Lecture 6: Joint Distributions

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Outline

1. Discrete Multivariate R.V.s
2. Continuous Multivariate R.V.s
3. Covariance and Correlation
4. Multinomial Distribution
5. Multivariate Normal

① event ⟶ prob. ⟶ conditional prob.

prior / posterior.

② random variable. (discrete / continuous)

Single : distribution. Story

⇓

two random variables (independence)

⇓

≥ 3 r.v.s joint distribution

Marginal distribution

Multivariate Distribution

③ Conditional Expectation. → estimation / learning (prediction)
(Signal processing

→ Statistics < Classical / Bayesian → Monte Carlo; Markov chain

- Joint distribution provides complete information about how multiple r.v.s interact in high-dimensional space
- Marginal distribution is the individual distribution of each r.v.
- Conditional distribution is the updated distribution for some r.v.s after observing other r.v.s

Outline

3 Key themes:

① Decomposition / Integration. { First-step. Conditioning. prob / expectation

② Transformation (invariance) PGF MGF

③ Approximation { Asymptotic approximation CLT: $n \to \infty$; Finite-size approximation

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2. Continuous Multivariate R.V.s
3. Covariance and Correlation
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Joint CDF

Definition

The *joint CDF* of r.v.s X and Y is the function $F_{X,Y}$ given by

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y).$$

The joint CDF of n r.v.s is defined analogously.

Joint PMF

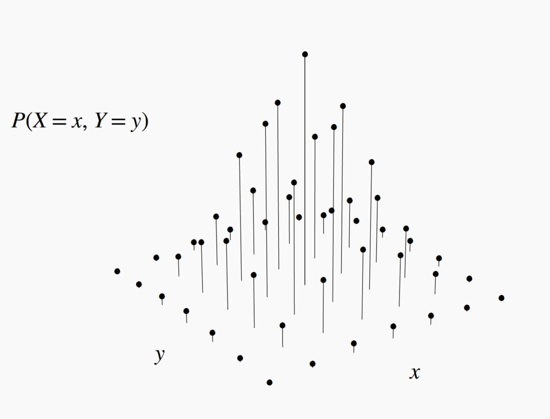
Definition

The joint PMF of discrete r.v.s X and Y is the function $p_{X,Y}$ given by

$$p_{X,Y}(x, y) = P(X = x, Y = y).$$

The joint PMF of n discrete r.v.s is defined analogously.

Joint PMF



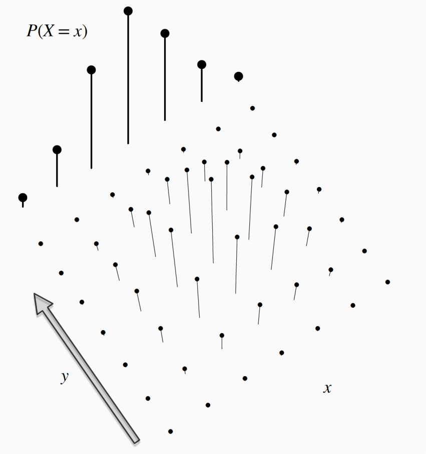
Marginal PMF

Definition

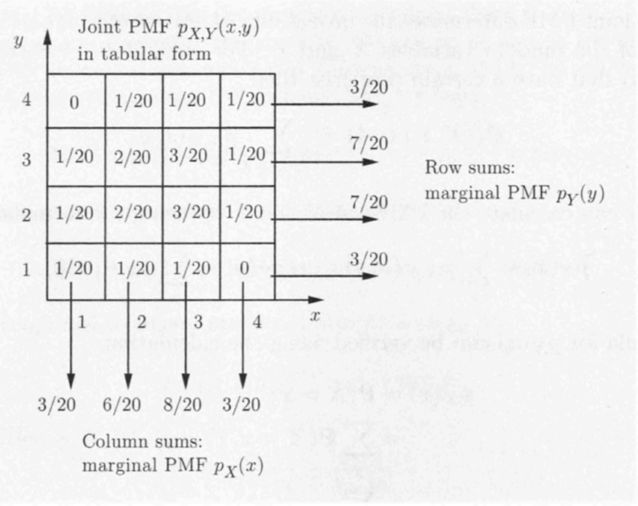
For discrete r.v.s X and Y, the *marginal PMF* of X is

$$P(X = x) = \sum_{y} P(X = x, Y = y).$$

Marginal PMF



Example


Joint PMF $p_{X,Y}(x,y)$
in tabular form
y
4
3
2
1
0 1/20 1/20 1/20
1/20 2/20 3/20 1/20
1/20 2/20 3/20 1/20
1/20 1/20 1/20 0
3/20
7/20
7/20
3/20
Row sums:
marginal PMF $p_Y(y)$
1 2 3 4
x
3/20 6/20 8/20 3/20
Column sums:
marginal PMF $p_X(x)$

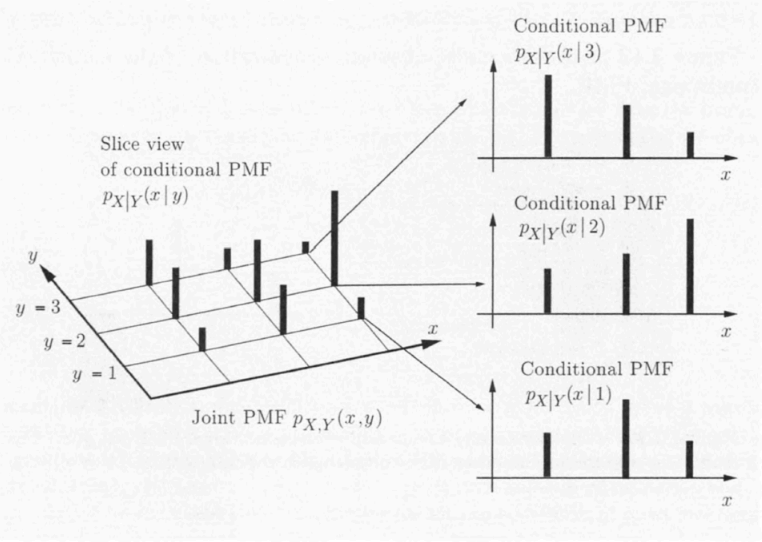
Conditional PMF

Definition

For discrete r.v.s X and Y , the *conditional PMF* of X given $Y = y$ is

$$P_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}.$$

Conditional PMF



Independence of Discrete R.V.s

Definition

Random variables X and Y are *independent* if for all x and y,

$$F_{X,Y}(x, y) = F_X(x) F_Y(y)$$

If X and Y are discrete, this is equivalent to the condition

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

for all x and y, and it is also equivalent to the condition

$$P(Y = y|X = x) = P(Y = y)$$

for all y and all x such that $P(X = x) > 0$.

Example: Chicken-egg

Suppose a chicken lays a random number of eggs, N, where $N \sim \mathrm{Pois}(\lambda)$. Each egg independently hatches with probability p and fails to hatch with probability $q = 1 - p$. Let X be the number of eggs that hatch and Y the number that do not hatch, so $X + Y = N$. What is the joint PMF of X and Y?

Solution

Solution

Related Theorem

Theorem

If $X \sim \mathrm{Pois}\,(\lambda p)$, $Y \sim \mathrm{Pois}\,(\lambda q)$, *and* X *and* Y *are independent, then* $N = X + Y \sim \mathrm{Pois}(\lambda)$ *and* $X|N = n \sim \mathrm{Bin}(n, p)$.

Related Theorem

Theorem

If $N \sim \text{Pois}(\lambda)$ *and* $X|N = n \sim \text{Bin}(n, p)$, *then* $X \sim \text{Pois}(\lambda p)$, $Y = N - X \sim \text{Pois}(\lambda q)$, *and* X *and* Y *are independent.*

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Conditional PDF Given an Event

Conditional PDF Given an Event

- The conditional PDF $f_{X|A}$ of a continuous random variable X, given an event A with $\mathbf{P}(A) > 0$, satisfies

$$\mathbf{P}(X \in B \,|\, A) = \int_B f_{X|A}(x)\,dx.$$

- If A is a subset of the real line with $\mathbf{P}(X \in A) > 0$, then

$$f_{X|\{X\in A\}}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbf{P}(X \in A)}, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

- Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space, and assume that $\mathbf{P}(A_i) > 0$ for all i. Then,

$$f_X(x) = \sum_{i=1}^{n} \mathbf{P}(A_i) f_{X|A_i}(x)$$

(a version of the total probability theorem).

Proof

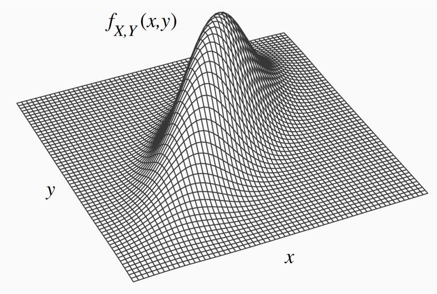
Joint PDF

Definition

If X and Y are continuous with joint CDF $F_{X,Y}$, their joint PDF is the derivative of the *joint CDF* with respect to x and y:

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y).$$

Joint PDF



Marginal PDF

Definition

For continuous r.v.s X and Y with joint PDF $f_{X,Y}$, the *marginal PDF* of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$$

This is the PDF of X, viewing X individually rather than jointly with Y.

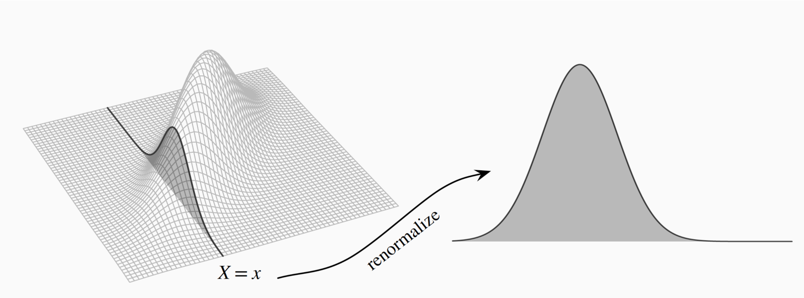
Conditional PDF

Definition

For continuous r.v.s X and Y with joint PDF $f_{X,Y}$, the *conditional PDF* of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}.$$

Conditional PDF



Technique Issue

- What is the meaning of conditioning on zero-probability event $X = x$ for a continuous r.v. X.
- We are actually conditioning on the event that X falls within a small interval of x: $X \in (x - \epsilon, x + \epsilon)$ and then taking a limit as $\epsilon \to 0$.

Example

The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} \frac{12x(2-x-y)}{5} & \text{if } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the conditional PDF of X given that $Y = y$, where $0 < y < 1$.

Example

Suppose that the joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} \frac{e^{-x/y-y}}{y} & \text{if } 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find $P\{X > 1 | Y = y\}$.

Continuous form of Bayes' Rule and LOTP

Theorem

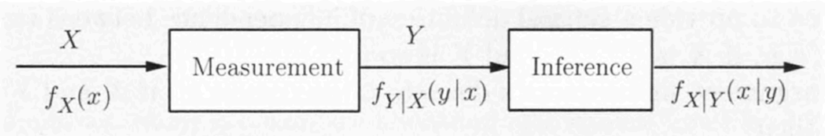
For continuous r.v.s X and Y,

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)\, f_Y(y)}{f_X(x)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)\, f_Y(y)\, dy$$

Proof

Bayes' Rule: Inference Perspective



Example

A light bulb produced by the GE company is known to have an exponential distributed lifetime Y. However, the company has been experiencing quality control problems. On any given day, the parameter λ of the PDF of Y is actually a random variable. uniformly distributed in the interval$[1, 3/2]$. We test a light bulb and record its lifetime. What we can say about the underlying parameter λ?

General Bayes' Rule

$f_X(x) \cdot 2\varepsilon \approx P(X \in (x-\varepsilon, x+\varepsilon))$

	Y **discrete**	Y **continuous**
X **discrete**	$P(Y=y \mid X=x) = \frac{P(X=x \mid Y=y)P(Y=y)}{P(X=x)}$	$f_Y(y \mid X=x) = \frac{P(X=x \mid Y=y) f_Y(y)}{P(X=x)}$
X **continuous**	$P(Y=y \mid X=x) = \frac{f_X(x \mid Y=y)P(Y=y)}{f_X(x)}$	$f_{Y \mid X}(y \mid x) = \frac{f_{X \mid Y}(x \mid y) f_Y(y)}{f_X(x)}$

class label

feature

$f_{Y|X}(y|x)$: Discriminative AI.

$f_{X,Y}(x,y)$: Generative AI

Proof

$X_1, \ldots X_n$

Stochastic processes

Markov : $\longrightarrow$ Markov chain

Stochastic information of $X_1, \ldots, X_n$

Martingale : $\longrightarrow$ Martingale

$\longrightarrow$ Branching processes

$\longrightarrow$ Poisson process.

Proof

General LOTP

(2): $\lim_{\varepsilon \to 0} \frac{P(X \in (x-\varepsilon, x+\varepsilon))}{2\varepsilon} = \sum_y \lim_{\varepsilon \to 0} \frac{P[X \in (x-\varepsilon, x+\varepsilon) | Y=y]}{2\varepsilon} \cdot P(Y=y)$

$f_X(x) = \sum_y f_X(x|Y=y) \cdot P(Y=y)$

	Y **discrete**	Y **continuous**
X **discrete**	$P(X=x) = \sum_y P(X=x\|Y=y)P(Y=y)$	$P(X=x) = \int_{-\infty}^{\infty} P(X=x\|Y=y) f_Y(y)dy$ (1)
X **continuous**	$f_X(x) = \sum_y f_X(x\|Y=y)P(Y=y)$ (2) ✓	$f_X(x) = \int_{-\infty}^{\infty} f_{X\|Y}(x\|y) f_Y(y)dy$

$f_X(x) \cdot 2\varepsilon$

Proof

① $P(X=x) = \int_{-\infty}^{+\infty} \underbrace{P(X=x|Y=y)}\, \underbrace{f_Y(y)}\, dy$

X: discrete ; Y: Continuous

$$P(X=x|Y=y) = \frac{f_Y(y|X=x)}{f_Y(y)} \cdot P(X=x)$$

$$\Rightarrow \quad P(X=x|Y=y) \cdot f_Y(y) = f_Y(y|X=x) \cdot P(X=x)$$

$$\Rightarrow \int_{-\infty}^{\infty} P(X=x|Y=y) f_Y(y) dy = \int_{-\infty}^{\infty} f_Y(y|X=x) \cdot P(X=x) dy$$

$$= P(X=x) \cdot \underline{\int_{-\infty}^{\infty} f_Y(y|X=x) dy}$$

$$= P(X=x) \cdot 1 = P(X=x)$$

Example

① $Y = N + S$; Conditioning on $S = s$ ($s = 1$ or -1)

$Y|_{S=s} \sim N(s, 1)$

discrete continuous

② $P(S=1 \mid Y=y) = \dfrac{f_{Y|S}(y|1)}{f_Y(y)} \cdot P(S=1) = \dfrac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-1)^2} \cdot p}{f_Y(y)}$

$s=1$

A binary signal S is transmitted, and we are given that $P(S = 1) = p$ and $P(S = -1) = 1 - p$. The received signal is $Y = N + S$, where N is normal noise, with zero mean and unit variance, independent of S. What is the probability that $S = 1$, as a function of the observed value y of Y?

LOTP

$S=-1$, $N(-1,1)$

③ $f_Y(y) = f_{Y|S}(y|1) \cdot P(S=1) + f_{Y|S}(y|-1) \cdot P(S=-1)$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-1)^2} \cdot p + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y+1)^2} \cdot (1-p)$

$y>0$

$e^{-2y}<1$

$p+(1-p)e^{-2y}$

$<p+1-p=1$

$P(S=1)=p$ prior.

④ $P(S=1 \mid Y=y) = \dfrac{p \cdot e^{y}}{p \cdot e^{y} + (1-p)e^{-y}} = \dfrac{p}{p+(1-p)e^{-2y}} = \begin{cases} >p & y>0 \\ =p & y=0 \\ <p & y<0 \end{cases}$

posterior.

Example: Comparing Exponentials of Different Rates

Let $T_1 \sim Expo(\lambda_1)$, $T_2 \sim Expo(\lambda_2)$, T_1 and T_2 are independent. Find $P(T_1 < T_2)$.

① $P(T_1 < T_2) \overset{\text{LOTP}}{=} \int_0^\infty P(T_1 < T_2 \mid T_2 = t) \cdot f_{T_2}(t) \, dt$

$= \int_0^\infty P(T_1 < t \mid T_2 = t) \cdot \lambda_2 e^{-\lambda_2 t} \, dt$

$= \int_0^\infty P(T_1 < t) \cdot \lambda_2 e^{-\lambda_2 t} \, dt$

$= \int_0^\infty (1 - e^{-\lambda_1 t}) \cdot \lambda_2 e^{-\lambda_2 t} \, dt$

$= \int_0^\infty \lambda_2 e^{-\lambda_2 t} dt - \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2) t} dt$

$= 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}$

② $P(T_1 = T_2) = 0$

$P(T_1 < T_2) = P(T_1 = \min(T_1, T_2))$

$T_1, \dots T_n \mid Expo(\lambda_1), \dots Expo(\lambda_n)$

independent.

$P(T_i = \min(T_1, \dots T_n)) = \frac{\lambda_i}{\lambda_1 + \dots + \lambda_n}$

$\min(T_1, \dots T_n) \sim Expo(\lambda_1 + \dots + \lambda_n)$

Independence of Continuous R.V.s

Definition

Random variables X and Y are *independent* if for all x and y,

$$F_{X,Y}(x, y) = F_X(x) F_Y(y)$$

If X and Y are continuous with joint PDF $f_{X,Y}$, this is equivalent to the condition

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

for all x and y, and it is also equivalent to the condition

$$f_{Y|X}(y|x) = f_Y(y)$$

for all y and all x such that $f_X(x) > 0$.

Proposition

$$f_{X,Y}(x,y) = \begin{cases} 8xy & ; 0 \le x \le y \le 1 \\ 0 & ; \text{otherwise.} \end{cases}$$

domain (x,y) couple

$$\ne f_X(x) f_Y(y)$$

$$f_X(x) = \begin{cases} 4x(1-x^2) & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}, \quad f_Y(y) = \begin{cases} 4y^3, & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

Suppose that the joint PDF $f_{X,Y}$ of X and Y factors as

pre. decouple

domain of X and Y

$$f_{X,Y}(x,y) = g(x)\,h(y)$$

$(-\infty, +\infty)$

for all x and y, where g and h are nonnegative functions. Then X and Y are independent. Also, if either g or h is a valid PDF, then the other one is a valid PDF too and g and h are the marginal PDFs of X and Y, respectively. (The analogous result in the discrete case also holds.)

Proof

$f_{X,Y}(x,y) = g(x)h(y) = \underline{c \cdot g(x) \cdot \frac{1}{c}h(y)}$

$c>0$ is a constant.

① Let $c = \int_{-\infty}^{\infty} h(y)dy \Rightarrow 1 = \int_{-\infty}^{\infty} \frac{1}{c}h(y)dy$

renormalization

Key idea:

$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_{-\infty}^{\infty} c\,g(x) \cdot \frac{1}{c}h(y)dy$

$= c\,g(x) \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{c}h(y)dy}_{1} = c\,g(x).$

② $\int_{-\infty}^{\infty} f_X(x)dx = 1 \Rightarrow \int_{-\infty}^{\infty} c\,g(x)dx = 1$

③ $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_{-\infty}^{\infty} c\,g(x) \cdot \frac{h(y)}{c}dx$

$= \frac{1}{c}h(y) \cdot \underbrace{\int_{-\infty}^{\infty} c\,g(x)dx}_{1} = \frac{1}{c}h(y).$

if g is a valid PDF, $\int_{-\infty}^{\infty} g(x)dx = 1$

$\Rightarrow c = 1$

$\Rightarrow \int_{-\infty}^{\infty} h(y)dy = 1$

$\Rightarrow h$ is a valid PDF.

④ $f_{X,Y}(x,y) = c\,g(x) \cdot \frac{1}{c}h(y) = f_X(x) \cdot f_Y(y)$ $\Rightarrow X, Y$ are independent.

2D LOTUS

No LOTUS: $(X, Y) \longrightarrow g(X, Y) \longrightarrow E[g(X,Y)]$

LOTUS: $(X, Y) \longrightarrow E[g(X,Y)]$

Theorem

Let g be a function from $\mathbb{R}^2$ to $\mathbb{R}$. If X and Y are discrete, then

$$E\left(g\left(X,Y\right)\right)=\sum_x\sum_y g\left(x,y\right)P\left(X=x,Y=y\right).$$

If X and Y are continuous with joint PDF $f_{X,Y}$, then

$$E\left(g\left(X,Y\right)\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(x,y\right)f_{X,Y}\left(x,y\right)dxdy.$$

Expected Distance between Two Uniforms

① $E(|X-Y|) \overset{\text{LOTUS}}{=} \int_0^1 \int_0^1 |x-y| f_X(x) \cdot f_Y(y)\, dy = \int_0^1 \int_0^1 |x-y|\, dx\, dy$

$= \int_0^1 \int_y^1 (x-y)\, dx\, dy + \int_0^1 \int_0^y (y-x)\, dx\, dy = \frac{1}{3}.$

②

For $X, Y \overset{\text{i.i.d.}}{\sim} \text{Unif}(0,1)$, find $E(|X-Y|)$, $E(\max(X,Y))$, and $E(\min(X,Y))$.

$E(\max(X,Y))$: $\frac{2}{3}$; $E(\min(X,Y))$: $\frac{1}{3}$

$M = \max(X,Y)$; $L = \min(X,Y)$;

$M + L = X + Y.$

$\Rightarrow E(M+L) = E(X+Y) \Rightarrow E(M) + E(L) = E(X) + E(Y)$

$= \frac{1}{2} + \frac{1}{2} = 1$;

③ $M - L = \max(X,Y) - \min(X,Y) = \begin{cases} X - Y & \text{if } X \geq Y. \\ Y - X & \text{if } X < Y \end{cases} = |X-Y|$

$\Rightarrow E(M-L) = E(|X-Y|) = \frac{1}{3}$

$E(M) - E(L) = \frac{1}{3}.$ $\Rightarrow$ ④ $E(M) = \frac{2}{3}$; $E(L) = \frac{1}{3}$.

Expected Distance between Two Normals

① Method 1: $E(|X-Y|) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y| \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dx\,dy$

② Method 2: $Y \sim N(0,1)$; $-Y \sim N(0,1)$.

$X-Y \sim N(0,2)$ $\quad X-Y = \sqrt{2}Z$, $Z \sim N(0,1)$

For $X, Y \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, find $E(|X-Y|)$.

$\Rightarrow \quad E(|X-Y|) = E(|\sqrt{2}Z|) = \sqrt{2}\,E(|Z|)$.

$E(|Z|) = \int_{-\infty}^{\infty} |z| \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 2\int_{0}^{\infty} z \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$

$= \sqrt{\frac{2}{\pi}}$.

$\Rightarrow \quad E(|X-Y|) = \frac{2}{\sqrt{\pi}}$.

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2. Continuous Multivariate R.V.s
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Covariance

Definition

The *covariance* between r.v.s X and Y is

$$\mathrm{Cov}(X, Y) = E\left((X - EX)(Y - EY)\right).$$

Multiplying this out and using linearity, we have an equivalent expression:

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y).$$

Key Properties of Covariance

- ① $Cov(X, X) = Var(X)$.
- ② $Cov(X, Y) = Cov(Y, X)$.
- $Cov(X, c) = 0$ for any constant c.
- $Cov(a \cdot X, Y) = a \cdot Cov(X, Y)$ for any constant a.
- $Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)$.
- ④ $Cov(X + Y, Z + W) =$
  $Cov(X, Z) + Cov(X, W) + Cov(Y, Z) + Cov(Y, W)$.
- $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$. ; proof
- For n r.v.s $X_1, \ldots, X_n$,

$$Var(X_1 + \ldots + X_n) = Var(X_1) + \ldots + Var(X_n) + 2\sum_{i<j} Cov(X_i, Y_j).$$

$Var(X+Y) \overset{①}{=} Cov(X+Y, X+Y)$
$\overset{④}{=} Cov(X,X) + Cov(X,Y) + Cov(Y,X) + Cov(Y,Y)$
$\overset{①}{=} Var(X) + Var(Y) + Cov(X,Y) + Cov(Y,X)$
$\overset{②}{=} Var(X) + Var(Y) + 2Cov(X,Y)$

Proof

Correlation

Definition

Linear Correlation

The correlation between r.v.s X and Y is

$$\operatorname{Corr}(X, Y) = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X) \operatorname{Var}(Y)}}.$$

(This is undefined in the degenerate cases $\operatorname{Var}(X) = 0$ or $\operatorname{Var}(Y) = 0$.)

Definition

Given r.v.s X and Y, if $\operatorname{Cov}(X, Y) = 0$ or $\operatorname{Corr}(X, Y) = 0$, X and Y are uncorrelated.

Uncorrelated

X, Y are independent.

$$\mathrm{Cov}(X,Y) = E[(X-EX)(Y-EY)]$$

$$= \underbrace{E[(X-EX)]}_{0} \cdot \underbrace{E[Y-EY]}_{0} = 0$$

$$\underbrace{E[X]-E[X]}_{0}$$

Theorem

If X and Y are independent, then they are uncorrelated.

Uncorrelated $\not\Rightarrow$ Independent

$Cov(X,Y) = \underbrace{E(XY)}_{0} - \underbrace{E(X)E(Y)}_{0} = 0.$

Example: $X \sim N(0,1)$; $Y = X^2$;

1°. $E(X) = 0$; $\Rightarrow E(X) \cdot E(Y) = 0$

2°. $E(XY) = E(X^3) = 0$

$3^\circ \Rightarrow Cov(X,Y) = 0$.

$E(X^3)$ $g(x)$

$= \int_{-\infty}^{+\infty} x^3 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 0$

$g(-x) = -g(x)$

$\int_{-\infty}^{\infty} g(x)dx = \int_0^{\infty} g(x)dx + \int_{-\infty}^{0} g(x)dx$

$\int_{\infty}^{0} g(x)dx$ $-\int_0^{\infty} g(x)dx$

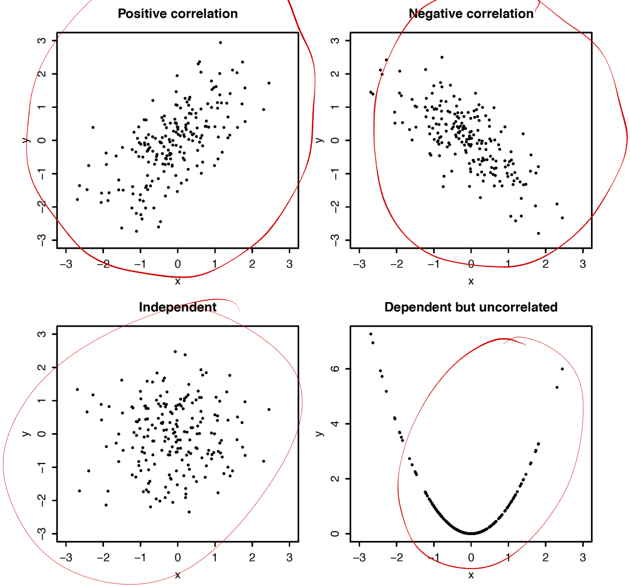
X, Y are linear uncorrelated (potentially nonlinear correlated)

but, X, Y are dependent.

Covariance & Correlation

- Measure a tendency of two r.v.s X & Y to go up or down together
- Positive covariance (Correlation): when X goes up, Y also tends to go up
- Negative covariance (Correlation): when X goes up, Y tends to go down

Correlation



Correlation Bounds

Cauchy-Schwarz Inequality

$E^2[X \cdot Y] \le E[X^2] \cdot E[Y^2]$

$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2$

$\left(\int_a^b f(x) g(x) dx\right)^2 \le \int_a^b f^2(x) dx \cdot \int_a^b g^2(x) dx$

1°. $\underbrace{f(t)}_{} \ge 0$: $f(t) = E[(X - tY)^2] = E[X^2 - 2tXY + t^2Y^2]$

$= E[X^2] - 2t E[XY] + t^2 E[Y^2]$

$= t^2 E[Y^2] - 2t E[XY] + E[X^2]$

Theorem

$\Delta = (2E[XY])^2 - 4 \cdot E[X^2] \cdot E[Y^2] \le 0 \Rightarrow E^2[XY] \le E[X^2] \cdot E[Y^2]$

For any r.v.s X and Y,

$$-1 \le \text{Corr}(X, Y) \le 1.$$

2° $X - E[X]$, $Y - E[Y]$ $\xrightarrow{C.S.}$ $E^2[(X - EX)(Y - EY)] \le E[(X - EX)^2] \cdot E[(Y - EY)^2]$

$\Rightarrow \text{Cov}^2(X, Y) \le \text{Var}(X) \cdot \text{Var}(Y)$

$\vec{a} \cdot \vec{b} \le \|a\| \cdot \|b\|$

$|\cos\theta| = \dfrac{\vec{a} \cdot \vec{b}}{\|a\| \cdot \|b\|} \le 1$

$\Rightarrow$ $\dfrac{\text{Cov}^2(X, Y)}{\text{Var}(X) \cdot \text{Var}(Y)} \le 1 \Rightarrow \text{Corr}^2(X, Y) \le 1$

$\Rightarrow -1 \le \text{Corr}(X, Y) \le 1$

Outline

1 Discrete Multivariate R.V.s

2 Continuous Multivariate R.V.s

3 Covariance and Correlation

Binomial Dis.

4 Multinomial Distribution

5 Multivariate Normal

Story

$K=2$ $\rightarrow$ Binomial.

Each of n objects is independently placed into one of k categories. An object is placed into category j with probability p_j, where the p_j are nonnegative and $\sum_{j=1}^{k} p_j = 1$. Let X_1 be the number of objects in category 1, X_2 the number of objects in category 2, etc., so that $X_1 + \ldots + X_k = n$. Then $X = (X_1, \ldots, X_k)$ is said to have the Multinomial distribution with parameters n and $\mathbf{p} = (p_1, \ldots, p_k)$. We write this as $\mathbf{X} \sim Mult_k(n, \mathbf{p})$.

$P = (P_1, P_2)$ $(P_1, 1-P_1)$

$P_1 + P_2 = 1$

$P_1 =$

Multinomial Joint PMF

Method 1°. $\binom{n}{n_1}\cdot\binom{n-n_1}{n_2}\cdot\binom{n-n_1-n_2}{n_3}\cdots\binom{\overbrace{n-n_1-\cdots-n_{k-1}}^{=n_k}}{n_k} = \frac{n!}{n_1!\cdots n_k!}$

Method 2°. $\frac{n!}{n_1!\cdot n_2!\cdot\cdots n_k!}$

Theorem

If $\mathbf{X} \sim \mathrm{Mult}_k(n, \mathbf{p})$, then the joint PMF of $\mathbf{X}$ is

n_i $p_1 \cdots p_i$

$$P(X_1 = n_1, \dots, X_k = n_k) = \frac{n!}{n_1! n_2! \dots n_k!} \cdot p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

for $n_1, \dots, n_k$ satisfying $n_1 + \cdots + n_k = n$.

Proof

Multinomial Marginals

Successful event : object → j th category.

Theorem

If $\mathbf{X} \sim \mathrm{Mult}_k(n, \mathbf{p})$*, then* $X_j \sim \mathrm{Bin}(n, p_j)$*.*

Multinomial Lumping

event A = " fall into i^{th} category "

B = " ...... j^{th} ...... "

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $P(A \cap B) = 0$

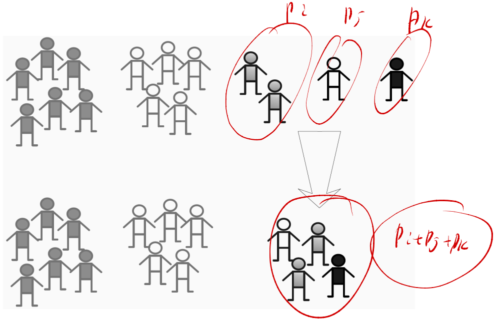
$= P(A) + P(B) = p_i + p_j$

Theorem

If $\mathbf{X} \sim \mathrm{Mult}_k(n, p)$*, then for any distinct* i *and* j*,* $X_i + X_j \sim \mathrm{Bin}(n, p_i + p_j)$*. The random vector of counts obtained from merging categories* i *and* j *is still Multinomial. For example, merging categories 1 and 2 gives*

$$(X_1 + X_2, X_3, ..., X_k) \sim \mathrm{Mult}_{k-1}(n, (p_1 + p_2, p_3, ..., p_n)) .$$

Multinomial Lumpling



Multinomial Conditioning

1°. Given n_1 objects in Category 1, the remaining $n-n_1$ objects landing into categories $2, \dots, k$: $\text{Multi}_{k-1}(n-n_1, \underline{p'})$

2°. $\underset{j \neq 1}{\underline{p_j'}}$ = Prob ($\underbrace{\text{landing into category } j}_{A}$ | $\underbrace{\text{not landing into category 1}}_{B}$)

$A \cap B = A$

= $\dfrac{\text{Prob. (landing into Category } j \text{)}}{\text{Prob. (not landing into Category 1)}}$

$= \dfrac{p_j}{1-p_1} = \dfrac{p_j}{p_2+\dots+p_k} = p_j'$

Theorem

If $\mathbf{X} \sim \mathrm{Mult}_k(n, p)$, *then*

$$(X_2, \dots, X_k) \mid X_1 = n_1 \sim \mathrm{Mult}_{k-1}\left(n - n_1, (p_2', \dots, p_k')\right),$$

where $p_j' = p_j / (p_2 + \dots + p_k)$.

$p_1+p_2+\dots+p_k = 1$

$p_2' + \dots + p_k' = 1$

renormalization

Covariance in A Multinomial

$Z \sim Bin(n,p)$; $Var(Z) = np(1-p)$.

1°. w.l.o.g. $i=1$; $j=2$.

$X_1 \sim Bin(n, p_1)$

$X_2 \sim Bin(n, p_2)$

$X_1 + X_2 \sim Bin(n, p_1+p_2)$

Theorem

Let $(X_1, ..., X_k) \sim \mathrm{Mult}_k(n, \mathbf{p})$, *where* $\mathbf{p} = (p_1, ..., p_k)$. *For* $i \neq j$, $\mathrm{Cov}(X_i, X_j) = -np_ip_j$. < 0

2°. $Var(X_1+X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$

$n(p_1+p_2)(1-p_1-p_2) = np_1(1-p_1) + np_2(1-p_2) + 2Cov(X_1, X_2)$

$\Rightarrow \quad Cov(X_1, X_2) = -np_1p_2 < 0$

Proof

Outline

1. Discrete Multivariate R.V.s
2. Continuous Multivariate R.V.s
3. Covariance and Correlation
4. Multinomial Distribution
5. Multivariate Normal

Multivariate Normal Distribution

Definition

A random vector $\mathbf{X} = (X_1, ..., X_k)$ is said to have a *Multivariate Normal* (MVN) distribution if every linear combination of the X_j has a Normal distribution. That is, we require

$$t_1 X_1 + \cdots + t_k X_k$$

to have a Normal distribution for any choice of constants $t_1, ..., t_k$. If $t_1 X_1 + \cdots + t_k X_k$ is a constant (such as when all $t_i = 0$), we consider it to have a Normal distribution, albeit a degenerate Normal with variance 0. An important special case is $k = 2$; this distribution is called the *Bivariate Normal* (BVN).

Non-example of MVN

$X = (X_1, \dots, X_k)$ if $X_j \sim N(\mu_j, \sigma_j^2)$

$X \neq MVN$

① $X \sim N(0,1)$, $S = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$; S and X are independent.

② $Y = S \cdot X \sim N(0,1)$: $\forall y \in \mathbb{R}$,

$P(Y \le y) = P(S \cdot X \le y)$

LOTP

$= P(SX \le y \mid S = 1) \cdot P(S = 1) + P(SX \le y \mid S = -1) P(S = -1)$

$= P(X \le y \mid S = 1) \cdot \frac{1}{2} + P(-X \le y \mid S = -1) \cdot \frac{1}{2}$

$= P(X \le y) \cdot \frac{1}{2} + P(-X \le y) \cdot \frac{1}{2}$

$P(X \ge -y) \cdot \frac{1}{2}$

$P(X \le y) \cdot \frac{1}{2}$

$= P(X \le y)$

$\Rightarrow Y \sim N(0,1)$

③ $P(X + Y = 0)$

$= P(S = -1) = \frac{1}{2}$.

X+Y is NOT Continuous.

$\Rightarrow$ Normal.

$(X, Y) \neq MVN$

Actual MVN

$\underline{\text{MGF}}$

Sum of independent Normal $\rightarrow$ Normal.

① $Z, W \overset{\text{i.i.d.}}{\sim} N(0,1)$

② $(Z+2W,\ 3Z+5W)$ is a MVN.

$\forall t_1, t_2 \in \mathbb{R}$, $\quad t_1(Z+2W) + t_2(3Z+5W)$

$$= (t_1+3t_2)\underline{Z} + (2t_1+5t_2)\underline{W}$$

$\sim$ Normal.

Theorem

$(X_1, X_2, X_3) \xrightarrow{MVN} \forall t_1, t_2, t_3 \in R.$

$t_1X_1 + t_2X_2 + t_3X_3 \sim$ Normal.

Theorem

Let $t_3 = 0$, $\forall t_1, t_2 \in R$.

If (X_1, X_2, X_3) is Multivariate Normal, then so is the subvector (X_1, X_2).

Theorem

Theorem

If $\mathbf{X} = (X_1, \dots, X_n)$ *and* $\mathbf{Y} = (Y_1, \dots, Y_m)$ *are MVN vectors with* $\mathbf{X}$ *independent of* $\mathbf{Y}$, *then the concatenated random vector* $\mathbf{W} = (X_1, \dots, X_n, Y_1, \dots, Y_m)$ *is Multivariate Normal.*

Parameters of MVN

$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)\right\}$ ✓

Joint PDF

$= \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left\{-\frac{1}{2}\left((x,y)\Sigma^{-1}(x,y)^T\right)\right\}$

$\rho=0$ $f_{X,Y}(x,y) = h(x) \cdot g(y)$

X and Y are independent!

Parameters of an MVN random vector $(X_1, \ldots, X_k)$ are:

- the mean vector $(\mu_1, \ldots, \mu_k)$, where $E(X_j) = \mu_j$.
- the covariance matrix, which is the $k \times k$ matrix of covariance between components, arranged so that the row i, column j entry is $Cov(X_i, X_j)$.

Standard Bivariate Normal. (X, Y)

$X \sim N(0,1)$

$Y \sim N(0,1)$

$\mathrm{Corr}(X,Y) = \rho \in (-1,1)$

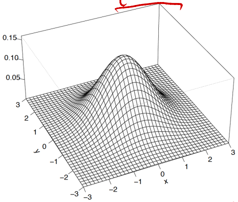
mean vector $(0, 0)$

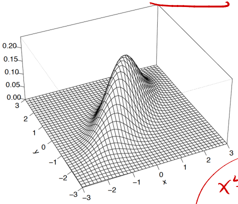
Covariance matrix $\Sigma = \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(Y,X) & Var(Y) \end{pmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$

Joint PDF of Bivariate Normal Distributions

$\rho = 0$ $\qquad$ $\rho = 0.75 > 0.$

Joint PDF





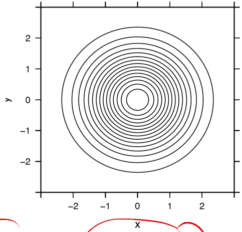
$X_1 = \frac{1}{\sqrt{2}}(x+y)$

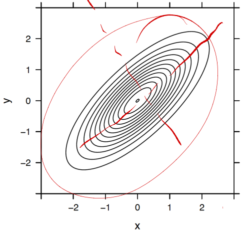
$y_1 = \frac{1}{\sqrt{2}}(x-y)$

$\Rightarrow \frac{x_1^2}{1+\rho} + \frac{y_1^2}{1-\rho} = C > 0$

$\frac{x^2+y^2-2\rho xy}{1-\rho^2} = C > 0$

contour





$\rho > 0$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\rho = 0$; $x^2 + y^2 = C > 0$

Joint MGF

① $W \sim$ Normal. MGF. $= e^{E[tW] + \frac{1}{2}Var(tW)}$

$E[e^{tW}] = e^{E[W]\cdot t + \frac{1}{2}Var(W)\cdot t^2}$ ✓

② $(X_1, \ldots X_k)$ MVN. $\Rightarrow$ Joint MGF. $E[e^{t_1X_1 + \cdots + t_kX_k}]$

Definition

$= \exp\{t_1E[X_1] + \cdots + t_kE[X_k] + \frac{1}{2}Var(t_1X_1 + \cdots + t_kX_k)\}$

The *joint MGF* of a random vector $\mathbf{X} = (X_1, \ldots, X_k)$ is the function which takes a vector of constants $\mathbf{t} = (t_1, \ldots, t_k)$ and returns

$$M(\mathbf{t}) = E\left(e^{\mathbf{t}'\mathbf{X}}\right) = E\left(e^{t_1X_1 + \cdots + t_kX_k}\right).$$

We require this expectation to be finite in a box around the origin in $\mathbb{R}^k$; otherwise we say the joint MGF does not exist.

Theorem

Theorem

Within an MVN random vector, uncorrelated implies independent. That is, if $\mathbf{X} \sim \mathrm{MVN}$ *can be written as* $\mathbf{X} = (\mathbf{X_1}, \mathbf{X_2})$*, where* $\mathbf{X_1}$ *and* $\mathbf{X_2}$ *are subvectors, and every component of* $\mathbf{X_1}$ *is uncorrelated with every component of* $\mathbf{X_2}$*, then* $\mathbf{X_1}$ *and* $\mathbf{X_2}$ *are independent.*
In particular, if (X, Y) *is Bivariate Normal and* $\mathrm{Corr}(X, Y) = 0$*, then* X *and* Y *are independent.*

Proof

① Recall fact: $X \sim N(\mu, v)$, $M_X(t) \overset{MGF}{=} e^{\mu t + \frac{1}{2}v^2 t^2}$

② Bivariate Normal (X, Y), $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$

$\mathrm{Corr}(X, Y) = \rho$.

$\Rightarrow \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$

$= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\rho\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}$

③ Joint MGF $M_{X,Y}(s,t) = E[e^{sX+tY}] = \mathrm{Exp}\{s\mu_1 + t\mu_2 + \frac{1}{2}\mathrm{Var}(sX+tY)\}$

$= \mathrm{Exp}\{s\mu_1 + t\mu_2 + \frac{1}{2}(s^2\sigma_1^2 + t^2\sigma_2^2 + 2st\sigma_1\sigma_2\rho)\}$

④ $\rho = 0$ $\Rightarrow M_{X,Y}(s,t) = \mathrm{Exp}\{s\mu_1 + t\mu_2 + \frac{1}{2}s^2\sigma_1^2 + \frac{1}{2}t^2\sigma_2^2\}$

Uncorrelated

$= \mathrm{Exp}\{s\mu_1 + \frac{1}{2}s^2\sigma_1^2\} \cdot \mathrm{Exp}\{t\mu_2 + \frac{1}{2}t^2\sigma_2^2\}$

$= M_X(s) \cdot M_Y(t)$ $\Rightarrow$ X and Y are independent.

Bivariate Normal Generation

1°. $Z = aX + bY$. $(a,b,c,d) \in \mathbb{R}$? $\Rightarrow$ (Z,W) bivariate Normal.

$W = cX + dY$

$Z, W \sim \mathcal{N}(0,1)$

$\text{Corr}(Z,W) = \rho$

2°. $E(Z) = E(aX+bY) = a\underbrace{E[X]}_{0} + b\underbrace{E[Y]}_{0} = 0$;

$E[W] = 0$;

Suppose that we have access to i.i.d. r.v.s $X, Y \sim \mathcal{N}(0, 1)$, but want to generate a Bivariate Normal (Z, W) with $\text{Corr}(Z, W) = \rho$ and Z, W marginally $\mathcal{N}(0, 1)$, for the purpose of running a simulation. How can we construct Z and W from linear combinations of X and Y ?

3°. $Var(Z) = Var(aX+bY) = Var(aX) + Var(bY) = a^2 Var(X) + b^2 Var(Y)$

$= a^2 + b^2 = 1$.

$Var(W) = Var(cX+dY) = c^2 + d^2 = 1$

4°. $Corr(Z,W) = Cov(Z,W) = \rho \Rightarrow Cov(aX+bY, cX+dY) = \rho$

$\Rightarrow ac\, Cov(X,X) + bd\, Cov(Y,Y) + 0 = \rho \Rightarrow ac + bd = \rho$

Solution

5° $\begin{cases} a^2+b^2=1 \\ c^2+d^2=1 \\ ac+bd=\rho \end{cases}$ Find one solution is enough.

$b=0$; $\Rightarrow a^2=1 \Rightarrow$ pick $a=1$; $c=\rho$.

$\Rightarrow d^2=1-\rho^2 \Rightarrow$ pick $d=\sqrt{1-\rho^2}$

6°. $Z = aX+bY = \underline{X}$.

$W = cX+dY = \underline{\rho X + \sqrt{1-\rho^2}\, Y}$.

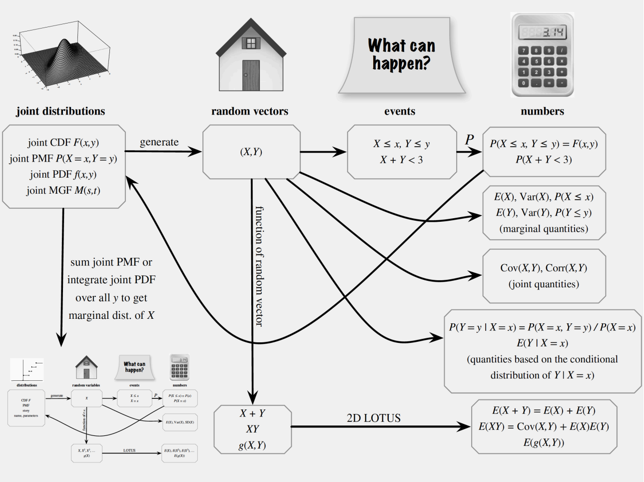
$\underline{(Z,W)}$ is the desired bivariate Normal distribution

<u>Joint PDF ?</u> $(Z,W) = \underline{g(X,Y)}$.

Summary 1: Discrete & Continuous

	Two discrete r.v.s	Two continuous r.v.s
Joint CDF	$F_{X,Y}(x,y) = P(X \le x, Y \le y)$	$F_{X,Y}(x,y) = P(X \le x, Y \le y)$
Joint PMF/PDF	$P(X = x, Y = y)$ • Joint PMF is nonnegative and sums to 1: $\sum_x \sum_y P(X = x, Y = y) = 1$.	$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$ • Joint PDF is nonnegative and integrates to 1: $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$. • To get probability, integrate joint PDF over region of interest.
Marginal PMF/PDF	$P(X = x) = \sum_y P(X = x, Y = y)$ $= \sum_y P(X = x \mid Y = y)P(Y = y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$ $= \int_{-\infty}^{\infty} f_{X\mid Y}(x\mid y)f_Y(y)dy$
Conditional PMF/PDF	$P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)}$ $= \frac{P(X = x \mid Y = y)P(Y = y)}{P(X = x)}$	$f_{Y\mid X}(y\mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$ $= \frac{f_{X\mid Y}(x\mid y)f_Y(y)}{f_X(x)}$
Independence	$P(X \le x, Y \le y) = P(X \le x)P(Y \le y)$ $P(X = x, Y = y) = P(X = x)P(Y = y)$ for all x and y. $P(Y = y \mid X = x) = P(Y = y)$ for all x and y, $P(X = x) > 0$.	$P(X \le x, Y \le y) = P(X \le x)P(Y \le y)$ $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all x and y. $f_{Y\mid X}(y\mid x) = f_Y(y)$ for all x and y, $f_X(x) > 0$.
LOTUS	$E(g(X,Y)) = \sum_x \sum_y g(x,y)P(X = x, Y = y)$	$E(g(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dxdy$

Summary 2: Multivariate Distribution



References

- Chapter 7 of **BH**
- Chapters 2 & 3 & 4 of **BT**