Lecture 5
Frequency Domain Filtering (1)

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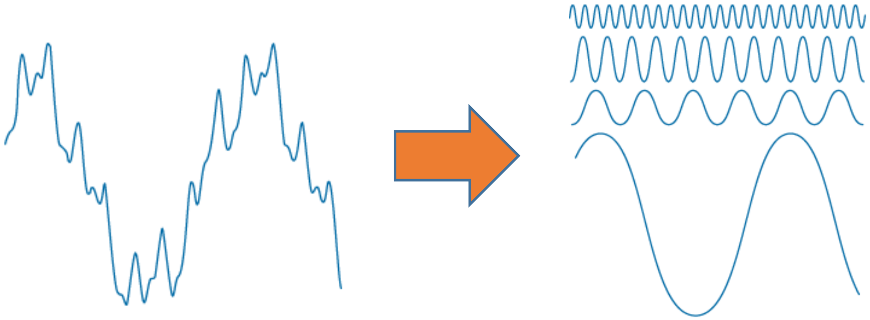
Outline

- Why Fourier Transform in DIP
- Introduction (recap) of Fourier Transform (CTFT, DTFT, DFT)
- Discrete Fourier Transform and its Inverse (1D & 2D)
- Sampling Theorem & Aliasing (1D & 2D)
- Properties of 2D DFT and IDFT
- 2D DFT implementation

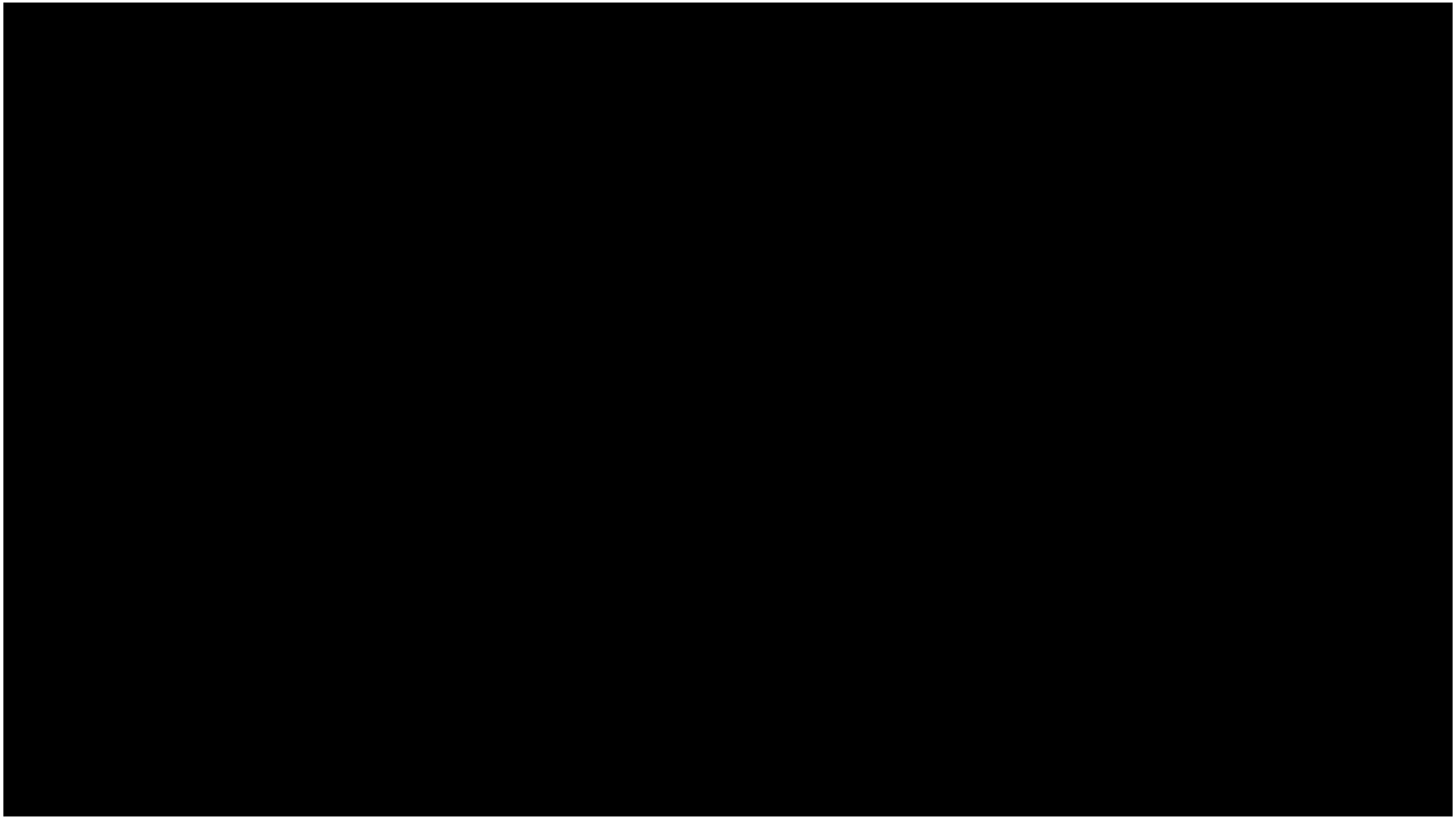
Jean Baptiste Joseph Fourier (1768-1830)

- **Had a crazy idea (1807)**
  - Any periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies
- Who didn’t believe it?
  - Lagrange, Laplace, Poisson…
  - Not translated into English until 1878!
- But it’s true!
  - We call it Fourier Series



Fourier Series

https://www.youtube.com/watch?v=cUD1gMAl6W4

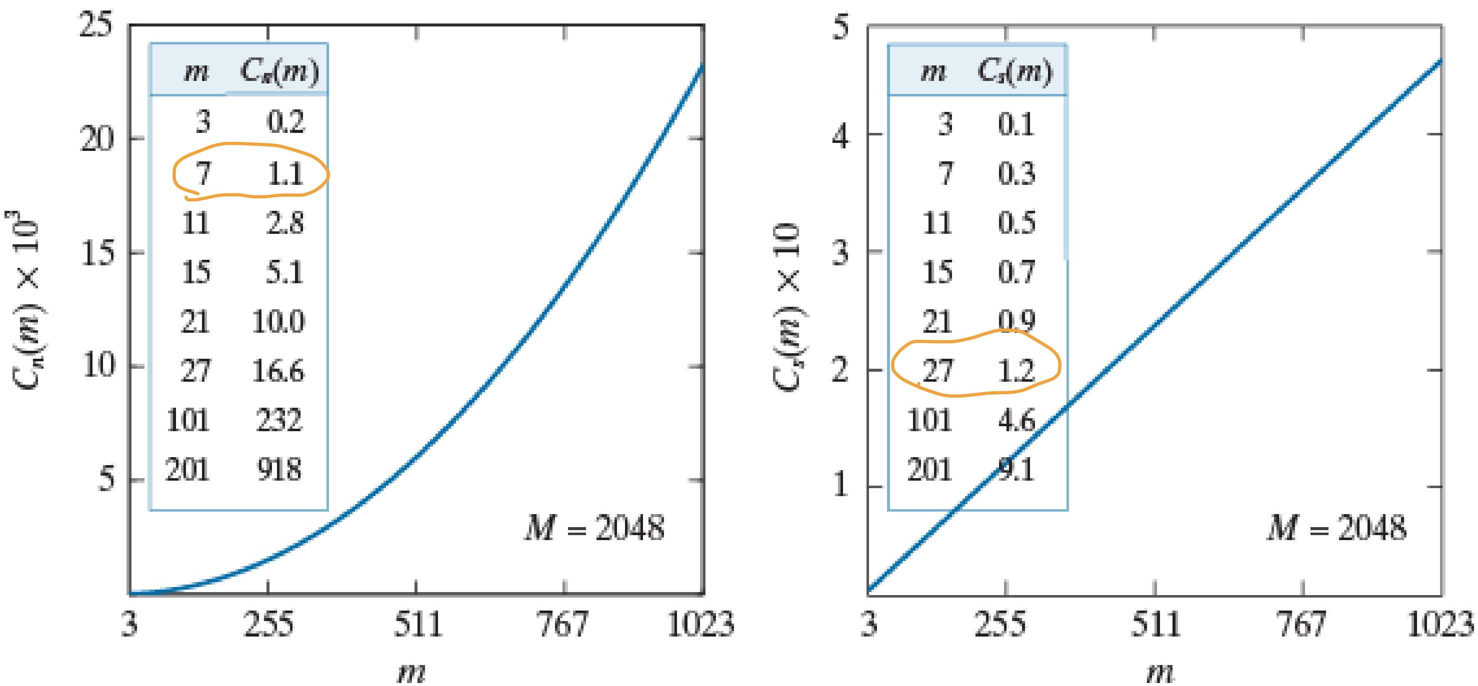


Why Fourier Transform in DIP

- Exposing image features not visible in spatial domain
- Image filtering in frequency domain (sometimes even faster!)
- Image compression or encoding
- Image reconstruction or recovery
- Classification
- …
- *Or simply to provide another perspective to look at the image*

Why Fourier Transform in DIP

❑ The importance of FFT



a b

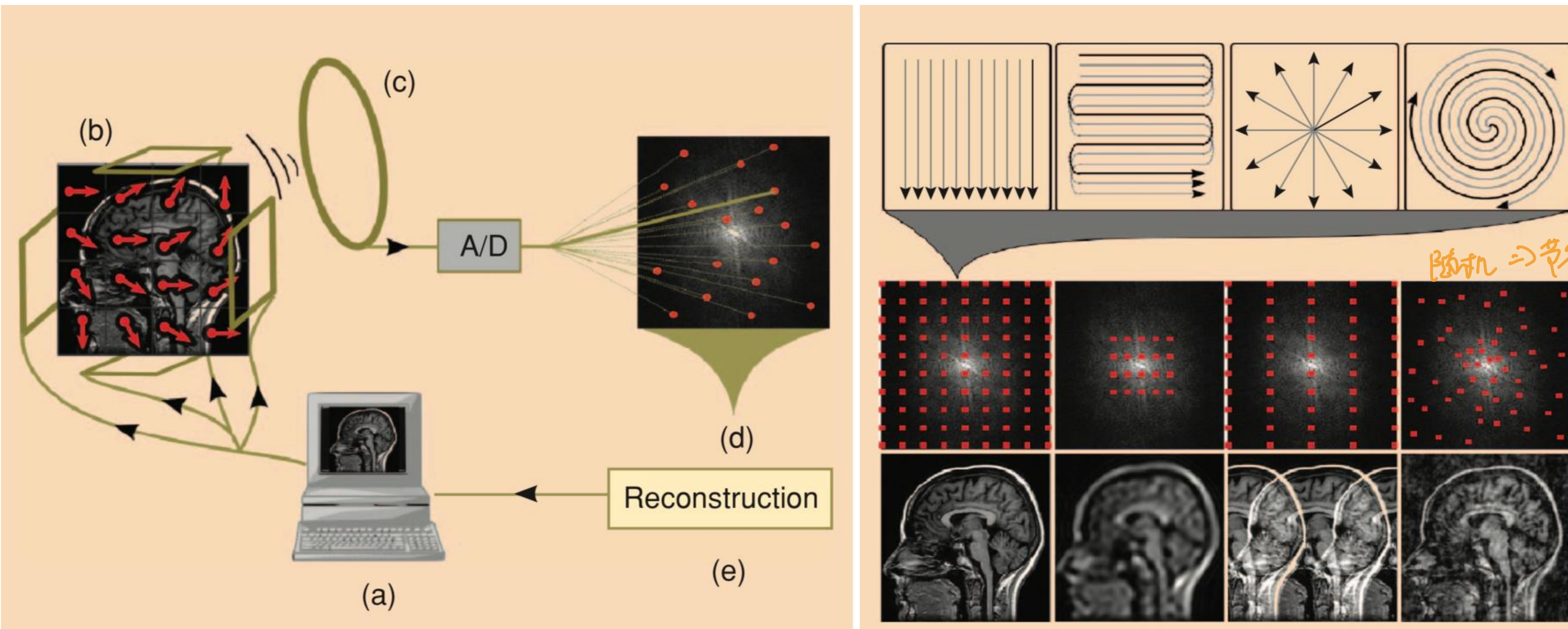
FIGURE 4.2 (a) Computational advantage of the FFT over non-separable spatial kernels. (b) Advantage over separable kernels. The numbers for $C(m)$ in the inset tables are not to be multiplied by the factors of 10 shown for the curves.

$$\frac{MNmn}{n}$$

$$2MN\log_2 MN$$

Why Fourier Transform in DIP

❑ A great example: Fourier Transform in MRI



随机 → 节省运算量

The Nyquist criterion sets the required k-space coverage, which can be achieved using various sampling trajectories.

IEEE SIGNAL PROCESSING MAGAZINE, 72, MARCH 2008

Introduction (recap) of Fourier Transform

- **Fourier series**

求系数

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(nt) + b_n \sin(nt)]$$

- **Continuous-time Fourier transform (CTFT)**

频率连续

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

- **Discrete-time Fourier Transform (DTFT)**

$$F(\omega) = \sum_{n=1}^{\infty} x[n]\, e^{-j\omega n}, \omega \in [0, 2\pi], \omega = 2\pi/T$$

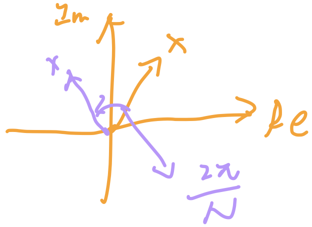
- **Discrete Fourier Transform (DFT)** ← 数字图像处理

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}}, (k = 0, 1, 2, \cdots, N-1)$$

Four Types of Fourier Transform

	Time Domain Non-Periodic	Time Domain Periodic
Frequency Domain Non-Periodic	CTFT (Both domains are continuous)	Fourier Series (Time domain continuous, Frequency domain discrete)
Frequency Domain Periodic	DTFT (Time domain discrete, Frequency domain continuous)	DFT (Both domains are discrete, and finite duration)

DFT and IDFT



DFT: $$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$= X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N}, \qquad k = 0, 1, \dots, N-1$$

Using: $\boldsymbol{W_N = e^{-j2\pi/N}}$, we can rewrite the DFT as

W_N 旋转因子

$$X[k] = \sum_{n=0}^{N-1} \underbrace{x[n]}_{} W_N^{kn}, \qquad k = 0, 1, \dots, N-1.$$

复数

IDFT: $$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad k = 0, 1, \dots, N-1.$$

2D DFT and IDFT

- **2D Discrete Fourier Transform (DFT)**

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

- **2D Inverse Discrete Fourier Transform (IDFT)**

$$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

  - $f(x,y)$: M*N input image
  - (x,y): spatial variables, $(x=0,1,2,\cdots,M-1; y=0,1,2,\cdots,N-1)$
  - (u,v): frequency variables, defines the continuous frequency domain $(u=0,1,2,\cdots,M-1; v=0,1,2,\cdots,N-1)$.
  - **Examples of basis function.**

Separability

2D DFT to 1D DFT

两个不同方向上的1D

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}=\sum_{x=0}^{M-1} e^{-j2\pi\frac{ux}{M}}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\frac{vy}{N}}$$

$$=\mathcal{F}_x\{\mathcal{F}_y\{f(x,y)\}\}$$

Calculate IDFT by DFT

basis function

$$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\boxed{e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}}$$

复数取共轭

共轭

$$MNf^*(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^*(u,v)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

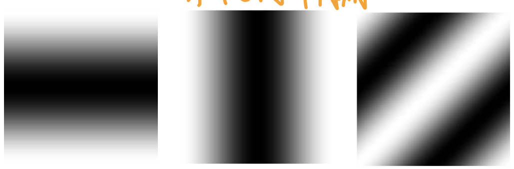
Basis Functions in 2D DFT

- [0,0]: Constant

在水平方向intensity常数，y方向上以1周期

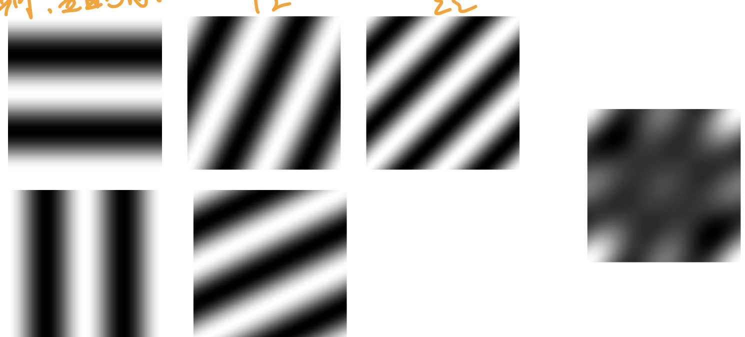
- [0,1] [1,0] [1,1]

水平方向一个周期



- [0,2] [2,0] [1,2] [2,1] [2,2]

2个周期，竖直方向



12 22

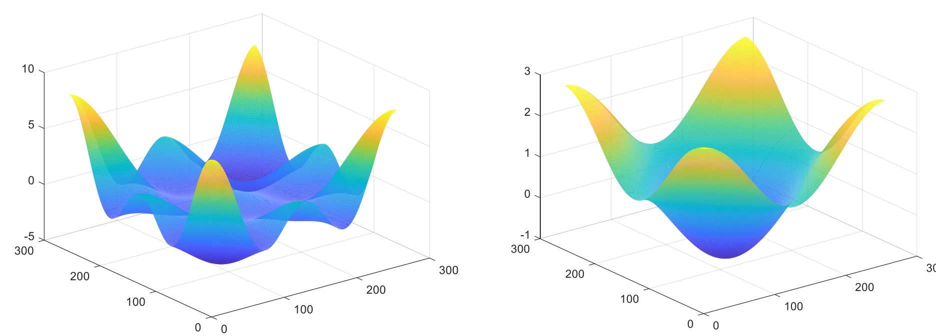
水平 21

```
function im = bf(m,n)

N = 256;
[x,y] = meshgrid(0:(N-1),0:(N-1));
im = real(exp(-j*2*pi*(m*x/N +n*y/N)));

if (m==0) && (n==0)
    im = round(im);
end

figure; imshow(im,[]);
```



Spectrum and Phase angle

2D DFT in polar form: $F(u,v) = |F(u,v)|e^{-j\Phi(u,v)}$ **, then**

某一基函数系数 实 虚

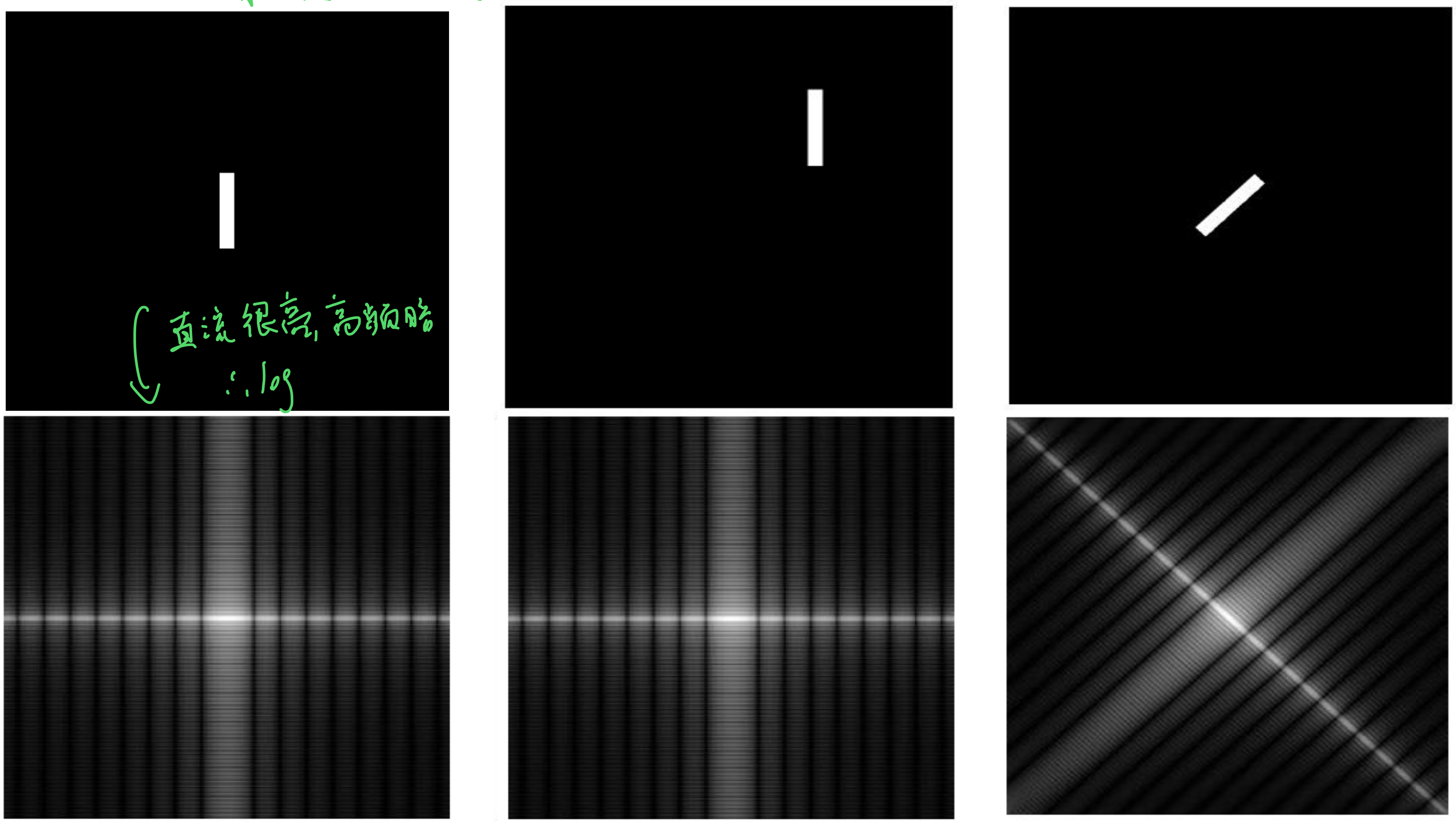
- **Fourier spectrum（频谱）:** $|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{\frac{1}{2}}$
- **Phase angle（相角）:** $\Phi(u,v) = \arctan\frac{I(u,v)}{R(u,v)}$
- **Power spectrum(功率谱):** $P(u,v) = |F(u,v)|^2$
- **DC component(直流分量):** $F(0,0) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = MN\overline{f(x,y)}$

Fourier Spectrum（频谱）

f(t)

sinc 函数

脉冲越短主瓣越高

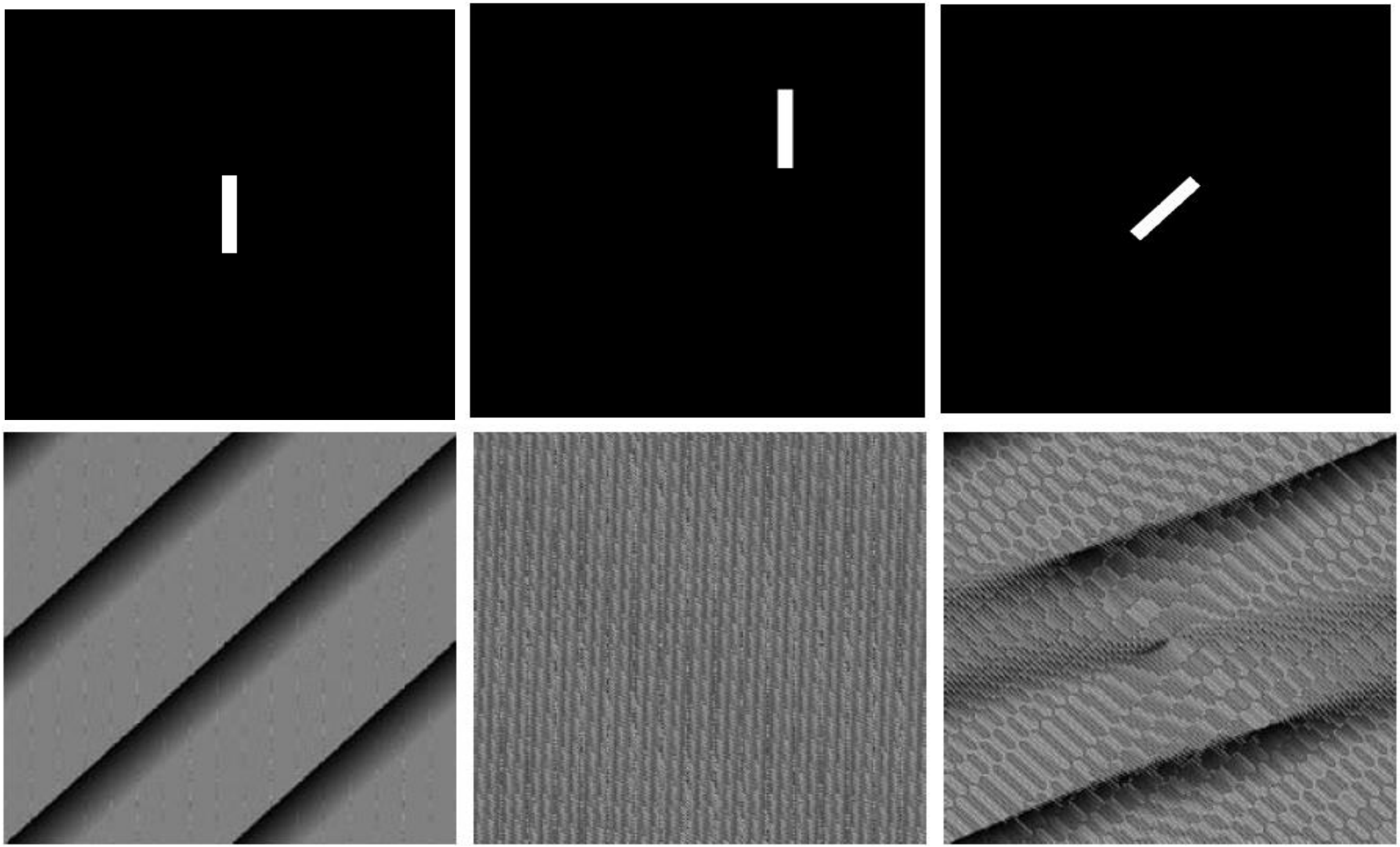


直流很亮，高频暗

∴ log

Phase angle（相角）

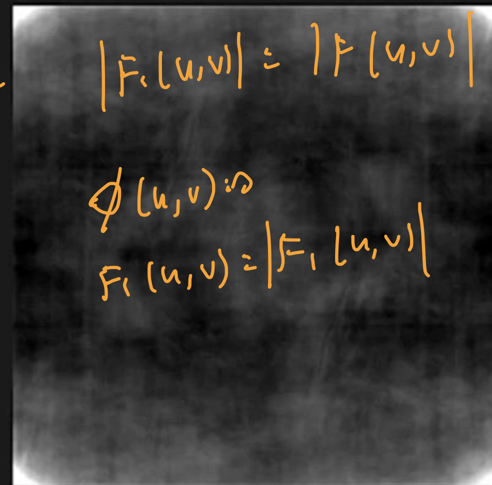
反映空间位置所在信息



The Importance of Phase

$|F_2(u,v)| = F(0,0)$

$\Leftrightarrow F(u,v) =$



$|F_1(u,v)| = |F(u,v)|$

$\phi(u,v) = 0$

$F_1(u,v) = |F_1(u,v)|$

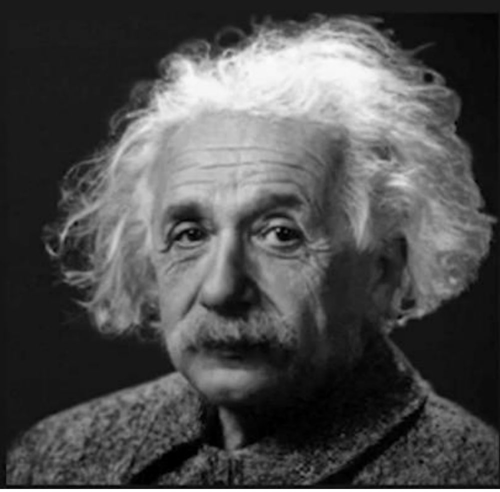
$\phi_2(u,v) = \phi(u,v)$

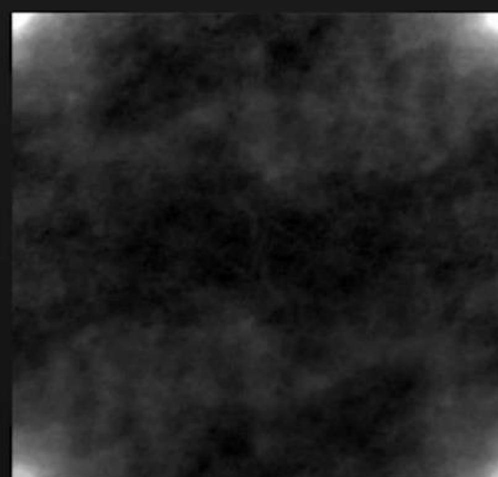
$F_2(u,v) = F(0,0)\, e^{i\phi(u,v)} / MN$

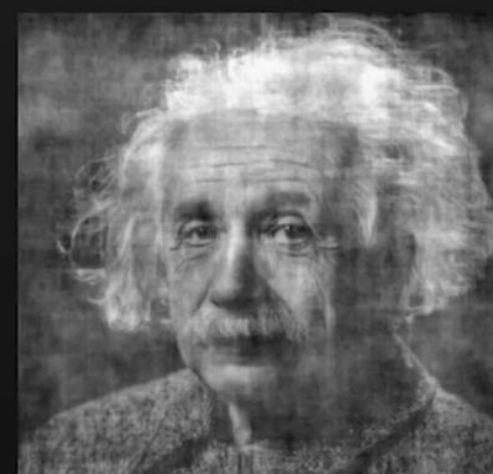
Original Image

Magnitude Preserved, Phase Set to Zero

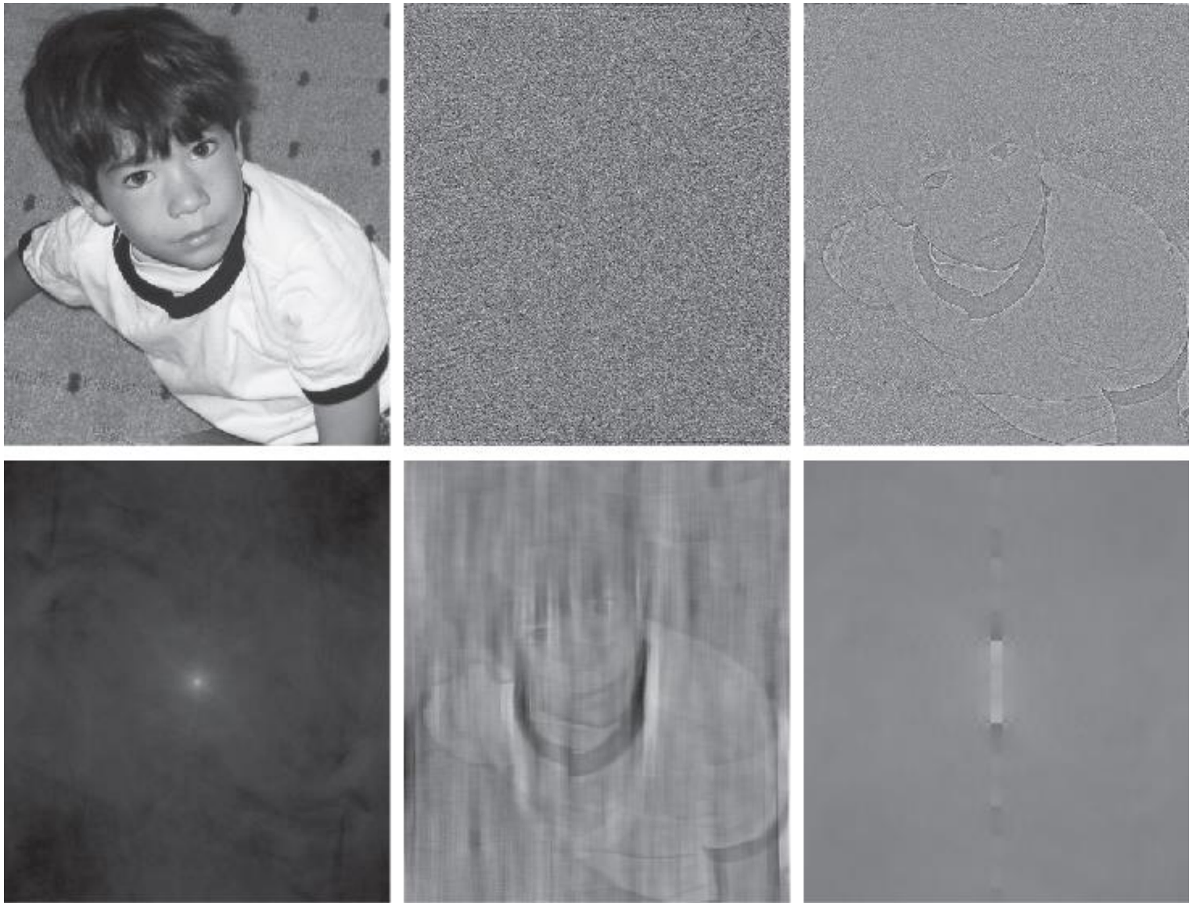
Phase Preserved, Magnitude Set to Average of Natural Images







Spectrum and Phase angle



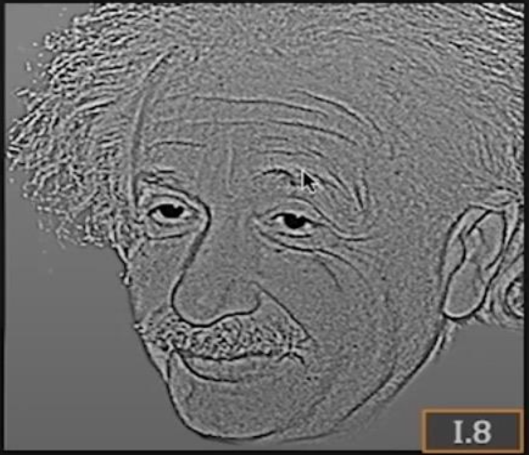
a b c
d e f

相角

FIGURE 4.26 (a) Boy image. (b) Phase angle. (c) Boy image reconstructed using only its phase angle (all shape features are there, but the intensity information is missing because the spectrum was not used in the reconstruction). (d) Boy image reconstructed using only its spectrum. (e) Boy image reconstructed using its phase angle and the spectrum of the rectangle in Fig. 4.23(a). (f) Rectangle image reconstructed using its phase and the spectrum of the boy's image.

Monroe or Einstein?

Low Freq Only



High Freq Only

物体轮廓 → 低频
特征 → 高频

Hybrid (Sum) Image

Coding task 1

1. Load image vallay-house2.jpeg.
2. fft_im = fft2(im);
3. Try to show the magnitude of DFT.

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

Coding task 1

0 - N-1

$[0 - \frac{f_s}{2}, -\frac{f_s}{2} : -0]$

$F(u,v)$

0频率成分

把0频率成分移到$\frac{N}{2}$

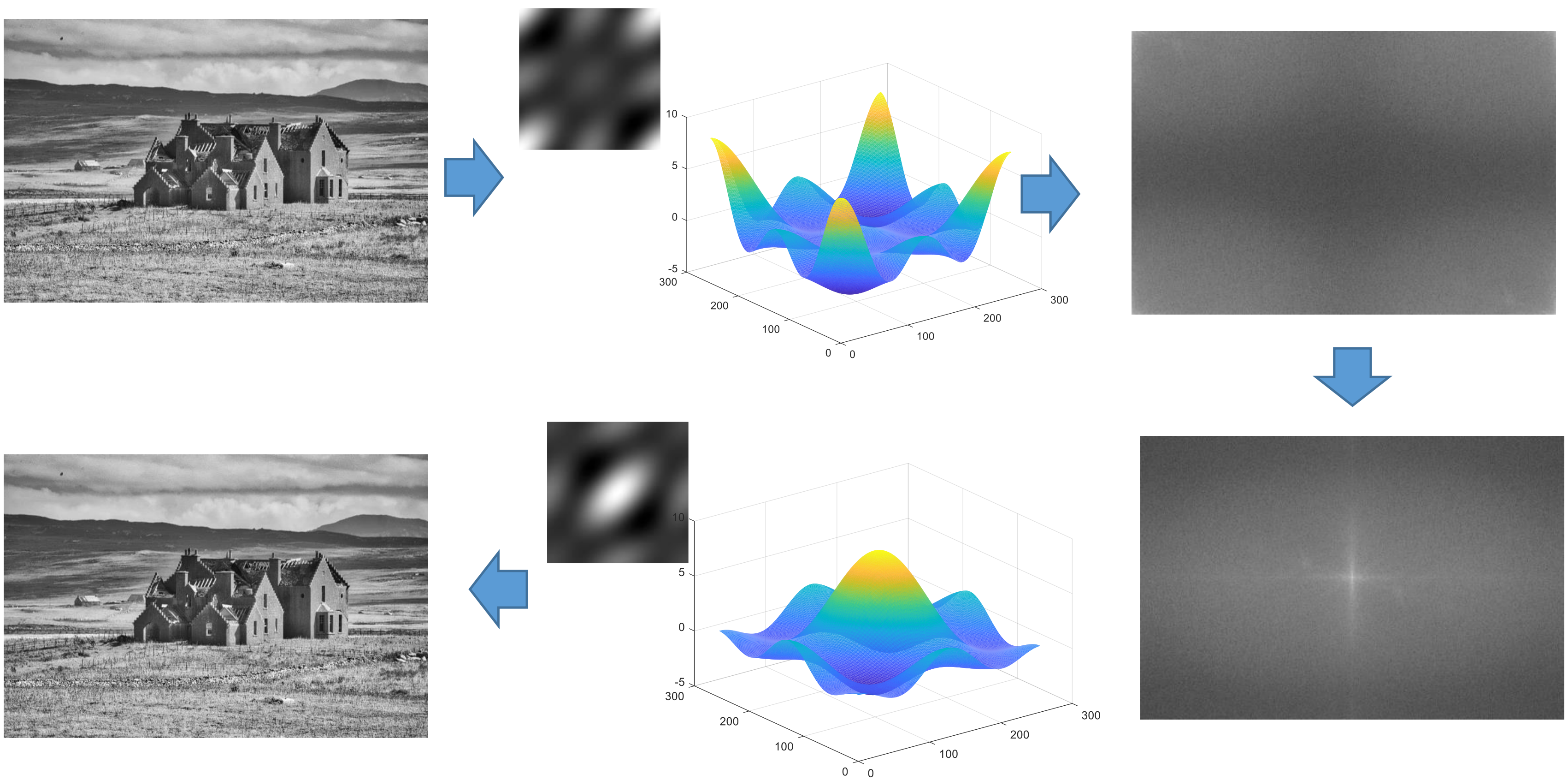
$F(u-\frac{M}{2}, v-\frac{N}{2})$

- Load image vallay-house2.jpeg.
  - fft_im = fft2(im);

离散 Fourier 变换后

- Try to show the magnitude of DFT.
  - fft_im_shifted = fftshift(fft_im);
  - im_recover = ifft2(fft_im);
- Try to show the inverse DFT image.

Visualization of DFT



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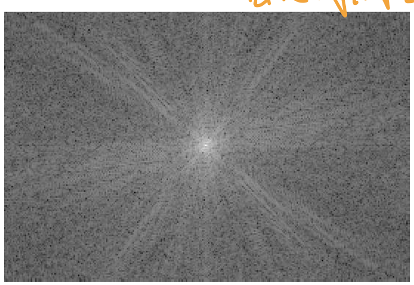
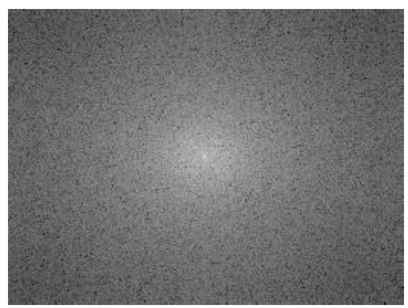
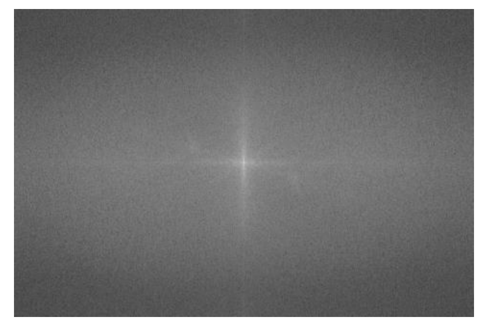
Visualization of DFT

固定周期图像特征





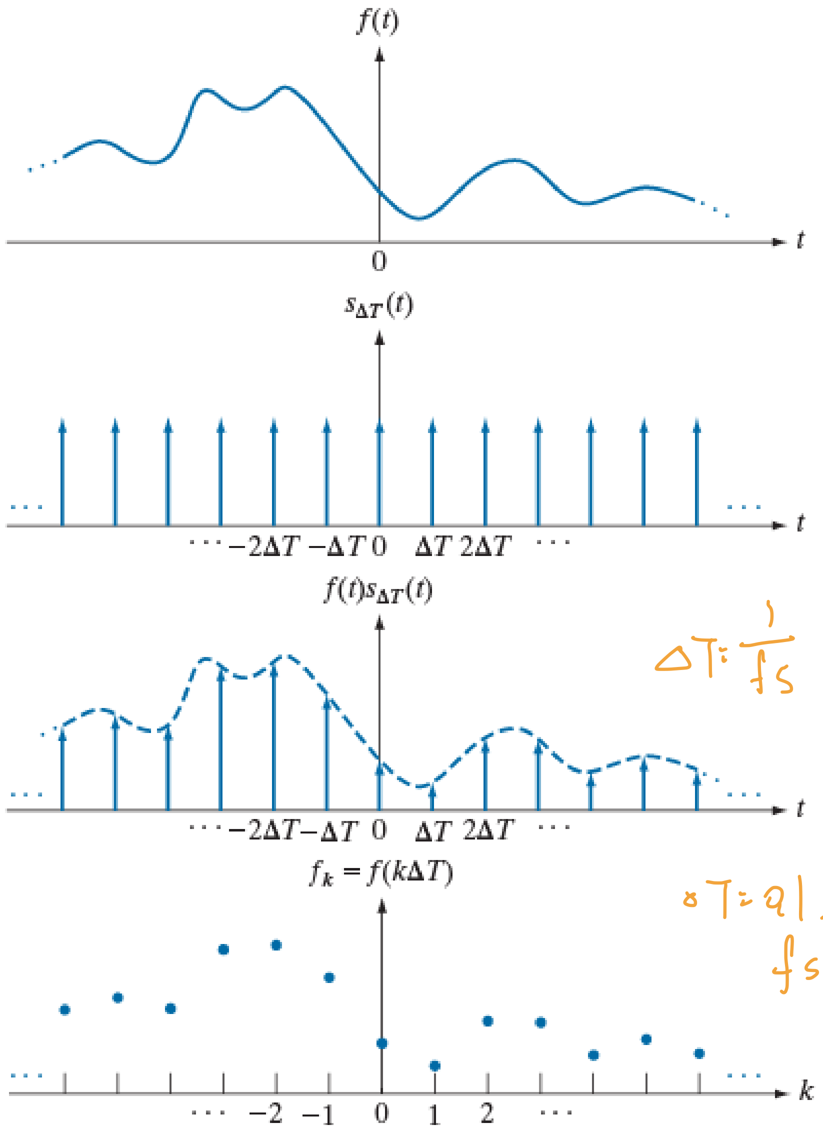
高频也像空间排布杂乱



1D Sampling

采样

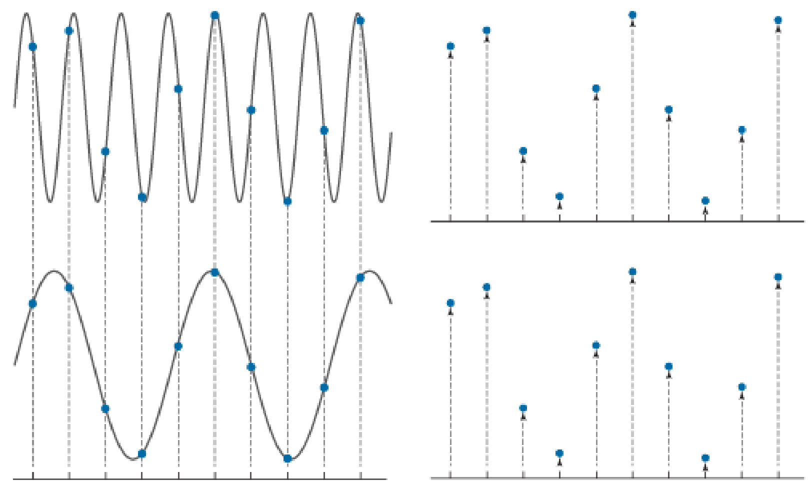
Nyquist → 最高频率 2倍以上



$\Delta T = \frac{1}{f_s}$

$\Delta T = 0.1\,s$

$f_s = 10\,Hz$



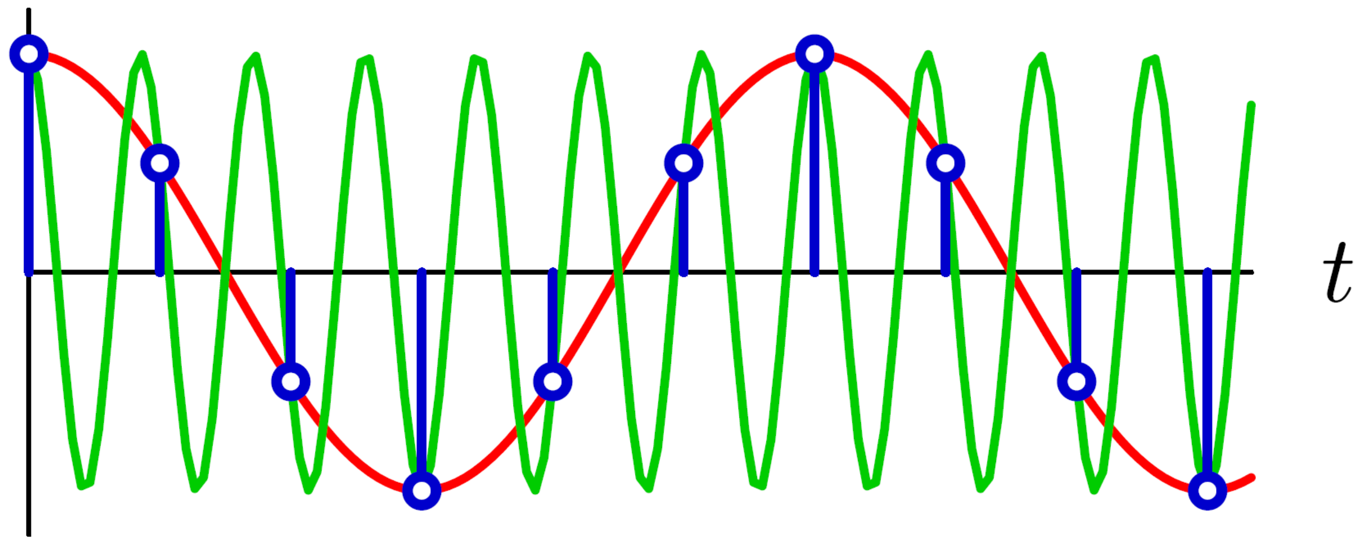
Aliasing

信号合起来，用蓝色采样一样

$\Rightarrow$ 不能还原 origin

$\cos \frac{7\pi}{3} n$?　　$\cos \frac{\pi}{3} n$?

$\Rightarrow$ 混叠



1D Sampling Theorem

$f(t) \xleftrightarrow{FT} F(\mu)$

$\hat{f}(t) = f(t) S_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t-k\Delta T)$

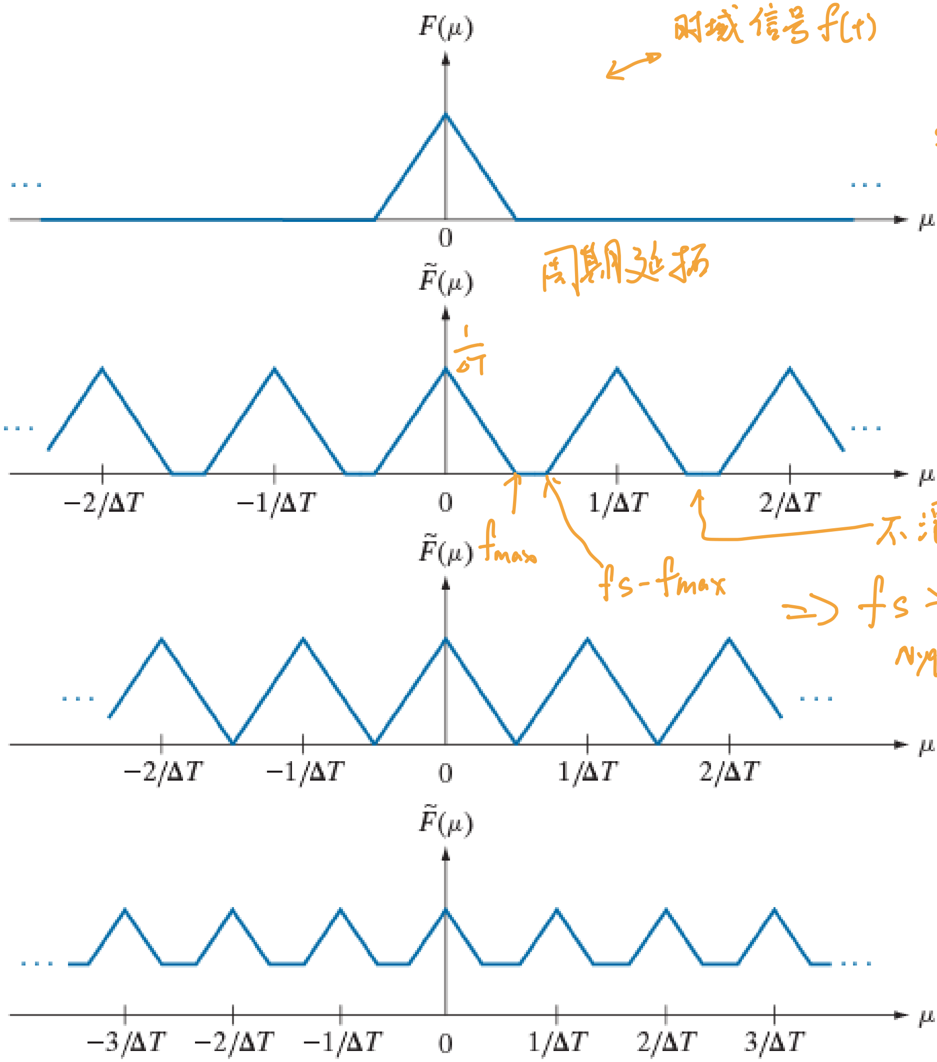
$S_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} \delta(t-k\Delta T) \xleftrightarrow{FT} \frac{1}{\Delta T}\sum_{k=-\infty}^{\infty}\delta(\mu-\frac{k}{\Delta T})$

$\mu-\tau-\frac{k}{\Delta T}=0$

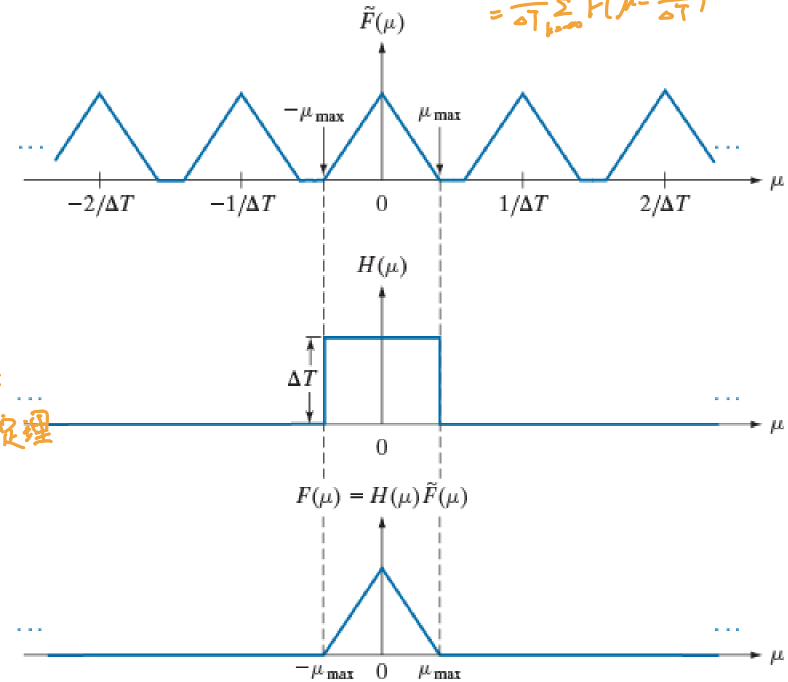
$\tau=\mu-\frac{k}{\Delta T}$

$\hat{F}(\mu) = \mathcal{F}[\underset{时}{f(t)S_{\Delta T}(t)}] = \underset{频}{F(\mu) * S(\mu)} = \int_{-\infty}^{\infty} F(\tau)S(\mu-\tau)d\tau = \frac{1}{\Delta T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}F(\tau)\delta(\mu-\tau-\frac{k}{\Delta T})d\tau$

$= \frac{1}{\Delta T}\sum_{k=-\infty}^{\infty} F(\mu-\frac{k}{\Delta T})$



Non-aliasing v.s. Aliasing

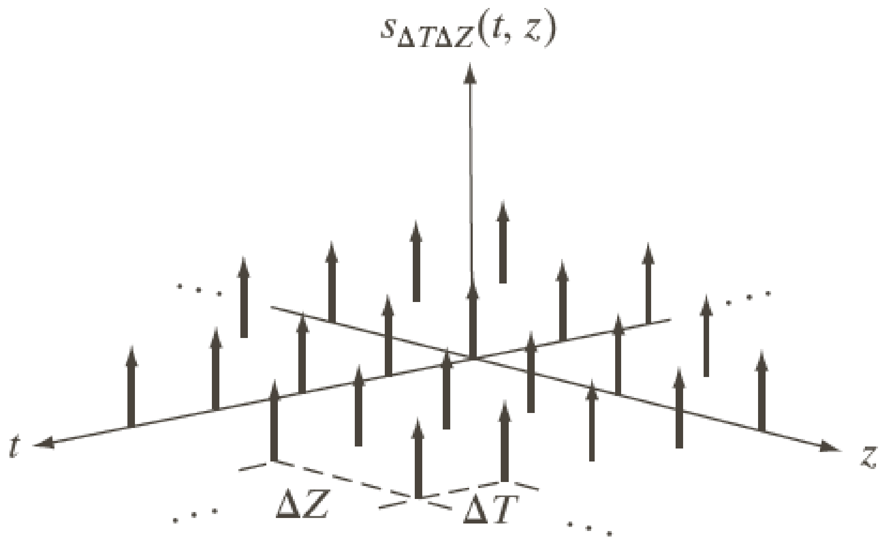


Recovery of the signal

2D Sampling

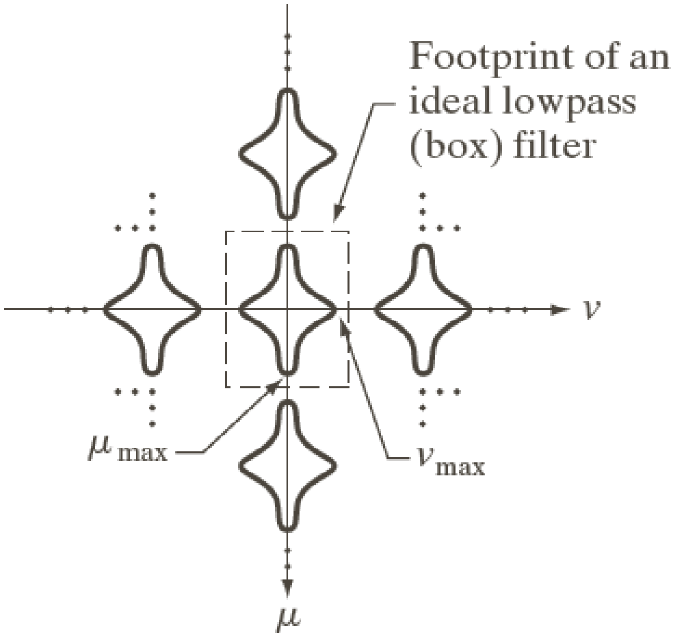
2D Sampling function (二维取样函数)

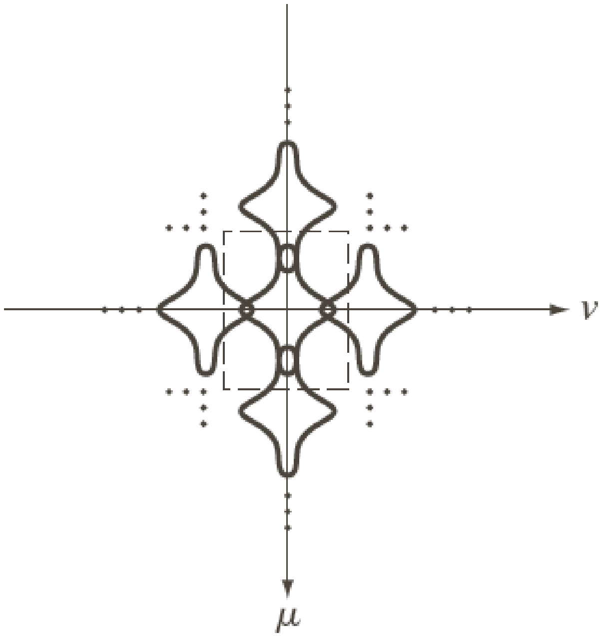
$$s_{\Delta T \Delta Z}(t,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$



2D Sampling Theorem

- $f(t,z)$ is band-limited (带限函数) if $F(\mu,\nu)=0,\ |\mu|\geq\mu_{\max}$ and $|\nu|\geq\nu_{\max}$
- The sampling rate: $\frac{1}{\Delta T}>2\mu_{\max}$, $\frac{1}{\Delta Z}>2\nu_{\max}$





Spatial Aliasing（空间混淆）

down sample 2 times

高频信息丢失（混叠）

3x3 均值滤波

纹理方向不正确

Properties of 2D DFT and IDFT

- Translation （平移）
- Periodicity（周期性）
- Rotation（旋转）
- Separability (可分性）
- Symmetry（对称性）
- 2D Convolution theorem（卷积定理）

Translation（平移）

Translation

$f(x,y) \overset{FT}{\longleftrightarrow} F(u,v)$

$$f(x,y)e^{j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)}$$

When $u_0 = \frac{M}{2}, v_0 = \frac{N}{2}$

fftshift

$$F\left(u-\frac{M}{2}, v-\frac{N}{2}\right) \Leftrightarrow f(x,y)e^{j\pi(x+y)} = f(x,y)(-1)^{(x+y)}$$

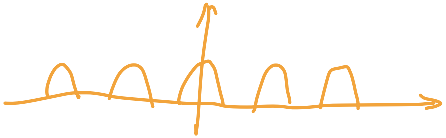
$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)}e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{u-u_0}{M}x+\frac{v-v_0}{N}y\right)}$$

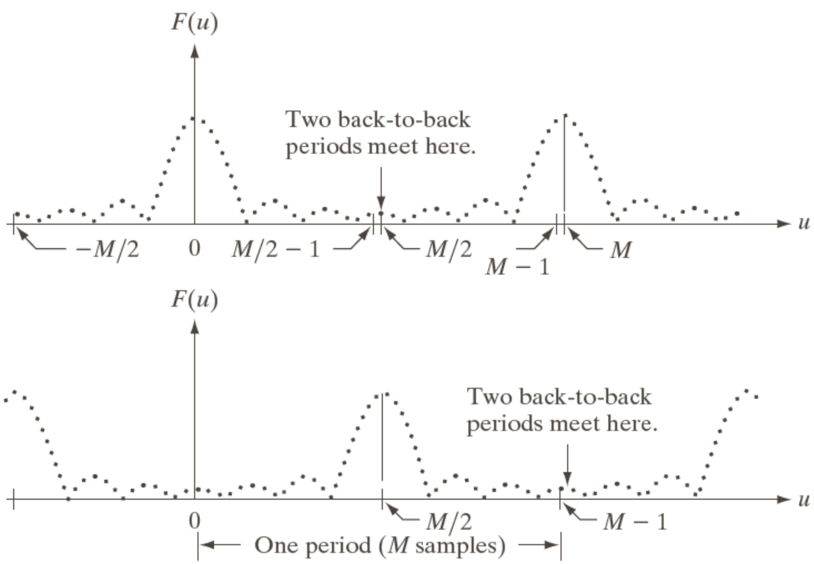
$$= F(u-u_0, v-v_0)$$

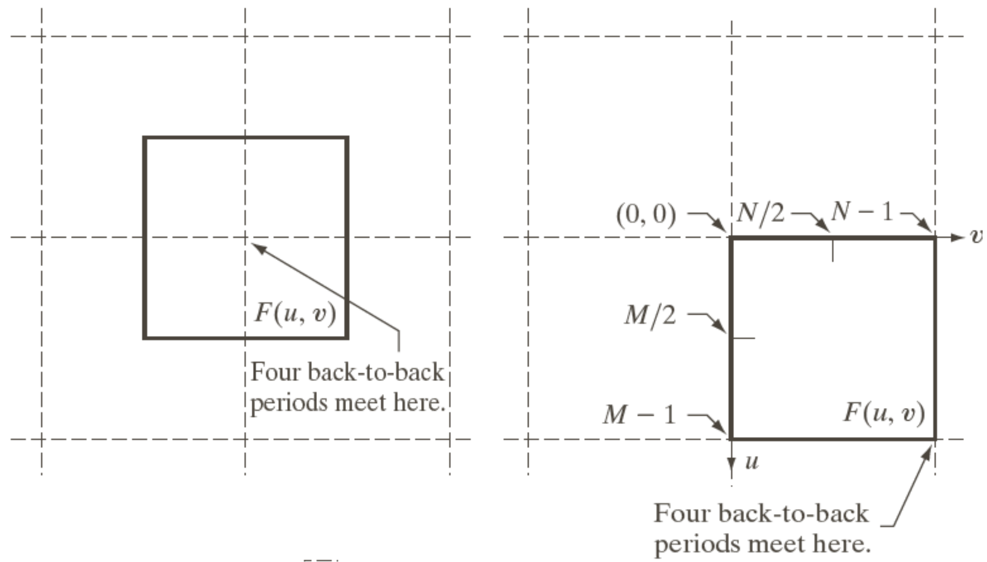
Periodicity（周期性）



- $f(x,y) = f(x+k_1M, y) = f(x, y+k_2N) = f(x+k_1M, y+k_2N)$
- $F(u,v) = F(u+k_1M, v) = F(u, v+k_2N) = F(u+k_1M, v+k_2N)$

Where k_1 and k_2 are integers



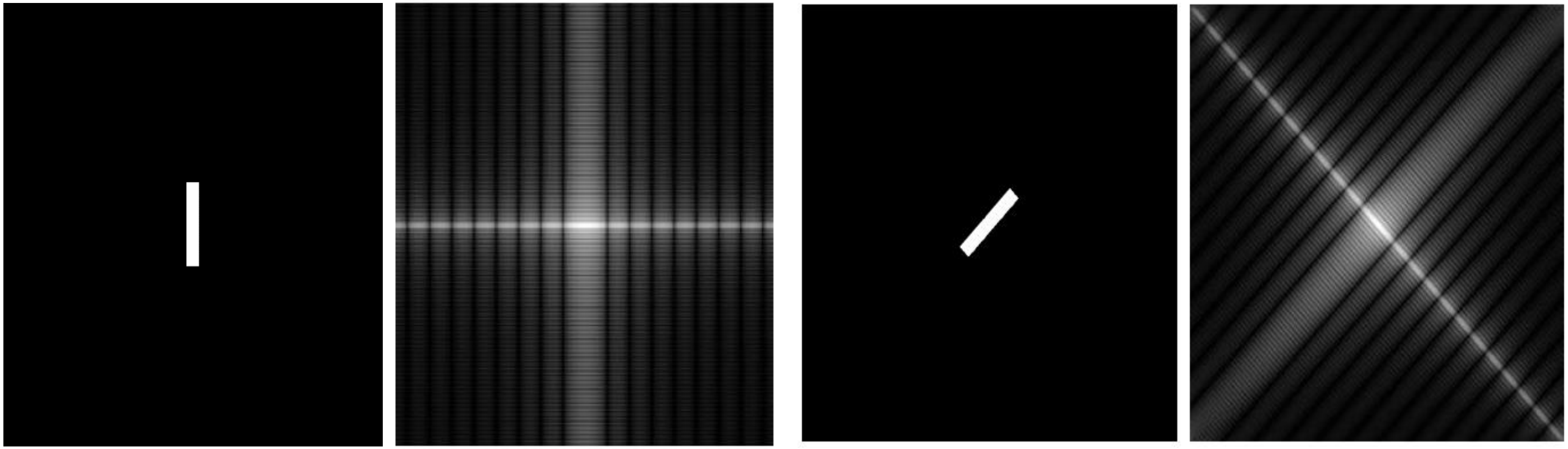


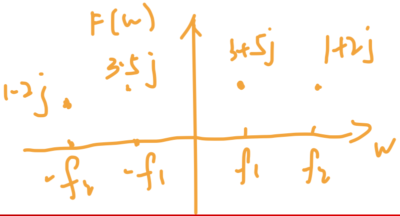
Rotation（旋转）

❑ Rotation

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$$

Where $x = r\cos\theta,\ \ y = r\sin\theta,\ \ u = \omega\cos\varphi,\ \ v = \omega\sin\varphi$



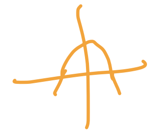


Symmetry（对称性）

$[0, \ldots, M-1]$

$\frac{M}{2}$

$f(x)=f(-x)$



在$\frac{M}{2}$, $\frac{N}{2}$对称

$y: \frac{M}{2}$

- **Even Function (偶函数)**

$$w_e(x,y) = w_e(-x,-y) \qquad w_e(x,y) = w_e(M-x, N-y)$$

- **Odd Function (奇函数)** $f(x) = -f(-x)$

$$w_o(x,y) = -w_o(-x,-y) \qquad w_o(x,y) = -w_o(M-x, N-y)$$

- **Conjugate symmetric (共轭对称)** $f(x,y)$ real

$f(x,y)$在正频率取共轭，得负频率频谱系数

$$F^*(u,v) = F(-u,-v) \qquad F^*(u,v) = F(M-u, N-v)$$

- **Conjugate antisymmetric (共轭反对称)** $f(x,y)$ 虚

$$F^*(u,v) = -F(-u,-v) \qquad F^*(u,v) = -F(M-u, N-v)$$

$$F^*(u,v) = \left[\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}\right]^*$$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f^*(x,y) e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

since $f(x,y)$ real $\Rightarrow f(x,y) = f^*(x,y)$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{(-u)x}{M}+\frac{(-v)y}{N}\right)}$$

$$= F(-u,-v)$$

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2D Discrete Fourier Transform (DFT)

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

Q1:

$$f(x,y) \quad \Leftrightarrow \quad F(u,v)$$

$$f(x,y)e^{j2\pi(\frac{u_0x}{M}+\frac{v_0y}{N})} \quad \Leftrightarrow \quad ?$$

Q2: Given that $f(x,y)$ is pure imaginary

Prove that $F^*(u,v)=-F(-u,-v)$

Symmetry（对称性）

	Spatial Domain†		Frequency Domain†
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$ 共轭对称
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

2D Convolution theorem（卷积定理）

f(x) A×B　P×Q

- **Convolution theorem**

卷积

$$f(x,y) \star h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

$$\textbf{or } f(x,y)\, h(x,y) \Leftrightarrow \frac{1}{MN} F(u,v) \star H(u,v)$$

- **Zero padding (零填充)**

$$f_p(x,y) = \begin{cases} f(x,y), & 0 \le x \le A-1, 0 \le y \le B-1 \\ 0, & A \le x \le P, B \le y \le Q \end{cases}$$

$$h_p(x,y) = \begin{cases} h(x,y), & 0 \le x \le C-1, 0 \le y \le D-1 \\ 0, & C \le x \le P, D \le y \le Q \end{cases}$$

Where $f(x,y)$**:** $A \times B$ image**;** $h(x,y)$**:** $C \times D$ image; $P \ge A + C - 1$; $Q \ge B + D - 1$

2D DFT implementation (FFT)

- **Fast Fourier transform (FFT):** an algorithm that computes the discrete Fourier transform (DFT) of a sequence, or its inverse (IDFT).

取 2^k

- The basic idea is to **decompose** successively the N-point DFT into smaller-size DFTs.
- Improvement of speed $\boldsymbol{O(N^2)}$ to $\boldsymbol{O(N log_2(N))}$
- There are many FFT algorithms!
- A classical one is Cooley-Tukey FFT (published in 1965; for more generalized complex FFT)

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$x[n]: N$

$X[k]: N$

DFT and IDFT

$x[0], x[1], \dots x[N-1]$

N:次乘法 N 数据 $O(N^2)$

DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$= X(e^{j\omega})\Big|_{\omega = 2\pi k/N}, \qquad k = 0, 1, \dots, N-1$$

Using: $W_N = e^{-j2\pi/N}$, we can rewrite the DFT as

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0, 1, \dots, N-1.$$

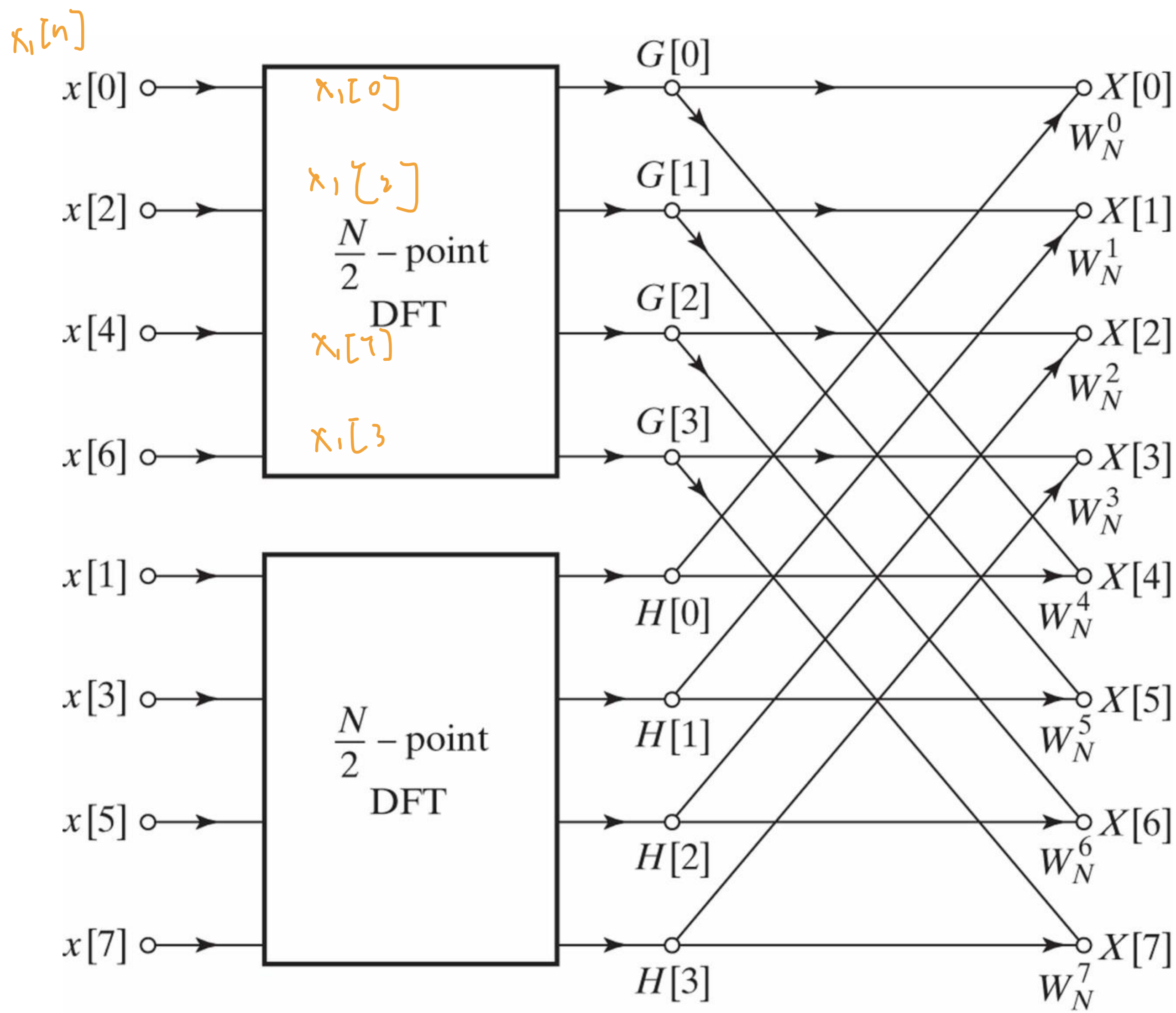
IDFT:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad k = 0, 1, \dots, N-1.$$

2D DFT implementation (FFT)

m:4

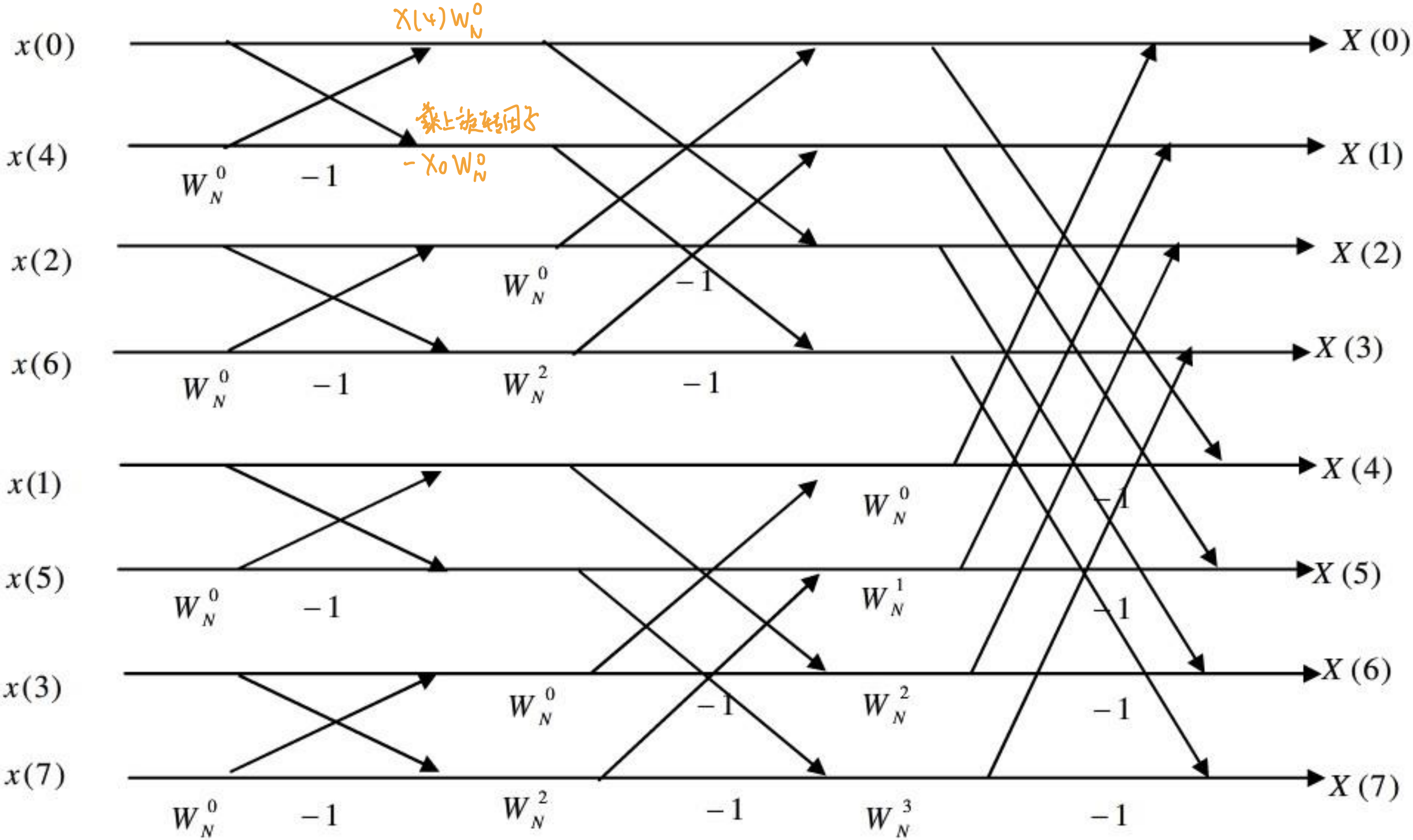
N:8



2D DFT implementation (FFT)

$3\times 8 = N \times \log_2 N$ $O(N\log_2 N)$

$N=8$ 1×8 1×8 1×8



Appendix 1: DFT definition and expressions

TABLE 4.3 Summary of DFT definitions and corresponding expressions.

	Name	Expression(s)
1)	Discrete Fourier transform (DFT) of $f(x,y)$	$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
2)	Inverse discrete Fourier transform (IDFT) of $F(u,v)$	$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
3)	Spectrum	$\lvert F(u,v)\rvert=\left[R^2(u,v)+I^2(u,v)\right]^{1/2}$ $\quad R=\text{Real}(F);\ I=\text{Imag}(F)$
4)	Phase angle	$\phi(u,v)=\tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
5)	Polar representation	$F(u,v)=\lvert F(u,v)\rvert e^{j\phi(u,v)}$
6)	Power spectrum	$P(u,v)=\lvert F(u,v)\rvert^2$
7)	Average value	$\bar{f}=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)=\frac{1}{MN}F(0,0)$
8)	Periodicity (k_1 and k_2 are integers)	$F(u,v)=F(u+k_1M,v)=F(u,v+k_2N)$ $=F(u+k_1,v+k_2N)$ $f(x,y)=f(x+k_1M,y)=f(x,y+k_2N)$ $=f(x+k_1M,y+k_2N)$
9)	Convolution	$(f\star h)(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$
10)	Correlation	$(f\,☆\,h)(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^*(m,n)h(x+m,y+n)$
11)	Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.
12)	Obtaining the IDFT using a DFT algorithm	$MNf^*(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$ This equation indicates that inputting $F^*(u,v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x,y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.

Appendix 2: DFT properties

TABLE 4.4 Summmary of DFT pairs. The closed-form expressions in 12 and 13 are valid only for continuous variables. They can be used with discrete variables by sampling the continuous expressions.

Name		DFT Pairs
1)	Symmetry properties	See Table 4.1
2)	Linearity	$af_1(x,y)+bf_2(x,y) \Leftrightarrow aF_1(u,v)+bF_2(u,v)$
3)	Translation (general)	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0,v-v_0)$ $f(x-x_0,y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$
4)	Translation to center of the frequency rectangle, $(M/2,N/2)$	$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2,v-N/2)$ $f(x-M/2,y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$
5)	Rotation	$f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$ $r=\sqrt{x^2+y^2} \quad \theta=\tan^{-1}(y/x) \quad \omega=\sqrt{u^2+v^2} \quad \varphi=\tan^{-1}(v/u)$
6)	Convolution theorem†	$f\star h)(x,y) \Leftrightarrow (F\cdot H)(u,v)$ $(f\cdot h)(x,y) \Leftrightarrow (1/MN)[(F\star H)(u,v)]$
7)	Correlation theorem†	$(f\,☆\,h)(x,y) \Leftrightarrow (F^*\cdot H)(u,v)$ $(f^*\cdot h)(x,y) \Leftrightarrow (1/MN)[(F\,☆\,H)(u,v)]$
8)	Discrete unit impulse	$\delta(x,y) \Leftrightarrow 1$ $1 \Leftrightarrow MN\delta(u,v)$
9)	Rectangle	$\text{rec}[a,b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
10)	Sine	$\sin(2\pi u_0x/M+2\pi v_0y/N) \Leftrightarrow \dfrac{jMN}{2}[\delta(u+u_0,v+v_0)-\delta(u-u_0,v-v_0)]$
11)	Cosine	$\cos(2\pi u_0x/M+2\pi v_0y/N) \Leftrightarrow \dfrac{1}{2}[\delta(u+u_0,v+v_0)+\delta(u-u_0,v-v_0)]$
	The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.	
12)	Differentiation (the expressions on the right assume that $f(\pm\infty,\pm\infty)=0$.	$\left(\dfrac{\partial}{\partial t}\right)^m\left(\dfrac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m(j2\pi\nu)^n F(\mu,\nu)$ $\dfrac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu,\nu); \quad \dfrac{\partial^n f(t,z)}{\partial z^m} \Leftrightarrow (j2\pi\nu)^n F(\mu,\nu)$
13)	Gaussian	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2} \quad (A \text{ is a constant})$

† Assumes that $f(x,y)$ and $h(x,y)$ have been properly padded. Convolution is associative, commutative, and distributive. Correlation is distributive (see Table 3.5). The products are elementwise products (see Section 2.6).