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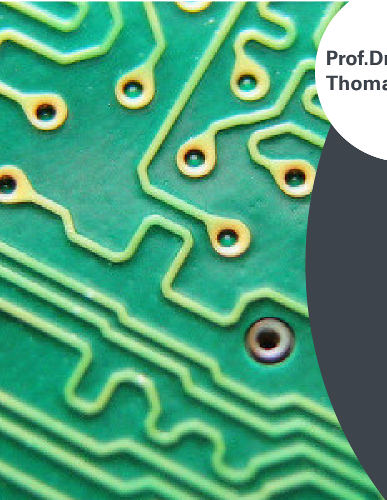

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Integrated Positioning and Navigation

2

Parameterization of the DCM

Parameterization of the DCM

Equation (1.10) constitutes a set of 6 condition equations for the nine elements of a DCM: $\Rightarrow$ only 3 independent parameters are needed to fully describe a DCM. Particular simple are single axis rotations.

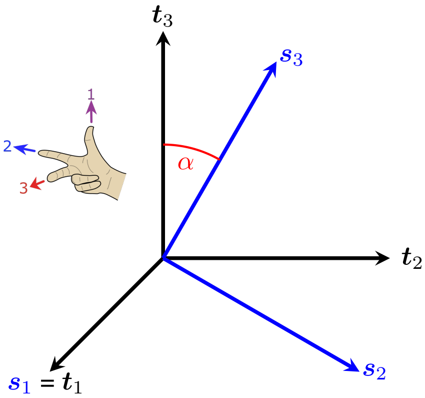


Figure 2.1: Single axis rotation

If the t-system is related to the s-system by a rotation a about the s_1-axis, then

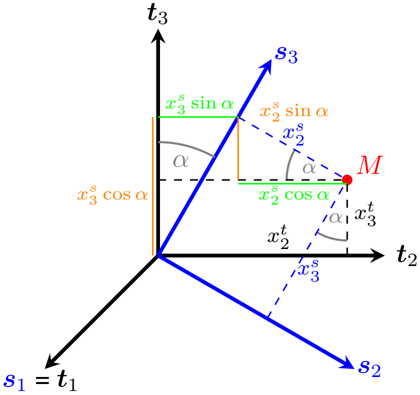
$$C_t^s(1,\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \quad (2.1)$$

Similarly:

$$C_t^s(2,\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \quad (2.2)$$

$$C_t^s(3,\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.3)$$

Parameterization of the DCM - cont'd



$$
\begin{aligned}
x^t_{1(M)} &= x^s_{1(M)} \\
x^t_{2(M)} &= x^s_{2(M)} \cdot \cos\alpha + x^s_{3(M)} \cdot \sin\alpha \\
x^t_{3(M)} &= -x^s_{2(M)} \cdot \sin\alpha + x^s_{3(M)} \cdot \cos\alpha
\end{aligned}
$$

$$
\begin{bmatrix} x^t_{1(M)} \\ x^t_{2(M)} \\ x^t_{3(M)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} x^s_{1(M)} \\ x^s_{2(M)} \\ x^s_{3(M)} \end{bmatrix}
$$

Parameterization of the DCM - cont'd

Euler angles

Two arbitrarily oriented coordinate systems can always be transformed into each other by 3 subsequent single axis rotations:

Case 1: rotation about a particular axis, followed by a rotation about another axis, followed a rotation about the axis, which was used for the first rotation:

Example:

$$\boldsymbol{C}_t^s = \boldsymbol{C}(3, \gamma) \cdot \boldsymbol{C}(1, \beta) \cdot \boldsymbol{C}(3, \alpha) \tag{2.4}$$

$$\boldsymbol{C}_t^s = \begin{bmatrix} \cos\gamma\cos\alpha - \cos\beta\sin\gamma\sin\alpha & \cos\gamma\sin\alpha + \cos\beta\sin\gamma\cos\alpha & \sin\beta\sin\gamma \\ -\sin\gamma\cos\alpha - \cos\beta\cos\gamma\sin\alpha & -\sin\gamma\sin\alpha + \cos\beta\cos\gamma\cos\alpha & \sin\beta\cos\gamma \\ \sin\beta\sin\alpha & -\sin\beta\cos\alpha & \cos\beta \end{bmatrix} \tag{2.5}$$

There is a total of six different rotation sequences possible.

Parameterization of the DCM - cont'd

Euler angles

Case 2: Subsequent rotation about all 3 axes.

Example:

$$\boldsymbol{C}_t^s = \boldsymbol{C}(1,\alpha)\cdot\boldsymbol{C}(2,\beta)\cdot\boldsymbol{C}(3,\gamma) \tag{2.6}$$

$$\boldsymbol{C}_t^s = \begin{bmatrix} \cos\beta\cos\gamma & \cos\beta\sin\gamma & -\sin\beta \\ -\sin\gamma\cos\alpha + \sin\beta\cos\gamma\sin\alpha & \cos\gamma\cos\alpha + \sin\beta\sin\gamma\sin\alpha & \cos\beta\sin\alpha \\ \sin\gamma\sin\alpha + \sin\beta\cos\gamma\cos\alpha & -\cos\gamma\sin\alpha + \sin\beta\sin\gamma\cos\alpha & \cos\beta\cos\alpha \end{bmatrix} \tag{2.7}$$

For this case 2, there is also a total of six different rotation sequences possible. In this case 2, if the angles α, β, γ are small, the DCM can, independent of the sequence of rotations, be approximated by

$$\boldsymbol{C}_t^s \approx \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix} \tag{2.8}$$

Parameterization of the DCM - cont'd

Extraction of rotation parameters from a DCM: In the computations to be performed by an Inertial Navigation System it becomes necessary, to calculate 3 independent rotation parameters from a numerically given DCM.

Case 1: extraction of Euler angles using the DCM representation (2.7):

$$\begin{aligned} \alpha &= \arctan\left(\frac{\boldsymbol{C}_t^s[2,3]}{\boldsymbol{C}_t^s[3,3]}\right), \\ \beta &= \arcsin\left(-\boldsymbol{C}_t^s[1,3]\right), \\ \gamma &= \arctan\left(\frac{\boldsymbol{C}_t^s[1,2]}{\boldsymbol{C}_t^s[1,1]}\right) \end{aligned} \tag{2.9}$$

Determination of angles not unique for DCM-parameterizations with Euler angles!

What kind of problems can occur?

Parameterization of the DCM - cont'd

Problems with Euler angles
If we have a transformation according to Equ. (2.7):

$$\boldsymbol{C}_t^s = \begin{bmatrix} \cos\beta\cos\gamma & \cos\beta\sin\gamma & -\sin\beta \\ -\sin\gamma\cos\alpha + \sin\beta\cos\gamma\sin\alpha & \cos\gamma\cos\alpha + \sin\beta\sin\gamma\sin\alpha & \cos\beta\sin\alpha \\ \sin\gamma\sin\alpha + \sin\beta\cos\gamma\cos\alpha & -\cos\gamma\sin\alpha + \sin\beta\sin\gamma\cos\alpha & \cos\beta\cos\alpha \end{bmatrix}$$

and

$$\beta = 90^\circ \;\Rightarrow \cos\beta = 0,\; \sin\beta = 1$$

then $\boldsymbol{C}_t^s$ becomes

$$\boldsymbol{C}_t^s = \begin{bmatrix} 0 & 0 & -1 \\ -\sin\gamma\cos\alpha + \cos\gamma\sin\alpha & \cos\gamma\cos\alpha + \sin\gamma\sin\alpha & 0 \\ \sin\gamma\sin\alpha + \cos\gamma\cos\alpha & -\cos\gamma\sin\alpha + \sin\gamma\cos\alpha & 0 \end{bmatrix}$$

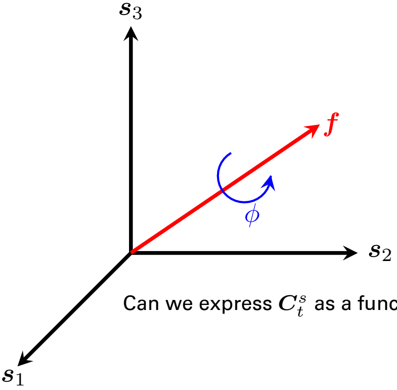
Can you recover the rotation angles from this matrix?

Parameterization of the DCM - cont'd

Euler Symmetric Parameters

According to *Euler's Theorem*, two arbitrarily oriented coordinate systems can always be transformed into each other by a single rotation about a well defined rotation axis, represented by an unit vector. This unit vector $\boldsymbol{f}$ has identical coordinates in both coordinate systems!

$$\boldsymbol{f} = \boldsymbol{t} \cdot \boldsymbol{f}^t = \boldsymbol{s} \cdot \boldsymbol{f}^s \quad \Rightarrow \quad \boldsymbol{f}^t = \boldsymbol{f}^s \tag{2.10}$$



Can we express $\boldsymbol{C}_t^s$ as a function of $\boldsymbol{f}$ and ϕ ?;

Parameterization of the DCM - cont'd

Euler Symmetric Parameters

If the coordinates of $\boldsymbol{f}$ are denoted by f_1, f_2, and f_3, and if the rotation angle is denoted by ϕ, then the DCM is:

$$\boldsymbol{C}_t^s = \begin{bmatrix} \cos\phi + f_1^2(1-\cos\phi) & f_1 f_2(1-\cos\phi) + f_3\sin\phi & f_1 f_3(1-\cos\phi) - f_2\sin\phi \\ f_1 f_2(1-\cos\phi) - f_3\sin\phi & \cos\phi + f_2^2(1-\cos\phi) & f_2 f_3(1-\cos\phi) + f_1\sin\phi \\ f_1 f_3(1-\cos\phi) + f_2\sin\phi & f_2 f_3(1-\cos\phi) - f_1\sin\phi & \cos\phi + f_3^2(1-\cos\phi) \end{bmatrix} \tag{2.11}$$

The Euler Symmetric Parameters are defined by:

$$q_0 = \cos\frac{\phi}{2}, \quad q_1 = f_1\sin\frac{\phi}{2}, \quad q_2 = f_2\sin\frac{\phi}{2}, \quad q_3 = f_3\sin\frac{\phi}{2} \tag{2.12}$$

The Euler Symmetric Parameters may be regarded as the components of a Quaternion $\boldsymbol{q}$; the Quaternion algebra is applicable for subsequent rotations! aufgefasst werden; dadurch läßt sich die Quaternionen Algebra für aufeinander folgende Rotationen anwenden!

$$\boldsymbol{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T = \begin{bmatrix} q_0 \\ \tilde{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} q_0 \\ \tilde{\boldsymbol{q}} \end{bmatrix} \begin{bmatrix} \cos(\phi/2) \\ \boldsymbol{f}\cdot\sin(\phi/2) \end{bmatrix} \tag{2.13}$$

Parameterization of the DCM - cont'd

Quaternion Algebra

Conjugate

$$\bar{\boldsymbol{q}} = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \end{bmatrix}^T$$

Scalar product

$$\boldsymbol{p} \cdot \boldsymbol{q} = p_0 \cdot q_0 + p_1 \cdot q_1 + p_2 \cdot q_2 + p_3 \cdot q_3$$

Multiplication

$$\begin{aligned} \boldsymbol{p} \circ \boldsymbol{q} &= \begin{bmatrix} p_0 \\ \tilde{\boldsymbol{p}} \end{bmatrix} \circ \begin{bmatrix} q_0 \\ \tilde{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} p_0 q_0 - \tilde{\boldsymbol{p}} \cdot \tilde{\boldsymbol{q}} \\ p_0 \tilde{\boldsymbol{q}} + q_0 \tilde{\boldsymbol{p}} + \tilde{\boldsymbol{p}} \times \tilde{\boldsymbol{q}} \end{bmatrix} \\ &= \begin{bmatrix} p_0 & -\tilde{\boldsymbol{p}} \\ \tilde{\boldsymbol{p}} & p_0 \cdot \boldsymbol{I} + \tilde{\boldsymbol{p}} \times \end{bmatrix} = \boldsymbol{\Pi} \cdot \boldsymbol{q} \end{aligned}$$

The two arbitrarily oriented coordinate systems can be transformed with the help of quaternions by:

$$\begin{aligned} \boldsymbol{x}^t &= \boldsymbol{q} \circ \boldsymbol{x}^s \circ \bar{\boldsymbol{q}} \\ \boldsymbol{x}^s &= \bar{\boldsymbol{q}} \circ \boldsymbol{x}^t \circ \bar{\boldsymbol{q}} \end{aligned} \qquad \text{with} \quad \boldsymbol{x} = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \end{bmatrix}^T$$

Parameterization of the DCM - cont'd

Using the abbreviations (2.12), the DCM of eqn. (2.11) can be re-written as:

$$\boldsymbol{C}_t^s = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_3q_0) & 2(q_1q_3-q_2q_0) \\ 2(q_1q_2-q_3q_0) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_1q_0) \\ 2(q_1q_3+q_2q_0) & 2(q_2q_3-q_1q_0) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix} \tag{2.14}$$

Of the 4 Euler Symmetric Parameters only 3 are independent, since:

$$q_0^2+q_1^2+q_2^2+q_3^2 = 1 \tag{2.15}$$

For a small rotation angle ϕ, the DCMs (2.11) and (2.14) may be approximated:

$$\boldsymbol{C}_t^s \approx \begin{bmatrix} 1 & \phi f_3 & -\phi f_2 \\ -\phi f_3 & 1 & \phi f_1 \\ \phi f_2 & -\phi f_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2q_3 & -2q_2 \\ -2q_3 & 1 & 2q_1 \\ 2q_2 & -2q_1 & 1 \end{bmatrix} \tag{2.16}$$

Parameterization of the DCM - cont'd

Extraction of rotation parameters from a DCM: In the computations to be performed by an Inertial Navigation System it becomes necessary, to calculate 3 independent rotation parameters from a numerically given DCM.

Case 1: extraction of Euler angles using the DCM representation (2.7):

$$\alpha = \arctan\left(\frac{\boldsymbol{C}_t^s[2,3]}{\boldsymbol{C}_t^s[3,3]}\right), \ \beta = \arcsin\left(-\boldsymbol{C}_t^s[1,3]\right), \ \gamma = \arctan\left(\frac{\boldsymbol{C}_t^s[1,2]}{\boldsymbol{C}_t^s[1,1]}\right)$$

Determination of angles not unique for DCM-parameterizations with Euler angles!

Case 2: extraction of Euler Symmetric Parameters from DCM (2.14):

$$\begin{gathered} q_0 = \frac{1}{2}\sqrt{\boldsymbol{C}_t^s[1,1] + \boldsymbol{C}_t^s[2,2] + \boldsymbol{C}_t^s[3,3] + 1}, \quad q_1 = \frac{\boldsymbol{C}_t^s[2,3] - \boldsymbol{C}_t^s[3,2]}{4q_0}, \\ q_2 = \frac{\boldsymbol{C}_t^s[3,1] - \boldsymbol{C}_t^s[1,3]}{4q_0}, \quad q_3 = \frac{\boldsymbol{C}_t^s[1,2] - \boldsymbol{C}_t^s[2,1]}{4q_0} \end{gathered} \tag{2.17}$$

The Euler Symmetric Parameters can always be extracted from a DCM without any ambiguities, independent of the rotations involved!

Parameterization of the DCM - cont'd

Comparison	Euler Angles	vs.	Quaternions
Singularities	yes $(\beta = \pm 90°)$		no
Ensure Orthogonality	complex		simple
Use of trigonom. functions	yes		no
Number of parameters (indep.)	3(3)		4(3)
Algebra	matrix calculations		Vector algebra

Ensure Orthogonality (important due to rounding errors)

- **Euler Angles:**
  - scalar product of different rows has to be 0
  - scalar product of each row with itself must be 1
  - the same criterions hold for each column

- **Quaternions:** normalize $\boldsymbol{q}$ to $\boldsymbol{q} = \dfrac{\boldsymbol{q}}{\sqrt{\boldsymbol{q}^T \boldsymbol{q}}}$

Parameterization of the DCM - cont'd

Example for the non-uniqueness of Euler angle extraction:

(DCM representation (2.7))

$$\beta = 135^\circ,\, \alpha = 135^\circ, \gamma = 135^\circ \rightarrow \boldsymbol{C}_t^s = \begin{bmatrix} 0.50000 & -0.50000 & -0.70711 \\ 0.14645 & 0.85355 & -0.50000 \\ 0.85355 & 0.14645 & 0.50000 \end{bmatrix}$$

Extraction: $\beta = 45^\circ,\, \alpha = -45^\circ, \gamma = -45^\circ$

$$\beta = 45^\circ,\, \alpha = -45^\circ, \gamma = -45^\circ \rightarrow \boldsymbol{C}_t^s = \begin{bmatrix} 0.50000 & -0.50000 & -0.70711 \\ 0.14645 & 0.85355 & -0.50000 \\ 0.85355 & 0.14645 & 0.50000 \end{bmatrix}$$

Extraction: $\beta = 45^\circ,\, \alpha = -45^\circ, \gamma = -45^\circ$

To achieve uniqueness, the parameter range must be restricted:

Example: $-90^\circ < \beta \leq 90^\circ,\, -180^\circ < \alpha \leq 180^\circ, -180^\circ < \gamma \leq 180^\circ$