Learning in Modal Space: Solving Time-Dependent Stochastic PDEs Using Physics-Informed Neural Networks

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December 2, 2021

1 Problem Setup

We consider the following time-dependent SPDE:

$$\frac{\partial u}{\partial t} = \mathcal{N}_x[u(x,t;\omega)], \quad x \in D, t \in [0,T], \omega \in \Omega \tag{1}$$

with initial and boundary conditons

$$u(x,t_0;\omega) = u_0(x;\omega) \tag{2}$$

$$\mathcal{B}_x[u(x,t;\omega)] = h(x,t;\omega) \tag{3}$$

2 An overview of the Dynamically Orthogonal decomposition methods

For a random field $u(x,t;\omega)$ that evoles in time, the generalized KL expansion at a given time t is

$$u(x,t;\omega) = \bar{u}(x,t) + \sum_{i=0}^{\infty} \sqrt{\lambda_i}\phi_i(x,t)\xi_i(t;\omega)$$

$$\int_D C_{u(x_1,t)u(x_2,t)}\phi_i(x_2,t)dx_2 = \lambda_i\phi_i(x_1,t)$$

$$C_{u(x_1,t)u(x_2,t)} = \mathbf{E}[(u(x_1,t;\omega) - \bar{u}(x_1,t))(u(x_2,t;\omega) - \bar{u}(x_2,t))]$$

Next, we consider a generalized expansion

$$u(x,t;\omega) = \bar{u}(x,t) + \sum_{i=1}^{\infty} u_i(x,t)Y_i(t;\omega), \quad \omega \in \Omega$$

$$\bar{u}(x,t) = \mathbf{E}[u(x,t;\omega)] = \int_\Omega u(x,t;\omega)dP(\omega)$$

$$\langle u_i, u_j \rangle = 0 \quad i \neq j$$

$$E[Y_i] = 0$$

We define the linear subspace $V_S = span\{u_i(x,t)\}_{i=1}^N$.

2.1 Dynamically orthogonal representation

A natural constraint to overcome redundancy is that the evolution of bases be orthogonal to the space V_S, this can be expressed through the following DO condition:

$$\frac{dV_S}{dt} \perp V_S \iff \langle \frac{\partial u_i(x,t)}{\partial t}, u_j(x,t) \rangle = 0 \quad i,j = 1,\ldots,N$$

here $\langle u(x,t), v(x,t) \rangle = \int_D u(x,t)v(x,t)dx$. The DO condition preserves the orthonormality and the length of the bases $\{u_i(x,t)\}_{i=1}^N$ since

$$\frac{\partial}{\partial t}\langle u_i(\cdot,t), u_j(\cdot,t) \rangle = \langle \frac{\partial u_i(\cdot,t)}{\partial t}, u_j(\cdot,t) \rangle + \langle u_i(\cdot,t), \frac{\partial u_j(\cdot,t)}{\partial t} \rangle = 0 \quad i,j = 1,\ldots,N$$

Theorem 2.1.

$$\frac{\bar{u}(x,t)}{\partial t} = \mathbf{E}[\mathcal{N}_x[u(\cdot,t;\omega)]$$

$$\frac{dY_i(t;\omega)}{dt} = \langle \mathcal{N}_x[u(\cdot,t;\omega)] - \mathbf{E}[\mathcal{N}_x[u(\cdot,t;\omega)]], u_i(\cdot,t)\rangle$$

$$\sum_{i=1}^{N} C_{Y_i(t)Y_j(t)} \frac{\partial u_i(x,t)}{\partial t} = \Pi_{V_s^\perp} \mathbf{E}[\mathcal{N}_x[u(\cdot,t;\omega)]Y_j]$$

The projection in the orthogonal complement of the linear usbspace V_S is defined as:

$$\Pi_{V_s^\perp} F(x) = F(x) - \Pi_{V_s^\perp} = F(x) - \sum_{k=1}^{N} \langle F(\cdot), u_k(\cdot,t)\rangle u_k(\cdot,t)$$

The boundary conditions are :

$$\mathcal{B}_x[\bar{u}(x,t;\omega)] = \mathbf{E}[h(x,t;\omega)]$$

$$\mathcal{B}_x[u_i(x,t)] = \mathbf{E}[Y_j(t;\omega)h(x,t;\omega)]C^{-1}_{Y_i(t)Y_j(t)}$$

The initial condition:

$$\bar{u}(x,t_0) = \mathbf{E}[u_0(x;\omega)]$$

$$Y_i(t_0;\omega) = \langle u_0(\cdot,\omega) - \bar{u}(x,t_0), v_i(\cdot)\rangle$$

$$u_i(x,t_0) = v_i(x)$$

where $v_i(x)$ is the eigenfield of the standard KL expansion of $u_0(x;\omega)$.

$$\int_D C_{u(\cdot,t_0)u(\cdot,t_0)}(x,y)v_i(x)dx = \lambda^2 v_i(x)$$