Part I. Map Projection

MAP

- Surface geometry: How to parametrize a surface and how to derive the ~~matrix~~ metric G

- How are the standard metric matrices? How is the $\boxed{\text{arc element}}$? $ds^2 = G_{11}du^2 + G_{22}dv^2 + 2G_{12}du\,dv$
  Jacobian matrix J — area element $dF = \sqrt{\det(G)}\,du\,dv$
- How are the general mapping equations for azimuthal, cylindrical and conical mapping?
  $C = J^T g J$ — f — $\alpha = \lambda$, $r = f(\varphi)$ — $x = \lambda$, $y = f(\varphi)$ — $\alpha = \sin\varphi_0 \cdot \lambda$, $r = f(\varphi)$
- How to implement oblique/transverse map projections? (Rotations!)
  Metapole $(\lambda_0, \varphi_0, \underline{\Omega_0})$ $P_2 = R_3(\Omega_0) R_2(90° - \varphi_0) R_3(\lambda_0)\, P_1$
- How to find unknown function f in the general mapping equations? (not for conical mappings)
- How to compute $\underset{C = J^T g J}{C}$ and principle distortions λ_1, λ_2? $\lambda^2 = \frac{ds^2}{dS^2} = \frac{1}{2}\left(tr(CG^{-1}) \pm \sqrt{[tr(CG^{-1})]^2 - 4 \cdot \det(CG^{-1})}\right)$
  $\det(C - \Lambda^2 G) = 0$
- Remember special computation rules: Conformality $\lambda_1 = \lambda_2 \Leftrightarrow (tr\, CG^{-1})^2 = 4 \det(CG^{-1})$

Equivalence (equal area) $\lambda_1 \lambda_2 = 1 \Leftrightarrow \det(CG^{-1}) = 1$.

$\Leftrightarrow \det(J^T g J) / \det(G) = 1$

$\boxed{\Leftrightarrow (\det J)^2 \cdot \det g / \det G = 1}$

principle distortions in direction of parameter line $\boxed{C, G \text{ diagonal}} \Rightarrow \lambda_1^2 = \frac{C_{11}}{G_{11}},\ \lambda_2^2 = \frac{C_{22}}{G_{22}}$

φ = const, parallel circles — λ = const, meridians

Equidistance $\lambda_1 = 1$ or $\lambda_2 = 1$. $(dS^2 = G_{11}d\lambda^2 + G_{22}d\varphi^2)$

Part II. Coordinates Systems (GK/UTM)

- Geometry of ellipsoid, length of a meridional arc, arc element
  $G = \begin{bmatrix} N^2\cos^2 B & 0 \\ 0 & M^2 \end{bmatrix}$ $N = \frac{A}{\sqrt{1 - E^2\sin^2 B}}$ $M = \frac{A(1-E^2)}{\sqrt{(1 - E^2\sin^2 B)^3}}$ $dS^2 = N^2\cos^2 B\, dL^2 + M^2 dB^2$
- Structure of bivariate conformal series (meaning of x, y, l, b or q, L_0, B_0, or Q_0)
  $(L, B) \xrightarrow{\text{iso}} (L, Q)$ $l = L - L_0$, $q = Q - Q_0$ $\{x(l,q) = \sum C_{ij} q^i l^j$ $x(l,b)$, $y(l,b)$ $\rightarrow x, y$
- Construction of Gauß-Krüger GK and UTM coordinates (+ 10^6 if $x<0$) from x, y (conformal already).
  GK: $H = X_0 + x(l,b)$ $R = y(l,b) + 10^6 \cdot L_0 / 3 + 500\,000$ Zone $= [\frac{L_0}{3°}]$ UTM: $N = X_0 + x(l,b)$ $E = y(l,b) + 500\,000$ Zone $= [\frac{L_0 + 3°}{6°}] + 30$
- Differences between GK and UTM.
  Gauß-Krüger, Universal Transverse Mercator; 3°/6°; 1/0.9996; tangent/secant
- Why strip system?
  ① $l = L - L_0$ is limited ② distortions on the boundary
- Interpretation of coordinates (E, N, R, H) with respect to location on ellipsoid.
  GK. $L = L_0 + l(x,y)$, $B = B_0 + b(x,y)$ UTM: $L = (\text{Zone} - 30) \cdot 6° - 3° + l(x,y)$ $B = B_0 + b(x,y)$
- Meridian convergence? Meaning?
- Relevance of GK/UTM? (Conformality, limit the distortion, cover the entire globe, ...)

Part III. Datum transformations

- Commutative diagram with 3 model types (Cartesian 3D, Cartesian 2D, Curvilinear 2D/3D)

$$\begin{tikzcd}
R, H \arrow[r, "Cur\,2D"] \arrow[d] & E, N \\
l, b, h \arrow[r, "T_x, T_y, T_z, \alpha, \beta, \gamma, \lambda"', "\delta a, \delta e^2"] \arrow[d] & L, B, H \arrow[u] \\
x, y, z \arrow[r, "T_x, T_y, T_z, \alpha, \beta, \gamma, \lambda"] & X, Y, Z \arrow[u]
\end{tikzcd}$$

- 7 parameter transformation

(parameters, rotation matrices, CTI-case, analysis, synthesis, magnitude of parameters) be th...

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \lambda \cdot R_3(\gamma) R_2(\beta) R_1(\alpha) \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

CTI-case: close to identity, $\lambda = 1$, $\alpha, \beta, \gamma = 0$

analysis ⇓ parameters

synthesis ⇓ coordinates

α, β, γ — seconds

$T_x T_y T_z$ — (meters)

λ — 1

o ap...

- 2D model: 4/6 parameter transformation

$$\begin{bmatrix} E \\ N \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ w & \lambda \end{bmatrix} \begin{bmatrix} R \\ H \end{bmatrix} + \begin{bmatrix} T_E \\ T_N \end{bmatrix}$$

2 rotation. 2 translation. 2 scale.

1. ① parameterize a surface : $\underset{\sim}{X}(u,v) = X_1(u,v)\underset{\sim}{E_1} + X_2(u,v)\underset{\sim}{E_2} + X_3(u,v)\underset{\sim}{E_3}$

② Metriz matrix (度量矩阵) : $\underline{G} = \begin{pmatrix} \langle G_1, G_1\rangle, & \langle G_1, G_2\rangle \\ \langle G_2, G_1\rangle, & \langle G_2, G_2\rangle \end{pmatrix} = \begin{pmatrix} G_{11}, & G_{12} \\ G_{21}, & G_{22} \end{pmatrix}$ dot product

where $G_1 = \frac{\partial \underset{\sim}{X}}{\partial u^1}$, $G_2 = \frac{\partial \underset{\sim}{X}}{\partial u^2}$ (tangent vector)

③ $\langle u, v\rangle \to \langle u', v'\rangle$: $G' = J^T G J$. $J = \begin{bmatrix} \frac{\partial u}{\partial u'} & \frac{\partial u}{\partial v'} \\ \frac{\partial v}{\partial u'} & \frac{\partial v}{\partial v'} \end{bmatrix}$ "Jacobian matrix".

2. Arc element : $ds^2 = G_{11}du^2 + 2G_{12}du\,dv + G_{22}dv^2 = [u, v]\,\underline{G}\begin{bmatrix} u \\ v \end{bmatrix}$

area element : $dF = \sqrt{\det(\underline{G})}\,du\,dv$

3. General mapping equations for azimuthal, cylindrical and conical mapping:

① Earth-Model (spherical):

$\underset{\sim}{X}(\lambda, \phi) = R\cos\phi\cos\lambda\,\underset{\sim}{E_1} + R\cos\phi\sin\lambda\,\underset{\sim}{E_2} + R\sin\phi\,\underset{\sim}{E_3}$

$G_1 = \frac{\partial \underset{\sim}{X}}{\partial \lambda} = -R\cos\phi\sin\lambda\,\underset{\sim}{E_1} + R\cos\phi\cos\lambda\,\underset{\sim}{E_2} + 0\cdot \underset{\sim}{E_3}$

$G_2 = \frac{\partial \underset{\sim}{X}}{\partial \phi} = -R\sin\phi\cos\lambda\,\underset{\sim}{E_1} - R\sin\phi\sin\lambda\,\underset{\sim}{E_2} + R\cos\phi\,\underset{\sim}{E_3}$

$\Rightarrow G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} R^2\cos^2\phi & 0 \\ 0 & R^2 \end{bmatrix}$

$dS^2 = R^2\cos^2\phi\, d\lambda^2 + R^2 d\phi^2$, $dF = R^2\cos\phi\, d\lambda\, d\phi$

comparison dS^2 & ds^2:

$\underline{C} = \underline{J}^T\underline{g}\,\underline{J}$

"Cauchy-Green Deformation tensor"

② ~~azimuthal~~ Plane :

1° (x, y) . $g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $ds^2 = d\phi^2 + d\lambda^2$

2° (α, r) . $g = \begin{bmatrix} r^2 & 0 \\ 0 & 1 \end{bmatrix}$, $ds^2 = r^2 d\alpha^2 + dr^2$

③ azimuthal : $\alpha = \lambda$, $r = f(\phi)$. $J = \begin{pmatrix} \frac{\partial \alpha}{\partial \lambda} & \frac{\partial \alpha}{\partial \phi} \\ \frac{\partial r}{\partial \lambda} & \frac{\partial r}{\partial \phi} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & f'(\phi) \end{bmatrix}$. conformal $\Downarrow$ $\lambda_1 = \lambda_2$

cylindrical : $x = R\lambda$, $y = Rf(\phi)$ $J = \begin{bmatrix} R & 0 \\ 0 & R\cdot f'(\phi) \end{bmatrix}$

conical : $\alpha = n\lambda$, $n = \sin\phi_0$. $r = f(\phi)$? $J = \begin{bmatrix} n & 0 \\ 0 & f'(\phi) \end{bmatrix}$ meridians no distortion $\Downarrow$ $\lambda_2 = 1$

4. Principle Distortions

① $\Phi(U, V) \to (u, v)$: $\quad C = \underline{J}^T \underline{g}\, \underline{J}$

② Distortion analysis. $\cancel{\lambda^2 = \frac{ds^2}{dS^2} = \frac{g_{kl}\,du^k du^l}{G_{kl}\,dU^k dU^l} = \frac{g_{kl}\frac{\partial u^k}{\partial U^k}\frac{\partial u^l}{\partial U^l}\,dU^k dU^l}{G_{kl}\,dU^k dU^l} = \frac{C_{kl}}{G_{kl}}}$

$$\lambda^2 = \frac{ds^2}{dS^2} = \frac{d\underline{u}^T \underline{g}\, d\underline{u}}{d\underline{U}^T \underline{G}\, d\underline{U}} = \frac{d\underline{U}^T \underline{J}^T \underline{g}\, \underline{J}\, d\underline{U}}{d\underline{U}^T \underline{G}\, d\underline{U}} = \frac{d\underline{U}^T \underline{C}\, d\underline{U}}{d\underline{U}^T \underline{G}\, d\underline{U}}$$

$$\frac{d}{d\underline{U}}(\lambda^2) = \frac{d}{dU}\left(\frac{2C\,dU\,dS^2 - 2ds^2 G\,dU}{dS^4}\right) \doteq 0 \Rightarrow (\underline{C} - \Lambda^2 \underline{G}) \cdot d\underline{U} \doteq 0 \Rightarrow \det(\underline{C} - \Lambda^2 \underline{G}) = 0$$

$$\Lambda^2_{min,max} = \frac{1}{2}\left\{ tr(CG^{-1}) \pm \sqrt{[tr(CG^{-1})]^2 - 4\det(CG^{-1})} \right\}$$

special cases : 1° conformality: $\Lambda_1 = \Lambda_2 \iff [tr(CG^{-1})]^2 = 4\det(CG^{-1})$

2° equivalence : $\Lambda_1 \cdot \Lambda_2 = 1 \iff \det(C \cdot G^{-1}) = 1$
(equal area)

or $\frac{\det(J^T g J)}{\det(G)} = 1$ or $\frac{(\det J)^2 \cdot \det g}{\det(G)} = 1$

3°. equal distance : $\Lambda_1 = 1$ or $\Lambda_2 = 1$

4° if G and C are diagonal : $\Lambda_1 = \sqrt{\frac{C_{11}}{G_{11}}}$, $\Lambda_2 = \sqrt{\frac{C_{22}}{G_{22}}}$
(max & min distortion on parameter lines)

5° $\Lambda_1 = \sqrt{\frac{C_{11}}{G_{11}}}$, $\Lambda_2 = \sqrt{\frac{C_{22}}{G_{22}}}$ distortion on parameter lines

$\Downarrow$ ϕ = const parallel circles

$\Downarrow$ λ = const meridians.

5. Oblique / transverse Projection.

① meta north pole $M(\lambda_0, \phi_0, \Omega_0)$. 1°. Given traditional coordinates (λ, ϕ)

$$B = \arcsin(\sin\phi \sin\phi_0 + \cos\phi \cos\phi_0 \cos\Delta\lambda) \quad (-\frac{\pi}{2}, \frac{\pi}{2})$$

$$A = \arctan\left(\frac{\sin\Delta\lambda}{\sin\phi_0 \cos\Delta\lambda - \cos\phi_0 \tan\phi}\right) - \Omega_0 \quad (0, 2\pi)$$

2°. Given meta coordinates (A, B)

$$\Lambda = \Lambda_0 + \arctan\frac{\sin(A + \Omega_0)}{\sin\phi_0 \cos(A + \Omega_0) + \cos\phi_0 \tan B}$$

2 $\quad \phi = \arcsin(\sin\phi_0 \sin B - \cos\phi_0 \cos(A + \Omega_0) \cos B)$

$$\begin{bmatrix} X^* \\ Y^* \\ Z^* \end{bmatrix} = R_3(\Omega_0)\, R_2(90^\circ - \phi_0)\, R_3(\lambda_0) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$A = \arctan \frac{Y^*}{X^*} \quad , B = \arctan \frac{Z^*}{(1-E^2)\sqrt{X^{*2}+Y^{*2}}}$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix} \quad R_2 = \begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix} \quad R_3 = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

rotate axis

II. Coordinate Systems (GK / UTM)

1. Geometry of ellipsoid

$$G = \begin{bmatrix} N^2\cos^2 B & 0 \\ 0 & M^2 \end{bmatrix} \quad N = \frac{A}{\sqrt{1-E^2\sin^2 B}} \quad M = \frac{A(1-E^2)}{(1-E^2\sin^2 B)^{3/2}} \quad dS^2 = N^2\cos^2 B\, dL^2 + M^2 dB^2$$

2. structure of bivariate conformal series

$(U, V) \xrightarrow[\text{cylindrical}]{\text{conformal}} (x, y)$:

ellipsoidal Longitude / Latitude L, B $\xrightarrow[\text{too difficult}]{\Lambda_1=\Lambda_2 \Leftrightarrow (tr(CG^{-1}))^2 = 4\det(CG^{-1})}$ conformal coordinates x, y

$\downarrow$

$\boxed{\text{ellipsoidal isometric coordinates } L, Q}$ $\xrightarrow[\text{bivariate conformal series}]{\text{Cauchy-Riemann \& Laplace Equation}}$ special points / conditions L_0, Q_0

spherical isometric ?

$$\begin{cases} x(L-L_0, Q-Q_0) = x(l,q) = (n,0)_x q^n + (n-1,1)_x q^{n-1} l + \cdots + (0,n)_x l^n \\ y(L-L_0, Q-Q_0) = y(l,q) = \ldots \end{cases}$$

3. Gauß Krüger and UTM (universal transverse mercator) coordinates : Structure

① GK : Northing & False Easting (H, R)

$$R = y(l,b) + 10^6 \cdot \left[\frac{L_0}{3^\circ}\right] + 500\,000 \quad [m]$$

$$H = X_0 + x(l,b)$$

strip number $= \left[\frac{L_0}{3^\circ}\right]$

③ UTM: False Northing & False Easting

$$N=\begin{cases} x_0 + x(l,b) & (x>0) \\ x_0 + x(l,b) + 10\,000\,000 & (x<0) \end{cases}$$

$$E = y(l,b) + 500\,000$$

$$\text{strip number} = \left[\frac{L_0+3^\circ}{6^\circ}\right] + 30$$

4. Difference between GK and UTM

1° Gauß-Krüger & universal transverse mercartor

2° transverse tangent cylinder & transverse secant cylinder

3° strip width 3° & 6°

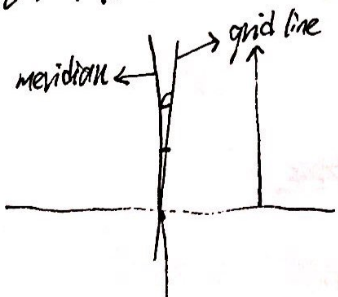
4°. scale of reference meridian 1 & 0.9996

5°. strip extension / boundary distortion / length distortion.

5. Why strip system?

Because $l = L - L_0$ is limited, and distortions on the boundary of the mapped region should be as small as possible.

6. Meridian convergence.



C: meridian convergence

λ: scale factor, distortion

7. Relevance of GK/UTM.

1° conformality 2° strips (limit distortion) 3° cover the entire globe

4°. transverse cylinder

1. Commutative diagram with 3 model types

local (DHDN) — global (ETRS89)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
 & R, H \arrow[rr, "\text{other methods}"] \arrow[d, "\text{map projection}", "a, e^2"'] & & E, N \\
h \arrow[r] & l, b, h \arrow[rr, "\text{curvilinear datum transformation}", "T_x, T_y, T_z, \alpha, \beta, \gamma, \lambda;\ \delta a, \delta e^2\ ?"'] \arrow[d, "\text{geometry}", "a, e^2"'] & & L, B, H \arrow[u, "\text{map projection}"', "A, E^2; L_0"] \\
 & x, y, z \arrow[rr, "\text{cartesian datum transformation}", "T_x, T_y, T_z, \alpha, \beta, \gamma, \lambda"'] & & X, Y, Z \arrow[u, "\text{geometry}"', "A, E^2"]
\end{tikzcd}
$$

2. 7-parameter-transformation

$$
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \lambda \cdot R_3(\gamma) R_2(\beta) R_1(\alpha) \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}
$$

$$
R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}, \quad R_2 = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \quad R_3 = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

CTI-transformation (close-to the identity): $\lambda = 1$, $\alpha = \beta = \gamma = 0$.

magnitude of parameters: α, β, γ — a few arc seconds

scale $\lambda \approx 1$

T_x, T_y, T_z — up to several 100 m.

analysis: Determine the unknown parameters from a certain number of homologous points in both

synthesis: transform points from system 1 to 2 using estimated parameters

3. 2D model (4/6 parameter transformation)

6-parameter ~~affine~~ transformation:

Given: a planar geodetic network with both local and global coordinates

unknown: translation, orientation, scale

Wanted: 6 parameters (2 rotations, 2 translations, 2 scale factors).

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

5-parameter: 2 translations, 2 rotations, 1 scale

4-parameter: 2 translations, 1 rotation, 1 scale

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

3-parameter: 2 translations, 1 rotation.