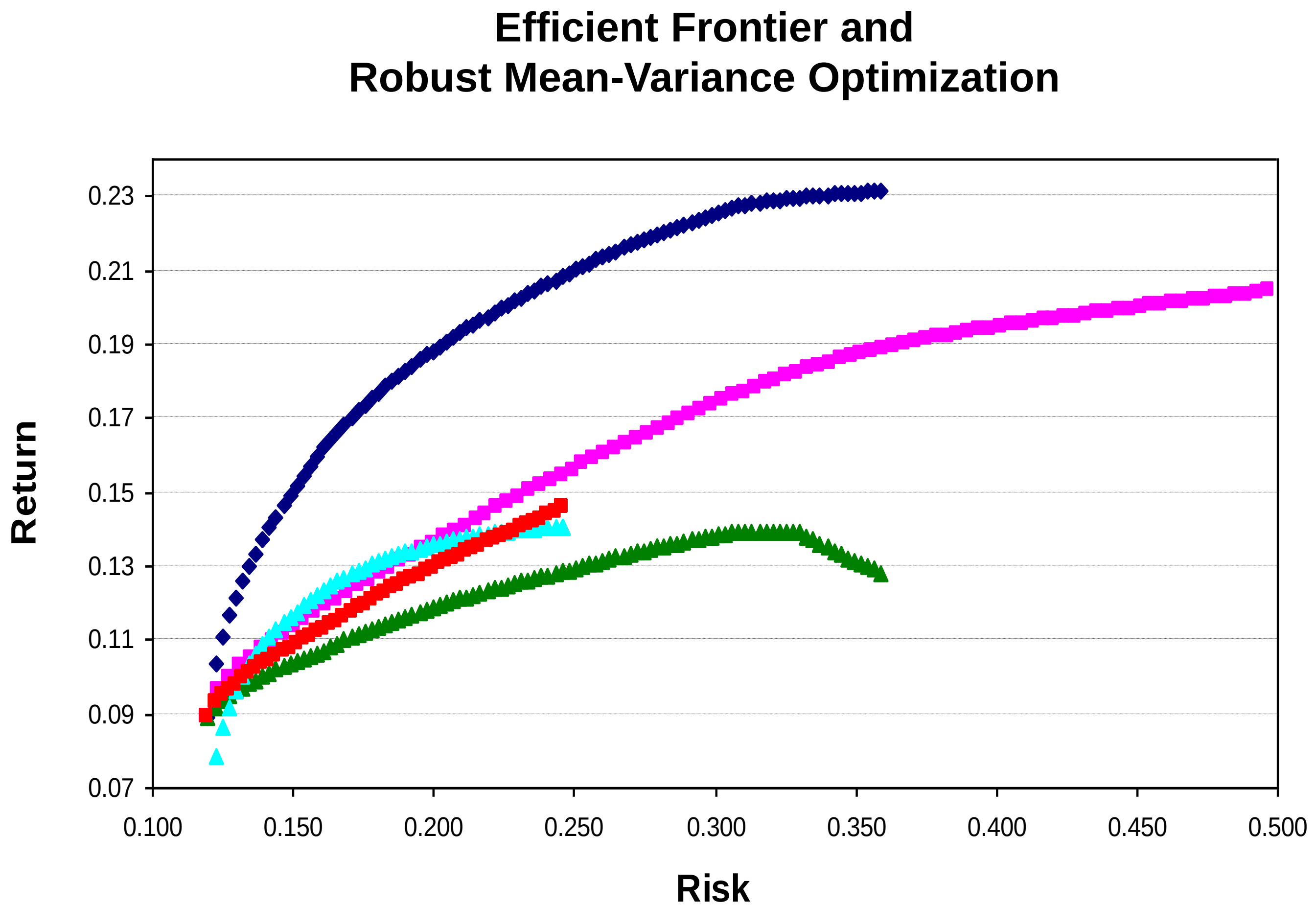
Source: S. Ceria. Robust portfolio construction, 2006

Reducing overestimation/underestimation

Estimated Robust Frontier
Robust Efficient Frontier computed using the estimated expected returns and the estimation error

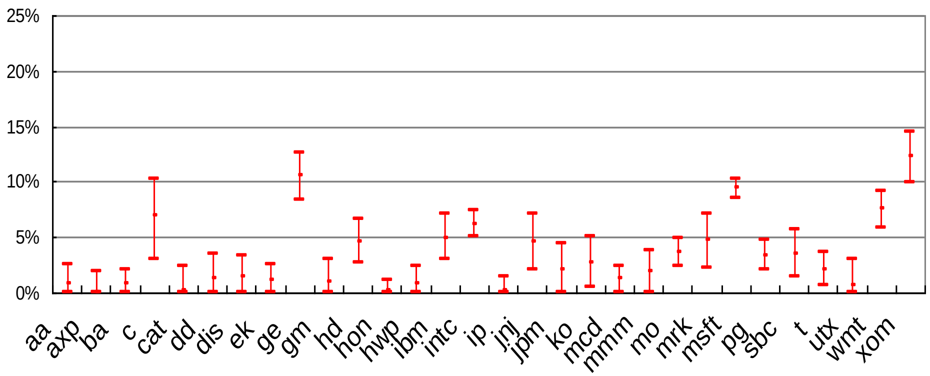
True Frontier
Efficient Frontier computed using the true expected returns

Actual Robust Frontier
Return for the portfolios in the Estimated Frontier using the true expected returns

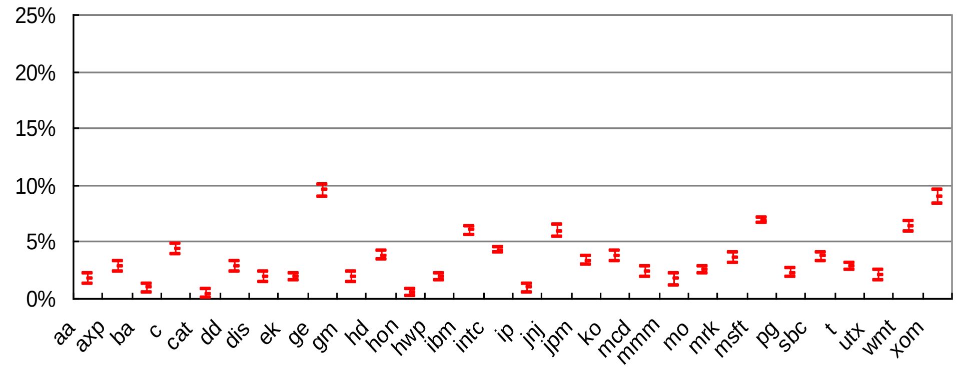


Source: S. Ceria. Robust portfolio construction, 2006

Improving optimal portfolio stability and intuition



Range of asset weights obtained from mean-variance optimization



Range of asset weights for robust mean-variance optimization

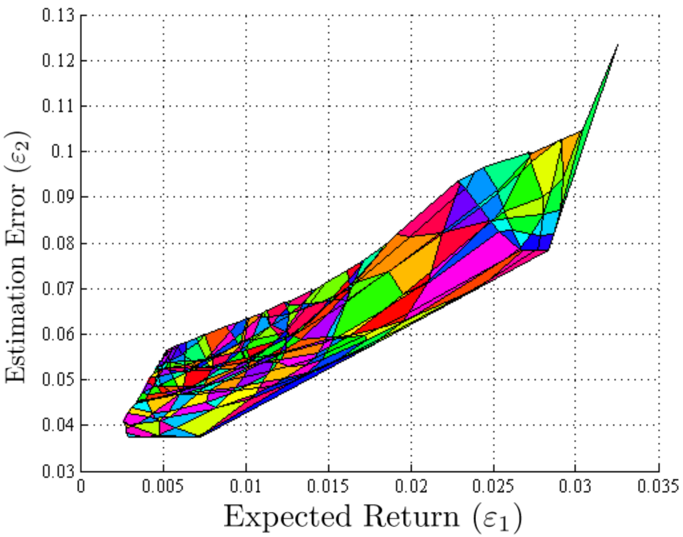
- Resulting asset weights using classical mean-variance optimization vs. robust mean-variance optimization for the prior Dow 30 example:
  - Lower ranges in asset weights
  - Less variability across asset weights

Source: S. Ceria. Robust portfolio construction, 2006

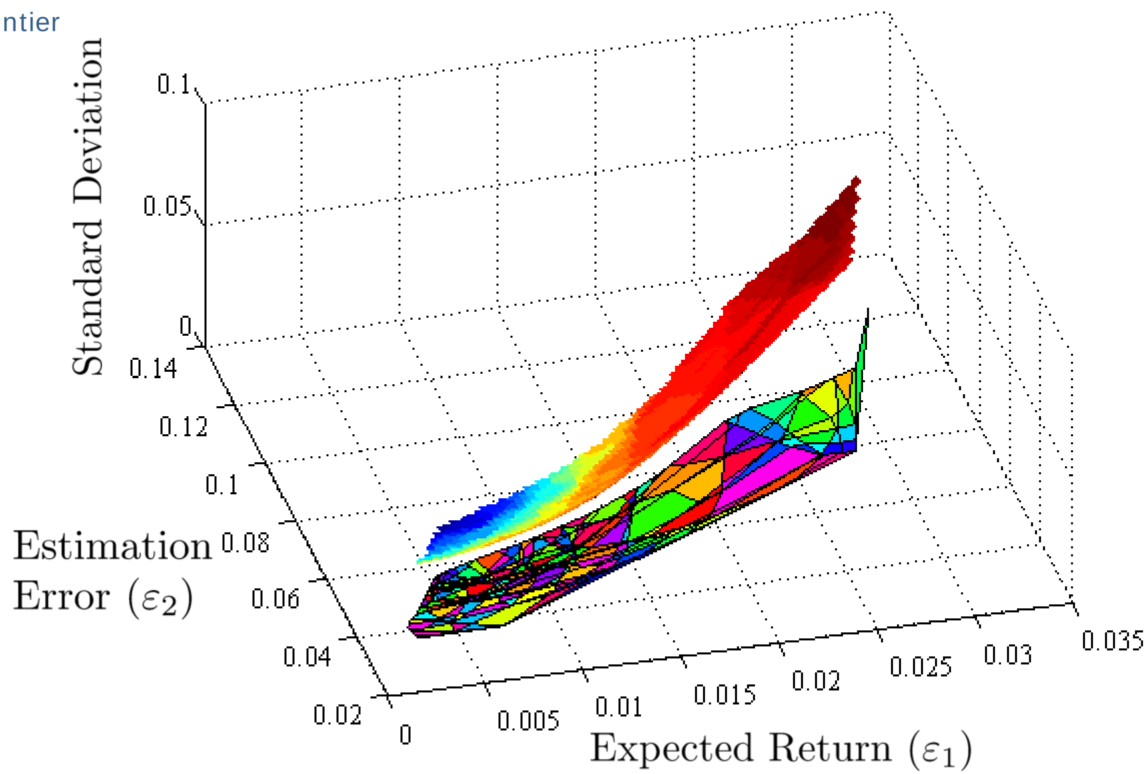
Robust optimization – box uncertainty set

Multi-objective robust optimization:

$$
\begin{aligned}
\min \quad & w^T Q w \\
\text{s.t.} \quad & \mu^T w \geq \varepsilon_1 \\
& \delta^T |w| \leq \varepsilon_2 \\
& \textstyle\sum_{i=1}^{100} w_i = 1 \\
& w \geq 0
\end{aligned}
$$



3-dimension frontier

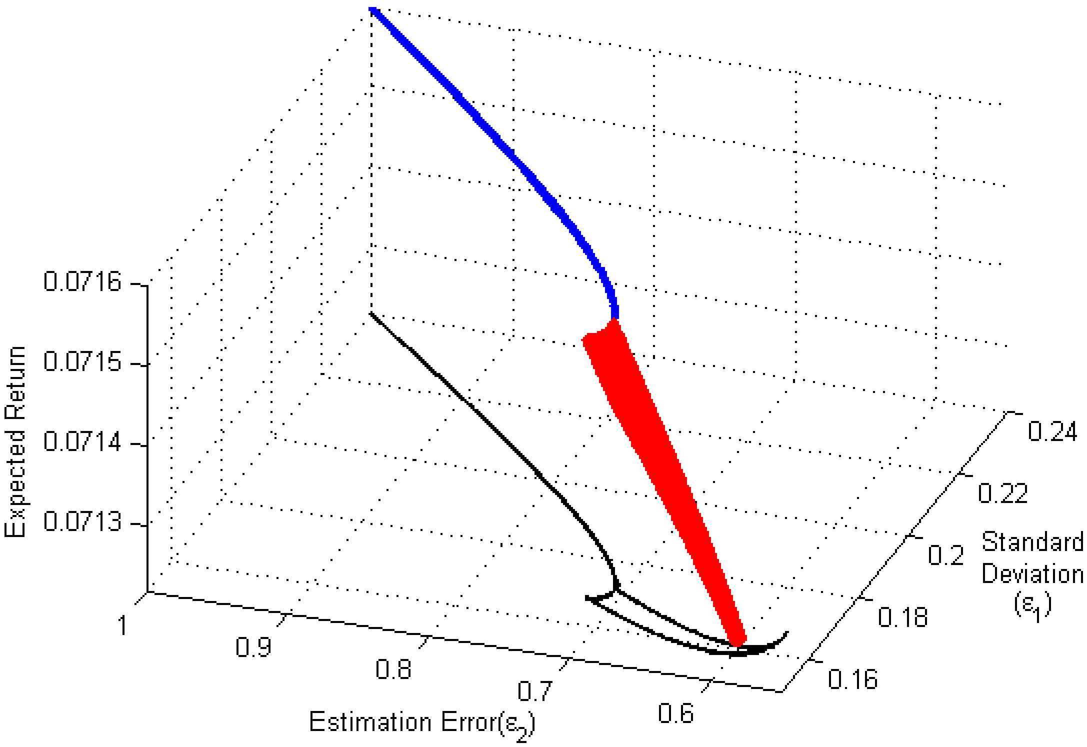


Robust optimization – ellipsoidal uncertainty set

Multi-objective robust optimization:

$$\begin{array}{ll} \max & \mu^T w \\ \text{s.t.} & w^T Q w \le \varepsilon_1 \\ & \|\Theta^{1/2} w\| \le \varepsilon_2 \\ & \sum_{i=1}^n w_i = 1 \\ & w \ge 0 \end{array}$$

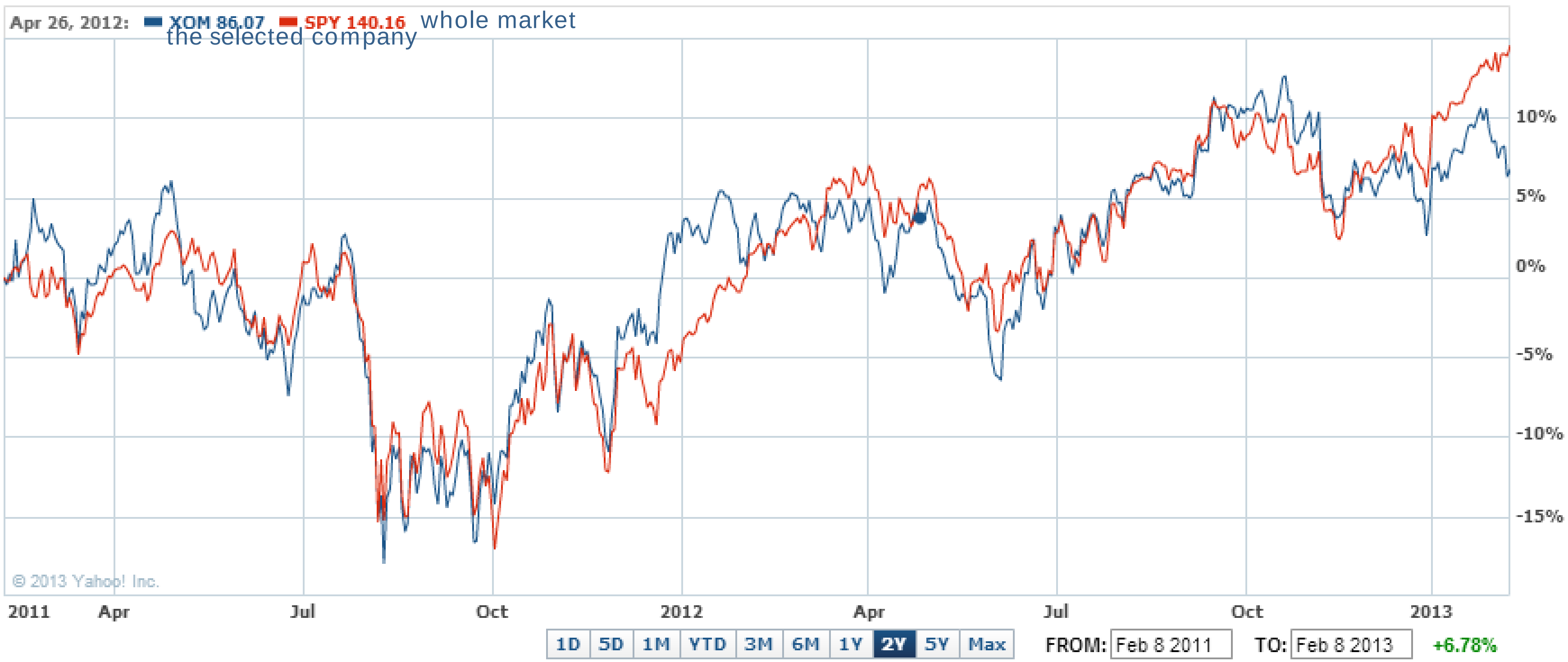
Robust portfolio optimization problem solution – **efficient surface**



📖 Factor Models

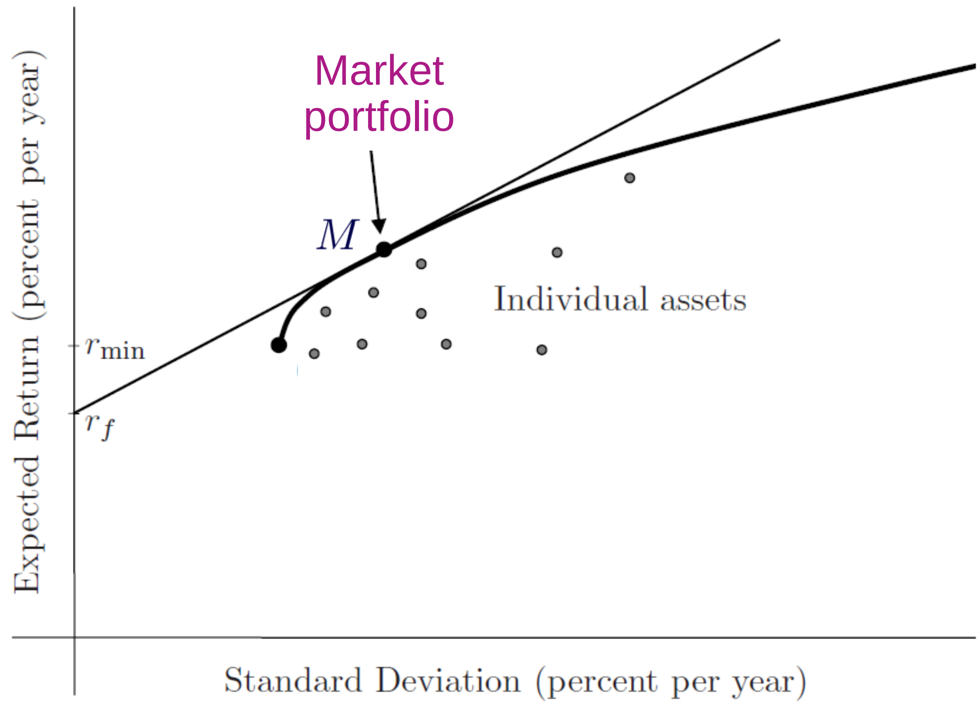
Factor models

- Factor analysis attempts to simplify the joint behaviour of assets by looking at key drivers:
  - Statistical factor analysis: identify principal components via principal-component analysis (PCA), but those components are not interpretable
  - Fundamental factor analysis: uses real drivers (indexes, interest rates, exchange rates, commodities), but may fail to include key drivers and is sensitive to data



Capital Market Line (CML)

- Capital Market Line (CML):
  - Capital Allocation Line (CML) is a ray in (r_P, σ_P) coordinates that starts at the point $(r_f, 0)$
  - A portfolio that maximizes Sharpe ratio corresponds to the point where CML is tangential to the Markowitz efficient frontier
  - In the presence of a risk-free asset, all efficient frontier portfolios lie on the CML and it gives the trade-off between portfolio risk and return



- CML describes all possible mean-variance efficient portfolios that are a combination of the risk free asset and market portfolio:
  - buy risk free asset (between *M* and r_f) or
  - sell risk free asset (beyond point *M*) and
  - hold the same portfolio *M* of risky assets

- Every portfolio on CML is an efficient fund of risky assets (market portfolio) and a risk free asset (a bond that matures at the end of investment horizon)

Capital Market Line (CML)

- Equation describes all portfolios on CML:

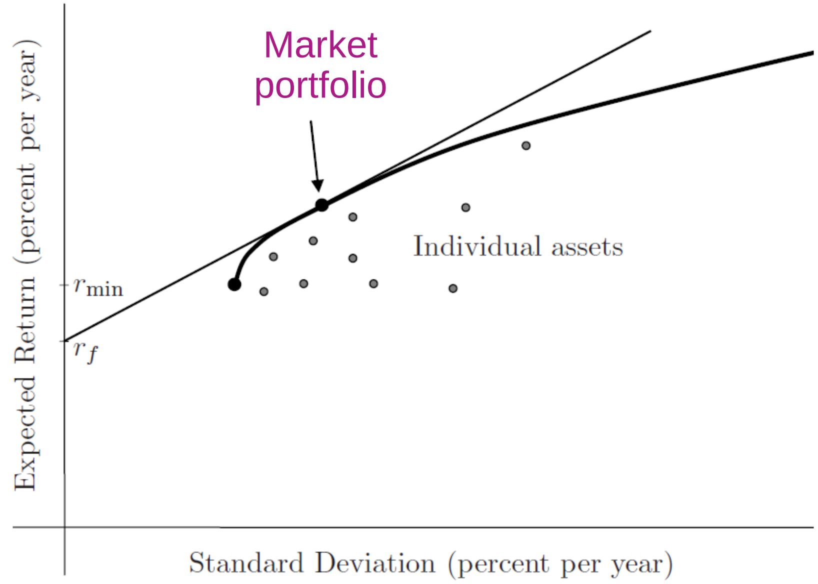
$$\overline{r}_P = r_f + \frac{\overline{r}_M - r_f}{\sigma_M} \cdot \sigma_P$$

$\mathbb{E}[r_M] = \overline{r}_M$ expected market rate of return

σ_M standard deviation of market rate of return

- CML relates the expected rate of return of an efficient portfolio to its standard deviation

- The slope the CML is called the price of risk



Capital Asset Pricing Model (CAPM)

- How does the expected rate of return of an individual asset relate to its individual risk? (*model developed by Sharpe, Lintner and Mossin in the 60's*)
- CAPM relates the random return on the *i*-th investment r_i to the random return on the market r_M by

we can derive the future price by current price times the return on stock

$$r_i - r_f = \beta_i(r_M - r_f) + \epsilon_i$$

assumptions

ϵ_i is a random variable with zero mean and σ_{ϵ_i} standard deviation

ϵ_i is uncorrelated with the market return r_M and $\mathrm{cov}(\epsilon_i, \epsilon_j) = 0$

- Taking the expectation we get the CAMP equation:

$$\mathbb{E}[r_i] - r_f = \beta_i(\mathbb{E}[r_M] - r_f)$$
$$\overline{r}_i - r_f = \beta_i(\overline{r}_M - r_f)$$

σ_M is the standard deviation of market return, $\mathbb{E}[\epsilon_i] = 0$

$\mathrm{cov}(\epsilon_i, r_M) = \mathrm{cov}(r_i, r_M) - \beta_i \cdot \mathrm{var}(r_M) = \sigma_{iM} - \beta_i\sigma_M^2 = 0$

- Beta of an asset (risk premium):

$$\beta_i = \frac{\mathrm{cov}(r_i, r_M)}{\mathrm{var}(r_M)} = \frac{\sigma_{iM}}{\sigma_M^2}$$

Capital Asset Pricing Model (CAPM)

- CAPM describes relationship between risk and expected return of asset
- Expected excess rate of return of an asset is proportional to the expected excess rate of return of the market portfolio – proportional factor is the beta of the asset

$$\overline{r}_i - r_f = \beta_i(\overline{r}_M - r_f)$$

- Beta of an asset:
  - beta of an asset measures the risk of the asset with respect to the market portfolio *M*
  - high beta assets earn higher average return in equilibrium because of $\beta_i(\overline{r}_M - r_f)$
  - beta of market portfolio – average risk of all assets

$$\beta_M = \frac{\mathrm{cov}(r_M, r_M)}{\mathrm{var}(r_M)} = \frac{\sigma_M^2}{\sigma_M^2} = 1$$

- If the betas of the individual assets are known, then the beta of the portfolio is

$$\beta_P = \sum_{i=1}^{n} \beta_i w_i$$

Show it by using $r_P = \sum_{i=1}^{n} r_i w_i$ and $\mathrm{cov}(r_P, r_M) = \sum_{i=1}^{n} w_i \cdot \mathrm{cov}(r_i, r_M)$

Systematic and specific risk

- CAPM divides total risk of holding risky assets into two parts:
  - systematic risk (risk of holding the market portfolio)
  - specific risk

- CAPM relates the random return on the *i*-th investment r_i to the random return on the market r_M by

$$r_i - r_f = \beta_i (r_M - r_f) + \epsilon_i$$

- Total risk of holding risky asset *i* is

$$\underbrace{\sigma_i^2}_{\text{total risk}} = \mathrm{var}[r_i] = \underbrace{\beta_i^2 \cdot \sigma_M^2}_{\text{systematic risk}} + \underbrace{\sigma_{\epsilon_i}^2}_{\text{specific risk}} = \beta_i \cdot \sigma_{iM} + \sigma_{\epsilon_i}^2$$

- For a portfolio:

$$\bar{r}_P = r_f + \beta_P (\bar{r}_M - r_f)$$

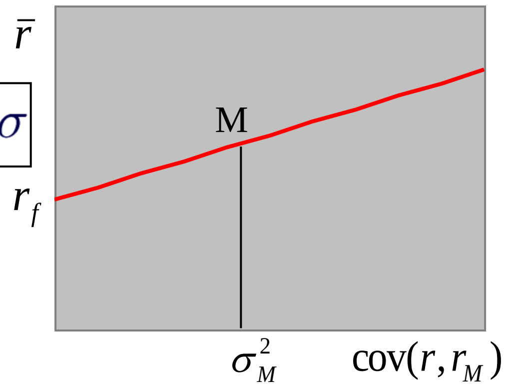
$$\boxed{\sigma_P = |\beta_P| \sigma_M} \qquad \beta_P = \sum_{i=1}^n \beta_i w_i$$

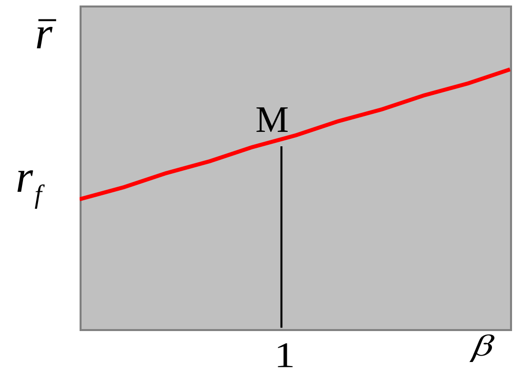
Beta of the market

- Average risk of all assets is 1 (beta of the market portfolio)
- Beta of market portfolio is used as a reference point to measure risk of other assets

Capital Market Line

$$\bar{r} = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma$$





Security Market Line

$$\bar{r}_i = r_f + \beta_i(\bar{r}_M - r_f)$$
$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

- Assets or portfolios with betas greater than 1 are above average risk: tend to move more than market. Example:
  - If risk free rate is 5% per year and market rises by 10%, then assets with a beta of 2 will tend to increase by 15%
  - If market falls by 10%, then assets with a beta of 2 will tend to fall by 25% on average
- Assets or portfolios with betas less than 1 are of below average risk: tend to move less than market

Single-Factor Models and Multi-Factor Models

- Consider n assets with rates of return r_i for i=1,2,…,n and one factor f which is a random quantity such as inflation, interest rate
- Assume that the rates of return and single factor are linearly related

$$r_i = \underbrace{\alpha_i}_{\text{constant}} + \underbrace{\beta_i}_{\text{constant}} \cdot f + \underbrace{\epsilon_i}_{\text{random}} \qquad i = 1, \ldots, n$$

Intercept **Factor loadings** **Error**

- Errors:
  - have zero mean $\mathbb{E}[\epsilon_i] = 0$
  - are uncorrelated with the factor $\mathbb{E}[f\epsilon_i] = 0$
  - are uncorrelated with each other $\mathbb{E}[\epsilon_i \epsilon_j] = 0, \; i \neq j$

- Single-factor model can be extended to have more than one factor (multi-factor):

$$r_i = \alpha_i + \sum_{k=1}^{m} \beta_{ik} \cdot f_k + \epsilon_i$$

Multi-Factor Models and Mean-Variance Analysis

- Multi-factor model: $r_i = \alpha_i + \sum_{k=1}^{m} \beta_{ik} \cdot f_k + \epsilon_i$

- Multi-factor model in vector form:

$$\mathbf{r} = \boldsymbol{\alpha} + \mathbf{B} \cdot \mathbf{f} + \boldsymbol{\epsilon}$$

$\mathbf{r}$ *n*-dimensional vector of returns
$\boldsymbol{\alpha}$ *n*-dimensional vector of mean returns
$\mathbf{f}$ *m*-dimensional vector of factors
$\mathbf{B}$ *m* x *n* matrix of factor loadings
$\boldsymbol{\epsilon}$ *n*-dimensional vector of residual errors

- Expected portfolio return:

Expected excess portfolio return

$$\boldsymbol{\alpha}^T \cdot \boldsymbol{w}$$

Expected portfolio return

$$\boldsymbol{\alpha}^T \cdot \boldsymbol{w} + \mathbf{B} \cdot \mathbf{f} \cdot \boldsymbol{w}$$

- Variance of the portfolio return:

$$\boldsymbol{w}^T \cdot (\mathbf{B}^T \cdot \mathbf{Q_f} \cdot \mathbf{B} + \mathbf{D}) \cdot \boldsymbol{w}$$

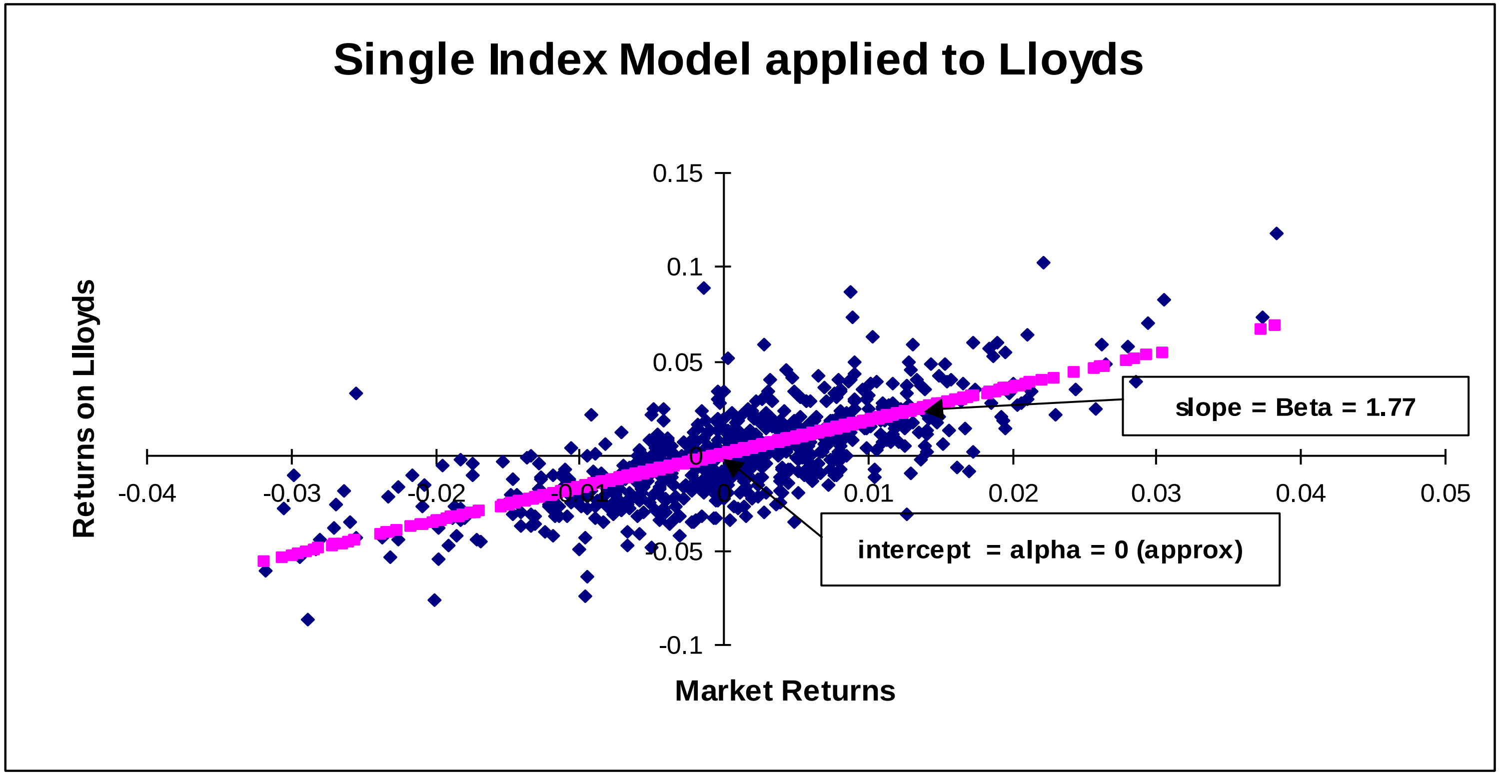
$\mathbf{Q_f}$ factor covariance matrix
$\mathbf{D}$ variance of error terms (diagonal matrix)

How to select factors?

- Factors are external to securities, e.g., economic factors:
  - gross domestic product (GDP)
  - consumer price index (CPI)
  - unemployment rate
  - credit spreads on bonds
- Factors are extracted from known information about security returns:
  - rate of return on the market portfolio (market risk)
  - average of the return of stocks in a particular country and industry (utilities, transportation, aerospace, financial, materials, manufacturing, etc.)
- Firm characteristics:
  - price earning ratio
  - dividend payout ratio
  - earnings growth forecast
- **How to select factors: It is part science and part art**
  - Statistical approach – principal component analysis
  - Fundamental approach – uses real factors, i.e., beta, inflation rate, interest rate, industrial production, etc.

CAPM as a Factor Model – example

- Single factor model equation defines a linear fit to data
  - Imagine several independent observations of the rate of return and factor
  - Straight line, defined by single factor model equation, is fitted through these points such that average value of errors is zero
  - Error is measured by the vertical distance from a point to the line



Source: P. Parpas. Computational Finance

Estimating betas in CAPM model

- Security's beta can be estimated from a set of observed returns for the security and the market return using linear regression:

$$r_{it} - r_{ft} = \alpha_i + \beta_i \cdot (r_{Mt} - r_{ft}) + e_{it}$$

r_{it} observed return on security *i* for time *t*

r_{ft} observed return on the risk-free asset for time *t*

r_{Mt} observed return on the market portfolio for time *t*

e_{it} error term for time *t*

Intercept term α_i should be statistically 0 for this equation to be consistent with CAPM

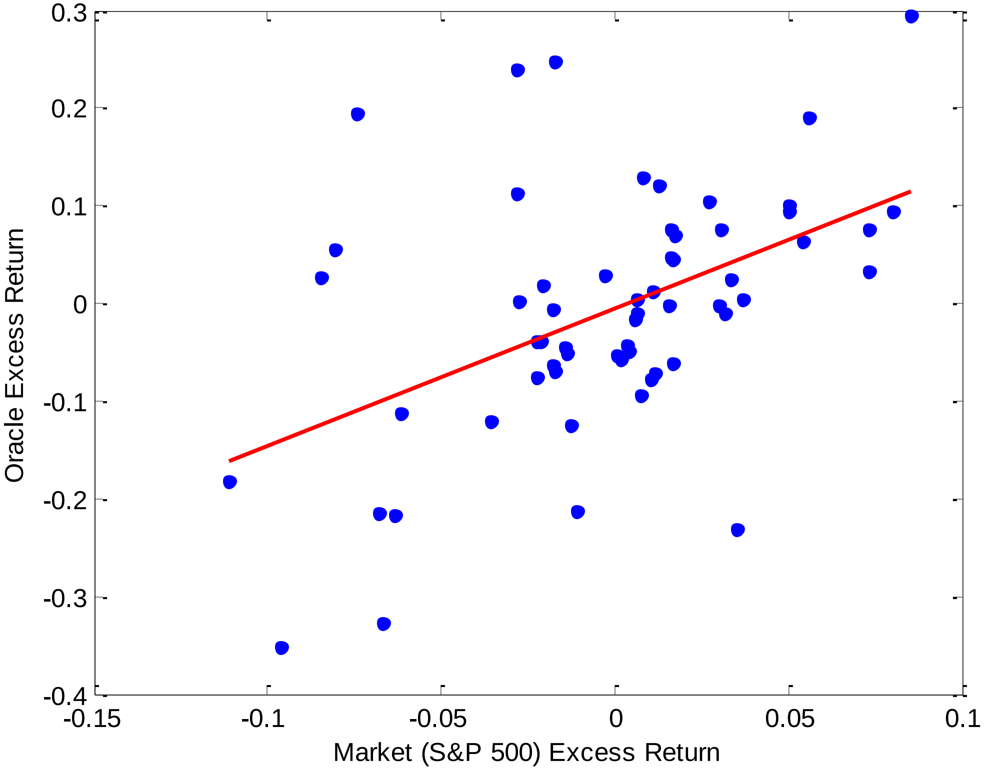
- Sample data:

Date	S&P return	Risk-free rate	S&P - risk-free rate	Oracle return	Oracle excess return
01/12/2000	0.03464	0.00473	0.02990	0.00206	-0.00267
01/01/2001	-0.09229	0.00413	-0.09642	-0.34753	-0.35165
01/02/2001	-0.06420	0.00393	-0.06813	-0.21158	-0.21550
01/03/2001	0.07681	0.00357	0.07325	0.07877	0.07521
01/04/2001	0.00509	0.00321	0.00188	-0.05322	-0.05643

Estimating betas in CAPM model in Matlab

```
SP500ExcessRet = xlsread('Beta.xls', 'Data', 'D4:D63');
OracleExcessRet = xlsread('Beta.xls', 'Data', 'F4:F63');

% Run linear regression
[beta, betaint] = regress(OracleExcessRet, [ones(length(OracleExcessRet),1) SP500ExcessRet])

% Run linear regression with CPLEX
beta_cpx = cplexlsqlin([ones(length(OracleExcessRet),1) SP500ExcessRet], OracleExcessRet, [], [])

plot(SP500ExcessRet, OracleExcessRet, 'b.', 'MarkerSize', 15); hold on;
line([min(SP500ExcessRet) max(SP500ExcessRet)], ...
     [beta(1)+beta(2)*min(SP500ExcessRet) beta(1)+beta(2)*max(SP500ExcessRet)]);
xlabel('Market (S&P 500) Excess Return')
ylabel('Oracle Excess Return')
```

```
beta =

  -0.00506278
   1.40556642

beta_cpx =

  -0.00506278
   1.40556642

betaint =

  -0.03417932  0.02405376
   0.72623617  2.08489667
```



Arbitrage Pricing Theory (APT)

- CAPM is criticised for two assumptions:
  - investors are mean-variance optimizers
  - model is single-period
- Stephen Ross ("The Arbitrage Pricing Theory of Capital Asset Pricing", Journal of Economic Theory, 1976) developed an alternative model based on arbitrage arguments
- APT vs. CAPM:
  - APT is a more general approach to asset pricing than CAPM
  - CAPM considers variances and covariance's as possible measures of risk while APT allows for a number of risk factors
  - APT postulates that a security's expected return is influenced by a variety of factors, as opposed to just the single market index of CAPM
  - APT does not specify what factors are, but assumes that the relationship between security returns and factors is linear

$$r_i = \mathbb{E}[r_i] + \beta_{i1} \cdot f_1 + \ldots + \beta_{iK} \cdot f_K$$

Robust portfolio selection

Mitigate return estimation errors

- Risk parity, equal risk contributions
- Robust mean-variance optimization

Factor models

- Capital Asset Pricing Model (CAPM)
- Arbitrage Pricing Theory (APT)

Next lecture:

- Introduction to simulations modelling
- Introduction to risk management