Geomatics Engineering
International Master Program

Universität
Stuttgart

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Examination Spring 2012
March 9, 2012

Geomatics Methodology

Module 2

Module Section **Signal Processing**

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Student's Surname	Other Names

Date	Student's Signature

Examination result
Grading in percentage

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%

Date	Examiner's Signature

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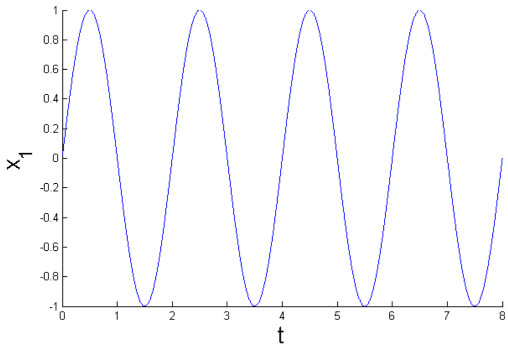
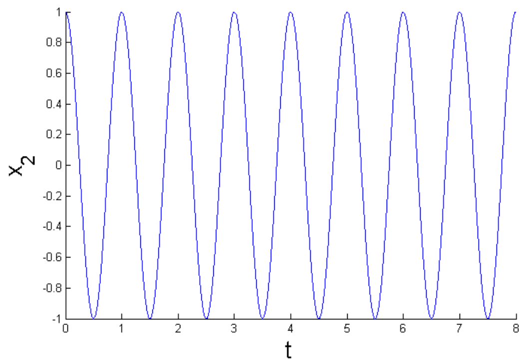
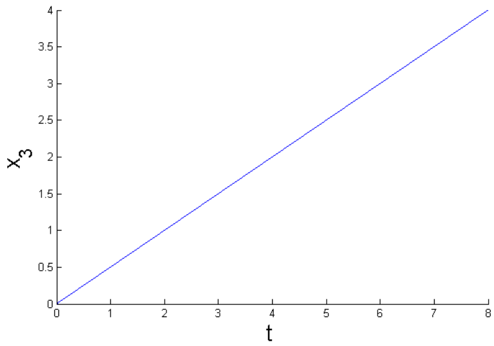
Examination **Signal Processing** - Spring 2012

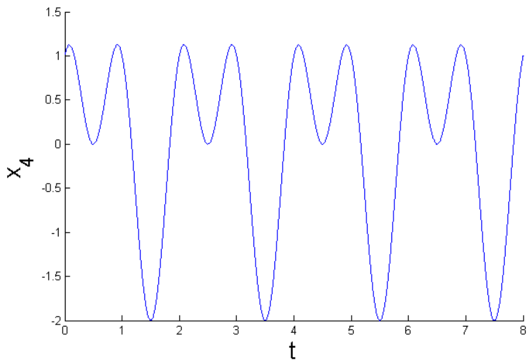
Question 1: (25%)

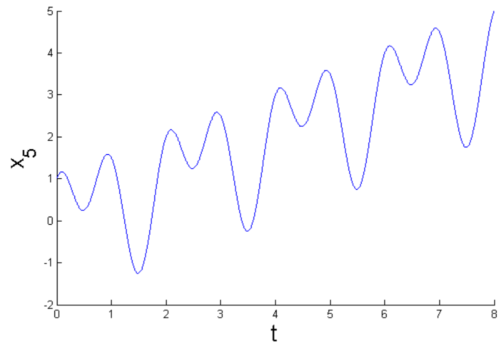
Given are five deterministic signals $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t) = x_1(t) + x_2(t)$ and $x_5(t) = x_1(t) + x_2(t) + x_3(t)$ (figures below).











(a) Determine the periods T_1, T_2, T_3, the frequencies f_1, f_2, f_3, and angular frequencies ω_1, ω_2, ω_3 of the signals x_1, x_2 and x_3.

(b) Write down the general formula for the Fourier Series.

(c) Is it possible to fully express the signal x_5 by the Fourier Series? If yes how many coefficients are required?

(d) Now the Fourier Series of the signal x_4 is developed by parameter estimation using

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the Gauss-Markov model.

a. Write down the general formulation of the Gauss-Markov model, the observation equations and the normal equation system.
b. Write down explicitly the design matrix A for K=2 and $\omega_0=\pi$.
c. How many samples/observations are required to solve the system? Name two criteria for choosing the samples to derive A'A possessing band structure.
d. Write down the coefficients A_0, A_k and B_k for k=1,2.

Question 2: (25%)

Given is the rectangular function

$$x(t) = \begin{cases} 1 & for \quad t \in [-0.5, 0.5] \\ 0 & otherwise \end{cases}.$$

(a) Write down the equations for the continuous Fourier transformation (FT) and the inverse Fourier transformation (IFT).

(b) Calculate the Fourier transformation of x(t). What is the name of this function? Make a sketch of the function x(t).

(c) As learned in the lecture the Fourier transform of the triangular function

$y(t) = 1 - |t|$

is defined by

$$Y(j\omega) = \frac{4}{\omega^2}\sin^2\frac{\omega}{2}$$

Show that the result of the convolution $x(t) * x(t)$ is given by $y(t)$.

Question 3: (15%)

Random signal processing needs methods and algorithms of statistical inference. Let be given a $n \times 1$ random vector $\mathbf{y}$ represented by

$\mathbf{y} = \mathbf{B}\mathbf{x} + \mathbf{s}$

Where $\mathbf{B}$ is a $n \times u$ matrix of fixed coefficients, $\mathbf{x}$ a $u \times 1$ random vector with dispersion matrix $D(x)$ and $\mathbf{s}$ is a $n \times 1$ vector of constants. Derive the dispersion matrix $D(y)$ simply by using E and D operators (E Expectation, D Dispersion).

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Question 4: (35%)

Let be given a discrete random signal

$x(m) = y(m) + r(m) \quad \forall m = 0,1,2$

with $y(m)$ as true/unknown signal and $r(m)$ as observation noise. Apply a Wiener Filter to derive an estimation $\hat{y}(m)$.

(a) Write down the objective function of the Wiener filter.
(b) Derive the filter equation in the time/signal domain.
(c) Derive the filter equation in the frequency/Fourier domain
(d) Sketch a typical frequency response of a Wiener filter. What about an interpretation reflecting pass- and stop bands, if any?

Exercise 1 (7 Pts)

a) $T_1 = 2 \quad f_1 = 1/2 \quad \omega = \pi$

$T_2 = 1 \quad f_1 = 1 \quad \omega = 2\pi$

$T_3 = \infty \quad f_1 = 0 \quad \omega = 0$ (1 Pkt)

b) $x(t) = A_0 + \sum_{k=0}^{\infty} A_k \cos k\omega_0 t + B_k \sin \omega_0 k t$ (1 Pkt)

c) No, not periodic (1 Pkt)

d)

1) $l + e = A \cdot x$ (1 Pkt)

$\hat{x} = (A^T P A)^{-1} A^T P l$

2) $\begin{pmatrix} 0.5 & \cos \pi t_0 & \sin \pi t_0 & \cos 2\pi t_0 & \sin 2\pi t_0 \\ 0.5 & \cos \pi t_1 & \sin \pi t_1 & \cos 2\pi t_1 & \sin 2\pi t_1 \\ 0.5 & \cos \pi t_2 & \sin \pi t_2 & \cos 2\pi t_2 & \sin 2\pi t_2 \\ 0.5 & \cos \pi t_3 & \sin \pi t_3 & \cos 2\pi t_3 & \sin 2\pi t_3 \\ 0.5 & \cos \pi t_4 & \sin \pi t_4 & \cos 2\pi t_4 & \sin 2\pi t_4 \end{pmatrix}$

$N \geq 2K + 1$ (1 Pkt)

3) ~~At~~ last 5 samples (1 Pkt)

$\hookrightarrow$ equidistant sampling

$\hookrightarrow$ neglect sample at $t = 2.7$

4) $B_1 = 1 \quad A_2 = 1$ rest 0 (1 Pkt)

Exercise 2 (7 Pts)

a) $X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} \, d\omega$$ (2 Pkt)

b) $X(j\omega) = \int_{-0.5}^{0.5} e^{-j\omega t} \, dt$

$$= -\frac{1}{j\omega} \left[e^{-j\omega t} \right]_{-0.5}^{0.5}$$

$$= -\frac{1}{j\omega} \left(e^{-j\omega 0.5} - e^{+j\omega 0.5} \right)$$

$$= -\frac{1}{j\omega} \left(\cos 0.5\omega - j \sin 0.5\omega - (\cos 0.5\omega + \sin 0.5\omega) \right)$$

$$= \frac{2}{\omega} \sin \frac{\omega}{2} = \operatorname{sinc} \frac{\omega}{2}$$ (2 Pkt)

name "sinc" + sketch (1 Pkt)

c) $\mathcal{F}\{x(t) * x(t)\} = X(j\omega) \cdot X(j\omega)$

$$= \frac{4}{\omega^2} \cdot \sin^2 \frac{\omega}{2} = Y(j\omega)$$ (1 Pkt)

As stated $\mathcal{F}^{-1}\{Y(j\omega)\} = |1 - t| = y(t)$ $\square$

(1 Pkt)

Exercise 3 4 Pts

using $\hat{x} = E[x]$ $y = Bx + s$

$$
\begin{aligned}
D(y) &= E\{(y - E(y))(y - E(y))^T\} \\
&= E\{(Bx + s - B\hat{x} - s)(Bx + s - B\hat{x} - s)^T\} \\
&= B\,E\,((\hat{x} - x)(\hat{x} - x)^T)\,B^T \\
&= B\,D(x)\,B^T
\end{aligned}
$$

Calculation 3 Pts
Result 1 Pt)

Exercise 4 (9 Pts)

Objective funktion

a) $\varepsilon^2 = E\left[\left(\varepsilon(m) - E[\varepsilon(m)]\right)^2\right] \overset{!}{=} \min$ (1 Pkt)

b) $\varepsilon^2 = E\left[\left(y(m) - \sum_{k=0}^{\infty} h(k) \cdot x(m-k)\right)^2\right] \overset{!}{=} \min$

orthogonality principle (1 Pkt)

$$E\left[\left[y(m) - \sum_{k=0}^{\infty} h(k)\, x(m-k)\right] \cdot x(m-l)\right] = 0$$

$$E\left[y(m)\, x(m-l)\right] = \sum_{k=0}^{\infty} h(k) \cdot E\left[x(m-k)\, x(m-l)\right]$$

$R_{yx} = E\left[y(m)\, x(m-l)\right]$ (1 Pkt)

$R_{xx} = E\left[x(m-k)\, x(m-l)\right]$

$$\Rightarrow R_{yx}(l) = \sum_{k=0}^{\infty} h(k)\, R_{xx}(l-k)$$ (1 Pkt)

3 Pkt

c) using $R_{yx}(l) = R_{yy}(l)$ (1 Pkt)

$R_{xx}(l) = R_{yy}(l) + R_{rr}(l)$

and the Fourier transform:

$\mathcal{F}\{R_{yx}\} = S_{yx}(j\omega)$ (1 Pkt)

$\mathcal{F}\{R_{xx}\} = S_{xx}(j\omega)$

$R_{yx}(l) = h(l) * R_{xx}(l)$

$$H(j\omega) = \frac{S_{yx}(j\omega)}{S_{xx}(j\omega)} = \frac{S_{yy}(j\omega)}{S_{yy}(j\omega) + S_{rr}(j\omega)}$$

(1 Pkt)