Universität Stuttgart

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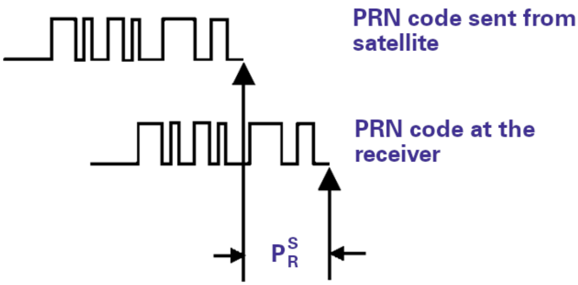
Satellite Navigation

Observables and error budget

6

Observables and error budget



- A GNSS receiver measures the time offset it needs to apply to its replica of the code to reach maximum correlation with received signal
- Thus, it is measuring the time difference between when a signal was transmitted (based on satellite clock) and when it was received (based on receiver clock).
- If the satellite and receiver clocks were synchronized, this would be a measure of range
- since they are not synchronized, it is called "pseudorange" P_R^S

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Pseudorange Pseudorange P_k^p is defined as

$$P_k^p = (\tilde{t}_k - \tilde{t}^p) \cdot c \tag{6.1}$$

where subscript k denotes the receiver and superscript p the satellite. $\tilde{t}_k$ is the reception in receiver clock time, $\tilde{t}^p$ is the transmission epoch and c is the speed of light.

However, the true range (expressed in units of time) between the satellite and the receiver is not equal to P_k^p since the receiver and satellite clocks are offset by small time differences due to clock offsets.

$$\begin{aligned} \tilde{t}_k &= t_k + \tau_k \\ \tilde{t}^p &= t^p + \tau^p \end{aligned} \tag{6.2}$$

t_k and t^p are true times; τ_k and τ^p are the clock offsets.

Substituting (6.2) into (6.1) yields

$$P_k^p = [(t_k - t^p) + (\tau_k - \tau^p)] \cdot c \tag{6.3}$$

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Since signals propagate slower than the speed of light in the ionosphere and troposphere, Eq. (6.3) can be written as

$$\boxed{P_k^p = \rho_k^p + (\tau_k - \tau^p) \cdot c + \underbrace{I_k^p}_{\text{ionosphere delay}} + \underbrace{A_k^p}_{\text{troposphere delay}}} \tag{6.4}$$

where ρ_k^p is the true (geometrical) range between receiver k and satellite p. This range is equal to the Euclidian distance computed from the position of the satellite at transmission time t_p and the position of the receiver at the time t the signal was captured.

Dropping now the index k, i.e. focusing on a single receiver, we obtain

$$\rho^p(t, t^p) = \sqrt{[X^p(t^p) - x(t)]^2 + [Y^p(t^p) - y(t)]^2 + [Z^p(t^p) - z(t)]^2} \tag{6.5}$$

which implies that we four unknowns (three position components + the receiver clock)

Observables and error budget

Ignoring tropospheric and ionospheric delays, we can set up a non-linear equation system for N pseudorange observations collected at epoch t

$$
\begin{aligned}
P^1 &= \sqrt{(X^1-x)^2+(Y^1-y)^2+(Z^1-z)^2}+c\tau-c\tau^1+\epsilon_1\\
P^2 &= \sqrt{(X^2-x)^2+(Y^2-y)^2+(Z^2-z)^2}+c\tau-c\tau^2+\epsilon_2\\
&\vdots\\
P^N &= \sqrt{(X^N-x)^2+(Y^N-y)^2+(Z^N-z)^2}+c\tau-c\tau^N+\epsilon_N
\end{aligned}
\tag{6.6}
$$

which implies that we have four unknowns (three position components + the receiver clock).

Eq. (6.6) is non-linear in position coordinates (x, y, z) and the terms ϵ_k denote the (random) noise contribution to a pseudorange observation. Linearization of (6.5) of the pseudorange equations yields,

$$
\begin{aligned}
P^k(x,y,z,\tau) \quad = \quad & \underbrace{P^k(x_0,y_0,z_0,\tau_0)}_{P^k_{\text{computed}}}+\underbrace{(x-x_0)}_{\Delta x}\frac{\partial P^k}{\partial x}+\underbrace{(y-y_0)}_{\Delta y}\frac{\partial P^k}{\partial y}+\\
& +\underbrace{(z-z_0)}_{\Delta z}\frac{\partial P^k}{\partial z}+\underbrace{(\tau-\tau_0)}_{\Delta \tau}\frac{\partial P^k}{\partial \tau}
\end{aligned}
\tag{6.7}
$$

Observables and error budget

$$\begin{aligned}\Delta P^k &= P^k_{\text{observed}} - P^k_{\text{computed}} \\ &= \frac{\partial P^k}{\partial x}\Delta x + \frac{\partial P^k}{\partial y}\Delta y + \frac{\partial P^k}{\partial z}\Delta z + \frac{\partial P^k}{\partial \tau}\Delta \tau + \epsilon\end{aligned} \tag{6.8}$$

In matrix notation

$$\Delta P^k = \begin{pmatrix} \frac{\partial P^k}{\partial x} & \frac{\partial P^k}{\partial y} & \frac{\partial P^k}{\partial z} & \frac{\partial P^k}{\partial \tau} \end{pmatrix} \cdot \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta \tau \end{pmatrix} + \epsilon \tag{6.9}$$

Thus, the equation array (6.6) can be written as

$$\boldsymbol{\Delta P} = \underbrace{\begin{pmatrix} \frac{x_0 - X^1}{\rho^1} & \frac{y_0 - Y^1}{\rho^1} & \frac{z_0 - Z^1}{\rho^1} & c \\ \frac{x_0 - X^2}{\rho^2} & \frac{y_0 - Y^2}{\rho^2} & \frac{z_0 - Z^2}{\rho^2} & c \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x_0 - X^N}{\rho^N} & \frac{y_0 - Y^N}{\rho^N} & \frac{z_0 - Z^N}{\rho^N} & c \end{pmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta \tau \end{pmatrix}}_{\boldsymbol{\Delta x}} + \underbrace{\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_N \end{pmatrix}}_{\boldsymbol{\epsilon}} \tag{6.10}$$

Observables and error budget

In case of $N > 4$ Eq. (6.10) poses an over-determined problem and $\boldsymbol{\Delta x}$ can be solved my a least-squares adjustment

$$\boldsymbol{\Delta x} = \underbrace{(\boldsymbol{A}^T\boldsymbol{A})^{-1}}_{\boldsymbol{Q}_{xx}}\boldsymbol{A}^T\boldsymbol{\Delta P} \tag{6.11}$$

where $\boldsymbol{Q}_{xx}$ is the cofactor matrix of the corrections $\boldsymbol{\Delta x}$ which relates to the covariance matrix $\boldsymbol{\Sigma}_{xx}$ by

$$\boldsymbol{\Sigma}_{xx} = \sigma_r^2\boldsymbol{Q}_{xx} = \sigma_r^2 \begin{bmatrix} q_{xx} & q_{xy} & q_{xz} & q_{x\tau} \\ q_{yx} & q_{yy} & q_{yz} & q_{y\tau} \\ q_{zx} & q_{zy} & q_{zz} & q_{z\tau} \\ q_{\tau x} & q_{\tau y} & q_{\tau z} & q_{\tau\tau} \end{bmatrix} \tag{6.12}$$

where σ_r is the measurement accuracy of a single pseudorange observation.

In GNSS applications (especially in real-time applications in which positions are determined ”instantaneously”), precision is represented by Dilution of Precision (DOP) values.

Observables and error budget

- PDOP: Overall 3D position precision

$$\sigma_P = \sigma_r \underbrace{\sqrt{q_{xx} + q_{yy} + q_{zz}}}_{\text{PDOP}}$$

- HDOP: 2D position precision (NB: you need to transform the covariance in the local coordinate system!)

$$\sigma_H = \sigma_r \underbrace{\sqrt{q_{nn} + q_{ee}}}_{\text{HDOP}}$$

- VDOP: vertical position precision (NB: you need to transform the covariance in the local coordinate system!)

$$\sigma_V = \sigma_r \underbrace{\sqrt{q_{uu}}}_{\text{VDOP}}$$

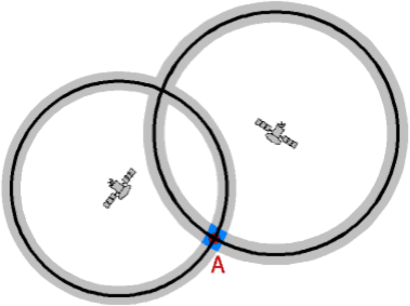
- TDOP: time precision

$$\sigma_T = \sigma_r \underbrace{\sqrt{q_{\tau\tau}}}_{\text{TDOP}}$$

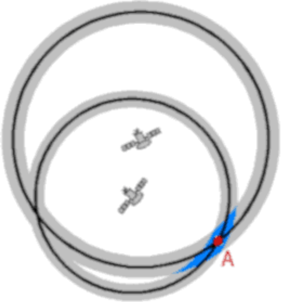
How is DOP related to the geometry? What is a ”good” or ”bad” geometry?

Observables and error budget

Geometric considerations



"good' geometry

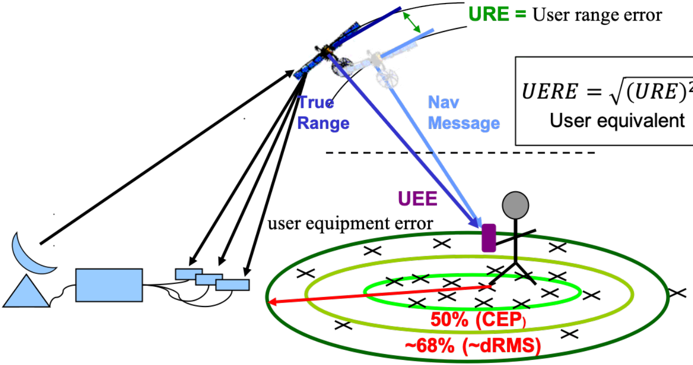


"bad' geometry

The DOP factors can be computed from the geometry.
E.g. see this planning tool `https://www.gnssplanning.com/`

Observables and error budget

Accuracy



$$UERE = \sqrt{(URE)^2 + (UEE)^2}$$

User equivalent range error

Observables and error budget

Typical Pre-2000 GPS Error Budget σ_r (with Selective Availability)

	One-sigma Error (meters)		
Error Source	Bias	Random	Total
Ephemeris	1.0	0.0	1.0
Satellite clock	20.0	0.7	20.0
Ionosphere	4.0	0.5	4.0
Troposphere	0.5	0.0	0.5
Multipath	0.2	0.2	0.3
Receiver noise	0.0	0.1	0.1
User equivalent range error, RME	20.5	0.9	20.5
Filtered UERE, RMS	20.5	0.4	20.5
Vertical one-sigma errors - VDOP = 1.7			34.8
Horizontal one-sigma errors - HDOP = 1.0			20.5

Observables and error budget

Typical Single-Frequency Error Budget σ_r (no Selective Availability)

Error Source	Bias	Random	Total
	One-sigma Error (meters)		
Ephemeris	0.8	0.0	0.8
Satellite clock	1.0	0.0	1.0
Ionosphere*	7.0	0.0	7.0
Troposphere	0.2	0.0	0.2
Multipath	0.2	0.2	0.3
Receiver noise	0.0	0.1	0.1
User equivalent range error, RME	7.1	0.2	7.1
Filtered UERE, RMS	7.1	0.1	7.1
Vertical one-sigma errors - VDOP = 1.7			12.1
Horizontal one-sigma errors - HDOP = 1.0			7.1

* Note that residual ionospheric delay errors tend to be highly correlated among satellites, and thus observed position-domain errors tend to be less than predicted by DOP UERE.

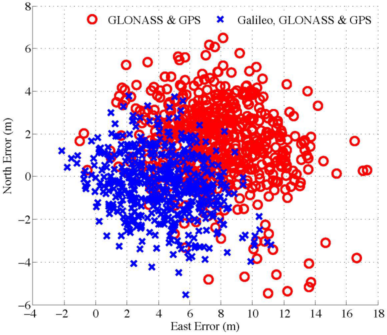
Observables and error budget

- The equation for the pseudorange uses the true range and corrections applied for propagation delays because the propagation velocity is not the in-vacuum value, c, 299 792 458 m/s
- To convert times to distance c is used and then corrections applied for the actual velocity not equaling c.
- The true range is related to the positions of the ground receiver and satellite.
- Also need to account for noise in measurements
- Pseudorange noise (random and not so random errors in measurements) contributions:
    - Correlation function width: The width of the correlation is inversely proportional to the bandwidth of the signal. Therefore the 1 MHz bandwidth of C/A code produces a peak 1 μs wide (300m) compared to the P(Y) code 10 MHz bandwidth which produces 0.1 μs peak (30 m) Rough rule is that peak of correlation function can be determined to 1% of width (with care). Therefore 3 m for C/A code and 0.3 m for P(Y) code.
    - Thermal noise: Effects of other random radio noise in the GPS bands Since C/A code has narrower bandwidth, tracking it in theory has less thermal noise power than the P(Y) code. Thermal noise is general smallest effect
    - Multipath: see previous lecture

Observables and error budget

Summary: Code phase/pseudorange observations

- The main noise sources are related to reflected signals and tracking approximations.
- High quality receiver: noise about 10 cm
- Low cost receiver: noise is a few meters (depends on surroundings and antenna)
- In general: C/A code pseudoranges are of similar quality to P(Y) code ranges. C/A can use narrowband tracking which reduces amount of thermal noise



Observables and error budget

Carrier phase measurements

- Carrier phase measurements are similar to pseudorange in that they are the difference in phase between the transmitting and receiving oscillators. Integration of the oscillator frequency gives the clock time.
- Basic notion in carrier phase (in cycles):

$$\varphi = f\Delta t \tag{6.13}$$

where ϕ is phase and f is frequency

- The "Big problem" is how to know the number of cycles in the phase measurements

$$\Phi_k^p + N_k^p = \varphi_k - \varphi^p \tag{6.14}$$

where φ corresponds to the true phase state of the receiver k and satellite p and N represents the (unknown) number of integer cycles.
Dropping the index for the receiver and re-arranging Eq. (6.14) we obtain

$$\Phi^p(\tilde{t}) = \varphi(\tilde{t}) - \varphi^p(\tilde{t}^p) - N^p \tag{6.15}$$

whereas N is constant over time as long as the receiver does not loose phase lock.

Observables and error budget

Carrier phase measurements In general both, the receiver and the satellite phases are expresses as the sum of a time varying part and a constant phase offset, i.e.

$$\varphi(\tilde{t}) = f_0 \cdot \tilde{t} + \varphi_0 \qquad \varphi^p(\tilde{t}^p) = f_0 \cdot \tilde{t}^p + \varphi_0^p$$

Thus, the observed carrier phase can be denoted as

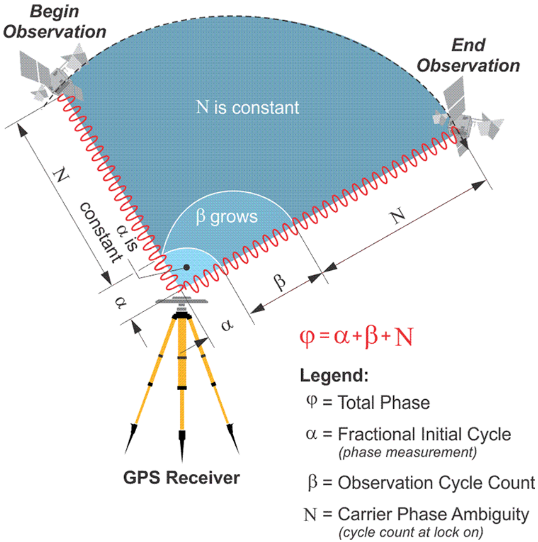
$$\begin{aligned} \Phi^p(\tilde{t}) &= f_0 \cdot \tilde{t} + \varphi_0 - f_0 \cdot \tilde{t}^p - \varphi_0^p - N^p \\ &= f_0(\tilde{t} - \tilde{t}^p) + \varphi_0 - \varphi_0^p - N^p \end{aligned} \tag{6.16}$$

Multiplication of (6.16) with the corresponding wavelength λ_0 yields the carrier phase observation in units of distance, i.e.

$$\begin{aligned} L^p(\tilde{t}) &= \lambda_0 \Phi^p(\tilde{t}) \\ &= \lambda_0 f_0(\tilde{t} - \tilde{t}^p) + \lambda_0(\varphi_0 - \varphi_0^p - N^p) \\ &= c(\tilde{t} - \tilde{t}^p) + \underbrace{\lambda_0(\varphi_0 - \varphi_0^p - N^p)}_{B^p} \\ &= \rho^p(t, t^p) + c\tau - c\tau^p + A^p - I^p + B^p \end{aligned} \tag{6.17}$$

Observables and error budget

The carrier phase observations



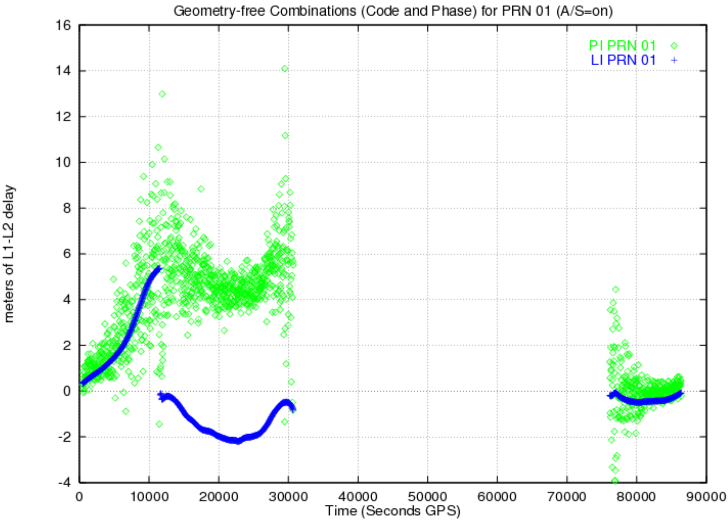
Observables and error budget

The carrier phase observations

- When carrier phase is used it is converted to distance using the standard L1 and L2 frequencies and vacuum speed of light.
- Clock terms are introduced to account for difference between true frequencies and nominal frequencies. As with range ionospheric and atmospheric delays account for propagation velocity
- Nominally carrier phase can be measured to 1% of wavelength (2mm L1 and 2.4 mm L2)
- Also effected by multipath, ionospheric delays (30m), atmospheric delays (3-30m).
- Since carrier phase is more precise than range, more effects need to be carefully accounted for with phase.
- Precise and consistent definition of time of events is one the most critical areas
- In general, carrier phase can be treated like pseudorange (code phase) measurements with an unknown offset due to cycles and offsets of oscillator phases.

Observables and error budget

Code vs. carrier phase observations (NB: Shown here are the L1-L2 / P1-P2 differences!)



Observables and error budget

GNSS data formats

- National Marine Electronics Association (NMEA) Format
  - NMEA is used to output measurement data from a sensor in a pre-defined ASCII format
  - in the case of GNSS, it outputs position, velocity, time and satellite related data
  - NMEA sentences (output) begins with a "Talker ID" and "Message Description"
  - Example:
    `$GPGGA,123519,4807.038,N,01131.000,E,1,08,0.9,545.4,M,46.9,M,,*47`

    which decodes to:

    Global Positioning System Fix Data Fix (1) taken at 12:35:19 UTC Latitude 48 deg 07.038' N Longitude 11 deg 31.000' E,

    other fields:
    `08` ... Number of satellites being tracked
    `0.9` ... Horizontal dilution of position
    `545.4,M` ... Altitude, Meters, above mean sea level
    `46.9,M` ... Height of geoid (mean sea level) above WGS84 ellipsoid

Observables and error budget

GNSS <u>observation</u> formats

- The Receiver Independent Exchange (RINEX) format allows to exchange raw satellite data among different types of receivers
- RINEX only provides Raw Data. It does not provide position output.
- Raw data consists of pseudorage, carrier phase, Doppler, SNR
- RINEX basically consists of two data types
  - `.*N` files for satellite ephemeris and related data. Also called Navigation Data.
  - `.*O` files for observational data like pseudorange, carrier Phase, doppler, SNR. Also called Observation Data

- Two format versions in use currently
- RINEX 2.11, still used for many applications, does not support all modern GNSS signals
  Format description: `ftp://igs.org/pub/data/format/rinex211.txt`
- RINEX 3.XX, very flexible, support all GNSS signal types
  Format description: `ftp://igs.org/pub/data/format/rinex303.pdf`

Observables and error budget

RINEX 2.11 navigation file example

```
     2.11           N: GPS NAV DATA                         RINEX VERSION / TYPE
XXRINEXN V2.10      AIUB                3-SEP-99 15:22      PGM / RUN BY / DATE
EXAMPLE OF VERSION 2.11 FORMAT                              COMMENT
     .1676D-07   .2235D-07  -.1192D-06  -.1192D-06          ION ALPHA
     .1208D+06   .1310D+06  -.1310D+06  -.1966D+06          ION BETA
     .133179128170D-06  .107469588780D-12   552960     1025 DELTA-UTC: A0,A1,T,W
    13                                                      LEAP SECONDS
                                                            END OF HEADER
 6 99  9  2 17 51 44.0 -.839701388031D-03 -.165982783074D-10  .000000000000D+00
     .910000000000D+02  .934062500000D+02  .116040547840D-08  .162092304801D+00
     .484101474285D-05  .626740418375D-02  .652112066746D-05  .515365489006D+04
     .409904000000D+06 -.242143869400D-07  .329237003460D+00 -.596046447754D-07
     .111541663136D+01  .326593750000D+03  .206958726335D+01 -.638312302555D-08
     .307155651409D-09  .000000000000D+00  .102500000000D+04  .000000000000D+00
     .000000000000D+00  .000000000000D+00  .000000000000D+00  .910000000000D+02
     .406800000000D+06  .000000000000D+00
13 99  9  2 19  0  0.0  .490025617182D-03  .204636307899D-11  .000000000000D+00
     .133000000000D+03 -.963125000000D+02  .146970407622D-08  .292961152146D+01
    -.498816370964D-05  .200239347760D-02  .928156077862D-05  .515328476143D+04
     .414000000000D+06 -.279396772385D-07  .243031939942D+01 -.558793544769D-07
     .110192796930D+01  .271187500000D+03 -.232757915425D+01 -.619632953057D-08
    -.785747015231D-11  .000000000000D+00  .102500000000D+04  .000000000000D+00
     .000000000000D+00  .000000000000D+00  .000000000000D+00  .389000000000D+03
     .410400000000D+06  .000000000000D+00
```

Observables and error budget

RINEX 2.11 observation data - header

```
     2.11           OBSERVATION DATA    G (GPS)             RINEX VERSION / TYPE
teqc  2011Oct11                         20111023 09:34:07UTCPGM / RUN BY / DATE
Linux 2.4.20-8|Pentium IV|gcc -static|Linux|486/DX+         COMMENT
BIT 2 OF LLI FLAGS DATA COLLECTED UNDER A/S CONDITION       COMMENT
INSA (COGO code)                                            COMMENT
INSA                                                        MARKER NAME
INSA                                                        MARKER NUMBER
-Unknown-           -Unknown-                               OBSERVER / AGENCY
4925K35627          TRIMBLE NETR8       4.14                REC # / TYPE / VERS
n                   TRM55971.00     NONE                    ANT # / TYPE
  4157188.6232   671202.3189  4774769.4135                  APPROX POSITION XYZ
        0.0000        0.0000        0.0000                  ANTENNA: DELTA H/E/N
     1     1                                                WAVELENGTH FACT L1/2
     4    C1    L1    D1    S1                              # / TYPES OF OBSERV
    15                                                      LEAP SECONDS
 SNR is mapped to RINEX snr flag value [0-9]                COMMENT
  L1 & L2: min(max(int(snr_dBHz/6), 0), 9)                  COMMENT
  2011    10    21    21     0    0.0000000     GPS         TIME OF FIRST OBS
                                                            END OF HEADER
```

Observables and error budget

RINEX 2.11 observation data - observations

```
:
                                                            END OF HEADER
 11 10 21 21  0  0.0000000  0 11G13G32G17G20G04G07G23G31G10G30G02
  20155765.0784  105919197.42048       817.1844         49.6004
  24807272.2424  130363271.39747     -2984.4964         42.7004
  24583721.500   129188341.904 6     -3155.723          41.100
  22156274.0784  116432358.94747     -2734.5274         47.8004
  20893261.2584  109794764.97748       562.8324         50.5004
  24006470.539   126154961.773 7      3728.004          42.000
  20438748.2114  107406456.87448      -978.6214         51.4004
  25302810.977   132967108.753 6     -3258.719          40.000
  21910621.5554  115141275.85048      2638.1604         48.8004
  24859456.7274  130637340.19546      -315.5234         37.8004
  23734534.2194  124725786.40647      3218.8484         44.8004
 11 10 21 21  0  1.0000000  0 11G13G32G17G20G04G07G23G31G10G30G02
  20155610.1094  105918380.42148       816.9064         49.1004
  24807840.1954  130366255.89247     -2984.3124         42.6004
  24584321.805   129191497.666 6     -3155.578          40.800
  22156794.0004  116435093.43448     -2734.3914         48.0004
:
```

Observables and error budget

RINEX 3.03 observation data

Header hold table that lists observation type for each GNSS

```
E    8 C1X C5X D1X D5X L1X L5X S1X S5X                          SYS / # / OBS TYPES
G   20 C1C C1W C2W C2X C5X D1C D1W D2W D2X D5X L1C L1W L2W  SYS / # / OBS TYPES
       L2X L5X S1C S1W S2W S2X S5X                          SYS / # / OBS TYPES
J   17 C1C C1Z C2X C5X D1C D1X D1Z D2X D5X L1C L1Z L2X L5X  SYS / # / OBS TYPES
       S1C S1Z S2X S5X                                      SYS / # / OBS TYPES
R   16 C1C C1P C2C C2P D1C D1P D2C D2P L1C L1P L2C L2P S1C  SYS / # / OBS TYPES
       S1P S2C S2P                                          SYS / # / OBS TYPES
```

Data section with one line for each satellite

```
> 2018 01 01 00 00 00.0000000  0 30
E02  24174379.051 0  24174382.216 0       783.933 0       585.401 0 127037224.150 8  94865462.237 9        49.750 0   ...
E07  26585495.945 0  26585500.951 0     -1630.526 0     -1217.585 0 139707745.275 7 104327210.887 7        43.250 0   ...
:
G02  24133922.029 0  24133920.426 0  24133916.354 0                                      -2482.863 0     -2482.863 0   ...
G04  20880431.607 0  20880431.627 0  20880429.176 0                                       -666.495 0      -666.495 0   ...
:
J01  42043537.415 0  42043534.463 0  42043535.474 0  42043539.625 0       617.664 0       617.658 0       617.527 0   ...
R03  22141543.842 0  22141544.405 0  22141548.497 0  22141551.286 0     -2894.487 0     -2894.487 0     -2251.255 0   ...
:
```

Observables and error budget

Single point positioning (SPP) - summary

- Signal, tagged with time from satellite clock, transmitted.
- About 66 msec (20,000 km) later the signal arrives at GNSS receiver. Satellite has moved about 66 m during the time it takes signal to propagate to receiver.
- Time the signal is received is given by clock in receiver. Difference between transmit time and receive time is pseudorange.
- During the propagation, signal passes through the ionosphere (10-100 m of delay, phase advance), and neutral atmosphere (2.3-30 m depending on elevation angle).
- To determine an accurate position from range data, we need to account for all these propagation effects (see following lectures) and time offsets.
- Basic clock treatment in GPS
  - True time of reception of signal needed
  - True time of transmission needed (af0, af1 from broadcast ephemeris initially is good enough for SPP)
  - Position of satellite when signal transmitted

Observables and error budget

Single point positioning (SPP) - considerations

- Satellites move at about 1 km/s, therefore an error of 1 ms in time results in 1 m satellite position (and therefore in range estimate and receiver position).
- For pseudo-range positioning (SPP), 1 ms errors OK. For phase positioning (1 mm), time is needed to 1μs.
- For low precision positioning (tens of meters) the satellite clocks are assumed known and given by the broadcast ephemeris.
  - Receiver clock can be estimated along with 3-D positions if 4 or more satellites are visible.

In the following lectures we will study the main error sources due to signal propagation and then discuss techniques that provide us more precise positions solutions from GNSS