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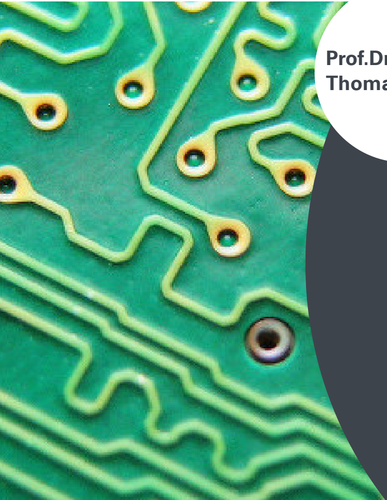

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Integrated Positioning and Navigation

Integration of the velocity and position equations in the e-system

9

Integration of the velocity and position equations

Differential equations (6.17) from Module 6:

$$\begin{aligned}\dot{\boldsymbol{x}}^e &= \boldsymbol{v}^e \\ \dot{\boldsymbol{v}}^e &= \boldsymbol{C}_p^e \boldsymbol{a}^p - 2\boldsymbol{\Omega}_{ie}^e \boldsymbol{v}^e - \boldsymbol{\Omega}_{ie}^e \boldsymbol{\Omega}_{ie}^e \boldsymbol{x}^e + \boldsymbol{g}^e\end{aligned} \tag{9.1}$$

Attitude equation has been integrated (Module 8); DCM is known!
Second, third and fourth term on r.h.s. are slowly varying terms and can be approximated by constant values in the interval $t_{k-1} < \tau < t_k$

$$\boldsymbol{v}^e(t_k) = \int_{t_{k-1}}^{t_k} \boldsymbol{C}_p^e \cdot \boldsymbol{a}^p d\tau - \left[2\boldsymbol{\Omega}_{ie}^e \boldsymbol{v}^e + \boldsymbol{\Omega}_{ie}^e \boldsymbol{\Omega}_{ie}^e \boldsymbol{x}^e - \boldsymbol{g}^e\right]_{t_{k-1}} \cdot (t_k - t_{k-1}) \tag{9.2}$$

Accelerometer output (Module 7):

$$\Delta \boldsymbol{v}^p(t_k) = \int_{t_{k-1}}^{t_k} \boldsymbol{a}^p(\tau) d\tau \tag{9.3}$$

Integration of the velocity and position equations - cont'd

Integration of equation (9.2) using *Simpson's Rule*:

$$\dot{y} = f(t) \Rightarrow$$
$$y(t_k) = y(t_{k-1}) + \frac{h}{6}\left(f(t_{k-1}) + 4f(t_{k-1} + \frac{h}{2}) + f(t_{k-1} + h)\right) \tag{9.4}$$
$$h = t_k - t_{k-1}, \quad f(x) = \boldsymbol{C}_p^e \cdot \boldsymbol{a}^p$$

Question: How can we obtain the $\boldsymbol{a}^p$ needed in equation (9.4) from the accelerometer output (equation (9.3))? In general, we can express $\boldsymbol{a}^p$ in a Taylor expansion in the interval $[t_{k-2}, t_k]$:

$$\boldsymbol{a}^p(t) = \boldsymbol{a}^p(t_{k-2}) + \dot{\boldsymbol{a}}^p(t_{k-2}) \cdot (t - t_{k-2}) + O(\delta t^2), \;\; t - t_{k-2} \leq \delta t \tag{9.5}$$

Integration of (9.5):

$$\Delta \boldsymbol{v}^p(t_{k-1}) = \int_{t_{k-2}}^{t_{k-1}} \boldsymbol{a}^p(\tau) d\tau = \boldsymbol{a}^p(t_{k-2})\Delta t + \frac{1}{2}\dot{\boldsymbol{a}}^p(t_{k-2})\Delta t^2 + \dots \tag{9.6}$$

$$\begin{aligned} \Delta \boldsymbol{v}^p(t_k) \quad &= \boldsymbol{a}^p(t_{k-2})\Delta t + \dot{\boldsymbol{a}}^p(t_{k-2}) \int_{t_{k-1}}^{t_k} (\tau - t_{k-1}) + (t_{k-1} - t_{k-2}) d\tau + \dots \\ &= \boldsymbol{a}^p(t_{k-2})\Delta t + \frac{3}{2}\dot{\boldsymbol{a}}^p(t_{k-2})\Delta t^2 \end{aligned} \tag{9.7}$$

Integration of the velocity and position equations - cont'd

From equations (9.6) and (9.7):

$$\boxed{\begin{aligned} \boldsymbol{a}^p(t_{k-2}) &= \frac{1}{2\Delta t}\left(3\Delta\boldsymbol{v}^p(t_{k-1}) - \Delta\boldsymbol{v}^p(t_k)\right) + \ldots \\ \dot{\boldsymbol{a}}^p(t_{k-2}) &= \frac{1}{\Delta t^2}\left(\Delta\boldsymbol{v}^p(t_k) - \Delta\boldsymbol{v}^p(t_{k-1})\right) + \ldots \end{aligned}} \tag{9.8}$$

Equations (9.8) inserted in (9.5) and integrated:

$$\begin{aligned} \boldsymbol{a}^p(t_{k-2}) \quad &= \frac{1}{2\Delta t}\left(3\Delta\boldsymbol{v}^p(t_{k-1}) - \Delta\boldsymbol{v}^p(t_k)\right) + \ldots \\ \boldsymbol{a}^p(t_{k-1}) \quad &= \frac{1}{2\Delta t}\left(3\Delta\boldsymbol{v}^p(t_{k-1}) - \Delta\boldsymbol{v}^p(t_k)\right) + \frac{1}{\Delta t}\left(\Delta\boldsymbol{v}^p(t_k) - \Delta\boldsymbol{v}^p(t_{k-1})\right) + \ldots \\ \boldsymbol{a}^p(t_k) \quad &= \frac{1}{2\Delta t}\left(3\Delta\boldsymbol{v}^p(t_{k-1}) - \Delta\boldsymbol{v}^p(t_k)\right) + \frac{2}{\Delta t}\left(\Delta\boldsymbol{v}^p(t_k) - \Delta\boldsymbol{v}^p(t_{k-1})\right) + \ldots \end{aligned} \tag{9.9}$$

Integration of the velocity and position equations - cont'd

Set $\delta t = 2\Delta t$ and indicate approximation by $\hat{}$ (remove higher order terms)

$$\begin{aligned}
\hat{\boldsymbol{a}}^p(t_{k-2}) &= \frac{3\Delta \boldsymbol{v}^p(t_{k-1}) - \Delta \boldsymbol{v}^p(t_k)}{\delta t} \\
\hat{\boldsymbol{a}}^p(t_{k-1}) &= \frac{\Delta \boldsymbol{v}^p(t_{k-1}) + \Delta \boldsymbol{v}^p(t_k)}{\delta t} \\
\hat{\boldsymbol{a}}^p(t_k) &= \frac{3\Delta \boldsymbol{v}^p(t_k) - \Delta \boldsymbol{v}^p(t_{k-1})}{\delta t}
\end{aligned} \tag{9.10}$$

$\boldsymbol{a}^p$ is a function of accelerometer output and known quantities!

Use Simpson's Rule (9.4) to integrate equation (9.2):

$$\int_{t_{k-1}}^{t_k} \boldsymbol{C}_p^e \boldsymbol{a}^p d\tau = \frac{\delta t}{6} \left[\left(\boldsymbol{C}_p^e \boldsymbol{a}^p \right)(t_{k-2}) + 4 \left(\boldsymbol{C}_p^e \boldsymbol{a}^p \right)(t_{k-1}) + \left(\boldsymbol{C}_p^e \boldsymbol{a}^p \right)(t_k) \right] \tag{9.11}$$

Integration of the velocity and position equations - cont'd

Use DCM as computed in Module 8, use $\boldsymbol{a}^p$ as computed in equation (9.10))

$$
\begin{aligned}
\boldsymbol{v}^e(t_k) = \quad & \hat{\boldsymbol{v}}^e(t_{k-2}) \\
& + \Big[\hat{\boldsymbol{C}}_p^e(t_{k-2})\left(3\Delta\boldsymbol{v}^p(t_{k-1}) - \Delta\boldsymbol{v}^p(t_k)\right) \\
& + 4\hat{\boldsymbol{C}}_p^e(t_{k-1})\left(\Delta\boldsymbol{v}^p(t_{k-1}) + \Delta\boldsymbol{v}^p(t_k)\right) \\
& + \hat{\boldsymbol{C}}_p^e(t_k)\left(3\Delta\boldsymbol{v}^p(t_k) - \Delta\boldsymbol{v}^p(t_{k-1})\right)\Big] / 6 \\
& - \Big[2\boldsymbol{\Omega}_{ie}^e \hat{\boldsymbol{v}}^e(t_{k-2}) + \boldsymbol{\Omega}_{ie}^e \boldsymbol{\Omega}_{ie}^e \hat{\boldsymbol{x}}^e(t_{k-2}) - \boldsymbol{g}^e\Big] \cdot \delta t
\end{aligned}
\tag{9.12}
$$

And finally

$$
\hat{\boldsymbol{x}}^e(t_k) = \hat{\boldsymbol{x}}^e(t_{k-1}) + \hat{\boldsymbol{v}}^e(t_{k-1}) \cdot \Delta t
\tag{9.13}
$$