Probability & Statistics for EECS: Homework #07

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Problem 1

Solution

(a) First check whether it is valid:
(1)increasing:$F'(x) = \frac{1}{\pi \sqrt[2]{x} \sqrt[2]{1-x}}$, and $1 < x < 0$, so $F'(x) > 0$. Therefore, F is monotone increasing.
(2)right-continuous: Because $argsinx$ is continuous when $0 < x < 1$ without doubts, then so do $F(x)$.
(3)limit: when x is 0^+, $F(x) \approx \frac{2}{\pi}x$, then $F(0^+) = 0$.
and when x is 1^-, $F(x) = \frac{2}{\pi}\frac{\pi}{2} = 1$. Therefore it satisfies the limitation.
And then $f(x) = \frac{1}{\pi \sqrt[2]{x} \sqrt[2]{1-x}}$ for $0 < x < 1$. Otherwise 0.

(b) That is because PDF means Probability desity function, and when we want to calculate the probability of any concrete point, it is equal to 0 anyway. Therefore, even though f(x) goes to ∞ as x approaches 0 or 1, it doesnot matter, because it is just one point. It is still possible for us to calculate the area of the PDF and let it be equal to 1.

Problem 2

a Because F is a CDF which is continuous and strictly increasing, then we can let $U \sim Unif(0, 1)$, and $X = F^{-1}(U)$.
And the meantime, $F(X) = F(F^{-1}(U)) = U$, so we can have $u = F(x)$.
And what we want is the area under the curve of the quantile function from 0 to 1.
So $\int_0^1 F^{-1}(u)\, du = \int_{-\infty}^{\infty} F^{-1}(F(x))\, dF(x) = \int_{-\infty}^{\infty} xf(x)\, dx = \mu$.
Therefore we get it.

Problem 3

Solution

(a) First for U_i, $1 <= i <= n$, they are i.i.d. $Unif(0, 1)$, then we can have:

$$
\begin{aligned}
P(X < x) &= P(X <= x) \\
&= P(max(U_1, \cdots, U_n) <= x) \\
&= P(U_1 <= x, \cdots, U_n <= x) \\
\textit{Because they are independent} \\
&= P(U_1 <= x) \cdots P(U_n <= x) \\
&= \frac{x-0}{1} \cdots \frac{x-0}{1} \\
&= x^n
\end{aligned}
$$

So because we have got CDF of X, then PDF of X is nx^{n-1}, when $0 < x < 1$. Otherwise is 0.

Problem 3 continued on next page. . .

At the same time

$$\begin{aligned}E[X] &= \int_0^1 x * n * x^{n-1}\, dx \\ &= n\int_0^1 x^n\, dx \\ &= \frac{n}{n+1}\end{aligned}$$

Problem 4

Solution

(a) Obviously, $X + Y = 1$, so we can have $R = \frac{X}{Y} = \frac{X}{1-X}$. And we also have the support of R is $(0, 1)$
Then for $F(R) = P(R <= x) = P(\frac{X}{1-X} <= x) = P(X <= \frac{x}{1+x})$, and $0 <= x <= 1$.
And for X, we have $P(X <= x) = 2x$ for $0 <= x <= 0.5$.
Therefore, we can have $P(R <= x) = \frac{2x}{1+x}$, for $0 <= x <= 1$.
And then, we can also get PDF of R, that is $f(R) = F'(R) = \frac{2}{(1+x)^2}$, for $0 <= x <= 1$.

(b) $E[R] = \int_0^1 x\frac{2}{(1+x)^2}\, dx = 2ln2 - 1$.

(c) $E[\frac{1}{R}] = \int_0^1 \frac{2}{x(1+x)^2}\, dx = \infty$.

Problem 5

Solution

(a) $T = G\Delta t$.

(b)

$$\begin{aligned}P(T >= t) &= P(G\Delta t >= t) \\ &= P(G >= \frac{t}{\Delta t}) \\ &= P(G >= \left\lfloor \frac{t}{\Delta t} \right\rfloor) \\ &= (1 - \lambda\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor}\end{aligned}$$

Therefore, the CDF of T is $1 - (1 - \lambda\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor}$.

(c)

$$
\begin{aligned}
1-(1-\lambda\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor} &= \lim_{\Delta t \to 0} 1-(1-\lambda\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor} \\
&= 1-\lim_{\Delta t \to 0}(1-\lambda\Delta t)^{\frac{t}{\Delta t}} \\
&= 1-\lim_{\Delta t \to 0}(1-\lambda\Delta t)^{\frac{\lambda t}{\lambda\Delta t}} \\
&= 1-(\frac{1}{e})^{\lambda t} \\
&= 1-e^{-\lambda t}
\end{aligned}
$$

Therefore, the CDF of T converges to the Expo(λ) CDF.

Problem 6

Solution

(a) For $E[max(Z-c,0)]$,

$$
\begin{aligned}
E[max(Z-c,0)] &= \int_{-\infty}^{c} 0\,dz + \int_{c}^{\infty}(z-c)\varphi(z)\,dz \\
&= 0 + \int_{-\infty}^{-c} -z\varphi(z)\,dz - c\int_{c}^{\infty}\varphi(z)\,dz \\
&= \varphi(-c)-\varphi(-\infty)-c(1-\Phi(c)) \\
\textit{Because of the symmetry of PDF}& \\
&= \varphi(c)-0-c(1-\Phi(c)) \\
&= \varphi(c)-c(1-\Phi(c))
\end{aligned}
$$