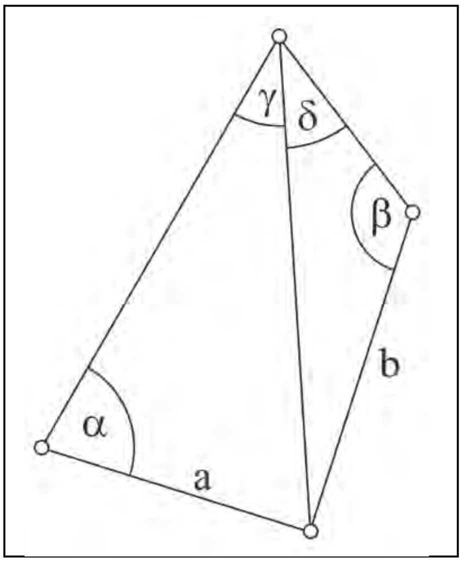
Exam Statistical Inference (WS 14/15)

Problem 1

The network in the picture (observed by 4 angles and 2 distances) is to be adjusted using the B-model ("Condition adjustment") with model formulation $w - B'e = 0$.



a) How is the redundancy r ?

b) In general, the conditions/condition equations in the B-model have to satisfy two postulates. Which are the postulates ?

c) How many condition equations have to be set up for this network and how are they ?

d) How is the condition equation matrix B' (specify explicitly the partial derivatives) and the vector w of misclosures (general formulation sufficient) ?

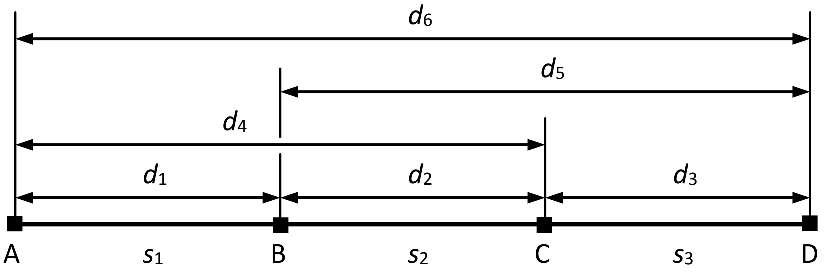
e) Since the problem is non-linear, iteration will be required. Which numbers are assigned to the initial approximate values (the Taylor point) ?

f) How do we switch from the model equations $w - B'e = 0$ to the normal equations to be solved, and how is the form of the normal equations ?

Problem 2

As shown in the figure below, the section AD of a straight line has been divided in three nearly equal parts. The distance measurements are given by

distance	d_1	d_2	d_3	d_4	d_5	d_6
in [m]	100,04	100,01	99,98	200,00	200,02	299,96



a) Set up the elements (y, A, x) of the linear system of observation equations (A-model)

b) Set up the elements (w, B') of the linear system of condition equations (B-model)

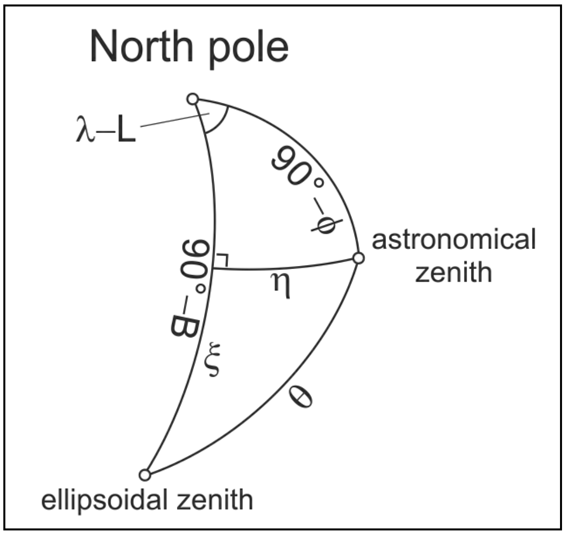
c) Calculate the adjusted inconsistencies $\hat{e}$ from the B-model.

d) How can you check the numerical results ?

Problem 3

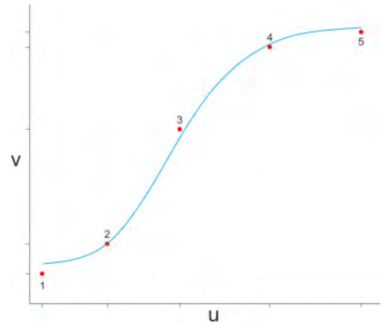
In order to determine the total deflection of the vertical $\theta = \sqrt{\xi^2 + \eta^2}$, astronomical latitude ϕ and astronomical longitude λ have been measured on a point with given ellipsoidal latitude B and ellipsoidal longitude L. According to the figure the east-west component of the deflection of the vertical is $\eta = (\lambda - L)\cos B$ while the north-east component is $\xi = \phi - B$. Calculate the standard deviation of $\theta = \sqrt{\xi^2 + \eta^2}$ for the numerical scenario given below:

	Value	Standard deviation σ
ϕ	52°00'05,3"	±0,3"
λ	6°14'20,6"	±0,7"
B	52°00'01,1"	---
L	6"14'24,4"	---



Exam Statistical Inference (WS 13/14)

Problem 1



In the picture, you see the function $v(u) = \frac{e^{\alpha+\beta u}}{1+e^{\alpha+\beta u}}$, which is assumed to model the characteristics of the five discrete points. While the u-coordinates of the points are fixed values, the corresponding v-coordinates are observed quantities. Our goal is to estimate the unknown model parameters α and β using the A-model. Approximate numbers α_0 and β_0 are given.

a) How is the formulation of the A-model ?

b) Do we have a datum problem ?

c) How is (in the A-model) the general equation for the redundancy and how is the redundancy of the problem here ?

d) Write down the matrix equation for the solution of $\hat{\alpha}$ and $\hat{\beta}$.

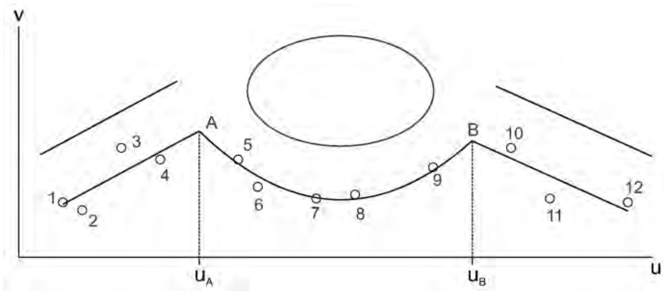
e) Show the detailed structure of the vectors/matrices involved in this A-model problem.

f) How would you treat the question that the final function should go through a given point $\tilde{u}, \tilde{v}$?

g) How will the square sum of estimated residuals $\hat{e}'\hat{e}$ change when you include the point constraint from f) in the adjustment ?

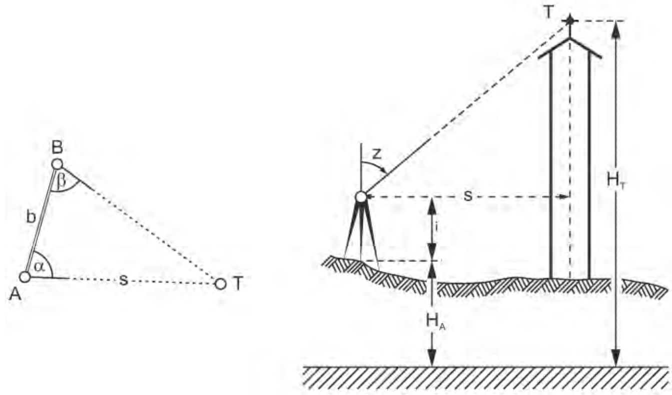
Problem 2

The figure sketches a roundabout and two roads leading from left and right into the roundabout. While the two roads are modelled as straight lines ($v_i = a_1 + b_1 u_i$, $i = 1,\ldots,4$ and $v_i = a_3 + b_3 u_i$, $i = 10,\ldots,12$), the lower arc of the roundabout is modelled as a parabola $v_i = a_2 + b_2 u_i + c_2 u_i^2$, $i = 5,\ldots,9$. Understandably, there shall be no gaps or jumps between the boundary lines of the roads and the roundabout; they shall be continuous in points A and B. In order to estimate the 7 parameters of the lines and the parabola, set up the quantities of the A-model under the requirement of continuity.



Problem 3

A typical problem for a surveyor is to determine the height H_T (above ground) of a tower (see figure) from measured zenith angle z and horizontal distance s. Reference point height H_A and height i of the instrument are given error-free. Unfortunately, distance s cannot be directly observed. For this reason base b and two angles α and β are measured using the horizontal auxiliary triangle ABT. Here, we are interested in the variance $\sigma^2_{H_T}$ of height H_T, only.



a) Specify the general (matrix) equation for $\sigma^2_{H_T}$ which results from the involved measurements.

b) Compute the necessary partial derivatives analytically.

c) Calculate the standard deviation of H_T from the data given in the table. Assume that there are no correlations between the measurements.

	Value	σ
z	50 gon	0,005 gon
α	32,3 gon	0,005 gon
β	67,7 gon	0,005 gon
b	53 m	0,01 m
i	1,5 m	0 m
H_A	253 m	0 m

Exam Statistical Inference (SS 13)

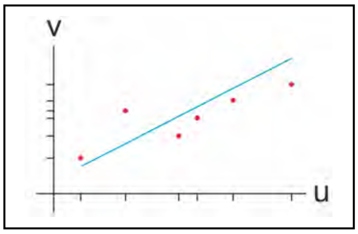
Problem 1: Same as WS0809, Problem 1

Problem 2 & 3: Same as SS10, Problems 2 & 3

Exam Statistical Inference (WS 12/13)

Problem 1

In order to represent a series of m>2 data points by a regression line coordinates u_i and v_i, i=1,...,m have been observed. Both sets of measurements are corrupted by random errors causing the necessity to introduce inconsistencies e_{u_i} and e_{v_i}: $v_i = a + b(u_i - e_{u_i}) + e_{v_i}$.



In order to estimate the regression line parameters a, b together with the inconsistencies the Gauß-Helmert approach is applied.

a) Specify its basic initial equation for the regression line problem.

b) Since the basic initial equation is nonlinear, write down its general Taylor series expansion up to the linear part.

c) For the regression line problem: How is the general Taylor point of approximation and how is the specific Taylor point in the first iteration ?

d) The linear(ized) model is most often written in the form $w + A\Delta\xi + B'e = 0$. How are the dimensions of the 6 vectors/matrices involved ?

e) How are the entries of the special matrices A and B for the regression line problem ?

Problem 2

It happens quite often in adjustment problems that constraints must be taken into account. Assume that a non-linear adjustment problem has been linearized properly. The basic equation of the A-model (Gauß-Markoff model) is therefore $\underset{m\times 1}{y} = \underset{m\times n}{A}\,\underset{n\times 1}{x} + \underset{m\times 1}{e}$ with y the vector of (reduced) observations, x the vector of unknown quantities, A the design matrix and e the vector of inconsistencies.

a) Specify the target function $\mathcal{L}_A(x,\lambda)$ which minimizes the square sum $\frac{1}{2}e'e$ and also takes into account constraints $D'x = c$ for the unknowns x.

b) Write down explicitly the necessary conditions for a minimum of $\mathcal{L}_A(x,\lambda)$?

c) Derive the normal equations.

d) Assume that A is of full column rank, i.e. its nullspace contains only the zero vector. How is the **explicit** solution for $\hat{\lambda}$?

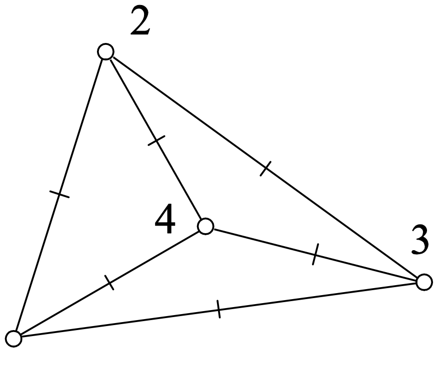
e) How is $\hat{x}$ obtained from the normal equations ? (**Explicit equation not necessary**)

Problem 3

See problem 3, WS 11/12

Exam Statistical Inference (WS 11/12)

Problem 1a



In order to compute the point coordinates in the simple planar trilateration network 1234 the indicated distances have been measured.

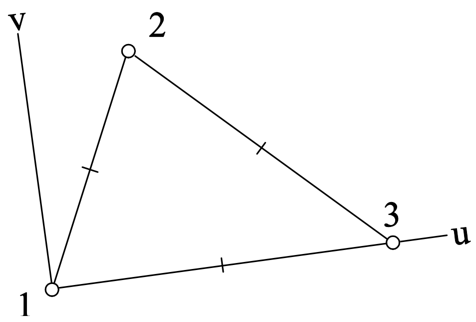
i) Determine the number m of observations, the number n of unknowns, the datum defect d and the redundancy r.

ii) If $d \neq 0$ then give reasons for a datum defect and describe its impact on the $m \times n$ design matrix A and the $n \times n$ normal equation matrix $N = A'PA$.

iii) How can the datum problem be fixed geometrically ? Specify two essentially different approaches !

iv) Starting from the linear model (A-model) $\underline{y} = Ax + \underline{e}$, rank of A = rk(A) < n, illustrate two different algebraic methods to remove the datum problem (no formulas required !). In other words, explain the term(s) "reduction/augmentation of the solution space".

Problem 1b



Now assume that the distance observations in the even simpler network 123 have to be processed and that the datum problem has been resolved by setting $u_1 = v_1 = v_3 = 0$. Furthermore, the distance observation s_{13} is taken as error free, while the distance observations $\underline{s}_{12}$ and $\underline{s}_{23}$ are stochastic variables with joint variance-covariance matrix

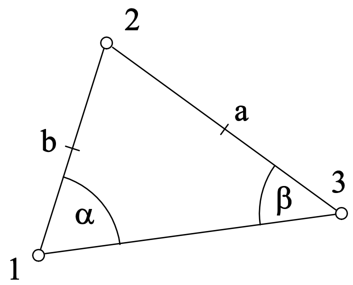
$$P^{-1} = Q_{[s_{12}, s_{23}]} = \begin{bmatrix} \sigma^2_{s_{12}} & 0 \\ 0 & \sigma^2_{s_{23}} \end{bmatrix}.$$

i) How is the variance $\sigma^2_{\hat{u}_2}$ of the estimated u-coordinate of point 2 (linear error propagation law) ?

$$\underline{\hat{u}}_2 = \frac{s_{13}^2 - \underline{s}_{23}^2 + \underline{s}_{12}^2}{2s_{13}}, \quad \underline{\hat{v}}_2 = \pm\sqrt{\underline{s}_{12}^2 - \underline{\hat{u}}_2^2}$$

ii) How is the variance $\sigma^2_{\hat{F}}$ of the estimated area $\hat{F}$ of the triangle 123 ?

Exam Statistical Inference (WS 11/12)



Problem 2

The figure shows a planar network where distance observations a, b and angle observations α, β have been carried out.

a) How many condition equations are required in order to adjust the network using the B-model. Specify the equation(s) !
b) Linearize the condition equation(s) if necessary and set up all quantities of the B-model $w - B^T e = 0$.
c) Comment on the datum problem.
d) Set up the target function $\mathcal{L}_B(e, \lambda)$ (Lagrange function) and derive the normal equations.
e) Solve the normal equations for $\hat{e}$.

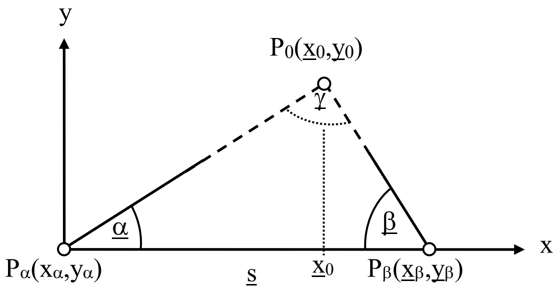
Problem 3

a) How are the projection matrices P_A and $P_A^{\perp}$ in a W-weighted least-squares problem, which project a vector orthogonally onto the column space of (WA) and onto a space orthogonal to the column space of (WA) ?
b) Which quantities are generated from the projections $P_A y$ and $P_A^{\perp} y$, respectively, if A is the design matrix and y the (reduced) vector of observations ?
c) How is the datum defect in a 2D-network being observed by distances, directions and azimuths ?
d) How is the datum defect in a 3D-network being observed by GPS-coordinates ?
e) How can symmetric 2×2 sub block matrices be interpreted geometrically, which stem from the diagonal of a large variance-covariance matrix of 2D-coordinates? How is the interpretation of corresponding 3×3 sub block matrices in 3D-space ?
f) How is the relation between a weight matrix P≡W of observations y and the variance-covariance matrix Q_y ?
g) Express the following variance-covariance matrices in terms of A, P and $\hat{\sigma}^2$?
$Q_{\hat{x}} \equiv D\{\underline{\hat{x}}\}$ (σ^2 known), $Q_{\hat{y}} \equiv D\{\underline{\hat{y}}\}$ (σ^2 known) and $\hat{Q}_{\hat{x}} \equiv \hat{D}\{\underline{\hat{x}}\}$ (σ^2 unknown) ?
h) Explain the terms Type-I-error probability and Type-II-error probability both graphically and in words.
i) How is the probability $P(a \le \underline{x} \le b)$ that a random variable $\underline{x}$ can take a value from the interval a, b ? Express the answer both in terms of the probability density function f(x) and the cumulative distribution function F(x) and draw two corresponding sketches.
j) Assume a stochastic quantity $\underline{x}$ which is normally distributed with mean value μ=3 and variance σ^2=0.25, i.e. $\underline{x} \sim N(\mu=3, \sigma^2=0.25)$. Find the critical value for α=0.01 and a one-sided test.
k) Which probability density function is connected with the quadratic form $\hat{e}^T P \hat{e}$? Sketch it !

Exam Statistical Inference (SS 11)

Problem 1

The graphics below shows a planar triangle in which observations $\underline{\alpha}$, $\underline{\beta}$ have been carried out in order to compute coordinates x_0, y_0 of point P_0 (in a local coordinate system x-y) from given coordinates $P_\alpha(x_\alpha, y_\alpha)$ und $P_\beta(\underline{x}_\beta, \underline{y}_\beta)$. This procedure is called "location by intersection". Here, given point $P_\alpha(x_\alpha, y_\alpha)$ is identical to the origin of the local coordinate system while given point $P_\beta(\underline{x}_\beta, \underline{y}_\beta)$ is located on the x-axis. As compared to fixed quantities stochastic quantities are underlined.



The direct solution for the unknown coordinate $\underline{x}_0$ is given through

$$\underline{x}_0 = \underline{s}\frac{\cos\underline{\alpha}\sin\underline{\beta}}{\sin(\underline{\alpha}+\underline{\beta})} \qquad \text{with} \quad \underline{s}^2 = (x_\alpha - \underline{x}_\beta)^2 + (y_\alpha - \underline{y}_\beta)^2.$$

Apply the linear error propagation law in order to derive the variance $\sigma^2_{x_0}$ of $\underline{x}_0$ using the following simplifying assumptions:

a) observations $\underline{\alpha}$, $\underline{\beta}$ are uncorrelated, and both pairs of coordinates $\underline{x}_\beta$, $\underline{y}_\beta$ and observations $\underline{\alpha}$, $\underline{\beta}$ are also uncorrelated, i.e. the common covariance matrix of all stochastic quantities

is specified through: $Q_{[x_\beta, y_\beta, \alpha, \beta]} = \left[\begin{array}{cc|cc} \sigma^2_{x_\beta} & \sigma_{x_\beta y_\beta} & 0 & 0 \\ \sigma_{x_\beta y_\beta} & \sigma^2_{y_\beta} & 0 & 0 \\ \hline 0 & 0 & \sigma^2_\alpha & 0 \\ 0 & 0 & 0 & \sigma^2_\beta \end{array}\right].$

b) $\underline{\alpha} + \underline{\beta} = 90°$.

Exam Statistical Inference (SS 11)

Problem 2

A given set of points $P_1,\dots,P_4$ is equipped with coordinates (u_i,v_i) and (x_i,y_i) , i=1,2,3,4, which refer to the base vectors of two different coordinate systems (u,v) and (x,y) (see figure).
(u_i,v_i) are called coordinates of the start system while (x_i,y_i) belong to the so-called target system. The start-to-target system transformation (2D similarity transformation) is defined by the equation(s)

$$\begin{bmatrix} \underline{x}_i \\ \underline{y}_i \end{bmatrix} = \lambda \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \quad \forall i = 1,\dots,4,$$

with stochastic quantities underlined. Scale λ, rotation angle α and translational parameters t_x and t_y are unknown. With the intention to circumvent the obvious non-linearity between unknowns λ and α the above problem is rewritten as

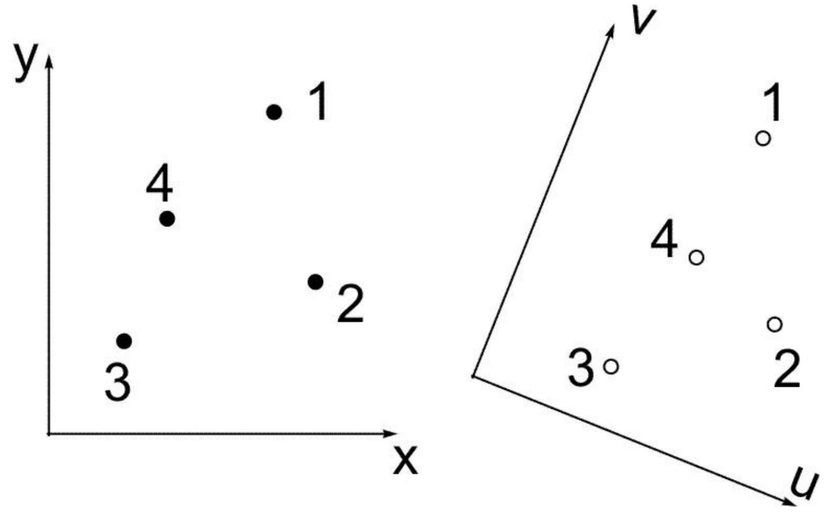
$$\begin{bmatrix} \underline{x}_i \\ \underline{y}_i \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \quad \forall i = 1,\dots,4.$$

a) Specify the number of observations m, the number of unknowns n, the datum defect d and the redundancy r.

b) For the adjustment within the linear model $E\{\underline{\ell}\} = A\xi$, $E\{\underline{\ell}\} \in \mathcal{R}(A)$, $D\{\underline{\ell}\} = Q_\ell$ ("A-model") specify the quantities "(reduced) observation vector $\underline{\ell}$", "vector of unknown quantities ξ" and "design matrix A".

c) How are the estimates $\hat{\underline{\lambda}}$, $\hat{\underline{\alpha}}$, $\hat{\underline{t}}_x$ and $\hat{\underline{t}}_y$ of the original transformation parameters computed from $\hat{\underline{\xi}}$?

d) How is the procedure for the variance-covariance matrix $Q_{[\hat{\lambda},\hat{\alpha}]}$ of $\hat{\underline{\lambda}}$ and $\hat{\underline{\alpha}}$ for given $Q_{\hat{\xi}}$?

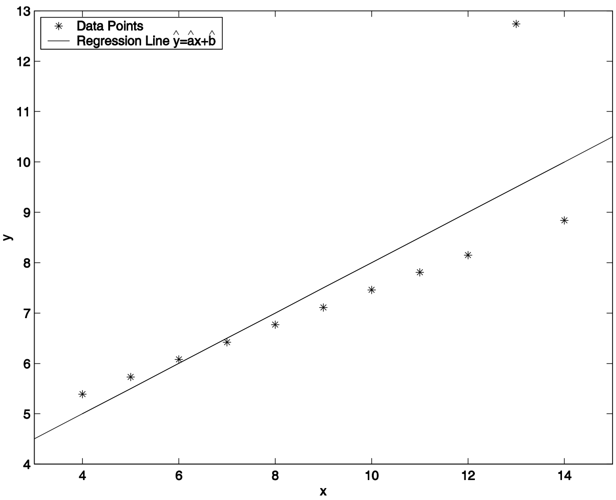


Exam Statistical Inference (SS 11)

Problem 3

The figure below displays a regression line $\underline{\hat{y}} = \underline{\hat{a}}x + \underline{\hat{b}}$ fitted to 11 data points. The variance-covariance matrix of the observations y was assumed as $D(y) = \sigma_0^2\, I_{11\times 11}$ with an a priori variance of the unit weight $\sigma_0^2 = 1.0$. The estimated/adjusted residuals $\underline{\hat{e}} = \underline{y} - \underline{\hat{y}}$ are

$$\underline{\hat{e}} = \begin{bmatrix} 0.389 \\ 0.229 \\ 0.079 \\ -0.081 \\ -0.230 \\ -0.390 \\ -0.540 \\ -0.689 \\ -0.849 \\ 3.241 \\ -1.159 \end{bmatrix}.$$



a) Calculate the a posteriori variance of the unit weight $\underline{\hat{\sigma}}_0^2$.

b) Does $\underline{\hat{\sigma}}_0^2$ significantly deviate from its a priori value σ_0^2 ("variance-ratio-test")? Set the level of significance for the estimation of the critical value to $\alpha = 5\%$.

c) In order to check the data for outliers, perform the global test assuming the type-I-error probability to be $\alpha = 5\%$, $\alpha = 25\%$ respectively. What does the choice of the level of significance mean for outlier detection?

d) Generally, what kind of conclusion is possible performing the global test? What conclusions can be drawn with regard to concrete gross errors in the data?

e) What does data snooping mean?

f) The local redundancy number (redundancy number of one individual observation i) is defined subject to $\underline{r}_i = 1 - \frac{\sigma_{\hat{y}_i}^2}{\sigma_{y_i}^2}$. What information provides the ratio $\frac{\sigma_{\hat{y}_i}^2}{\sigma_{y_i}^2}$ with regard to network control?

Exam Statistical Inference (WS 10/11)

Problem 1

The figure shows the diagram for a system of measured angles $\underline{w}_1,\ldots,\underline{w}_6$ about position S. Angles α, β and γ are unknown.

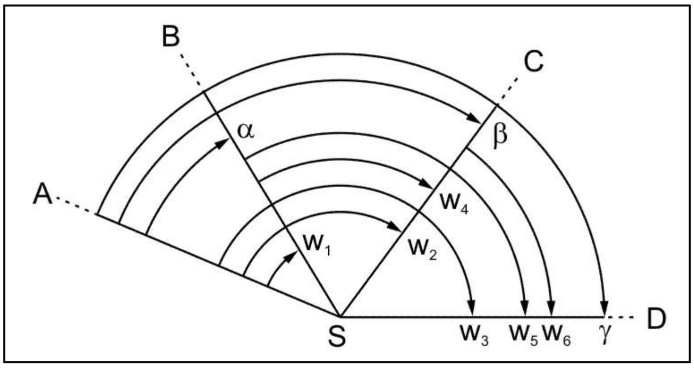
a) For the adjustment within the linear model ("A-model") $E\{\underline{y}\} = Ax$, $E\{\underline{y}\} \in \mathcal{R}(A), D\{\underline{y}\} = I_6$ specify the quantities "(reduced) observation vector $\underline{y}$", "vector of unknown quantities x" and "design matrix A".

b) Determine the adjusted angles $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ from observations $\underline{w}_1,\ldots,\underline{w}_6$.

c) In order to perform a condition adjustment ("B-model"), determine the redundancy of the problem and set up condition matrix B and vector of misclosures k.

d) Which equation must be satisfied in order to switch from the A-model (parameter adjustment) to the B-model (condition adjustment) ? Check it !

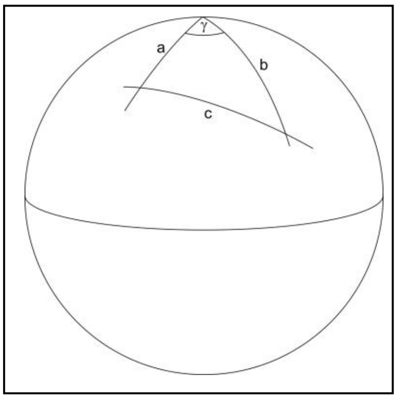
e) For the condition adjustment, formulate the Lagrange-Function $\mathcal{L}$ to be minimized, derive the necessary conditions for its minimum and specify the resulting normal equations.



f) How is the analytic solution for array $\hat{\underline{e}}$ of estimated inconsistencies in terms of B and k ?

Problem 2

If in a spherical triangle the lengths of two sides a, b and the angle γ opposite to c are given, then the length of c can be computed from the spherical law of cosines: $\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$. Assume that the joint 3×3-variance-covariance matrix Q of a, b und γ (all quantities in [rad]) is given. How is the variance of c using the linear error propagation law ?



a) Write down the linear error propagation law and explain the quantities involved.

b) Determine the necessary elements of the law. It is **not required** to give an explicit final equation for σ_c^2.

Exam Statistical Inference (WS 10/11)

Problem 3

Give short and concise answers to the following questions/statements

a) Consider the linear model $E\{\underline{y}\} = Ax$, $E\{\underline{y}\} \in \mathcal{R}(A)$, $D\{\underline{y}\} = \sigma_0^2 Q$ Are the estimated quantities (parameters and functions of the parameters as well as their variance-covariance matrices) dependent or independent on the variance component σ_0^2 ?

b) What is the variance-ratio test and how is the probability distribution of the ratio of two independent variances ?

c) Discuss the relation $\lambda_0 = \lambda(\alpha_L, q = 1, \gamma_0 = 1 - \beta_0) = \lambda(\alpha_G, q = m - n > 1, \gamma_0 = 1 - \beta_0)$ and explain its importance.

d) Comment on the terms "global test/overall model test" and "local test/individual test" and identify their difference(s).

e) Point out the meaning of the variate $\underline{w} = \sqrt{\underline{T}} = \dfrac{\hat{\underline{e}}_i}{\sigma_{\hat{e}_i}}$ and give its probability distribution.

f) Explain the terms "data snooping" and "minimal detectable bias $|\nabla_i|$". Which quantities influence $|\nabla_i|$?

g) Explain the term $r_i =$ "local redundancy" and show from which quantities it is computed. Which values can it take ?

h) Explain the meaning of $EV_i = 100\% \times r_i = 70\%$ in connection with its corresponding observation.

i) Give an example for a geodetic observation with EV_i close to zero.

Exam Statistical Inference (SS 10)

Problem 1

Describe concisely the testing procedure of how to detect blunders (gross errors, outliers). Give special attention to the following quantities/terms:

a) Model of the null hypothesis and corresponding square sum of (estimated) residuals
b) Model of the alternative hypothesis and corresponding square sum of (estimated) residuals
c) Minimal/maximal number of additional parameters in the model of the alternative hypothesis and resulting consequences for the square sum of (estimated) residuals
d) General test statistics, test statistics for the cases of minimal/maximal number of additional parameters, and probability distributions involved
e) DIA principle
f) Problems occurring during the overall model test and the local (individual) test, and reasons for them
g) Baarda's B-method

Problem 2

You are given a non-linear function y=f(x) which relates observations $y_i, i = 1,...,m$ to unknown parameters $x_j, j = 1,...,n$. Due to m>n this is an inconsistent problem. Describe the iterative adjustment algorithm for the A-model under the assumption that (a) the datum problem has been resolved before and (b) approximate values x_j^0 for x_j (j=1,...,n) are available.

Problem 3

During the adjustment of a planar geodetic network distance observations have to be processed. The observation equation between two points P_i and P_k is $d_{ik} = (1+m)\sqrt{(x_i - x_k)^2 + (y_i - y_k)^2} + c$ and contains – apart from the traditional 4 coordinate unknowns – an additional unknown offset c and an unknown scale factor 1+m.

Specify (reduced) observation vector ℓ, design matrix A and vector of unknowns ξ in the linear model ℓ=Aξ+e.

Exam Statistical Inference (SS 10)

Problem 4

The free adjustment of two subnets of a geodetic network has yielded the following results :

subnet	year	$\hat{\underline{\sigma}}$	degrees of freedom
1	2009/2010	±0,248	58
2	2008	±0,181	41

Perform a hypothesis test

$$H_0 : \sigma^2_{\text{Subnet 1}} = \sigma^2_{\text{Subnet 2}} \quad \leftrightarrow \quad H_a : \sigma^2_{\text{Subnet 1}} > \sigma^2_{\text{Subnet 2}}$$

using a level of significance of $\alpha = 5\%$ and come to a proper decision.

Attachments: Statistical tables

Normal Distribution: Computation of one sided level of significance

k	0	1	2	3	4	5	6	7	8	9
0,0	0,5000	0,4960	0,4920	0,4880	0,4840	0,4801	0,4761	0,4721	0,4681	0,4641
0,1	0,4602	0,4562	0,4522	0,4483	0,4443	0,4404	0,4364	0,4325	0,4286	0,4247
0,2	0,4207	0,4168	0,4129	0,4090	0,4052	0,4013	0,3974	0,3936	0,3897	0,3859
0,3	0,3821	0,3783	0,3745	0,3707	0,3669	0,3632	0,3594	0,3557	0,3520	0,3483
0,4	0,3446	0,3409	0,3372	0,3336	0,3300	0,3264	0,3228	0,3192	0,3156	0,3121
0,5	0,3085	0,3050	0,3015	0,2981	0,2946	0,2912	0,2877	0,2843	0,2810	0,2776
0,6	0,2743	0,2709	0,2676	0,2643	0,2611	0,2578	0,2546	0,2514	0,2483	0,2451
0,7	0,2420	0,2389	0,2358	0,2327	0,2296	0,2266	0,2236	0,2206	0,2177	0,2148
0,8	0,2119	0,2090	0,2061	0,2033	0,2005	0,1977	0,1949	0,1922	0,1894	0,1867
0,9	0,1841	0,1814	0,1788	0,1762	0,1736	0,1711	0,1685	0,1660	0,1635	0,1611
1,0	0,1587	0,1562	0,1539	0,1515	0,1492	0,1469	0,1446	0,1423	0,1401	0,1379
1,1	0,1357	0,1335	0,1314	0,1292	0,1271	0,1251	0,1230	0,1210	0,1190	0,1170
1,2	0,1151	0,1131	0,1112	0,1093	0,1075	0,1056	0,1038	0,1020	0,1003	0,0985
1,3	0,0968	0,0951	0,0934	0,0918	0,0901	0,0885	0,0869	0,0853	0,0838	0,0823
1,4	0,0808	0,0793	0,0778	0,0764	0,0749	0,0735	0,0721	0,0708	0,0694	0,0681
1,5	0,0668	0,0655	0,0643	0,0630	0,0618	0,0606	0,0594	0,0582	0,0571	0,0559
1,6	0,0548	0,0537	0,0526	0,0516	0,0505	0,0495	0,0485	0,0475	0,0465	0,0455
1,7	0,0446	0,0436	0,0427	0,0418	0,0409	0,0401	0,0392	0,0384	0,0375	0,0367
1,8	0,0359	0,0351	0,0344	0,0336	0,0329	0,0322	0,0314	0,0307	0,0301	0,0294
1,9	0,0287	0,0281	0,0274	0,0268	0,0262	0,0256	0,0250	0,0244	0,0239	0,0233
2,0	0,0228	0,0222	0,0217	0,0212	0,0207	0,0202	0,0197	0,0192	0,0188	0,0183
2,1	0,0179	0,0174	0,0170	0,0166	0,0162	0,0158	0,0154	0,0150	0,0146	0,0143
2,2	0,0139	0,0136	0,0132	0,0129	0,0125	0,0122	0,0119	0,0116	0,0113	0,0110
2,3	0,0107	0,0104	0,0102	0,0099	0,0096	0,0094	0,0091	0,0089	0,0087	0,0084
2,4	0,0082	0,0080	0,0078	0,0075	0,0073	0,0071	0,0069	0,0068	0,0066	0,0064
2,5	0,0062	0,0060	0,0059	0,0057	0,0055	0,0054	0,0052	0,0051	0,0049	0,0048
2,6	0,0047	0,0045	0,0044	0,0043	0,0041	0,0040	0,0039	0,0038	0,0037	0,0036
2,7	0,0035	0,0034	0,0033	0,0032	0,0031	0,0030	0,0029	0,0028	0,0027	0,0026
2,8	0,0026	0,0025	0,0024	0,0023	0,0023	0,0022	0,0021	0,0021	0,0020	0,0019
2,9	0,0019	0,0018	0,0018	0,0017	0,0016	0,0016	0,0015	0,0015	0,0014	0,0014
3,0	0,0013	0,0013	0,0013	0,0012	0,0012	0,0011	0,0011	0,0011	0,0010	0,0010
3,1	0,0010	0,0009	0,0009	0,0009	0,0008	0,0008	0,0008	0,0008	0,0007	0,0007
3,2	0,0007	0,0007	0,0006	0,0006	0,0006	0,0006	0,0006	0,0005	0,0005	0,0005
3,3	0,0005	0,0005	0,0005	0,0004	0,0004	0,0004	0,0004	0,0004	0,0004	0,0003
3,4	0,0003	0,0003	0,0003	0,0003	0,0003	0,0003	0,0003	0,0003	0,0003	0,0002

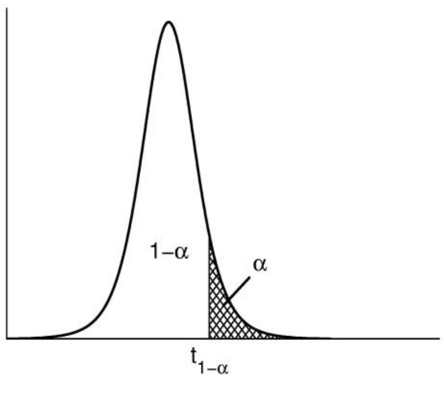
Central χ^2-Distribution: Computation of critical value $k_\alpha=\chi^2_{1-\alpha}(q,\lambda=0)$

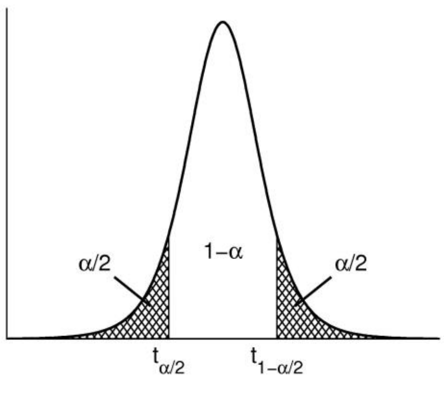
q\α	0.995	0.99	0.975	0.95	0.9	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.001
1	0,000	0,000	0,001	0,004	0,016	0,455	1,323	2,706	3,841	5,024	6,635	7,879	10,828
2	0,010	0,020	0,051	0,103	0,211	1,386	2,773	4,605	5,991	7,378	9,210	10,597	13,816
3	0,072	0,115	0,216	0,352	0,584	2,366	4,108	6,251	7,815	9,348	11,345	12,838	16,266
4	0,207	0,297	0,484	0,711	1,064	3,357	5,385	7,779	9,488	11,143	13,277	14,860	18,467
5	0,412	0,554	0,831	1,145	1,610	4,351	6,626	9,236	11,070	12,833	15,086	16,750	20,515
6	0,676	0,872	1,237	1,635	2,204	5,348	7,841	10,645	12,592	14,449	16,812	18,548	22,458
7	0,989	1,239	1,690	2,167	2,833	6,346	9,037	12,017	14,067	16,013	18,475	20,278	24,322
8	1,344	1,646	2,180	2,733	3,490	7,344	10,219	13,362	15,507	17,535	20,090	21,955	26,124
9	1,735	2,088	2,700	3,325	4,168	8,343	11,389	14,684	16,919	19,023	21,666	23,589	27,877
10	2,156	2,558	3,247	3,940	4,865	9,342	12,549	15,987	18,307	20,483	23,209	25,188	29,588
11	2,603	3,053	3,816	4,575	5,578	10,341	13,701	17,275	19,675	21,920	24,725	26,757	31,264
12	3,074	3,571	4,404	5,226	6,304	11,340	14,845	18,549	21,026	23,337	26,217	28,300	32,909
13	3,565	4,107	5,009	5,892	7,042	12,340	15,984	19,812	22,362	24,736	27,688	29,819	34,528
14	4,075	4,660	5,629	6,571	7,790	13,339	17,117	21,064	23,685	26,119	29,141	31,319	36,123
15	4,601	5,229	6,262	7,261	8,547	14,339	18,245	22,307	24,996	27,488	30,578	32,801	37,697
16	5,142	5,812	6,908	7,962	9,312	15,338	19,369	23,542	26,296	28,845	32,000	34,267	39,252
17	5,697	6,408	7,564	8,672	10,085	16,338	20,489	24,769	27,587	30,191	33,409	35,718	40,790
18	6,265	7,015	8,231	9,390	10,865	17,338	21,605	25,989	28,869	31,526	34,805	37,156	42,312
19	6,844	7,633	8,907	10,117	11,651	18,338	22,718	27,204	30,144	32,852	36,191	38,582	43,820
20	7,434	8,260	9,591	10,851	12,443	19,337	23,828	28,412	31,410	34,170	37,566	39,997	45,315
21	8,034	8,897	10,283	11,591	13,240	20,337	24,935	29,615	32,671	35,479	38,932	41,401	46,797
22	8,643	9,542	10,982	12,338	14,041	21,337	26,039	30,813	33,924	36,781	40,289	42,796	48,268
23	9,260	10,196	11,689	13,091	14,848	22,337	27,141	32,007	35,172	38,076	41,638	44,181	49,728
24	9,886	10,856	12,401	13,848	15,659	23,337	28,241	33,196	36,415	39,364	42,980	45,559	51,179
25	10,520	11,524	13,120	14,611	16,473	24,337	29,339	34,382	37,652	40,646	44,314	46,928	52,620
26	11,160	12,198	13,844	15,379	17,292	25,336	30,435	35,563	38,885	41,923	45,642	48,290	54,052
27	11,808	12,879	14,573	16,151	18,114	26,336	31,528	36,741	40,113	43,195	46,963	49,645	55,476
28	12,461	13,565	15,308	16,928	18,939	27,336	32,620	37,916	41,337	44,461	48,278	50,993	56,892
29	13,121	14,256	16,047	17,708	19,768	28,336	33,711	39,087	42,557	45,722	49,588	52,336	58,301
30	13,787	14,953	16,791	18,493	20,599	29,336	34,800	40,256	43,773	46,979	50,892	53,672	59,703
35	17,192	18,509	20,569	22,465	24,797	34,336	40,223	46,059	49,802	53,203	57,342	60,275	66,619
40	20,707	22,164	24,433	26,509	29,051	39,335	45,616	51,805	55,758	59,342	63,691	66,766	73,402
45	24,311	25,901	28,366	30,612	33,350	44,335	50,985	57,505	61,656	65,410	69,957	73,166	80,077
50	27,991	29,707	32,357	34,764	37,689	49,335	56,334	63,167	67,505	71,420	76,154	79,490	86,661
60	35,534	37,485	40,482	43,188	46,459	59,335	66,981	74,397	79,082	83,298	88,379	91,952	99,607
70	43,275	45,442	48,758	51,739	55,329	69,334	77,577	85,527	90,531	95,023	100,425	104,215	112,317
80	51,172	53,540	57,153	60,391	64,278	79,334	88,130	96,578	101,879	106,629	112,329	116,321	124,839
90	59,196	61,754	65,647	69,126	73,291	89,334	98,650	107,565	113,145	118,136	124,116	128,299	137,208
100	67,328	70,065	74,222	77,929	82,358	99,334	109,141	118,498	124,342	129,561	135,807	140,169	149,449

t-Distribution: Computation of critical value $k_\alpha = t_{1-\alpha}(q)$

$q \backslash \alpha$	0.1	0.05	0.025	0.01	0.005	0.001
1	3,078	6,314	12,706	31,821	63,657	318,309
2	1,886	2,920	4,303	6,965	9,925	22,327
3	1,638	2,353	3,182	4,541	5,841	10,215
4	1,533	2,132	2,776	3,747	4,604	7,173
5	1,476	2,015	2,571	3,365	4,032	5,893
6	1,440	1,943	2,447	3,143	3,707	5,208
7	1,415	1,895	2,365	2,998	3,499	4,785
8	1,397	1,860	2,306	2,896	3,355	4,501
9	1,383	1,833	2,262	2,821	3,250	4,297
10	1,372	1,812	2,228	2,764	3,169	4,144
11	1,363	1,796	2,201	2,718	3,106	4,025
12	1,356	1,782	2,179	2,681	3,055	3,930
13	1,350	1,771	2,160	2,650	3,012	3,852
14	1,345	1,761	2,145	2,624	2,977	3,787
15	1,341	1,753	2,131	2,602	2,947	3,733
16	1,337	1,746	2,120	2,583	2,921	3,686
17	1,333	1,740	2,110	2,567	2,898	3,646
18	1,330	1,734	2,101	2,552	2,878	3,610
19	1,328	1,729	2,093	2,539	2,861	3,579
20	1,325	1,725	2,086	2,528	2,845	3,552
21	1,323	1,721	2,080	2,518	2,831	3,527
22	1,321	1,717	2,074	2,508	2,819	3,505
23	1,319	1,714	2,069	2,500	2,807	3,485
24	1,318	1,711	2,064	2,492	2,797	3,467
25	1,316	1,708	2,060	2,485	2,787	3,450
26	1,315	1,706	2,056	2,479	2,779	3,435
27	1,314	1,703	2,052	2,473	2,771	3,421
28	1,313	1,701	2,048	2,467	2,763	3,408
29	1,311	1,699	2,045	2,462	2,756	3,396
30	1,310	1,697	2,042	2,457	2,750	3,385
35	1,306	1,690	2,030	2,438	2,724	3,340
40	1,303	1,684	2,021	2,423	2,704	3,307
45	1,301	1,679	2,014	2,412	2,690	3,281
50	1,299	1,676	2,009	2,403	2,678	3,261
60	1,296	1,671	2,000	2,390	2,660	3,232
70	1,294	1,667	1,994	2,381	2,648	3,211
80	1,292	1,664	1,990	2,374	2,639	3,195
90	1,291	1,662	1,987	2,368	2,632	3,183
100	1,290	1,660	1,984	2,364	2,626	3,174
200	1,286	1,653	1,972	2,345	2,601	3,131
500	1,283	1,648	1,965	2,334	2,586	3,107

$$t_\alpha(q) = -t_{1-\alpha}(q)$$





Central F-Distribution: Computation of critical value $k_\alpha = F_{1-\alpha}(q_1, q_2, \lambda=0)$

α=0.10

$q_2 \backslash q_1$	1	2	3	4	5	6	8	10	40	50	100
1	39,863	49,500	53,593	55,833	57,240	58,204	59,439	60,195	62,529	62,688	63,007
2	8,526	9,000	9,162	9,243	9,293	9,326	9,367	9,392	9,466	9,471	9,481
3	5,538	5,462	5,391	5,343	5,309	5,285	5,252	5,230	5,160	5,155	5,144
4	4,545	4,325	4,191	4,107	4,051	4,010	3,955	3,920	3,804	3,795	3,778
5	4,060	3,780	3,619	3,520	3,453	3,405	3,339	3,297	3,157	3,147	3,126
6	3,776	3,463	3,289	3,181	3,108	3,055	2,983	2,937	2,781	2,770	2,746
7	3,589	3,257	3,074	2,961	2,883	2,827	2,752	2,703	2,535	2,523	2,497
8	3,458	3,113	2,924	2,806	2,726	2,668	2,589	2,538	2,361	2,348	2,321
9	3,360	3,006	2,813	2,693	2,611	2,551	2,469	2,416	2,232	2,218	2,189
10	3,285	2,924	2,728	2,605	2,522	2,461	2,377	2,323	2,132	2,117	2,087
40	2,835	2,440	2,226	2,091	1,997	1,927	1,829	1,763	1,506	1,483	1,434
50	2,809	2,412	2,197	2,061	1,966	1,895	1,796	1,729	1,465	1,441	1,388
100	2,756	2,356	2,139	2,002	1,906	1,834	1,732	1,663	1,382	1,355	1,293

α=0.05

$q_2 \backslash q_1$	1	2	3	4	5	6	8	10	40	50	100
1	161,448	199,500	215,707	224,583	230,162	233,986	238,883	241,882	251,143	251,774	253,041
2	18,513	19,000	19,164	19,247	19,296	19,330	19,371	19,396	19,471	19,476	19,486
3	10,128	9,552	9,277	9,117	9,013	8,941	8,845	8,786	8,594	8,581	8,554
4	7,709	6,944	6,591	6,388	6,256	6,163	6,041	5,964	5,717	5,699	5,664
5	6,608	5,786	5,409	5,192	5,050	4,950	4,818	4,735	4,464	4,444	4,405
6	5,987	5,143	4,757	4,534	4,387	4,284	4,147	4,060	3,774	3,754	3,712
7	5,591	4,737	4,347	4,120	3,972	3,866	3,726	3,637	3,340	3,319	3,275
8	5,318	4,459	4,066	3,838	3,687	3,581	3,438	3,347	3,043	3,020	2,975
9	5,117	4,256	3,863	3,633	3,482	3,374	3,230	3,137	2,826	2,803	2,756
10	4,965	4,103	3,708	3,478	3,326	3,217	3,072	2,978	2,661	2,637	2,588
40	4,085	3,232	2,839	2,606	2,449	2,336	2,180	2,077	1,693	1,660	1,589
50	4,034	3,183	2,790	2,557	2,400	2,286	2,130	2,026	1,634	1,599	1,525
100	3,936	3,087	2,696	2,463	2,305	2,191	2,032	1,927	1,515	1,477	1,392

α=0.01

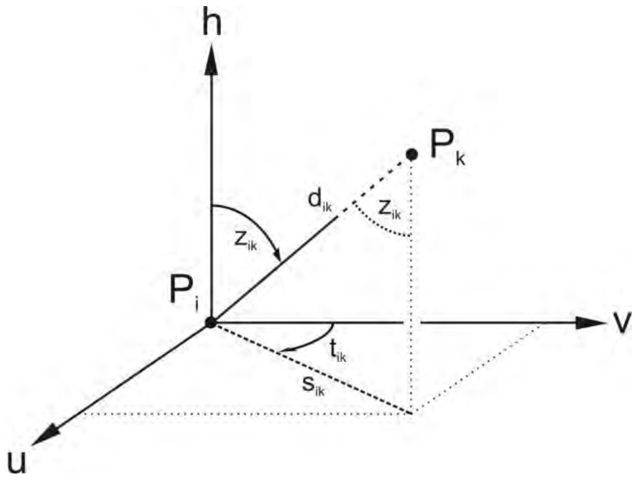
$q_2 \backslash q_1$	1	2	3	4	5	6	8	10	40	50	100
1	4052,181	4999,500	5403,352	5624,583	5763,650	5858,986	5981,070	6055,847	6286,782	6302,517	6334,110
2	98,503	99,000	99,166	99,249	99,299	99,333	99,374	99,399	99,474	99,479	99,489
3	34,116	30,817	29,457	28,710	28,237	27,911	27,489	27,229	26,411	26,354	26,240
4	21,198	18,000	16,694	15,977	15,522	15,207	14,799	14,546	13,745	13,690	13,577
5	16,258	13,274	12,060	11,392	10,967	10,672	10,289	10,051	9,291	9,238	9,130
6	13,745	10,925	9,780	9,148	8,746	8,466	8,102	7,874	7,143	7,091	6,987
7	12,246	9,547	8,451	7,847	7,460	7,191	6,840	6,620	5,908	5,858	5,755
8	11,259	8,649	7,591	7,006	6,632	6,371	6,029	5,814	5,116	5,065	4,963
9	10,561	8,022	6,992	6,422	6,057	5,802	5,467	5,257	4,567	4,517	4,415
10	10,044	7,559	6,552	5,994	5,636	5,386	5,057	4,849	4,165	4,115	4,014
40	7,314	5,179	4,313	3,828	3,514	3,291	2,993	2,801	2,114	2,058	1,938
50	7,171	5,057	4,199	3,720	3,408	3,186	2,890	2,698	2,007	1,949	1,825
100	6,895	4,824	3,984	3,513	3,206	2,988	2,694	2,503	1,797	1,735	1,598

$$F_\alpha(q_1, q_2, 0) = \frac{1}{F_{1-\alpha}(q_2, q_1, 0)}$$

Exam Statistical Inference (WS 09/10)

Problem 1

In the adjustment of three dimensional networks very often zenith angle observations z_{ik} (see figure) have to be processed. Here, it is assumed that P_i is a fixed trigonometric point with given coordinates u_i,v_i,h_i while point P_k shall be computed from the adjustment. Approximate coordinates u_k^0, v_k^0, h_k^0 are provided.



Q1: Set up three different formulations of the observation equation for zenith angle observation z_{ik} between points $P_i(u_i,v_i,h_i)$ and $P_k(u_k,v_k,h_k)$. (The observation equation should be a function of all six coordinates u_i,v_i,h_i and u_k,v_k,h_k)

Q2: Linearize the non-linear observation equation for z_{ik} with respect to the unknowns. **Important hint**: Implicit differentiation might be much simpler and less time consuming than explicit differentiation.

Q3: Specify which parts of the linearized observation equation enter the (reduced) observation vector **y** and which the design matrix **A** of the linear model **y**=**Ax**+**e**.

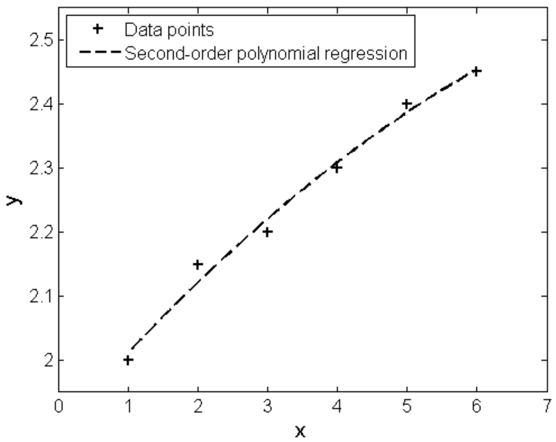
Q4: Comment on the physical units of the quantities entering **y**, **A** and **x**.

Exam Statistical Inference (WS 09/10)

Problem 2

In order to represent six data points by a regression curve, a second-order polynomial $y = ax^2 + bx + c$ has been fitted to the coordinates (see figure). The least-squares parameter estimate of the polynomial coefficients and its variance-covariance matrix are

$$\begin{aligned} \hat{a} &= -0.0054 \\ \hat{b} &= \ \ 0.1261 \\ \hat{c} &= \ \ 1.8900 \end{aligned} \quad , \qquad Q_{\hat{a},\hat{b},\hat{c}} = \begin{bmatrix} 0.0147 & -0.1027 & 0.1369 \\ -0.1027 & 0.7500 & -1.0679 \\ 0.1369 & -1.0679 & 1.7523 \end{bmatrix} \cdot 10^{-3}.$$



Q1: Determine the standard deviations $\sigma_{\hat{a}}, \sigma_{\hat{b}}, \sigma_{\hat{c}}$ of the adjusted polynomial coefficients.

Q2: Does the adjusted polynomial coefficient $\hat{a}$ significantly deviate from zero (two-sided "signal-to-noise ratio test")? Set the level of significance for critical value determination to $\alpha = 5\%$.

Q3: Is the adjusted polynomial coefficient $\hat{b}$ significantly larger than 0.05 (one-sided "signal-to-noise ratio test")? Set the level of significance for critical value determination to $\alpha = 5\%$.

Q4: What conclusion can be drawn from Q2 and Q3 with regard to the polynomial approximation of the data?

Q5: The square sum of adjusted residuals turns out to be $\hat{e}^T\hat{e} = 1.64 \cdot 10^{-3}$. Compute the a posteriori estimate of the unit weight $\hat{\sigma}_0^2$.

Q6: The a priori value of the unit weight was assumed to be $\sigma_0^2 = 1.0$. Do the a priori and a posteriori values significantly deviate from each other ("variance-ratio test" with $\alpha = 5\%$)?

Problem 2 (continued)

Attachment 1 for problem 2

Central F-distribution: Computation of the critical value $k_\alpha = F_{1-\alpha}(q_1, q_2, \lambda = 0)$

α=0.050

q_2/q_1	1	2	3	4	5	6	8	10	20	100	∞
1	161.448	199.500	215.707	224.583	230.162	233.986	238.883	241.882	248.013	253.041	254.302
2	18.513	19.000	19.164	19.247	19.296	19.330	19.371	19.396	19.446	19.486	19.496
3	10.128	9.552	9.277	9.117	9.013	8.941	8.845	8.786	8.660	8.554	8.527
4	7.709	6.944	6.591	6.388	6.256	6.163	6.041	5.964	5.803	5.664	5.628
5	6.608	5.786	5.409	5.192	5.050	4.950	4.818	4.735	4.558	4.405	4.365
6	5.987	5.143	4.757	4.534	4.387	4.284	4.147	4.060	3.874	3.712	3.669
8	5.318	4.459	4.066	3.838	3.687	3.581	3.438	3.347	3.150	2.975	2.928
10	4.965	4.103	3.708	3.478	3.326	3.217	3.072	2.978	2.774	2.588	2.538
20	4.351	3.493	3.098	2.866	2.711	2.599	2.447	2.348	2.124	1.907	1.844
100	3.936	3.087	2.696	2.463	2.305	2.191	2.032	1.927	1.676	1.392	1.284
∞	3.842	2.997	2.606	2.373	2.215	2.099	1.939	1.832	1.572	1.245	1.033

α=0.025

q_2/q_1	1	2	3	4	5	6	8	10	20	100	∞
1	647.789	799.500	864.163	899.583	921.848	937.111	956.656	968.627	993.103	1013.17	1018.20
2	38.506	39.000	39.165	39.248	39.298	39.331	39.373	39.398	39.448	39.488	39.498
3	17.443	16.044	15.439	15.101	14.885	14.735	14.540	14.419	14.167	13.956	13.903
4	12.218	10.649	9.979	9.605	9.364	9.197	8.980	8.844	8.560	8.319	8.258
5	10.007	8.434	7.764	7.388	7.146	6.978	6.757	6.619	6.329	6.080	6.016
6	8.813	7.260	6.599	6.227	5.988	5.820	5.600	5.461	5.168	4.915	4.850
8	7.571	6.059	5.416	5.053	4.817	4.652	4.433	4.295	3.999	3.739	3.671
10	6.937	5.456	4.826	4.468	4.236	4.072	3.855	3.717	3.419	3.152	3.081
20	5.871	4.461	3.859	3.515	3.289	3.128	2.913	2.774	2.464	2.170	2.086
100	5.179	3.828	3.250	2.917	2.696	2.537	2.321	2.179	1.849	1.483	1.349
∞	5.025	3.690	3.117	2.787	2.568	2.409	2.193	2.050	1.710	1.298	1.040

α=0.010

q_2/q_1	1	2	3	4	5	6	8	10	20	100	∞
1	4052.18	4999.50	5403.35	5624.58	5763.65	5858.98	5981.07	6055.84	6208.73	6334.11	6365.54
2	98.503	99.000	99.166	99.249	99.299	99.333	99.374	99.399	99.449	99.489	99.499
3	34.116	30.817	29.457	28.710	28.237	27.911	27.489	27.229	26.690	26.240	26.126
4	21.198	18.000	16.694	15.977	15.522	15.207	14.799	14.546	14.020	13.577	13.464
5	16.258	13.274	12.060	11.392	10.967	10.672	10.289	10.051	9.553	9.130	9.022
6	13.745	10.925	9.780	9.148	8.746	8.466	8.102	7.874	7.396	6.987	6.881
8	11.259	8.649	7.591	7.006	6.632	6.371	6.029	5.814	5.359	4.963	4.860
10	10.044	7.559	6.552	5.994	5.636	5.386	5.057	4.849	4.405	4.014	3.910
20	8.096	5.849	4.938	4.431	4.103	3.871	3.564	3.368	2.938	2.535	2.422
100	6.895	4.824	3.984	3.513	3.206	2.988	2.694	2.503	2.067	1.598	1.429
∞	6.637	4.607	3.784	3.321	3.019	2.804	2.513	2.323	1.880	1.361	1.048

$$F_\alpha(q_1, q_2, 0) = \frac{1}{F_{1-\alpha}(q_2, q_1, 0)}$$

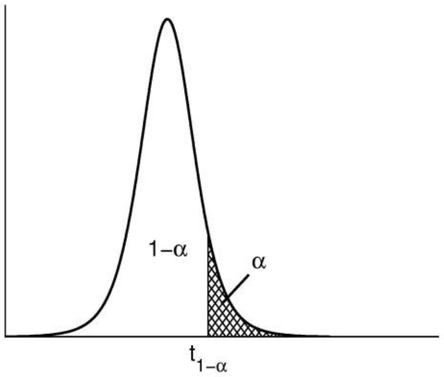
Problem 2 (continued)

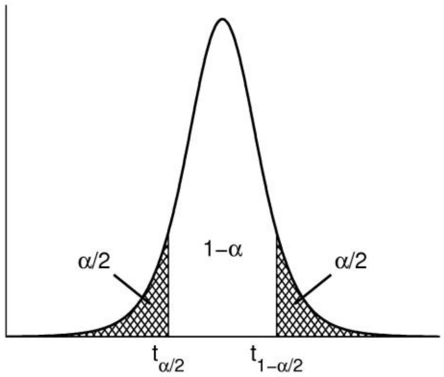
Attachment 2 for problem 2

t-distribution: Computation of the critical value $k_\alpha = t_{1-\alpha}(q)$

q/α	0.100	0.050	0.025	0.010	0.005
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
30	1.310	1.697	2.042	2.457	2.750
40	1.303	1.684	2.021	2.423	2.704
50	1.299	1.676	2.009	2.403	2.678
60	1.296	1.671	2.000	2.390	2.660
70	1.294	1.667	1.994	2.381	2.648
80	1.292	1.664	1.990	2.374	2.639
90	1.291	1.662	1.987	2.368	2.632
100	1.290	1.660	1.984	2.364	2.626
200	1.286	1.653	1.972	2.345	2.601
500	1.283	1.648	1.965	2.334	2.586
∞	1.282	1.645	1.960	2.327	2.576

$$t_\alpha(q) = -t_{1-\alpha}(q)$$





Exam Statistical Inference (WS 08/09)

Problem 1

For a given set of points, coordinates (u,v) have been observed in order to fit an exponential function $v = a\exp(bu)$ to the data. Coordinates u are taken as fixed, i.e. non stochastic variables, coordinates v as stochastic variables, uncorrelated with equal weights. Thus the A-model (parameter adjustment model) is appropriate to compute estimates for the unknown model parameters a and b.

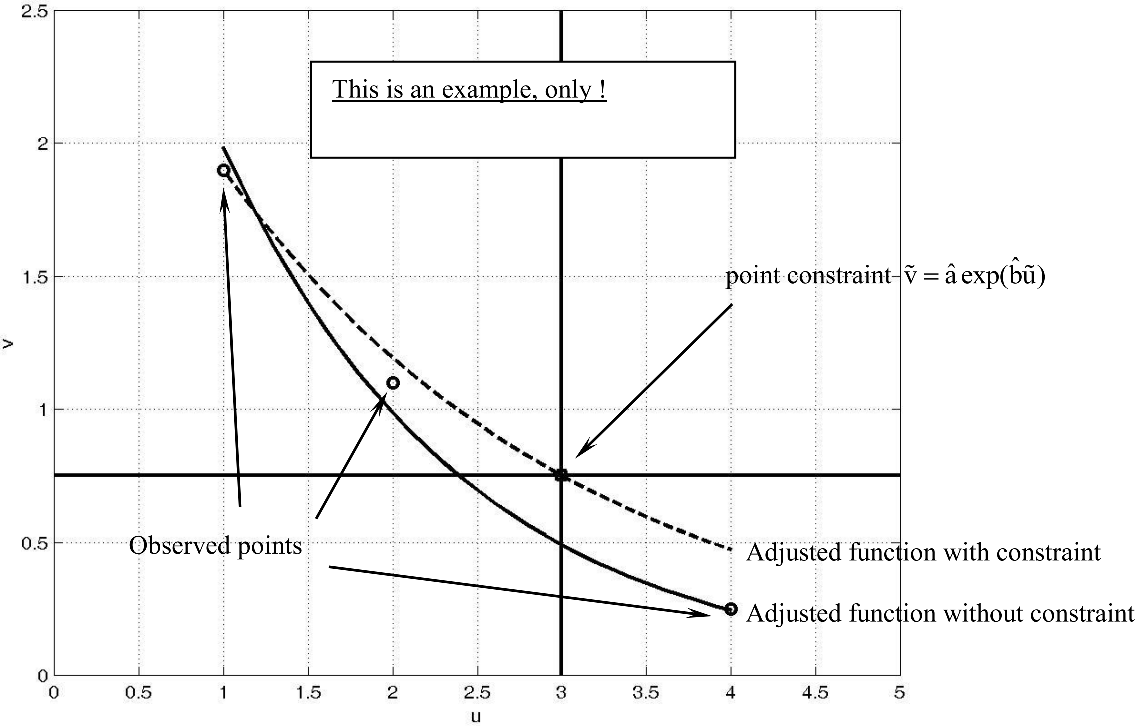
a) For the problem above and the linear model $E\{\underline{y}\} = Ax$, $E\{\underline{y}\} \in \mathcal{R}(A)$, $\mathrm{rk}\,A = 2$, $D\{\underline{y}\} = \sigma^2 I$ set up the *reduced vector of observations*, the *design matrix* and the *vector of unknown parameters*.

b) How would you include the non-linear constraint that the adjusted function should pass through the point $v(u = \tilde{u}) \equiv \tilde{v} = \hat{a}\exp(\hat{b}\tilde{u})$?

c) Set up the constrained Lagrangian function that has to be minimized in order to compute estimates $\hat{a}$ and $\hat{b}$.

d) Which are the necessary conditions for a minimum of the constrained Lagrangian ?

e) Specify the extended normal equation system including the point constraint.



Exam Statistical Inference (WS 08/09)

Problem 2

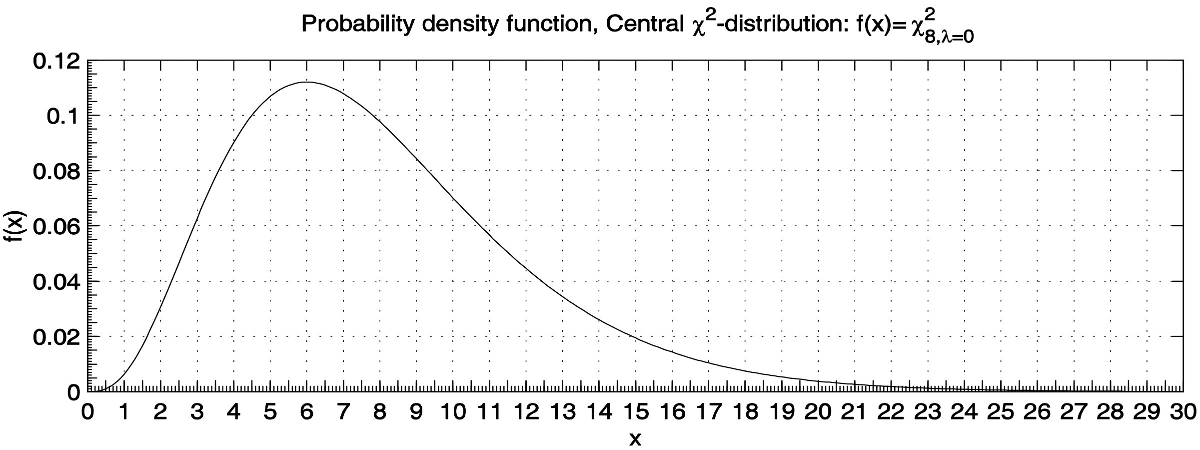
A stochastic variable $\underline{x}$ is assumed to obey a central χ^2-distribution with m=8 degrees of freedom, $\underline{x} \sim \chi^2_{8,\lambda=0}$ (null hypothesis).

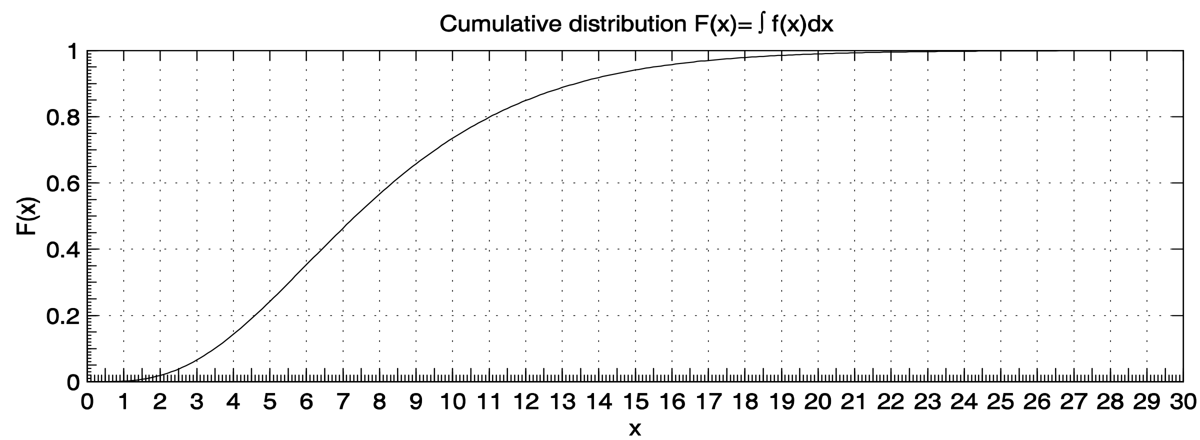
a) Use the graphics below to sketch
   - a1) the probability density function of a stochastic variable $\underline{x}'$ having a non-central χ^2-distribution, $\underline{x}' \sim \chi^2_{8,\lambda\neq 0}$ (alternative hypothesis).
   - a2) the critical value k_α, acceptance and rejection regions with respect to the null hypothesis.
   - a3) the type-I-error probability α and the type-II-error probability β

b) In order to arrive at a conclusion from the hypothesis test, both the critical value k_α and a sample value x are needed. Explain the relation between x and k_α with respect to the statistical inference. What are both used for ?

c) Use the graphics below in order to find the critical value k_α given a significance level (type-I-error probability) $\alpha = 10\%$. Specify k_α up to 1 digit after the decimal point.

d) Use the attached table for the non central χ^2-distribution with non-centrality parameter $\lambda = 15$ in order to calculate the type-II-error probability β (linear interpolation is permitted and recommended). Specify β in [%] with 2 digits after the decimal point.





Problem 2 (continued)

Non central χ^2-distribution with non-centrality parameter $\lambda = 15$
(q: degrees of freedom, α: Type-I error probability in right-hand tail)

q/α	0.900	0.800	0.500	0.250	0.100	0.050	0.025	0.010
1	6.716	9.189	15.000	20.680	26.569	30.447	34.023	38.432
2	7.544	10.085	16.011	21.772	27.728	31.642	35.250	39.693
3	8.375	10.983	17.022	22.862	28.884	32.835	36.474	40.951
4	9.209	11.882	18.031	23.951	30.038	34.026	37.695	42.206
5	10.046	12.782	19.041	25.038	31.190	35.214	38.913	43.459
6	10.885	13.683	20.049	26.124	32.340	36.400	40.129	44.709
7	11.727	14.586	21.057	27.209	33.487	37.584	41.343	45.956
8	12.571	15.490	22.065	28.292	34.633	38.765	42.555	47.201
9	13.418	16.394	23.072	29.374	35.778	39.945	43.764	48.443
10	14.267	17.300	24.079	30.454	36.920	41.123	44.971	49.683
11	15.118	18.207	25.086	31.534	38.061	42.298	46.176	50.921
12	15.971	19.115	26.092	32.612	39.200	43.472	47.378	52.156
13	16.825	20.025	27.098	33.690	40.338	44.644	48.579	53.389
14	17.682	20.935	28.103	34.766	41.474	45.814	49.778	54.621
15	18.541	21.846	29.109	35.841	42.608	46.983	50.975	55.850
16	19.401	22.758	30.114	36.916	43.741	48.149	52.170	57.077
17	20.263	23.671	31.119	37.989	44.873	49.314	53.363	58.301
18	21.126	24.585	32.123	39.062	46.004	50.478	54.555	59.525
19	21.992	25.499	33.128	40.134	47.133	51.640	55.744	60.746
20	22.858	26.415	34.132	41.205	48.261	52.801	56.932	61.965
21	23.726	27.331	35.136	42.275	49.387	53.960	58.119	63.182
22	24.596	28.248	36.140	43.344	50.513	55.117	59.304	64.398
23	25.467	29.166	37.144	44.413	51.637	56.273	60.487	65.612
24	26.339	30.085	38.147	45.481	52.760	57.428	61.669	66.825
25	27.212	31.004	39.151	46.548	53.882	58.582	62.849	68.035
26	28.087	31.924	40.154	47.614	55.003	59.734	64.028	69.244
27	28.963	32.845	41.157	48.680	56.123	60.885	65.206	70.452
28	29.840	33.767	42.160	49.746	57.241	62.035	66.382	71.658
29	30.718	34.689	43.163	50.810	58.359	63.184	67.556	72.862
30	31.598	35.612	44.166	51.874	59.476	64.331	68.730	74.065
40	40.446	44.873	54.191	62.485	70.596	75.747	80.397	86.019
50	49.379	54.185	64.209	73.051	81.639	87.070	91.958	97.852
60	58.380	63.538	74.223	83.580	92.620	98.315	103.430	109.584
70	67.438	72.927	84.234	94.078	103.548	109.496	114.827	121.229
80	76.543	82.345	94.243	104.551	114.431	120.621	126.159	132.800
90	85.689	91.790	104.251	115.002	125.275	131.697	137.435	144.305
100	94.871	101.257	114.257	125.435	136.085	142.730	148.660	155.751

Exam Statistical Inference (WS 07/08)

Problem 1

Many engineering disciplines are concerned with different kind of oscillations such as free oscillations, forced oscillations, damped and undamped oscillations.

Here, we assume that a certain oscillation phenomenon is modelled by the simple harmonic function $Z(t) = a\sin\omega t + b\cos\omega t$ and that signals Z_i have been measured at discrete times t_i (i=1,…,m>3) in order to determine amplitudes a and b as well as frequency ω using a least-squares approach.

Q 1: Determine the number of observations, the number of unknowns and the redundancy.

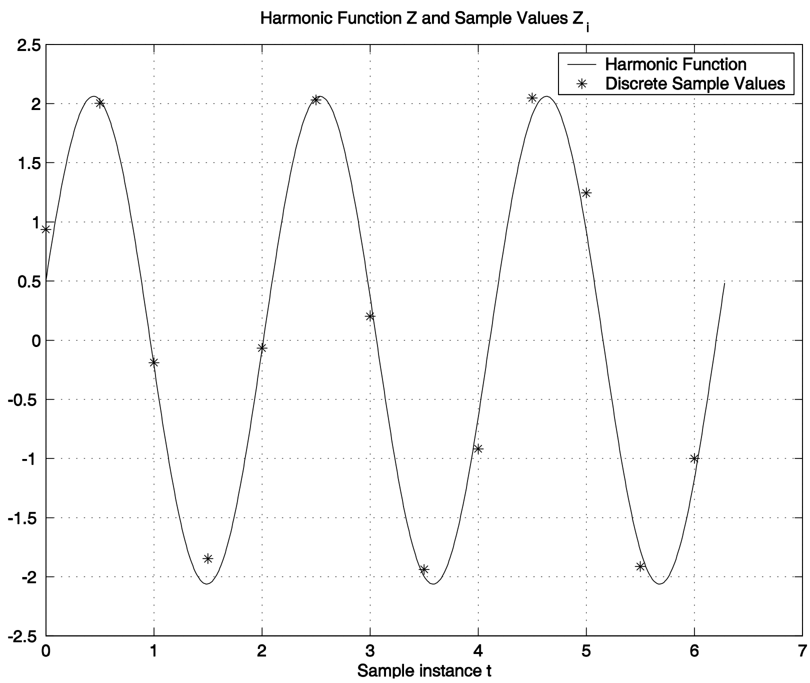
Q 2: For the least-squares approach (A-model), linearize the model function under the assumption of given approximate values a_0, b_0 and ω_0.

Q 3: Set up (i) array of reduced observations **y**, (ii) design matrix **A** and (iii) array of unknowns **x**.

Q 4: Are all unknowns estimable from the observations or is any datum problem (rank deficiency of **A**) encountered ?

Q5: Write down the equation for the estimates of the unknowns and their variance-covariance matrix, given **A** and **y**.

Q6: Derive – according to the linear error propagation law – the variance of the function $f := \sqrt{\hat{a}^2 + \hat{b}^2}$, given the variance-covariance matrix $\mathbf{D}(\hat{a}, \hat{b}) = \begin{bmatrix} \sigma_{\hat{a}}^2 & \sigma_{\hat{a}\hat{b}} = 0 \\ \sigma_{\hat{a}\hat{b}} = 0 & \sigma_{\hat{b}}^2 \end{bmatrix}$.

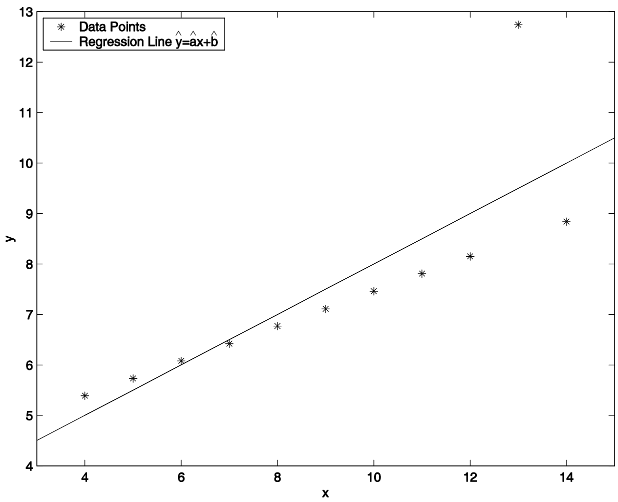


Exam Statistical Inference (WS 07/08)

Problem 2

The figure below displays a regression line $\hat{\mathbf{y}} = \hat{a}\mathbf{x} + \hat{b}$ fitted to 11 data points. The variance-covariance matrix of the observations $\mathbf{y}$ reads $\mathbf{D}(\mathbf{y}) = \sigma_0^2\, \mathbf{I}_{11\times 11}$ with the a priori variance of the unit weight $\sigma_0^2 = 1.0$. The adjusted residuals $\hat{\mathbf{e}} = \mathbf{y} - \hat{\mathbf{y}}$ are

$$\hat{\mathbf{e}} = \begin{bmatrix} 0.389 \\ 0.229 \\ 0.079 \\ -0.081 \\ -0.230 \\ -0.390 \\ -0.540 \\ -0.689 \\ -0.849 \\ 3.241 \\ -1.159 \end{bmatrix}.$$



Q1: Determine the a posteriori variance of the unit weight $\hat{\sigma}_0^2$.

Q2: Does the a posteriori estimate of the unit weight $\hat{\sigma}_0^2$ significantly deviate from the a priori value σ_0^2 ("variance-ratio-test")? Set the level of significance for the estimation of the critical value to $\alpha = 5\%$.

Q3: In order to check the data for outliers, perform the global test assuming the type-I-error probability to be $\alpha = 5\%$, $\alpha = 25\%$ respectively. What does the choice of the level of significance mean for outlier detection?

Q4: Generally, what kind of conclusion is possible performing the global test? What conclusions can be drawn with regard to concrete gross errors in the data?

Q5: What does data snooping mean?

Q6: The local redundancy number (redundancy number of one individual observation i) is defined subject to $r_i = 1 - \frac{\sigma_{\hat{y}_i}^2}{\sigma_{y_i}^2}$. What information provides the ratio $\frac{\sigma_{\hat{y}_i}^2}{\sigma_{y_i}^2}$ with regard to network control?

Problem 2 (continued)

Chi-square distribution: computation of critical value

q\α	0.500	0.250	0.100	0.050	0.025	0.010	0.005	0.001
1	0.455	1.323	2.706	3.841	5.024	6.635	7.879	10.83
2	1.386	2.773	4.605	5.991	7.378	9.210	10.60	13.82
3	2.366	4.108	6.251	7.815	9.348	11.34	12.84	16.27
4	3.357	5.385	7.779	9.488	11.14	13.28	14.86	18.47
5	4.351	6.626	9.236	11.07	12.83	15.09	16.75	20.52
6	5.348	7.841	10.64	12.59	14.45	16.81	18.55	22.46
7	6.346	9.037	12.02	14.07	16.01	18.48	20.28	24.32
8	7.344	10.22	13.36	15.51	17.53	20.09	21.95	26.12
9	8.343	11.39	14.68	16.92	19.02	21.67	23.59	27.88
10	9.342	12.55	15.99	18.31	20.48	23.21	25.19	29.59
11	10.34	13.70	17.28	19.68	21.92	24.72	26.76	31.26
12	11.34	14.85	18.55	21.03	23.34	26.22	28.30	32.91
13	12.34	15.98	19.81	22.36	24.74	27.69	29.82	34.53
14	13.34	17.12	21.06	23.68	26.12	29.14	31.32	36.12
15	14.34	18.25	22.31	25.00	27.49	30.58	32.80	37.70
16	15.34	19.37	23.54	26.30	28.85	32.00	34.27	39.25
17	16.34	20.49	24.77	27.59	30.19	33.41	35.72	40.79
18	17.34	21.60	25.99	28.87	31.53	34.81	37.16	42.31
19	18.34	22.72	27.20	30.14	32.85	36.19	38.58	43.82
20	19.34	23.83	28.41	31.41	34.17	37.57	40.00	45.31
21	20.34	24.93	29.62	32.67	35.48	38.93	41.40	46.80
22	21.34	26.04	30.81	33.92	36.78	40.29	42.80	48.27
23	22.34	27.14	32.01	35.17	38.08	41.64	44.18	49.73
24	23.34	28.24	33.20	36.42	39.36	42.98	45.56	51.18
25	24.34	29.34	34.38	37.65	40.65	44.31	46.93	52.62
26	25.34	30.43	35.56	38.89	41.92	45.64	48.29	54.05
27	26.34	31.53	36.74	40.11	43.19	46.96	49.64	55.48
28	27.34	32.62	37.92	41.34	44.46	48.28	50.99	56.89
29	28.34	33.71	39.09	42.56	45.72	49.59	52.34	58.30
30	29.34	34.80	40.26	43.77	46.98	50.89	53.67	59.70
40	39.34	45.62	51.81	55.76	59.34	63.69	66.77	73.40
50	49.33	56.33	63.17	67.50	71.42	76.15	79.49	86.66
60	59.33	66.98	74.40	79.08	83.30	88.38	91.95	99.61
70	69.33	77.58	85.53	90.53	95.02	100.4	104.2	112.3
80	79.33	88.13	96.58	101.9	106.6	112.3	116.3	124.8
90	89.33	98.65	107.6	113.1	118.1	124.1	128.3	137.2
100	99.33	109.1	118.5	124.3	129.6	135.8	140.2	149.4

Table B.2: Chi-square distribution. χ^2 (q, 0); given is k, critical value, for α, probability in right-hand tail, and q, degrees of freedom, e.g. α=0.010 and q=10 yield k = 23.21; $k = \chi^2_\alpha(q,0)$ for test (102) in Section 4.5.

Central *F*-distribution: computation of critical value

α = 0.10

$q_2 \backslash q_1$	1	2	3	4	5	6	8	10	20
1	39.86	49.50	53.59	55.83	57.24	58.20	59.44	60.19	61.74
2	8.526	9.000	9.162	9.243	9.293	9.326	9.367	9.392	9.441
3	5.538	5.462	5.391	5.343	5.309	5.285	5.252	5.230	5.184
4	4.545	4.325	4.191	4.107	4.051	4.010	3.955	3.920	3.844
5	4.060	3.780	3.619	3.520	3.453	3.405	3.339	3.297	3.207
6	3.776	3.463	3.289	3.181	3.108	3.055	2.983	2.937	2.836
8	3.458	3.113	2.924	2.806	2.726	2.668	2.589	2.538	2.425
10	3.285	2.924	2.728	2.605	2.522	2.461	2.377	2.323	2.201
20	2.975	2.589	2.380	2.249	2.158	2.091	1.999	1.937	1.794
100	2.756	2.356	2.139	2.002	1.906	1.834	1.732	1.663	1.494
∞	2.706	2.303	2.084	1.945	1.847	1.774	1.670	1.599	1.421

α = 0.05

$q_2 \backslash q_1$	1	2	3	4	5	6	8	10	20
1	161.4	199.5	215.7	224.6	230.2	234.0	238.9	241.9	248.0
2	18.51	19.00	19.16	19.25	19.30	19.33	19.37	19.40	19.45
3	10.13	9.552	9.277	9.117	9.013	8.941	8.845	8.786	8.660
4	7.709	6.944	6.591	6.388	6.256	6.163	6.041	5.964	5.803
5	6.608	5.786	5.409	5.192	5.050	4.950	4.818	4.735	4.558
6	5.987	5.143	4.757	4.534	4.387	4.284	4.147	4.060	3.874
8	5.318	4.459	4.066	3.838	3.687	3.581	3.438	3.347	3.150
10	4.965	4.103	3.708	3.478	3.326	3.217	3.072	2.978	2.774
20	4.351	3.493	3.098	2.866	2.711	2.599	2.447	2.348	2.124
100	3.936	3.087	2.696	2.463	2.305	2.191	2.032	1.927	1.676
∞	3.841	2.996	2.605	2.372	2.214	2.099	1.938	1.831	1.571

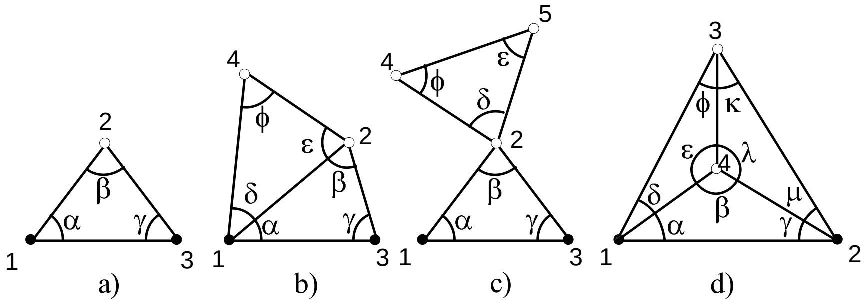
α = 0.01

$q_2 \backslash q_1$	1	2	3	4	5	6	8	10	20
1	4052.	5000.	5403.	5625.	5764.	5859.	5981.	6056.	6209.
2	98.50	99.00	99.17	99.25	99.30	99.33	99.37	99.40	99.45
3	34.12	30.82	29.46	28.71	28.24	27.91	27.49	27.23	26.69
4	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.55	14.02
5	16.26	13.27	12.06	11.39	10.97	10.67	10.29	10.05	9.553
6	13.75	10.92	9.780	9.148	8.746	8.466	8.102	7.874	7.396
8	11.26	8.649	7.591	7.006	6.632	6.371	6.029	5.814	5.359
10	10.04	7.559	6.552	5.994	5.636	5.386	5.057	4.849	4.405
20	8.096	5.849	4.938	4.431	4.103	3.871	3.564	3.368	2.938
100	6.895	4.824	3.984	3.513	3.206	2.988	2.694	2.503	2.067
∞	6.635	4.605	3.782	3.319	3.017	2.802	2.511	2.321	1.878

Table B.3: Central *F*-distribution. *F* (q_1, q_2, 0); given is k, critical value, for degrees of freedom, for some values of α, probability in right-hand tail, e.g. q_1=10, q_2=∞ yield k = 2.321; k=F_α (q, ∞, 0) for test (106) in Section 4.5.

Exam Statistical Inference (WS 06/07)

Consider the following 4 triangulation networks:



Only angles (Greek letters) have been measured in these networks. Black points denote the datum points; the coordinates of the white points are unknown.

i. Using network a), discuss why the coordinates of exactly two points need to be fixed in 2D triangulation networks.

ii. Suppose (just suppose!) you would have to adjust the networks a)–d) according to observation equations (A-model). Discuss for each network the number of observations (*m*), the number of unknowns (*n*), the redundancy (*r*) and whether all unknowns are estimable.

iii. Write down the non-linear observation equations for observation α in network a).

iv. Which steps are required to obtain the design matrix *A* for network a)? *(Remark: don't set up the A matrix. Just describe the required steps.)*

v. Explain two ways to deal mathematically with the datum problem in the A-model.

vi. Write down, for each of the 4 networks, the condition equations in the form $B^T y = b$. Does the number of conditions correspond to the redundancies of question *ii*?

vii. How does the B-model deal with the datum problem?

viii. Derive, only for network b), the normal matrix *N* of the B-model and the projector P_B. Show numerically whether these matrices are idempotent or not.

ix. Demonstrate for network b), using the formulae below, how the condition misclosures *w* are adjusted, i.e. distributed over the observations.

x. Briefly explain the concepts of internal and external reliability.

xi. The minimal detectable bias ∇ is given by:

$$\lambda_0 = c_i^T Q_y^{-1} Q_{\hat{e}} Q_y^{-1} c_i \nabla^2 \quad \Leftrightarrow \quad |\nabla_i| = \sqrt{\frac{\lambda_0}{c_i^T Q_y^{-1} Q_{\hat{e}} Q_y^{-1} c_i}} = \sigma_{y_i} \sqrt{\frac{\lambda_0}{r_i}}$$

Discuss each individual variable that occurs in these equations. Also explain the meaning and role of the local redundancies r_i.

Formulae

$$y = Ax + e$$

$$\hat{x} = (A^T A)^{-1} A^T y$$

$$\hat{y} = A(A^T A)^{-1} A^T y = P_A y$$

$$\hat{e} = y - \hat{y} = \left[I - P_A \right] y = P_A^{\perp} y$$

$$B^T(y-e)=b$$

$$B^T e = B^T y - b = w$$

$$\hat{e} = B\left(B^T B\right)^{-1} w = B\left(B^T B\right)^{-1} B^T y - B\left(B^T B\right)^{-1} b = P_B y - B\left(B^T B\right)^{-1} b$$

$$\hat{y} = y - \hat{e} = \left[I - P_B\right] y + B\left(B^T B\right)^{-1} b = P_B^{\perp} y + B\left(B^T B\right)^{-1} b$$