Probability & Statistics for EECS: Homework #04

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Problem 1

Solution

a First assume that event A is that Monty open door 2, and event B is that Monty open door 3.
And $P(A) = p, P(B) = (1-p)$.
Event D_i is that car is behind the i door, which is denoted by $P(D_i) = \frac{1}{3}$ and $i \in 1, 2, 3$.
Event C is that I win the game after switching, which is denoted by $P(C)$.
And from the question, the door we open at the beginning is door 1. So $P(C) = \sum_{i=1}^{3} P(C|D_i)P(D_i)$.
Therefore, if car is behind door 1, we cannot win, so $P(C|D_1) = 0$. And if car is behind door 2 or door 3, no matter which door Monty openes, we must win if we switch, so $P(C|D_2) = P(C|D_3) = 1$.
Finally, $P(C) = 0 * 1/3 + 1 * 1/3 + 1 * 1/3 = 2/3$.

b From what we have defined and calculated in (a), then what we need to calculate is $P(C|A)$.
And from bayes formula, we have $P(C|A) = \frac{P(C,A)}{P(A)}$. And for A, there are two cases.
First is when the car behind door 3, Monty has to open door 2, that is $P(A|D_3)P(D_3) = \frac{1}{3} * 1$.
Second is when the car behind door 1, Monty opens door door 2, that is $P(A|D_1)P(D_1) = \frac{1}{3}p$.
So $P(A) = P(A|D_3)P(D_3) + P(A|D_1)P(D_1) = \frac{1}{3}(p+1)$.
And for $P(C, A)$, it is equal to first case.
Therefore $P(C|A) = \frac{\frac{1}{3}}{\frac{1}{3}(p+1)} = \frac{1}{p+1}$.

c From what we have defined and calculated in (a), then what we need to calculate is $P(C|B)$.
And from bayes formula, we have $P(C|B) = \frac{P(C,B)}{P(B)}$. And for B, there are two cases.
First is when the car behind door 2, Monty has to open door 3, that is $P(B|D_2)P(D_2) = \frac{1}{3} * 1$.
Second is when the car behind door 1, Monty opens door door 3, that is $P(B|D_1)P(D_1) = \frac{1}{3}(1-p)$.
So $P(B) = P(B|D_2)P(D_2) + P(B|D_1)P(D_1) = \frac{1}{3}(2-P)$.
And for $P(C, B)$, it is equal to first case.
Therefore $P(C|B) = \frac{\frac{1}{3}}{\frac{1}{3}(2-p)} = \frac{1}{2-p}$.

Problem 2

Solution

(a) No. From the definition of the valid PMF, we can assume that $P(X = n) = a * \frac{1}{n}$. And then we can have :
$\sum_{k=1}^{\infty} P(X = k) = a * \sum_{k=1}^{\infty} \frac{1}{k}$.
From what we have learned, we know this series diverges, so even if multiplied by the constant a, it cannot be = 1. Therefore there is no valid PMF. And then we get it.

(b) Yes. From the definition of the valid PMF, we can assume that $P(X = n) = a * \frac{1}{n^2}$. And then we can have :
$\sum_{k=1}^{\infty} P(X = k) = a * \sum_{k=1}^{\infty} \frac{1}{k^2}$.
From what we have learned, we know this series converges to $\frac{\pi^2}{6}$. So we can let the constant a be equal to $\frac{6}{\pi^2}$. And then $\sum_{k=1}^{\infty} P(X = k)$ is equal to 1. Therefore there is a valid PMF. Finally we get it.

Problem 3

Solution

(a) X and Y have the same distribution. For X we can easily get X is a discrete uniform distribution. We can denote it as $X \sim DUnif(1,2,3,4,5,6,7)$. So $P(X=i)=\frac{1}{7}$ for every $1 <= i <= 7$. And from the question we can know Y is the next day after X. So if X=1 then Y=2. If X=2 then Y=3 $\cdots$ If X=7 then Y=1. At the same time, Both of them have the same probability at this moment. Therefore for Y, we have $P(Y=i)=\frac{1}{7}$ for every $1 <= i <= 7$. So Y is distributed as $Y \sim DUnif(1,2,3,4,5,6,7)$. Then X and Y has the same distribution.
For $P(X<Y)$, there is just one case $X>Y$, that is $X=7$ and $Y=1$. Therefore, $P(X<Y) = 1-P(X>=Y) = 1-\frac{1}{7} = \frac{6}{7}$.

Problem 4

Solution

(a) For the new r.v. X, we choose two coins randomly and the probability is 0.5. At the same time, the probability of first one landing head is p_1, and another one is p_2.
So, PMF of X is $P(X=k) = 1/2\binom{n}{k}p_1^k(1-p_1)^{n-k} + 1/2\binom{n}{k}p_2^k(1-p_2)^{n-k}$ for $0 <= k <= n$.

(b) Because PMF of X and $p_1 = p_2$, then we have PMF of X is $P(X=k) = \binom{n}{k}p_1^k(1-p_1)^{n-k}$. And this is obviously the binomial distribution. And it can be denoted by $X \sim Bin(n, p_1)$.

(c) For $p_1 \neq p_2$, if we choose different coin then the left part and the right part of the formula would not be equal, because the probability of landing heads are different, then we cannot seem it as binomial distribution after adding up.

Problem 5

Solution

(a) $X \oplus Y$ is distributed as Bernoulli distribution.
That is because when $X=Y$, $X \oplus Y = 0$, that is $P(X=Y) = P(X=Y=1) + P(X=Y=0) = \frac{1}{2}p + \frac{1}{2}(1-p) = 1/2$.
And when $X \neq Y$, $X \oplus Y = 1$, that is $P(X \neq Y) = P(X=0, Y=1) + P(X=1, Y=0) = \frac{1}{2}p + \frac{1}{2}(1-p) = 1/2$.
Therefore it is obviously the Bernoulli distribution.

(b) First $P(X \oplus Y|X) = \frac{P(X \oplus Y, X)}{P(X)}$.
So let we consider it at X=1 and $X \oplus Y = 1$.
$P(X \oplus Y = 1|X=1) = \frac{P(X \oplus Y=1, X=1)}{P(X=1)} = \frac{P(Y=0,X=1)}{P(X=1)} = \frac{\frac{1}{2}p}{p} = \frac{1}{2}$.
Then $X \oplus Y$ is independent of X. Second, $P(X \oplus Y|Y) = \frac{P(X \oplus Y, Y)}{P(Y)}$.
So let we consider it at $X \oplus Y = 1$ and Y=1.
$P(X \oplus Y = 1|Y=1) = \frac{P(X \oplus Y=1, Y=1)}{P(Y=1)} = \frac{P(Y=1,X=0)}{P(Y=1)} = \frac{\frac{1}{2}(1-p)}{\frac{1}{2}} = 1-p$.
Then if $p=\frac{1}{2}$, then $X \oplus Y$ is independent of Y. Otherwise it is not.

(c) First we need to consider about $Y_J \sim Bern(1/2)$, and $Y_J = \bigoplus_{j \in J} X_j$ and $X_j \sim Bern(1/2)$ for $j \in \{1, 2, \cdots, n\}$.
When J=1, obviously $X_1 = Y_1 \sim Bern(1/2)$.
Then we can assume that when $J=k$, we have $Y_k = \bigoplus_{j=1}^{k} X_j \sim Bern(1/2)$.
Then when $J=k+1$, we have $P(Y_{k+1}) = P((\bigoplus_{j=1}^{k} X_j) \oplus X_{k+1} = 1) = P(\bigoplus_{j=1}^{k} X_j = 0, X_{k+1} =$

$1) + P(\bigoplus_{j=1}^{k} X_j = 1, X_{k+1} = 0)$.
Therefore $P(Y_{k+1}) = \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} = \frac{1}{2}$ from we have known.
Then we get $Y_{k+1} \sim Bern(1/2)$.
Finally, we can get $Y_J \sim Bern(1/2)$ for $J >= 1$.
Second, choose two subsets of Y, and they are disjoint and unempty, denoted as M and N.
$P(Y_M = 1, Y_N = 1) = P(Y_a \bigoplus Y_b = 1, Y_a \bigoplus Y_c = 1)$
$= 0.5P(Y_a \bigoplus Y_b = 1, Y_a \bigoplus Y_c = 1|Y_a = 1) + 0.5P(Y_a \bigoplus Y_b = 1, Y_a \bigoplus Y_c = 1|Y_a = 0)$.
$= 0.5P(Y_b = 0, Y_c = 0|Y_a = 1) + 0.5P(Y_b = 1, Y_c = 1|Y_a = 1) = 0.5*0.5*0.5+0.5*0.5*0.5 = 0.5*0.5 = \frac{1}{4} = P(Y_M = 1)P(Y_N = 1)$.
Therefore, we can get $2^n - 1 R.V.s$ are pairwise independent. Third, choose three subsets called x_1, x_2, x_3, and x_1, x_2 are disjoint and $x_3 = x_1 \cup x_2$. And from the definition, we can get Y_1, Y_2, Y_3.
Then we have $P(Y_1 = 1, Y_2 = 1, Y_3 = 1) = 1/4$, but $P(Y_1 = 1)P(Y_2 = 1)P(Y_3 = 1) = 1/8$.
Therefore they are not independent.