Exercise on 30.10.2019

Task 1 (3 points)

Show, that for two consecutive rotations with angles ϕ_1 and ϕ_2 (around the same vector) the following equation holds:

$$\boldsymbol{R}_{\phi_1+\phi_2} = \boldsymbol{R}_{\phi_1} \cdot \boldsymbol{R}_{\phi_2}$$

Annotation: $\boldsymbol{R}_i \in SO(3)$

Solution:

Use $\boldsymbol{C}_t^s$ (equation (2.11)) from the lecture. Show:

$$\boldsymbol{C}_t^s(\alpha_1) \cdot \boldsymbol{C}_t^s(\alpha_2) = \boldsymbol{C}_t^s(\alpha_1 + \alpha_2)$$

This can be shown element wise. Here an example:

$$\begin{aligned}
[\boldsymbol{C}_t^s(\alpha_1+\alpha_2)]_{11} = &\left[\cos\alpha_1 + f_1^2(1-\cos\alpha_1)\right] \cdot \left[\cos\alpha_2 + f_1^2(1-\cos\alpha_2)\right] \\
&+ [f_1 f_2(1-\cos\alpha_1) + f_3\sin\alpha_1] \cdot [(f_1 f_2(1-\cos\alpha_2) - f_3\sin\alpha_2)] \\
&+ [f_1 f_2(1-\cos\alpha_1) - f_2\sin\alpha_1] \cdot [(f_1 f_3(1-\cos\alpha_2) + f_2\sin\alpha_2)] = \\
= &\cos\alpha_1\cos\alpha_2 \\
&+ f_1^2\cos\alpha_1 - f_1^2\cos\alpha_1\cos\alpha_2 + f_1^2\cos\alpha_2 - f_1^2\cos\alpha_2 - f_1^2\cos\alpha_1\cos\alpha_2 \\
&+ f_1^4 - f_1^4\cos\alpha_1 - f_1^4\cos\alpha_2 + f_1^4\cos\alpha_1\cos\alpha_2 \\
&+ f_1^2 f_2^2 - f_1^2 f_2^2\cos\alpha_1 - f_1^2 f_2^2\cos\alpha_2 + f_1^2 f_2^2\cos\alpha_2\cos\alpha_2 - f_3^2\sin\alpha_1\sin\alpha_2 \\
&+ f_1^2 f_3^2 - f_1^2 f_3^2\cos\alpha_1 - f_1^2 f_3^2\cos\alpha_2 + f_1^2 f_3^2\cos\alpha 1\cos\alpha 2 - f_2^2\sin\alpha_1\sin\alpha_2 = \\
= &\cos\alpha_1\cos\alpha_2 \\
&+ f_1^2\cos\alpha_1(1-f_2^2-f_3^2) + f_1^2(f_1^2+f_2^2+f_3^2) + f_1^2\cos\alpha_2(1-f_2^2-f_3^2) \\
&+ f_1^2\cos\alpha_1\cos\alpha_2(-2+f_2^2+f_3^2) \\
&- f_1^4\cos\alpha_1 - f_1^4\cos\alpha_2 + f_1^4\cos\alpha_1\cos\alpha_2 - f_3^2\sin\alpha_1\sin\alpha_2 - f_2^2\sin\alpha_1\sin\alpha_2 = \\
\overset{(1)(2)}{=} &\cos\alpha_1\cos\alpha_2 + f_1^2 - f_1^2\cos\alpha_1\cos\alpha_2 - f_3^2\sin\alpha_1\sin\alpha_2 - f_2^2\sin\alpha_1\sin\alpha_2 = \\
= &\cos\alpha_1\cos\alpha_2 + f_1^2 - f_1^2\cos\alpha_1\cos\alpha_2 - \underbrace{(f_3^2+f_2^2)}_{1-f_1^2} \cdot \sin\alpha_1\sin\alpha_2 = \\
\overset{(4)}{=} &\cos(\alpha_1+\alpha_2) + f_1^2 \cdot (1-\cos(\alpha_1+\alpha_2))
\end{aligned}$$

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Annotations:

$$f_1^2 + f_2^2 + f_3^2 = 1$$

$$\Leftrightarrow 1 - f_2^2 - f_3^2 = f_1^2 \tag{1}$$

$$\Leftrightarrow -2 + f_2^2 + f_3^2 = -1 - f_1^2 \tag{2}$$

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y \tag{3}$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y \tag{4}$$

Task 2 (4 points)

For the given matrix

$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & A_{13} \\ \frac{\sqrt{6}}{8} & \frac{5\sqrt{2}}{8} & -\frac{\sqrt{2}}{4} \\ \frac{3\sqrt{6}}{8} & -\frac{\sqrt{2}}{8} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

i) Calculate A_{13}, such that $\boldsymbol{A}$ is a rotation matrix.

ii) Calculate the euler angles using the parametrisation given in formula (2.6) from the lecture. Are the rotations unique?

iii) Explain the effect of $\boldsymbol{A}$ on a vector $\boldsymbol{v}$ if A_{13} differs from the solution above.

Solution:

Zu i) Different possibilities e.g.:

- Calculate $\boldsymbol{A} \cdot \boldsymbol{A}^T \stackrel{!}{=} \mathbb{1}$
- $\det(\boldsymbol{A}) \stackrel{!}{=} 1$

$\rightarrow \quad A_{13} = -\frac{\sqrt{3}}{2}$

Zu ii) See equation (2.9) from the lecture:

$\alpha = \arctan\left(\frac{A_{23}}{A_{33}}\right) = \arctan\left(\frac{-\frac{\sqrt{2}}{4}}{\frac{\sqrt{2}}{4}}\right) = -\frac{\pi}{4}$

$\beta = \arcsin\left(-A_{13}\right) = \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$\gamma = \arctan\left(\frac{A_{12}}{A_{11}}\right) = \arctan\left(\frac{-\frac{\sqrt{3}}{4}}{\frac{1}{4}}\right) = -\frac{\pi}{3}$

Zu iii) For $\det(\boldsymbol{A}) = -1$ the matrix represents mirroring, the other cases are mixtures of mirroring and dilations.

Task 3 (3 points)

Show, that the quaternion rotation defined by

$$\boldsymbol{p}' = \boldsymbol{q}\boldsymbol{p}\bar{\boldsymbol{q}}$$

corresponds to the DCM (2.14) from the lectures.

Annotation: $\boldsymbol{p} = \begin{pmatrix} 0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$ contains the coordinates of the vector $\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$

Solution:

$$\boldsymbol{p} = \begin{bmatrix} 0 \\ \tilde{p} \end{bmatrix} \qquad \boldsymbol{q} = \begin{bmatrix} q_0 \\ \tilde{q} \end{bmatrix} \qquad \bar{\boldsymbol{q}} = \begin{bmatrix} q_0 \\ -\tilde{q} \end{bmatrix}$$

$$\begin{aligned}
\boldsymbol{p} \circ \bar{\boldsymbol{q}} &= \begin{bmatrix} \tilde{p}\tilde{q} \\ q_0\tilde{p} - \tilde{p} \times \tilde{q} \end{bmatrix} \\
\boldsymbol{q} \circ (\boldsymbol{p} \circ \bar{\boldsymbol{q}}) &= \begin{bmatrix} (\tilde{p} \cdot \tilde{q})q_0 - \tilde{q} \cdot (q_0\tilde{p} - \tilde{p} \times \tilde{q}) \\ q_0\left(q_0\tilde{p} - \tilde{p} \times \tilde{q}\right) + (\tilde{p} \cdot \tilde{q}) \cdot \tilde{q} + \tilde{q} \times \left(q_0\tilde{p} - \tilde{p} \times \tilde{q}\right) \end{bmatrix} = \\
&= \begin{bmatrix} 0 \\ q_0^2\tilde{p} - (\tilde{p} \cdot \tilde{q})\tilde{q} - \tilde{q} \times (\tilde{p} \times \tilde{q}) \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \boldsymbol{p}' =& q_0^2\tilde{p} + 2q_0(\tilde{q} \times \tilde{p}) + (\tilde{p} \cdot \tilde{q})\tilde{q} - [\tilde{p}(\tilde{q} \cdot \tilde{q}) - \tilde{q}(\tilde{p} \cdot \tilde{q})] = \\
=& (q_0^2 - \tilde{q} \cdot \tilde{q})\tilde{p} + 2q_0(\tilde{q} \times \tilde{p}) + 2(\tilde{p} \cdot \tilde{q})\tilde{q} = \\
=& \begin{pmatrix} q_0^2 - q_1^2 - q_2^2 - q_3^2 & 0 & 0 \\ 0 & q_0^2 - q_1^2 - q_2^2 - q_3^2 & 0 \\ 0 & 0 & q_0^2 - q_1^2 - q_2^2 - q_3^2 \end{pmatrix} \tilde{p} \\
&+ \begin{pmatrix} 0 & -2q_0q_3 & 2q_0q_2 \\ 2q_0q_3 & 0 & -2q_0q_1 \\ -2q_0q_2 & 2q_0q_1 0 & \end{pmatrix} \tilde{p} + \underbrace{\begin{pmatrix} q_1(p_1q_1 + p_2q_2 + p_3q_3) \\ q_2(p_1q_1 + p_2q_2 + p_3q_3) \\ q_3(p_1q_1 + p_2q_2 + p_3q_3) \end{pmatrix}}_{= \begin{pmatrix} 2q_1^2 & 2q_1q_2 & 2q_1q_3 \\ 2q_1q_2 & 2q_2^2 & 2q_2q_3 \\ 2q_1q_3 & 2q_2q_3 & 2q_3^2 \end{pmatrix} \tilde{p}} \\
\Rightarrow \boldsymbol{p}' =& \begin{pmatrix} q_0^2 - q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 - q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 - q_3^2 \end{pmatrix} \\
=& [C_t^s]^T \text{ (Equation 2.14) from Lecture}
\end{aligned}$$

Transposed, because in the lecture the coordinate system is rotated, here a vector.