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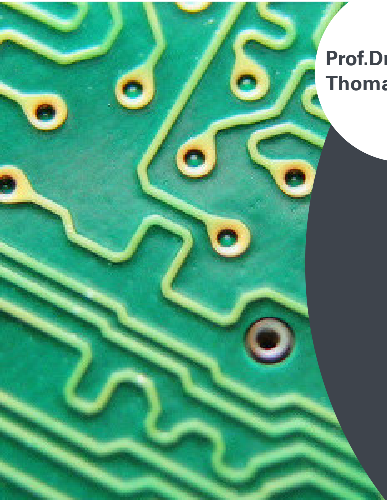

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Integrated Positioning and Navigation

6

Differential Equations for a Strap Down IMU

Differential Equations for a Strap Down IMU

Navigation in 3D – Strapdown concept

Two orthogonal sensor triads are needed

- An accelerometer triad
- A triad of gyroscopes

} **IMU**

Be careful:

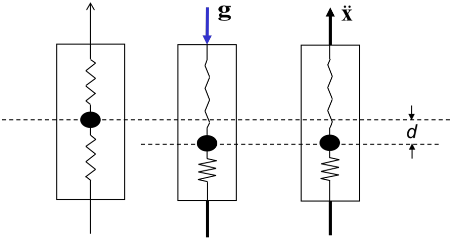
- Accelerometers measure also gravitational acc.

$$\boldsymbol{a} = \ddot{\boldsymbol{x}} - \boldsymbol{g}$$

$\boldsymbol{a}$: specific force (acc. output)

$\ddot{\boldsymbol{x}}$: acceleration with respect to the inertial space (needed)

$\boldsymbol{g}$: gravitational acceleration



Differential Equations for a Strap Down IMU - cont'd

- In a Strap Down Navigator IMU three accelerometers and three gyros are fixed to the platform carrying the IMU (fixed to the p-system)
- The accelerometers are sensitive to the specific force
- Specific force consists of kinematic acceleration of the IMU w.r.t. inertial space (i-system) and gravitational acceleration

specific force
(accelerometer measurement)

$$\boldsymbol{a} = \underbrace{\frac{d^2}{dt^2}\boldsymbol{x}}_{\text{kinem. acc.}} - \boldsymbol{g} \tag{6.1}$$

Differential Equations in the e-system

The coordinates of the specific force vector $\boldsymbol{a}$ are measured in the p-system. They can be directly transformed into the e-system

$$\boldsymbol{a}^e = \boldsymbol{C}_p^e \cdot \boldsymbol{a}^p \tag{6.2}$$

with the composite DCM

$$\boldsymbol{C}_p^e = \boldsymbol{C}_n^e \cdot \boldsymbol{C}_b^n \cdot \boldsymbol{C}_p^b \tag{6.3}$$

Differential Equations for a Strap Down IMU - cont'd

Transformation of the kinematic acceleration

$$\frac{d^2}{dt^2}\boldsymbol{x} = \frac{d^2}{dt^2}\boldsymbol{i} \cdot \boldsymbol{x}^i = \boldsymbol{i} \cdot \frac{d^2}{dt^2}\boldsymbol{x}^i \tag{6.4}$$

Relation between e-system and i-system

$$\boldsymbol{x}^i = \boldsymbol{C}_e^i \cdot \boldsymbol{x}^e \tag{6.5}$$

Therefore:

$$\frac{d^2}{dt^2}\boldsymbol{x}^i = \frac{d^2}{dt^2}\left(\boldsymbol{C}_e^i \cdot \boldsymbol{x}^e\right) \tag{6.6}$$

First derivative (use equ. (3.12)):

$$\frac{d}{dt}\left(\boldsymbol{C}_e^i \cdot \boldsymbol{x}^e\right) = \boldsymbol{C}_e^i \cdot \frac{d}{dt}\boldsymbol{x}^e + \boldsymbol{C}_e^i \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{x}^e \tag{6.7}$$

Differential Equations for a Strap Down IMU - cont'd

Second derivative (use equ. (3.12) again):

$$\frac{d^2}{dt^2}\left(\boldsymbol{C}_e^i \cdot \boldsymbol{x}^e\right) = \boldsymbol{C}_e^i \left(\frac{d^2}{dt^2}\boldsymbol{x}^e + 2\boldsymbol{\Omega}_{ie}^e \cdot \frac{d}{dt}\boldsymbol{x}^e + \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{x}^e\right) \tag{6.8}$$

Insert equ. (6.6) and (6.8) into equ. (6.4):

$$\frac{d^2}{dt^2}\boldsymbol{x} = \boldsymbol{i} \cdot \boldsymbol{C}_e^i \left(\frac{d^2}{dt^2}\boldsymbol{x}^e + 2\boldsymbol{\Omega}_{ie}^e \cdot \frac{d}{dt}\boldsymbol{x}^e + \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{x}^e\right) \tag{6.9}$$

Use equ. (1.8) to transform the base vectors:

$$\frac{d^2}{dt^2}\boldsymbol{x} = \boldsymbol{e} \cdot \left(\frac{d^2}{dt^2}\boldsymbol{x}^e + 2\boldsymbol{\Omega}_{ie}^e \cdot \frac{d}{dt}\boldsymbol{x}^e + \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{x}^e\right) \tag{6.10}$$

This means:
in the parentheses on the r.h.s. are the coordinates (in the e-system) of the kinematic acceleration of the IMU with respect to inertial space.

Differential Equations for a Strap Down IMU - cont'd

Use dots above the symbols to denote time derivatives in the e-system
Combine equ. (6.1), (6.2) and (6.10):

$$\ddot{\boldsymbol{x}}^e = \boldsymbol{C}_p^e \cdot \boldsymbol{a}^p - 2\boldsymbol{\Omega}_{ie}^e \cdot \dot{\boldsymbol{x}}^e - \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{x}^e + \boldsymbol{g}^e \tag{6.11}$$

Coordinates of gravity vector in e-system: $\boldsymbol{g}^e$

This is a second order differential equation for the position (coordinates) of the IMU in the e-system.

Second order DGL can be transformed into first order DGL by substituting the velocity coordinates $\boldsymbol{v}^e$ for the first derivative of the position coordinates.:

$$\begin{aligned} \dot{\boldsymbol{x}}^e &= \boldsymbol{v}^e \\ \dot{\boldsymbol{v}}^e &= \underbrace{\boldsymbol{C}_p^e}_{?} \cdot \boldsymbol{a}^p - 2\boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{v}^e - \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{x}^e + \boldsymbol{g}^e \end{aligned} \tag{6.12}$$

Differential Equations for a Strap Down IMU - cont'd

The gyros in a Strap Down Navigator IMU measure the rotational velocity of the platform system (p-system) with respect to inertial space. These measurements can be used to determine the DCM from the p-system to the e-system.

Decompose the DCM from the p-system to the i-system:

$$\boldsymbol{C}_p^i = \boldsymbol{C}_e^i \cdot \boldsymbol{C}_p^e \tag{6.13}$$

and take the time derivative on both sides

$$\boldsymbol{C}_e^i \cdot \boldsymbol{C}_p^e \cdot \boldsymbol{\Omega}_{ip}^p = \boldsymbol{C}_e^i \cdot \dot{\boldsymbol{C}}_p^e + \boldsymbol{C}_e^i \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{C}_p^e \tag{6.14}$$

and re-order

$$\dot{\boldsymbol{C}}_p^e = \boldsymbol{C}_p^e \cdot \boldsymbol{\Omega}_{ip}^p - \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{C}_p^e \tag{6.15}$$

An alternative formulation can be obtained by taking directly the time derivative of the DCM from the p-system to the e-system:

$$\dot{\boldsymbol{C}}_p^e = \boldsymbol{C}_p^e \boldsymbol{\Omega}_{ep}^p = \boldsymbol{C}_p^e \cdot \left(\boldsymbol{\Omega}_{ip}^p - \boldsymbol{\Omega}_{ie}^p \right) \tag{6.16}$$

Use equ. (3.6) to show equivalence!

Differential Equations for a Strap Down IMU - cont'd

Complete system of DGL:

$$
\begin{aligned}
\dot{\boldsymbol{x}}^e &= \boldsymbol{v}^e \\
\dot{\boldsymbol{v}}^e &= \boldsymbol{C}_p^e \cdot \boldsymbol{a}^p - 2\boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{v}^e - \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{x}^e + \boldsymbol{g}^e \\
\dot{\boldsymbol{C}}_p^e &= \boldsymbol{C}_p^e \cdot \left(\boldsymbol{\Omega}_{ip}^p - \boldsymbol{\Omega}_{ie}^p\right)
\end{aligned}
\tag{6.17}
$$

with $\boldsymbol{C}_p^e = \boldsymbol{C}_n^e \cdot \boldsymbol{C}_b^n \cdot \boldsymbol{C}_p^b$

The third equation is de-coupled from the first two equations! Theoretically it can be integrated separately from the position-velocity equations.

Differential Equations for a Strap Down IMU - cont'd

Why do we prefer modelling body motion in the local level system?

- The attitude angles yaw, pitch and roll can be obtained directly as an output of the mechanization equations because the local level frame is aligned with the north east and down direction
- We can directly obtain geographic coordinate differences $\Delta\phi$, $\Delta\lambda$, Δh (can easily separate between horizontal and vertical components)
- Simple representation of gravity vector $\boldsymbol{g}$
- Due to the so-called Schuler effect the computational errors in the navigation parameters on the horizontal plane are bounded (errors oscillate with Schuler frequency)

Differential equations in the n-system

Position parameterized by longitude λ, latitude ϕ and height h.

Velocity in the n-system:

$$\boldsymbol{v}^n = \boldsymbol{C}_e^n \cdot \dot{\boldsymbol{x}}^e \tag{6.18}$$

Differential Equations for a Strap Down IMU - cont'd

Relation between cartesian coordinates and longitude, latitude and height:

$$\begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \end{bmatrix} = \begin{bmatrix} (N+h)\cos\phi\cos\lambda \\ (N+h)\cos\phi\sin\lambda \\ [N(1-e^2)+h]\sin\phi \end{bmatrix} \tag{6.19}$$

Meaning of N, M, e? Take time derivatives on both sides:

$$\begin{bmatrix} \dot{x}_1^e \\ \dot{x}_2^e \\ \dot{x}_3^e \end{bmatrix} = \begin{bmatrix} -\dot{\phi}(M+h)\sin\phi\cos\lambda - \dot{\lambda}(N+h)\cos\phi\sin\lambda + \dot{h}\cos\phi\cos\lambda \\ -\dot{\phi}(M+h)\sin\phi\sin\lambda + \dot{\lambda}(N+h)\cos\phi\cos\lambda + \dot{h}\cos\phi\sin\lambda \\ \dot{\phi}(M+h)\cos\phi + \dot{h}\sin\phi \end{bmatrix} \tag{6.20}$$

Insert equ. (6.20) in equ. (6.18):

see module 5:
$$\begin{bmatrix} v_N \\ v_E \\ v_D \end{bmatrix} = \begin{bmatrix} \dot{\phi}(M+h) \\ \dot{\lambda}(N+h)\cos\phi \\ -\dot{h} \end{bmatrix} \tag{6.21}$$

Differential Equations for a Strap Down IMU - cont'd

Equ. (6.20) can be directly inverted to yield a ODE for the position:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\lambda} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} \dfrac{v_N}{(M+h)} \\ \dfrac{v_E}{(N+h)\cos\phi} \\ -v_D \end{bmatrix} \tag{6.22}$$

For the ODE for velocity we transform the corresponding equation in the e-system (equ. (6.12). Replace l.h.s. from equ. (6.18):

$$\dot{\boldsymbol{v}}^e = \frac{d}{dt}\dot{\boldsymbol{x}}^e = \frac{d}{dt}\left(\boldsymbol{C}_n^e \cdot \boldsymbol{v}^n\right) = \boldsymbol{C}_n^e \left(\frac{d}{dt}\boldsymbol{v}^n + \boldsymbol{\Omega}_{en}^n \boldsymbol{v}^n\right) \tag{6.23}$$

Substitute (6.18) and equ. (6.19) on the r.h.s.:

$$\frac{d}{dt}\boldsymbol{v}^n + \boldsymbol{\Omega}_{en}^n \boldsymbol{v}^n = \boldsymbol{C}_e^n \cdot \left[\boldsymbol{C}_p^e \cdot \boldsymbol{a}^p - 2\boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{C}_n^e \cdot \boldsymbol{v}^n - \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{x}^e(\phi, \lambda, h) + \boldsymbol{g}^e\right] \tag{6.24}$$

Differential Equations for a Strap Down IMU - cont'd

Apply equ. (3.6)

$$\boldsymbol{C}_e^n \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{C}_n^e = \boldsymbol{\Omega}_{ie}^n \tag{6.25}$$

$$\frac{d}{dt}\boldsymbol{v}^n = \boldsymbol{C}_p^n \cdot \boldsymbol{a}^p - (2\boldsymbol{\Omega}_{ie}^n + \boldsymbol{\Omega}_{en}^n) \cdot \boldsymbol{v}^n - \boldsymbol{C}_e^n \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{x}^e(\phi, \lambda, h) + \boldsymbol{g}^n \tag{6.26}$$

The DGL for the attitude:

$$\dot{\boldsymbol{C}}_p^n = \boldsymbol{C}_p^n \cdot \boldsymbol{\Omega}_{np}^p = \boldsymbol{C}_n^p(\boldsymbol{\Omega}_{ip}^p - \boldsymbol{\Omega}_{in}^p) \tag{6.27}$$

System of ODE

$$\begin{bmatrix} \dot{\phi} \\ \dot{\lambda} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} \dfrac{v_N}{(M+h)} \\ \dfrac{v_E}{(N+h)\cos\phi} \\ -v_D \end{bmatrix} \tag{6.28}$$

$$\frac{d}{dt}\boldsymbol{v}^n = \boldsymbol{C}_p^n \cdot \boldsymbol{a}^p - (2\boldsymbol{\Omega}_{ie}^n + \boldsymbol{\Omega}_{en}^n) \cdot \boldsymbol{v}^n - \boldsymbol{C}_e^n \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{\Omega}_{ie}^e \cdot \boldsymbol{x}^e(\phi, \lambda, h) + \boldsymbol{g}^n$$

$$\dot{\boldsymbol{C}}_p^n = \boldsymbol{C}_p^n \cdot \boldsymbol{\Omega}_{np}^p = \boldsymbol{C}_n^p(\boldsymbol{\Omega}_{ip}^p - \boldsymbol{\Omega}_{in}^p)$$

System of ODE is now coupled! Compare equ. (6.17).

Differential Equations for a Strap Down IMU - cont'd

Replace Diff. Equ. for DCM with Diff. Equ. for Quaternion Elements?

From (2.16) with $\boldsymbol{q} = \boldsymbol{q}_t^s$

$$\begin{aligned} & 4q_0q_1 = \boldsymbol{C}_t^s(2,3) - \boldsymbol{C}_t^s(3,2) \\ & \Rightarrow 4(\dot{q}_0q_1 - q_0\dot{q}_1) = \dot{\boldsymbol{C}}_t^s(2,3) - \dot{\boldsymbol{C}}_t^s(3,2) \end{aligned} \tag{6.29}$$

From (3.12)

$$\begin{gathered} \dot{\boldsymbol{C}}_t^s(2,3) = \boldsymbol{C}_t^s(2,1)\omega_{st2}^t - \boldsymbol{C}_t^s(2,2)\omega_{st1}^t \\ \dot{\boldsymbol{C}}_t^s(3,2) = -\boldsymbol{C}_t^s(3,1)\omega_{st3}^t + \boldsymbol{C}_t^s(3,3)\omega_{st1}^t \\ \Rightarrow 4(\dot{q}_0q_1 - q_0\dot{q}_1) = -\left(\boldsymbol{C}_t^s(2,2) + \boldsymbol{C}_t^s(3,3)\right)\omega_{st1}^t + \boldsymbol{C}_t^s(2,1)\omega_{st2}^t + \boldsymbol{C}_t^s(3,1)\omega_{st3}^t \end{gathered} \tag{6.30}$$

Replace DCM-elements from equ. (2.12):

$$\begin{aligned} 2(\dot{q}_0q_1 - q_0\dot{q}_1) = \quad & -q_0^2\omega_{st1}^t - q_0q_3\omega_{st2}^t + q_0q_2\omega_{st3}^t + \\ & q_1^2\omega_{st1}^t + q_1q_2\omega_{st2}^t + q_1q_3\omega_{st3}^t \end{aligned} \tag{6.31}$$

Differential Equations for a Strap Down IMU - cont'd

Equation (6.31) must be valid for arbitrary q_1, q_0

$$\begin{aligned} \dot{q}_0 &= \frac{1}{2}\left(q_1\omega^t_{st1} + q_2\omega^t_{st2} + q_3\omega^t_{st3}\right) \\ \dot{q}_1 &= \frac{1}{2}\left(-q_0\omega^t_{st1} - q_3\omega^t_{st2} + q_2\omega^t_{st3}\right) \end{aligned} \tag{6.32}$$

Similarly we obtain:

$$\begin{aligned} \dot{q}_2 &= \frac{1}{2}\left(q_3\omega^t_{st1} - q_0\omega^t_{st2} - q_1\omega^t_{st3}\right) \\ \dot{q}_3 &= \frac{1}{2}\left(-q_2\omega^t_{st1} + q_1\omega^t_{st2} - q_0\omega^t_{st3}\right) \end{aligned} \tag{6.33}$$

In compact form:

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & \omega^t_{st1} & \omega^t_{st2} & \omega^t_{st3} \\ -\omega^t_{st1} & 0 & \omega^t_{st3} & -\omega^t_{st2} \\ -\omega^t_{st2} & -\omega^t_{st3} & 0 & \omega^t_{st1} \\ -\omega^t_{st3} & \omega^t_{st2} & -\omega^t_{st1} & 0 \end{bmatrix}\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}, \quad \dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{A}\boldsymbol{q} \tag{6.34}$$

Differential Equations for a Strap Down IMU - cont'd

With equ. (6.34) the Diff. Equ. for the DCM in equ. (6.17) can be replaced

$$\begin{bmatrix} \dot{q}^e_{p0} \\ \dot{q}^e_{p1} \\ \dot{q}^e_{p2} \\ \dot{q}^e_{p3} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \omega^p_{ip1} - \omega^p_{ie1} & \omega^p_{ip2} - \omega^p_{ie2} & \omega^p_{ip3} - \omega^p_{ie3} \\ -\omega^p_{ip1} + \omega^p_{ie1} & 0 & \omega^p_{ip3} - \omega^p_{ie3} & -\omega^p_{ip2} + \omega^p_{ie2} \\ -\omega^p_{ip2} + \omega^p_{ie2} & -\omega^p_{ip3} + \omega^p_{ie3} & 0 & \omega^p_{ip1} - \omega^p_{ie1} \\ -\omega^p_{ip3} + \omega^p_{ie3} & \omega^p_{ip2} - \omega^p_{ie2} & -\omega^p_{ip1} + \omega^p_{ie1} & 0 \end{bmatrix} \begin{bmatrix} q^e_{p0} \\ q^e_{p1} \\ q^e_{p2} \\ q^e_{p3} \end{bmatrix} \tag{6.35}$$

and the third equation in (6.28) transforms to:

$$\begin{bmatrix} \dot{q}^n_{p0} \\ \dot{q}^n_{p1} \\ \dot{q}^n_{p2} \\ \dot{q}^n_{p3} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \omega^p_{ip1} - \omega^p_{in1} & \omega^p_{ip2} - \omega^p_{in2} & \omega^p_{ip3} - \omega^p_{in3} \\ -\omega^p_{ip1} + \omega^p_{in1} & 0 & \omega^p_{ip3} - \omega^p_{in3} & -\omega^p_{ip2} + \omega^p_{in2} \\ -\omega^p_{ip2} + \omega^p_{in2} & -\omega^p_{ip3} + \omega^p_{in3} & 0 & \omega^p_{ip1} - \omega^p_{in1} \\ -\omega^p_{ip3} + \omega^p_{in3} & \omega^p_{ip2} - \omega^p_{in2} & -\omega^p_{ip1} + \omega^p_{in1} & 0 \end{bmatrix} \begin{bmatrix} q^n_{p0} \\ q^n_{p1} \\ q^n_{p2} \\ q^n_{p3} \end{bmatrix} \tag{6.36}$$