Writing down equation for an K_9-Dessins d'enfants

Yanshu Wang

Joint with Polymath Jr Dessin equation group: Ruizhen Liu, ...

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Background Knowledge and Notation

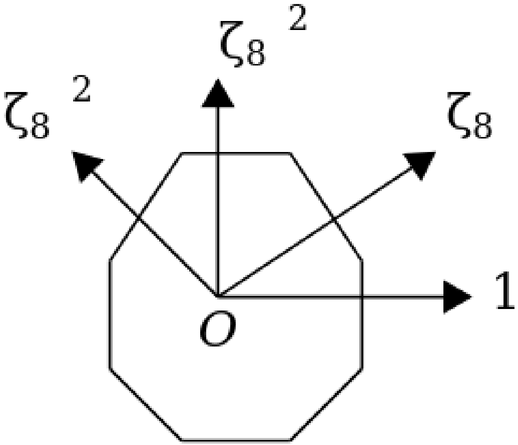
Goal: It can be shown that the affine model for K_9 dessin is a $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ cover of the Bolza surface, whose fundamental domain on Poincaré disk is an octagon. So we can color the 8-gon tessellation to make a visualization of K_9 dessin.

- There is a surfective morphism $\pi_1(P_8/\sim) \mapsto \mathbb{Z}[\zeta_8]/(1+\sqrt{-2})$. ($P_8/\sim$ is the regular 8-gon with the opposite side identified.)
- Label each octagon an element of the residue field $\mathbb{Z}[\zeta_8]/(1+\sqrt{-2})$.

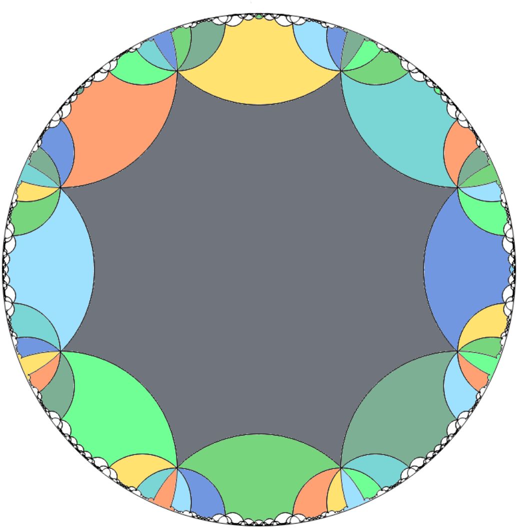
Element in the Residue Field	Color	Color Codes
0	Navy	#001F3F
1	Blue	#0074D9
2	Aqua	#7FDBFF
ζ_8	Teal	#39CCCC
ζ_8+1	Olive	#3D9970
ζ_8+2	Green	#2ECC40
$2\zeta_8$	Lime	#01FF70
$2\zeta_8+1$	Yellow	#FFDC00
$2\zeta_8+2$	Orange	#FF851B

Background Knowledge and Notation

This is the correspondence of $\pi_1(P_8/\sim) \mapsto \mathbb{Z}[\zeta_8]/(1+\sqrt{-2})$.



Visualization



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History

Biggs (1971) constructed complete regular maps $K_n \hookrightarrow \Sigma_g$ for any prime power $n = p^f$ as a Cayley map associated to the finite fields $\mathbb{F}_n = \mathbb{F}_{p^f}$.
James and Jones (1985) proved all complete regular maps are given by Biggs' construction.
Affine models for $K_2, K_3, K_4, K_5, K_7, K_8$ have already been obtained.

Observation

Let n be odd. Then the K_n-dessin is an abelian étale cover of the Wiman surface.

In particular, when $n = 9$, we have an $(\mathbb{Z}/(3))^2$-cover of the Bolza surface

$$B\colon y^2 = x(x^4 - 1).$$

In algebraic geometry and more generally in life, a way to obtain finite étale cover of a curve C is to base change from isogenies of Jacobian $J_C = \mathrm{Pic}^0_C$.

Hope

The K_9-dessin fits into the following Cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & J_B \arrow[d, "\phi"] \\ B \arrow[r, "\mathrm{AJ}"'] & J_B \end{tikzcd}$$

We are interested in the left-hand vertical arrow, so we need to compute

- AJ, the Abel-Jacobi map.
- ϕ, a degree-9 isogeny

We can proceed in two ways: algebraically, or analytically.

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Analytic interpretation of Jacobians of curves

Let C be a Riemann surface of genus g.

- Let $K_C = \Omega^1_C$ be the sheaf of differentials. Integration along 1-cycles gives embedding

$$\iota \colon H_1(C,\mathbb{Z}) \hookrightarrow H^0(K_C)^\vee; \quad [\gamma] \mapsto (\eta \mapsto \int_\gamma \eta).$$

- Define the Jacobian of C by

$$J_C(\mathbb{C}) = H^0(K_C)^\vee / \iota H_1(C,\mathbb{Z}).$$

- If you pick basis for $H^0(K_B)$ and $H_1(B,\mathbb{Z})$, ι can be expressed as a g-by-$2g$ matrix.

Period matrix of Bolza surface

- Let $\beta_i, i = 0, 1, 2, 3$, be classes of H_1-systoles that generates $H_1(B(\mathbb{C})^{an}, \mathbb{Z})$.
- Let $\omega_0 = dx/y$ and $\omega_1 = \zeta^3 x\, dx/y$, where, be the representative of basis of $H^0(K_B)$.
- The map ι is a $(2,4)$-matrix given by $A_{ij} = \int_{\beta_j} \omega_i$. This is computed by Quine's paper [1]:

$$A = \alpha \begin{pmatrix} \zeta + \zeta^2 & -1 - \zeta & -\zeta^3 + 1 & \zeta^2 + \zeta^3 \\ \zeta^3 - \zeta^2 & -1 - \zeta^3 & -\zeta + 1 & \zeta^6 + \zeta \end{pmatrix}, \quad \alpha = \frac{\pi\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})e^{i\pi/8}}{4\pi i}$$

- Let M be the Minkowski embedding of $\mathbb{Q}(\zeta_8)$ given by $a \mapsto (a, \tau(a))$ where $\tau : \zeta_8 \mapsto \zeta_8^3$. Linear algebra says

$$J_B(\mathbb{C}) \cong \mathbb{C}^2 / M(\mathbb{Z}[\sqrt{-2}]) \cong (\mathbb{C}/\mathbb{Z}[\sqrt{-2}])^2$$

.

Analytic approach to isogenies of E times E

Let $\kappa = \frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{4\sqrt{6\pi}}, \Lambda = \kappa\mathbb{Z}[\sqrt{-2}]$. Uniformization: we have an isomorphism

$$\mathbb{C}/\Lambda \longrightarrow E/\mathbb{C}\colon y^2 = x^3 - 30x - 56 \quad z \mapsto [\wp_\Lambda(z) : \frac{\wp'_\Lambda(z)}{2} : 1]$$

Here $\wp_\Lambda$ is the Weierstrass-$\wp$ function of Λ.

Goal

1. Show $\wp_{\alpha\Lambda}(z) + \wp_{\alpha\Lambda}(z+\kappa) + \wp_{\alpha\lambda}(z-\kappa) = \wp_\Lambda(z) + \frac{2\sqrt{-2}}{1+\sqrt{-2}}$, where $\alpha = 1 + \sqrt{-2}$,
2. Compute the endmorphism of elliptic curve $\phi : E \longrightarrow E$ given by cm $1+\sqrt{-2}$. The product map $\phi \times \phi$ gives the isogeny.

Sketch of Isogeny Computation

- Use Liouville's theorem to prove $\wp_{\alpha\Lambda}(z) + \wp_{\alpha\Lambda}(z+\kappa) + \wp_{\alpha\lambda}(z-\kappa) - \wp_\Lambda(z)$ is constant, where $\alpha = 1 + \sqrt{-2}$.
- Its value at 0 is $\frac{2}{\alpha^2}\wp_\Lambda(\frac{\kappa}{\alpha})$. And $\wp_\Lambda(\frac{\kappa}{\alpha})$ is 3-torsion point of the corresponding elliptic curve. According to exercise 3.7 of *The Arithmetic of Elliptic Curves* by Silverman [3], it is a root of a degree-3 division polynomial.
- Use the identity above and addition formula of Weierstrass elliptic function to compute the map ϕ:$[\wp_\Lambda(\frac{z}{\alpha}), \frac{\wp'_\Lambda(\frac{z}{\alpha})}{2}, 1] \mapsto [\wp_\Lambda(z), \frac{\wp'_\Lambda(z)}{2}, 1]$. This gives the complex multiplication by $\alpha = 1 + \sqrt{-2}$ on the elliptic curve of lattice Λ.

Express $\wp_\Lambda(z)$ using $\wp_{\alpha\Lambda}(z)$:

$\wp_\Lambda(z) = \wp_{\alpha\Lambda}(z) + \wp_{\alpha\Lambda}(z+\kappa) + \wp_{\alpha\Lambda}(z-\kappa) - \frac{2\sqrt{-2}}{1+\sqrt{-2}} =$
$\frac{1}{4}\frac{2\wp'^2_{\alpha\Lambda}(z)+2\wp'^2_{\alpha\Lambda}(\kappa)}{(\wp_{\alpha\Lambda}(z)-\wp_{\alpha\lambda}(\kappa))^2} - 2\wp_{\alpha\Lambda}(z) - 2\wp_{\alpha\Lambda}(\kappa) + \wp_{\alpha\Lambda}(z) - \frac{2\sqrt{-2}}{1+\sqrt{-2}}$

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Change-of-variable from a pencil of conics

It turns out one can obtain the splitting of J_B purely algebraically!

- Write the defining equation for B as

$$y^2 = x(x^2 + (i-1)x - i)(x^2 - (i-1)x - i),$$

- The three quadratics move in the pencil

$$E_{[\lambda:\mu]} : \lambda(x-\alpha)^2 + \mu(x+\alpha)^2 = 0,$$

- Making a change of variable, we can write B in the following way, with obvious maps to two isom. elliptic curves with j-invariant 8000:

$$\begin{tikzcd}
B\colon y^2 = x^6 + 5x^4 - 5x^2 - 1. \arrow[dr, "(x,y)\mapsto(1/x^2,y/x^3)"] \arrow[dd, "(x,y)\mapsto(x^2,y)"'] \arrow[ddr, dotted, "\Phi"] & \\
 & E'\colon y^2 = -x^3 - 5x^2 + 5x + 1. \\
E\colon y^2 = x^3 + 5x^2 - 5x - 1. & E \times E' \cong E \times E \arrow[u, "\pi_1"'] \arrow[l, "\pi_2"']
\end{tikzcd}$$

Obtaining the Abel-Jacobi map

Claim

The map $\Phi \colon B \mapsto E \times E$ is the Abel-Jacobi map.

By universal property of Abel-Jacobi,

$$\begin{tikzcd}
 & J_B \arrow[dr, "g"] & \\
B \arrow[ur, hook, "\text{cl. imm.}", "\text{AJ}"'] \arrow[rr, "\Phi"] & & E \times E
\end{tikzcd}$$

Check Φ is 1-1 & unramified by computation, so Φ and hence g are closed immersions as well. Exercise: dimensionality and integrality of $E \times E$ are already letting us win.

Algebraically computing a deg-9 isogeny of $E \times E$

- Want a degree-3 isogeny of E given the cm $1+\sqrt{-2}$.
- There are explicit parameterisation of deg-2 isogenies elliptic curves.
- The equation of multiplication by $\sqrt{-2}$ has been already computed in Proposition 2.3.1 of [2], and then use addition formula to get multiplication by $(1+\sqrt{-2})$.

In the Weierstraß model $E'': y^2 = x^3 + 4x^2 + 2x$, put $P = (x, y) \in E''$,

$$x([1+\sqrt{-2}]P) = -\frac{1}{2}x + \frac{\left(\sqrt{-2}y\left(\frac{2}{x^2}-1\right)+4y\right)^2}{4\left(3x+\frac{2}{x}+4\right)^2} + \frac{1}{x} - 2$$

$$y([1+\sqrt{-2}]P) = \frac{\left(\sqrt{-2}y\left(\frac{2}{x^2}-1\right)+4y\right)\left(2x - \frac{\left(\sqrt{-2}y\left(\frac{2}{x^2}-1\right)+4y\right)^2}{\left(3x+\frac{2}{x}+4\right)^2} - \frac{4}{x} + 8\right)}{8\left(3x+\frac{2}{x}+4\right)} + \frac{\sqrt{-2}xy\left(\frac{2}{x^2}-1\right) - 2\left(x+\frac{2}{x}+4\right)y}{2\left(3x+\frac{2}{x}+4\right)}$$

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$$\begin{tikzcd} X \arrow[r] \arrow[d] & J_B \arrow[d, "\varphi"] \\ X \arrow[r, "AJ"'] & J_B \end{tikzcd}$$

Obtaining the deg-9 cover

Let $C_{\mathfrak{p}}$ be the fibre product of B and $E \times E$, namely the 6-tuples (x,y,a,b,c,d) satisfying $AJ(x, y) = \phi_{\mathfrak{p}}(a, b, c, d)$. By chapter 9 of [**Milne**], $C_{\mathfrak{p}}$ is the cover X we want.

Obtaining the belyi function

An affine model for D_4 is (C_4, β_4) where

$$C_4 : y^2 = x(x^4 - 1) \text{ and } \beta_4(x, y) = \frac{1}{x^4}$$

.

So $\beta = \beta_4 \circ f_{\mathfrak{p}}$ where $f_{\mathfrak{p}} : C_{\mathfrak{p}} \mapsto C_4$ is the projection.

So we get the affine model for K_9 dessin. Thanks!

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References

[1] J. R. Quine. "Systoles of two extremal Riemann surfaces". en. In: *The Journal of Geometric Analysis* 6.3 (Sept. 1996), pp. 461–474. ISSN: 1559-002X. DOI: 10.1007/BF02921661. URL: https://doi.org/10.1007/BF02921661 (visited on 08/24/2024).

[2] Joseph H. Silverman. *Advanced Topics in the Arithmetic of Elliptic Curves*. Vol. 151. Graduate Texts in Mathematics. New York, NY: Springer, 1994. ISBN: 9780387943282 9781461208518. DOI: 10.1007/978-1-4612-0851-8. URL: http://link.springer.com/10.1007/978-1-4612-0851-8 (visited on 08/24/2024).

[3] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*. Vol. 106. Graduate Texts in Mathematics. New York, NY: Springer, 2009. ISBN: 9780387094939 9780387094946. DOI: 10.1007/978-0-387-09494-6. URL: http://link.springer.com/10.1007/978-0-387-09494-6 (visited on 07/30/2024).