Exercise on 09.07.2019

Task 1 (5 points)

A stochastic process (random walk)

$$\dot{x} = 0 + w(t).$$

is implicitly observed by two instruments. The first instrument provides measurements z, which are related to the state x as follows:

$$z(t) = 0.5 \cdot x(t).$$

The second instrument uses another kind of measurement which has the following relationship between measurement z and state x.

$$z(t) = \cos(p) \cdot x(t) \quad \text{with } p = 1 + t/90 \text{ (in rad)}.$$

In order to estimate the state at discrete times t_i a Kalman filter is used. Your task now is to compute the uncertainty, i.e. the standard deviation of x over time for $t = 1 : 200$ with a step size of $\Delta t = 1$ when only measurements of instrument 1 or instrument 2 are available. Plot the standard deviation over time for both solutions. One can use $\Sigma_0 = 100$ for the first epoch and set the variance of the process noise to $\sigma^2 = 4$. The variance of the measurement noise is $\sigma^2 = 1$ for both instruments. You will recognize a strange behaviour in the uncertainty of the state when using the second instrument. Explain the reason for that.

Task 2 (5 points)

A damped mass-spring oscillator is defined by

$$m \cdot \ddot{x}(t) + b \cdot \dot{x}(t) + k \cdot x(t) = 0$$

where

- $x(t)$ denotes the position of the mass at time t
- m is the mass
- b is the damping coefficient
- k is the spring constant

Write this equations as a first-order linear system with two states (x_1 and x_2), where x_1 is the position x and x_2 is the velocity $\dot{x}$.

The position x_1 is observed during $t = 0 : 50$ in time steps of $\Delta t = 0.25$. The measurements are available from the file `KF_task2.txt`. The data shall be processed by means of Kalman filtering assuming the following parameters:

```
m = 20; % mass
k = 7;  % spring constant
b = 2;  % damping coefficient
x0 = [-1, 0]; % initial state vector (position and velocity)
R = 0.09; % variance of measurement noise
```

The covariance matrix of the first state is given by

$$\Sigma_0 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

and the covariance matrix of the process noise is

$$Q = 0.0004 \cdot \begin{bmatrix} \Delta t & 0 \\ 0 & \Delta t \end{bmatrix}$$

The true trajectory (position) of the mass can by computed (Matlab) by defining the function:

```
rhs = @(t,x) F*x;  % rhs of function
```

and using

```
[~,trueTrajectory] = ode45(rhs,t,[1,0]);
```

where the function `ode45` integrates a system of differential equations using the time vector t and the true initial state vector `[1,0]`. F is the coefficient matrix for the first-order linear system.

Plot your results and compare the filtered positions with the true positions.

In addition, discuss the influence on your results, when either Q or R is 100 times larger. Explain the oscillating behaviour of the standard deviation as well as the obtained time series of the position for the case that the measurement noise matrix is 100 times larger, i.e. using a value $R = 9.0$.