Thesis outline, errata, and comments*

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1 Thesis outline

(The page `wangyangvictor.github.io/thesis_links.html` has links to all relevant papers and drafts. Questions, comments, corrections, and suggestions are all welcome.)

For $n, X \in \mathbb{Z}_{\geq 0}$, let $r_3(n) := \#\{x, y, z \in \mathbb{Z}_{\geq 0} : x^3 + y^3 + z^3 = n\}$ and $M_2(X) := \sum_{a \leq X^3} r_3(a)^2$. Conditionally on Langlands-type hypotheses and GRH (for certain Hasse–Weil L-functions), Hooley (1997) and Heath-Brown (1998) proved $M_2(X) \ll_\epsilon X^{3+\epsilon}$. Furthermore, Hooley (1986) conjectured $M_2(X) \sim c_{\mathrm{HLH}} X^3$ (as $X \to \infty$) for a specific constant $c_{\mathrm{HLH}} \in \mathbb{R}_{>0}$, which is *strictly greater* than the Hardy–Littlewood constant $c_{\mathrm{HL}} \in \mathbb{R}_{>0}$.

My thesis consists of three parts:

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*The link `tinyurl.com/hooley33` will always point to the latest version of this document.

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1. Paper I: *Diagonal cubic forms and the large sieve* (42 pages).

   This shows that Hooley's (and Heath-Brown's) hypotheses can be replaced with a large sieve hypothesis a la Bombieri–Vinogradov.

2. Paper II: *Isolating special solutions in the delta method: The case of a diagonal cubic equation in evenly many variables over* $\mathbb{Q}$ (34 pages).

   Heath-Brown's work, and morally also Hooley's work, is based on the "delta method" for $M_2(X)$. One can easily "extract" $c_{\mathrm{HL}}X^3$ from the delta method. Paper II extracts $(c_{\mathrm{HLH}} - c_{\mathrm{HL}})X^3$ in a natural way.

3. Paper III: *Approaching cubic Diophantine statistics via mean-value L-function conjectures of Random Matrix Theory type* (136 pages).

   Building on Paper II, we prove (i) a general localized form of Hooley's conjecture and (ii) that asymptotically 100% of integers $a \not\equiv \pm 4 \bmod 9$ are sums of three cubes, conditionally on some standard number theory conjectures—the main additions (relative to Hooley and Heath-Brown) being conjectures of Random Matrix Theory and Square-free Sieve type. To reduce (i) to these conjectures, we introduce several new *unconditional* ingredients. For example, certain complete exponential sums "fail square-root cancellation" quite badly—and thus do not fall under "standard" conjectural frameworks—and we prove new results that help to control such behavior.

(Thanks to Nick Katz for helpful suggestions on wording.)

2 Papers I–III: Errata and comments

For now, all errata and comments refer to the versions of August 7, 2021:

1. `arXiv:2108.03395v1` for I,

2. `arXiv:2108.03396v1` for II, and

3. `arXiv:2108.03398v1` for III.

Thanks below are given in parentheses.

2.1 Paper I (`arXiv:2108.03395v1`)

- Paragraph before Definition I.1.11: This is OK, but it would be more canonical to say "if $\boldsymbol{c} \in \mathbb{Z}^m$ and $p \nmid F^\vee(\boldsymbol{c})$, then $(\mathcal{V}_{\boldsymbol{c}})_{\mathbb{F}_p}$ is a smooth complete intersection in $\mathbb{P}^{m-1}_{\mathbb{F}_p}$ of dimension m_* and multi-degree $(3,1)$".

- Definition I.1.11: "$\dim H^{m_*}_{\mathrm{prim}}(V_{\boldsymbol{c}} \times \mathbb{C})$" should be "$\operatorname{rank} H^{m_*}_{\mathrm{prim}}(V_{\boldsymbol{c}} \times \mathbb{C})$".

- After Definition I.1.11: Add a remark that (i) "for each $\boldsymbol{c}$ above, $V_{\boldsymbol{c}}$ is a *subvariety* of $\mathbb{P}^{m-1}_{\mathbb{Q}}$ by definition, so $H^{m_*}_{\mathrm{prim}}(V_{\boldsymbol{c}} \times \mathbb{C})$ is well-defined"; and (ii) "each $L_p(s, V_{\boldsymbol{c}})$ above is well-defined: for any $\boldsymbol{c}, \boldsymbol{c}' \in \mathbb{Z}^m$ with $V_{\boldsymbol{c}} = V_{\boldsymbol{c}'}$, one can show that $E_{\boldsymbol{c}}(q) = E_{\boldsymbol{c}'}(q)$ holds for all prime powers q coprime to $F^\vee(\boldsymbol{c})F^\vee(\boldsymbol{c}')$".

  (To avoid discussing (i)–(ii), we could write $-\mathbf{1}_{2|m_*} + \operatorname{rank} H^{m_*}(V_{\boldsymbol{c}} \times \mathbb{C})$ in place of $\operatorname{rank} H^{m_*}_{\mathrm{prim}}(V_{\boldsymbol{c}} \times \mathbb{C})$, and $L_p(s, \boldsymbol{c})$ in place of $L_p(s, V_{\boldsymbol{c}})$. But the current notation is more transparent and suggestive.)

2.2 Paper III (`arXiv:2108.03398v1`)

- Paragraph after Remark III.1.9: Remove "essentially". (Sarnak)

- Definition III.3.8: Replace "Also let" with "And for each prime p, let". Write "diff" instead of "prim", to avoid conflict with the usual definition of $H^\bullet_{\mathrm{prim}}$. Also (for convenience), generalize "$H^d(\mathbb{P}^{1+d})$" and "*hypersurface* $W/\mathbb{Q}$ of dimension $d \ge 0$" to "$H^d(\mathbb{P}^{r+d})$" and "*complete intersection* $W/\mathbb{Q}$ of dimension $d \ge 0$ and codimension $r \ge 1$".

  Then modify the following accordingly:

  - Remark 3.10, Definition 3.11, Conjecture 3.18, Observation 3.21, paragraph after Observation 4.2, point (2) on p. 73 (before Remark A.4), and Remark A.11(2).

- Definition III.3.11: Add "And for all p, j, let $\tilde{\alpha}_{\boldsymbol{c},j}(p) := \tilde{\alpha}_{M,j}(p)$, where $M := H^{m_*}_{\mathrm{prim}}(V_{\boldsymbol{c}})$".

- §III.4.1: Replace "Definition 3.8" (both times) with "Definition 3.11".

- Definition III.C.2: Add "projective" before "variety Y/k", so that Remark C.3 holds as written. (Katz)

Also, the current notion of "error-relevant" is extrinsic (Katz), would be better termed "non-planar", and is not symmetric enough for us (given that $H^i(\mathbb{P}^n_k) \to H^i(X)$, for $i \le 2\dim X$, could presumably fail to be injective for some embedded projective variety $X \subseteq \mathbb{P}^n_k$).

To resolve these issues robustly and cleanly, *append* ", and for each $i \ge 0$, let $\mathcal{E}^i(Y)$ denote the multiset of (geometric) Frobenius eigenvalues on $H^i(Y)$" to the *second* sentence of C.2, and *replace* the *third* with "Now fix a projective variety X/k, let $\mathcal{E}^i_\triangle(X,\mathbb{P}) := (\mathcal{E}^i(X) \cup \mathcal{E}^i(\mathbb{P}^{\dim X})) \setminus (\mathcal{E}^i(X) \cap \mathcal{E}^i(\mathbb{P}^{\dim X}))$ for $i \ge 0$ (so that if $\alpha \in \overline{\mathbb{Q}}_\ell$ has multiplicities j, l in $\mathcal{E}^i(X), \mathcal{E}^i(\mathbb{P}^{\dim X})$, then it has multiplicity $|j-l|$ in $\mathcal{E}^i_\triangle(X,\mathbb{P})$), and let $\mathcal{E}_\triangle(X,\mathbb{P}) := \bigsqcup_{i\ge 0} \mathcal{E}^i_\triangle(X,\mathbb{P})$." Then do the following:

- In C.4, replace "all of the error-relevant Frobenius eigenvalues on $H^\bullet(X)$" with "all $\alpha \in \mathcal{E}_\triangle(X,\mathbb{P})$".
- In C.4 and C.6, generalize "projective hypersurface $X \subseteq \mathbb{P}^n_{\mathbb{F}_q}$" to "projective complete intersection $X \subseteq \mathbb{P}^n_{\mathbb{F}_q}$ with $\operatorname{codim} X \ge 1$".
- In C.6: To be safe, replace the "$18(3+\deg X)^{n+1}$" in (1) with "$18(3+\operatorname{codim} X \deg X)^{n+1} 2^{\operatorname{codim} X}$".

  (Then in Lemma III.4.1, proof of (1), replace "$18(3+3)^{m-1}$" with "$72(3+6)^m$"; in Problem 4.16, "$18(3+k)^s$" with "$72(3+2k)^s$"; in 4.16 and 4.19, "$M_{d,m-1}$" with "$M_{d,m}$".)
- In C.6 and its proof: Replace "$\mathbb{P}^{n-1}(k_r)$" (all three times) with "$\mathbb{P}^{\dim X}(k_r)$".
- In C.6, proof that (2) implies (1): Replace "$(n, 1, \deg X)$" with "$(n, \operatorname{codim} X, \|\boldsymbol{d}\|_\infty)$, if X has multi-degree $\boldsymbol{d} \in \mathbb{Z}^{\operatorname{codim} X}_{\ge 1}$; here $\|\boldsymbol{d}\|_\infty \le d_1 \cdots d_{\operatorname{codim} X} = \deg X$".

  And explicitly state the LTF and Betti bounds involved.
- In C.6, proof of equivalence of (2)–(3): Replace "the only error-relevant eigenvalues can come from… [through footnote 40]" with "$\mathcal{E}^i_\triangle(X,\mathbb{P}) = \emptyset$ for all $i \ge 2 + \dim X$ [Hoo91b, Katz's Appendix, assertion (2) in the proof of Theorem 1]". (Katz)

  And replace "If $\dim X = 1$… $\dim X \ge 2$… *with* $\dim(\operatorname{Sing}(X_{\overline{k}})) = 0$, so" with "Now assume $\dim X \ge 1$. Then the hypothesis $\dim(\operatorname{Sing}(X_{\overline{k}})) = 0$ implies that".
- In C.7: Replace "after viewing $V_{\boldsymbol{c}}$ as a projective hypersurface…" with "since $V_{\boldsymbol{c}}$ is a complete intersection in $\mathbb{P}^{m-1}_k$".