

# Thesis outline

Victor Wang

For  $n, X \in \mathbb{Z}_{\geq 0}$ , let  $r_3(n) := \#\{x, y, z \in \mathbb{Z}_{\geq 0} : x^3 + y^3 + z^3 = n\}$  and  $M_2(X) := \sum_{a \leq X^3} r_3(a)^2$ . Conditionally on Langlands-type hypotheses and GRH (for certain Hasse–Weil  $L$ -functions), Hooley (1997) and Heath-Brown (1998) proved  $M_2(X) \ll_{\epsilon} X^{3+\epsilon}$ . Furthermore, Hooley (1986) conjectured  $M_2(X) \sim c_{\text{HLH}} X^3$  (as  $X \rightarrow \infty$ ) for a specific constant  $c_{\text{HLH}} \in \mathbb{R}_{>0}$ , which is *strictly greater* than the Hardy–Littlewood constant  $c_{\text{HL}} \in \mathbb{R}_{>0}$ .

My thesis consists of three parts:

1. Paper I: *Diagonal cubic forms and the large sieve* (42 pages).

This shows that Hooley’s (and Heath-Brown’s) hypotheses can be replaced with a large sieve hypothesis a la Bombieri–Vinogradov.

2. Paper II: *Isolating special solutions in the delta method: The case of a diagonal cubic equation in evenly many variables over  $\mathbb{Q}$*  (34 pages).

Heath-Brown’s work, and morally also Hooley’s work, is based on the “delta method” for  $M_2(X)$ . One can easily “extract”  $c_{\text{HL}} X^3$  from the delta method. Paper II extracts  $(c_{\text{HLH}} - c_{\text{HL}}) X^3$  in a natural way.

3. Paper III: *Approaching cubic Diophantine statistics via mean-value  $L$ -function conjectures of Random Matrix Theory type* (136 pages).

Building on Paper II, we prove (i) a general localized form of Hooley’s conjecture and (ii) that 100% of integers  $a \not\equiv \pm 4 \pmod{9}$  are sums of three cubes, conditionally on certain standard conjectures—the main additions (relative to Hooley and Heath-Brown) being conjectures of Random Matrix Theory and Square-free Sieve type. To reduce Hooley’s conjecture to standard conjectures, we introduce several new *unconditional* ingredients. For example, certain complete exponential sums “fail square-root cancellation” quite badly—and thus do not fall under standard conjectural frameworks—and we prove new results that help to “control” such behavior.