

# **Probability & Statistics for EECS:**

## **Homework #03**

Due on March 05, 2023 at 23:59

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## Problem 1

### Solution

1. We can directly calculate the probability of the system working and divide it into the following 5 cases.  
 First, five devices all function and there is just one situation, that is  $P(5) = 1 * p^5$ .  
 Second, four devices function and there are  $\binom{5}{4} = 5$  situations, that is  $P(4) = 5 * p^4 * (1 - p)$ .  
 Third, two devices function and there are two situations, those are 1 with 3 and 2 with 4. So  $P(2) = 2 * p^2 * (1 - p)^3$ .  
 Fourth, there devices function and there are eight situations. On the basis of 1 and 3 we have 3 cases, on the basis of 2 and 4 we also have 3 cases and on the crossover we have 2 cases namely 1, 4, 5 and 2, 3, 5.  
 Therefore,  $P(3) = 8 * p^3 * (1 - p)^2$ .  
 Fifth, one device cannot let system function, and it is similar with the condition of zero device.  
 Finally,  $P(\text{system functions}) = P(2) + P(3) + P(4) + P(5) = 2p^5 - 5p^4 + 2p^3 + 2p^2$ .

## Problem 2

### Solution

- (a) Assume event C: admit an applicant.  
 Assume event A: applicant is good at Apex.  
 Assume event B: applicant is good at Genshin Impact.  
 And from the question, we have  $C = A \text{ or } B$ .  
 Then we have  $1 = P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C)$ . And then we can assume two events are disjoint, then we can explain it intuitively, because the sum of the probability of two events is 1. Due to the increasement of  $P(A)$ ,  $P(B)$  must decrease. And we can also intuitively explain similarly when two events are joint, because we donnot need to care the part where they are joint.
- (b) From the question and the Bayes' formula, we have  $P(A \cap B \cap C) = P(A \cap B|C)P(C) = P(A|B \cap C)P(B|C)P(C)$ .  
 and then we can eliminate  $P(C)$  at both sides, and then have  $P(A \cap B|C) = P(A|B \cap C)P(B|C)$ .  
 And we have  $P(A|B \cap C) < P(A|C)$ . Therefore we can prove  $P(A \cap B|C) \neq P(B|C)P(A|C)$ .  
 So A and B are conditionally dependent given C.

## Problem 3

### Solution

- (a)  $p_n = \frac{1}{6}(p_{n-1} + p_{n-2} + p_{n-3} + p_{n-4} + p_{n-5} + p_{n-6})$  for  $n > 0$ . At the meantime,  $p_0 = 1$  and  $p_k = 0$  for  $k < 0$ .
- (b) First, we need to calculate  $p_1, p_2, p_3, p_4, p_5, p_6$ .  
 $p_1 = \frac{1}{6}(p_{-1} + p_{-2} + p_{-3} + p_{-4} + p_{-5} + p_0) = \frac{1}{6}$ .  
 $p_2 = \frac{1}{6}(p_1 + p_{-1} + p_{-2} + p_{-3} + p_{-4} + p_0) = \frac{7}{36}$ .  
 $p_3 = \frac{1}{6}(p_1 + p_2 + p_{-1} + p_{-2} + p_{-3} + p_0) = \frac{49}{216}$ .  
 $p_4 = \frac{1}{6}(p_1 + p_2 + p_3 + p_{-1} + p_{-2} + p_0) = \frac{343}{1296}$ .  
 $p_5 = \frac{1}{6}(p_1 + p_2 + p_3 + p_4 + p_{-1} + p_0) = \frac{2401}{7776}$ .  
 $p_6 = \frac{1}{6}(p_1 + p_2 + p_3 + p_4 + p_5 + p_0) = 0.3602$ .  
 $p_7 = \frac{1}{6}(p_1 + p_2 + p_3 + p_4 + p_5 + p_6) = 0.2536$ .

- (c) The expectation for each throw of the dice is  $E[\text{one throw of dice}] = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$ .

And from the definition of the expectation we can interpret this as meaning the roll will hit 1 in every 3.5 times.

Therefore, the probability of  $p_n = \frac{1}{3.5} = \frac{2}{7}$  when  $n$  converging to infinity.

## Problem 4

### Solution

- (a)  $p_{i,j} = \frac{n-j+1}{n}p_{i-1,j-1} + \frac{j}{n}p_{i-1,j}$  for  $i \geq 2$  and  $1 \leq j \leq n$ .

Also for the purpose of explanation in b we define other cases here:

First,  $p_{i,j} = 0$  for  $j > i$ .

Second,  $p_{i,j} = 0$  for  $j = 0$  or  $i = 0$ .

Third,  $p_{1,1} = 1$ .

- (b) From the recursive equation and the definition, we can calculate  $p_{i,j}$  by considering  $p_{i,j}$  as a matrix, in which  $i$  is the column and  $j$  is row.

At the beginning, we can calculate  $p_{i,1}$  firstly for  $i \geq 2$ , because  $p_{i,0} = 0$ .

Therefore, we can use  $p_{1,1}$  to column-by-column calculate  $p_{i,1}$  for  $i \geq 2$  by recursive equation.

Secondly, we can use  $p_{i,1}$  for  $i \geq 2$  which we have calculated and  $p_{1,2} = 0$  to column-by-column calculate  $p_{i,2}$  for  $i \geq 2$  by recursive equation.

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Finally, we can step by step to calculate the  $j$ th row, and get  $p_{i,j}$  similarly for  $i \geq 2$ .

So we can get it by calculating layer-by-layer.

## Problem 5

### Solution

- (a)  $p_k = p * p_{k-1} + q * p_{k+1}$  for all  $k \geq 0$ . And for  $p_0$ , because we must start at the original point, then  $p_0 = 1$ .

- (b) First we can simplify the recursive equation to obtain the isoperimetric series, that is

$$\frac{p}{q} = \frac{p_{k+1} - p_k}{p_k - p_{k-1}}, \text{ and we can assume } \alpha_k = \frac{p}{q} \alpha_{k-1}.$$

We can express  $p_k$  by summing the equiprobable series as  $p_k - p_0 = \frac{(p_1 - p_0) * (1 - (\frac{p}{q})^k)}{1 - \frac{p}{q}}$ .

So what we need to get is just  $p_1$ .

Because if we finally reach 1 and assume one process has been finished, and then we start another process to reach  $k$ . Therefore through this way we can also get  $p_k$ , by considering the new original point as 1 not 0. So we have a new formula  $p_k = p_1 * p_{k-1}$ . And then we can have  $p_k = p_1^k$ .

And then we can calculate  $p_1$ , through the formula and the recursive equation, and we have  $p_1 = p_0 * p + p_2 * q = p + p_1^2 q$ .

And then we have one equation about  $p_1$ , that is  $p_1^2 q - p_1 + p = 0$ . And then we can use the root formula  $p_{1,1 \text{ or } 2} = \frac{1 \pm \sqrt{1 - 4pq}}{2q}$ .

And then we can simplify it as  $p_{1,1 \text{ or } 2} = \frac{1 \pm |2p-1|}{2q}$ .

For  $p = 0.5$ ,  $p_{1,1 \text{ or } 2} = 1$ , and then  $p_1 = 1$ . So  $p_k = 1$ .

For  $p > 0.5$ , and because  $0 \leq p_1 \leq 1$ , we have  $p_1 = 1$ . So  $p_k = 1$ .

For  $p < 0.5$ , and because  $0 \leq p_1 \leq 1$ , we have  $p_1 = \frac{p}{q}$  or  $p_1 = 1$ . So when  $p_1 = 1$ ,  $p_k = 1$ . And for

another case,  $p_k = \left(\frac{p}{q}\right)^k$ .