Probability & Statistics for EECS: Homework #11

Due on April 30, 2023 at 23:59

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Problem 1

Solution

(a) Yes, that is because $t_1X + t_2Y + t_3(X + Y) = (t_1 + t_3)X + (t_2 + t_3)Y$ and X, Y are i.i.d. Normal distributions. And what we have known is that the sum of independent Normal distribution is also a Normal distribution.
So $(t_1 + t_3)X + (t_2 + t_3)Y$ is normal for any constant number t_1,t_2,t_3.

(b) No, that is because $X + Y + (SX + SY) = (1 + S)X + (1 + S)Y$, at the mean time S is a random sign, with 0.5 probability making $(1 + S)X + (1 + S)Y$ be equal to 0 when $S = -1$. So from the definition of MVN, we cannot get a MVN.

(c) Yes, that is because $t_1SX + t_2SY = S(t_1X + t_2Y)$, from the threom mentioned in (a), we can have $(t_1X + t_2Y) \sim N(0, t_1^2 + t_2^2)$.
And from what we have proved in class, we can know $Z = SW$ and $Z \sim N(0, 1)$ when $W \sim N(0, 1)$ and S is sign random and is independent of W.
So under this situation, we can let $W = \frac{(t_1X+t_2Y)}{\sqrt{t_1^2+t_2^2}}$, then $W \sim N(0, 1)$, and we have known S is independent of (X, Y).
Therefore $Z = SW = S\frac{(t_1X+t_2Y)}{\sqrt{t_1^2+t_2^2}}$, then $\sqrt{t_1^2 + t_2^2}Z = S(t_1X + t_2Y)$ and $Z \sim N(0, 1)$. Finally $S(t_1X + t_2Y) \sim N(0, t_1^2 + t_2^2)$.

Problem 2

(1) Method 1 MVN
$t_1(X+Y)+t_2(X-Y)=(t_1+t_2)X+(t_1+t_2)Y$, because X, Y are i.i.d. Normal distribution and because the sum of the Normal distribution is also Normal. Therefore we can get it.

(2) Method 2 change of variables
First denote $(X+Y, X-Y)=(Z, W)$, then $Z=X+Y,\ W=X-Y$.
Then we can get $z=x+y,\ w=x-y$, and then $x=\frac{z+w}{2},\ y=\frac{z-w}{2}$.
Therefore we can get $\frac{\partial(x,y)}{\partial(z,w)}$, $\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$ Then $\left|\frac{\partial(x,y)}{\partial(z,w)}\right|=\frac{1}{2}$.
Then $f_{Z,W}(z,w)=f_{X,Y}(x,y)*\left|\frac{\partial(x,y)}{\partial(z,w)}\right|=\frac{1}{2}f_{X,Y}(x,y)$.
Because X, Y are i.i.d. Normal distribution, then we can get $f_{Z,W}(z,w)=\frac{1}{2}f_X(x)f_Y(y)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}=\frac{1}{4\pi}e^{-\frac{z^2}{4}}e^{-\frac{w^2}{4}}$.
Obviously we can simply it as $h(z)g(w)$, so they are independent.

Problem 3

Solution

(a) For this question we can use change of variables method too.
From the picture we can have $R = \sqrt{X^2+Y^2}$ and $\Theta = arctan(\frac{Y}{X})$.
Then we have $r = \sqrt{x^2+y^2}$ and $\theta = arctan(\frac{y}{x})$.
Then we can $\frac{\partial(r,\theta)}{\partial(x,y)}$ more easily. $\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$ Therefore $\left|\frac{\partial(r,\theta)}{\partial(x,y)}\right| = \frac{1}{\sqrt{x^2+y^2}}$.
Then $f_{R,\Theta}(r,\theta) = f_{X,Y}(x,y) * \left|\frac{1}{\frac{\partial(r,\theta)}{\partial(x,y)}}\right| = \sqrt{x^2+y^2} f_{X,Y}(x,y)$.
Because X, Y are i.i.d. Normal distribution, then we can get $f_{R,\Theta}(r,\theta) = \sqrt{x^2+y^2} f_X(x) f_Y(y) = r\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$.
And we have $x^2 = \frac{r^2}{1+tan^2\theta}$, $y^2 = \frac{r^2 tan^2\theta}{1+tan^2\theta}$, then we have,
$f_{R,\Theta}(r,\theta) = \frac{r}{2\pi}e^{-\frac{r^2}{2}}$.
And obviously θ is distributed as Uniform distribution, so they are independent.

Problem 4

Solution

(a) For this question we can use change of variables method too.
From the question we can have $T = X + Y$, $W = \frac{X}{Y}$.
Then we have $t = x + y$, $w = \frac{x}{y}$.
Then $x = \frac{wt}{w+1}$, $y = \frac{t}{w+1}$.
Then we can $\frac{\partial(t,w)}{\partial(x,y)}$ more easily. $\begin{pmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix}$ Therefore $\left|\frac{\partial(t,w)}{\partial(x,y)}\right| = \frac{x+y}{y^2}$.
Then $f_{T,W}(t,w) = f_{X,Y}(x,y) * \left|\frac{1}{\frac{\partial(t,w)}{\partial(x,y)}}\right| = \frac{y^2}{x+y} f_{X,Y}(x,y)$.
Because X, Y are i.i.d. exponential distribution, then we can get $f_{T,W}(t,w) = \frac{y^2}{x+y} f_X(x) f_Y(y) = \frac{y^2}{x+y}\lambda e^{-\lambda x}\lambda e^{-\lambda y}$
Therefore $f_{T,W}(t,w) = \frac{1}{(w+1)^2}\lambda^2 t e^{-\lambda t}$.
Obviously it can be denoted as $h(w)g(t)$, then they are independent.
Because $\int_0^\infty \frac{1}{(1+w)^2}\, dw = 1$.
Finally we can have $f_W(w) = \frac{1}{(1+w)^2}$, and $f_T(t) = \lambda^2 t e^{-\lambda t}$.

(b) First let $T = X + Y$, then we can have $f_T(t) = t$ when $0 < t < 1$, and $f_T(t) = 2 - t$ when $1 < t < 2$, otherwise =0.
Then $W = X + Y + Z = T + Z$, with $0 < w - t < 1$ and by using convolution method, we have,
$f_W(w) = \int_{-\infty}^{+\infty} f_T(t) f_Z(w-t)\, dt = \int_{w-1}^{w} f_T(t) * 1\, dt$.
When $0 < w < 1$,
$\int_{w-1}^{w} f_T(t)\, dt = \int_0^w t\, dt = \frac{w^2}{2}$.
When $1 < w < 2$,
$\int_{w-1}^{w} f_T(t)\, dt = \int_{w-1}^{1} t\, dt + \int_1^w 2 - t\, dt = -w^2 + 3w - \frac{3}{2}$.
When $2 < w < 3$,
$\int_{w-1}^{w} f_T(t)\, dt = \int_{w-1}^{2} 2 - t\, dt = \frac{(w-3)^2}{2}$.
Otherwise =0.

(c) method 1 convolution:
$F_M(t) = P(M <= t) = P(X <= t, Y <= t)$.
Because they are i.i.d. exponential distribution.
Then $F_M(t) = P(X <= t)P(Y <= t) = (1 - e^{-\lambda t})^2$.
Then we can get $f_M(t) = 2\lambda e^{-2\lambda t} * (e^{\lambda t} - 1)$ by derivation.
$f_{X+\frac{1}{2}Y}(t) = \int_{-\infty}^{+\infty} f_X(x) f_{\frac{1}{2}Y}(t-x)\, dx = \int_0^t \lambda e^{-\lambda x} 2\lambda e^{-2\lambda(t-x)}\, dx = 2\lambda e^{-2\lambda t}\int_0^t \lambda e^{\lambda x}\, dx$.
$= 2\lambda e^{-2\lambda t} * (e^{\lambda t} - 1) = 2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t}$.
Then we get it.
method 2 exponential properties:
Let $L = min(X,Y)$, then we can denote M as $M = L + (M - L)$, and obviously $L \sim Expo(2\lambda)$ because we have known the PDF of M and $M + L = X + Y$.
That is $2\lambda e^{-\lambda t} = 2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t} + L$. Then $L = 2\lambda e^{-2\lambda t}$.
At the same time L can be seen as the subject finishing the first thing. And $M - L \sim Expo(\lambda)$, can be seen as the time of the second thing being finished.
Because of the property of memorylessness, then we can add them up and it is equal to M. Meanwhile, $X \sim Expo(\lambda)$ and $\frac{1}{2}Y \sim Expo(2\lambda)$.
Therefore, we get it.

Problem 5

Solution

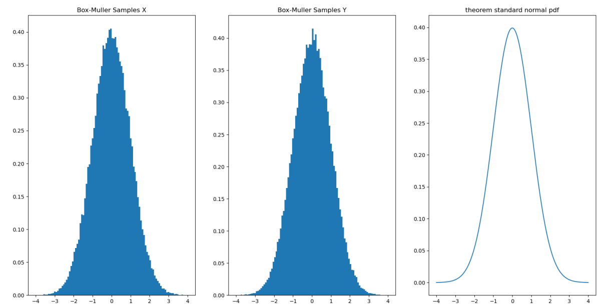


Figure 1: figure 1

(a)

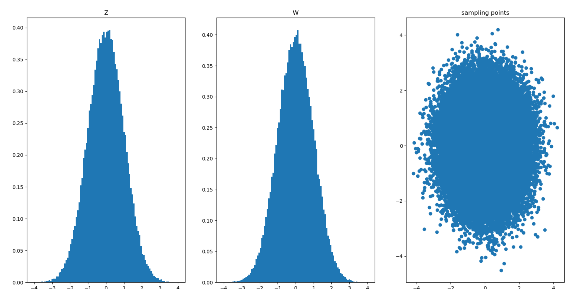
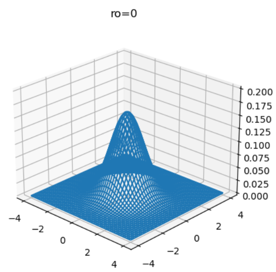


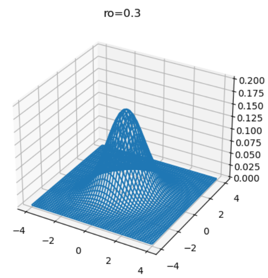
Figure 2: figure 2

(b)

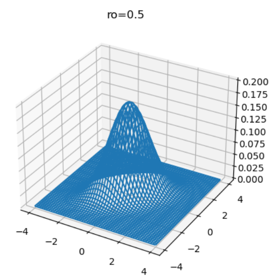
(c)



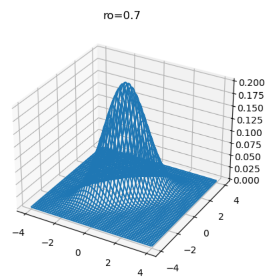
(a) Fig1



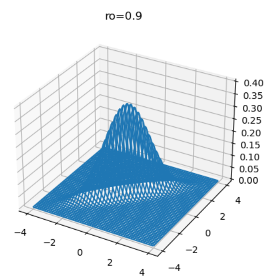
(b) Fig2



(c) Fig3



(d) Fig4



(e) Fig5

Figure 3: figure 3 3D

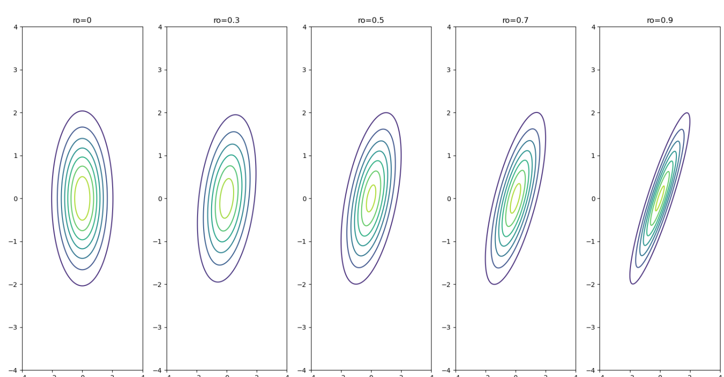


Figure 4: figure 4 contour

```
import numpy as np
import matplotlib.pyplot as plt
import math
from scipy.stats import multivariate_normal
##(1)
###unif
u1=np.random.random(100000)
u2=np.random.random(100000)
x1 = np.linspace(0 - 4 , 0 + 4, 1000)
R=np.sqrt(-2*np.log(u1))
theta=2*np.pi*u2
x=R*np.cos(theta)
y=R*np.sin(theta)
theorem=(1/np.sqrt(2*np.pi))*np.exp(-(x1**2)/2)
plt.figure(1)
plt.subplot(1,3,1)
plt.hist(x,bins=100,range=(-4,4),density=True)
plt.title("Box-Muller Samples X")
plt.subplot(1,3,2)
plt.hist(y,bins=100,range=(-4,4),density=True)
plt.title("Box-Muller Samples Y")
plt.subplot(1,3,3)
plt.title("theorem standard normal pdf")
plt.plot(x1,theorem)
plt.show()
##(2)
##because x y are independent so ro=0
ro=0
##(Z,W) is the MVN which we have learned in class
z=x
w=ro*x+np.sqrt(1-ro**2)*y
plt.figure(2)
plt.subplot(1,3,1)
plt.hist(z,bins=100,range=(-4,4),density=True)
plt.title("Z")
plt.subplot(1,3,2)

plt.hist(w,bins=100,range=(-4,4),density=True)
plt.title("W")
plt.subplot(1,3,3)
plt.scatter(z,w)
plt.title("sampling points")
plt.show()
##(3)
##set ro
ro_vector=[0,0.3,0.5,0.7,0.9]
##initial other parameters
cov=[[1,0],[0,1]]
u=[0,0]
```

```
x2 = np.linspace(0 - 4 , 0 + 4, 1000)
x3 = np.linspace(0 - 4 , 0 + 4, 1000)
Z,W=np.meshgrid(x2,x3)
space=np.empty(Z.shape+(2,))
space[:,:,0]=Z
space[:,:,1]=W
plt.figure(3)
for i in range(5):
    ##set parameters
    ro=ro_vector[i]
    cov[1][0]=ro
    cov[0][1]=ro
    #generate mvn
    generate_mvn=multivariate_normal(u,cov)
    U=generate_mvn.pdf(space)
    plt.subplot(1,5,i+1)
    plt.contour(Z,W,U)
    plt.title(f"ro={ro}")
plt.show()
fig=plt.figure()
ax=fig.add_subplot(projection='3d')
ax.set_zlim(0,0.2)
wframe=None
##set parameters
ro=ro_vector[0]
cov[1][0]=ro
cov[0][1]=ro
#generate mvn
generate_mvn=multivariate_normal(u,cov)
U=generate_mvn.pdf(space)
wframe=ax.plot_wireframe(Z,W,U)

plt.title(f"ro={ro}")
plt.show()
fig=plt.figure()
ax=fig.add_subplot(projection='3d')
ax.set_zlim(0,0.2)
wframe=None
##set parameters
ro=ro_vector[1]
cov[1][0]=ro
cov[0][1]=ro
#generate mvn
generate_mvn=multivariate_normal(u,cov)
U=generate_mvn.pdf(space)
wframe=ax.plot_wireframe(Z,W,U)

plt.title(f"ro={ro}")
plt.show()
```

```
fig=plt.figure()
ax=fig.add_subplot(projection='3d')
ax.set_zlim(0,0.2)
wframe=None
##set parameters
ro=ro_vector[2]
cov[1][0]=ro
cov[0][1]=ro
#generate mvn
generate_mvn=multivariate_normal(u,cov)
U=generate_mvn.pdf(space)
wframe=ax.plot_wireframe(Z,W,U)

plt.title(f"ro={ro}")
plt.show()
fig=plt.figure()
ax=fig.add_subplot(projection='3d')
ax.set_zlim(0,0.2)
wframe=None
##set parameters
ro=ro_vector[3]
cov[1][0]=ro
cov[0][1]=ro
#generate mvn
generate_mvn=multivariate_normal(u,cov)
U=generate_mvn.pdf(space)
wframe=ax.plot_wireframe(Z,W,U)

plt.title(f"ro={ro}")
plt.show()
fig=plt.figure()
ax=fig.add_subplot(projection='3d')
ax.set_zlim(0,0.4)
wframe=None
##set parameters
ro=ro_vector[4]
cov[1][0]=ro
cov[0][1]=ro
#generate mvn
generate_mvn=multivariate_normal(u,cov)
U=generate_mvn.pdf(space)
wframe=ax.plot_wireframe(Z,W,U)
plt.title(f"ro={ro}")
plt.show()
```