Lecture 9
Wavelet and Other Image Transforms

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Outline

- **2D Unitary transform** 酉变换
- **Frequency Domain Extension**
  - Discrete Cosine Transform（余弦变换）
  - Hadamard Transform （哈德马变换）
  - Discrete Wavelet Transform （小波变换）
- **Discrete Wavelet Transform (DWT)**
  - An example for 1D-DWT
  - Generalization of 1D-DWT
  - 2D-DWT

Unitary Transform

- **Forward Transform:**

$AA^T = I$ unitary Transform

$A^T = A^{-1}$

$$t = Af$$

$A^{-1}t = f$ $f = A^{-1}t$

$$t[k] = \sum_{n=0}^{N-1} A[k,n]f[n]$$

- **Inverse Transform:** 存在复数

$$f = A^H t \quad \text{if } A^H = (A^T)^* \text{ and } AA^H = I$$

Example for 1D Unitary Transform

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -sc + sc & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

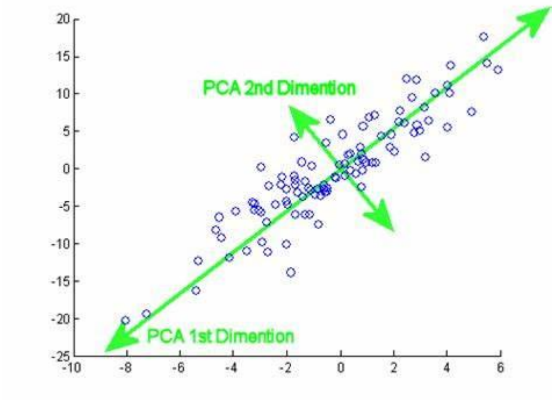
- **Image rotation:**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}$$

- **Principle Component Analysis (PCA):**

$Y = PX$ that satisfies $C = XX^T$ $D = PCP^T$

and $PP^T = I$



Discrete Fourier Transform

➢ **Forward Transform:**

$$t = Af; \quad t[k] = \sum_{n=0}^{N-1} A[k,n]f[n]$$

➢ **Inverse Transform:**

$$f = A^H t; \quad f[n] = \sum_{k=0}^{N-1} A^H[k,n]t[k]$$

a_1 $a_1 a_1^H = I$

a_2 $a_1 a_2^H = 0$

$\vdots$ $a_1 a_3^H = 0$

a_n 完备正交基

basis function

➢ **1-D DFT**

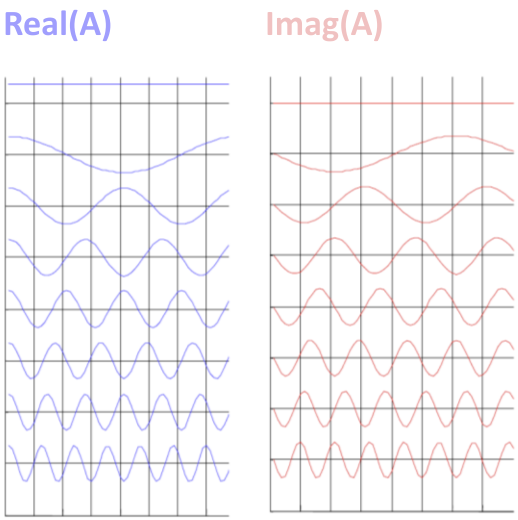
$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}}, (k = 1,2,\cdots,N)$$

$$A[k,n] = e^{-j\frac{2\pi kn}{N}} = cos(2\pi\frac{kn}{N})\text{-}jsin(2\pi\frac{kn}{N})$$

$A = e^{-j\frac{2\pi kn}{N}}$

$A^H = e^{j\frac{2\pi kn}{N}}$

$AA^H = I$



2D Unitary Transform

- Forward Transform:

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

$$= A_M f A_N$$

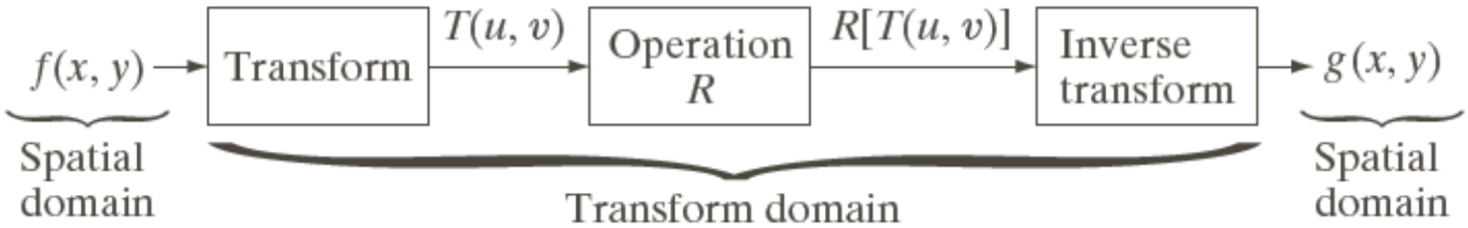
- Inverse Transform

$A_M A_M^H = I$ $A_N A_N^H = I$

$$f = A_M^H F A_N^H \qquad AA^H = I$$

Image Transform

- The general approach for operating in linear transform domain



- The unitary transform satisfies

$$\sum_{x=0}^{M}\sum_{y=0}^{N}(\boldsymbol{f}[\boldsymbol{x},\boldsymbol{y}])^2 = \sum_{u=0}^{M}\sum_{v=0}^{N}(\boldsymbol{F}[\boldsymbol{u},\boldsymbol{v}])^2$$

  - i.e., the energy is preserved.

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Good and Bad things about DFT

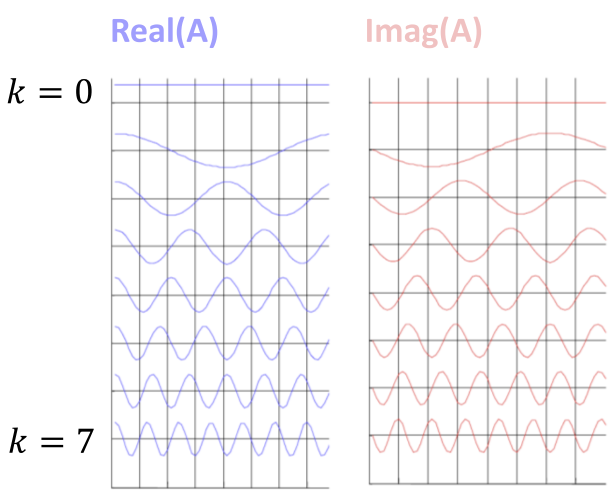
- **Positive:**
  - Energy is usually packed into low-frequency coefficients
  - Convolution property：频率域 空间滤波
  - Fast implementation FFT
- **Negative:**
  - Transform is complex, even if image is real
  - The basis function span image height/width

DFT vs. DCT (Discrete Cosine Transform)

- **1D-DFT**

$$A[k,n] = e^{-j\frac{2\pi kn}{N}}$$

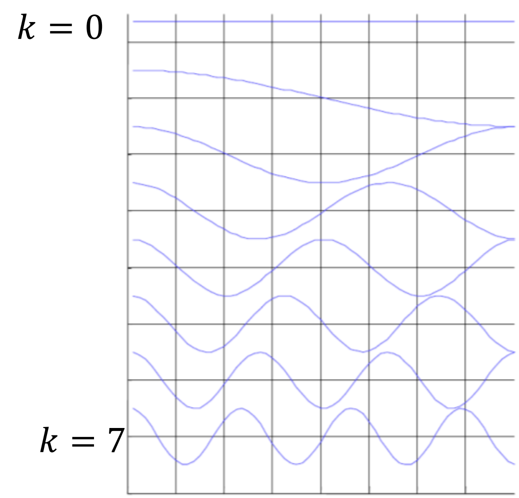
$$= cos\left(2\pi\frac{kn}{N}\right) + jsin\left(2\pi\frac{kn}{N}\right)$$



- **1D-DCT**

$$A[k,n] = \sqrt{\frac{2}{N}}cos\frac{\pi(2n+1)k}{2N}$$

不需在实和复数空间切换



What's the difference???

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2D DCT

Forward Transform

$$F(0,0)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)$$

$$F(u,0)=\frac{\sqrt{2}}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\cos\frac{(2x+1)u\pi}{2N}$$

$$F(0,v)=\frac{\sqrt{2}}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\cos\frac{(2y+1)v\pi}{2N}$$

$$F(u,v)=\frac{2}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\cos\frac{(2x+1)u\pi}{2N}\cos\frac{(2y+1)v\pi}{2N}$$

2D IDCT

- Inverse Transform

$$f(x,y) = \frac{1}{N} F(0,0)$$

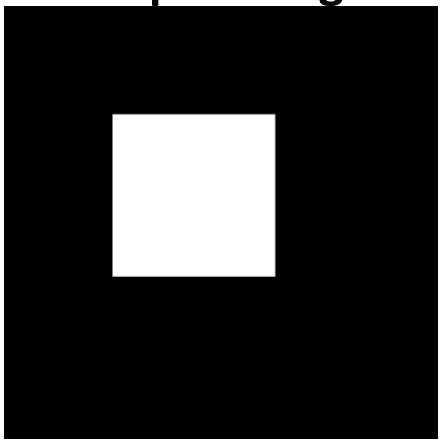
$$+ \frac{\sqrt{2}}{N} \sum_{u=1}^{N-1} F(u,0) \cos \frac{(2x+1)u\pi}{2N}$$

$$+ \frac{\sqrt{2}}{N} \sum_{v=1}^{N-1} F(0,v) \cos \frac{(2y+1)v\pi}{2N}$$

$$+ \frac{2}{N} \sum_{x=1}^{N-1} \sum_{y=1}^{N-1} F(u,v) \cos \frac{(2x+1)u\pi}{2N} \cos \frac{(2y+1)v\pi}{2N}$$

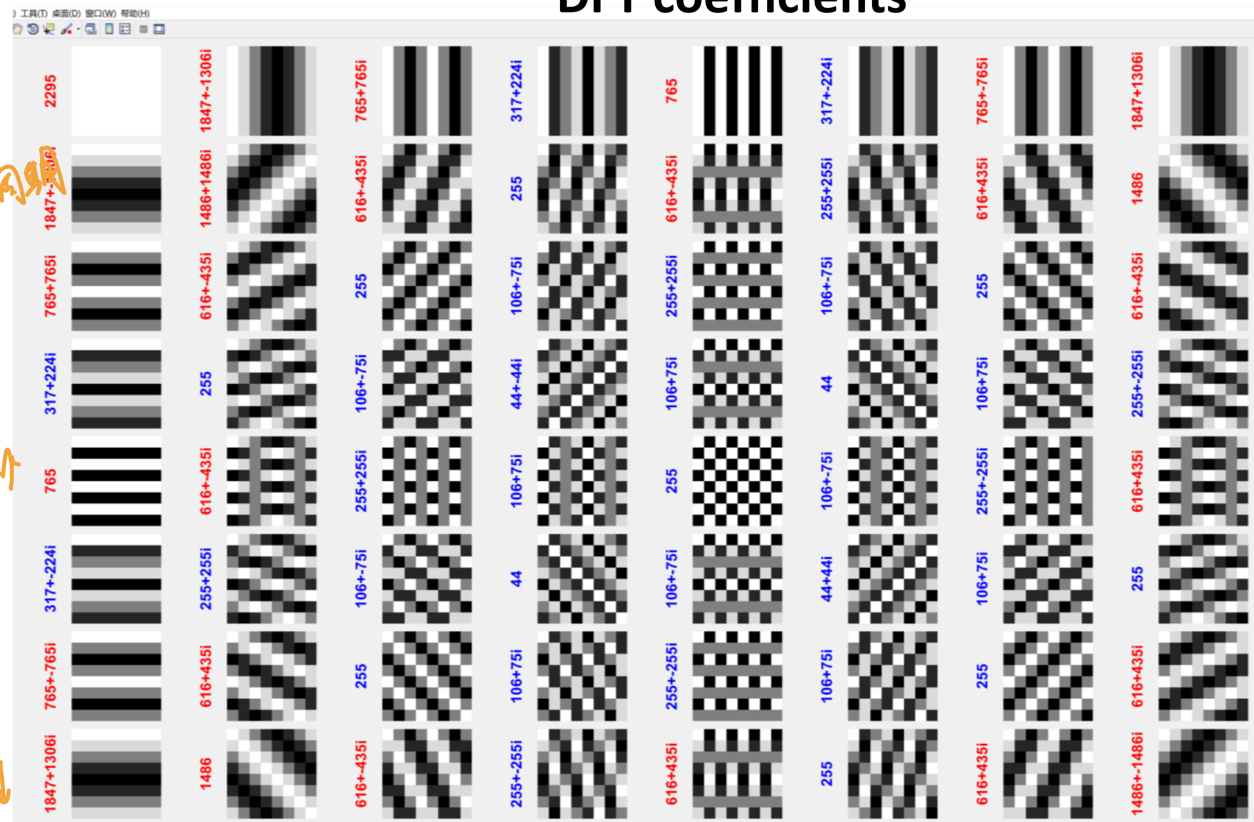
DFT example

Input image



0：均值以上

DFT coefficients



一个周期

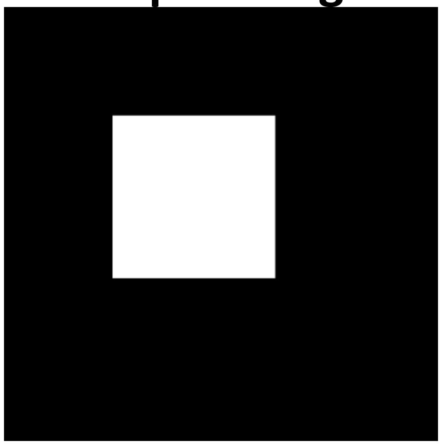
升

降

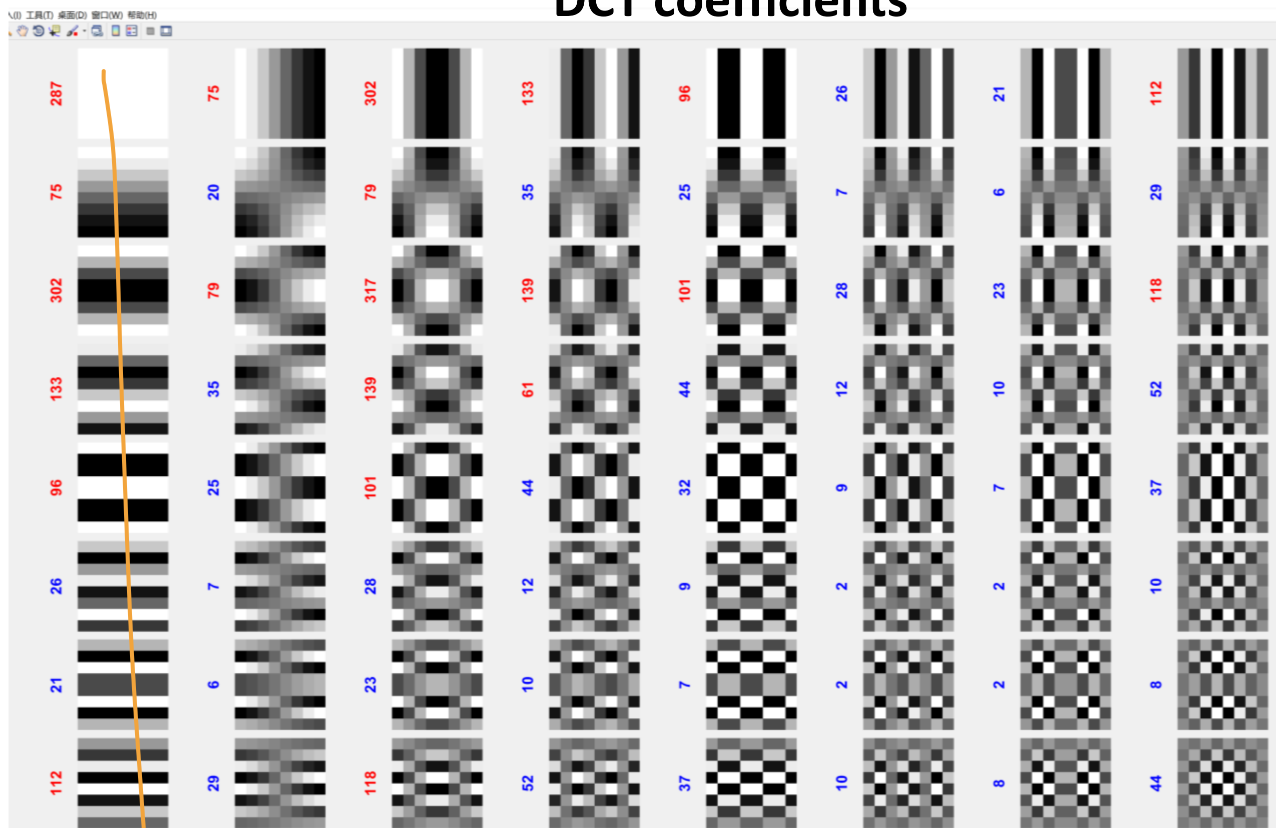
basis function 振荡频率，
小→大→小

DCT example

Input image



DCT coefficients

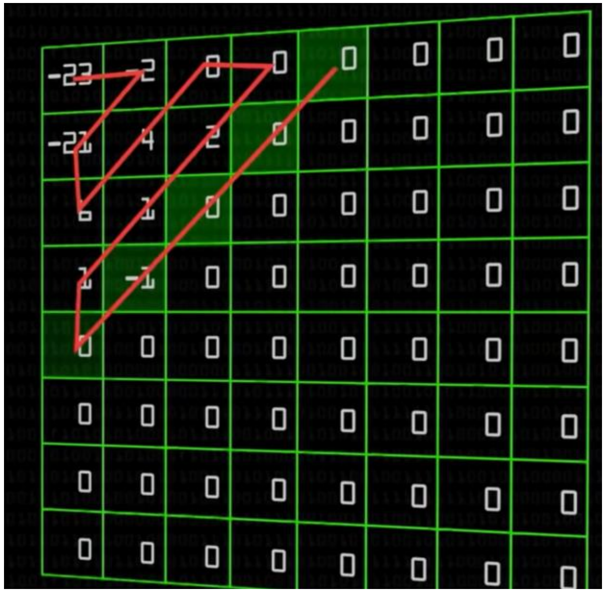


低频到高频

(更 compact 紧密的)

Good things about DCT

- **Positive:**
  - Transform is real, $C^{-1} = C^{T}$ (unitary transform).
  - Excellent energy compaction for nature images.
  - Fast transform.
  - JPEG algorithms. 能量在左上角



Walsh Transform

- Consist of ± 1 arranged in a checkerboard pattern.
- Transform:

$$W(i) = \frac{1}{N}\sum_{t=0}^{N-1} f(t)\, Wal(i,t)$$

$$f(t) = \sum_{t=0}^{N-1} W(i)\, Wal(i,t)$$

- Types of Wal(i, t).
  - Walsh Ordering (沃尔什定序)
  - Paley Ordering (佩利定序)
  - Hadamard Matrix Ordering (哈达玛矩阵定序)

Hadamard Matrix Ordering

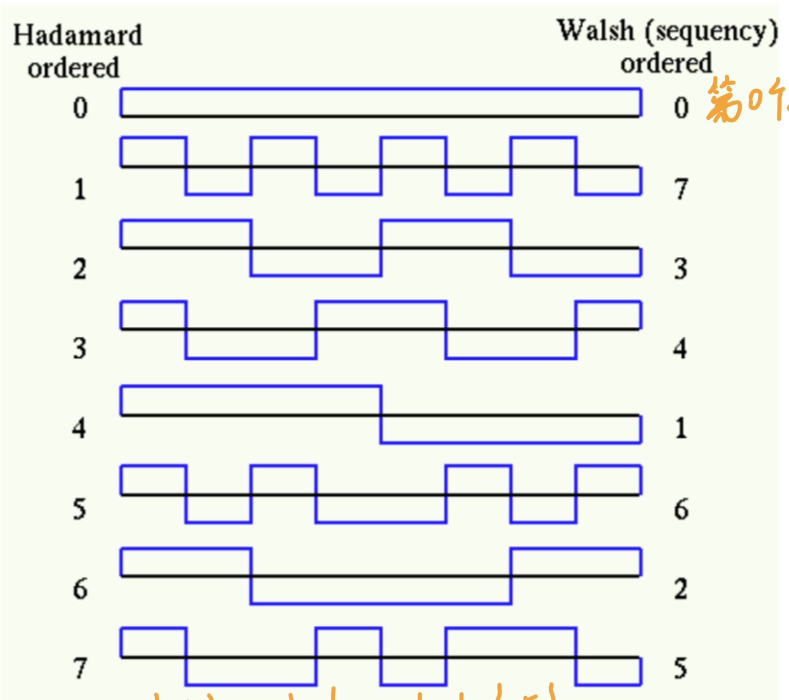
$W_2 W_2^T = I$ $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$

$W_2 = \begin{bmatrix} w_1 & w_1 \\ w_1 & -w_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

要保持能量不变，2消掉

$\times \frac{1}{\sqrt{2}}$

宽度为8



第0个基函数

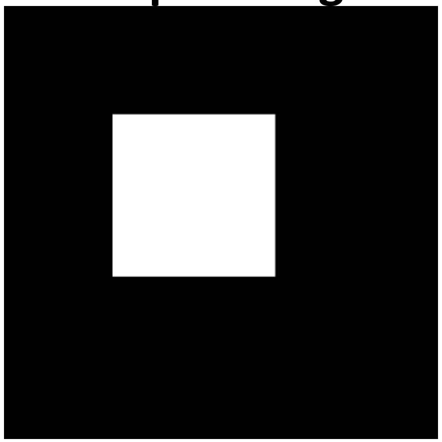
8×8

$$V_8 = \begin{pmatrix} W_4 & W_4 \\ W_4 & -W_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

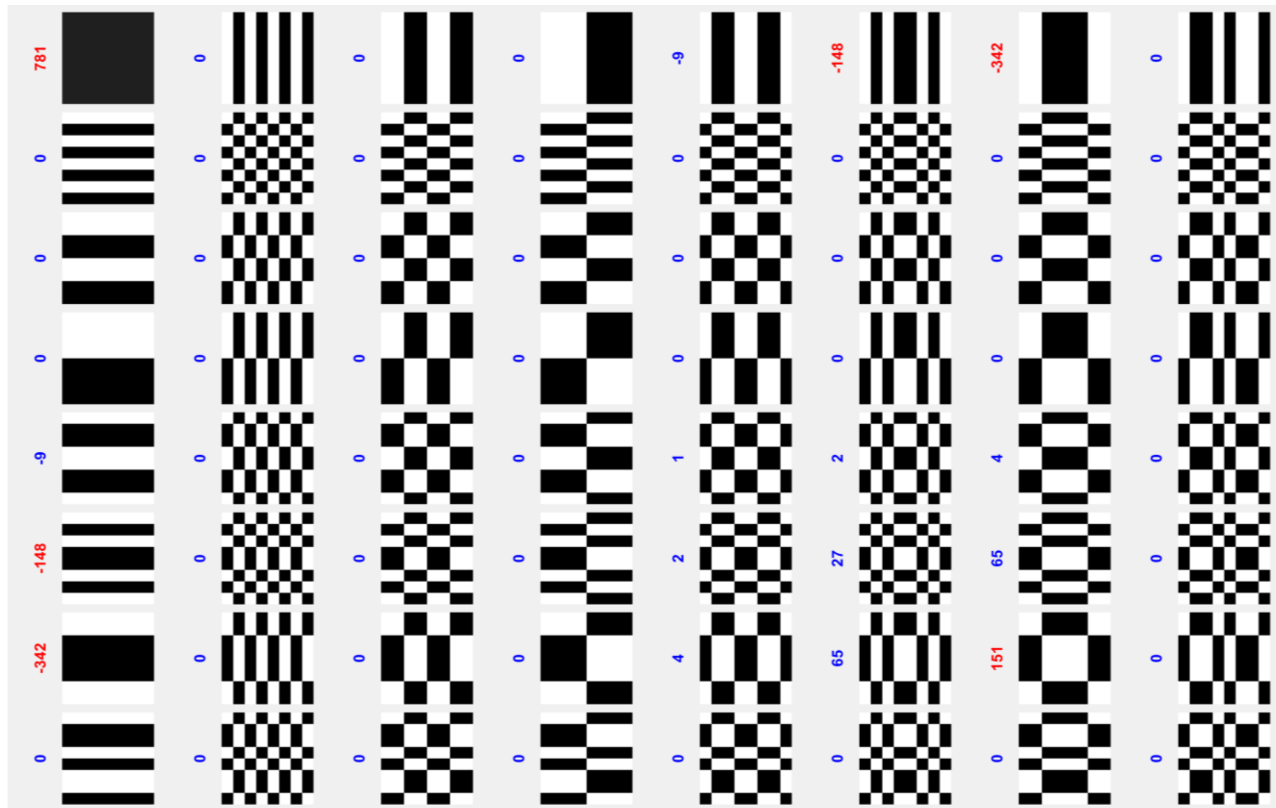
W_4 $-W_4$

Hadarmad Transform

Input image



Hadarmad coefficients



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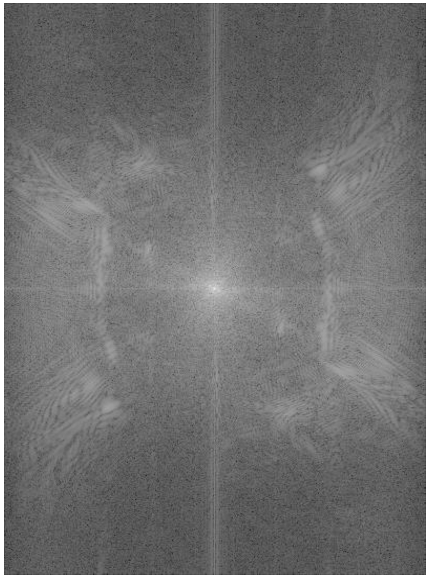
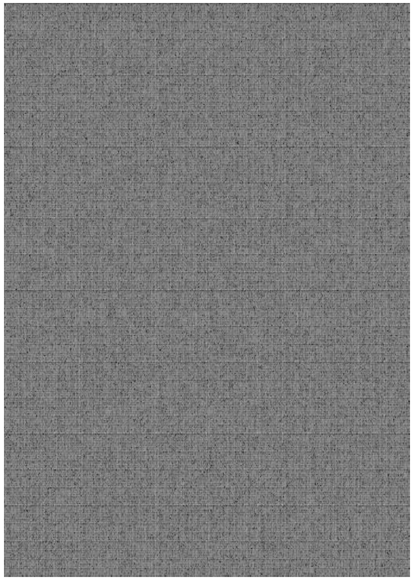
DFT, DCT, & Hadamard

DFT



DCT

Hadamard



Any connection
between DFT and DCT? 某个象限相似

Take home message

- The key idea for unitary transform is to find a proper basis for data decomposition.
- DCT provides better frequency consistency than DFT.
- Hadamard transform is able to present a simple image with simple coefficients. But can not keep energy compact for image full of details.

Wavelet transform Outline

- Discrete Wavelet Transform (DWT)（小波变换）
  - An example for 1D-DWT
  - Generalization of 1D-DWT
  - 2D-DWT

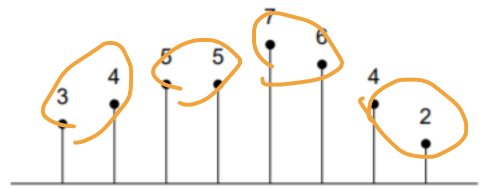
Discrete Wavelet Transform (DWT)

jpeg 压缩，视频压缩

- Based on small waves called Wavelets-1) limited; 2) oscillation.
- Key idea: Translation & Scaling.
- Localized both time/space and frequency.
- Efficient for noise reduction and image compression.
- Two types of DWT one for image processing (easy invertible) and one for signal processing (invertible but computational expensive).

A simple example

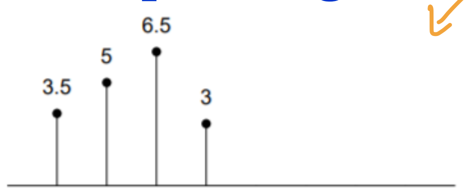
- We can decompose an eight-point signal x(n):



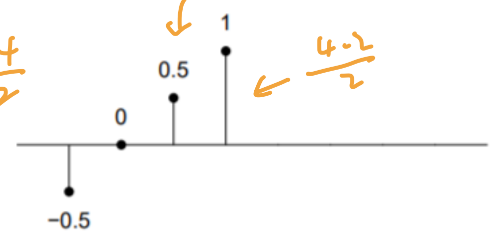
两两求差/2

into two four-point signals: 两两配对平均

$\frac{7-6}{2}$



$\frac{3-4}{2}$



$\frac{4-2}{2}$

$$c(n) = 0.5\,x(2n) + 0.5\,x(2n+1) \qquad d(n) = 0.5\,x(2n) - 0.5\,x(2n+1)$$

$c[n]+d[n] = x[2n]$

$c[n]-d[n] = x[2n+1]$ $\Rightarrow x[n]$

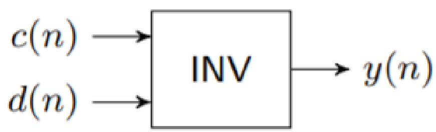
A simple example

- The above process can be represented by a block diagram:

It is clear that this decomposition can be easily reversed:

$$y(2n) = c(n) + d(n)$$
$$y(2n+1) = c(n) - d(n)$$

Which is also represented by a block diagram:

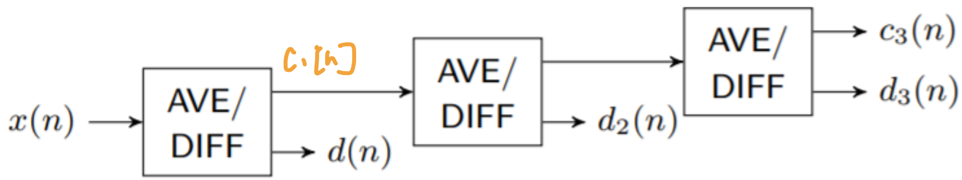


A simple example

t: Af

Forward transform

- When we repeat the simple AVE/DIFF signal decomposition:



$$x(n) = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8] = [3, 4, 5, 5, 7, 6, 4, 2]$$

Level 1

求平均->低频分量

diff->高频分量

$$c_1 = \frac{1}{2}[x_1 + x_2, x_3 + x_4, x_5 + x_6, x_7 + x_8]$$

$$d = d_1 = \frac{1}{2}[x_1 - x_2, x_3 - x_4, x_5 - x_6, x_7 - x_8]$$

Level 2

$$c_2 = \frac{1}{4}[x_1 + x_2 + x_3 + x_4, x_5 + x_6 + x_7 + x_8]$$

c_3+d_3

$$d_2 = \frac{1}{4}[x_1 - x_3 + x_2 - x_4, x_5 - x_7 + x_6 - x_8]$$

c_3-d_3

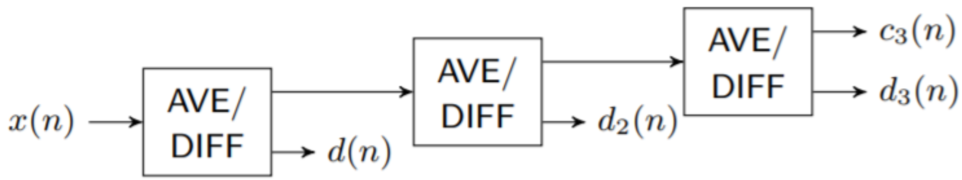
Level 3

$$c_3 = \frac{1}{8}[x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8]$$

$$d_3 = \frac{1}{8}[x_1 + x_2 + x_3 + x_4 - x_5 - x_6 - x_7 - x_8]$$

A simple example

- When we repeat the simple AVE/DIFF signal decomposition:



The Haar wavelet representation of the eight-point signal x[n] is simply the set of four output signals produced by this three-level operation :

$c_3 = H_8 \cdot X = x_1 + \cdots + x_8$

$$c_3 = [4.5]$$
$$d_3 = [-0.25]$$
$$d_2 = [-0.75, \quad 1.75]$$
$$d = [-0.5, \quad 0, \quad 0.5, \quad 1]$$

$$\mathbf{H}_8 = \frac{1}{\sqrt{8}} \begin{array}{cl} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} & \begin{array}{l} c_3 \\ d_3 \\ d_2(1) \\ d_2(2) \\ d(1) \\ d(2) \\ d(3) \\ d(4) \end{array} \end{array}$$

$H_8 \cdot H_8^T$ 保证为单位阵

酉变换

Haar Transform matrix

- **When N=2 we have:**

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- **When N=4 we have:**

$$\mathbf{H}_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

← level 3 的向[illegible]

- **When N=8 we have:**

$$H_4H_4^T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0, & 0, & 0 \\ & 4 & & \\ & & 0 & 4 \end{bmatrix}$$

$$= I$$

$$\mathbf{H}_8 = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Haar Transform matrix

- The family of N Haar functions $h_u(x)$, $(u = 0, \dots, N-1)$ are defined on the interval $0 \le x < 1$. The shape of the specific function $h_u(x)$ of a given index u depends on two parameters p and q:

u: 7 = 4+3 = 2^2+3 (p=2 q=3)

N=4

u=0 [1, 1, 1, 1] = $h_0(x)$

u=1 [1, 1, -1, -1] = $h_1(x)$

u=2 [$\sqrt{2}$, $-\sqrt{2}$, 0, 0] = $h_2(x)$

$$\boldsymbol{u = 2^p + q}$$

u	p	q
1	0	0
2	1	0
3	1	1

- The Haar basis functions are defined by:

$$h_u(x) = \begin{cases} 1 & u = 0 \text{ and } 0 \le x < 1 \\ 2^{p/2} & u > 0 \text{ and } q/2^p \le x < (q+0.5)/2^p \\ -2^{p/2} & u > 0 \text{ and } (q+0.5)/2^p \le x < (q+1)/2^p \\ 0 & \text{otherwise} \end{cases}$$

u=1 => p=0 q=0

=> $0/2^0 = 0$ $0.5/2^0 = 0.5$

$0 \le x < 0.5$

$0.5 \le x < 1$

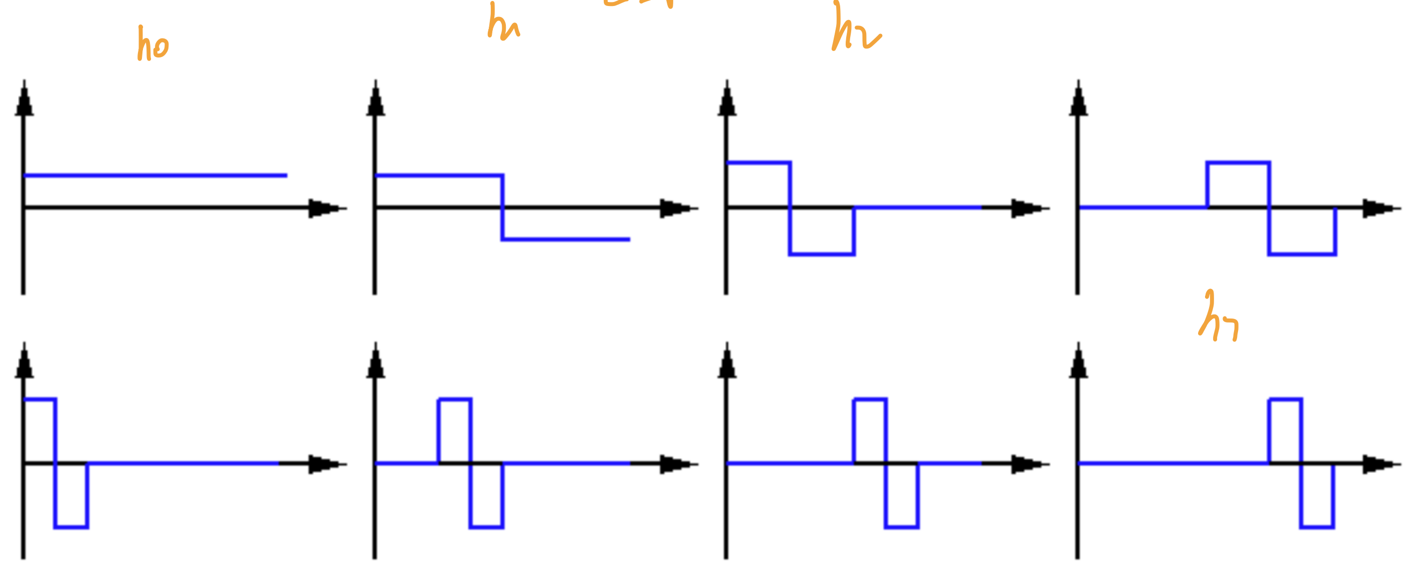
u: 2 => p=1 q=0

$0 \le x < 0.25$

$0.25 \le x \le 0.5$

Haar Transform matrix

把h1平移+尺度变换得其它
[母小波 mother wavelet]



$$
\mathbf{H}_8 = \frac{1}{\sqrt{8}} \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -2
\end{bmatrix}
$$

Generalization of 1D-DWT

- **Discrete Wavelet Transform (DWT):**

小波变换系数
(对应 c和d)

$$W_\varphi(j_0,k) = \frac{1}{\sqrt{M}}\sum_n f(n)\,\underbrace{\varphi_{j_0,k}(n)}_{\text{basis function}}$$

$$W_\psi(j,k) = \frac{1}{\sqrt{M}}\sum_n f(n)\,\psi_{j,k}(n) \qquad j \ge j_0$$

- **Inverse Discrete Wavelet Transform (IDWT):**

$$f(n) = \frac{1}{\sqrt{M}}\sum_k W_\varphi(j_0,k)\,\varphi_{j_0,k}(n) + \frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_k W_\psi(j,k)\,\psi_{j,k}(n)$$

Where

$\varphi_{j_0,k}(n)$: scaling function (尺度函数)

$\psi_{j,k}(n)$: Wavelet (小波)

$W_\varphi(j_0,k)$: Approximation coefficients (近似系数)

[求平均]

$W_\psi(j,k)$: detail coefficients (细节系数)

[求差]

参数里第几个

2D-DWT

- **Define 2D wavelet function: Directionally sensitive wavelet**

y方向求和　　vertical方向细节

$$\psi^H(x,y) = \underbrace{\psi(x)}_{\text{x方向求差}}\varphi(y) \qquad \psi^V(x,y) = \varphi(x)\psi(y) \qquad \psi^D(x,y) = \psi(x)\psi(y)$$

- **2D-DWT**

$$W_\varphi(j_0,m,n) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\varphi_{j_0,m,n}(x,y)$$

$$W_\psi(j,m,n) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\psi_{j,m,n}^i(x,y) \qquad i = \{H,V,D\}$$

- **2D-IDWT**

$$f(x,y) = \frac{1}{\sqrt{MN}}\sum_m\sum_n W_\varphi(j_0,m,n)\,\varphi_{j_0,m,n}(x,y)$$

$$+\frac{1}{\sqrt{MN}}\sum_{i=\{H,V,D\}}\sum_{j=j_0}^{\infty}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} W_\psi(j,m,n)\,\psi_{j,m,n}^i(x,y)$$

Haar Transform matrix

Input: image size 8X8 I_{in}

Generate a Haar matrix of 8X8 as shown right
Then clip it into 4 part:

$$H_{L1}(dim = 4 * 8); H_{L2}(dim = 2 * 8);$$
$$H_{L3}(dim = 1 * 8); L_{L3}(dim = 1 * 8);.$$

$$\mathbf{H}_8 = \frac{1}{\sqrt{8}} \begin{array}{c} \varphi \\ \psi \\ \\ \\ \\ \\ \\ \\ \end{array} \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \hline \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ \hline 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{array}\right] \begin{array}{l} \\ H_{L3} \\ H_{L2} \\ \\ \\ H_{L1} \\ \\ \\ \end{array}$$

For computing *level 1* components:

分解

LL_1 is down-sample of I_{in} on both X and Y direction	HL_1=H_{L1}*I_{in} + downsample on Y direction
LH_1= $I_{in} * H_{L1}$ + downsample on X direction	HH_1= H_{L1}*$I_{in} * H_{L1}$

diff不管

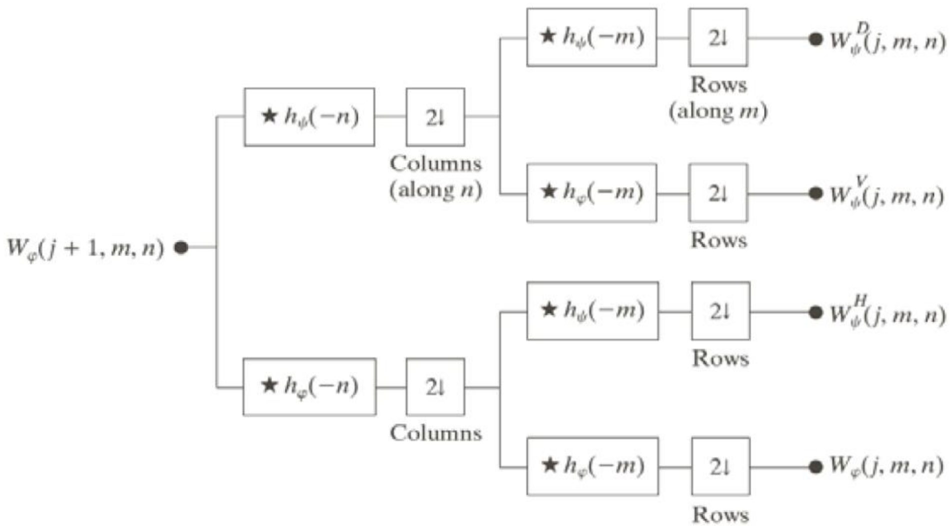
For computing *level 2* components:

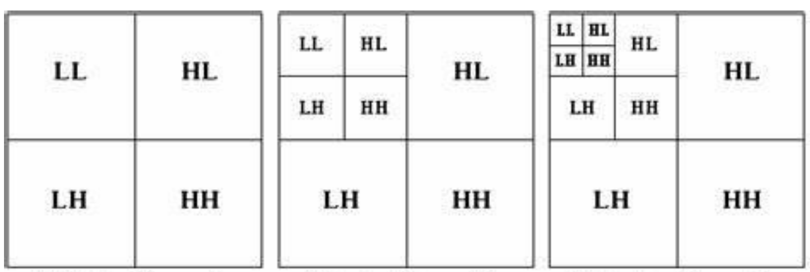
LL_2 is down-sample of LL_1 on both X and Y direction	HL_2=H_{L2}*I_{in} + downsample **twice** on Y direction
LH_2= $I_{in} * H_{L2}$ + downsample **twice** on X direction	HH_2= H_{L2}*$I_{in} * H_{L2}$

For computing *level 3* components:

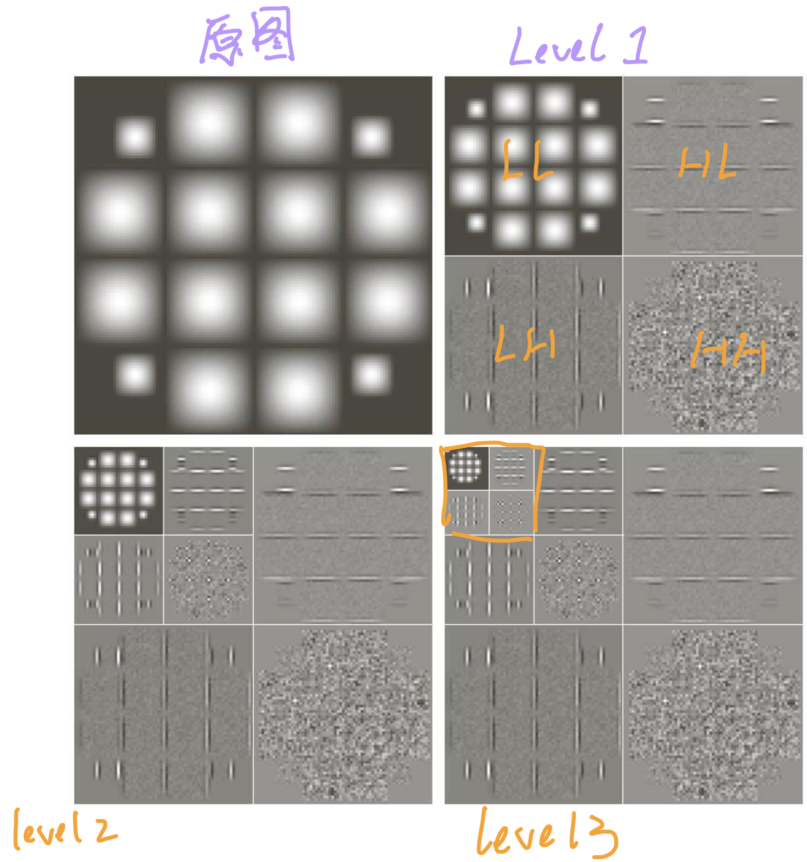
LL_3 is down-sample of LL_2 on both X and Y direction	HL_3=H_{L3}*I_{in} + downsample **3 times** on Y direction
LH_3= $I_{in} * H_{L3}$ + downsample **3 times** on X direction	HH_3= H_{L3}*$I_{in} * H_{L3}$

2D-DWT





(a) Single Level Decomposition (b) Two Level Decomposition (c) Three Level Decomposition



2D Haar Transform

8×8

$$A = \begin{pmatrix} 88 & 88 & 89 & 90 & 92 & 94 & 96 & 97 \\ 90 & 90 & 91 & 92 & 93 & 95 & 97 & 97 \\ 92 & 92 & 93 & 94 & 95 & 96 & 97 & 97 \\ 93 & 93 & 94 & 95 & 96 & 96 & 96 & 96 \\ 92 & 93 & 95 & 96 & 96 & 96 & 96 & 95 \\ 92 & 94 & 96 & 98 & 99 & 99 & 98 & 97 \\ 94 & 96 & 99 & 101 & 103 & 103 & 102 & 101 \\ 95 & 97 & 101 & 104 & 106 & 106 & 105 & 105 \end{pmatrix}$$

$A' = \begin{bmatrix} 91.75 & -3 & -0.75 & \cdots & -0.5 \\ \vdots & \vdots & \vdots & & \end{bmatrix}$

竖直方向分解(每列)

next page

3-level Haar Transform for the first row

$r_1 = (88 \quad 88 \quad 89 \quad 90 \quad 92 \quad 94 \quad 96 \quad 97)$

$\frac{89+90}{2}$

Group r_1 in pair $[88, 88], [89, 90], [92, 94], [96, 97]$

$r_1 h_1 = (88 \quad 89.5 \quad 93 \quad 96.5 \quad 0 \quad -0.5 \quad -1 \quad -0.5)$

Approximation coefficients **Detail coefficients**

近似 细节

copy 下来

Group the first 4 columns in pair $[88, 89.5], [93, 96.5]$

$r_1 h_1 h_2 = (88.75 \quad 94.75 \quad -0.75 \quad -1.75 \quad 0 \quad -0.5 \quad -1 \quad -0.5)$

近似 细节

Group the first 2 columns in pair $[\overset{88.75}{\cancel{88}}, 94.75]$

$r_1 h_1 h_2 h_3 = (91.75 \quad -3 \quad -0.75 \quad -1.75 \quad 0 \quad -0.5 \quad -1 \quad -0.5)$

detail

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2D Haar Transform

Repeat the same processing for all the columns and for the rows of the resulting matrix, we get

$$
\begin{pmatrix}
96 & -2.0312 & -1.5312 & -0.2188 & -0.4375 & -0.75 & -0.3125 & 0.125 \\
-2.4375 & -0.0312 & 0.7812 & -0.7812 & 0.4375 & 0.25 & -0.3125 & -0.25 \\
-1.125 & -0.625 & 0 & -0.625 & 0 & 0 & -0.375 & -0.125 \\
-2.6875 & 0.75 & 0.5625 & -0.0625 & 0.125 & 0.25 & 0 & 0.125 \\
-0.6875 & -0.3125 & 0 & -0.125 & 0 & 0 & 0 & -0.25 \\
-0.1875 & -0.3125 & 0 & -0.375 & 0 & 0 & -0.25 & 0 \\
-0.875 & 0.375 & 0.25 & -0.25 & 0.25 & 0.25 & 0 & 0 \\
-1.25 & 0.375 & 0.375 & 0.125 & 0 & 0.25 & 0 & 0.25
\end{pmatrix}
$$

x方向做完的基础上再在y方向分解　类似傅立叶

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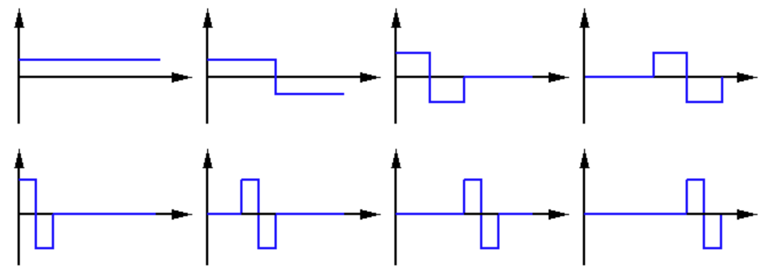
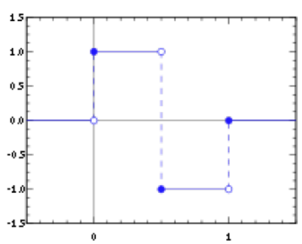
ShanghaiTech University

Mother Wavelet (母小波)

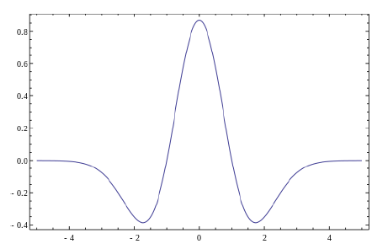
- **Mother Wavelet should satisfy:**
  - $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$
  - $\int_{-\infty}^{\infty} |\psi(t)| dt < \infty$
  - $\int_{-\infty}^{\infty} \psi(t) dt = 0$ 不改变信号本身自带能量



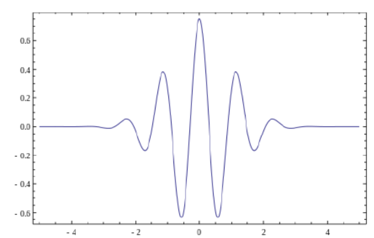
Haar



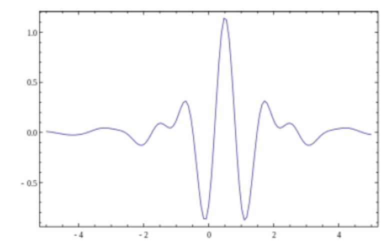
Mexican Hat



Morlet



Meyer



Take home message

- Based on small waves called 摆动 Wavelets-1) limited; 2) oscillation.
- Key idea: Translation & Scaling.
- Localized both time/space and frequency.
- Efficient for noise reduction and image compression.
- JPEG2000, FBI finger printing databased.