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Dynamic System Estimation

Sequential Least Squares Parameter Estimation

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Sequential Least Squares Parameter Estimation

Assumption: the system of (linear) observation equations can be divided into two sets of equations which are uncorrelated.

$$\begin{bmatrix} \boldsymbol{y_1} \\ \boldsymbol{y_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A_1} \\ \boldsymbol{A_2} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \boldsymbol{\varepsilon_1} \\ \boldsymbol{\varepsilon_2} \end{bmatrix} \qquad \boldsymbol{P} = \begin{bmatrix} \boldsymbol{P_1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P_2} \end{bmatrix} \tag{1.1}$$

The solution of the first set of equations has already been computed and is available.

$$\begin{aligned} \hat{\boldsymbol{x}}_{(1)} &= \left(\boldsymbol{A_1^T P_1 A_1}\right)^{-1} \boldsymbol{A_1^T P_1 y_1} \\ \boldsymbol{\varepsilon_{1(1)}} &= \boldsymbol{y_1} - \boldsymbol{A_1} \hat{\boldsymbol{x}}_{(1)} \\ \hat{\sigma}_{0(1)}^2 &= \frac{\boldsymbol{\varepsilon_{1(1)}^T P_1 \varepsilon_{1(1)}}}{n_{(1)} - u} \\ \boldsymbol{\Sigma}(\hat{\boldsymbol{x}}_{(1)}) &= \hat{\sigma}_{0(1)}^2 \left(\boldsymbol{A_1^T P_1 A_1}\right)^{-1} \end{aligned} \tag{1.2}$$

Subscripts 1, 2 for first and second set of equations; subscripts (1), (2) for estimation based on first set of equations and for estimation based on both sets of equations!

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Question: Can the solution of eqns. (1.1) be obtained by updating the solution of the first set of equations 1.2 instead of solving the complete system?

Formally the solution to (1.1) reads:

$$\begin{aligned}\hat{\boldsymbol{x}}_{(2)} &= \left[\begin{bmatrix}\boldsymbol{A}_1^T & \boldsymbol{A}_2^T\end{bmatrix}\begin{bmatrix}\boldsymbol{P}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{P}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{A}_1\\ \boldsymbol{A}_2\end{bmatrix}\right]^{-1}\begin{bmatrix}\boldsymbol{A}_1^T & \boldsymbol{A}_2^T\end{bmatrix}\begin{bmatrix}\boldsymbol{P}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{P}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{y}_1\\ \boldsymbol{y}_2\end{bmatrix}\\ &= \left[\boldsymbol{A}_1^T\boldsymbol{P}_1\boldsymbol{A}_1 + \boldsymbol{A}_2^T\boldsymbol{P}_2\boldsymbol{A}_2\right]^{-1}\left[\boldsymbol{A}_1^T\boldsymbol{P}_1\boldsymbol{y}_1 + \boldsymbol{A}_2^T\boldsymbol{P}_2\boldsymbol{y}_2\right] \qquad (1.3)\end{aligned}$$

if

$$\hat{\boldsymbol{x}}_{(2)} = \hat{\boldsymbol{x}}_{(1)} + \Delta\hat{\boldsymbol{x}} \qquad (1.4)$$

then

$$\left[\boldsymbol{A}_1^T\boldsymbol{P}_1\boldsymbol{A}_1 + \boldsymbol{A}_2^T\boldsymbol{P}_2\boldsymbol{A}_2\right]\left[\hat{\boldsymbol{x}}_{(1)} + \Delta\hat{\boldsymbol{x}}\right] = \left[\boldsymbol{A}_1^T\boldsymbol{P}_1\boldsymbol{y}_1 + \boldsymbol{A}_2^T\boldsymbol{P}_2\boldsymbol{y}_2\right]$$

$$\Rightarrow \underbrace{\boldsymbol{A}_1^T\boldsymbol{P}_1\boldsymbol{A}_1\hat{\boldsymbol{x}}_{(1)}}_{\boldsymbol{A}_1^T\boldsymbol{P}_1\boldsymbol{y}_1} + \boldsymbol{A}_2^T\boldsymbol{P}_2\boldsymbol{A}_2\hat{\boldsymbol{x}}_{(1)} + \left[\boldsymbol{A}_1^T\boldsymbol{P}_1\boldsymbol{A}_1 + \boldsymbol{A}_2^T\boldsymbol{P}_2\boldsymbol{A}_2\right]\Delta\hat{\boldsymbol{x}} = \left[\boldsymbol{A}_1^T\boldsymbol{P}_1\boldsymbol{y}_1 + \boldsymbol{A}_2^T\boldsymbol{P}_2\boldsymbol{y}_2\right]$$

$$\Rightarrow \cancel{\boldsymbol{A}_1^T\boldsymbol{P}_1\boldsymbol{y}_1} + \boldsymbol{A}_2^T\boldsymbol{P}_2\boldsymbol{A}_2\hat{\boldsymbol{x}}_{(1)} + \left[\boldsymbol{A}_1^T\boldsymbol{P}_1\boldsymbol{A}_1 + \boldsymbol{A}_2^T\boldsymbol{P}_2\boldsymbol{A}_2\right]\Delta\hat{\boldsymbol{x}} = \left[\cancel{\boldsymbol{A}_1^T\boldsymbol{P}_1\boldsymbol{y}_1} + \boldsymbol{A}_2^T\boldsymbol{P}_2\boldsymbol{y}_2\right] \qquad (1.5)$$

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Parameter update

$$\Delta\hat{\boldsymbol{x}} = \left[\boldsymbol{A_1^T P_1 A_1} + \boldsymbol{A_2^T P_2 A_2}\right]^{-1} \boldsymbol{A_2^T P_2} \left[\boldsymbol{y_2} - \boldsymbol{A_2}\hat{\boldsymbol{x}}_{(1)}\right]$$

$$\boxed{\begin{aligned} \Delta\hat{\boldsymbol{x}} &= \left[\hat{\sigma}^2_{0(1)}\boldsymbol{\Sigma}(\hat{\boldsymbol{x}}_{(1)})^{-1} + \boldsymbol{A_2^T P_2 A_2}\right]^{-1} \boldsymbol{A_2^T P_2} \left[\boldsymbol{y_2} - \boldsymbol{A_2}\hat{\boldsymbol{x}}_{(1)}\right] \\ \hat{\boldsymbol{x}}_{(2)} &= \hat{\boldsymbol{x}}_{(1)} + \Delta\hat{\boldsymbol{x}} \end{aligned}} \tag{1.6}$$

Updating the variance of unit weight:

$$\hat{\sigma}^2_{0(2)} = \frac{\boldsymbol{\varepsilon}^T_{\boldsymbol{1}(2)}\boldsymbol{P_1}\boldsymbol{\varepsilon}_{\boldsymbol{1}(2)} + \boldsymbol{\varepsilon}^T_{\boldsymbol{2}(2)}\boldsymbol{P_2}\boldsymbol{\varepsilon}_{\boldsymbol{2}(2)}}{n_{(2)} - u}$$

$$\boldsymbol{\varepsilon}_{\boldsymbol{1}(2)} = \boldsymbol{y_1} - \boldsymbol{A_1}\hat{\boldsymbol{x}}_{(2)} = \boldsymbol{y_1} - \boldsymbol{A_1}\hat{\boldsymbol{x}}_{(1)} - \boldsymbol{A_1}\Delta\hat{\boldsymbol{x}} = \boldsymbol{\varepsilon}_{\boldsymbol{1}(1)} - \boldsymbol{A_1}\Delta\hat{\boldsymbol{x}}$$

$$\boldsymbol{\varepsilon}^T_{\boldsymbol{1}(2)}\boldsymbol{P_1}\boldsymbol{\varepsilon}_{\boldsymbol{1}(2)} = \boldsymbol{\varepsilon}^T_{\boldsymbol{1}(1)}\boldsymbol{P_1}\boldsymbol{\varepsilon}_{\boldsymbol{1}(1)} - 2\Delta\hat{\boldsymbol{x}}^T \underbrace{\boldsymbol{A_1^T P_1}\boldsymbol{\varepsilon}_{\boldsymbol{1}(1)}}_{=0} + \Delta\hat{\boldsymbol{x}}^T\boldsymbol{A_1^T P_1 A_1}\Delta\hat{\boldsymbol{x}}$$

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$$\boxed{\hat{\sigma}^2_{0(2)} = \frac{1}{n_{(2)} - u} \left(\hat{\sigma}^2_{0(1)} \left(n_{(1)} - u + \Delta\hat{\boldsymbol{x}}^T \boldsymbol{\Sigma}(\hat{\boldsymbol{x}}_{(1)})^{-1} \Delta\hat{\boldsymbol{x}}\right) + \boldsymbol{\varepsilon}^T_{\mathbf{2}(2)} \boldsymbol{P_2} \boldsymbol{\varepsilon}_{\mathbf{2}(2)}\right)} \tag{1.7}$$

Updating the covariance matrix of the estimated parameters:

$$\boxed{\boldsymbol{\Sigma}(\hat{\boldsymbol{x}}_{(2)}) = \hat{\sigma}^2_{0(2)} \left[\hat{\sigma}^2_{0(1)} \boldsymbol{\Sigma}(\hat{\boldsymbol{x}}_{(1)})^{-1} + \boldsymbol{A}_{\mathbf{2}}^T \boldsymbol{P_2} \boldsymbol{A_2}\right]^{-1}} \tag{1.8}$$

Eqns. (1.6) - (1.8) update the parameter estimation based on the previous solution and the new measurements and the corresponding design matrix.

Sequential Least Squares Parameter Estimation - cont'd

Typical application example: Time series of measurements related to a common set of parameters, uncorrelated between measurement epochs.

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}(t_1) \\ \boldsymbol{y}(t_2) \\ \vdots \\ \boldsymbol{y}(t_n) \end{bmatrix}, \ \boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}(t_1) \\ \boldsymbol{A}(t_2) \\ \vdots \\ \boldsymbol{A}(t_n) \end{bmatrix}, \ \boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}(t_1) & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}(t_2) & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{P}(t_n) \end{bmatrix} \tag{1.9}$$

Denoting $\hat{\boldsymbol{x}}(t_k) = \hat{\boldsymbol{x}}_t$, a sequential solution of equation system (1.9) can be written as

$$\boxed{\begin{aligned} \hat{\boldsymbol{x}}_k &= \hat{\boldsymbol{x}}_{k-1} + \left[\hat{\sigma}^2_{0k-1}\boldsymbol{\Sigma}(\hat{\boldsymbol{x}}_{k-1})^{-1} + \boldsymbol{A}_{\boldsymbol{k}}^T\boldsymbol{P}_{\boldsymbol{k}}\boldsymbol{A}_{\boldsymbol{k}}\right]^{-1} \boldsymbol{A}_{\boldsymbol{k}}^T\boldsymbol{P}_{\boldsymbol{k}}\left[\boldsymbol{y}_{\boldsymbol{k}} - \boldsymbol{A}_{\boldsymbol{k}}\hat{\boldsymbol{x}}_{k-1}\right] \\ \hat{\sigma}^2_{0k} &= \frac{\hat{\sigma}^2_{0k-1}\left(n_{k-1} - u + \Delta\hat{\boldsymbol{x}}^T\boldsymbol{\Sigma}(\hat{\boldsymbol{x}}_{(k-1)})^{-1}\Delta\hat{\boldsymbol{x}}\right) + [\boldsymbol{y}_{\boldsymbol{k}} - \boldsymbol{A}_{\boldsymbol{k}}\hat{\boldsymbol{x}}_k]^T\boldsymbol{P}_{\boldsymbol{k}}[\boldsymbol{y}_{\boldsymbol{k}} - \boldsymbol{A}_{\boldsymbol{k}}\hat{\boldsymbol{x}}_k]}{n_k - u} \\ \boldsymbol{\Sigma}(\hat{\boldsymbol{x}}_k) &= \hat{\sigma}^2_{0k}\left[\hat{\sigma}^2_{0k-1}\boldsymbol{\Sigma}(\hat{\boldsymbol{x}}_{k-1})^{-1} + \boldsymbol{A}_{\boldsymbol{k}}^T\boldsymbol{P}_{\boldsymbol{k}}\boldsymbol{A}_{\boldsymbol{k}}\right]^{-1} \end{aligned}} \tag{1.10}$$

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You will find the examples discussed in this lecture as Jupyter notebook under

`https://github.com/spacegeodesy/ParameterEstimationDynamicSystems/blob/master/example01.ipynb`