Universität Stuttgart

Prof.Dr.
Thomas Hobiger

# Satellite Navigation

# Orbits

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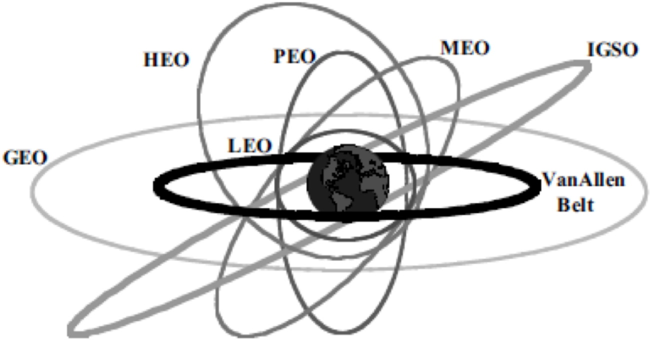
# Orbits

**Satellite Orbits – Types of Earth Orbits**

- LEO - Low Earth Orbit
- MEO - Medium Earth Orbits
- GEO - Geostationary Earth Orbit
- IGSO - Inclined geosynchronous Orbit
- HEO - Highly (inclined) Elliptical Orbit
- PEO - Polar Earth Orbit



# Orbits

**Newtons Equation of Motion (Second Newton Law, 1687)**

> The acceleration of a particle in a force field is proportional to the applied force and inverse proportional to the particle mass:
> $$\ddot{\boldsymbol{x}}(t) = \frac{1}{m(t)} \boldsymbol{F}(t) \tag{3.1}$$

In this simple form, equ. (3.1) is valid in Inertial Space only! If written w.r.t. a rotating coordinate system, the equation must be transformed and it gets more complicated.

If initial values for position and velocity at an epoch $t_0$ are given, and if the force field is known, the Equation of Motion (3.1) can be integrated to yield the position and velocity at arbitrary epochs $t$:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \dot{\boldsymbol{x}}_0(t) + \int_{t_0}^{t} \ddot{\boldsymbol{x}}(\tau) d\tau \,, \\ \boldsymbol{x}(t) &= \boldsymbol{x}_0(t) + \int_{t_0}^{t} \dot{\boldsymbol{x}}(\tau) d\tau \end{aligned} \tag{3.2}$$

# Orbits

**Special case: Equation of motion for a spherically symmetric force field**

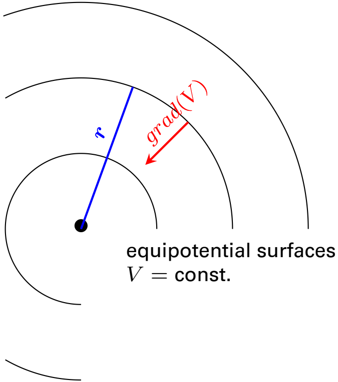
Assumption:

The force field is spherically symmetric and results from a gravitational potential $V$ of a spherically symmetric attracting body (e.g. spherical earth model).

$$V(r) = \frac{GM}{r} \qquad (3.3)$$

$GM$ is the product of the gravitational constant $G$ and the mass of the earth $M_E$

$\rightarrow\ GM = 3.98600415 \times 10^{14}\text{m}^3\text{s}^{-2}.$



# Orbits

**Special case: Equation of motion for a spherically symmetric force field**

The acceleration of a particle in this gravitational force field is equal to the gradient of the gravitational potential:

$$\ddot{\boldsymbol{x}}(t) = gradV(r(t)) = -\frac{GM}{(r(t))^3}\boldsymbol{x}(t) \tag{3.4}$$

$$\text{with } r(t) = |\boldsymbol{x}(t)|$$

The magnitude of the acceleration is inversely proportional to the square of the distance between satellite and attracting body $r(t)$.

The resulting trajectories are plane conical sections.

$\rightarrow$ Satellites that do not leave the gravitational field of the attracting body move in elliptical orbits.

# Orbits

**Special case: Equation of motion for a spherically symmetric force field**

Based on observations of the orbits of the planets of the solar system *Johannes Kepler* derived empirically the three Kepler Laws (1st and 2nd law in 1609, 3rd law in 1619):

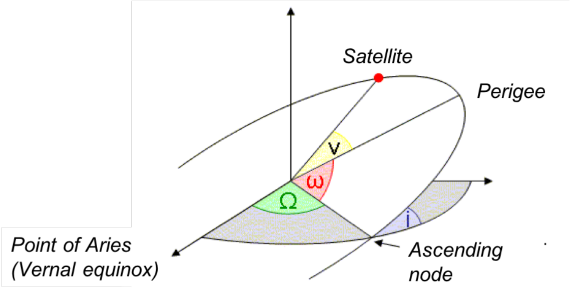
- LAW 1: The orbit of a planet around the Sun is an ellipse with the Sun's center of mass at one focal point
- LAW 2: A line joining a planet and the Sun sweeps out equal areas in equal intervals of time
- LAW 3: The squares of the periods of the planets are proportional to the cubes of their semi-major axes

Kepler derived these laws without knowing anything about Newton's laws.

# Orbits

**Special case: Equation of motion for a spherically symmetric force field**

Such elliptical orbits can be described completely by the six Keplerian Elements:




| | | |
|---|---|---|
| $a$: | semi-major axis | Size and shape |
| $e$: | eccentricity | |
| $i$: | inclination | Orientation of orbital plane in space |
| $\Omega$: | right ascension of the ascending node | |
| $\omega$: | argument of perigee | Orientation of ellipse w.r.t. the nodal line |
| $T_0$: | time of perigee passing | position of the satellite in its orbit |

# Orbits

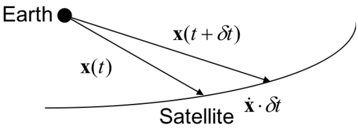
**Special case: Equation of motion for a spherically symmetric force field**

Two body problem (Kepler's problem):
The satellite moves in a plane orbit:

$$\boldsymbol{x}(t) \times \ddot{\boldsymbol{x}}(t) = 0 \;\Rightarrow\; \frac{d}{dt}\left(\boldsymbol{x}(t) \times \dot{\boldsymbol{x}}(t)\right) = 0 \;\Rightarrow\; \boldsymbol{x}(t) \times \dot{\boldsymbol{x}}(t) = \boldsymbol{h} \qquad (3.5)$$

$\boldsymbol{h}$ is the specific angular momentum of the satellite. It is constant in time; its direction is perpendicular to the plane of position and velocity vector $\rightarrow$ this plane is constant $\rightarrow$ the satellite moves in a plane orbit.

Derivation of Kepler's second law




$$\begin{aligned} \delta F &= \frac{1}{2}|\boldsymbol{x} \times \dot{\boldsymbol{x}} \cdot \delta t| \\ &= \frac{1}{2}|\boldsymbol{h}|\delta t \end{aligned} \qquad (3.6)$$

# Orbits

**Special case: Equation of motion for a spherically symmetric force field**

Two body problem (Kepler's problem):

The shape of the orbit: multiply both sides of equ. (3.4) by $\boldsymbol{h}$

$$\boldsymbol{h}(t) \times \ddot{\boldsymbol{x}} = -\frac{GM}{r^3}(\boldsymbol{h} \times \boldsymbol{x}) = -\frac{GM}{r^3}((\boldsymbol{x} \times \dot{\boldsymbol{x}}) \times \boldsymbol{x}) = -\frac{GM}{r^3}(\dot{\boldsymbol{x}}(\boldsymbol{x} \cdot \boldsymbol{x}) - \boldsymbol{x}(\boldsymbol{x} \cdot \dot{\boldsymbol{x}})) \quad (3.7)$$

Using $\frac{d}{dt}\left(\frac{\boldsymbol{x}}{r}\right) = \frac{1}{r}\dot{\boldsymbol{x}} - \frac{\dot{r}}{r^2}\boldsymbol{x} = \frac{1}{r^3}(\dot{\boldsymbol{x}}(\boldsymbol{x} \cdot \boldsymbol{x}) - \boldsymbol{x}(\boldsymbol{x} \cdot \dot{\boldsymbol{x}}))$ we get

$$\boldsymbol{h} \times \ddot{\boldsymbol{x}} = -GM\frac{d}{dt}\left(\frac{\boldsymbol{x}}{r}\right) \quad (3.8)$$

which we can integrate to

$$\boldsymbol{h} \times \dot{\boldsymbol{x}} = -GM\left(\frac{\boldsymbol{x}}{r}\right) - \boldsymbol{A} \quad (3.9)$$

The three vectors of this equation are all in the orbit plane!

# Orbits

**Special case: Equation of motion for a spherically symmetric force field**

Two body problem (Kepler's problem):

Multiply both sides of equ. (3.9) by $\boldsymbol{x}$

$$(\boldsymbol{h} \times \dot{\boldsymbol{x}}) \cdot \boldsymbol{x} = -GMr - \boldsymbol{A} \cdot \boldsymbol{x} \tag{3.10}$$

Using the basic relation $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = -(\boldsymbol{c} \times \boldsymbol{b}) \cdot \boldsymbol{a}$ we can re-arrange the l.h.s to

$$\begin{aligned} &-(\boldsymbol{x} \times \dot{\boldsymbol{x}}) \cdot \boldsymbol{h} = -GMr - \boldsymbol{A} \cdot \boldsymbol{x} \\ \Rightarrow\ &-|\boldsymbol{h}|^2 = -GMr - \boldsymbol{A} \cdot \boldsymbol{x} \\ \Rightarrow\ &h^2 = GMr + Ar \cdot \cos(\boldsymbol{A}, \boldsymbol{x}) \end{aligned} \tag{3.11}$$

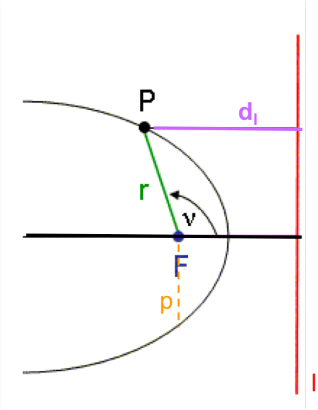
Denoting the angle between $\boldsymbol{A}$ and $\boldsymbol{x}$ by "true anomaly, $\nu$ ", the orbit becomes a conic section:

$$r = \frac{p}{1 + e \cdot \cos\nu} \quad \text{where} \quad p = \frac{h^2}{GM} \ \text{and}\ e = \frac{A}{GM} \tag{3.12}$$

# Orbits

**Special case: Equation of motion for a spherically symmetric force field**

Two body problem (Kepler's problem):




Geometric definition of a conic section

Numeric Eccentricity: $e = r/d_l$
Semi latus rectum: $p = b^2/a$

Elliptical: $0 \leq e < 1$
Parabolic: $e = 1$
Hyperbolic: $e > 1$

*Special case*:
Circle: $e = 0$

**In central force fields satellite orbits are plane conic sections. For satellites that do not leave the gravitational field they are ellipses.**

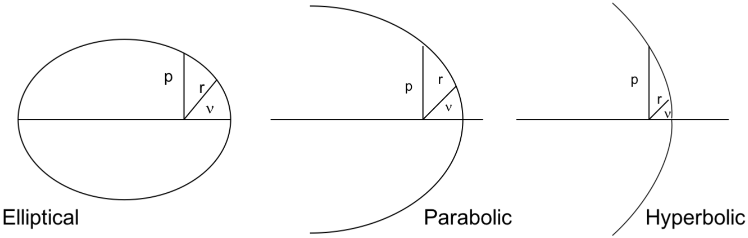
# Orbits

**Special case: Equation of motion for a spherically symmetric force field**

Two body problem (Kepler's problem):

| Type of orbit | eccentricity | $r_{min}$ | $r_{max}$ |
|---|---|---|---|
| Elliptical | $0 \leq e < 1$ | $p/(1+e)$ | $p/(1-e)$ |
| Parabolic | $e = 1$ | $p/2$ | infinite ($\nu = \pi$) |
| Hyperbolic | $e > 1$ | $p/(1+e)$ | infinite ($\cos\nu = -1/e$) |

Special case $e = 0$ is a circular orbit.



# Orbits

**Special case: Equation of motion for a spherically symmetric force field**

Two body problem (Kepler's problem):

The semi-major axis $a$ for the elliptical orbit is the average of the maximum and the minimum distance from the central body:

$$a = \frac{1}{2}(r_{min} + r_{max}) = \frac{1}{2}\left(\frac{p}{1+e} + \frac{p}{1-e}\right) = \frac{p}{1-e^2} = \frac{h^2}{GM(1-e^2)} \qquad (3.13)$$

This definition can be formally also applied to parabolic and hyperbolic orbits. Parabolic: $1/a = 0$; hyperbolic: $a < 0$

The Energy Integral: take the square of equ. (3.9)

$$\begin{aligned}(\boldsymbol{h} \times \dot{\boldsymbol{x}})^2 &= (GM)^2 + 2GM\left(\frac{\boldsymbol{x}\cdot\boldsymbol{A}}{r}\right) + A^2 \\ &= (GM)^2\left(1 + 2\left(\frac{\boldsymbol{x}\cdot\boldsymbol{A}}{GMr}\right) + \frac{A^2}{(GM)^2}\right) \\ &= (GM)^2\left(1 + 2e\cos\nu + e^2\right) \\ &= (GM)^2\left(2(1+e\cos\nu) - (1-e^2)\right)\end{aligned} \qquad (3.14)$$

# Orbits

**Special case: Equation of motion for a spherically symmetric force field**

Two body problem (Kepler's problem):
The l.h.s. of equ. (3.14) is the square of the cross product of two orthogonal vectors. Because of the orthogonality, this is equivalent to the product of the squares of the absolute values of the vectors

$$|\boldsymbol{h}|^2|\dot{\boldsymbol{x}}|^2 = h^2v^2 = (GM)^2\left(2(1+e\cos\nu)-(1-e^2)\right) \tag{3.15}$$

Making use of the equations for the conic section (3.12) and the semi major axis (3.13) we obtain the vis-viva law

$$\boxed{v^2 = GM\left(\frac{2}{r}-\frac{1}{a}\right)} \tag{3.16}$$

This is equivalent to the energy law stating that the sum of kinetic and potential energy remains constant

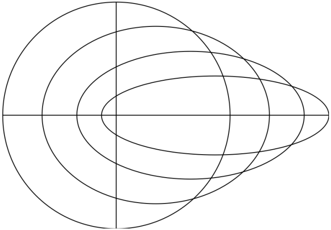
$$E_{kin} = \frac{1}{2}mv^2,\; E_{pot} = -m\frac{GM}{r} \;\Rightarrow E = m\left(\frac{1}{2}v^2-\frac{GM}{r}\right) \;\Rightarrow E = -GM\frac{m}{2a} \tag{3.17}$$

# Orbits

**Special case: Equation of motion for a spherically symmetric force field**

Two body problem (Kepler's problem):

- The energy depends on the semi major axis, not the eccentricity
- The energy of an elliptical orbit is negative ($1/a > 0$)
- The energy of a parabolic orbit is zero ($1/a = 0$)
- The energy of a hyperbolic orbit is positive ($1/a < 0$)

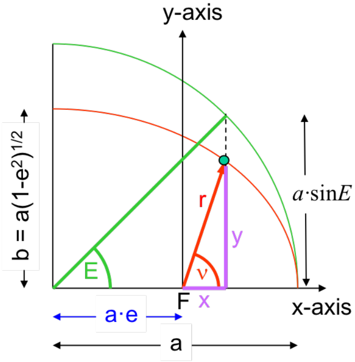


Elliptical orbits of equal energy

- same semi major axis
- different eccentricities

# Orbits

**Keplerian elements (K.E.) to Cartesian coordinates and velocities**




Two step procedure:

1. From K.E. to x, y coordinates in orbital plane; z coordinate is zero
2. From orbital plane coordinates to geocentric X,Y, Z coordinates

$F$: focal point (centre of earth)
$E$: eccentric anomaly
$\nu$: true anomaly
$M$: mean anomaly (not a geom. quantity)
$n$: mean motion
$T$: orbital period of the satellite

The 3rd Kepler law:

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^3}{GM} \Rightarrow n = \frac{2\pi}{T} \Rightarrow n = \sqrt{\frac{GM}{a^3}} \tag{3.18}$$

# Orbits

**Keplerian elements (K.E.) to Cartesian coordinates and velocities**

The mean anomaly $M(t)$ is the integral of the mean motion $n$ since the time of perigee passing $T_0$

$$M(t) = \int_{T_0}^{t} n\,dt = n \int_{T_0}^{t} dt = n(t - T_0) \tag{3.19}$$

$M(t)$ cannot be shown graphically; it does not have a geometric meaning. During one orbital period $M(t)$ increases uniformly from $0$ to $2\pi$. $M(t)$ is computed from the Kepler elements $a$ and $T_0$ !

The mean anomaly is related to the eccentric anomaly $E(t)$ through Kepler's Equation (this now includes the K.E. $e$)

$$\boxed{M(t) = E(t) - e \sin E(t)} \tag{3.20}$$

For a given $E(t)$ equ. (3.20) enables the computation of $M(t)$. For the inverse computation, equ. (3.20) must be solved iteratively. Example for iteration:

$$E_{i+1}(t) = M(t) + e \sin E_i(t); \;\; E_0(t) = M(t) \tag{3.21}$$

# Orbits

**Keplerian elements (K.E.) to Cartesian coordinates and velocities**

The relation between eccentric anomaly $E(t)$ and the true anomaly $n(t)$:

$$\tan\frac{\nu(t)}{2} = \sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{E(t)}{2} \tag{3.22}$$

With $E(t)$ computed from equ. (3.21) we obtain the position coordinates in the orbital plane coordinate system.

$$\boxed{\begin{aligned} x(t) &= a(\cos E(t) - e) \\ y(t) &= a\sqrt{1-e^2} \cdot \sin E(t) \\ z(t) &= 0 \end{aligned}} \tag{3.23}$$

The computation of $x(t)$, $y(t)$, $z(t)$ involves the K.E. $a$, $e$, $T_0$ and the parameter $t$.

# Orbits

**Keplerian elements (K.E.) to Cartesian coordinates and velocities**

The distance from the focal point of the ellipse:

$$r(t) = \sqrt{x(t)^2 + y(t)^2 + z(t)^2} = a(1 - e\cos E(t)) \tag{3.24}$$

The velocity in the orbital plane coordinate system is obtained by taking the time derivative of equ. (3.23)

$$\begin{aligned} \dot{x}(t) &= a\sin E(t) \cdot \dot{E} \\ \dot{y}(t) &= a\sqrt{1-e^2} \cdot \cos E(t) \cdot \dot{E} \\ \dot{z}(t) &= 0 \end{aligned} \tag{3.25}$$

This can be further simplified with the time derivative of Kepler's equation (3.20).

$$n = \dot{E} \cdot (1 - e\cos E) \ \Rightarrow \dot{E} = n\frac{a}{r} \tag{3.26}$$

# Orbits

**Keplerian elements (K.E.) to Cartesian coordinates and velocities**

With (3.26) in (3.25) we finally get:

$$\boxed{\begin{aligned} \dot{x}(t) &= -n\frac{a^2}{r(t)}\sin E(t) \\ \dot{y}(t) &= n\frac{a^2}{r(t)}\sqrt{1-e^2}\cdot\cos E(t) \\ \dot{z}(t) &= 0 \end{aligned}} \tag{3.27}$$

Taking the second derivative

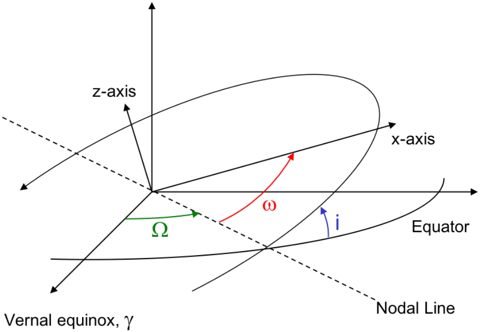
$$\begin{aligned} \ddot{x}(t) &= -\frac{n^2a^3}{r^3}a(\cos E - e) \\ \ddot{y}(t) &= -\frac{n^2a^3}{r^3}a\sqrt{1-e^2}\sin E \\ \ddot{z}(t) &= 0 \end{aligned} \tag{3.28}$$

It can be shown using (3.18) and (3.23), that (3.28) is equivalent to (3.4); i.e. (3.23) is indeed a solution of the equation of motion (3.4).

**Keplerian elements (K.E.) to Cartesian coordinates and velocities**
The transformation from the orbital plane coordinate system to the geocentric coordinate system:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \boldsymbol{R}_3(-\Omega) \cdot \boldsymbol{R}_1(-i) \cdot \boldsymbol{R}_3(-\omega) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{3.29}$$

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \boldsymbol{R}_3(-\Omega) \cdot \boldsymbol{R}_1(-i) \cdot \boldsymbol{R}_3(-\omega) \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \tag{3.30}$$



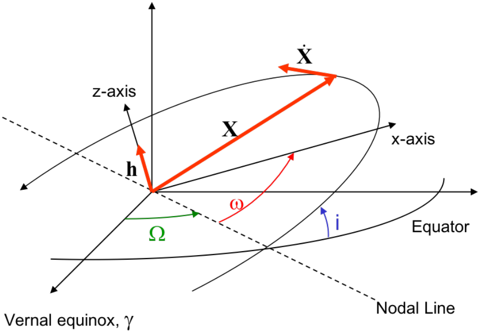
# Orbits

**Cartesian coordinates and velocities to Keplerian elements (K.E.)**
Given are the Cartesian coordinates and velocities for a specified epoch; the transformation to Keplerian elements starts from the vector product of position and velocity:

$$\boldsymbol{h} = \boldsymbol{X} \times \dot{\boldsymbol{X}} \tag{3.31}$$

According to equ. (3.5)), $\boldsymbol{h}$ is the specific angular momentum of the satellite, perpendicular to its orbital plane.

$$\tan\Omega = h_x / - h_y \;\Rightarrow \Omega; \qquad \tan i = \sqrt{h_x^2 + h_y^2}/h_z \;\Rightarrow i \tag{3.32}$$



# Orbits

**Cartesian coordinates and velocities to Keplerian elements (K.E.)**
From the coordinates and velocities we also can compute the geocentric distance $r$ of the satellite and its speed $v$ (the length of the velocity vector):

$$\begin{aligned} r &= \sqrt{X^2 + Y^2 + Z^2} \\ v &= \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2} \end{aligned} \tag{3.33}$$

Insert equ. (3.33) into equ. (3.16), the vis-viva law, to compute the semi major axis $a$ of the satellite orbit:

$$\boxed{a = \left(\frac{2}{r} - \frac{v^2}{GM}\right)^{-1}} \tag{3.34}$$

Compute the length of $\boldsymbol{h}$ from equ. (3.31))

$$h = \sqrt{h_x^2 + h_y^2 + h_z^2} \tag{3.35}$$

and obtain the eccentricity $e$ of the orbit from equ. (3.13)

$$e = \sqrt{1 - \frac{h^2}{a \cdot GM}} \tag{3.36}$$

# Orbits

**Cartesian coordinates and velocities to Keplerian elements (K.E.)**
From equ. (3.23) and (3.27) we obtain for the scalar product of position and velocity vector

$$\boldsymbol{x} \cdot \dot{\boldsymbol{x}} = \boldsymbol{X} \cdot \dot{\boldsymbol{X}} = -a(\cos E - e)n\frac{a^2}{r}\sin E + a(1-e^2)\sin E \cdot n\frac{a^2}{r}\cos E \tag{3.37}$$

With the help of equ. (3.24) this can be reduced to read:

$$e \sin E = \boldsymbol{X} \cdot \dot{\boldsymbol{X}}/(a^2 n) \tag{3.38}$$

From (3.24) we have

$$e \cos E = 1 - r/a \tag{3.39}$$

which then gives a unique solution for the eccentric anomaly $E$

$$\tan E = \frac{\boldsymbol{X} \cdot \dot{\boldsymbol{X}}/(a^2 n)}{1 - r/a} \Rightarrow E \tag{3.40}$$

The eccentric anomaly can be converted to the mean anomaly using Kepler's Equation (3.20), and the time of perigee passing $T_0$ is then obtained from equ. (3.19)

$$T_0 = t - M/n \tag{3.41}$$

# Orbits

**Cartesian coordinates and velocities to Keplerian elements (K.E.)**
The last Keplerian element to be determined is the argument of perigee. To determine $\omega$, we first compute the argument of latitude $u$, which is the sum of the argument of perigee and the true anomaly.

The argument of latitude $u$ can be computed from the scalar product of a unit vector pointing towards the ascending node, and the unit vector pointing to the satellite:

$$\cos u = \frac{1}{r}\left(X\cos\Omega + Y\sin\Omega\right) \tag{3.42}$$

If the Z-coordinate is negative, the computed $u$ must be corrected:

$$\text{if } Z < 0 \text{ then } u = 2\pi - u \tag{3.43}$$

The eccentric anomaly is converted to the true anomaly with equation (3.22), and the argument of perigee $\omega$ is obtained from

$$\omega = u - \nu \tag{3.44}$$

$\Rightarrow$ This completes the computation of K. E. from position and velocity.

# Orbits

**Relation between Cartesian coordinates and velocities and K.E.**

Spherically symmetric force field
satellite moves in an elliptical orbit

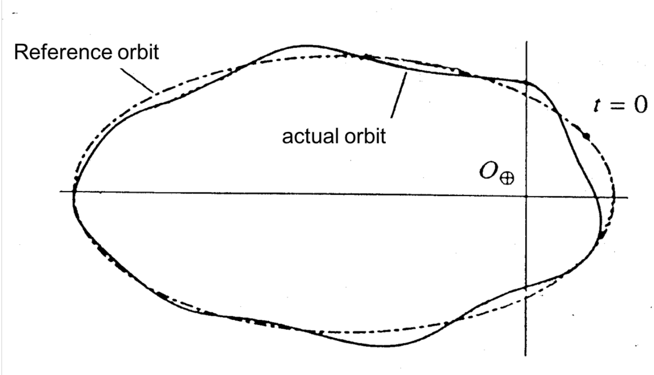
- The elliptical orbit can be described by 6 Keplerian Elements (K.E.).
- The K.E. in this case are constant in time.
- For any particular epoch t, Cartesian coordinates and velocities can be computed from the K.E.; the coordinates and velocities are functions of time.
- The Cartesian coordinates and velocities can be re-converted to K.E.; the K.E. remain constant in time.

Non-spherically symmetric force field
satellite moves in a non-elliptical orbit

- The satellite's orbit can be described by Cartesian coordinates and velocities, both being functions of time.
- At any particular epoch t, the Cartesian coordinates and velocities can be converted to K.E.; but since the satellite does not move along an ellipse, the K.E. in this case are functions of time.
- The time-varying K.E. are called Osculating K.E.
- The amount of variation of the Osculating K.E. depends on the deviation of the force field from spherical symmetry.

# Orbits

## The disturbed Keplerian motion



# Orbits

### The disturbed Keplerian motion

In general, the force field acting on a satellite consists of the spherically symmetric force field according to equ. (3.4) plus a number of additional (smaller) accelerations.

$$\ddot{\boldsymbol{X}}(t) = -\frac{GM}{(r(t))^3}\boldsymbol{X}(t) + \boldsymbol{F}_{as}(t) + \boldsymbol{F}_S(t) + \boldsymbol{F}_M(t) + \boldsymbol{F}_R(t) + \boldsymbol{F}_A(t) \qquad (3.45)$$

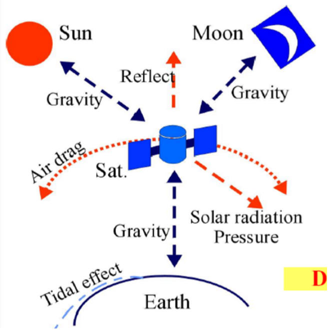
$\boldsymbol{F}_{as}(t)$: gradient of the non-symmetrical part of the gravitational potential

$\boldsymbol{F}_S(t)$: gradient of the gravitational potential of the sun

$\boldsymbol{F}_M(t)$: gradient of the gravitational potential of the moon

$\boldsymbol{F}_R(t)$: acceleration due to the pressure from solar radiation

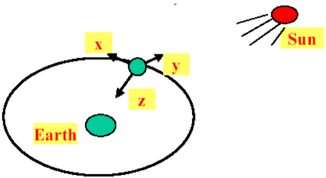
$\boldsymbol{F}_A(t)$: acceleration due to the resistance of the atmosphere



# Orbits

**The disturbed Keplerian motion**

- Pressure from solar radiation (points away from the sun)
- Resistance of atmosphere (points in negative velocity direction)




The largest of these additional forces is the effect of the equatorial bulge of the earth in $\boldsymbol{F}_{as}(t)$.

If

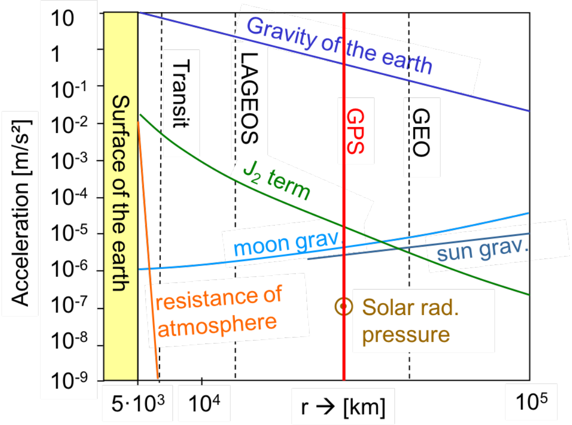
- appropriate models of the different terms of the force field are known
- initial conditions for position and velocity at time $t_0$ are known,

the equations (3.45) can be used to (numerically) compute position and velocity for any time $t > t_0$.

# Orbits

### The disturbed Keplerian motion

Size of the disturbance terms



# Orbits

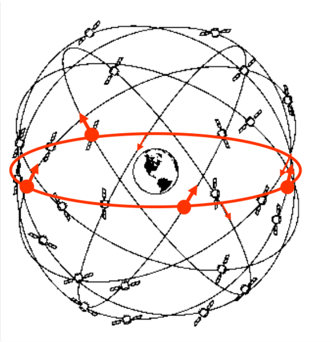
**The disturbed Keplerian motion**

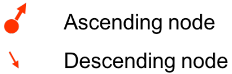
Size of the disturbance terms on a GPS satellite

| Disturbance term | Acceleration [m/s$^2$] | Effect on orbit 3-hour-orbit | Effect on orbit 3-day-orbit |
|---|---|---|---|
| Central field (comparison) | $0.56$ | | |
| Flattening of earth | $5 \cdot 10^{-5}$ | 2000 m | 14,000 m |
| Attraction of moon and sun | $5 \cdot 10^{-6}$ | 5-150 m | 1,000-3,000 m |
| Other terms of gravitation potential | $3 \cdot 10^{-7}$ | 50-80 m | 100-1,500 m |
| Solar rad. pressure | $1 \cdot 10^{-7}$ | 5-10 m | 100-800 m |
| Albedo | $1 \cdot 10^{-9}$ | — | 1.0-1.5 m |
| Solid earth tides | $1 \cdot 10^{-9}$ | — | 0.5-1.0 m |
| Ocean loading effects | $1 \cdot 10^{-9}$ | — | 0.0-2.0 m |
| Atmospheric Drag | 0 | negligible | negligible |

# Orbits

**The (nominal) GPS satellite constellation**

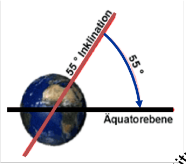


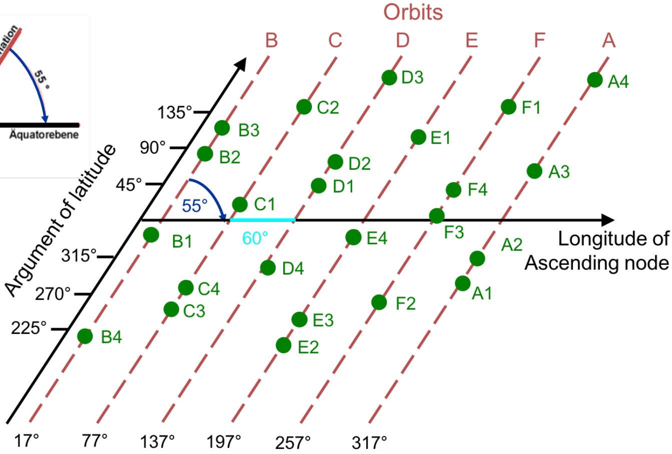



- 24 satellites in 6 orbital planes
  (since June 2011: "Expandable 24" =24+6)
- Orbital planes are numbered A, B, . . . , F
- Orbital planes have a difference in $\Omega$ of 60°
- Unequal distribution of 4 satellites in each orbital plane
- Inclination of orbital planes $i$: 55°
- Semi major axis $a$: 26500 km
- Eccentricity $e$: <0.02 (circular orbits)
- Orbital period: 11h 58m (1/2 sidereal day)

# Orbits

### The (nominal) GPS satellite constellation



# Orbits

**The (current) GNSS satellite constellation**

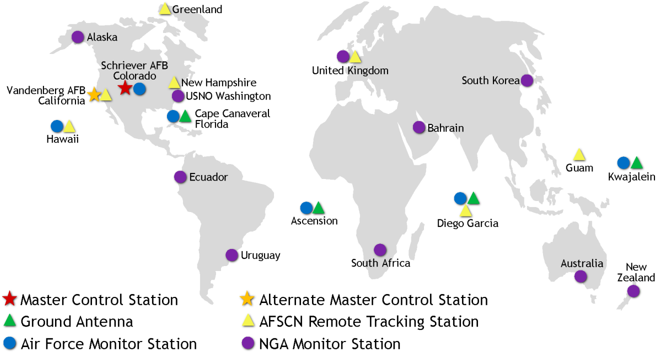
- Latest GPS Constellation Status
  `http://www2.unb.ca/gge/Resources/GPSConstellationStatus.txt`
- Latest GLONASS Constellation Status
  `http://www2.unb.ca/gge/Resources/GLONASSConstellationStatus.txt`
- GPS Constellation Plot
  `http://www2.unb.ca/gge/Resources/GPSConstellationPlot.pdf`
- GLONASS Constellation Plot
  `http://www2.unb.ca/gge/Resources/GLONASSConstellationPlot.pdf`
- Live sky plot, for example `https://in-the-sky.org/satmap_worldmap.php`

# Orbits

## The description of the GPS satellite orbits

### Broadcast Ephemeris

- Transmitted by the satellite as part of the broadcast message
- Based on the observations of monitor stations
- Master Control Station is responsible for the calculation of the broadcast ephemeris and for the upload to the satellites
- Will be updated each hour and should be used not more than 4 hours
- Accuracy: < 5 m with 3 uploads per day / 10 m with 1 upload per day



# Orbits

**The description of the GPS satellite orbits**

Almanac (ID/$t_0$, week, $\sqrt{a}$ , $e$, $M$, $\omega_0$, $i_0$, $\Omega_0$, $\dot{\Omega}$; $Af_0$, $Af_1$ )

- Transmitted by each satellite as part of the message and also available on the internet
- For planning purposes (visibility diagrams)
- Used by the receiver for the satellite tracking
- Update minimum every 6 days

Precise Ephemeris ($\boldsymbol{x}(t)$,$\dot{\boldsymbol{x}}(t)$)

- Based on measured data (in the whole IGS network)
- Available on request only post mission (few days up to 2 weeks)
- IGS delivers the most accurate orbits
- Less accurate data are available for real-time applications (ultra-rapid and rapid orbits)
- Format: Satellite positions and velocities in equidistant time intervals (typical: 15min, SP3)

# Orbits

## IGS orbit product overview

**GPS Satellite Ephemerides / Satellite & Station Clocks**

| Type | | Accuracy | Latency | Updates | Sample Interval |
|---|---|---|---|---|---|
| Ultra-Rapid (predicted half) | orbits | ~5 cm | real time | at 03, 09, 15, 21 UTC | 15 min |
| | Sat. clocks | ~3 ns RMS<br>~1.5 ns SDev | | | |
| Ultra-Rapid (observed half) | orbits | ~3 cm | 3 – 9 hours | at 03, 09, 15, 21 UTC | 15 min |
| | Sat. clocks | ~150 ps RMS<br>~50 ps SDev | | | |
| Rapid | orbits | ~2.5 cm | 17 – 41 hours | at 17 UTC daily | 15 min |
| | Sat. & Stn. clocks | ~75 ps RMS<br>~25 ps SDev | | | 5 min |
| Final | orbits | ~2.5 cm | 12 – 18 days | every Thursday | 15 min |
| | Sat. & Stn. clocks | ~75 ps RMS<br>~20 ps SDev | | | Sat.: 30s<br>Stn.: 5 min |

(modified after `http://www.igs.org/products` )

# Orbits

### Almanac in YUMA-Format

```
******** Week 453 almanac for PRN-03 ********
ID:                          03
Health:                      000
Eccentricity:                0.1052618027E-001 ← O.K.E
Time of Applicability(s):    319488.0000       ← seconds in the week
Orbital Inclination(rad):    0.9258728027      ← O.K.E
Rate of Right Ascen(r/s):   -0.8083588909E-008 ← O.K.E
SQRT(A)  (m 1/2):            5153.626953.      ← O.K.E
Right Ascen at Week(rad):   -0.8138453960E+000 ← O.K.E
Argument of Perigee(rad):    0.810930967       ← O.K.E
Mean Anom(rad):              0.1102093339E+001 ← O.K.E
Af0(s):                      0.2212524414E-003 ← satellite clock offset
Af1(s/s):                    0.3637978807E-011 ← satellite clock drift rate
week:                        453               ← (0000-1024) since 22.8.1999
```

# Orbits

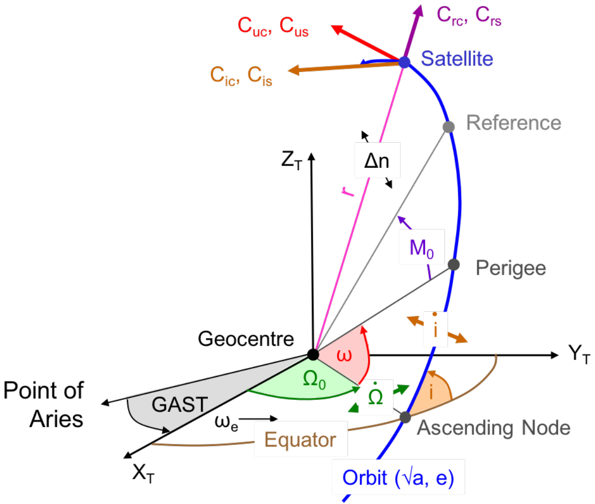
**The description of the GPS satellite orbits**

Broadcast Ephemeris Message Parameter

- $t_{0e}$: ephemeris data reference time of week
- $e$: eccentricity
- $\sqrt{a}$: square root of semi-major axis
- $M_0$: mean anomaly at reference time
- $\omega_0$: argument of perigee for ephemeris reference time
- $\Omega_0$: longitude of the ascending node at beginning of GPS week
- $i_0$: inclination angle at ephemeris reference time
- $\Delta n$: mean motion difference from computed value
- $\dot{\Omega}$: rate of right ascension difference
- $\dot{i}$: rate of inclination angle
- $C_{uc}, C_{us}$: amplitude for cos- and sin-correction of argument of latitude
- $C_{rc}, C_{rs}$: amplitude for cos- and sin-correction correction of orbital radius
- $C_{ic}, C_{is}$: amplitude for cos- and sin-correction correction of inclination

# Orbits

**The description of the GPS satellite orbits**



# Orbits

**The description of the GPS satellite orbits**

Broadcast Ephemeris Message Parameter for details see pages 104-105 of the IS-GPS-200, `https://www.gps.gov/technical/icwg/IS-GPS-200J.pdf` )

The underlined quantities are transmitted in the GPS signal

$$a(t) = \underline{a_0} \tag{3.46}$$

$$M(t) = \underline{M_0} + (n_0 + \underline{\Delta n})(t - \underline{t_0}) \Rightarrow E(t) \Rightarrow \nu(t) \tag{3.47}$$

$$u(t) = \underline{\omega_0} + \nu(t) + \underline{C_{uc}} \cos[2(\underline{\omega_0} + \nu(t)] + \underline{C_{us}} \sin[2(\underline{\omega_0} + \nu(t)] \tag{3.48}$$

$$r(t) = a(1 - \underline{e} \cos E) + \underline{C_{rc}} \cos[2(\underline{\omega_0} + \nu(t)] + \underline{C_{rs}} \sin[2(\underline{\omega_0} + \nu(t)] \tag{3.49}$$

$$i(t) = \underline{i_0} + \underline{\dot{i}}(t - \underline{t_0}) + \underline{C_{ic}} \cos[2(\underline{\omega_0} + \nu(t)] + \underline{C_{is}} \sin[2(\underline{\omega_0} + \nu(t)] \tag{3.50}$$

$$\Omega(t) = \underline{\Omega_0} + (\underline{\dot{\Omega}} - \dot{\Omega}_e)(t - \underline{t_0}) - \dot{\Omega}_e \underline{t_0} \tag{3.51}$$

where

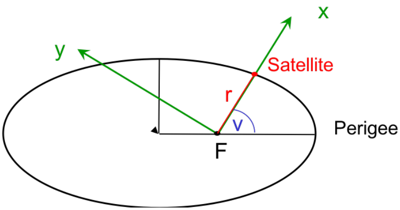
$$\dot{\Omega}_e = 7.2921151467 \cdot 10^{-5} \text{ rad/s} \tag{3.52}$$

is the WGS84 Earth rotation rate.

# Orbits

**Computation of XYZ satellite position from broadcast ephemeris parameters - approach I**

1. Compute $n_0$ from Kepler's third law
   $n_0 = \sqrt{GM/a^3}$
2. Add $\Delta n$ and compute mean, eccentric and true anomalies for a specified time $t$
3. Compute $u(t)$, $r(t)$ and $i(t)$
4. Compute $\Omega(t)$, $\lambda(t)$
5. Compute orbital coordinates $(r(t), 0)$
6. Compute Cartesian coordinates for the satellite position in ECEF

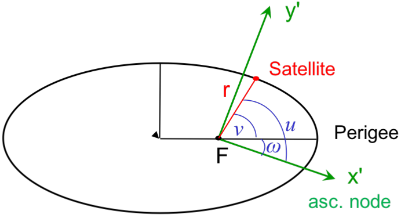



$$\begin{bmatrix} X(t) \\ Y(t) \\ Z(t) \end{bmatrix} = \boldsymbol{R}_3(-\lambda(t)) \cdot \boldsymbol{R}_1(-i(t)) \cdot \boldsymbol{R}_3(-u(t)) \begin{bmatrix} r(t) \\ 0 \\ 0 \end{bmatrix} \tag{3.53}$$

# Orbits

**Computation of XYZ satellite positon from broadcast ephemeris parameters - approach II (cf. IS-GPS-200)**

1. Compute $n_0$ from Kepler's third law
   $n_0 = \sqrt{GM/a^3}$
2. Add $\Delta n$ and compute mean, eccentric and true anomalies for a specified time $t$
3. Compute $u(t)$, $r(t)$ and $i(t)$
4. Compute $\Omega(t)$, $\lambda(t)$
5. Compute orbital coordinates $(x'(t), y'(t))$
6. Compute Cartesian coordinates for the satellite position in ECEF




$$\begin{bmatrix} X(t) \\ Y(t) \\ Z(t) \end{bmatrix} = \boldsymbol{R}_3(-\lambda(t)) \cdot \boldsymbol{R}_1(-i(t)) \begin{bmatrix} x'(t) \\ y'(t) \\ 0 \end{bmatrix} \tag{3.54}$$