R M 1

1/ $\partial_t u = \Delta_p u^q$

$u=(x,t)$

$x\in R^n, t>0$

$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ $(p=2,\ \Delta_2 u=\Delta u)$

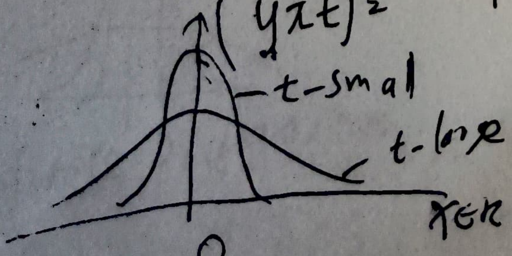
$p>1, q>0$

if $p=2, q=1$

$\partial_t u = \Delta u$ heat eq

Fundamental solution

$$u(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(-\frac{|x|^2}{4t}\right)$$



$u(x,t)>0, \forall x,t$

Filtration liquid in porous medium

$u(x,t)$ is fraction of liquid at x at time t. $0\le u\le 1$

L.S. Leibenson

Barenblatt solution

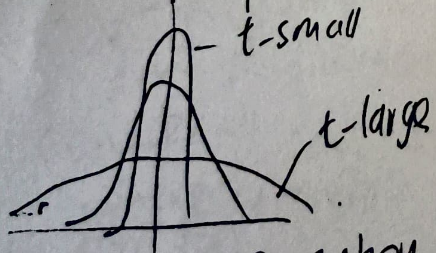
$q(p-1)>1$ (1) has solutions

$$u(x,t) = \frac{1}{t^{n/\beta}}\left(C - k\left(\frac{|x|}{t^{1/\beta}}\right)^{\frac{p}{p-1}}\right)_+^{\gamma}$$ → take positive part

$\forall C>0$, $\beta = p + n(q(p-1)-1)$

$$\gamma = \frac{p-1}{q(p-1)-1} > 0$$

$$k = \frac{q(p-1)-1}{pq}\beta^{-\frac{1}{p-1}} > 0$$



finite propagation speed

$\Leftrightarrow$ $u(\cdot,t)$ has compact support, $\forall t>0$

if $q(p-1)<1$, the same formula, k is neg, γ is neg

$\Rightarrow$ infinite prop speed of B. solution

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if $q(p-1)=1$

$$u(x,t)=\frac{1}{t^{n/p}}\exp\left(-\left(\frac{|x|}{t^{1/p}}\right)^{\frac{q}{p-1}}\right)$$

infinite prop speed

$$C=(p-1)^2 p^{-\frac{p}{p-1}}$$

contais Gauss-weiestrass solution

Equ on Riemanian manifolds

let M be a R manifold, geod complete

$\Rightarrow$ all geodeic balls are precompact sets

∇ - R gradient

div - R. divergence

consider (1) $\partial_t u = \Delta_p(u^q)$

where $u(x,t)\geq 0$, $x\in M$, $t>0$

~~Thm~~ $p>1$, $q>0$

Thm 1. let $q(p-1)>1$

Then any bounded solution u of (1) with compact supported $u(\cdot,0)$ has finite prop. speed, that is, $u(\cdot,t)$ is also compactly supported, $\forall t\in(0,T)$ $T\in(0,+\infty)$

chain rule for $\Delta_p f(u)$

$u \geq 0$, $f: \mathbb{R} \to \mathbb{R}$, $f \in C^2(\mathbb{R})$

$f \geq 0$, $f' \leq 0$

$$\Delta_p v = \mathrm{div}(|\nabla v|^{p-2} \nabla v)$$

$$v = f(u) \quad \nabla v = f'(u)\nabla u$$

$$\Delta_p v = \mathrm{div}(|f'(u)\nabla u|^{p-2} f'(u) \nabla u)$$

$$= -\mathrm{div}(\underbrace{|f'(u)|^{p-1}}_{\text{function}} \underbrace{|\nabla u|^{p-2} \nabla u}_{\text{vector field}})$$

$$\mathrm{div}(a v) = \nabla a \cdot v + a \,\mathrm{div}\, v$$

$$= -|f'(u)|^{p-1} \underbrace{\mathrm{div}(|\nabla u|^{p-2} \nabla u)}_{\Delta_p u} - \nabla(|f'(u)|^{p-1}) |\nabla u|^{p-2} \nabla u$$

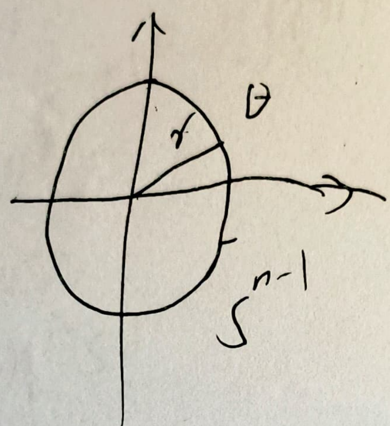
$$= -|f'(u)|^{p-1} \Delta_p u$$

$$-\nabla((-f'(u))^{p-1})(\nabla u)^{p-2} \nabla u$$

$$= -|f'(u)|^{p-1} \Delta_p u + (p-1)|f'(u)|^{p-2} f''(u) |\nabla u|^p$$

~~Mold~~ Model manifold

$M = R_+ \times S^{n-1}$



$x \mapsto (r, \theta)$

$r = |x|$

$\theta = \frac{x}{|x|} \in S^{n-1}$

$R^n \setminus \{0\} \cong R_+ \times S^{n-1}$

R metric

$ds^2 = dr^2 + \psi^2(r)\, d\theta^2$

$d\theta^2$ is standard R metric on S^{n-1}

$\Delta_p f(r)$?

~~$\Delta_p r =$~~

On model M:

$$\Delta v = \partial_r^2 v + \frac{S'(r)}{S(r)} \partial_r v + \frac{1}{\psi^2(r)} \overbrace{\Delta_\theta v}^{\text{on } S^{n-1}}$$

for $v = v(r)$

$$\Delta v = \partial_r^2 v + \frac{S'(r)}{S(r)} \partial_r v$$

$$\Delta r = \partial_r^2 r + \frac{S'}{S} \partial_r r = \frac{S'}{S}$$

$$\Delta_p r = \mathrm{div}(|\nabla r|^{p-2} \nabla r) = \Delta r = \frac{S'(r)}{S(r)}$$

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By (2)

$$\Delta_p f(r) = -|f'(r)|^{p-1}\frac{S'}{S} + (p-1)|f'(r)|^{p-2}\cdot f''(r)$$

$$(|f'(r)|^{p-1}S)' = |f'(r)|^{p-1}S' - (p-1)|f'(r)|^{p-2}f''(r)S$$

$$\Delta_p f(r) = -\frac{1}{S}(|f'(r)|^{p-1}S)'$$

On model manifold. look for solution $u=u(r,t)$ of $\partial_t u = \Delta_p(u^q)$,

$u \geq 0$, $\partial_r u \leq 0$

$$\partial_t u = -\frac{1}{S}\partial_r(S|-\partial_r u^q|^{p-1})$$

specific $S(r)$: $S(r) = r^{\alpha-1}$, $\alpha > 0$

$$\partial_t u = -\frac{1}{r^{\alpha-1}}\partial_r(r^{\alpha-1}(-\partial_r u^q)^{p-1})$$

Look for solution in form $u(r,t) = t^a f(rt^b)$

$f \geq 0$, $f' \leq 0$, $a, b < 0$, constants

Additional condition $\quad d\mu(x) = \mu(r)^{n-1}dr\,d\theta = \frac{1}{\omega_n}S(r)\,dr\,d\theta$

$$\int_M u(x,t)\,d\mu(x) = \cancel{cst}\ \text{const.}$$

$$= \int_0^\infty\int_{S^{n-1}} t^a f(rt^b)\frac{1}{\omega_n}S(r)\,d\theta\,dr$$

$$= \int_0^\infty t^a f(s)\left(\frac{s}{t^b}\right)^{\alpha-1}\frac{ds}{t^b} = t^{a-\alpha b}\int_0^\infty f(s)s^{\alpha-1}ds$$

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$\Rightarrow a = \alpha b \quad (5)$

$\cancel{\partial_t u = \partial_t (t^\alpha f(t r^b)) = a t^{a-1} f(t r^b)}$

$$\partial_t u = \partial_t (t^\alpha f(r t^b)) = a t^{a-1} f(r t^b) + t^a f'(r t^b) \, r b t^{b-1}$$

$$= b t^{a-1} (\alpha f(r t^b) + f'(r t^b) r t^b) = b t^{a-1}(\alpha f(s) + f'(s) s)$$

$$= b t^{a-1} (\alpha f(s) + s f'(s))$$

$$= \frac{b t^{a-1}}{s^{\alpha-1}} (\alpha s^{\alpha-1} f(s) + s^\alpha f'(s))$$

$$\partial_t u = \frac{b t^{a-1}}{s^{\alpha-1}} (s^\alpha f(s))' \quad (6)$$

$\cancel{\partial_t u}$

$$\partial_r u^q = q u^{q-1} \partial_r u$$

$$= q (t^a f(r t^b))^{q-1} \partial_r (t^a f(r t^b))$$

$$= q t^{a(q-1)} f(r t^b)^{q-1}$$

$$= q t^{aq+b} f(s)^{q-1} f'(s)$$

$$(4) \Leftrightarrow \frac{b t^{a-1}}{s^{\alpha-1}} (s^\alpha f(s))' = -\frac{1}{r^{\alpha-1}} \partial_r (r^{\alpha-1} (-q t^{aq+b} f(s)^{q-1} f'(s))^{p-1}$$

$$= \frac{q^{p-1} t^{(aq+b)(p-1)}}{(s/t^b)^{\alpha-1}} \partial_r \left(\left(\frac{s}{t^b}\right)^{\alpha-1} (-f(s)^{q-1} f'(s))^{p-1} \right)$$

$$= -q^{p-1} \frac{t^{(aq+b)(p-1)}}{s^{\alpha-1}} \frac{t^{b\alpha-b}}{t^{b(\alpha-1)}} t^{b} \partial_s \left(s^{\alpha-1} \left(-f(s)^{q-1} f'(s) \right)^{p-1} \right)$$

$$a-1 = (aq+b)(p-1)+b$$

$$-1 = (\alpha q+1)b(p-1)+b-\alpha b$$

$$= ((\alpha q+1)(p-1)+1-\alpha)b$$

$$= (\alpha \underbrace{(q(p-1)-1)}_{=:\delta>0}+p)b$$

$$(\alpha\delta+p)b = -1$$

$$b = -\frac{1}{\alpha\delta+p}$$

$$= -q^{p-1} \left(s^{\alpha-1} (-f(s))^{q-1} f'(s) \right)^{p-1} \Big)$$

$$\left(\text{to sum: } b\,(s^{\alpha} f(s))' \right)$$

R M 2

$M = R_+ \times S^{n-1}$

$ds^2 = dr^2 + \psi(r)^2 d\theta^2$

$S(r) = \omega_n \psi(r)^{n-1}$

$\partial_t u = \Delta_p u^q$

$S(r) = r^{\alpha-1}$

$u(r,t) = t^a f(rt^b)$, $a, b < 0$, $f \geq 0$, $f' \leq 0$

$p > 1$, $q > 0$

$a = \alpha b$

$\delta = q(p-1) - 1 > 0$

$$b = -\frac{1}{\alpha\delta + p}$$

$$b(s^\alpha f(s))' = -q^{p-1}\left(s^{\alpha-1}\left(-f(s)^{q-1} f'(s)\right)^{p-1}\right)'$$

$$b s^\alpha f(s) = -q^{p-1} s^{\alpha-1}\left(-f(s)^{q-1} f'(s)\right)^{p-1} + C$$

let $C = 0$

$$|b| s f(s) = q^{p-1} f(s)^{(q-1)(p-1)} (-f')^{p-1}$$

$$\frac{|b| s}{q^{p-1}} = f(s)^{(q-1)(p-1)-1} (-f')^{p-1}$$

let $r = q - 1 - \frac{1}{p-1}$

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$$\left(f(s)^{\gamma}\right)' = -\frac{\gamma}{q}|b|^{\frac{1}{p-1}} s^{\frac{1}{p-1}}$$

$$f(s)^{\gamma} = C - K s^{\frac{p}{p-1}}, \quad k = \frac{\gamma |b|^{\frac{1}{p-1}}}{q} \frac{p-1}{p}$$

$$f(s) = \left(C - K s^{\frac{p}{p-1}}\right)^{\frac{1}{\gamma}}$$

$$f(s) := 0, \ \forall s > s_0$$

$$u(x,t) = t^{\alpha b} f(x t^{b}) = \frac{1}{t^{\alpha/\beta}} \left(C - K\left(\frac{x}{t^{1/\beta}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{1}{\gamma}}$$

$$\alpha b = -\frac{\alpha}{2\delta + p} = -\frac{\alpha}{\beta}$$

$$\beta = 2\delta + p$$

$$b = -\frac{1}{\beta}$$

weak solution

M - any R manifold → $M \times I$ time

$\partial_t u \underset{\le}{=} \Delta_p u^q$

sub solution

are understood weakly

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$v \geq 0$
in the following class

$v \in C(I, L^2(\Omega))$ and $v^q \in L^q_{loc}(I, W^{1,p}(\Omega))$

$v(\cdot, t) \in L^2(\Omega)$

$t \longmapsto v(\cdot, t)$: continous in $L^2(\Omega)$

$t_k \to t \Rightarrow v(\cdot, t_k) \xrightarrow{L^2(\Omega)} v(\cdot, t)$

$v^q \in L^p_{loc}(I, W^{1,p}(\Omega))$ means

$v^q(\cdot, t) \in W^{1,p}(\Omega)$
$\Downarrow$
$v^q(\cdot, t) \in L^p(\Omega)$

$\Leftrightarrow \int_\Omega (v^{qp} + |\nabla v^q|^p) < \infty$

$\forall t_1 < t_2,\ t_1, t_2 \in I$

$\int_{t_1} \int_\Omega (v^{qp} + |\nabla v^q|^p)\, d\mu(x)\, dt < \infty$

μ - measure

Eq $\partial_t v = \Delta_p v^q$ is understood weakly

using test function in certain class

$\Delta_p v^q = \mathrm{div}(\nabla v^q)$

$|\partial_t v - \Delta_p v^q| h = 0$