Stochastic Galerkin Method

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1 Definition of Single Random Variable

Let z be a random variable with a distribution function $F(z) = P(z \leq Z)$, The generalized polynomial chaos basis functions are the orthogonal polynomial functions satisfying

$$\mathbf{E}[\Phi_m(z)\Phi_n(z)] = \gamma_n \delta_{nm}$$

where

$$\gamma_n = \mathbf{E}[\Phi_n^2(z)]$$

are the normalization factors.

If z is continuous, then its probability density function (PDF) exists such that $dF(z) = \rho(z)dz$ and the orthogonality can be written as

$$\mathbf{E}[\Phi_m(z)\Phi_n(z)] = \int \Phi_m(z)\Phi_n(z)\rho(z)dz = \gamma_n \delta_{nm}$$

Similarly, when z is discrete, the orthogonality can be written as

$$\mathbf{E}[\Phi_m(z)\Phi_n(z)] = \sum_i \Phi_m(z_i)\Phi_n(z_i)\rho_i$$

1.1 Hermite polynomial chaos

Let z is a standard Gaussian random variable with zero mean and unit variance. Its PDF is

$$\rho(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

we employ the Hermite polynomials as the basis functions,

$$H_0(z) = 1,\ H_1(z) = z,\ H_2(z) = z^2 - 1,\ H_3(z) = z^3 - 3z$$

This is the classical Wiener-Hermite polynomial chaos basis.

1.2 Legendre polynomial chaos

2 General Procedure

2.1 Stochastic diffusion equation

$$-\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) = f(\omega, x)\ (\omega, x) \in \Omega \times D$$

$$u(\omega, x) = 0\ x \in \partial D$$

2.2 Tensor Product

$$(f \otimes g)(x, y) = f(x)g(y)$$

$$((A \otimes B)_{i_1 j_1})_{i_2 j_2} = A_{i_1 j_1} B_{i_2 j_2}$$

2.3 Solution

$$\mathbf{E}[\int_D a\nabla u \nabla v dx] = E[\int_D f v dx]$$

$$v(x) = \sum_{i=1}^{N} V_i \phi_i(x)$$

$$\int_D a\nabla(\sum_{i=1}^{N} U_i\phi_i(x)) \cdot \nabla\phi_j dx = \int_D f\phi_j dx$$

$$\sum_{i=1}^{N} U_i \int_D a\nabla\phi_i \cdot \nabla\phi_j dx = \int_D f\phi_j dx$$

$$KU = F$$

$$a(\omega, x) = \bar{a} + \sum_{l=1}^{\infty} \varphi_l(x) y_l(\omega)$$

where the function $\varphi_l(x)$ are determined by the eigenvalues and eigenfunctions of the covariance function of $a(\omega, x)$, and $y_l(\omega)$ are mutually independent.

$$u(\omega, x) = \sum_{n=1}^{Q}\sum_{i=1}^{N} (U_n)_i \psi_n(\omega)\phi_i(x)$$

$$\mathbf{E}[\int_D a\nabla \sum_{i=1}^{N}\sum_{n=1}^{Q} (U_n)_i \psi_n\phi_i \cdot \nabla\psi_m\phi_j dx] = \mathbf{E}[\int_D f\psi_m\phi_j dx]$$

$$\sum_{i=1}^{N}\sum_{n=1}^{Q} (U_n)_i \mathbf{E}[\int_D a\nabla\psi_n\phi_i \cdot \nabla\psi_m\phi_j dx = \mathbf{E}[\int_D f\psi_m\phi_j dx]$$

$$\sum_{i=1}^{N}\sum_{n=1}^{Q} (U_n)_i \mathbf{E}[\psi_n\psi_m \int_D a\nabla\phi_i \cdot \nabla\phi_j dx] = \mathbf{E}[\psi_m \int_D f\phi_j dx]$$

$$a(\omega, x) = \sum_{l=0}^{S} a_l(x) y_l(\omega)$$

$$\sum_{l=0}^{S}\sum_{i=1}^{N}\sum_{n=1}^{Q} (U_n)_i \mathbf{E}[y_l\psi_n\psi_m \int_D a_l\nabla\phi_i \cdot \nabla\phi_j dx] = \mathbf{E}[\psi_m \int_D f\phi_j dx]$$

$$\sum_{l=0}^{S}\sum_{i=1}^{N}\sum_{n=1}^{Q} (U_n)_i \mathbf{E}[y_l\psi_n\psi_m] \int_D a_l\nabla\phi_i \cdot \nabla\phi_j dx = \mathbf{E}[\psi_m \int_D f\phi_j dx]$$

$\mathbf{E}[y_l\psi_n\psi_m]$ can pe can be evaluated prior to any computations.

$$\sum_{l=0}^{S}\sum_{i=1}^{N}\sum_{n=1}^{Q} (U_n)_i (G_l)_{nm} (K_l)_{ij} = (F_m)_j$$

$$\sum_{l=0}^{S}\sum_{i=1}^{N}\sum_{n=1}^{Q} (U_n)_i ((G_l \otimes K_l)_{nm})_{ij} = (F_m)_j$$

$$\sum_{l=0}^{S} (G_l \otimes K_l) U = F$$

$$G_{lnm} = \mathbf{E}[y_l y_n y_m$$

$$G_{lnm} = \begin{cases} \mathbf{E}[y] & n = m = 0 \\ \mathbf{E}[y]^2 & \textbf{one of } n, m = 0 \textbf{ and the other } \neq l \\ \mathbf{E}[y^2] & \textbf{one of } n, m = 0 \textbf{ and the other } = l \\ \mathbf{E}[y]^3 & n \neq m \neq l \neq n \\ \mathbf{E}[y^2]\mathbf{E}[y] & n, m, l \neq 0 \textbf{ and exactly 2 are equal,} \\ \mathbf{E}[y^3] & n = m = l \end{cases}$$