Thesis outline, errata, and comments*

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1 Thesis outline

(The page `wangyangvictor.github.io/thesis_links.html` has links to all relevant papers and drafts. Questions, comments, corrections, and suggestions are all welcome.)

For $n, X \in \mathbb{Z}_{\geq 0}$, let $r_3(n) := \#\{x, y, z \in \mathbb{Z}_{\geq 0} : x^3 + y^3 + z^3 = n\}$ and $M_2(X) := \sum_{a \leq X^3} r_3(a)^2$. Conditionally on Langlands-type hypotheses and GRH (for certain Hasse–Weil L-functions), Hooley (1997) and Heath-Brown (1998) proved $M_2(X) \ll_\epsilon X^{3+\epsilon}$. Furthermore, Hooley (1986) conjectured $M_2(X) \sim c_{\mathrm{HLH}} X^3$ (as $X \to \infty$) for a specific constant $c_{\mathrm{HLH}} \in \mathbb{R}_{>0}$, which is *strictly greater* than the Hardy–Littlewood constant $c_{\mathrm{HL}} \in \mathbb{R}_{>0}$.

My thesis consists of three parts:

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*The link `tinyurl.com/hooley33` will always point to the latest version of this document.

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1. Paper I: *Diagonal cubic forms and the large sieve* (42 pages).

   This shows that Hooley's (and Heath-Brown's) hypotheses can be replaced with a large sieve hypothesis a la Bombieri–Vinogradov.

2. Paper II: *Isolating special solutions in the delta method: The case of a diagonal cubic equation in evenly many variables over* $\mathbb{Q}$ (34 pages).

   Heath-Brown's work, and morally also Hooley's work, is based on the "delta method" for $M_2(X)$. One can easily "extract" $c_{\mathrm{HL}}X^3$ from the delta method. Paper II extracts $(c_{\mathrm{HLH}} - c_{\mathrm{HL}})X^3$ in a natural way.

3. Paper III: *Approaching cubic Diophantine statistics via mean-value L-function conjectures of Random Matrix Theory type* (136 pages).

   Building on Paper II, we prove (i) a general localized form of Hooley's conjecture and (ii) that asymptotically 100% of integers $a \not\equiv \pm 4 \bmod 9$ are sums of three cubes, conditionally on some standard number theory conjectures—the main additions (relative to Hooley and Heath-Brown) being conjectures of Random Matrix Theory and Square-free Sieve type. To reduce (i) to these conjectures, we introduce several new *unconditional* ingredients. For example, certain complete exponential sums "fail square-root cancellation" quite badly—and thus do not fall under "standard" conjectural frameworks—and we prove new results that help to control such behavior.

(Thanks to Nick Katz for helpful suggestions on wording.)

2 Papers I–III: Errata and comments

For now, all errata and comments refer to the versions of August 7, 2021:

1. `arXiv:2108.03395v1` for I,
2. `arXiv:2108.03396v1` for II, and
3. `arXiv:2108.03398v1` for III.

Thanks below are given in parentheses.

2.1 Paper I (`arXiv:2108.03395v1`)

- Paragraph after Definition I.1.3: It would be good to state Vaughan's record, of the form $N_F(X) \ll X^{7/2}/(\log X)^c$. (Wooley)

- Paragraph before Definition I.1.11: This is OK, but it would be more canonical to say "if $\boldsymbol{c} \in \mathbb{Z}^m$ and $p \nmid F^\vee(\boldsymbol{c})$, then $(\mathcal{V}_{\boldsymbol{c}})_{\mathbb{F}_p}$ is a smooth complete intersection in $\mathbb{P}^{m-1}_{\mathbb{F}_p}$ of dimension m_* and multi-degree $(3,1)$".

- Definition I.1.11: Replace "$\dim H^{m_*}_{\mathrm{prim}}(V_{\boldsymbol{c}} \times \mathbb{C})$" (which should have been "$\operatorname{rank} H^{m_*}_{\mathrm{prim}}(V_{\boldsymbol{c}} \times \mathbb{C})$") with "$\operatorname{rank}(H^{m_*}_{\mathrm{sing}}(V_{\boldsymbol{c}}(\mathbb{C}), \mathbb{Z})/H^{m_*}_{\mathrm{sing}}(\mathbb{P}^{m-1}(\mathbb{C}), \mathbb{Z}))$".

  Then delete "where $H^{m_*}_{\mathrm{prim}}$... Betti... (*primitive*...)".

- After Definition I.1.11: Add a remark that (i) "for each $\boldsymbol{c}$ above, $V_{\boldsymbol{c}}$ is a *subvariety* of $\mathbb{P}^{m-1}_{\mathbb{Q}}$ by definition, so $H^{m_*}_{\mathrm{sing}}(V_{\boldsymbol{c}}(\mathbb{C}), \mathbb{Z})/H^{m_*}_{\mathrm{sing}}(\mathbb{P}^{m-1}(\mathbb{C}), \mathbb{Z})$ is well-defined"; and (ii) "each $L_p(s, V_{\boldsymbol{c}})$ above is well-defined: for any $\boldsymbol{c}, \boldsymbol{c}' \in \mathbb{Z}^m$ with $V_{\boldsymbol{c}} = V_{\boldsymbol{c}'}$, one can show that $E_{\boldsymbol{c}}(q) = E_{\boldsymbol{c}'}(q)$ holds for all prime powers q coprime to $F^\vee(\boldsymbol{c})F^\vee(\boldsymbol{c}')$".

  (To avoid discussing (i)–(ii), we could write $-\mathbf{1}_{2|m_*} + \operatorname{rank} H^{m_*}_{\mathrm{sing}}(V_{\boldsymbol{c}}(\mathbb{C}), \mathbb{Z})$ in place of $\operatorname{rank}(\cdots/\cdots)$, and $L_p(s, \boldsymbol{c})$ in place of $L_p(s, V_{\boldsymbol{c}})$. But the current notation is more transparent and suggestive.)

2.2 Paper II (`arXiv:2108.03396v1`)

- Remark II.1.20 and §II.5.2: Some (purely expository) comments are missing obvious hypotheses. In 1.20, $I_{\boldsymbol{c}}(n)$ is only "morally positive" if $w \geq 0$. In §5.2, some of the comments only apply if $\sigma_{\infty, L^\perp, w} \neq 0$ (and in particular, $L \cap (\operatorname{Supp} w) \neq \emptyset$).

- §II.5.2: Some of the n's should be q's.

2.3 Paper III (`arXiv:2108.03398v1`)

- Paragraph after Remark III.1.9: Remove "essentially". (Sarnak)

- Definition III.3.8: Replace "Also let" with "And for each prime p, let". Write "diff" instead of "prim", to avoid conflict with the usual definition of $H^{\bullet}_{\mathrm{prim}}$. Also (for convenience), generalize "$H^d(\mathbb{P}^{1+d})$" and "*hypersurface* $W/\mathbb{Q}$ of dimension $d \geq 0$" to "$H^d(\mathbb{P}^{r+d})$" and "*complete intersection* $W/\mathbb{Q}$ of dimension $d \geq 0$ and codimension $r \geq 1$".

Then modify the following accordingly: Remark 3.10, Definition 3.11, Conjecture 3.18, Observation 3.21, paragraph after Observation 4.2, point (2) on p. 73 (before Remark A.4), and Remark A.11(2).

- Definition III.3.11: Add "And for all p, j, let $\tilde{\alpha}_{\boldsymbol{c},j}(p) := \tilde{\alpha}_{M,j}(p)$, where $M := H^{m_*}_{\mathrm{prim}}(V_{\boldsymbol{c}})$".

- §III.4.1: Replace "Definition 3.8" (both times) with "Definition 3.11".

- §III.7.5, statement of 7.25 (RA1') and derivation of 7.26 (RA1'E): Replace "$n_0 \leq Z^{\hbar}$" (both times) with "$n_0 \leq Z^{\hbar^2}$", and "$n_0 \geq Z^{\hbar}$" with "$n_0 \geq Z^{\hbar^2}$". Then in the derivation of (RA1'), explicitly choose $\hbar \lesssim 1$ sufficiently small so that in item (2) in the second paragraph, we have $n_0^{O(1)} \leq Z^{\hbar/2}$.

  (These changes are important for §10, but not for §9.)

  One could also remark that *morally*, when applying (RA1) here (towards (RA1')) and elsewhere (via (RA1'), (RA1'E), and (RA1'E')), it suffices to work with boxes $\mathcal{B}(\boldsymbol{Z})$ of *intermediate* "lopsidedness" (say $\leq Z/Z^{1-\hbar} = Z^{\hbar}$), and with residue classes $\boldsymbol{a} \bmod n_0$ of *small* modulus (say $\leq Z^{\hbar^2}$).

- Definition III.C.2: Add "projective" before "variety Y/k", so that Remark C.3 holds as written. (Katz)

  Also, the current notion of "error-relevant" is extrinsic (Katz), would be better termed "non-planar", and is not symmetric enough for us (given that $H^i(\mathbb{P}^n_k) \to H^i(X)$, for $i \leq 2\dim X$, could presumably fail to be injective for some embedded projective variety $X \subseteq \mathbb{P}^n_k$).

  To resolve these issues robustly and cleanly, *append* ", and for each $i \geq 0$, let $\mathcal{E}^i(Y)$ denote the multiset of (geometric) Frobenius eigenvalues on $H^i(Y)$" to the *second* sentence of C.2, and *replace* the *third* with "Now fix a projective variety X/k, let $\mathcal{E}^i_{\triangle}(X, \mathbb{P}) := (\mathcal{E}^i(X) \cup \mathcal{E}^i(\mathbb{P}^{\dim X}_k)) \setminus (\mathcal{E}^i(X) \cap \mathcal{E}^i(\mathbb{P}^{\dim X}_k))$ for $i \geq 0$ (so that if $\alpha \in \overline{\mathbb{Q}}_\ell$ has multiplicities j, l in $\mathcal{E}^i(X), \mathcal{E}^i(\mathbb{P}^{\dim X}_k)$, then it has multiplicity $|j - l|$ in $\mathcal{E}^i_{\triangle}(X, \mathbb{P})$), and let $\mathcal{E}_{\triangle}(X, \mathbb{P}) := \bigsqcup_{i \geq 0} \mathcal{E}^i_{\triangle}(X, \mathbb{P})$."

  Then right after C.3, add the following theorem (and proof sketch). Below, we will refer to the theorem as "Theorem P".

Theorem (Deligne, Katz, Skorobogatov, and Ghorpade–Lachaud)**.** *Let $k := \mathbb{F}_q$. Fix integers $n, N \geq 1$, and a complete intersection $X \subseteq \mathbb{P}^n_k$ with $\dim X = N$ and $\operatorname{codim} X \geq 1$. Let $D := \dim(\operatorname{Sing}(X_{\overline{k}}))$, with the convention $\dim(\emptyset) := -1$. Then for $i \in \mathbb{Z}$, the following hold.*

1. *If $i \geq N + D + 2$, then $\mathcal{E}^i_{\triangle}(X, \mathbb{P}) = \emptyset$.*
2. *If $i = N + D + 1$ and $2 \mid i$, then $\{q^{i/2}\} = \mathcal{E}^i(\mathbb{P}^N_k) \subseteq \mathcal{E}^i(X)$.*

Proof sketch. Claim (1) follows from [Hoo91b, Katz's Appendix, assertion (2) in the proof of Theorem 1]. And if $D = -1$, then (2) follows from weak Lefschetz. Now assume $D \geq 0$, let $i := N + D + 1$, and suppose $2 \mid i$. Then $i \leq 2N$ by generic smoothness, so $\mathcal{E}^i(\mathbb{P}^N_k) = \{q^{i/2}\}$. It remains to show that $q^{i/2} \in \mathcal{E}^i(X)$; [`arXiv:0808.2169v1`, first sentence of Remark 3.5] essentially states this without proof,[1] so it seems appropriate to sketch one. Let $T := \mathbb{A}^1_k$. Following Katz (essentially), we can reduce to the case in which there exists a closed subscheme $Z \subseteq \mathbb{P}^n_T$, flat over T, such that (i) $Z_0 = X$ and (ii) $Y := Z_1$ is a smooth complete intersection in $\mathbb{P}^n_k$ with $\dim Y = N$. In this case, [SGA 7 I, Deligne's Exposé I, Corollaire 4.3] implies that the specialization map $H^i(Z_t) \to H^i(Z \times_T \overline{k(T)}, \mathbb{Q}_\ell)$ is an isomorphism at $t = 1$ (since $D \geq 0$), and a surjection at $t = 0$. By G_k-equivariance, it follows that $\mathcal{E}^i(Y) \subseteq \mathcal{E}^i(X)$. But $i \geq N + 1$ (since $D \geq 0$), so $\mathcal{E}^i(Y) = \mathcal{E}^i(\mathbb{P}^N_k)$ (by (1) for Y). Thus $\{q^{i/2}\} = \mathcal{E}^i(\mathbb{P}^N_k) \subseteq \mathcal{E}^i(X)$, as desired. □

Then do the following:

- In C.4, replace "all of the error-relevant Frobenius eigenvalues on $H^\bullet(X)$" with "all $\alpha \in \mathcal{E}_\triangle(X, \mathbb{P})$".
- In C.4 and C.6, generalize "projective hypersurface $X \subseteq \mathbb{P}^n_{\mathbb{F}_q}$" to "projective complete intersection $X \subseteq \mathbb{P}^n_{\mathbb{F}_q}$ with $\operatorname{codim} X \geq 1$".
- In C.6: Convert "$\dim(\operatorname{Sing}(X_{\overline{k}})) = 0$" to "$\dim(\operatorname{Sing}(X_{\overline{k}})) \leq 0$". (Then make corresponding changes elsewhere.)

[1] though (1), [Poo17, Corollary 7.5.21], and [`arXiv:0808.2169v1`, Theorem 2.4 or Skorobogatov (1992), after using a Veronese embedding] might allow for a proof by induction on $\operatorname{codim} X$, which the authors may have had in mind

- In C.6: To be safe, replace the "$18(3+\deg X)^{n+1}$" in (1) with "$18(3+\operatorname{codim} X \deg X)^{n+1} 2^{\operatorname{codim} X}$".

  (Then in Lemma III.4.1, proof of (1), replace "$18(3+3)^{m-1}$" with "$72(3+6)^{m}$"; in Problem 4.16, "$18(3+k)^{s}$" with "$72(3+2k)^{s}$"; in 4.16 and 4.19, "$M_{d,m-1}$" with "$M_{d,m}$".)

- In C.6 and its proof: Replace "$\mathbb{P}^{n-1}(k_r)$" (all three times) with "$\mathbb{P}^{\dim X}(k_r)$".

- In C.6, proof that (2) implies (1): Replace "$(n, 1, \deg X)$" with "$(n, \operatorname{codim} X, \|\boldsymbol{d}\|_\infty)$, if X has multi-degree $\boldsymbol{d} \in \mathbb{Z}_{\geq 1}^{\operatorname{codim} X}$; here $\|\boldsymbol{d}\|_\infty \leq d_1 \cdots d_{\operatorname{codim} X} = \deg X$".

  And explicitly state the LTF and Betti bounds involved.

- In C.6, proof of equivalence of (2)–(3): Replace "the only error-relevant eigenvalues can come from… [through footnote 40]" with "the multiset $\{\alpha \in \mathcal{E}_\triangle(X, \mathbb{P}) : \text{weight}(\alpha) \geq 1 + \dim X\}$ is a sub-multiset of $\mathcal{E}^{1+\dim X}(X) \setminus \mathcal{E}^{1+\dim X}(\mathbb{P}_k^{\dim X})$ (by Theorem P)". (Katz)

  And replace "If $\dim X = 1 \ldots \dim X \geq 2 \ldots$ *with* $\dim(\operatorname{Sing}(X_{\overline{k}})) = 0$, so" with "Now assume $\dim X \geq 1$. Then the hypothesis $\dim(\operatorname{Sing}(X_{\overline{k}})) \leq 0$ implies that".

- At the end of C.6: Add the sentence "Furthermore, (1)–(3) hold *if* $\dim H^{1+\dim X}(X) = \dim H^{1+\dim X}(\mathbb{P}_k^{\dim X})$." For proof, use Theorem P.

  Then in Appendix C.1.1, proof of Proposition C.9(2): Replace "$H^i(V_{\boldsymbol{c}})/H^i(\mathbb{P}_k^{m-1}) \neq 0$" (both times) with "$\dim H^{1+m_*}(V_{\boldsymbol{c}}) \neq \dim H^{1+m_*}(\mathbb{P}_k^{m_*})$", and justify this (the first time) using the new "final sentence" of C.6.

  And similarly, in Appendix C.1.2, proof of Proposition C.9(1): Make similar changes, appealing to [Poo17, Corollary 7.5.21] and the new "final sentence" of C.6.

- In C.7: Replace "after viewing $V_{\boldsymbol{c}}$ as a projective hypersurface…" with "since $V_{\boldsymbol{c}}$ is a complete intersection in $\mathbb{P}_k^{m-1}$".

• Sentence before Appendix III.C.3.1: Replace "as shown in the proof of Observation C.6" with "by Theorem P(1) and [Poo17, Corollary 7.5.21]".

• Proposition III.C.13, proof of second part (giving an alternative approach to Proposition C.9(2) when $m = 6$): The H^2's should be H^4's, and we should consider all eigenvalues on $H^4(X)$ to be safe (in case $H^4(\mathbb{P}^4_k) \to H^4(X)$ fails to be injective).

Then one should replace each of the three expressions $\tilde{\alpha}_1^? + \cdots + \tilde{\alpha}_b^?$ with $\tilde{\alpha}_1^? + \cdots + \tilde{\alpha}_b^? - 1$. The Dirichlet argument now gives $b = 1$. One then needs to prove that $\tilde{\alpha}_1 = 1$; this can be done by appealing to Theorem P(2). (Alternatively, one might try using the $G_{\mathbb{Q}}$-invariance of the multiset $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_b\}$ to obtain $\tilde{\alpha}_1 \in \mathbb{Q}$, i.e. $\tilde{\alpha}_1^2 = 1$. But without P(2), it seems hard to rule out the possibility that $\tilde{\alpha}_1 = -1$; taking $r \equiv 1 \bmod 2$ large does not seem to help here.)

Some of the references to C.6 should also be replaced by appropriate references to Theorem P(1).