Probability & Statistics for EECS: Homework #12

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Problem 1

Solution

(a) First p is given a $Unif(0,1)$, then we can model it as a r.v.$\in [0,1]$. Therefore, the prior distribution $p \sim Beta(1,1)$.
Second we can use Beta-Binomial Conjugacy which we have proved in class so that we can intuitively explain the process with the chain rule of bayes rule.
At the begining, depending on the data $X_1 = x_1$, we can update the prior distribution and get a middle distribution $Beta(1+x_1, 1+(1-x1))$. And then we can seem this as a new prior distribution and update it untill using every data we have.
So finally, we can get the posterior distribution $p|X_1 = x_1, \cdots, X_n = x_n \sim Beta(1 + x_1 + x_2 + \cdots + x_n, n + 1 - (x_1 + x_2 + \cdots + x_n))$.
Therefore, from the posterior distribution, we can prove that we just need one-dimensional quantity $x_1 + x_2 + \cdots + x_n$ to obtain the posterior distribution.

(b) For this question we can use LOTP to solve it,

$$\begin{aligned}
P(X_{n+1}|X_1 + X_2 + \cdots + X_n = k) &= \int_0^1 P(X_{n+1}|X_1 + X_2 + \cdots + X_n = k, p) f(p|X_1 + X_2 + \cdots + X_n)\, dp \\
&= \frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(n+1-k)} \int_0^1 p * p^k (1-p)^{n-k}\, dp \\
&= \frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(n+1-k)} \frac{\Gamma(k+2)\Gamma(n+1-k)}{\Gamma(n+3)} \\
&= \frac{k+1}{n+2}
\end{aligned}$$

Therefore we get it.

(c) From we have learned about Beta-Binomial Conjugacy, Laplace's law of succession actually is Bayesian average. We have known the prior distribution $p \sim Unif(0,1)$, and the data model $X_1 + X_2 + \cdots + X_n$ is a conditional Binomial distribution $Binomial(n,p)$.
Then we can get the posterior distribution when $X_1 + X_2 + \cdots + X_n = k$, that is $Beta(a+k, b+n-k)$. And then we can consider about the parameters in posterior distribution. Let $a + k$ is the times of success, and $b + n - k$ is the times of failure.
Then we can predict the condition next day.
Then $E[p|X_1 + X_2 + \cdots + X_n = k] = \frac{a+k}{a+b+n}$, and $a = b = 1$.
Therefore $E[p|X_1 + X_2 + \cdots + X_n = k] = \frac{k+1}{n+2}$, that is the probability that we predict depending on the posterior distribution of the same event happening next day.

Problem 2

(a) From the question, we have,

$$\begin{aligned} E[p^2(1-p)^2] &= \int_0^1 p^2(1-p)^2 f(p)\, dp \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p^2(1-p)^2 p^{a-1}(1-p)^{b-1}\, dp \end{aligned}$$

(b)

(c)

(d)

(e)

Problem 3

Solution

(a) Assume n balls to be put in the interval [0,1] randomly, so the distribution of the ball landing in the interval can be regarded as $Unif(0,1)$.
And assume $U_{(1)} = x_1, U_{(2)} = x_2, \cdots U_{(n)} = x_n$, then there are no other balls falling into points except $x_1, x_2, \cdots, x_n$.
And then we can use the story of multinomial to explain this, and get the joint PDF of $U_{(1)}, U_{(2)}, \cdots U_{(n)}$.

$$f_{U_{(1)},U_{(2)},\cdots U_{(n)}} = 0$$

(b) Assume n balls to be put in the interval [0,1] randomly, so the distribution of the ball landing in the interval can be regarded as $Unif(0,1)$.
And assume $U_{(j)} = x, U_{(k)} = y$, then we can know there are j-1 balls on the left of the point x, and n-k balls on the right of the point y.
Then we can know there are k-j+1 points between point x and point y, including them simultaneously.
Then we can use the story of multinomial to explain this too.
Then we can have the joint PDF of $U_{(j)}, U_{(k)}$,

$$f_{U_{(j)},U_{(k)}}(x,y) = \frac{n!}{(j-1)!(k-j-1)!(n-k)!} x^{j-1}(y-x)^{k-j-1}(1-y)^{n-k}$$

(c) We already have $U_1, \cdots, U_n$ are i.i.d. $Unif(0,1)$. Then we can define a new Bernoulli trial, which means when $U_j <= p$, the trial is thought of a success.
Then Let X mean the number of success, then $X >= j$ means $U_{(j)} <= p$.
Therefore we can have the following formula,

$$\begin{aligned}
P(X >= j) &= P(U_{(j)} <= p) \\
&= \frac{n!}{(j-1)!(n-j)!}(p)^{j-1}(1-p)^{n-j} = \frac{1}{\beta(j, n-j+1)}(p)^{j-1}(1-p)^{n-j} \\
&= P(B <= p)
\end{aligned}$$

Therefore we get it.

(d) From what we have proved in (c), let p=x, then we can have $P(X >= j) = P(B <= x)$, and $X \sim Bin(n,x), B \sim Beta(j, n-j+1)$.
So $P(X >= j) = \sum_{k=j}^{n} \binom{n}{k} x^k (1-x)^{n-k}$.
And $P(B <= x) = P(U_{(j)} <= x) = \int_0^x \frac{n!}{(j-1)!(n-j)!} t^{j-1}(1-t)^{n-j}\, dt$.
Then $\sum_{k=j}^{n} \binom{n}{k} x^k (1-x)^{n-k} = \int_0^x \frac{n!}{(j-1)!(n-j)!} t^{j-1}(1-t)^{n-j}\, dt$. Therefore we get it.

Problem 4

Solution

(a)

Problem 5

Solution

Figure 1: figure 1

(a)

Figure 2: figure 2

(b)

(a) (b) (c) (d) (e)
Fig1Fig2Fig3Fig4Fig5

Figure 3: figure 3 3D

(c)

Problem 5 continued on next page. . .

Figure 4: figure 4 contour

```
import numpy as np
import matplotlib.pyplot as plt
import math
from scipy.stats import multivariate_normal
##(1)
###unif
u1=np.random.random(100000)
u2=np.random.random(100000)
x1 = np.linspace(0 - 4 , 0 + 4, 1000)
R=np.sqrt(-2*np.log(u1))
theta=2*np.pi*u2
x=R*np.cos(theta)
y=R*np.sin(theta)
theorem=(1/np.sqrt(2*np.pi))*np.exp(-(x1**2)/2)
plt.figure(1)
plt.subplot(1,3,1)
plt.hist(x,bins=100,range=(-4,4),density=True)
plt.title("Box-Muller Samples X")
plt.subplot(1,3,2)
plt.hist(y,bins=100,range=(-4,4),density=True)
plt.title("Box-Muller Samples Y")
plt.subplot(1,3,3)
plt.title("theorem standard normal pdf")
plt.plot(x1,theorem)
plt.show()
##(2)
##because x y are independent so ro=0
ro=0
##(Z,W) is the MVN which we have learned in class
z=x
w=ro*x+np.sqrt(1-ro**2)*y
plt.figure(2)
plt.subplot(1,3,1)
plt.hist(z,bins=100,range=(-4,4),density=True)
plt.title("Z")
plt.subplot(1,3,2)

plt.hist(w,bins=100,range=(-4,4),density=True)
plt.title("W")
plt.subplot(1,3,3)
plt.scatter(z,w)
plt.title("sampling points")
plt.show()
##(3)
##set ro
ro_vector=[0,0.3,0.5,0.7,0.9]
```

Problem 5 continued on next page...

```
##initial other parameters
cov=[[1,0],[0,1]]
u=[0,0]
x2 = np.linspace(0 - 4 , 0 + 4, 1000)
x3 = np.linspace(0 - 4 , 0 + 4, 1000)
Z,W=np.meshgrid(x2,x3)
space=np.empty(Z.shape+(2,))
space[:,:,0]=Z
space[:,:,1]=W
plt.figure(3)
for i in range(5):
    ##set parameters
    ro=ro_vector[i]
    cov[1][0]=ro
    cov[0][1]=ro
    #generate mvn
    generate_mvn=multivariate_normal(u,cov)
    U=generate_mvn.pdf(space)
    plt.subplot(1,5,i+1)
    plt.contour(Z,W,U)
    plt.title(f"ro={ro}")
plt.show()
fig=plt.figure()
ax=fig.add_subplot(projection='3d')
ax.set_zlim(0,0.2)
wframe=None
##set parameters
ro=ro_vector[0]
cov[1][0]=ro
cov[0][1]=ro
#generate mvn
generate_mvn=multivariate_normal(u,cov)
U=generate_mvn.pdf(space)
wframe=ax.plot_wireframe(Z,W,U)

plt.title(f"ro={ro}")
plt.show()
fig=plt.figure()
ax=fig.add_subplot(projection='3d')
ax.set_zlim(0,0.2)
wframe=None
##set parameters
ro=ro_vector[1]
cov[1][0]=ro
cov[0][1]=ro
#generate mvn
generate_mvn=multivariate_normal(u,cov)
U=generate_mvn.pdf(space)
wframe=ax.plot_wireframe(Z,W,U)
```

Problem 5 continued on next page. . .

```
plt.title(f"ro={ro}")
plt.show()
fig=plt.figure()
ax=fig.add_subplot(projection='3d')
ax.set_zlim(0,0.2)
wframe=None
##set parameters
ro=ro_vector[2]
cov[1][0]=ro
cov[0][1]=ro
#generate mvn
generate_mvn=multivariate_normal(u,cov)
U=generate_mvn.pdf(space)
wframe=ax.plot_wireframe(Z,W,U)

plt.title(f"ro={ro}")
plt.show()
fig=plt.figure()
ax=fig.add_subplot(projection='3d')
ax.set_zlim(0,0.2)
wframe=None
##set parameters
ro=ro_vector[3]
cov[1][0]=ro
cov[0][1]=ro
#generate mvn
generate_mvn=multivariate_normal(u,cov)
U=generate_mvn.pdf(space)
wframe=ax.plot_wireframe(Z,W,U)

plt.title(f"ro={ro}")
plt.show()
fig=plt.figure()
ax=fig.add_subplot(projection='3d')
ax.set_zlim(0,0.4)
wframe=None
##set parameters
ro=ro_vector[4]
cov[1][0]=ro
cov[0][1]=ro
#generate mvn
generate_mvn=multivariate_normal(u,cov)
U=generate_mvn.pdf(space)
wframe=ax.plot_wireframe(Z,W,U)
plt.title(f"ro={ro}")
plt.show()
```