Writing down equation for an K_9-Dessins d'enfants

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Background Knowledge and Notation

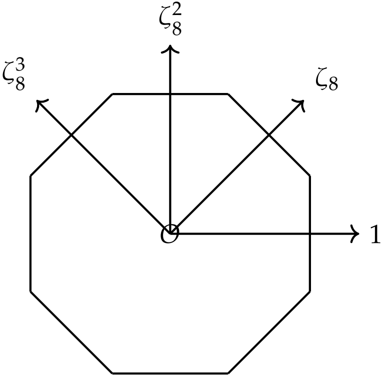
It can be shown that the affine model for K_9 dessin is a $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ cover of the Bolza surface, whose fundamental domain on Poincaré disk is an octagon. So we can color the 8-gon tessellation to make a visualization of K_9 dessin.

- There is a surjective morphism $\pi_1(P_8/\sim) \mapsto \mathbb{Z}[\zeta_8]/(1+\sqrt{-2})$. ($P_8/\sim$ is the regular 8-gon with the opposite side identified.)
- Label each octagon an element of the residue field $\mathbb{Z}[\zeta_8]/(1+\sqrt{-2})$.

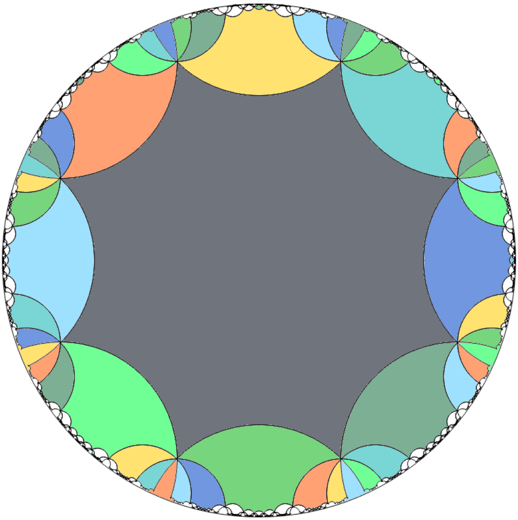
Element in the Residue Field	Color	Hex Colors
0	Navy	#001F3F
1	Blue	#0074D9
2	Aqua	#7FDBFF
ζ_8	Teal	#39CCCC
ζ_8+1	Olive	#3D9970
ζ_8+2	Green	#2ECC40
$2\zeta_8$	Lime	#01FF70
$2\zeta_8+1$	Yellow	#FFDC00
$2\zeta_8+2$	Orange	#FF851B

Background Knowledge and Notation

This is the correspondence between $\pi_1(P_8/\sim)$ and $\mathbb{Z}[\zeta_8]/(1+\sqrt{-2})$ and the hyperbolic translation on the 8-gon tessellation of the Poincaré disk.



Visualization



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History

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Biggs (1971) [1] constructed complete regular maps $K_n \hookrightarrow \Sigma_g$ for any prime power $n = p^f$ as a Cayley map associated to the finite fields $\mathbb{F}_n = \mathbb{F}_{p^f}$.
James and Jones (1985) [2] proved all complete regular maps are given by Biggs' construction.
Affine models for $K_2, K_3, K_4, K_5, K_7, K_8$ have already been obtained.

Observation

Let n be odd. Then the K_n-dessin is an abelian étale cover of the Wiman surface.

In particular, when $n = 9$, we have an $\mathbb{Z}/(3)^{\oplus 2}$-cover of the Bolza surface

$$B\colon y^2 = x(x^4 - 1).$$

In algebraic geometry and more generally in life, a way to obtain finite étale cover of a curve C is to base change from isogenies of Jacobian $J_C = \mathrm{Pic}^0_C$.

Big Pictures

- Let $n = p^f$ be odd prime power. According to the results of the arithmetic group, each K_n-dessin is an unramified abelian cover of a Wiman surface.
- In particular, each K_9-dessin is a $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$-cover of the **Bolza surface**

$$B\colon y^2 = x(x^4 - 1) \quad \text{equipped with the Belyi function} \quad \beta_B = 1/x^4.$$

The K_9-dessin fits into the following Cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & J_B \arrow[d, "\phi"] \\ B \arrow[r, "\mathrm{AJ}"'] & J_B \end{tikzcd}$$

We are interested in the left-hand vertical arrow, so we need to compute

- AJ, the Abel-Jacobi map.
- ϕ, a degree-9 isogeny

We can proceed in two ways: algebraically, or analytically.

Main results

- **Theorem**: An affine model for a K_9-dessin over its minimal field of definition $\mathbb{Q}(\sqrt{-2})$ is (C_{K_9}, β_{K_9}) where C_{K_9} is the genus ten curve given by the equations

$$\frac{x^3+\left(4-2i\sqrt{2}\right)x^2+\left(-46-28i\sqrt{2}\right)x-80i\sqrt{2}-112}{\left(i\sqrt{2}x+x+i\sqrt{2}+4\right)^2}=-\frac{\left(32-26i\sqrt{2}\right)u^4-\left(88+32i\sqrt{2}\right)u^3-\left(-24-60i\sqrt{2}\right)u^2+8\left(1-4i\sqrt{2}\right)u+6i\sqrt{2}-8}{\left(-\left(\left(-4+i\sqrt{2}\right)u^2\right)+\left(-4-2i\sqrt{2}\right)u+i\sqrt{2}\right)^2}$$

$$\frac{\left(x^3+\left(6-3i\sqrt{2}\right)x^2+6\left(9+2i\sqrt{2}\right)x+150i\sqrt{2}+76\right)y}{\left(i\sqrt{2}x+x+i\sqrt{2}+4\right)^3}=\frac{\left(u+\frac{1}{3}\left(1+i\sqrt{2}\right)\right)^3 v}{\left(\left(-4+i\sqrt{2}\right)u^2-\left(-4-2i\sqrt{2}\right)u-i\sqrt{2}\right)^3}$$

$$\frac{z^3+\left(4-2i\sqrt{2}\right)z^2+\left(-46-28i\sqrt{2}\right)z-80i\sqrt{2}-112}{\left(i\sqrt{2}z+z+i\sqrt{2}+4\right)^2}=-\frac{\left(-8-6i\sqrt{2}\right)u^4+8\left(1+4i\sqrt{2}\right)u^3-\left(-24+60i\sqrt{2}\right)u^2-8\left(11-4i\sqrt{2}\right)u+26i\sqrt{2}+32}{\left(-i\sqrt{2}u^2+2\left(-2+i\sqrt{2}\right)u+i\sqrt{2}+4\right)^2}$$

$$\frac{\left(2-3\sqrt{2}\right)\left(z^3+\left(6-3i\sqrt{2}\right)z^2+6\left(9+2i\sqrt{2}\right)z+150i\sqrt{2}+76\right)}{\left(i\sqrt{2}z+z+i\sqrt{2}+4\right)^3}=-\frac{\left(u-i\sqrt{2}-1\right)^3 v}{\left(-i\sqrt{2}u^2+2\left(-2+i\sqrt{2}\right)u+i\sqrt{2}+4\right)^3}$$

$$v^2=u^6-5u^4-5u^2+1,\quad (-23-10i\sqrt{2})y^2=x^3-30x-56,\quad -1728w^2=z^3-30z-56$$

and β_{K_9} is the rational function

$$\beta_{K_9}(x,y,z,w,u,v)=-\frac{(u+1)^4}{(u-1)^4}.$$

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Analytic interpretation of Jacobians of curves

Let C be a Riemann surface of genus g.

- Let $K_C = \Omega^1_C$ be the sheaf of differentials. Integration along 1-cycles gives embedding (Exercise: convince yourself that this map is injective)

$$\iota \colon H_1(C, \mathbb{Z}) \hookrightarrow H^0(K_C)^\vee; \quad [\gamma] \mapsto (\eta \mapsto \int_\gamma \eta).$$

- Define the Jacobian of C by

$$J_C(\mathbb{C}) = H^0(K_C)^\vee / \iota H_1(C, \mathbb{Z}).$$

- If you pick basis for $H^0(K_B)$ and $H_1(B, \mathbb{Z})$, ι can be expressed as a g-by-$2g$ matrix.

Period matrix of Bolza surface

- Let $\beta_i, i = 0, 1, 2, 3$, be classes of H_1-systoles that generates $H_1(B(\mathbb{C})^{an}, \mathbb{Z})$.
- Let $\omega_0 = dx/y$ and $\omega_1 = \zeta^3 x\, dx/y$, where, be the representative of basis of $H^0(K_B)$.
- The map ι is a $(2, 4)$-matrix given by $A_{ij} = \int_{\beta_j} \omega_i$. This is computed by Quine [4]:

$$A = \alpha \begin{pmatrix} \zeta + \zeta^2 & -1 - \zeta & -\zeta^3 + 1 & \zeta^2 + \zeta^3 \\ \zeta^3 - \zeta^2 & -1 - \zeta^3 & -\zeta + 1 & \zeta^6 + \zeta \end{pmatrix}, \quad \alpha = \frac{\pi\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})e^{i\pi/8}}{4\pi i}$$

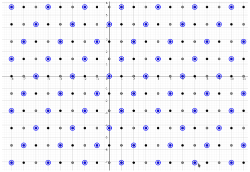
- Let M be the Minkowski embedding of $\mathbb{Q}(\zeta_8)$ given by $a \mapsto (a, \tau(a))$ where $\tau : \zeta_8 \mapsto \zeta_8^3$. Linear algebra says

$$J_B(\mathbb{C}) \cong \mathbb{C}^2/M(\mathbb{Z}[\sqrt{-2}]) \cong (\mathbb{C}/\mathbb{Z}[\sqrt{-2})^2$$

.

Analytic approach to isogenies of $E \times E$

- Let E be the elliptic curve $E : y^2 = x^3 - 30x - 56$. Let $\kappa = \frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{4\sqrt{6\pi}}, \Lambda = \kappa\mathbb{Z}[\sqrt{-2}]$.
- **Proposition**: We have an isomorphism of Riemann surfaces $\mathbb{C}/\Lambda \to E(\mathbb{C})$ given by $z \mapsto [\wp_\Lambda(z) : \frac{\wp'_\Lambda(z)}{2} : 1]$ where $\wp_\Lambda$ is the Weierstrass-$\wp$ function of Λ.



- This lattice picture shows the poles of $\wp_{\alpha\Lambda}(z) + \wp_{\alpha\Lambda}(z+\kappa) + \wp_{\alpha\lambda}(z-\kappa)$, where $\alpha = 1 + \sqrt{-2}$. Using Liouville's theorem, we can show it equals to $\wp_\Lambda(z) + \frac{2\sqrt{-2}}{1+\sqrt{-2}}$.
- **Proposition:** $\wp_{\alpha\Lambda}(z) + \wp_{\alpha\Lambda}(z+\kappa) + \wp_{\alpha\lambda}(z-\kappa) = \wp_\Lambda(z) + \frac{2\sqrt{-2}}{1+\sqrt{-2}}$.
- Using this proposition, we can compute the endomorphism ϕ of the elliptic curve E given by complex multiplication by $1 + \sqrt{-2}$. The product map $\phi \times \phi$ gives the isogeny ϕ

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Change-of-variable from a pencil of conics

It turns out one can obtain the splitting of J_B purely algebraically!

- Write the defining equation for B as

$$y^2 = x(x^2 + (i-1)x - i)(x^2 - (i-1)x - i),$$

- The three quadratics move in the pencil

$$E_{[\lambda:\mu]} : \lambda(x-\alpha)^2 + \mu(x+\alpha)^2 = 0,$$

- Making a change of variable, we can write B in the following way, with obvious maps to two isom. elliptic curves with j-invariant 8000:

$$\begin{tikzcd}
B\colon y^2 = x^6 + 5x^4 - 5x^2 - 1. \arrow[dd, "{(x,y)\mapsto(x^2,y)}"'] \arrow[dr, "{(x,y)\mapsto(1/x^2,y/x^3)}"] \arrow[ddr, dotted, "\Phi"] & \\
& E'\colon y^2 = -x^3 - 5x^2 + 5x + 1. \\
E\colon y^2 = x^3 + 5x^2 - 5x - 1. & E \times E' \cong E \times E \arrow[l, "\pi_2"'] \arrow[u, "\pi_1"']
\end{tikzcd}$$

Obtaining the Abel-Jacobi map

Claim

The map $\Phi\colon B \mapsto E \times E$ is the Abel-Jacobi map.

By universal property of Abel-Jacobi,

$$\begin{tikzcd}
 & J_B \arrow[dr, "g"] & \\
B \arrow[ur, hook, "\text{cl. imm.}", "\mathrm{AJ}"'] \arrow[rr, "\Phi"] & & E \times E
\end{tikzcd}$$

Check Φ is 1-1 & unramified by computation, so Φ and hence g are closed immersions as well. Exercise: dimensionality and integrality of $E \times E$ are already letting us win.

Algebraically computing a deg-9 isogeny of $E \times E$

- Want a degree-3 isogeny of E given the cm $1+\sqrt{-2}$.
- There are explicit parameterisation of deg-2 isogenies elliptic curves.
- Solve for the equation of multiplication-by-$\sqrt{-2}$ (Proposition 2.3.1 of Silverman [5]), and then use addition formula to get multiplication-by-$(1+\sqrt{-2})$.

In the Weierstraß model $E'' \colon y^2 = x^3 + 4x^2 + 2x$, put $P = (x, y) \in E''$,

$$x([1+\sqrt{-2}]P) = -\frac{1}{2}x + \frac{\left(\sqrt{-2}y\left(\frac{2}{x^2}-1\right)+4y\right)^2}{4\left(3x+\frac{2}{x}+4\right)^2} + \frac{1}{x} - 2$$

$$y([1+\sqrt{-2}]P) = \frac{\left(\sqrt{-2}y\left(\frac{2}{x^2}-1\right)+4y\right)\left(2x - \frac{\left(\sqrt{-2}y\left(\frac{2}{x^2}-1\right)+4y\right)^2}{\left(3x+\frac{2}{x}+4\right)^2} - \frac{4}{x} + 8\right)}{8\left(3x+\frac{2}{x}+4\right)} + \frac{\sqrt{-2}xy\left(\frac{2}{x^2}-1\right) - 2\left(x+\frac{2}{x}+4\right)y}{2\left(3x+\frac{2}{x}+4\right)}$$

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Obtaining Affine Model for K_9 Dessin

- Let $C_{\mathfrak{p}}$ be the fiber product of B and $E \times E$. By chapter 9 of Milne [3], $C_{\mathfrak{p}}$ is the cover we want. Thus $(C_{\mathfrak{p}}, \beta_{K_9})$ is an affine model for a K_9 dessin.
- Make a suitable coordinate change. We get an affine model over $\mathbb{Q}(\sqrt{-2})$.

Future work

- Compute affine models for K_n dessins for $n > 9$.
- Try to find a uniform method for the case when n is odd.

References

[1] N. Biggs. “Automorphisms of imbedded graphs”. In: *J. Comb. Theory Ser. B* 11.2 (1971), pp. 132–138.

[2] L. D. James and G. A. Jones. “Regular orientable imbeddings of complete graphs”. In: *J. Comb. Theory Ser. B* 39.3 (1985), pp. 353–367.

[3] J. S. Milne. “Jacobian Varieties”. In: *Arithmetic Geometry*. Ed. by Gary Cornell and Joseph H. Silverman. New York, NY: Springer New York, 1986, pp. 167–212. ISBN: 978-1-4613-8655-1. DOI: 10.1007/978-1-4613-8655-1_7. URL: https://doi.org/10.1007/978-1-4613-8655-1_7.

[4] J. R. Quine. “Systoles of two extremal Riemann surfaces”. en. In: *J. Geom. Anal.* 6.3 (Sept. 1996), pp. 461–474. ISSN: 1559-002X. DOI: 10.1007/BF02921661. URL: https://doi.org/10.1007/BF02921661 (visited on 08/24/2024).

[5] J. H. Silverman. *Advanced Topics in the Arithmetic of Elliptic Curves*. New York, NY: Springer, 1994. ISBN: 9780387943282 9781461208518. DOI: 10.1007/978-1-4612-0851-8. URL: http://link.springer.com/10.1007/978-1-4612-0851-8 (visited on 08/24/2024).