Thesis outline

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For $n, X \in \mathbb{Z}_{\geq 0}$, let $r_3(n) := \#\{x, y, z \in \mathbb{Z}_{\geq 0} : x^3 + y^3 + z^3 = n\}$ and $M_2(X) := \sum_{a \leq X^3} r_3(a)^2$. Conditionally on Langlands-type hypotheses and GRH (for certain Hasse–Weil L-functions), Hooley (1997) and Heath-Brown (1998) proved $M_2(X) \ll_\epsilon X^{3+\epsilon}$. Furthermore, Hooley (1986) conjectured $M_2(X) \sim c_{\mathrm{HLH}} X^3$ (as $X \to \infty$) for a specific constant $c_{\mathrm{HLH}} \in \mathbb{R}_{>0}$, which is *strictly greater* than the Hardy–Littlewood constant $c_{\mathrm{HL}} \in \mathbb{R}_{>0}$.

My thesis consists of three parts:

1. Paper I: *Diagonal cubic forms and the large sieve* (42 pages).

   This shows that Hooley's (and Heath-Brown's) hypotheses can be replaced with a large sieve hypothesis a la Bombieri–Vinogradov.

2. Paper II: *Isolating special solutions in the delta method: The case of a diagonal cubic equation in evenly many variables over* $\mathbb{Q}$ (34 pages).

   Heath-Brown's work, and morally also Hooley's work, is based on the "delta method" for $M_2(X)$. One can easily "extract" $c_{\mathrm{HL}} X^3$ from the delta method. Paper II extracts $(c_{\mathrm{HLH}} - c_{\mathrm{HL}}) X^3$ in a natural way.

3. Paper III: *Approaching cubic Diophantine statistics via mean-value L-function conjectures of Random Matrix Theory type* (136 pages).

   Building on Paper II, we prove (i) a general localized form of Hooley's conjecture, and (ii) that 100% of integers are sums of three cubes, all conditionally on certain standard conjectures—the main additions (relative to Hooley and Heath-Brown) being conjectures of Random Matrix Theory and Square-free Sieve type. To reduce Hooley's conjecture to standard conjectures, we introduce several new *unconditional* ingredients. For example, certain complete exponential sums "fail square-root cancellation" quite badly—and thus do not fall under standard conjectural frameworks—and we prove new results that help to "control" such behavior.