Learning in Modal Space: Solving Time-Dependent Stochastic PDEs Using Physics-Informed Neural Networks

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December 13, 2021

1 Karhunen-Loeve Expansion

1.1 Derivation

$$u(x,\theta) = \sum_{i=0}^{\infty} \sqrt{\lambda_n}\xi(\theta) f_n(x)$$

$C(x_1, x_2)$ denotes the covariance function, by definition of the covariance function, it is bounded, symmertic and positive definite. Thus, it has the spectral decomposition:

$$C(x_1, x_2) = \sum_{n=0}^{\infty} \lambda_n f_n(x_1) f_n(x_2)$$

λ_n and $f_n(x)$ are the solution of the integral equation

$$\int_D C(x_1, x_2) f_n(x) dx_1 = \lambda_n f_n(x_2)$$

$u(x,\theta)$ can be written as

$$u(x,\theta) = \bar{u}(x) + \alpha(x,\theta)$$

where $\alpha(x,\theta)$ is a process with zero mean and covariance function $C(x_1, x_2)$, the process $alpha(x,\theta)$ can be expanded in terms of eigenfunctions $f_n(x)$ as:

$$\alpha(x,\theta) = \sum_{n=0}^{\infty} \xi(\theta)\sqrt{\lambda_n} f_n(x)$$

By multiplying equation by $f_n(x)$ and integrating over the domain D, that is:

$$\xi(\theta) = \frac{1}{\lambda_n}\int_D \alpha(x,\theta) f_n(x) dx$$

The covariance function is given by

$$C(x_1, x_2) = e^{-|x_1 - x_2|/b}$$

where b is a parameter with the same units as x and is often termed the correlation length, since it reflects the rate at which the correlation decays between two points of the process.

$$\int_{-a}^{a} e^{-c|x_1 - x_2|} f(x_2) dx_2 = \lambda f(x_1)$$

where $c = \frac{1}{b}$. Equation can be written as

$$\int_{-a}^{x} e^{-c(x_1 - x_2)} f(x_2) dx_2 + \int_{x}^{a} e^{c|x_1 - x_2|} f(x_2) dx_2 = \lambda f(x_1)$$

Differentiating equation with respect to x and rearranging gives

$$\lambda f'(x_1) = -c\int_{-a}^{x} e^{-c(x_1 - x_2)} f(x_2) dx_2 + c\int_{x}^{a} e^{c(x_1 - x_2)} f(x_2) dx_2$$

Differentiating once more with respect to x_1, the following equation is obtained

$$\lambda f''(x) = (-2c + c^2\lambda) f(x)$$

Introducing the new variable

$$\omega^2 = \frac{2c - c^2\lambda}{\lambda}$$

then equation becomes

$$f''(x) + \omega^2 f(x) = 0 \quad x \in D$$

To find the boundary conditions associated with the differential equation, equations are evaluated at $x = -a$ and $x = a$

$$cf(a) + f'(a) = 0$$

$$cf(-a) - f'(-a) = 0$$

The solution is

$$f(x) = a_1 cos(\omega x) + a_2 sin(\omega x)$$

Further, applying the boundary conditions

$$a_1(c - \omega tan(\omega a) + a_2(\omega + ctan(\omega a)) = 0$$

$$a_1(c - \omega tan(\omega a) - a_2(\omega + ctan(\omega a)) = 0$$

Setting this determinant equal to zero gives the following transcendental equations

$$c - \omega tan(\omega a) = 0$$

$$\omega + ctan(\omega a) = 0$$

Denoting the solution of the second of these equations by ω^*, the resulting eigenfunctions are

$$f_n(x) = \frac{cos(\omega_n x)}{\sqrt{a + \frac{sin(2\omega_n a)}{2\omega_n}}}$$

and

$$f_n^*(x) = \frac{sin(\omega_n^* x)}{\sqrt{a - \frac{sin(2\omega_n^* a)}{2\omega_n^*}}}$$

for even n and odd n respectively. The corresponding eigenvalues are

$$\lambda_n = \frac{2c}{\omega_n^2 + c^2}$$

and

$$\lambda_n^* = \frac{2c}{\omega_n^{*2} + c^2}$$