Probability & Statistics for EECS: Homework #09

Due on April 16, 2023 at 23:59

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Problem 1

Solution

(a) X,Y are discrete

$$\begin{aligned} P(Y=y|X=x) &= \frac{P(Y=y,X=x)}{P(X=x)} \\ &= \frac{P(X=x|Y=y)P(Y=y)}{P(X=x)} \end{aligned}$$

This uses Bayes rule and the confine of it is $P(Y=y), P(X=x) > 0$.

(b) X is discrete, Y is continuous

$$\begin{aligned} P(Y \in (y-\varepsilon, y+\varepsilon)|X=x) &= \frac{P(X=x, Y \in (y-\varepsilon, y+\varepsilon))}{P(X=x)} \\ &= \frac{P(X=x|Y \in (y-\varepsilon, y+\varepsilon))P(Y \in (y-\varepsilon, y+\varepsilon))}{P(X=x)} \\ \frac{P(Y \in (y-\varepsilon, y+\varepsilon)|X=x)}{2\varepsilon} &= \frac{P(X=x|Y \in (y-\varepsilon, y+\varepsilon))\frac{P(Y \in (y-\varepsilon, y+\varepsilon))}{2\varepsilon}}{P(X=x)} \end{aligned}$$

$Because\ the\ definition\ of\ integral\ and\ when\ \varepsilon = 0$

$$f_Y(y|X=x) = \frac{P(X=x|Y=y)f_Y(y)}{P(X=x)}$$

This uses Bayes rule and the confine of it is $P(Y \in (y-\varepsilon, y+\varepsilon)), P(X=x) > 0$.

(c) X is continuous, Y is discrete

$$\begin{aligned} P(Y=y|X \in (x-\varepsilon, x+\varepsilon)) &= \frac{P(X \in (x-\varepsilon, x+\varepsilon), Y=y)}{P(X \in (x-\varepsilon, x+\varepsilon))} \\ &= \frac{P(X \in (x-\varepsilon, x+\varepsilon)|Y=y)P(Y=y)}{P(X \in (x-\varepsilon, x+\varepsilon))} \\ P(Y \in (y-\varepsilon, y+\varepsilon)|X=x) &= \frac{\frac{P(X \in (x-\varepsilon, x+\varepsilon)|Y=y)}{2\varepsilon}P(Y=y)}{\frac{P(X \in (x-\varepsilon, x+\varepsilon))}{2\varepsilon}} \end{aligned}$$

$Because\ the\ definition\ of\ integral\ and\ when\ \varepsilon = 0$

$$P(Y=y|X=x) = \frac{f_X(x|Y=y)P(Y=y)}{f_X(x)}$$

This uses Bayes rule and the confine of it is $P(Y=y), P(X \in (x-\varepsilon, x+\varepsilon)) > 0$.

(d) X,Y are continuous

$$\begin{aligned} P(Y \in (y-\varepsilon, y+\varepsilon)|X \in (x-\varepsilon, x+\varepsilon)) &= \frac{P(Y \in (y-\varepsilon, y+\varepsilon), X \in (x-\varepsilon, x+\varepsilon))}{P(X \in (x-\varepsilon, x+\varepsilon))} \\ \frac{P(Y \in (y-\varepsilon, y+\varepsilon)|X \in (x-\varepsilon, x+\varepsilon))}{2\varepsilon} &= \frac{\frac{P(Y \in (y-\varepsilon, y+\varepsilon), X \in (x-\varepsilon, x+\varepsilon))}{4\varepsilon^2}}{\frac{P(X \in (x-\varepsilon, x+\varepsilon))}{2\varepsilon}} \end{aligned}$$

$Because\ the\ definition\ of\ integral\ and\ when\ \varepsilon = 0$

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{f_{X|Y(x|y)f_Y(y)}}{f_X(x)} \end{aligned}$$

This uses Bayes rule and the confine of it is $P(Y \in (y-\varepsilon, y+\varepsilon)), P(X \in (x-\varepsilon, x+\varepsilon)) > 0$.

Problem 2

(a) Because $N = X + Y$, then $P(N = x + y|X = x, Y = y) = 1$ and we can have:
$P(X = x, Y = y, N = n) = P(X = x, Y = y)$
Because X and Y are i.i.d., then we can have:
$P(X = x, Y = y) = P(X = x)P(Y = y) = p * q^x * p * q^y = p^2 * q^{x+y} = p^2q^n$, for nonnegative integers x, y, n, and $n = x + y$.

(b) From(1), we can also have:

$$\begin{aligned} P(X = x, N = n) &= P(X = x, Y = n - x, N = n) \\ &= P(X = x, Y = n - x) \\ &= P(X = x)P(Y = n - x) \\ &= p * q^x * p * q^{n-x} = p^2q^n \end{aligned}$$

For $0 <= x <= n$.

(c)

$$\begin{aligned} & \textit{Bayes rule} \\ P(X = x|N = n) &= \frac{P(X = x, N = n)}{P(N = n)} \\ & LOTP \\ &= \frac{p^2q^n}{\sum_{x=0}^{n} P(N = n|X = x)P(X = x)} \\ N &= X + Y \\ &= \frac{p^2q^n}{\sum_{x=0}^{n} P(X + Y = n|X = x)P(X = x)} \\ &= \frac{p^2q^n}{\sum_{x=0}^{n} P(Y = n - x|X = x)P(X = x)} \\ & Because\ of\ X, Y\ are\ i.i.d. \\ &= \frac{p^2q^n}{\sum_{x=0}^{n} P(Y = n - x)P(X = x)} \\ &= \frac{p^2q^n}{\sum_{x=0}^{n} p^2q^n} \\ &= \frac{p^2q^n}{(n + 1)p^2q^n} \\ &= \frac{1}{n + 1} \end{aligned}$$

At the same time, obviously we can get $N \sim NBin(2, p)$. That is because X, Y are i.i.d. $Geom(p)$, and N=X+Y, then we can easily find that all we need to calculate is the number of the failture times before the second success. So we need to find one success and n failures in the first n+1 times. Then we have $P(N = n) = \binom{n+2-1}{2-1} p * p * q^n = (n + 1)p^2q^n$. Therefore we can also get it.
Moreover, when N is given, $P(X = x|N = n) = \frac{1}{n+1}$ is exact.

Problem 3

Solution

(a) From the question, we have

$$\begin{aligned} P(X <= x | X > c) &= \frac{P(X <= x, X > c)}{P(X > c)} \\ &= \frac{P(c < X <= x)}{P(X > c)} \\ &= \frac{P(X <= x) - P(X <= c)}{1 - P(X <= c)} \\ &= \frac{1 - e^{-\lambda x} - 1 + e^{-\lambda c}}{1 - 1 + e^{-\lambda c}} \\ &= 1 - e^{-\lambda(x-c)} \end{aligned}$$

Then the conditional PDF of X given $X > c$ is $\lambda e^{-\lambda(x-c)}$.

(b) First if $x <= c$, then $P(X <= x | X < c) = 1$.
On the other hand when $x > c$, we have

$$\begin{aligned} P(X <= x | X < c) &= \frac{P(X <= x, X < c)}{P(X < c)} \\ &= \frac{P(X <= x)}{P(X < c)} \\ &= \frac{1 - e^{-\lambda x}}{1 - e^{-\lambda c}} \end{aligned}$$

Then the conditional PDF of X given $X < c$ is $\frac{\lambda}{1-e^{-\lambda c}} e^{-\lambda x}$.

Problem 4

Solution

(a) Because $M = max(U_1, U_2, U_3)$, then $P(M <= m) = P(U_1 <= m, U_2 <= m, U_3 <= m)$.
Because U_1, U_2, U_3 are i.i.d. Unif(0,1), then

$$F_M(m) = P(M <= m) = P(U_1 <= m)P(U_2 <= m)P(U_3 <= m) = m^3$$

For, $0 <= m <= 1$.
Then

$$f_M(m) = 3m^2$$

For, $0 <= m <= 1$.
Because $P(M <= m) = P(L <= l, M <= m) + P(L > l, M <= m)$,
then $P(L <= l, M <= m) = P(M <= m) - P(L > l, M <= m)$.
So we need to calculate $P(L > l, M <= m)$ first and we also have the confine of l, which is $0 <= l <= m <= 1$.

$$\begin{aligned} P(L > l, M <= m) &= P(min(U_1, U_2, U_3) > l, max(U_1, U_2, U_3) <= m) \\ &= P(l < U_1 <= m, l < U_2 <= m, l < U_3 <= m) \\ &= (m - l)^3 \end{aligned}$$

Then

$$\begin{aligned} F_{L,M}(l, m) = (L <= l, M <= m) &= m^3 - (m - l)^3 \\ &= 3m^2 l - 3ml^2 + l^3 \end{aligned}$$

And then we can also get the PDF by respectively differentiating l and m, that is

$$f_{L,M}(l, m) = 6(m - l)$$

for $0 <= l <= m <= 1$.

(b) What we want to get is $f_{M|L}(m, l) = \frac{f_{M,L}(m,l)}{f_L(l)}$,
so we just need to calculate $f_L(l)$, for $0 <= l <= 1$.

$$\begin{aligned} 1 - F_L(l) = P(L > l) &= P(U_1 > l)P(U_2 > l)P(U_3 > l) = (1 - l)^3 \\ F_L(l) &= 1 - (1 - l)^3 \\ F_L(l) &= 3l - 3l^2 + l^3 \\ f_L(l) &= 3 - 6l + 3l^2 \\ f_L(l) &= 3(1 - l)^2 \end{aligned}$$

So $f_{M|L}(m, l) = \frac{2(m-l)}{(1-l)^2}$, for $0 <= l <= m <= 1$.

Problem 5

Solution

(a) First we can divide this problem into three cases.
case1, $l > m$, then $P(L = l, M = m) = 0$
case2, $l = m$, then $P(L = l, M = m) = P(X = l, Y = l)$
Because they are independent Geom distribution,
so $P(X = l, Y = l) = P(X = l)P(Y = l) = p * q^l * p * q^l = p^2 * q^{2l}$.
case3 is as follow,

$$\begin{aligned} P(L = l, M = m) &= P(X = l, Y = m) + P(X = m, Y = l) \\ \textit{Because } X, Y \textit{ are i.i.d.} \\ &= P(X = l)P(Y = m) + P(X = m)P(Y = l) \\ &= p * q^l * p * q^m + p * q^m * p * q^l \\ &= 2p^2 q^{m+l} \end{aligned}$$

From the cases we divied, obviously they are not independent. Because if we have known something about L, then we can also have some information about M, due to $L <= M$.

(b) From (1), we can have

$$\begin{aligned} P(L = l) &= \sum_{m=0}^{\infty} P(L = l, M = m) \\ &= P(L = l, M = l) + \sum_{m=l+1}^{\infty} P(L = l, M = m) \\ &= p^2 q^{2l} + 2p^2 q^l \sum_m m = l + 1^{\infty} q^m \\ &= p^2 q^{2l} + 2pq^{2l+1} \end{aligned}$$

By using story, all we need to do is to translate $L = min(X, Y)$.
L can be seemed as X,Y at least happen one for the first success when L=l. So L is also a Geom distribution and $p_l = 2p - p^2$.
That is $L \sim Geom(2p - p^2)$.
Then $P(L = l) = (2p - p^2)(1 - 2p + p^2)^l = (2p - p^2)((1 - p)^2)^l = (2p - p^2)q^{2l} = (p^2 + 2p(1 - p))q^{2l} = p^2 q^{2l} + 2pq^{2l+1}$

(c) Because $E[M + L] = E[X + Y]$, then we have

$$\begin{aligned} E[M] + E[L] &= E[X] + E[Y] \\ E[M] &= E[X] + E[Y] - E[L] \\ &= \frac{1-p}{p} + \frac{1-p}{p} - \frac{1 - 2p + p^2}{2p - p^2} \\ &= \frac{(1-p)(3-p)}{(2-p)p} \end{aligned}$$

(d) From the question and (a), and because $n >= 0$, then we have

$$\begin{aligned} P(L = l, M - L = n) &= P(L = l, M = n + l) \\ &= 2p^2 q^{l+n+l} \\ &= 2p^2 q^{n+2l} \end{aligned}$$

So for the joint PMF of L and M-L, and because n and l are nonnegative integers, then we can use the theorem and get
$f_{L,M-L}(l,n) = g(l)h(n)$, and $g(l) = a * q^{2l}$, and $h(n) = b * q^n$.
To get the valid PMF, $g(l) = a\sum_{l=0}^{\infty}(q^2)^l = a * \frac{1}{1-q^2} = 1$, then we can get $a = 1 - q^2$.
And for another one, this must be valid too, so $h(n) = \frac{2p^2q^n}{1-q^2}$.
Therefore $L, M - L$ are independent.