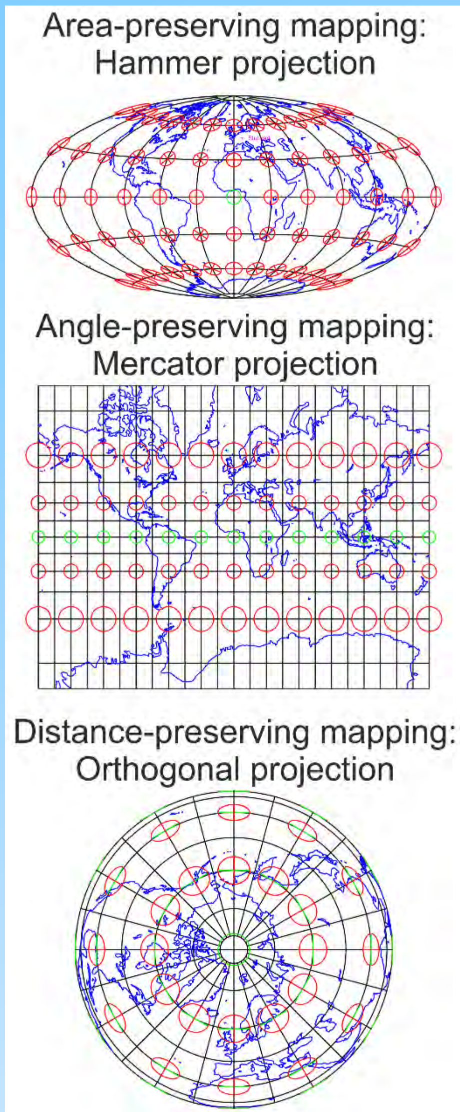
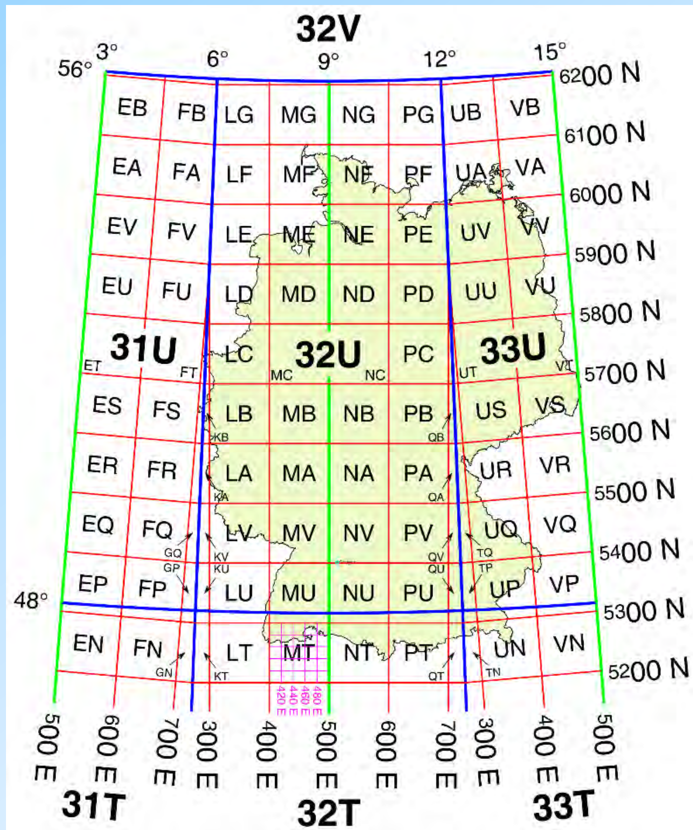
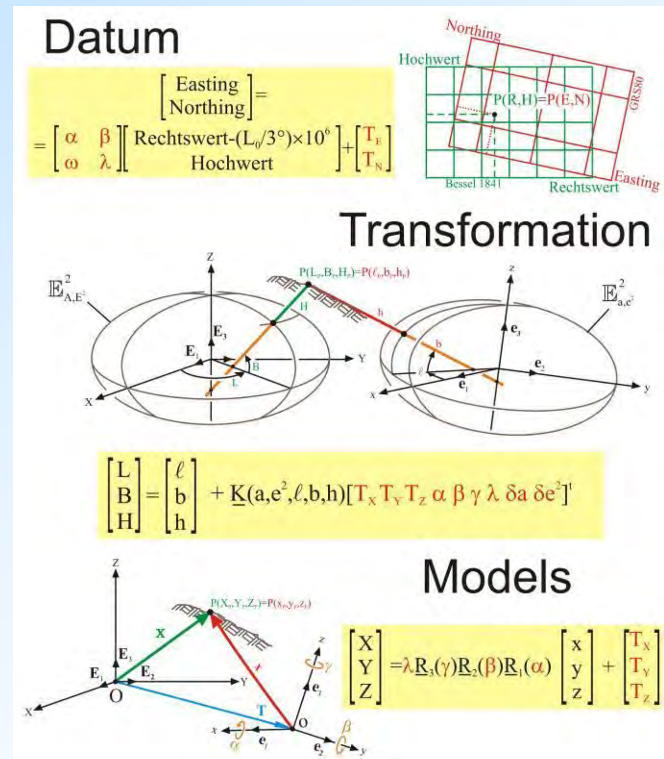
Map Projections and Geodetic Coordinate Systems







 GIS

Organization/Preliminary remarks

- Exam: 60 min, asap after summer term, together with "Physical Geodesy" (60 min) (no tools: "closed book"); assigned marks will be averaged with results of other module part. Exam admission requirements: Acknowledged labs → Lab guidelines
- Lecture Notes incl. numerical examples: available from website
- MATLAB stuff from website
- Got nothing to do ? Have too much spare time ? Want more work ? More explanations, more examples ? → Ask for a tutorial !

→ References

References

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→ Contents

Contents

Part I "Map Projections" (Theory and Practice)

→ Contents

Contents

Part II "Geodetic Coordinate Systems" (Applications)

→ Notation

Notation

- Scalar variables, numbers: t, x, x^1, X, X^n, c_{11}, … small or capital letters, with or without lower/upper index
- Vectors (Directed quantities): $\mathbf{x}$, $\mathbf{X}^1$, $\mathbf{a}$, $\mathbf{b}_1$, … small or capital bold face letters, with or without lower/upper index (on the blackboard: $\underset{\sim}{x}$, $\underset{\sim}{X}^1$, $\underset{\sim}{a}$, $\underset{\sim}{b}_1$, ... small or capital letters with a tilde)
- Arrays ("vectors", matrices): $\underline{a}$, $\underline{b}$, $\underline{C}$, $\underline{G}$, … small or capital letters, underlined. "Vectors" are always column arrays.
- Arrays ("vectors", matrices) built from scalars: $\underline{G}=[G_{ij}]$, $\underline{b}=[b_k]$
- Arrays ("vectors", matrices) built from vectors: $\underline{\mathbf{E}} = [\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3]^T$ (on the blackboard: $\underset{\sim}{\underline{E}} = [\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3]^T$)

transposed array

→ Local Curve Theory

Example 1: "Straight Line"

Let $\mathbf{x}_0$ and $\mathbf{a}$ be vectors in 3d Euclidean space $\mathbb{E}^3$ with $\mathbf{a} \neq \mathbf{0}$. The set of $\mathbf{x}$ in $\mathbb{E}^3$

$$\mathbf{x} : t \mapsto \mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{a}, \quad t \in I = \mathbb{R}$$

or

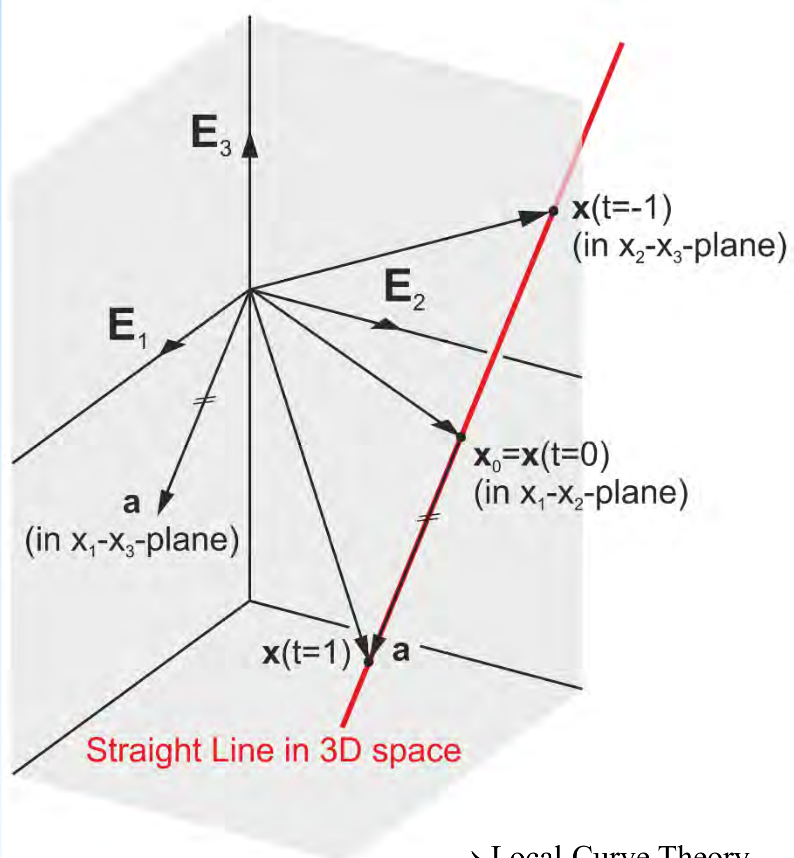
$$x_i(t) = x_{0i} + ta_i \qquad i = 1,2,3$$

is called a straight line through $\mathbf{x}_0$ and parallel to $\mathbf{a}$. The equations are called the parametric equations of the line. $\mathbf{x}$ generates the line as parameter t varies over interval I; it is a vector function of the real variable t.

Example:

$$\mathbf{x}_0 = \mathbf{E}_1 + 2\mathbf{E}_2, \quad \mathbf{a} = \mathbf{E}_1 - \mathbf{E}_3$$

$$\Rightarrow \mathbf{x} = (1+t)\mathbf{E}_1 + 2\mathbf{E}_2 - t\mathbf{E}_3.$$



→ Local Curve Theory

<u>Local Curve</u> and Surface <u>Theory</u> (vector functions of a real variable)

Example 2: "Parabola"

Let $\mathbf{x}_0$, **a** and **b** be fixed vectors in space, **a** and **b** linearly independent. The equation $\mathbf{x} : t \mapsto \mathbf{x}(t) = \mathbf{x}_0 - 2t\mathbf{a} + t^2\mathbf{b},\ \ t \in I = [-2, 2]$ defines a vector function of t with domain $-2 \leq t \leq 2$. It represents a parabola in 3d-Euclidean space and is called its parametric equation. With

$$\mathbf{x}_0 = \mathbf{E}_1 + 2\mathbf{E}_2,\ \mathbf{a} = \mathbf{E}_2 - \mathbf{E}_3,\ \mathbf{b} = \mathbf{E}_1 - \mathbf{E}_3$$

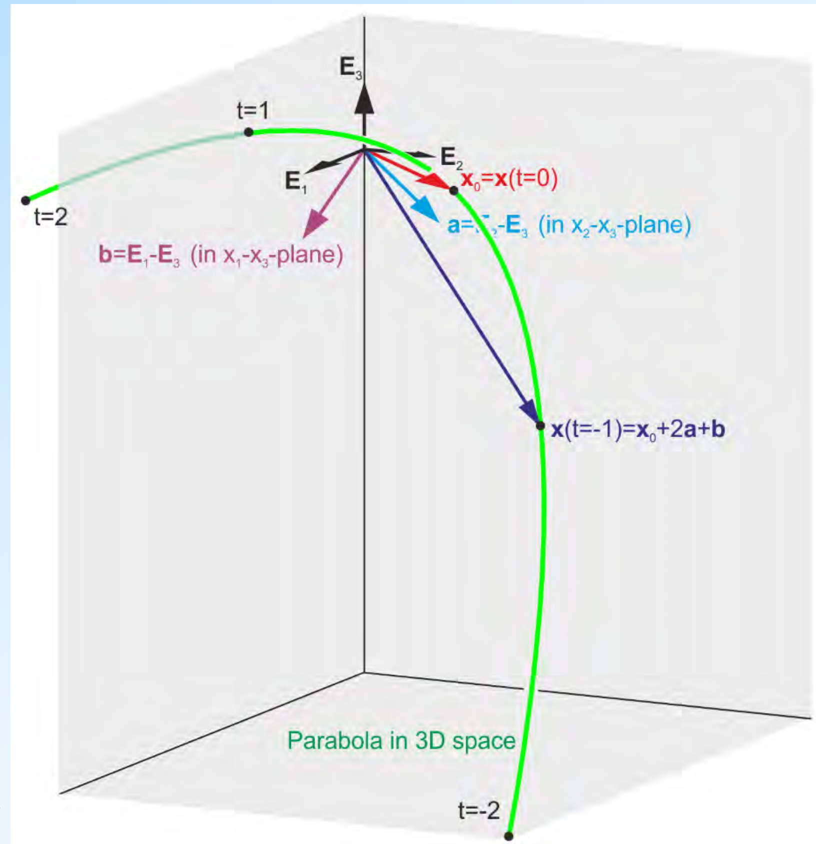
the representation

$$\mathbf{x}(t) = x_1(t)\mathbf{E}_1 + x_2(t)\mathbf{E}_2 + x_3(t)\mathbf{E}_3$$

is obtained, where

$$x_1(t) = 1 + t^2,\ x_2(t) = 2(1-t),\ x_3(t) = t(2-t)$$

are scalar functions of t. $\mathbf{x}(t)$ uniquely determines the functions $x_1(t)$, $x_2(t)$, $x_3(t)$, its coordinates with respect to the basis. Conversely, the scalar functions $x_1(t)$, $x_2(t)$, $x_3(t)$ on a common domain uniquely define the vector function $\mathbf{x}(t)$.



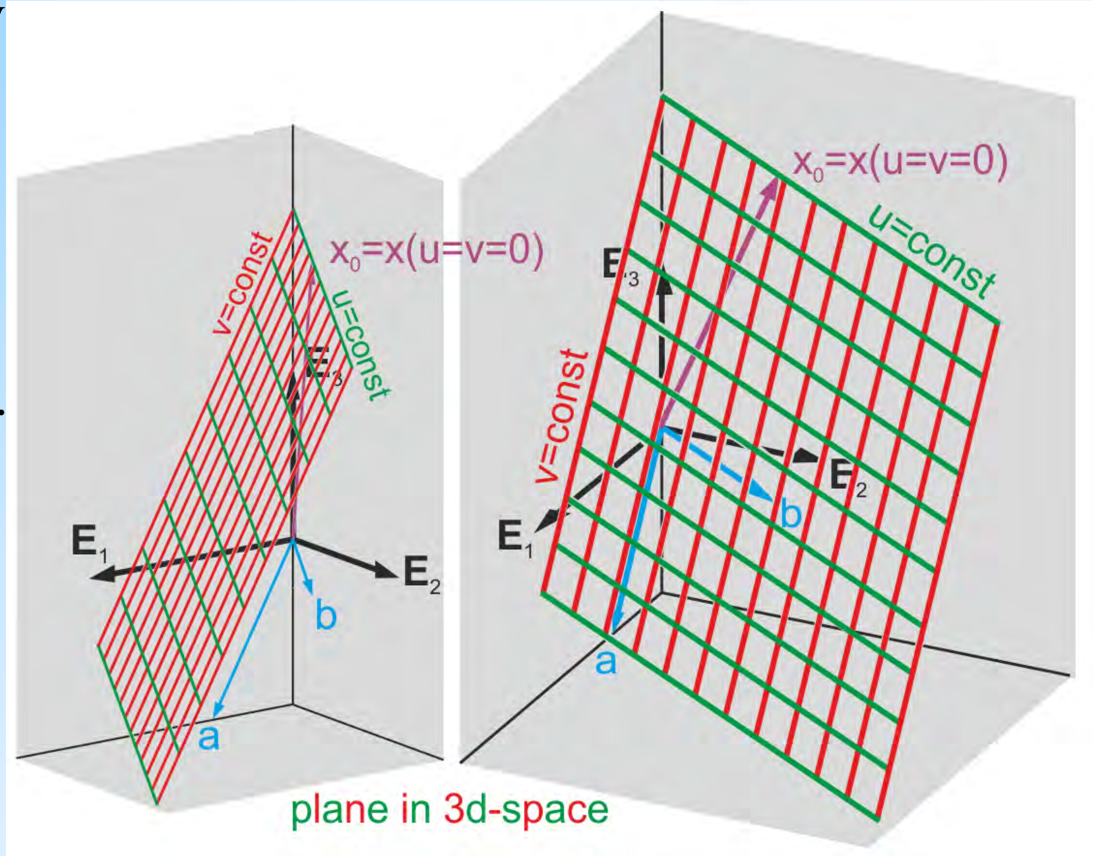
→ Local Surface Theory

Local Curve and **Surface Theory** (vector functions of two real variables)

In order to pass from curves to surfaces parameter t of the curve is replaced by two independent parameters $u^1 \equiv u$, $u^2 \equiv v$ which then describe a 2-dimensional object, the parameterized surface.

Example 3: "Plane"

Let $\mathbf{x}_0$, $\mathbf{a}$ and $\mathbf{b}$ be fixed vectors in space, $\mathbf{a}$ and $\mathbf{b}$ linearly independent.



The equation ($u^1, u^2 \in U = \mathbb{R}^2$)

$$\mathbf{x}(u^1, u^2) = \mathbf{x}_0 + u^1\mathbf{a} + u^2\mathbf{b}$$

defines a vector function of $u \equiv u^1$, $v \equiv u^2$, representing a plane in 3d-space. With $\mathbf{x}_0 = \mathbf{E}_1 + 2\mathbf{E}_2 + 2\mathbf{E}_3$, $\mathbf{a} = \mathbf{E}_1 - \mathbf{E}_3$, $\mathbf{b} = \mathbf{E}_1 + 2\mathbf{E}_2$ the form

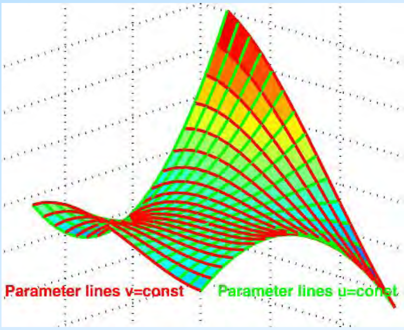
$$\mathbf{x} = (1 + u + v)\,\mathbf{E}_1 + 2(1 + v)\,\mathbf{E}_2 + (2 - u)\,\mathbf{E}_3$$

$$= x_1(u,v)\,\mathbf{E}_1 + x_2(u,v)\,\mathbf{E}_2 + x_3(u,v)\,\mathbf{E}_3 \quad \text{is obtained.}$$

→ Local Surface Theory

Local Curve and Surface Theory (vector functions of two real variables)

Example 4: "Monkey Saddle"



$$\mathbf{x}:(u,v)\mapsto \mathbf{x}=u\mathbf{E}_1+v\mathbf{E}_2+(u^3-3uv^2)\mathbf{E}_3 \quad (u,v)\in[-a,a]\times[-a,a]$$

Example 5: "Ellipsoid-of-Revolution $\mathbb{E}^2_{A,E^2}$" / "Sphere $\mathbb{S}^2_R$"

$$\mathbf{X}:(U,V)\mapsto \mathbf{X}=N\cos V\cos U\mathbf{E}_1+N\cos V\sin U\mathbf{E}_2+N(1-E^2)\sin V\mathbf{E}_3$$

$$N:=\frac{A}{\sqrt{1-E^2\sin^2 V}}=\frac{A\sqrt{1+E'^2}}{\sqrt{1+E'^2\cos^2 V}} \qquad U\in[-\pi,\pi],\ V\in[-\pi/2,\pi/2]$$

$\mathbb{E}^2_{A,E^2}$: A, B ... major, minor semi axis

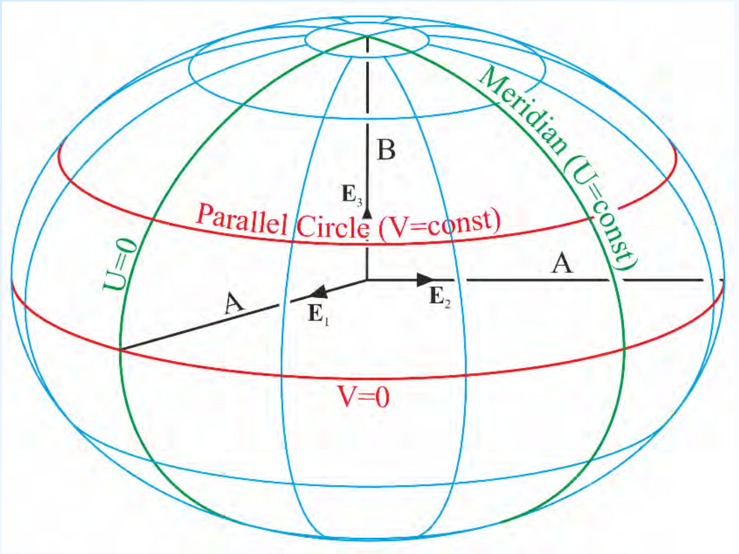
$F=(A-B)/A$ ... Flattening

E, E' ... first, second numerical eccentricity

$E^2=(A^2-B^2)/A^2,\ E'^2=(A^2-B^2)/B^2$

$\mathbb{S}^2_R$: $R=A=B=N$ ... Radius

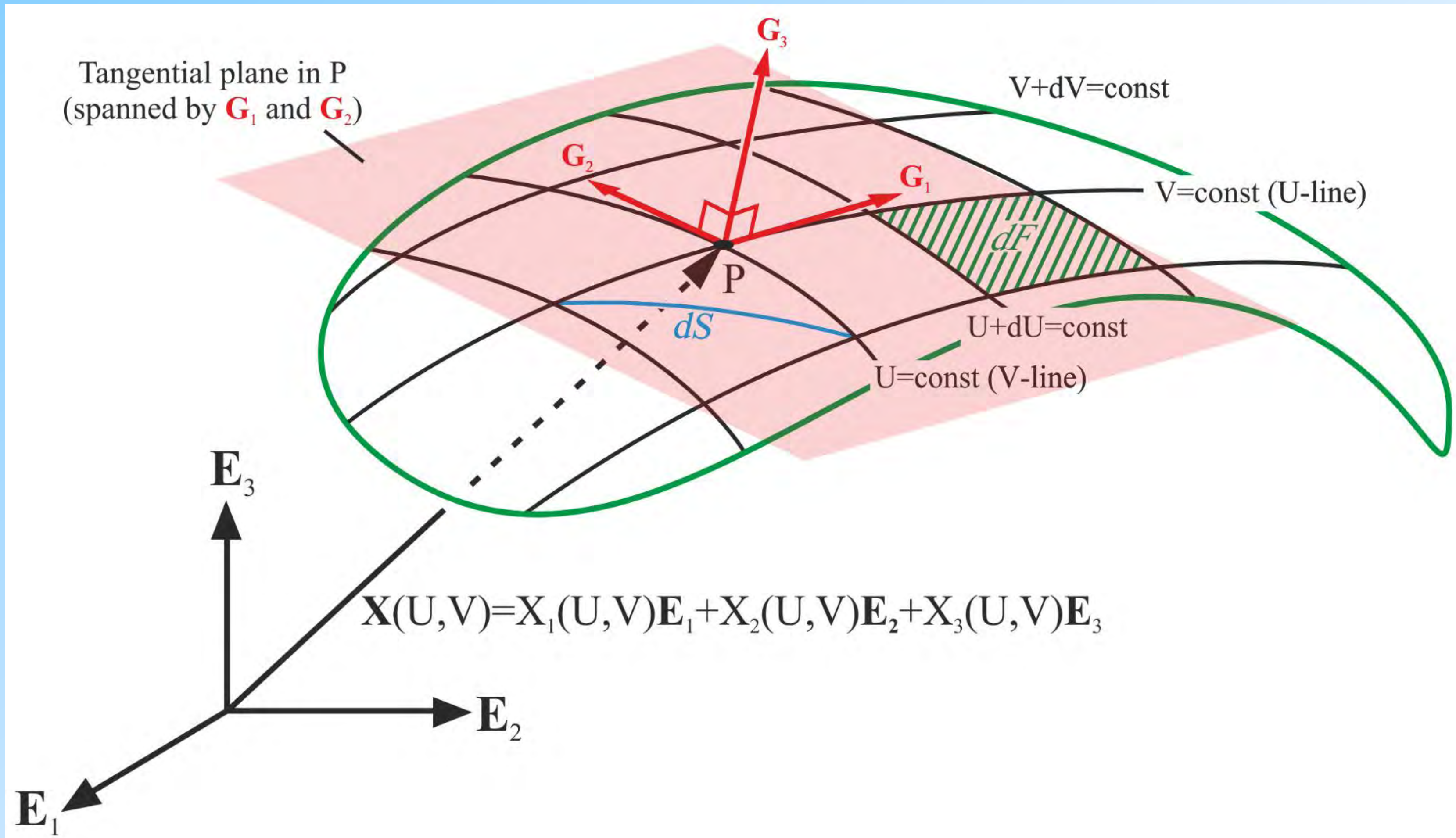
$F=E^2=E'^2=0$



→ Surface Elements

Surface Elements

Tangent vectors, normal vector, 1st and 2nd fundamental form, curvature matrix, principal curvatures, arc element, area element, angle



→ Surface Elements

Surface Elements

Tangent vectors (elements 1,2 of the Gauß 3-frame)

$$\mathbf{G}_1 := \frac{\partial \mathbf{X}}{\partial U^1} \equiv \frac{\partial \mathbf{X}}{\partial U}, \quad \mathbf{G}_2 := \frac{\partial \mathbf{X}}{\partial U^2} \equiv \frac{\partial \mathbf{X}}{\partial V}$$

not normalized, not (necessarily) orthogonal

Normal vector (element 3 of the Gauß 3-frame)

$$\mathbf{G}_3 := \frac{\mathbf{G}_1 \times \mathbf{G}_2}{\left\| \mathbf{G}_1 \times \mathbf{G}_2 \right\|}$$

"Cross product", "Vector product"; normalized, $\mathbf{G}_3 \perp \mathbf{G}_1$, $\mathbf{G}_3 \perp \mathbf{G}_2$

First fundamental form (2×2 metric matrix (symm.) → distances, angles, areas)

$$\underline{G} = [G_{KL}] = \begin{bmatrix} \langle \mathbf{G}_1, \mathbf{G}_1 \rangle & \langle \mathbf{G}_1, \mathbf{G}_2 \rangle \\ \langle \mathbf{G}_2, \mathbf{G}_1 \rangle & \langle \mathbf{G}_2, \mathbf{G}_2 \rangle \end{bmatrix}, \quad G_{KL} = \left\langle \frac{\partial \mathbf{X}}{\partial U^K}, \frac{\partial \mathbf{X}}{\partial U^L} \right\rangle, \quad K, L = 1, 2$$

Physical units !?

↑ "Scalar product", "Dot product" !

→ Surface Elements

Surface Elements

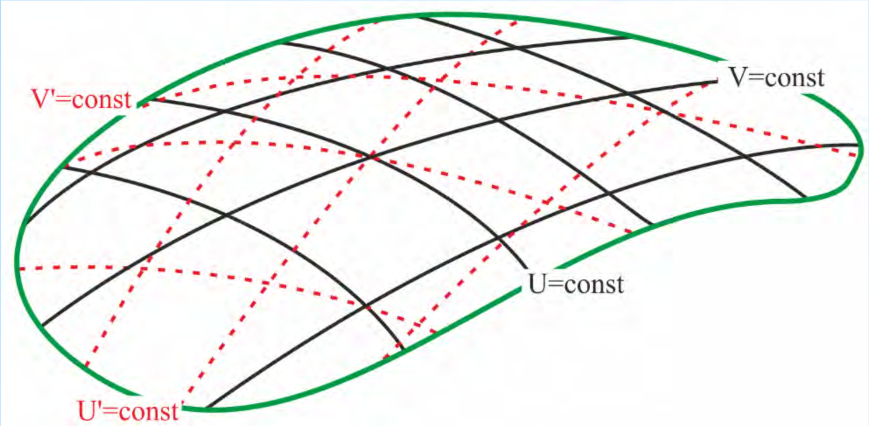
Change of G due to a change of surface parameters

$$(U,V) \mapsto (U',V') = [U'(U,V), V'(U,V)]$$

$$\underline{G} \mapsto \underline{G}' = \underline{J}^T \underline{G}\ \underline{J}$$

$$\underline{J} = \begin{bmatrix} \frac{\partial U'}{\partial U} & \frac{\partial U'}{\partial V} \\ \frac{\partial V'}{\partial U} & \frac{\partial V'}{\partial V} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\partial U}{\partial U'} & \frac{\partial U}{\partial V'} \\ \frac{\partial V}{\partial U'} & \frac{\partial V}{\partial V'} \end{bmatrix}$$

"Jacobian (matrix)"



Second fundamental form (2×2 Hessian matrix → curvatures)

$$\underline{H} = [H_{KL}] = \left[\left\langle \frac{\partial \mathbf{G}_K}{\partial U^L}, \mathbf{G}_3 \right\rangle \right] = \left[\left\langle \mathbf{G}_3, \frac{\partial^2 \mathbf{X}}{\partial U^K \partial U^L} \right\rangle \right], \quad K,L = 1,2$$

Physical units !?

Curvature matrix K and principle curvatures κ_1, κ_2

$$\underline{K} = -\underline{H}\,\underline{G}^{-1}$$

κ_1, κ_2: eigenvalues of $\underline{K}$ with directions=eigenvectors of $\underline{K}$, eigenvectors refer to $\mathbf{G}_1, \mathbf{G}_2$

→ Surface Elements

Surface Elements

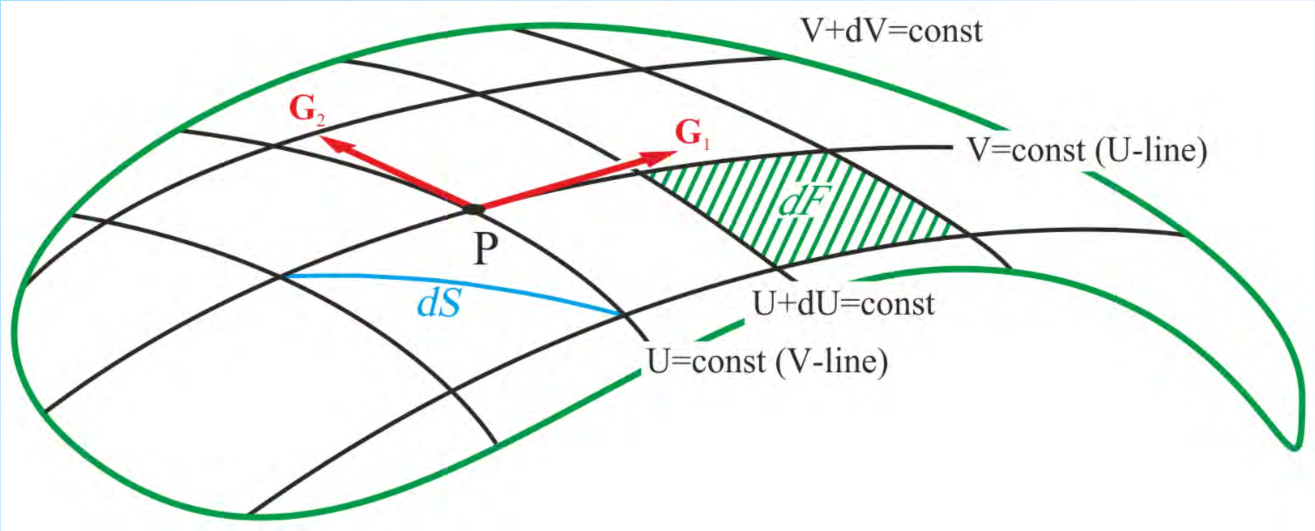
Arc element, line element dS (K,L=1,2 $U^1 \equiv U$, $U^2 \equiv V$)

$$dS^2 = G_{11}dU^2 + 2G_{12}dUdV + G_{22}dV^2 = \sum_{K=1}^{2}\sum_{L=1}^{2} G_{KL}dU^K dU^L = G_{KL}dU^K dU^L$$

"Einstein summation rule": summation over repeated indices

Area element (area of an infinitesimal piece of the surface)

$$dF = \sqrt{\det \underline{G}}\; dUdV$$



→ Surface Elements

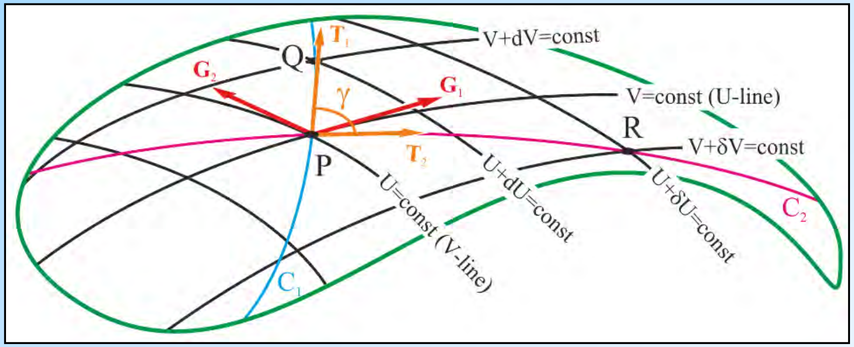
Surface Elements

Angle γ between two surface curves

$$\mathrm{P}:\mathbf{X}(\mathrm{U},\mathrm{V})$$

$$\mathrm{Q}:\mathbf{X}(\mathrm{U}+\mathrm{dU},\mathrm{V}+\mathrm{dV})=\mathbf{X}(\mathrm{U},\mathrm{V})+\mathrm{d}\mathbf{X}(\mathrm{dU},\mathrm{dV})$$

$$\mathrm{R}:\mathbf{X}(\mathrm{U}+\delta\mathrm{U},\mathrm{V}+\delta\mathrm{V})=\mathbf{X}(\mathrm{U},\mathrm{V})+\delta\mathbf{X}(\delta\mathrm{U},\delta\mathrm{V})$$



$$\mathbf{T}_1:=\mathrm{d}\mathbf{X}=\frac{\partial\mathbf{X}}{\partial\mathrm{U}}\mathrm{dU}+\frac{\partial\mathbf{X}}{\partial\mathrm{U}}\mathrm{dV}=\mathbf{G}_1\mathrm{dU}+\mathbf{G}_2\mathrm{dV}\ ,\ \mathbf{T}_2:=\delta\mathbf{X}=\frac{\partial\mathbf{X}}{\partial\mathrm{U}}\delta\mathrm{U}+\frac{\partial\mathbf{X}}{\partial\mathrm{U}}\delta\mathrm{V}=\mathbf{G}_1\delta\mathrm{U}+\mathbf{G}_2\delta\mathrm{V}$$

$\mathbf{T}_1,\mathbf{T}_2$: Linear combinations of $\mathbf{G}_1,\mathbf{G}_2$

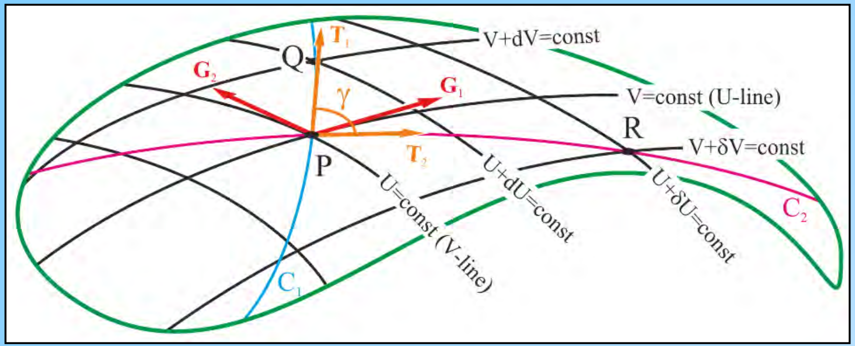
$$\cos\gamma=\frac{\left\langle\mathbf{T}_1,\mathbf{T}_2\right\rangle}{\left\|\mathbf{T}_1\right\|\left\|\mathbf{T}_2\right\|}=\frac{\left\langle\mathrm{d}\mathbf{X},\delta\mathbf{X}\right\rangle}{\left\|\mathrm{d}\mathbf{X}\right\|\left\|\delta\mathbf{X}\right\|}=\frac{\left\langle\mathbf{G}_1\mathrm{dU}+\mathbf{G}_2\mathrm{dV},\mathbf{G}_1\delta\mathrm{U}+\mathbf{G}_2\delta\mathrm{V}\right\rangle}{\mathrm{dS}\ \delta\mathrm{S}}$$

$$=\frac{\mathrm{dU}\delta\mathrm{U}\left\langle\mathbf{G}_1,\mathbf{G}_1\right\rangle+(\mathrm{dU}\delta\mathrm{V}+\mathrm{dV}\delta\mathrm{U})\left\langle\mathbf{G}_1,\mathbf{G}_2\right\rangle+\mathrm{dV}\delta\mathrm{V}\left\langle\mathbf{G}_2,\mathbf{G}_2\right\rangle}{\mathrm{dS}\ \delta\mathrm{S}}$$

$$=\frac{\mathrm{dU}\delta\mathrm{U}\mathrm{G}_{11}+(\mathrm{dU}\delta\mathrm{V}+\mathrm{dV}\delta\mathrm{U})\mathrm{G}_{12}+\mathrm{dV}\delta\mathrm{V}\mathrm{G}_{22}}{\sqrt{\mathrm{G}_{11}\mathrm{dU}^2+2\mathrm{G}_{12}\mathrm{dUdV}+\mathrm{G}_{22}\mathrm{dV}^2}\ \sqrt{\mathrm{G}_{11}\delta\mathrm{U}^2+2\mathrm{G}_{12}\delta\mathrm{U}\delta\mathrm{V}+\mathrm{G}_{22}\delta\mathrm{V}^2}}$$

→ Surface Elements

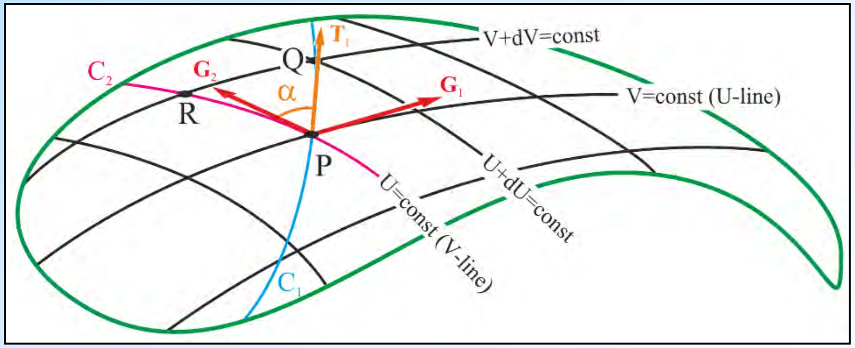
Surface Elements



Azimuth α (δU,δV) → (0,dV)

$$\cos\gamma = \frac{dUdVG_{12} + dV^2 G_{22}}{dS\sqrt{G_{22}}\,dV} = \frac{G_{12}dU + G_{22}dV}{\sqrt{G_{22}}\,dS}$$

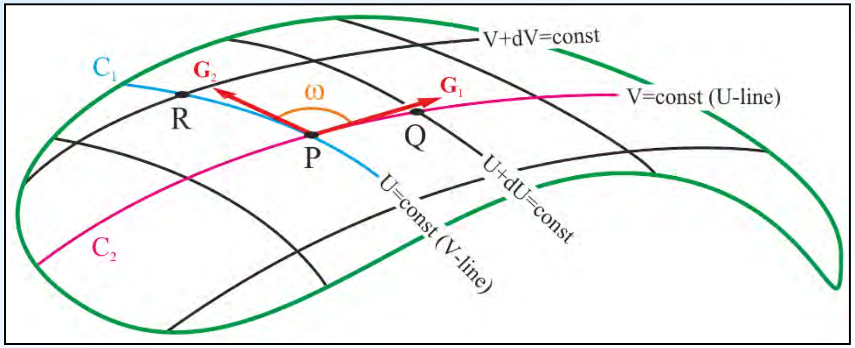
$$\equiv \cos\alpha \overset{G_{12}=0}{=} \sqrt{G_{22}}\,\frac{dV}{dS}$$



Angle ω between two parameter lines (dV,δU,δV) → (0,0,dV)

$$\cos\gamma = \frac{dUdVG_{12}}{\sqrt{G_{11}}\,dU\sqrt{G_{22}}\,dV} = \frac{G_{12}}{\sqrt{G_{11}G_{22}}} = \cos\omega$$

$$\tan\omega = \frac{\sqrt{G_{11}G_{22} - G_{12}^2}}{G_{12}} = \frac{\sqrt{\det \underline{G}}}{G_{12}}$$



→ Surface Elements

Surface Elements

Angle ω between two parameter lines U=const, V=const

$$dS^2 = G_{11}dU^2 + 2G_{12}dUdV + G_{22}dV^2 = \sum_{K=1}^{2}\sum_{L=1}^{2} G_{KL}dU^K dU^L = G_{KL}dU^K dU^L \qquad \text{"Einstein"}$$

Law of Cosines (planar trigonometry):

$$dS^2 = dS_U^2 + dS_V^2 - 2dS_U dS_V \cos(180° - \omega) = G_{11}dU^2 + G_{22}dV^2 + 2\sqrt{G_{11}G_{22}}\cos\omega\, dUdV$$

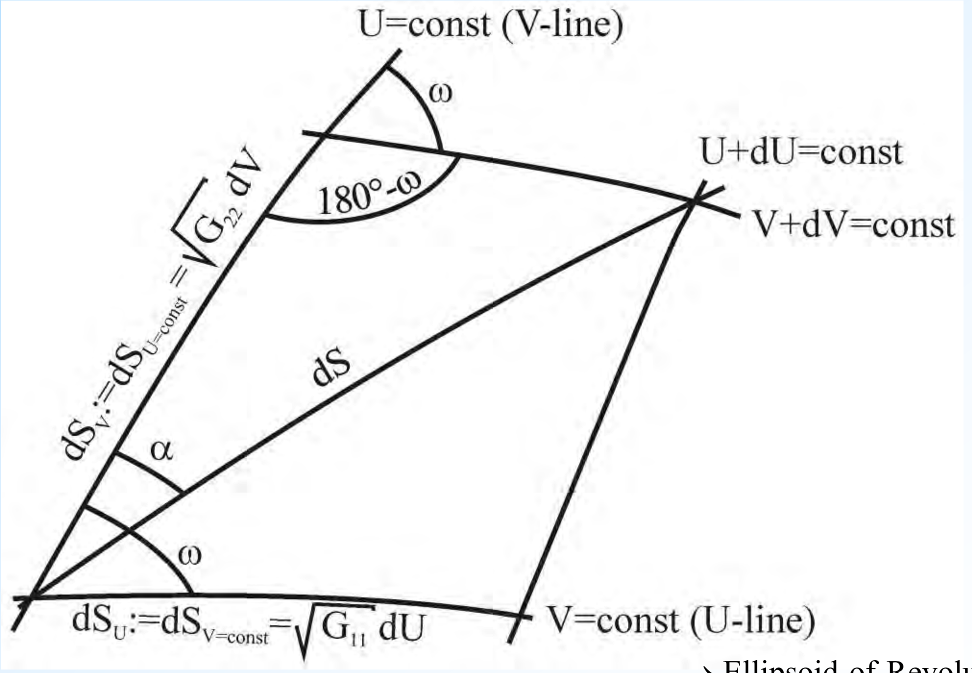
$$\Rightarrow \cos\omega = \frac{G_{12}}{\sqrt{G_{11}G_{22}}}$$

Interpretation: $G_{12}=0 \Leftrightarrow \omega=90°$

Orthogonal parameter lines !

$$\Rightarrow \cos\alpha = \sqrt{G_{22}}\frac{dV}{dS} \Leftrightarrow \frac{dV}{dS} = \frac{\cos\alpha}{\sqrt{G_{22}}}$$

$$\sin\alpha = \sqrt{G_{11}}\frac{dU}{dS} \Leftrightarrow \frac{dU}{dS} = \frac{\sin\alpha}{\sqrt{G_{11}}}$$

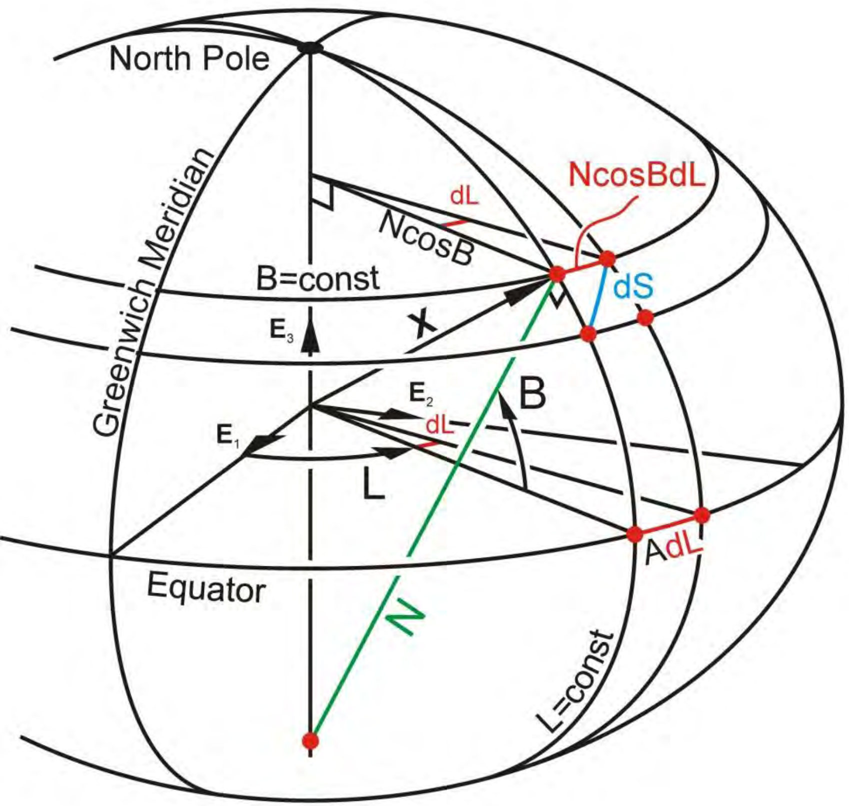


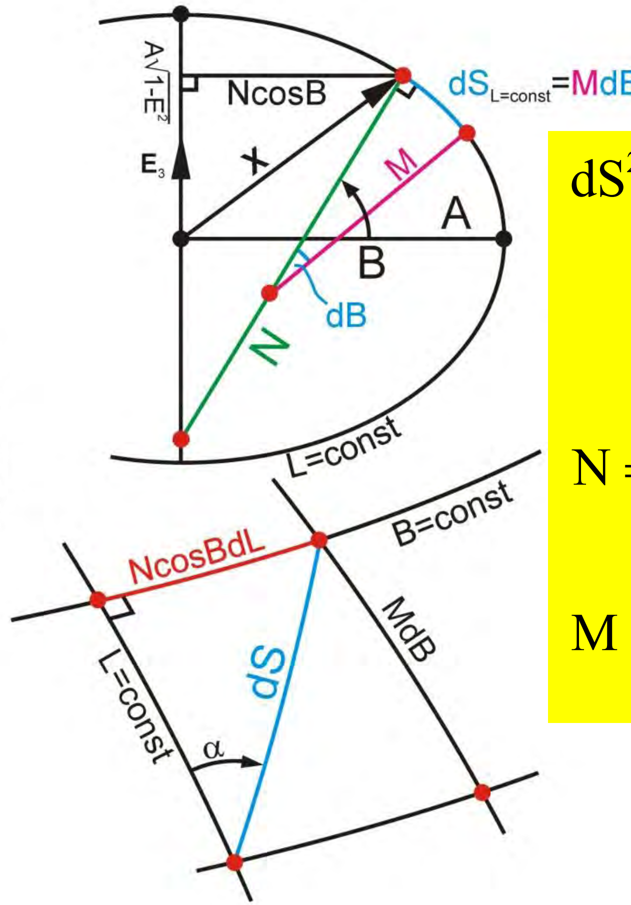
→ Ellipsoid-of-Revolution

Earth model "Ellipsoid-of-Revolution $\mathbb{E}^2_{A,E^2}$"

<u>Geometry:</u> Major semi axis A, squared first eccentricity E^2

<u>Parameter lines:</u> $U^1 \equiv U = L$ (ellipsoidal longitude), $U^2 \equiv V = B$ (ellipsoidal latitude)





$$dS^2 = N^2\cos^2 B\,dL^2 + \\ +M^2 dB^2 \\ = G_{11}dL^2 + G_{22}dB^2$$

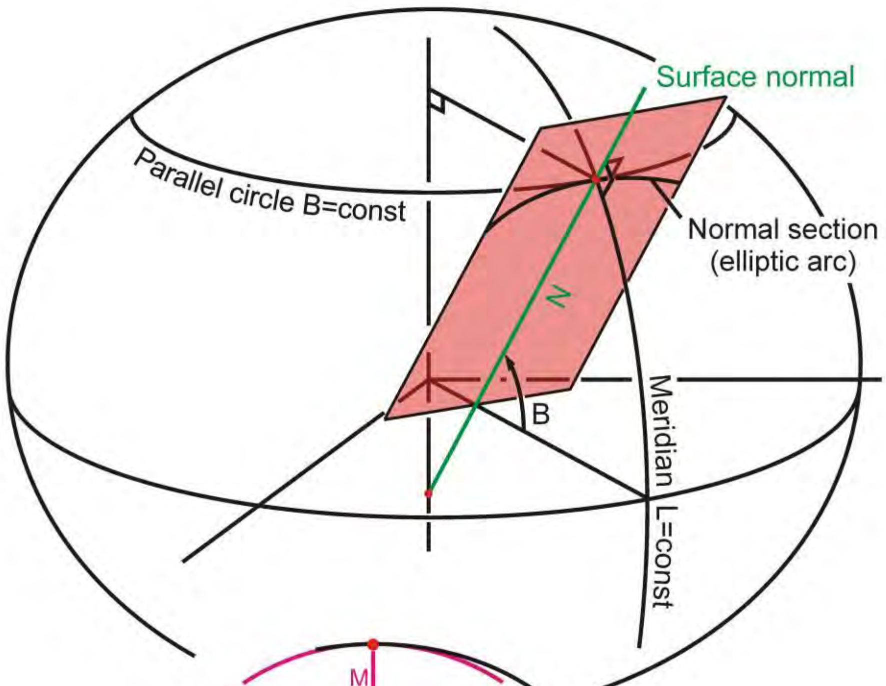
$$N = \frac{A}{\sqrt{1-E^2\sin^2 B}}$$

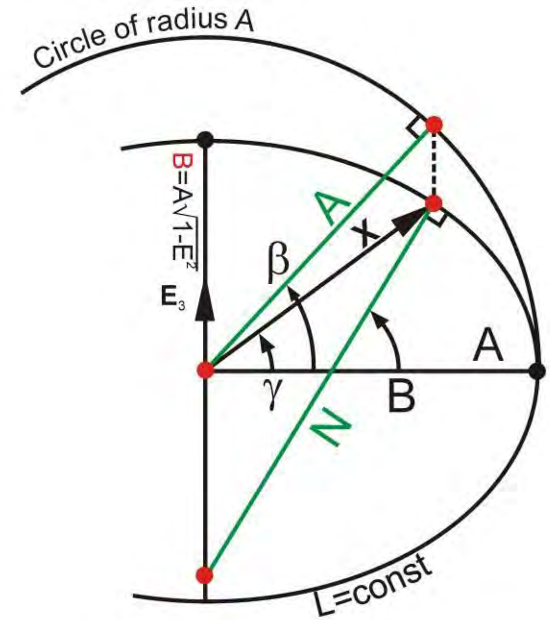
$$M = \frac{A(1-E^2)}{(1-E^2\sin^2 B)^{3/2}}$$

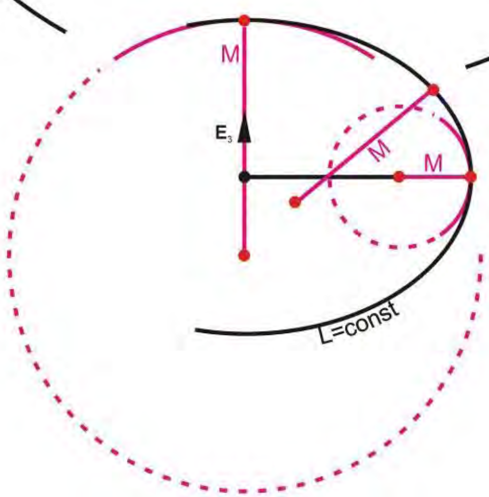
N, M... radii of curvature in the prime vertical, in the meridian

→ Ellipsoid-of-Revolution

Earth model "Ellipsoid-of-Revolution $\mathbb{E}^2_{A,E^2}$"







$\mathbf{X}=X\mathbf{E}_1+Y\mathbf{E}_2+Z\mathbf{E}_3$

$X=N\cos B\cos L = A\cos\beta\cos L$

$Y=N\cos B\sin L = A\cos\beta\sin L$

$Z=N(1-E^2)\sin B = B\sin\beta$

B... Minor semi axis

$\tan\gamma=(1-E^2)\tan B$

$\tan\beta=\sqrt{1-E^2}\tan B$

γ... geocentric latitude

β... reduced latitude

B… geodetic latitude

Conflict between B and B !

→ Map Projections: Procedure

Map Projections: Procedure

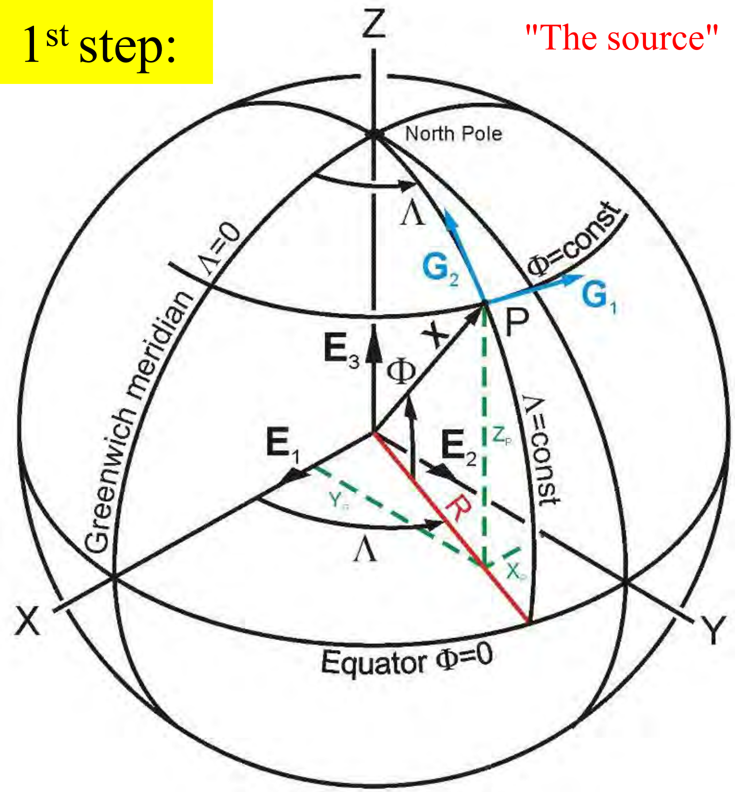
How to map one surface ("the source") onto another surface ("the chart"), which distortions occur and how can these be quantified, visualized and interpreted ?

1st step:	Define parameters $U^1 \equiv U$, $U^2 \equiv V$ used to describe a point on the "source", i.e. specify the position/radius vector (coordinates wrt to the base vectors)
2nd step:	Derive metric matrix $\underline{G}$ and arc element $dS^2 = G_{KL} dU^K dU^L$ for the source
3rd step:	Define parameters $u^1 \equiv u$, $u^2 \equiv v$ used to describe a point on the "chart", i.e. specify the position/radius vector (coordinates wrt to the base vectors)
4th step:	Derive metric matrix $\underline{g}$ and arc element $ds^2 = g_{k\ell} du^k du^\ell$ for the "chart"
5th step:	Define the mapping equations, i.e. the relation between source and chart parameters: $u = u(U,V)$, $v = v(U,V)$
6th step:	Quantify, visualize and interpret the distortion of the parameter lines (of the mapping in general): distortion analysis !

→ Map Projections: Example

Map Projections: Example Sphere $\rightarrow$ Plane

1st step:



$U^1 \equiv U = \Lambda$: spherical longitude, counted from Greenwich meridian, positive east

$U^2 \equiv V = \Phi$: spherical latitude, counted from the equator, positive north

$$\mathbf{X}_{\mathbb{S}^2_R} \equiv \mathbf{X} = R(\cos\Phi\cos\Lambda\mathbf{E}_1 + \cos\Phi\sin\Lambda\mathbf{E}_2 + \sin\Phi\mathbf{E}_3)$$

$$X = R\cos\Phi\cos\Lambda,\ Y = R\cos\Phi\sin\Lambda,\ Z = R\sin\Phi$$

capital letters for quantities of the source !

2nd step:

$$\underline{G} = [G_{KL}] = \begin{bmatrix} \langle \mathbf{G}_1, \mathbf{G}_1 \rangle & \langle \mathbf{G}_1, \mathbf{G}_2 \rangle \\ \langle \mathbf{G}_2, \mathbf{G}_1 \rangle & \langle \mathbf{G}_2, \mathbf{G}_2 \rangle \end{bmatrix}, \quad \mathbf{G}_1 = \frac{\partial \mathbf{X}}{\partial U^1} \equiv \frac{\partial \mathbf{X}}{\partial \Lambda}, \mathbf{G}_2 = \frac{\partial \mathbf{X}}{\partial U^2} \equiv \frac{\partial \mathbf{X}}{\partial \Phi}$$

$$= R^2 \begin{bmatrix} \cos^2\Phi & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \quad \begin{aligned} dS^2 &= G_{11} d\Lambda^2 + 2G_{12} d\Lambda d\Phi + G_{22} d\Phi^2 \\ &= R^2(\cos^2\Phi d\Lambda^2 + d\Phi^2) \end{aligned}$$

$\rightarrow$ Map Projections: Example

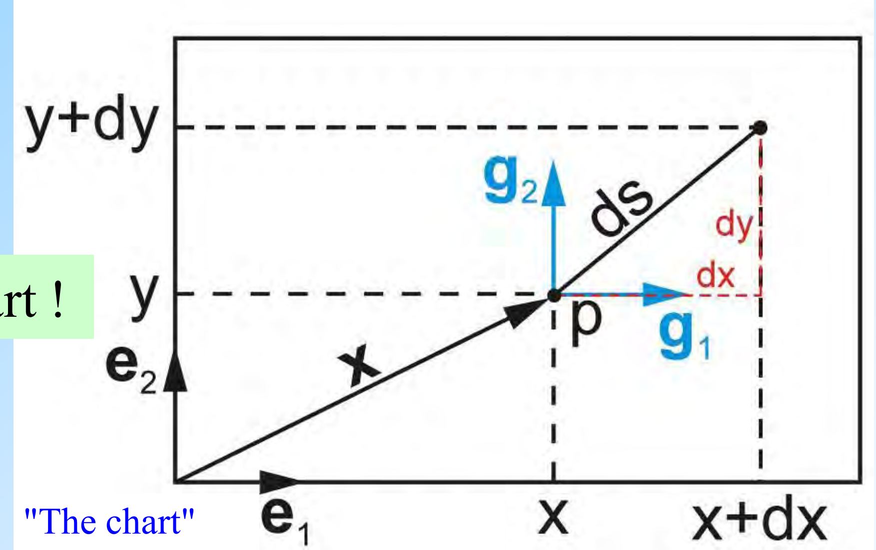
Map Projections: Example Sphere → Plane

3rd step: Cartesian coordinates

$u^1 \equiv u = x,\ u^2 \equiv v = y$

$$\mathbf{x}_{\mathbb{P}_O^2} \equiv \mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2$$

small letters for quantities of the chart !



"The chart"

4th step:

$$\underline{g} = [g_{k\ell}] = \begin{bmatrix} \langle \mathbf{g}_1, \mathbf{g}_1 \rangle & \langle \mathbf{g}_1, \mathbf{g}_2 \rangle \\ \langle \mathbf{g}_2, \mathbf{g}_1 \rangle & \langle \mathbf{g}_2, \mathbf{g}_2 \rangle \end{bmatrix}, \quad \mathbf{g}_1 = \frac{\partial \mathbf{x}}{\partial u^1} \equiv \frac{\partial \mathbf{x}}{\partial x}, \mathbf{g}_2 = \frac{\partial \mathbf{x}}{\partial u^2} \equiv \frac{\partial \mathbf{x}}{\partial y}$$

$$= \underline{I}_2 \Rightarrow ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2 = dx^2 + dy^2$$

5th step (Example): Isoparametric Mapping (Plate Carrée "Projection")

$$\underset{[m]}{x} = \underset{[m]}{R}\ \underset{[rad]}{\Lambda},\quad \underset{[m]}{y} = \underset{[m]}{R}\ \underset{[rad]}{\Phi}$$

Remark: In general, u=u(U,V), v=v(U,V) !

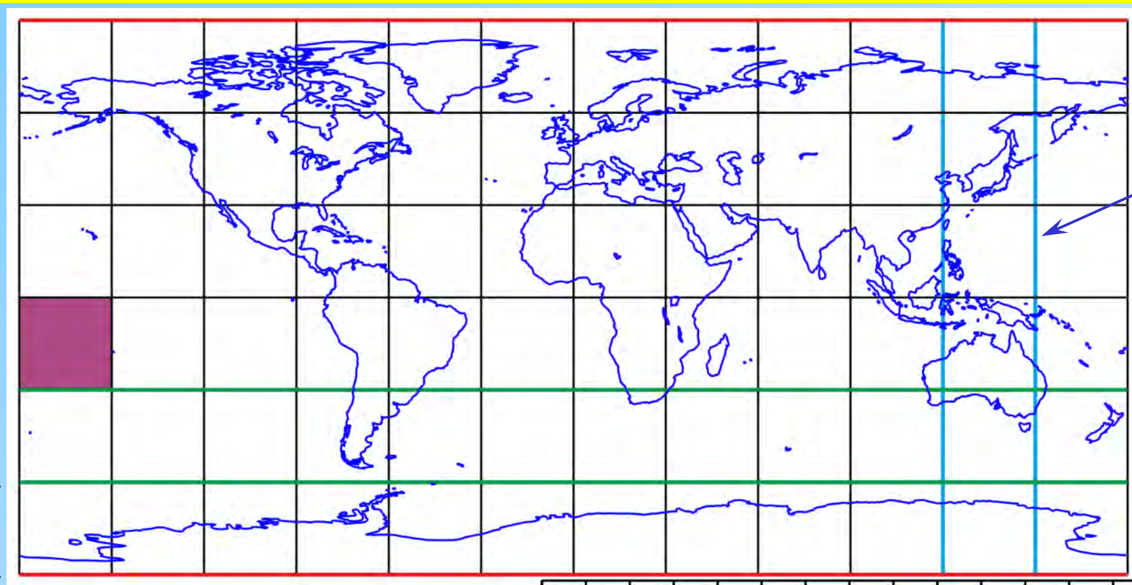
Attention: Scale factor

→ Map Projections: Example

Map Projections: Example Plate-Carrée "Projection"

$y = R\Phi$

Square



Poles: Straight lines !

Meridian images: Parallel straight lines !

Parallel circle images: Parallel straight lines !

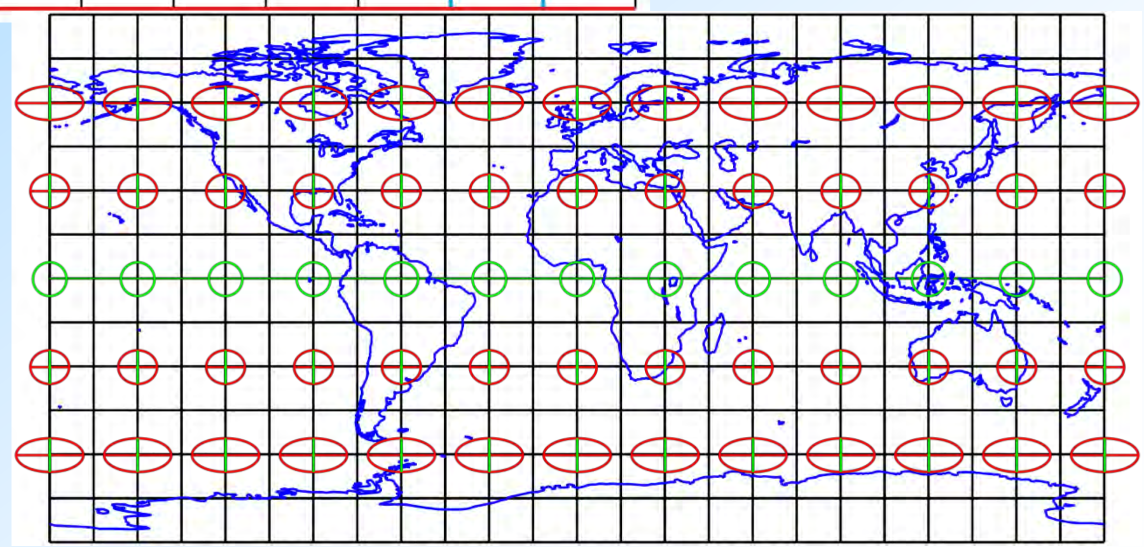
$\Delta\Phi = 30°$

$\Delta\Lambda = 30°$

$x = R\Lambda$

6th step:

Map with overlaid distortion ellipses (Tissot indicatrix)



See also:
http://www.gis.uni-stuttgart.de/lehre/campus-docs/PlateCarree.m

→ Distortion analysis

Map Projections: Distortion analysis

6^{th} step: Compare dS^2 and ds^2

$$dS^2 = R^2\cos^2\Phi d\Lambda^2 + R^2 d\Phi^2 \quad \leftrightarrow \quad ds^2 = dx^2 + dy^2$$

$$= G_{KL}\,dU^K dU^L \qquad\qquad = g_{k\ell}\,du^k du^\ell \qquad K, L, k, \ell = 1, 2$$

Comparison forbidden because $\underline{G}$, $\underline{g}$, dU's and du's have different magnitudes and/or physical units: $d\Lambda, d\Phi$ [rad, deg (°)], dx, dy [m]
→ Transform du^k, du^ℓ to dU^K, dU^L using the mapping equations

$$\begin{aligned} ds^2 &= g_{k\ell}\,du^k du^\ell \\ &= g_{k\ell} \quad ? \quad dU^K dU^L \\ &= g_{k\ell}\frac{\partial u^k}{\partial U^K}\frac{\partial u^\ell}{\partial U^L}dU^K dU^L \\ &= \qquad c_{KL} \qquad dU^K dU^L \end{aligned}$$

Comparison permitted using G_{KL} and c_{KL}, only !

$\underline{C}=[c_{KL}]$: 2×2 Cauchy-Green deformation tensor

→ Distortion analysis using matrices

Map Projections: Distortion analysis using matrices

6th step: Compare dS^2 and ds^2

$$\underline{dU} := \begin{bmatrix} dU^1 \\ dU^2 \end{bmatrix} = \begin{bmatrix} dU \\ dV \end{bmatrix} = \begin{bmatrix} d\Lambda \\ d\Phi \end{bmatrix}, \quad d\underline{u} = \begin{bmatrix} du^1 \\ du^2 \end{bmatrix} = \begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$d\underline{u} = \left[\frac{\partial u}{\partial U} dU + \frac{\partial u}{\partial V} dV, \frac{\partial v}{\partial U} dU + \frac{\partial v}{\partial V} dV \right]^T = \left\{ \begin{bmatrix} \frac{\partial u}{\partial U} & \frac{\partial u}{\partial V} \\ \frac{\partial v}{\partial U} & \frac{\partial v}{\partial V} \end{bmatrix} \begin{bmatrix} dU \\ dV \end{bmatrix} \right\} = \underline{J} d\underline{U}$$

$$dS^2 = R^2 \cos^2 \Phi d\Lambda^2 + R^2 d\Phi^2 \quad \leftrightarrow \quad ds^2 = dx^2 + dy^2$$

$$= d\underline{U}^T \underline{G} d\underline{U} \qquad\qquad = d\underline{u}^T \underline{g} d\underline{u}$$

$$= d\underline{U}^T \underline{J}^T \underline{g} \underline{J} \quad d\underline{U}$$

$$= d\underline{U}^T \quad \underline{C} \quad d\underline{U}$$

$\underline{C} = \underline{J}^T \underline{g}\ \underline{J}$ $\quad$ $\underline{J}$: Jacobian (matrix): change of (u,v) by change of (U,V) !

→ Properties of $\underline{G}$, $\underline{g}$ and $\underline{C}$

Properties of $\underline{G}$, $\underline{g}$ and $\underline{C}$

source only

(i) $\underline{G}$ is 2×2, pos. definite ⇔ eigenvalues > 0, geometrically representing an ellipse, not necessarily aligned with the coordinate axes. $\underline{G}$ depends on the parameters [U,V] used to describe the "source". A change of [U,V] → [U',V'] results in a change of $\underline{G} \rightarrow \underline{G}' = \underline{J}^T \underline{G}\ \underline{J}$. $\underline{J} = \underline{J}(U,V,U',V')$.

chart only

(ii) $\underline{g}$ is 2×2, pos. definite ⇔ eigenvalues > 0, geometrically representing an ellipse, not necessarily aligned with the coordinate axes. $\underline{g}$ depends on the parameters [u,v] used to describe the "chart". A change of [u,v] → [u',v'] results in a change of $\underline{g} \rightarrow \underline{g}' = \underline{j}^T \underline{g}\ \underline{j}$, $\underline{j} = \underline{j}(u,v,u',v')$.

mapping

(iii) $\underline{C} = \underline{J}^T \underline{g}\ \underline{J}$ is 2×2, pos. definite ⇔ eigenvalues > 0, geometrically representing an ellipse, not necessarily aligned with the coordinate axes. $\underline{C}$ depends on $\underline{g}$ and $\underline{J} = \underline{J}(u,v,U,V)$, i.e. on the parameter sets of both surfaces. In general, $\underline{C}$ is a full matrix. But $C_{12} = 0$ ⇔

- $g_{12} = 0 \wedge u^1 = u^1(U^1) \wedge u^2 = u^2(U^2)$ or
- $g_{12} = 0 \wedge u^1 = u^1(U^2) \wedge u^2 = u^2(U^1)$

→ Structure of $\underline{G}$ and $\underline{C}$

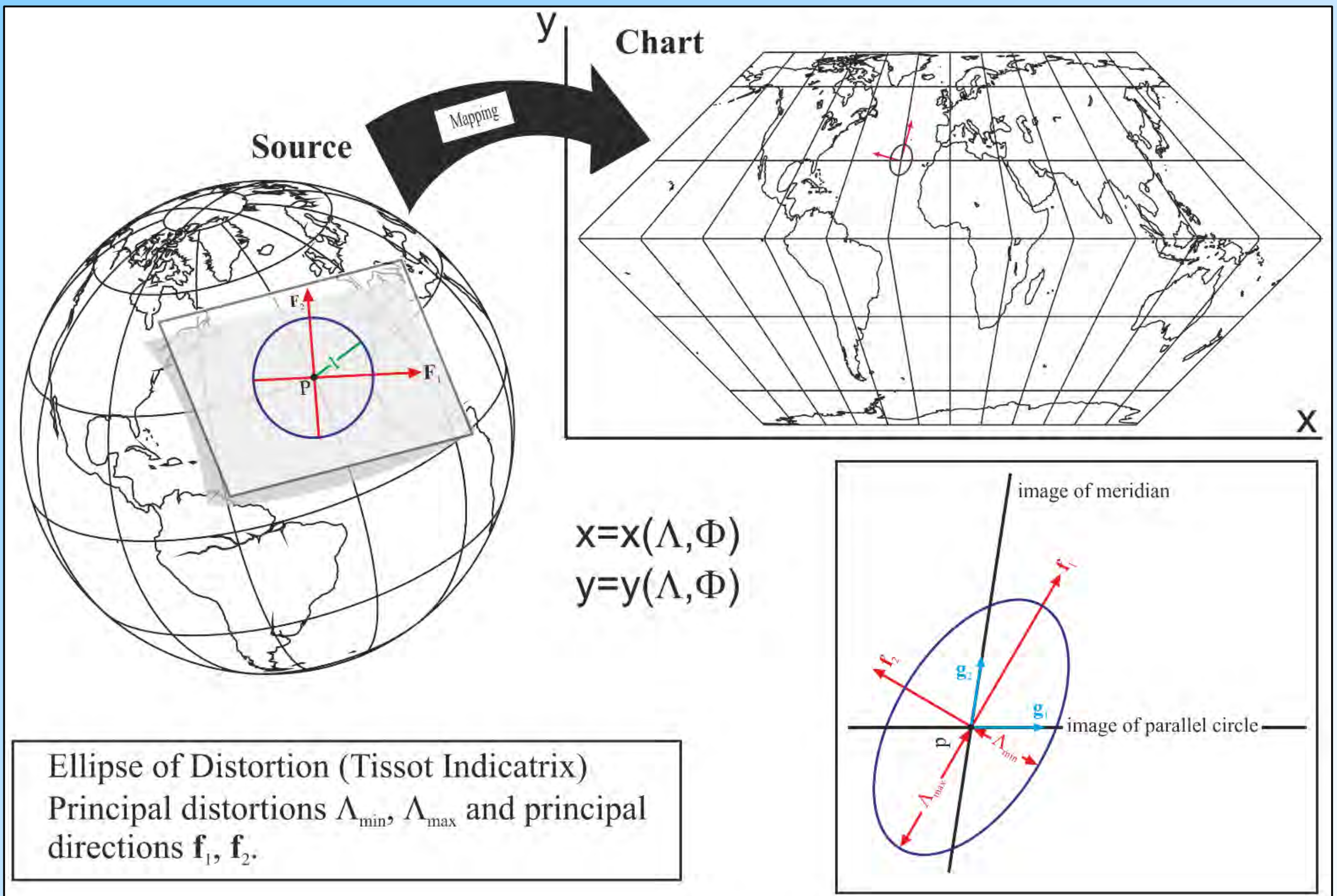
"Natural" choices of U,V,u,v and structure of $\underline{G}$ and

Ellipsoid $\mathbb{E}^2_{A,E^2}$	Sphere $\mathbb{S}^2_R / \mathbb{S}^2_r$	Plane $\mathbb{P}^2_O$	
"Source"	"Source"/"Chart"	"Chart"	
Ellipsoidal Longitude, Latitude	Spherical Longitude, Latitude	Cartesian Coordinates	Polar Coordinates
U=L, V=B	U=Λ,V=Φ / u=λ, v=φ	u=x, v=y	u=α, v=r
$\underline{G} = \begin{bmatrix} N^2\cos^2 B & 0 \\ 0 & M^2 \end{bmatrix}$ $N = \frac{A}{\sqrt{1-E^2\sin^2 B}}$ $M = \frac{A(1-E^2)}{(1-E^2\sin^2 B)^{3/2}}$	"Source" $\underline{G} = \begin{bmatrix} R^2\cos^2\Phi & 0 \\ 0 & R^2 \end{bmatrix}$ "Chart" $\underline{g} = \begin{bmatrix} r^2\cos^2\varphi & 0 \\ 0 & r^2 \end{bmatrix}$	$\underline{g} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\underline{g} = \begin{bmatrix} r^2 & 0 \\ 0 & 1 \end{bmatrix}$

→ Tissot concept – Tissot indicatrix

Tissot concept – Tissot indicatrix

Infinitesimal circle at point P on the source will suffer a deformation due to the mapping process (affine transformation) and will be mapped onto an infinitesimal ellipse in point p, the image of P on the chart. Principal axes $\Lambda_{min}, \Lambda_{max}$ of the ellipse (principal distortions) will not be aligned with the lines u=const, v=const, in general: the directions $\mathbf{f}_1$, $\mathbf{f}_2$ will coincide with those of $\mathbf{g}_1, \mathbf{g}_2$ as the images of $\mathbf{G}_1$, $\mathbf{G}_2$ in special cases, only !



Ellipse of Distortion (Tissot Indicatrix)
Principal distortions Λ_{min}, Λ_{max} and principal directions $\mathbf{f}_1$, $\mathbf{f}_2$.

Size and orientation of the indicatrix is a measure for map distortion !

→ Comparing G and C

Comparing $\underline{G}$ and $\underline{C}$

Tissot: Search extrema of stretch (length distortion) Λ^2 (!) in $ds^2/dS^2 = \Lambda^2(d\underline{U})$

$$\frac{d}{d\underline{U}}\Lambda^2 = \frac{d}{d\underline{U}}\frac{ds^2}{dS^2} = \frac{d}{d\underline{U}}\left(\frac{d\underline{U}^T\underline{C}\,d\underline{U}}{d\underline{U}^T\underline{G}\,d\underline{U}}\right) = \frac{2\underline{C}\,d\underline{U}dS^2 - 2ds^2\underline{G}\,d\underline{U}}{dS^4} = 0 \Rightarrow (\underline{C}-\Lambda^2\underline{G})d\underline{U} = 0$$

$$(\underline{C}-\Lambda_i^2\underline{G})\underline{F}_i = \underline{0}\ ,\quad i = 1,2$$

General Eigenvalue-Eigenvector Problem

$$(\underline{C}-\Lambda_i^2\,\underline{I}\,)\underline{F}_i = \underline{0}\ ,\quad i = 1,2$$

(Special) Eigenvalue-Eigenvector Problem $\underline{G}=\underline{I}_2$

$$\det(\underline{C}-\Lambda^2\underline{G}) = 0 \quad \Rightarrow$$

Characteristic Polynomial $\Rightarrow \Lambda_1^2, \Lambda_2^2$

$$(\underline{C}-\Lambda_1^2\underline{G})\underline{F}_1 = \underline{0}\ ,\quad (\underline{C}-\Lambda_2^2\underline{G})\underline{F}_2 = \underline{0}$$

For two 2×2 matrices $\underline{C}$ and $\underline{G}$ compute a diagonal matrix $\underline{D} = \mathrm{diag}(\Lambda_1^2, \Lambda_2^2)$ and a full 2×2 matrix $\underline{F}$ whose columns $\underline{F}=[\underline{F}_1, \underline{F}_2]$ are the corresponding "eigenvectors" so that $\underline{C}\ \underline{F}=\underline{G}\ \underline{F}\ \underline{D}$. This problem is also called "simultaneous diagonalization of two quadratic forms, Q_1 ($\underline{G}$) and Q_2 ($\underline{C}$)" and means "compare two ellipses G and C, which have different axis lengths and different axis orientations, in general". Matrix $\underline{F}$ is not orthonormal, but normalized in such a way that

$\underline{F}^T\underline{G}\underline{F} = \underline{I}_2$ "Circle on the source"

$\underline{F}^T\underline{C}\underline{F} = \underline{D}$ "Ellipse on the chart"

Orthonormality wrt $\underline{G}$, the metric of the source ~ $\underline{G}$-orthonormality of $\underline{F}$

→ Simultaneous diagonalization

Simultaneous Diagonalization of 2 quadratic forms

Semi major and minor axis, $\Lambda_1 = +\sqrt{\Lambda_1^2}$, $\Lambda_2 = +\sqrt{\Lambda_2^2}$ of Tissot distortion ellipse (indicatrix) are analytically specified through

$$\Lambda^2_{\min,\max} : \Lambda^2_{1,2} = \frac{1}{2}\left\{ \operatorname{tr}(\underline{C}\underline{G}^{-1}) \pm \sqrt{[\operatorname{tr}(\underline{C}\underline{G}^{-1})]^2 - 4\det(\underline{C}\underline{G}^{-1})} \right\} \quad .$$

Coordinate arrays $\underline{F}_1$, $\underline{F}_2$ of eigenvectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are given by the formulas

$$\underline{F}_1 = \frac{1}{\sqrt{(C_{22} - \Lambda_1^2 G_{22})^2 G_{11} - 2(C_{12} - \Lambda_1^2 G_{12})(C_{22} - \Lambda_1^2 G_{22}) G_{12} + (C_{12} - \Lambda_1^2 G_{12})^2 G_{22}}} \begin{bmatrix} C_{22} - \Lambda_1^2 G_{22} \\ -(C_{12} - \Lambda_1^2 G_{12}) \end{bmatrix}$$

$$\underline{F}_2 = \frac{1}{\sqrt{(C_{11} - \Lambda_2^2 G_{11})^2 G_{22} - 2(C_{12} - \Lambda_2^2 G_{12})(C_{11} - \Lambda_2^2 G_{11}) G_{12} + (C_{12} - \Lambda_2^2 G_{12})^2 G_{11}}} \begin{bmatrix} -(C_{12} - \Lambda_2^2 G_{12}) \\ C_{11} - \Lambda_2^2 G_{11} \end{bmatrix}.$$

$\underline{F}_1$ and $\underline{F}_2$ as representatives of $\mathbf{F}_1$ and $\mathbf{F}_2$ refer to the Gauß tangent vectors $\mathbf{G}_1$, $\mathbf{G}_2$ of the source. In order to display the distortion ellipses in the map, $\mathbf{F}_1$ and $\mathbf{F}_2$ must be transformed into $\mathbf{f}_1$ and $\mathbf{f}_2$ (represented by $\underline{f}_1$ and $\underline{f}_2$), which refer to the Gauß tangent vectors $\mathbf{g}_1$, $\mathbf{g}_2$ of the map.

$\underline{f} = [\underline{f}_1, \underline{f}_2] = \underline{J}\,\underline{F}$ (orthogonal $\underline{f}_1 \perp \underline{f}_2$, but not normalized)

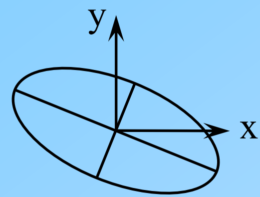
$\underline{f} = [\underline{f}_1, \underline{f}_2] = \underline{J}\,\underline{F}\,\underline{D}^{-1/2}$ (orthonormal: $\underline{f}^T \underline{f} = \underline{I}_2$)

→ Geometric Interpretation

Geometric Interpretation (same manifold !)

Quadratic form Q_1 given by $\underline{G}$

(not aligned with coordinate axes)

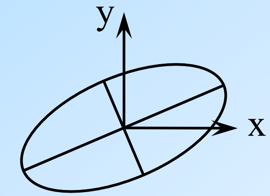


Comparison not permitted

(different axes orientation)

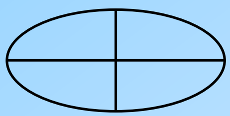
Quadratic form Q_2 given by $\underline{C}$

(not aligned with coordinate axes)



Quadratic form Q_1' (Q_1 rotated by $\underline{F}$)

(aligned with coordinate axes)

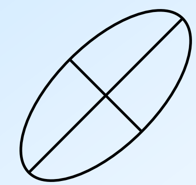


Comparison not permitted

(different axes orientation)

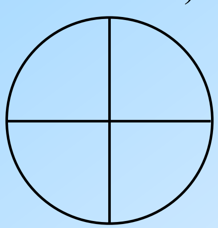
Quadratic form Q_2' (Q_2 rotated by $\underline{F}$)

(still not aligned with coordinate axes)

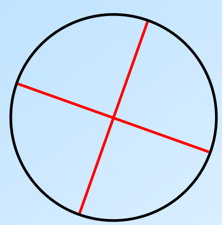


Quadratic form Q_1'' (Q_1' scaled by $\underline{S}$)

(circle, aligned with coordinate axes)

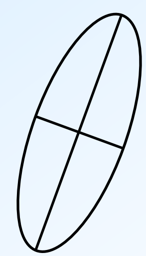


(Q_1'', aligned with Q_2'')



Quadratic form Q_2'' (Q_2' scaled by $\underline{S}$)

(not aligned with coordinate axes, changed orientation !)



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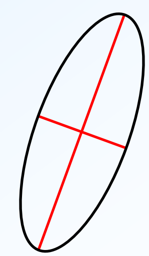


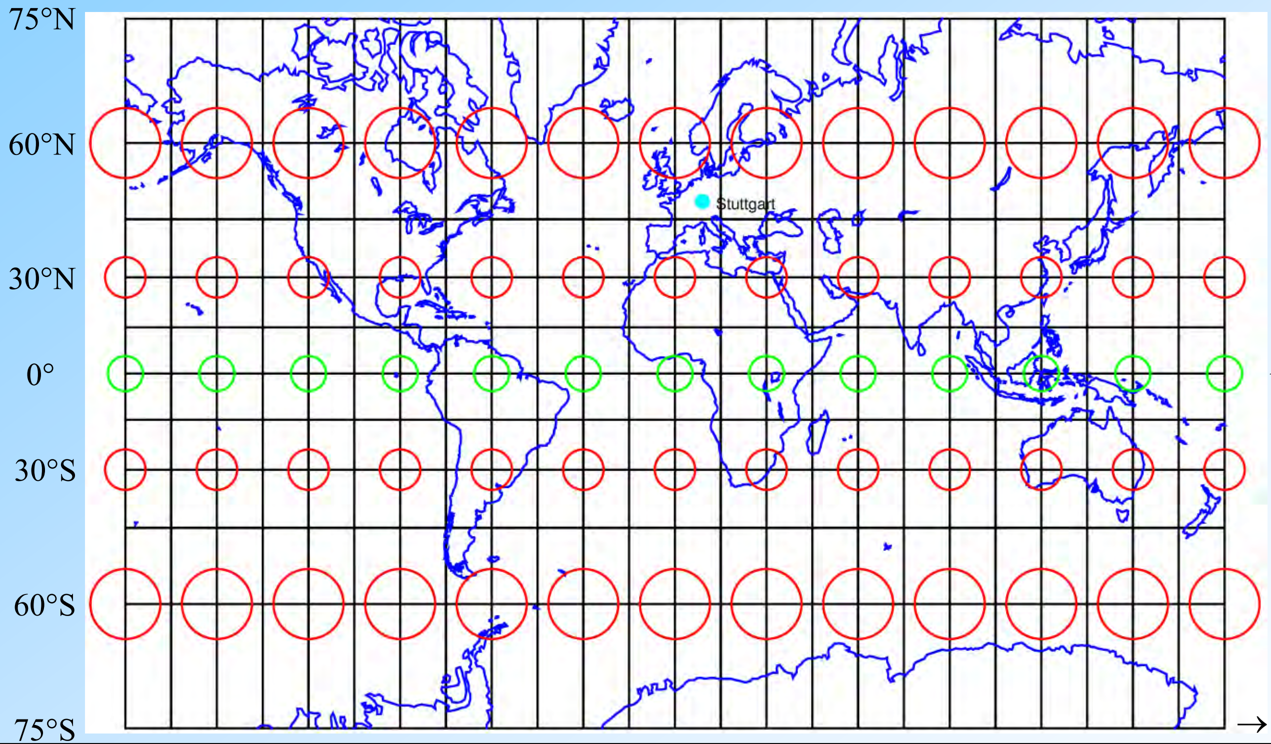
Comparison permitted !

(same axes orientation)

$\rightarrow$ Interpretation of $\Lambda_{1,2}$

Interpretation of $\Lambda_{1,2}$

$\Lambda_1=\Lambda_2 \Leftrightarrow [\mathrm{tr}(\underline{C}\ \underline{G}^{-1})]^2 = 4 \det(\underline{C}\ \underline{G}^{-1})$: "Angle preserving mapping", "distortion circles", "distortions independent on the direction in the map", "isotropic distortions", "***Conformal*** map". E.g. Gauß-Krüger maps, UTM maps, Mercator projection. Special case: $\Lambda_1=\Lambda_2 = 1 \Leftrightarrow$ "Isometry". No distortions whatsoever, but globally not possible.



← $\Lambda_1=\Lambda_2 = 1$

Distortion increasing with distance from equator, infinite at the poles

→ Intermezzo: Mercator postage stamp 2012

Intermezzo: Mercator postage stamp 2012

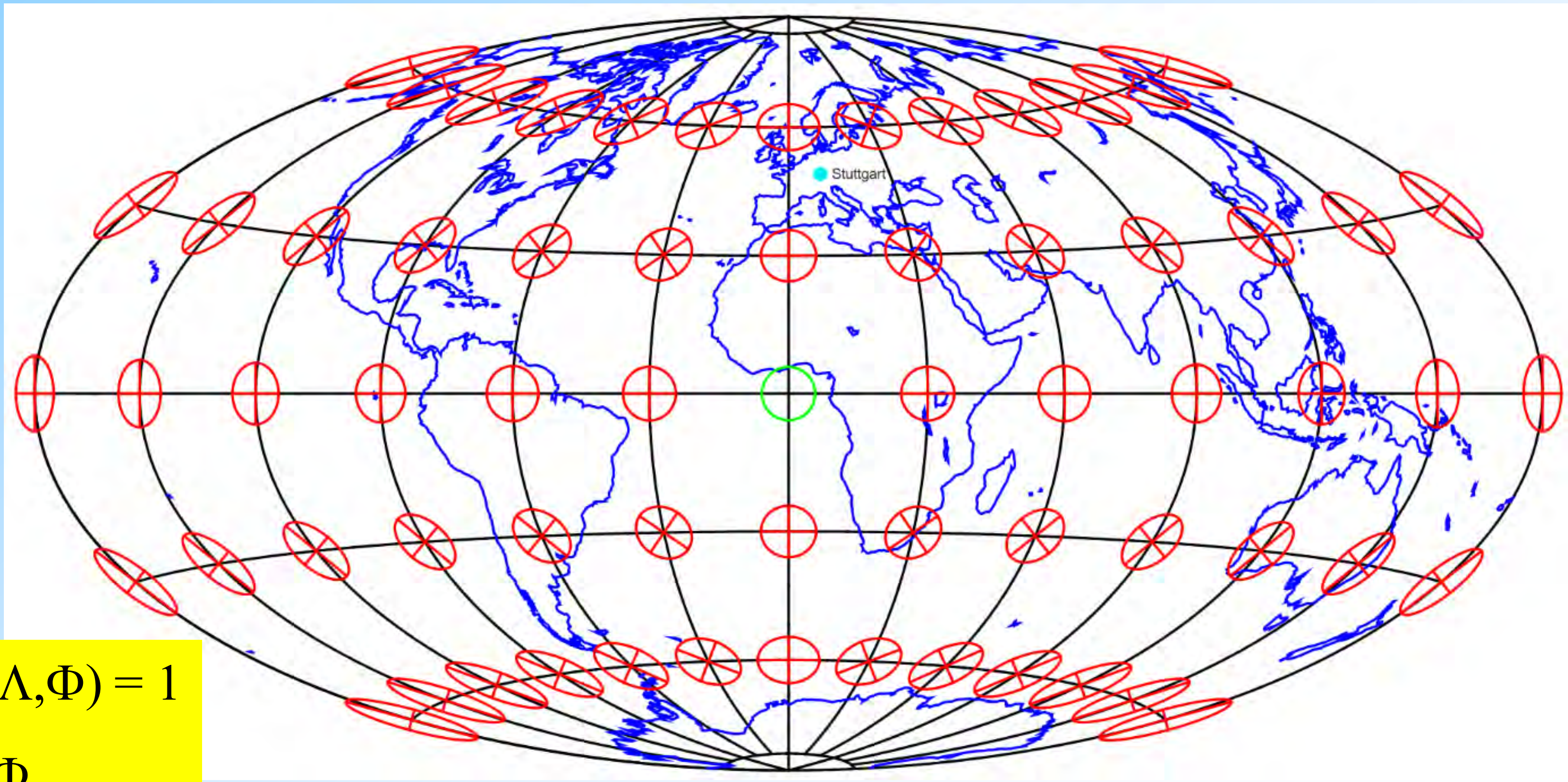
→ Interpretation of $\Lambda_{1,2}$

Interpretation of $\Lambda_{1,2}$

$$\Lambda_1\Lambda_2 = 1 \Leftrightarrow \det(\underline{C}\,\underline{G}^{-1}) = 1 \Leftrightarrow \det \underline{C} \det \underline{G}^{-1} = 1 \Leftrightarrow \det \underline{J}^{T} \underline{g} \underline{J} / \det \underline{G} = 1 \Leftrightarrow$$

$$\left(\det \underline{J}\right)^2 \det \underline{g} / \det \underline{G} \overset{\underline{g}=\underline{I}_2}{=} 1 \Leftrightarrow \det \underline{J} = \sqrt{\det \underline{G}}$$

"Area preserving mapping", "no areal distortion", e.g. Hammer Projection



$\Lambda_1(\Lambda,\Phi)\ \Lambda_2(\Lambda,\Phi) = 1$

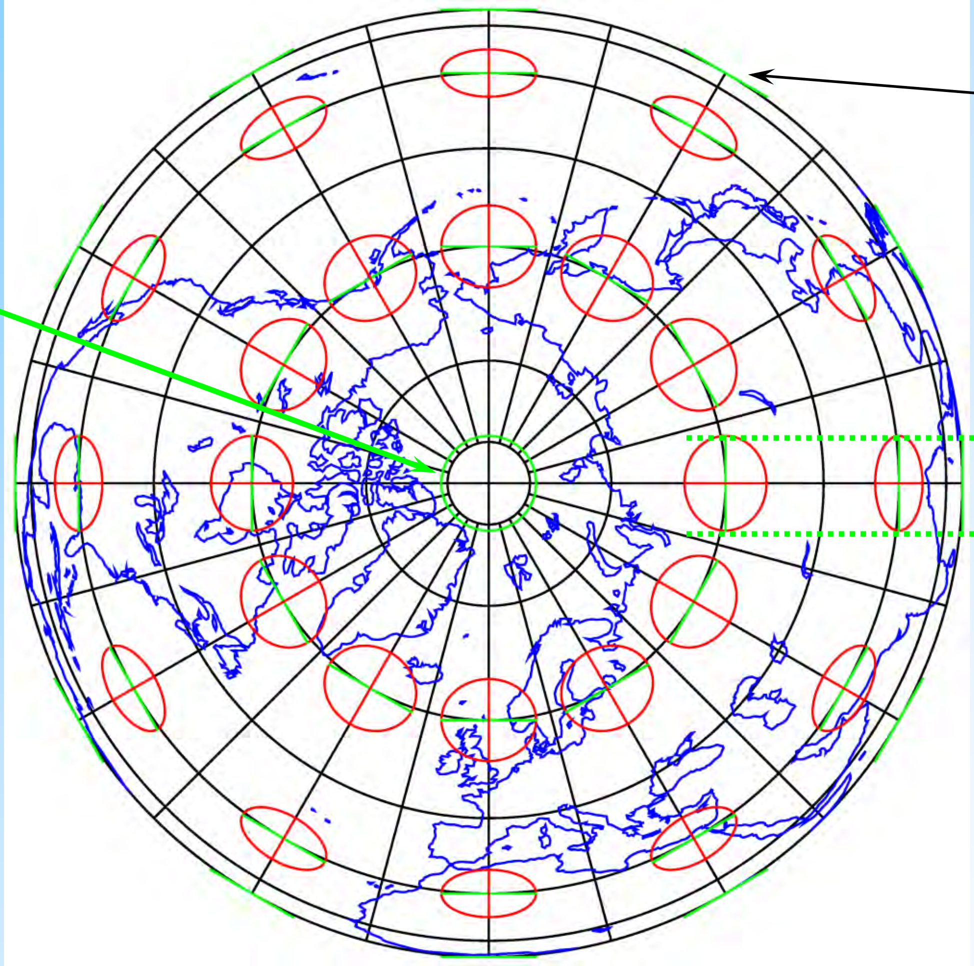
$\forall\ \Lambda,\Phi$

→ Interpretation of $\Lambda_{1,2}$

Interpretation of $\Lambda_{1,2}$

$\Lambda_1=1\ \forall\Lambda_2$ or $\Lambda_2=1\ \forall\Lambda_1$: "Distance preserving mapping" (globally possible), e.g.
Orthogonal Projection (Parallel Projection)

$\Lambda_1=\Lambda_2 = 1$:
no distortion
(isometry)



Extremely distorted meridians:
$\Lambda_1= 1,\ \Lambda_2= 0$!

$\Lambda_1= 1\ \forall\Lambda_2$:
Undistorted (length preserved) parallel circles

→ Special computation formulas

Special computation formulas

a) If both $\underline{G}$ and $\underline{C}$ are diagonal ($G_{12}=G_{21}=C_{12}=C_{21}=0$)

$$\Lambda_1 = \sqrt{\frac{C_{11}}{G_{11}}},\ \Lambda_2 = \sqrt{\frac{C_{22}}{G_{22}}},\ \underline{F}_1 = \frac{1}{\sqrt{G_{11}}}\begin{bmatrix}1\\0\end{bmatrix},\ \underline{F}_2 = \frac{1}{\sqrt{G_{22}}}\begin{bmatrix}0\\1\end{bmatrix}$$

Minimum and maximum distortions are in the directions of the parameter lines, which is not generally the case, see e.g. Hammer Projection.

b) $\Lambda_1 = \sqrt{C_{11}/G_{11}},\ \Lambda_2 = \sqrt{C_{22}/G_{22}}$ are <u>always</u> the distortions in the direction of the parameter lines. But, they are the extremal distortions only if both $\underline{G}$ and $\underline{C}$ are diagonal matrices.

$\Lambda_1 = \sqrt{C_{11}/G_{11}}$: Distortions of the parallel circles Φ=const

$\Lambda_2 = \sqrt{C_{22}/G_{22}}$: Distortions of the meridians Λ=const

How to remember this rule ?

c) Maximal angular distortion ω is given by the equation $\omega = 2\arcsin\dfrac{\Lambda_1 - \Lambda_2}{\Lambda_1 + \Lambda_2}$

→ Example: Plate Carrée

Example Plate-Carrée "Projection"

$$\Lambda_1 = \sqrt{\frac{C_{11}}{G_{11}}} = \frac{1}{\cos\Phi}$$

Distortions of the parallels grow with distance from equator; no distortion of the equator (length preserved equator).

$$\Lambda_2 = \sqrt{\frac{C_{22}}{G_{22}}} = 1$$

No distortions of meridians (length preserved meridians)

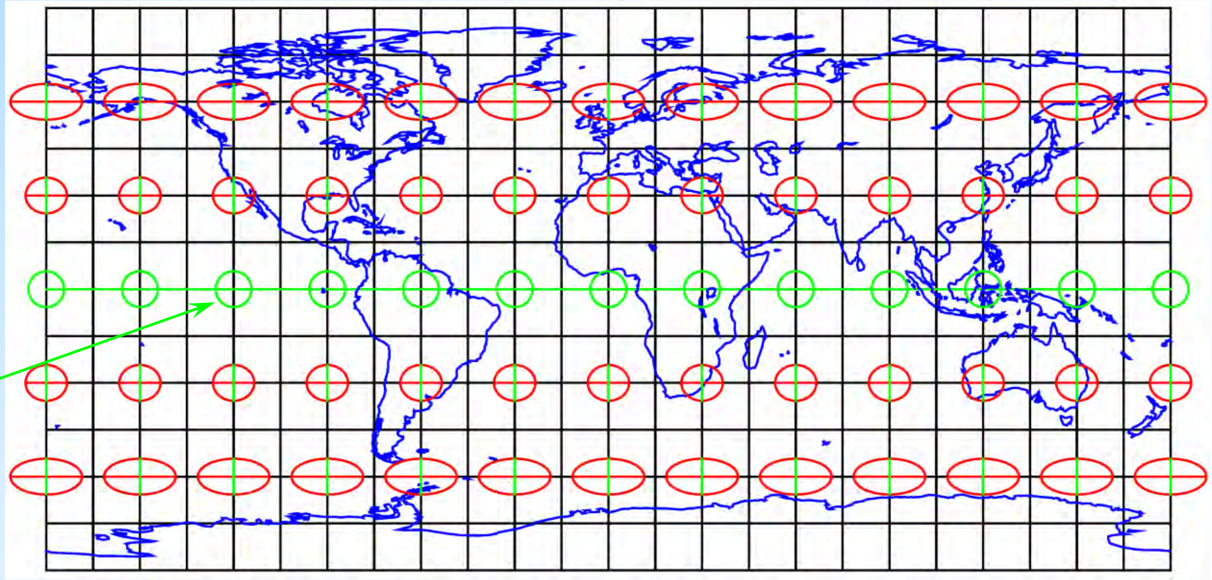
$$\underline{f} = \left[\underline{f}_1 \quad \underline{f}_2\right] = \underline{J}\underline{F}\underline{D}^{-1/2} = \underline{I}_2$$

$$\omega = 2\arcsin\frac{1-\cos\Phi}{1+\cos\Phi}$$

$$\omega\big|_{\Phi=90°} = \max., \quad \omega\big|_{\Phi=0°} = 0$$

Conformality on the equator !

Tissot circles !



→ General overview

General overview

Reference surface ("Source")
(Geometric Earth model: Sphere $\mathbb{S}^2_R$, Ellipsoid-of-Revolution $\mathbb{E}^2_{A,E^2}$)

[Global mappings: $\mathbb{S}^2_R$, regional, local, legal mappings: $\mathbb{E}^2_{A,E^2}$]

Mapping surface ("Chart")
(Ellipsoid-of-Revolution $\mathbb{E}^2_{a,e^2}$, Sphere $\mathbb{S}^2_r$, Plane $\mathbb{P}^2_O$)

- Fundamental properties (2)
- Special properties (1)
- Aspect of map projection
- General mapping equations (extract)

→ Special properties

Special properties (apply to all kinds of surfaces)

- Conformality (angle preserving): $\Lambda_1 = \Lambda_2 \Leftrightarrow \omega = 0 \Leftrightarrow$ no angular distortion $\Leftrightarrow$ circular Tissot distortion ellipses $\Leftrightarrow$ distortion independent on the direction (possible for entire map, example: Mercator projection)
- Equivalence (area preserving, equiareal, equal area): $\Lambda_1 \Lambda_2 = 1 \Leftrightarrow \Lambda_1 = 1/\Lambda_2 \Leftrightarrow$ no areal distortion (possible for entire map, example: Hammer projection)
- Equidistance 1 (length preserving): $\Lambda_1 = 1\ \forall \Lambda_2$ (possible for entire map, example: Orthogonal projection)
- Equidistance 2 (length preserving): $\Lambda_2 = 1\ \forall \Lambda_1$ (possible for entire map, example: Plate-Carrée projection)
- Isometry: $\Lambda_1 = \Lambda_2 = 1$ (not possible for entire map, only possible at selected points/lines)

→ Local and global distortion measures

Local and global (qualitative) distortions measures

Local measures of departure from isometry, conformality and equivalence

- Isometry: $\varepsilon_A^2 := \frac{1}{2}[(\Lambda_1 - 1)^2 + (\Lambda_2 - 1)^2]$ Airy criterion (G.B. Airy 1861)

  $\varepsilon_{AK}^2 := \frac{1}{2}[(\ln\Lambda_1)^2 + (\ln\Lambda_2)^2]$ Airy-Kavraiskiy criterion (V.V. Kavraiskiy 1958)
- Conformality: $\varepsilon_{conf}^2 := (\Lambda_1 - \Lambda_2)^2$
- Equivalence $\varepsilon_{areal}^2 := (\Lambda_1\Lambda_2 - 1)^2$

Global (domain) measures of ... (Integration over the area F to be mapped)

$$I_{\{A,AK,conf,areal\}} = \frac{1}{F}\int_F \varepsilon^2_{\{A,AK,conf,areal\}} dF$$ e.g.

$$I_{conf} = \frac{1}{F}\int_{\Lambda_{West}}^{\Lambda_{East}}\int_{\Phi_{South}}^{\Phi_{North}} \varepsilon^2_{conf}[\Lambda_1(\Lambda,\Phi),\Lambda_2(\Lambda,\Phi)]dF$$

$I_{\{A,AK,conf,areal\}}$ is also called distortion energy

with $F = \int_{\Lambda_{West}}^{\Lambda_{East}}\int_{\Phi_{South}}^{\Phi_{North}} dF, \quad dF = \sqrt{\det \underline{G}}\,d\Phi d\Lambda$

→ Fundamental properties

Fundamental properties I (apply to the plane as chart)

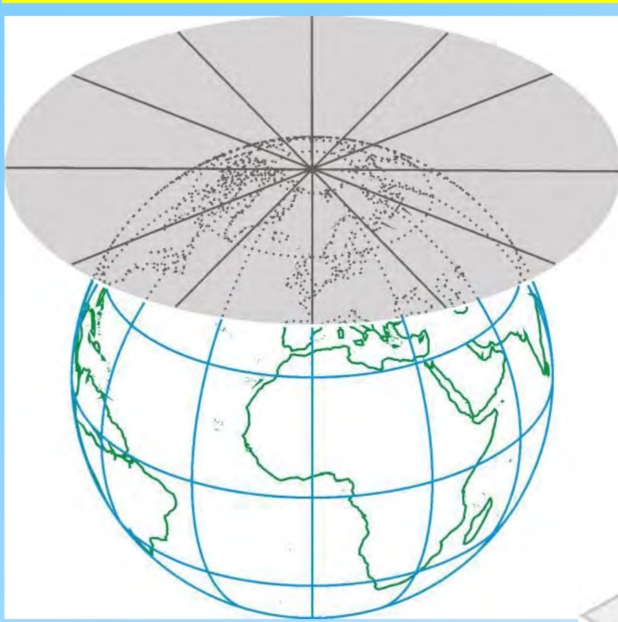
Azimuthal projections: A tangential plane is fixed to the sphere/ellipsoid so that the surface normal coincides with the rotations axis of the Earth model. The point where both surfaces meet ("point-of-contact") is chosen as origin of a Cartesian coordinate system in the map. Its x-axis is tangential to the Greenwich meridian, its y-axis tangential to the meridian $\Lambda = +90°$.

Parameters of the source are $U^1 \equiv U$=Longitude, $U^2 \equiv V$=Latitude, parameters in the chart are $u^1 \equiv u = \alpha$, $u^2 \equiv v = r$ (polar coordinates) or $u^1 \equiv u = x$, $u^2 \equiv v = y$ (Cartesian coordinates).

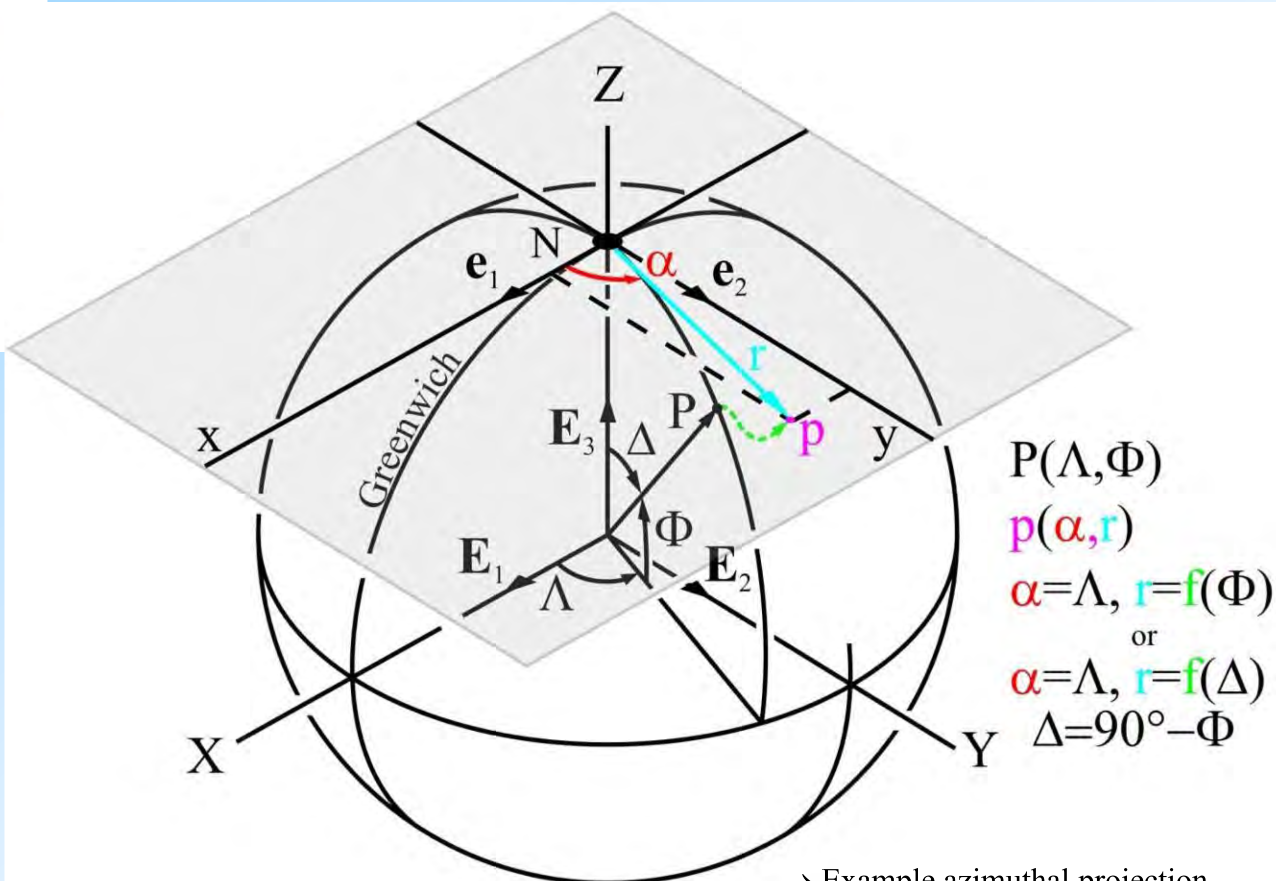
Attention: Metric matrix $\underline{g}$ depends on the choice of $u^1 \equiv u$, $u^2 \equiv v$; the transformations $x = r \cos\alpha$, $y = r \sin\alpha$ and $\underline{g}(\alpha, r) \rightarrow \underline{g}(x, y)$ apply. The term "azimuthal" stems from the first mapping equation $\alpha = \Lambda$. The second part of the mapping equations is $r = f(\Phi)$, which describes the distance of the image point from the origin. Sometimes $r = f(\Delta)$, $\Delta = 90° - \Phi$ is chosen in order to emphasize that r increases as the "polar distance" Δ is increasing from the pole to the equator. The unknown function $f(\Phi)$ is determined from one of the special properties "conformality", "equivalence" or "equidistance".

→ Azimuthal projections

Azimuthal Projections



Azimuthal Projections



$P(\Lambda,\Phi)$

$p(\alpha,r)$

$\alpha=\Lambda,\ r=f(\Phi)$

or

$\alpha=\Lambda,\ r=f(\Delta)$

$\Delta=90°-\Phi$

MATLAB©:
x to the right, y to the top → Greenwich meridian to the right !

→ Example azimuthal projection

Example azimuthal projection

$$\text{Sphere } \mathbb{S}^2_R : U = \Lambda, V = \Phi \rightarrow \text{ Plane } \mathbb{P}^2_O : u = \alpha, v = r, \quad \text{Conformal mapping}$$

Metric matrix source

$$\underline{G} = R^2 \text{diag}(\cos^2\Phi, 1)$$

Metric matrix chart

$$\underline{g} = \text{diag}(r^2, 1)$$

Mapping equations

$$\alpha = \Lambda, r = f(\Delta), \Delta = \pi/2 - \Phi$$

Jacobian matrix

$$\underline{J} = \text{diag}(\partial\alpha/\partial\Lambda, \partial r/\partial\Delta) = \text{diag}(1, f')$$

Cauchy-Green deformation tensor

$$\underline{C} = \underline{J}^T \underline{g}\, \underline{J} = \text{diag}(r^2, f'^2) = \text{diag}(f^2, f'^2)$$

Extremal distortions=distortions in direction of the parameter lines

$$\Lambda_1^2 = C_{11}/G_{11} = f^2/R^2\cos^2\Phi = f^2/R^2\sin^2\Delta \qquad \Lambda_2^2 = C_{22}/G_{22} = f'^2/R^2$$

Determine unknown f from conformality postulate $\Lambda_1 = \Lambda_2$

$$\Lambda_1 = \Lambda_2 \Leftrightarrow \frac{f}{\cancel{R}\sin\Delta} = \frac{f'}{\cancel{R}} \Leftrightarrow \frac{f}{\sin\Delta} = \frac{df}{d\Delta} \Leftrightarrow \frac{d\Delta}{\sin\Delta} = \frac{df}{f} \Rightarrow \int\frac{d\Delta}{\sin\Delta} = \int\frac{df}{f}$$

$$\Rightarrow \ln f = \ln\tan\frac{\Delta}{2} + \ln c = \ln\left(c\tan\frac{\Delta}{2}\right) \Leftrightarrow f = c\tan\frac{\Delta}{2} = c\tan\left(\frac{\pi}{4} - \frac{\Phi}{2}\right)$$

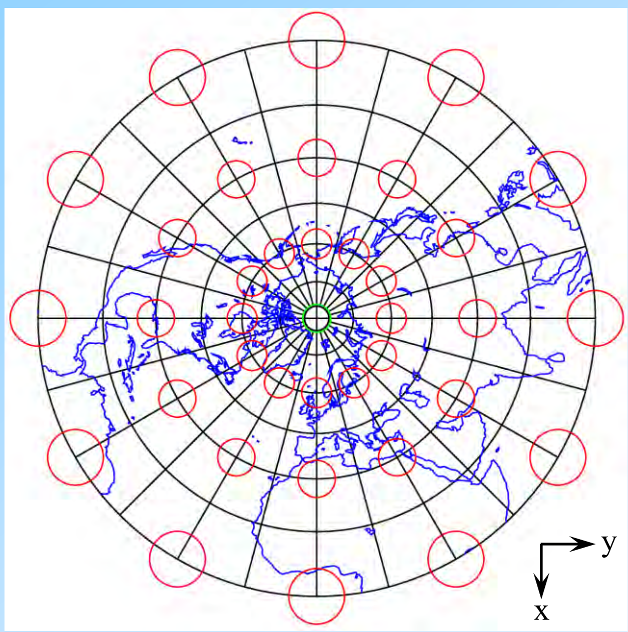
→ Example azimuthal projection

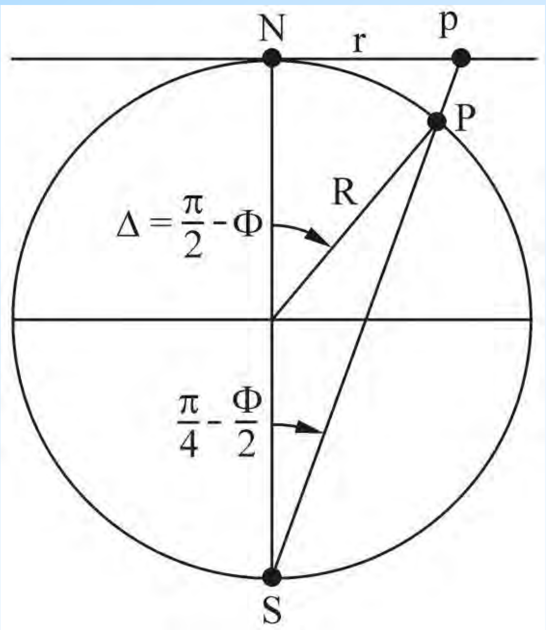
Example azimuthal projection

Fix integration constant c by an additional postulate, e.g. isometry – no distortion – at the North Pole: $\Lambda_1\big|_{\Phi=90^\circ \Leftrightarrow \Delta=0} = \Lambda_2\big|_{\Phi=90^\circ \Leftrightarrow \Delta=0} = 1\,!$

$$\Lambda_1 = f/R\sin\Delta = \frac{c\tan\frac{\Delta}{2}}{R\sin\Delta} \overset{\text{L'Hospital}}{\Rightarrow} \Lambda_1\big|_{\Delta=0} = \frac{c}{2R} \overset{!}{=} 1 \Leftrightarrow c = 2R$$

Final distortions and mapping equations: Universal Polar Stereographic (UPS)





$$\Lambda_1 = \Lambda_2 = 1/\cos^2\left(\frac{\pi}{4} - \frac{\Phi}{2}\right)$$

$$\alpha = \Lambda \;,\; r = 2R\tan\left(\frac{\pi}{4} - \frac{\Phi}{2}\right)$$

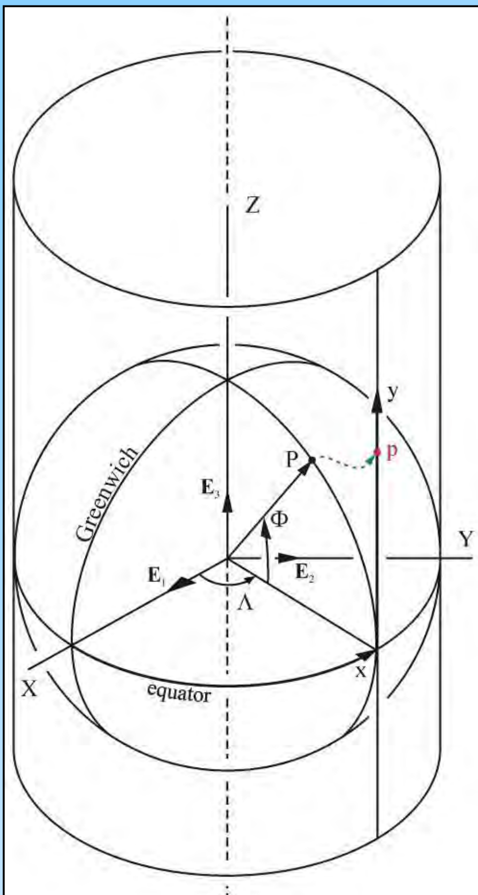
$$\Downarrow$$

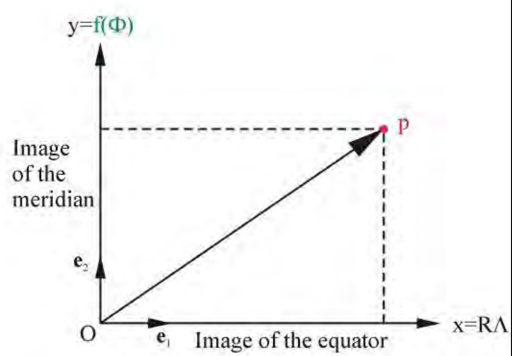
$$x = r\cos\alpha = \ldots,\; y = r\sin\alpha = \ldots$$

→ Fundamental properties

Fundamental properties I (apply to the plane as chart)

Cylindrical projections





A cylinder is wrapped around the sphere/ellipsoid in such a way that the axis of the cylinder coincides with the rotations axis of the Earth model. After the mapping the cylinder is sliced along a generator (German: "Mantellinie") and spread out as a plane. The line of contact, i.e. the line where the cylinder touches the sphere/ellipsoid, is the equator, which is mapped distortion-free. Parallels and meridians are mapped as straight lines orthogonal to each other. North and South pole also become straight lines. The

→ Example 1 cylindrical projection

Example 1 cylindrical projection

cylinder is exclusively parameterized in Cartesian coordinates $u^1 \equiv u = x$, $u^2 \equiv v = y$. While x is proportional to Λ, x=RΛ, the y-coordinate is a function of latitude Φ. This unknown function f(Φ) is determined from one of the special properties "conformality", "equivalence" or "equidistance".

$$\text{Sphere } \mathbb{S}^2_R : U = \Lambda, V = \Phi \rightarrow \text{ Plane } \mathbb{P}^2_O : u = x, v = y$$

equal area mapping

Metric matrix source

$$\underline{G} = R^2 \mathrm{diag}(\cos^2\Phi, 1)$$

Metric matrix chart

$$\underline{g} = \underline{I}_2$$

Mapping equations

$$x = R\Lambda, y = Rf(\Phi)$$

Jacobian matrix

$$\underline{J} = \mathrm{diag}(\partial x / \partial\Lambda, \partial y / \partial\Phi) = R\mathrm{diag}(1, f')$$

Cauchy-Green deformation tensor

$$\underline{C} = \underline{J}^T \underline{g}\, \underline{J} = R^2 \mathrm{diag}(1, f'^2)$$

Extremal distortions=distortions in direction of the parameter lines

$$\Lambda_1^2 = C_{11} / G_{11} = 1 / \cos^2\Phi \qquad \Lambda_2^2 = C_{22} / G_{22} = f'^2$$

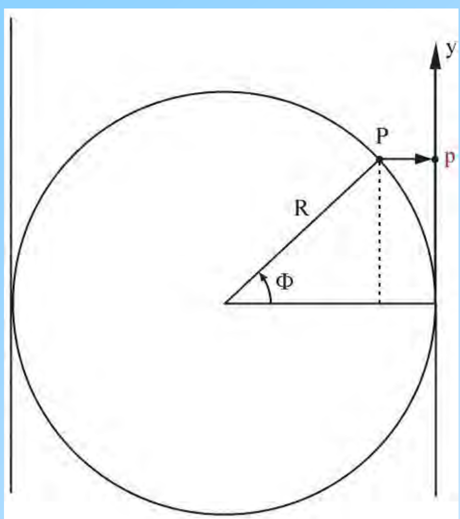
→ Example 1 cylindrical projection

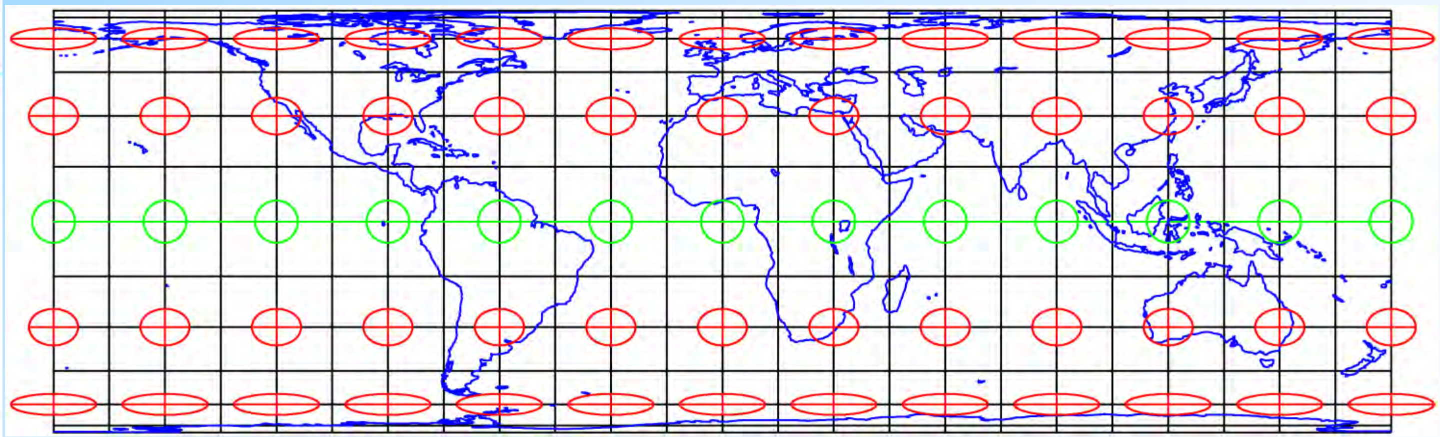
Example 1 cylindrical projection

Determine unknown f from equivalence postulate $\Lambda_1\Lambda_2 = 1$

$$\Lambda_1\Lambda_2 = 1 \Leftrightarrow f' = \cos\Phi \Leftrightarrow f = \sin\Phi + c \Rightarrow y = R\sin\Phi + Rc$$

Fix integration constant c by an additional postulate, e.g. for $\Phi=0°$ the value for y should also be zero (align x-axis with equator image): $y\big|_{\Phi=0} = 0 \Leftrightarrow c = 0.$





Final mapping equations and distortions: Lambert projection

$$x = R\Lambda \quad , \quad y = R\sin\Phi \quad , \quad \Lambda_1 = \Lambda_2^{-1} = \frac{1}{\cos\Phi}$$

(Conformality and isometry – no distortion – on the equator)

→ Example 2 cylindrical projection

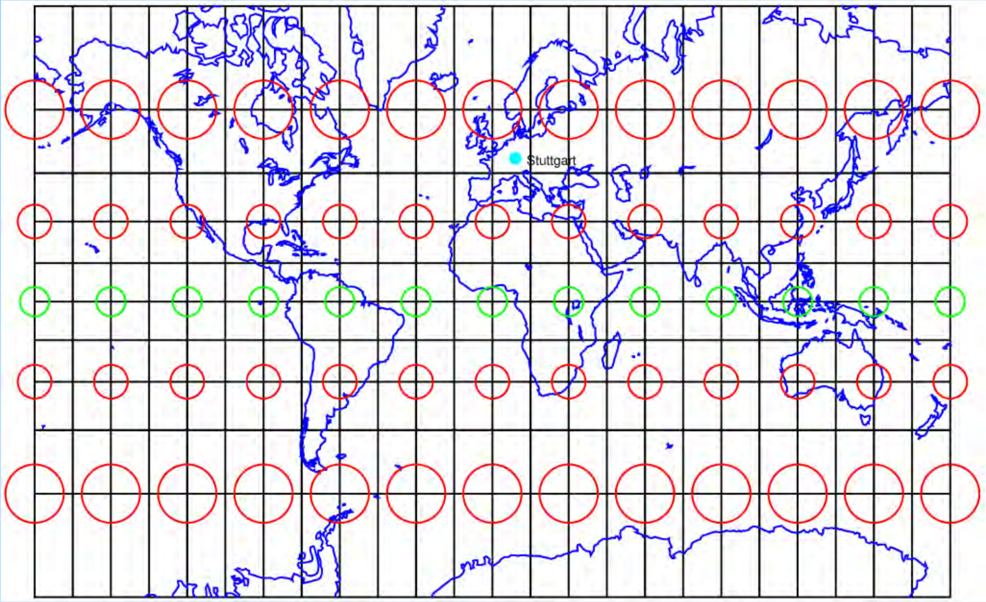
Example 2 cylindrical projection

Sphere $\mathbb{S}^2_R : U = \Lambda, V = \Phi \rightarrow$ Plane $\mathbb{P}^2_O : u = x, v = y$ conformal mapping

$$\ldots \quad \Lambda_1 = \Lambda_2 \Leftrightarrow \frac{1}{\cos\Phi} = f' = \frac{df}{d\Phi} \Leftrightarrow \frac{d\Phi}{\cos\Phi} = df \Rightarrow \int df = \int \frac{d\Phi}{\cos\Phi} \Rightarrow$$

$$f = \ln\tan\left(\frac{\pi}{4} + \frac{\Phi}{2}\right) + c \Rightarrow y = R\ln\tan\left(\frac{\pi}{4} + \frac{\Phi}{2}\right) + Rc \overset{y|_{\Phi=0} \overset{!}{=} 0}{=} R\ln\tan\left(\frac{\pi}{4} + \frac{\Phi}{2}\right)$$

Final mapping equations and distortions: Mercator projection



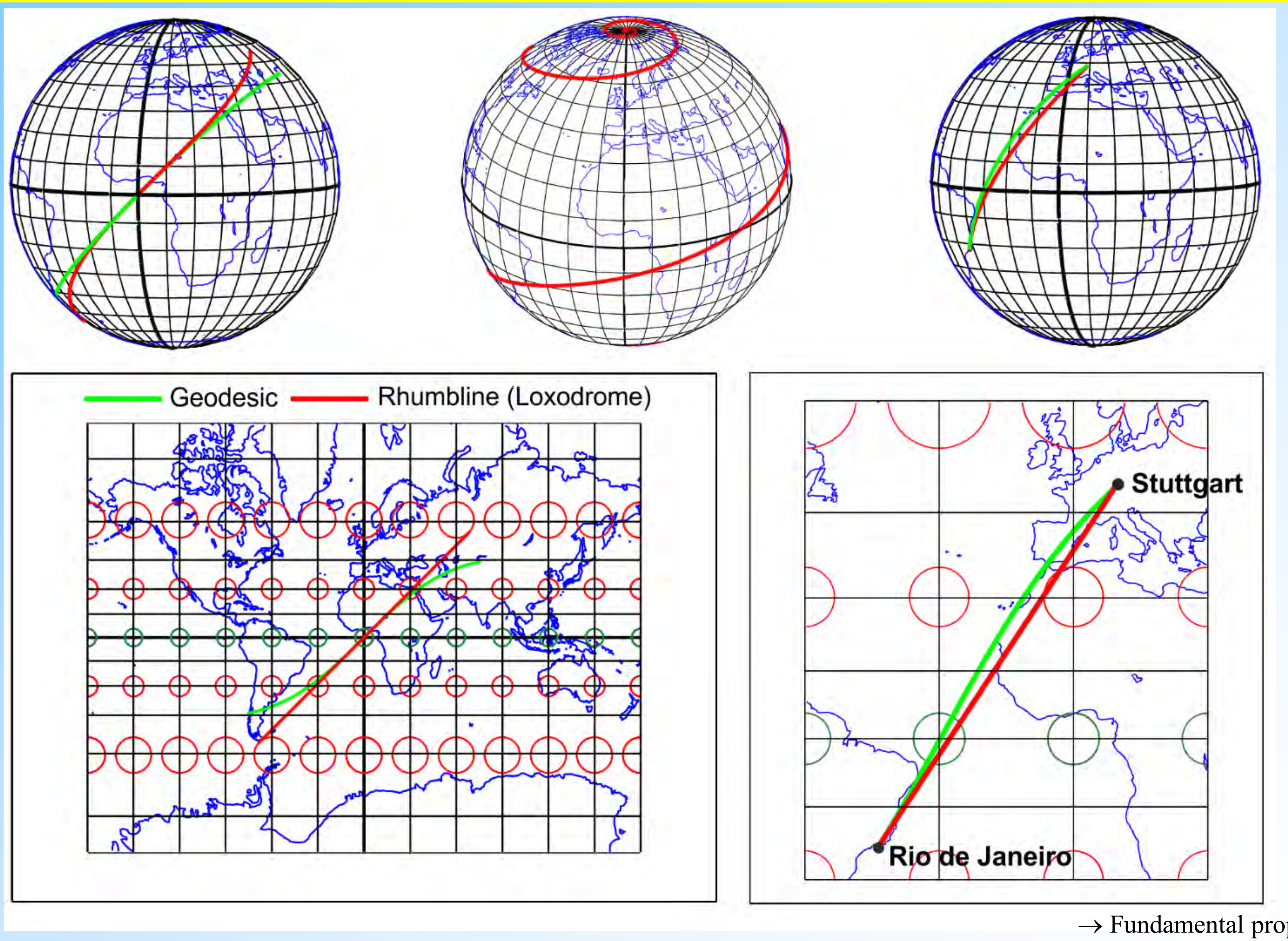
$$x = R\Lambda \quad , \quad y = R\ln\tan\left(\frac{\pi}{4} + \frac{\Phi}{2}\right)$$

$$\Lambda_1 = \Lambda_2 = \frac{1}{\cos\Phi}$$

(Isometry – no distortion – on the equator)

→ Fundamental properties

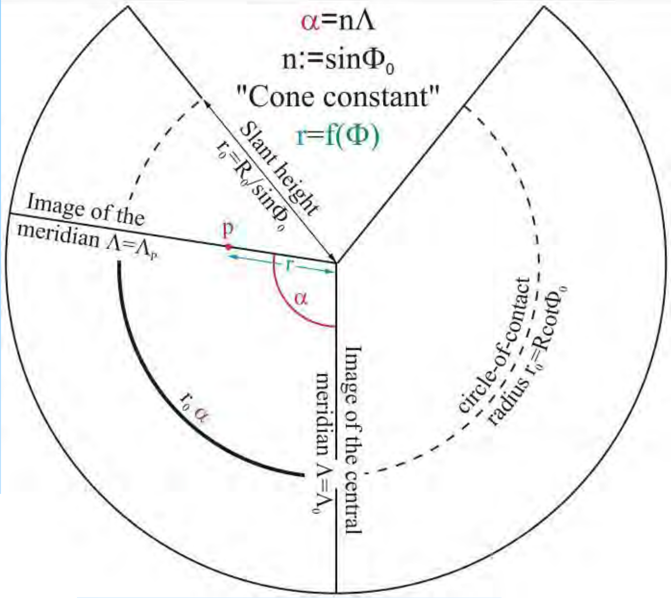
Example 2 cylindrical projection



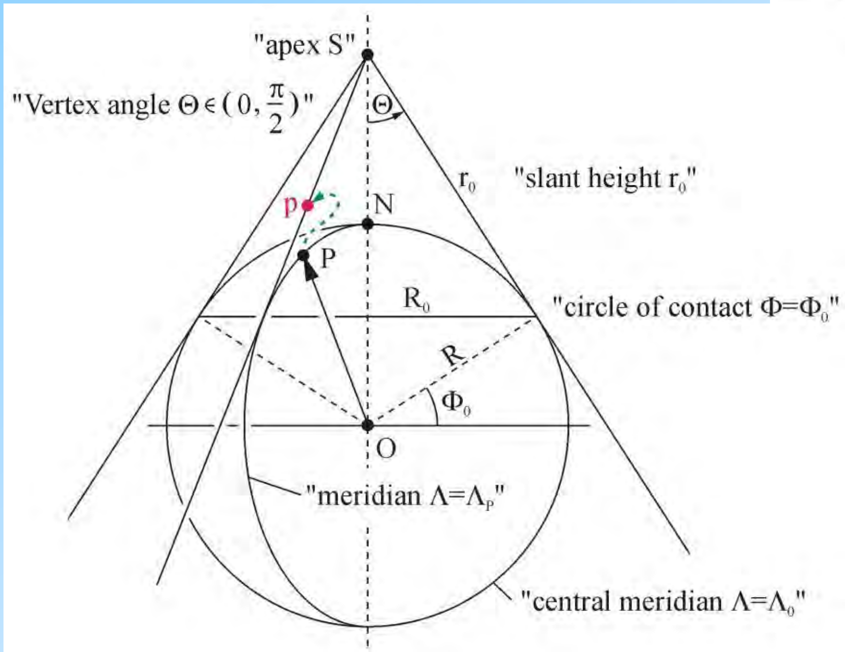
→ Fundamental properties

Fundamental properties I (apply to the plane as chart)

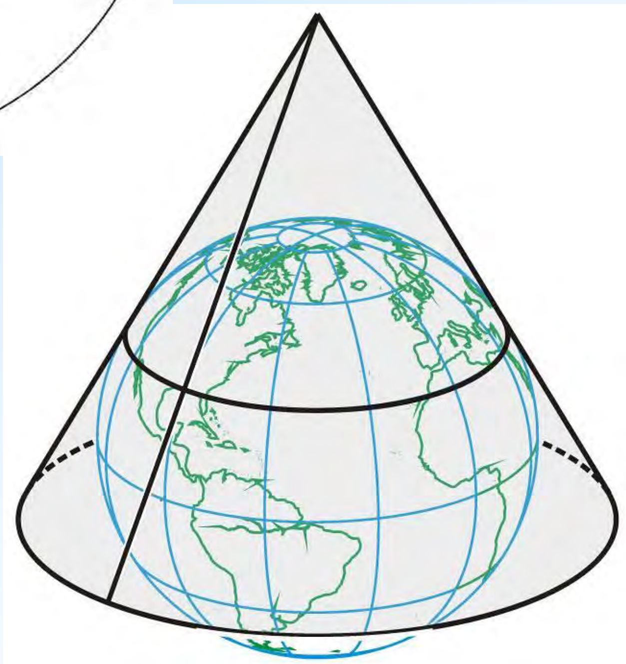
Conical projections: A cone is wrapped around the sphere/ ellipsoid in such a way that the axis of the cone coincides with the rotations axis of the Earth model. The vertex angle Θ is chosen from the interval $(0,\pi/2)$,



which leads to a circle-of-contact with latitude $\Phi_0 = \Theta$. The case $\Theta = 0°$ generates a cylindrical mapping, while



the case $\Theta = 90°$ produces an azimuthal mapping.



→ Example Conical projection

Example conical projection

Following the mapping procedure the cone is sliced along a generator, which is opposite to a certain central meridian $\Lambda=\Lambda_0$, and developed into the plane. The North Pole is mapped as a single point or as a line, meridians are straight lines intersecting the images of the parallel circles orthogonally. Parallels are displayed as concentric circles around the apex S. Usually a parameterization with $u^1 \equiv u=\alpha$, $u^2 \equiv v=r$ (polar coordinates) is applied. While α is proportional to Λ, $\alpha=n\Lambda$ ($n:=\sin\Phi_0$ "Cone constant"), r is a function of Φ or Δ, $r=f(\Delta)=f(\pi/2-\Phi)$. As in cylindrical mappings, f is determined from one of the postulates "conformality", "equivalence" or "equidistance". The image of the circle-of-contact is usually free from any distortion. The outline of conical mappings is always fan-shaped.

$$\text{Sphere } \mathbb{S}^2_R : U=\Lambda, V=\Phi \rightarrow \text{ Plane } \mathbb{P}^2_O : u=\alpha, v=r$$

equidistant meridians

Metric matrix source — Metric matrix chart — Mapping equations

$$\underline{G} = R^2 \operatorname{diag}(\cos^2\Phi, 1) \qquad \underline{g} = \operatorname{diag}(r^2, 1) \qquad \alpha = n\Lambda, n = \sin\Phi_0, r = f(\Delta), \Delta = \pi/2 - \Phi$$

Jacobian matrix — Cauchy-Green deformation tensor

$$\underline{J} = \operatorname{diag}(\partial\alpha/\partial\Lambda, \partial r/\partial\Delta) = \operatorname{diag}(n, f') \qquad \underline{C} = \underline{J}^T \underline{g}\, \underline{J} = \operatorname{diag}(n^2 r^2, f'^2) = \operatorname{diag}(n^2 f^2, f'^2)$$

→ Example Conical projection

Example conical projection

Extremal distortions=distortions in direction of the parameter lines

$$\Lambda_1^2 = C_{11}/G_{11} = n^2 f^2 / R^2 \cos^2\Phi = n^2 f^2 / R^2 \sin^2\Delta \qquad \Lambda_2^2 = C_{22}/G_{22} = f'^2 / R^2$$

Determine f from the postulate that all meridians shall be mapped distortion-free

$$\Lambda_2 = 1 \Leftrightarrow f' = R \Rightarrow r = f = R\Delta + c$$

if c > 0: Pole=circular arc

Final mapping equations and distortions

$$\alpha = n\Lambda, n = \sin\Phi_0, r = R(\pi/2 - \Phi) + c, c = \text{const}$$

$$\Lambda_1 = \frac{n[R(\pi/2-\Phi)+c]}{R\cos\Phi}, \Lambda_2 = 1$$

Fix integration constant c by an additional postulate, e.g. conformality on the circle-of-contact: $\Lambda_1\big|_{\Phi=\Phi_0} = \Lambda_2 = 1$

$$\Lambda_1\big|_{\Phi=\Phi_0} = \frac{n[R(\pi/2-\Phi_0)+c]}{R\cos\Phi_0} = 1 \Leftrightarrow n[R(\pi/2-\Phi_0)+c] = R\cos\Phi_0$$

$$\Rightarrow c = R[\cot\Phi_0 - (\pi/2 - \Phi_0)]$$

c > 0: Pole=circular arc !

→ Example conical projection

Example conical projection

Final mapping equations and distortions: Ptolemy projection (85-150 AD)

$$\alpha = n\Lambda, n = \sin\Phi_0$$

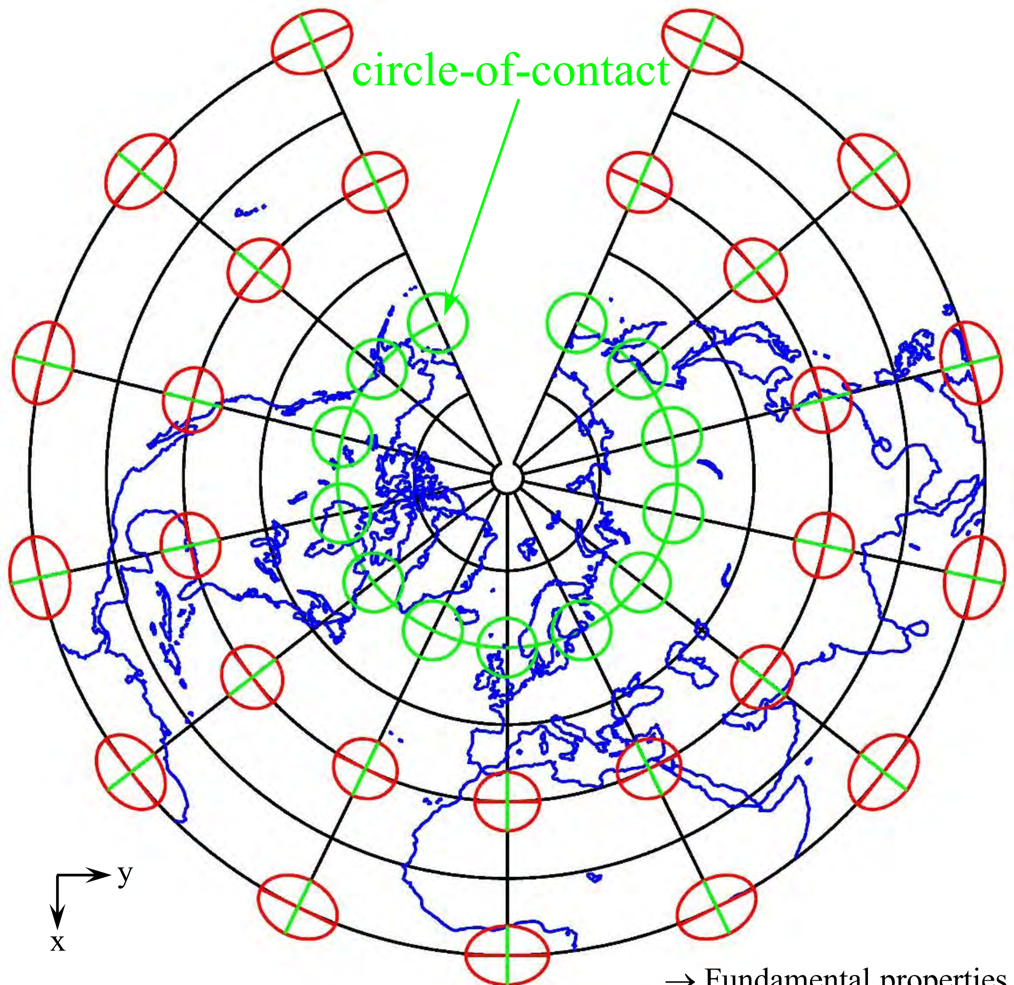
$$r = R(\cot\Phi_0 + \Phi_0 - \Phi)$$

$$x = r\cos\alpha$$

$$y = r\sin\alpha$$

$$\Lambda_1 = \frac{\sin\Phi_0(\cot\Phi_0 + \Phi_0 - \Phi)}{\cos\Phi}$$

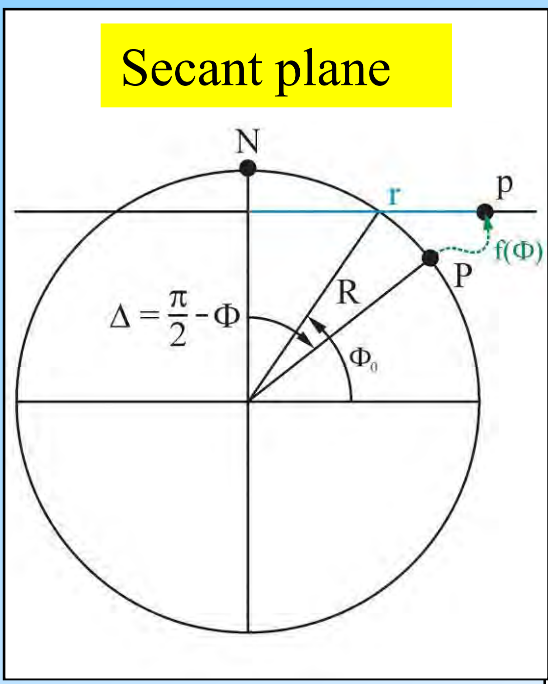
$$\Lambda_2 = 1$$



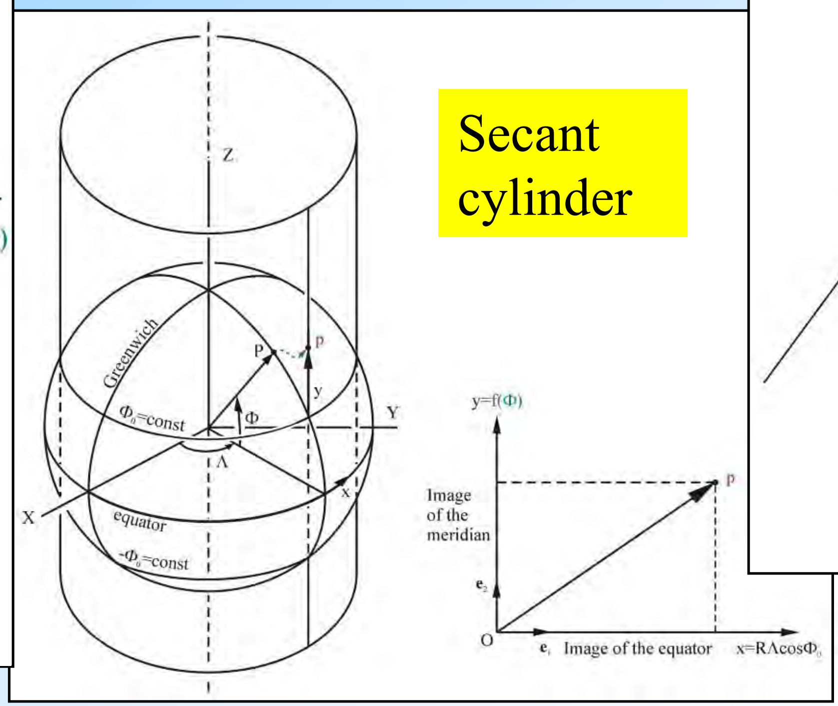
→ Fundamental properties

Fundamental properties II (apply to the plane as chart)

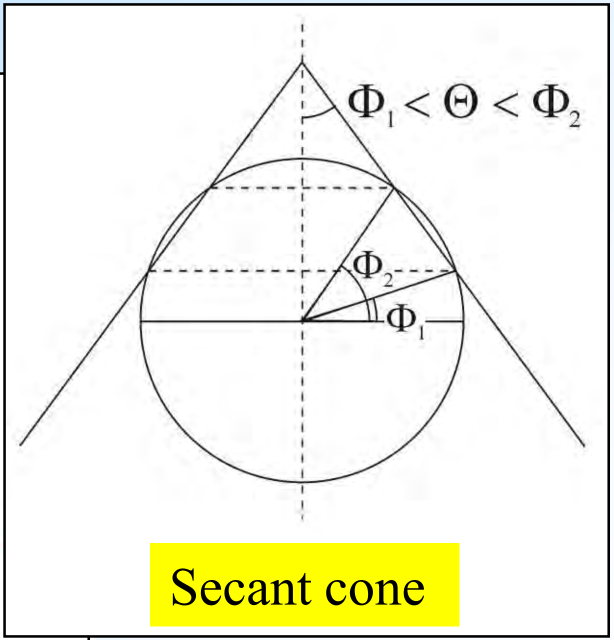
Azimuthal, cylindrical and conical projections are also generated when the mapping surface (plane, cylinder, cone) does not touch but intersect the sphere/ellipsoid. For azimuthal projections a circle of contact (intersection) $\Phi=\Phi_0$ is achieved, which is mapped distortion-free. For cylinder and cone even two parallel circles (cylinder: $\Phi = \pm\Phi_0$, cone: $\Phi = \Phi_1$ and $\Phi = \Phi_2$) will be generated which have no distortion at all.



Secant plane



Secant cylinder



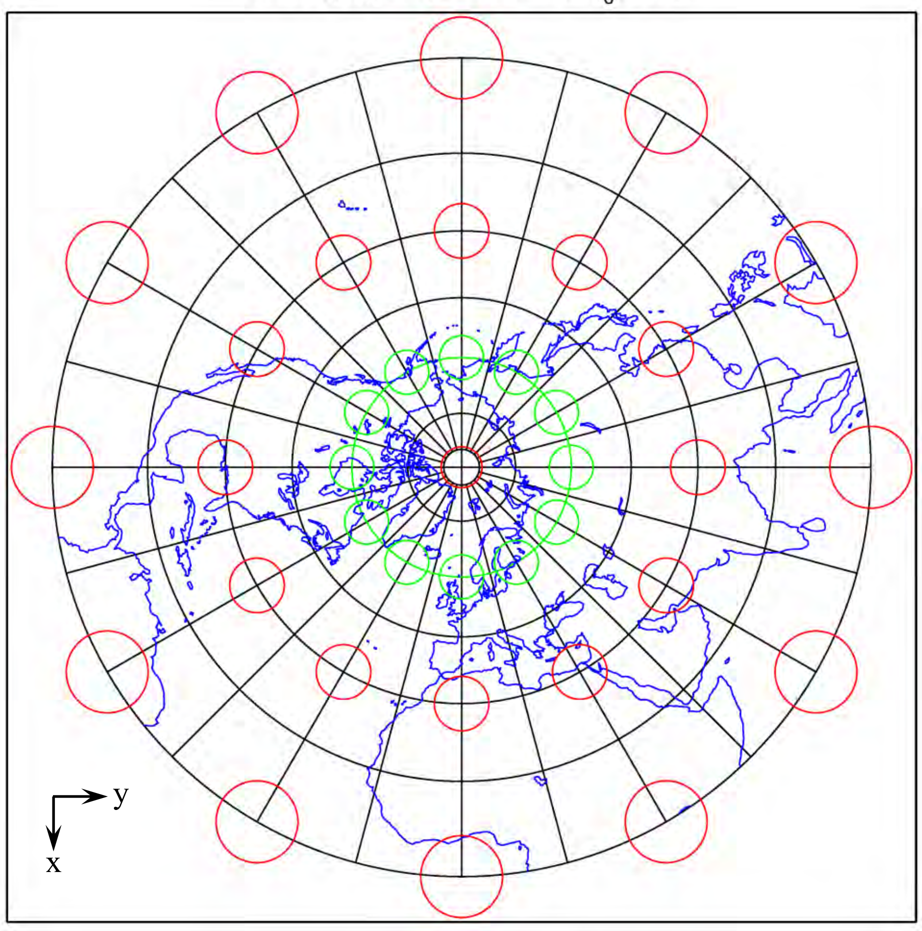
Secant cone

→ Example: Conformal azimuthal mapping with isometric parallel circle

GIS

Conformal azimuthal mapping with isometric parallel circle

Start from $\color{red}{f} = c\tan(\Delta/2)$ with $\Lambda_1 = \Lambda_2 = \dfrac{\color{red}{f}}{R\sin\Delta} = \dfrac{c\tan(\Delta/2)}{R\sin\Delta} = \dfrac{c}{2R\cos^2(\Delta/2)}$



and fix integration constant by the postulate

$$\Lambda_1\Big|_{\Delta=\Delta_0>0} = 1 = \frac{c}{2R\cos^2(\Delta_0/2)}$$

$$\Downarrow$$

$$c = 2R\cos^2\frac{\Delta_0}{2} = R(1+\cos\Delta_0)$$

$$\Downarrow$$

$$\alpha = \Lambda$$

$$r = f = R(1+\sin\Phi_0)\tan\left(\frac{\pi}{4}-\frac{\Phi}{2}\right)$$

$$\Lambda_1 = \Lambda_2 = \frac{(1+\sin\Phi_0)\tan\left(\frac{\pi}{4}-\frac{\Phi}{2}\right)}{\cos\Phi}$$

→ Example: Equivalent cylindrical mapping with 2 isometric parallel circles

Equivalent cylindrical mapping with 2 isometric parallel circles

Simply multiply $x = R\Lambda$ by $\cos\Phi_0$ where Φ_0 is the latitude of the parallel circle to be mapped free of distortion. Then, the general mapping equations are

$$x = R\cos\Phi_0\Lambda,\ y = Rf(\Phi)$$

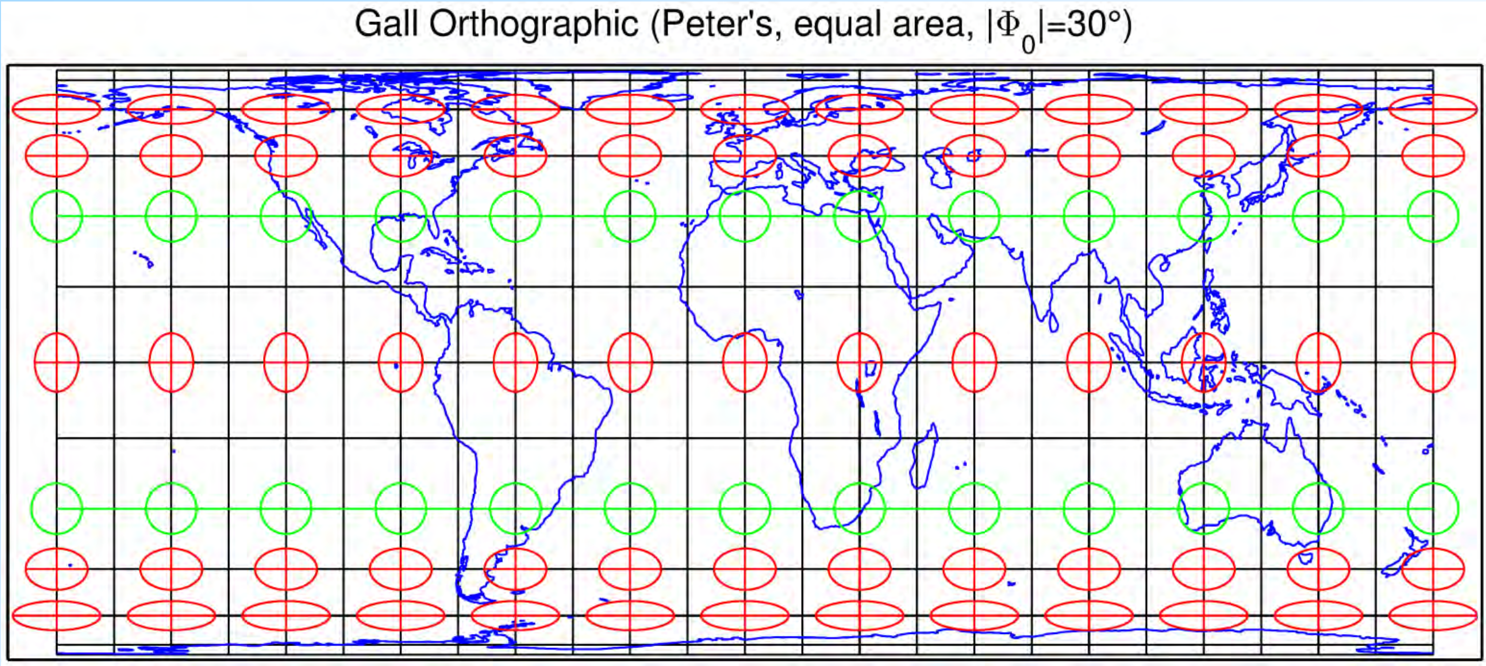
and $R\cos\Phi_0$ is the radius of that parallel circle. The principal distortions are $\Lambda_1 = \cos\Phi_0 / \cos\Phi$, $\Lambda_2 = f'(\Phi)$ and the implemented postulate of equivalence leads to $f = \sin\Phi / \cos\Phi_0$. The integration constant c has been set to zero again.

$$x = R\cos\Phi_0\Lambda$$

$$y = R\frac{\sin\Phi}{\cos\Phi_0}$$

$$\Lambda_1 = \frac{\cos\Phi_0}{\cos\Phi}$$

$$\Lambda_2 = \frac{\cos\Phi}{\cos\Phi_0}$$



→ Example: Conical projection, equidistant meridians, isometry on 2 parallels

Conical projection, equidistant meridians, isometry on 2 parallels

General mapping equations and distortions equidistant conical projection

$$\alpha = n\Lambda, n = \sin\Phi_0 \text{ , } r = R(\pi/2-\Phi)+c \text{ , } c = \text{const} \text{ , } \Lambda_1 = \frac{n[R(\pi/2-\Phi)+c]}{R\cos\Phi} \text{ , } \Lambda_2 = 1$$

Additional postulate: $\Lambda_1\big|_{\Phi=\Phi_1} = \Lambda_1\big|_{\Phi=\Phi_2} = 1$ (Isometry on two parallels !)

$$\Lambda_1\big|_{\Phi=\Phi_1} = \frac{n\left[R\left(\pi/2-\Phi_1\right)+c\right]}{R\cos\Phi_1} = \frac{n\left[R\left(\pi/2-\Phi_2\right)+c\right]}{R\cos\Phi_2} = \Lambda_1\big|_{\Phi=\Phi_2} \overset{!}{=} 1$$

$$\Rightarrow n = \frac{\cancel{R}\cos\Phi_1}{R\left(\pi/2-\Phi_1\right)+c} = \frac{\cancel{R}\cos\Phi_2}{R\left(\pi/2-\Phi_2\right)+c}$$

$$\Rightarrow \left[R\left(\pi/2-\Phi_2\right)+c\right]\cos\Phi_1 = \left[R\left(\pi/2-\Phi_1\right)+c\right]\cos\Phi_2$$

$$\Rightarrow c(\cos\Phi_1-\cos\Phi_2) = R\left[\left(\pi/2-\Phi_1\right)\cos\Phi_2-\left(\pi/2-\Phi_2\right)\cos\Phi_1\right]$$

$$\Rightarrow c = R\frac{\left(\pi/2-\Phi_1\right)\cos\Phi_2-\left(\pi/2-\Phi_2\right)\cos\Phi_1}{\cos\Phi_1-\cos\Phi_2}$$

→ Example: Conical projection, equidistant meridians, isometry on 2 parallels (cont.)

Conical projection, equidistant meridians, isometry on 2 parallels

$$\Rightarrow n := \sin\Phi_0 = \ldots = \frac{\cos\Phi_1 - \cos\Phi_2}{\Phi_2 - \Phi_1}$$ cone constant

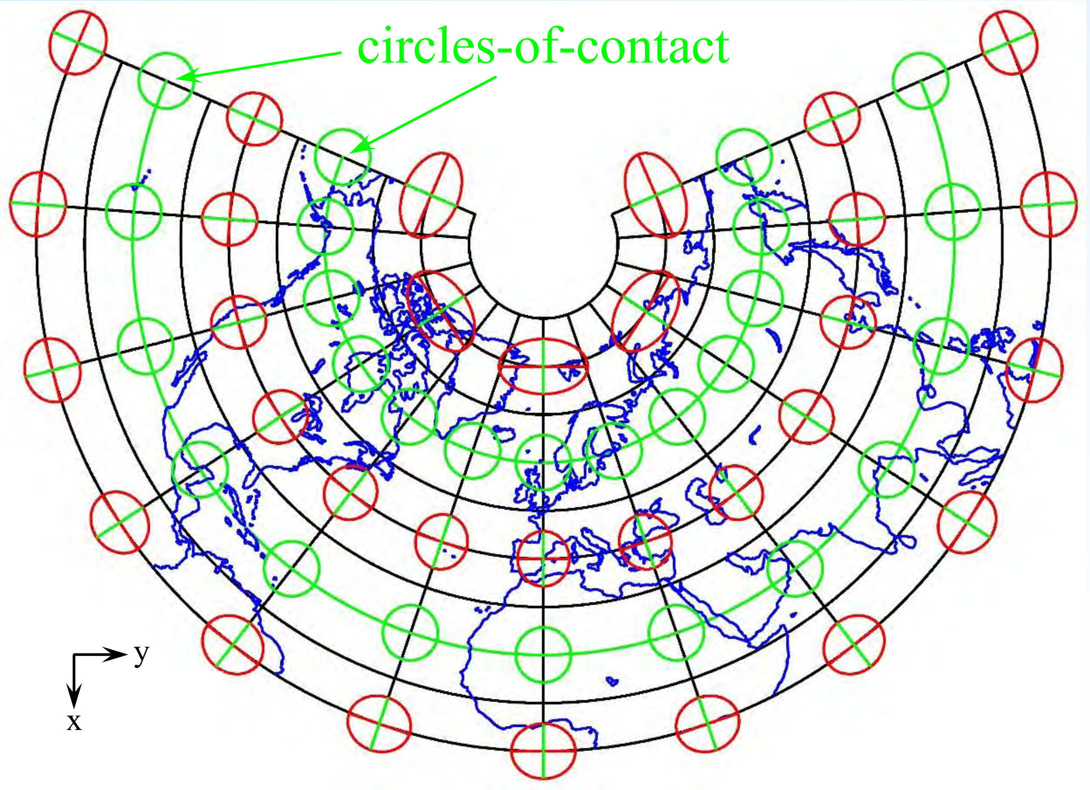
Final mapping equations and distortions: de L'Isle projection (J.N. de L'Isle 1745)

$$\alpha = \frac{\cos\Phi_1 - \cos\Phi_2}{\Phi_2 - \Phi_1}\Lambda$$

$$r = R\left(\frac{\Phi_1\cos\Phi_2 - \Phi_2\cos\Phi_1}{\cos\Phi_2 - \cos\Phi_1} - \Phi\right)$$

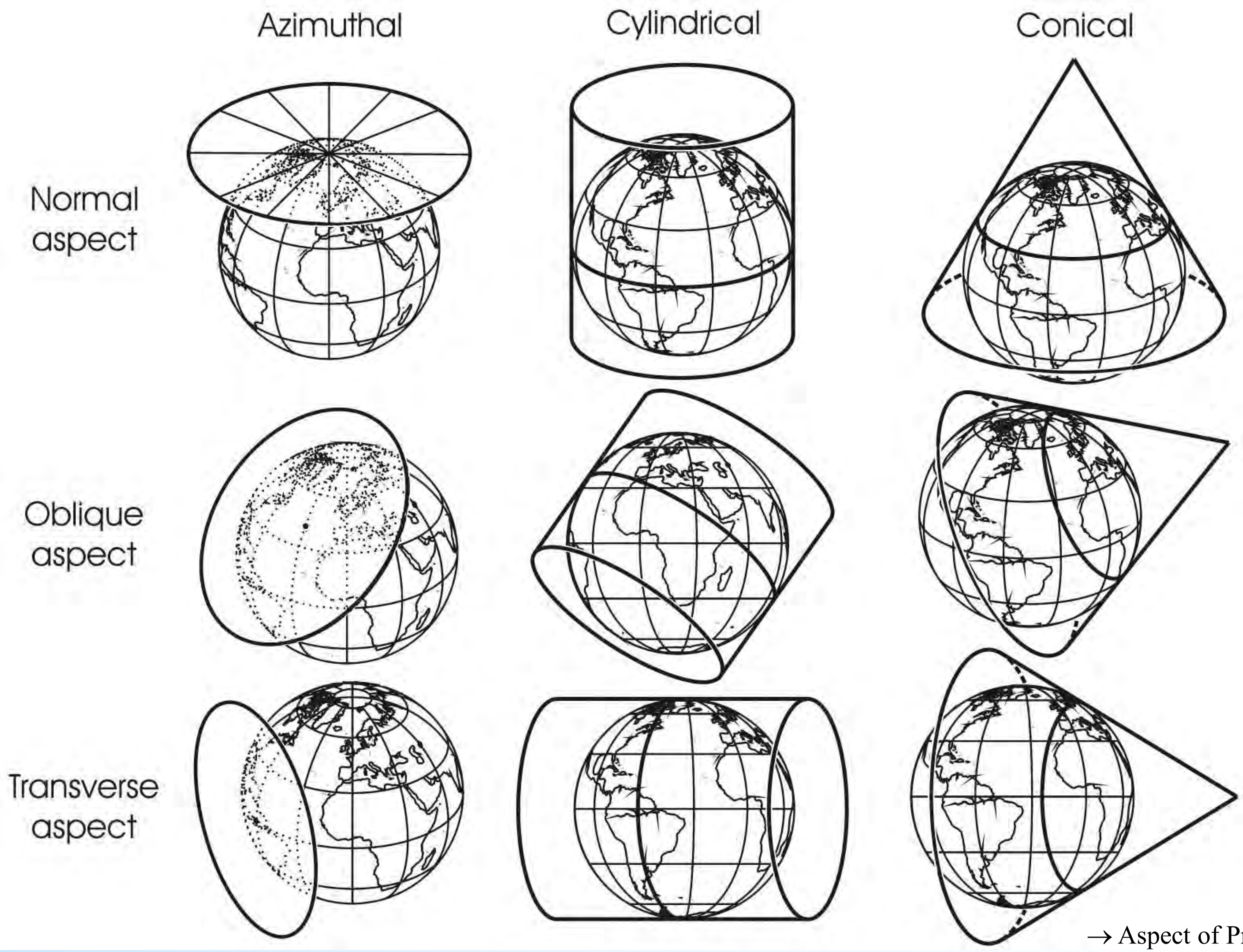
$$x = r\cos\alpha = \ldots \quad , \quad y = r\sin\alpha = \ldots$$

$$\Lambda_1 = \frac{\Phi_2\cos\Phi_1 - \Phi_1\cos\Phi_2}{(\Phi_2 - \Phi_1)\cos\Phi} + \frac{\Phi(\cos\Phi_2 - \cos\Phi_1)}{(\Phi_2 - \Phi_1)\cos\Phi} \quad , \quad \Lambda_2 = 1$$



$\rightarrow$ Aspect of projection

Aspect of Projection



→ Aspect of Projection

Aspect of Projection

The term aspect refers to the position of the normal of the plane, the axes of cylinder and cone, respectively, in relation to the rotation axis of the Earth.
Normal aspect: Earth rotation axis coincides with the normal of the plane in azimuthal projections, with the axes of cylinder or cone in cylindrical or conical projections, no matter if tangent or secant cases are concerned.
The transverse aspect treats the case where the normal of the plane, or the axes of cylinder or cone, lie in the equatorial plane of the Earth. As the normal aspect it is generated from the general case, the oblique aspect.

General formulas for the sphere
(the ellipsoid is treated in the polar/transverse aspect only)

The general formulas can be derived from spherical trigonometry or using rotation matrices. In the general case the mapping equations do not (anymore) involve spherical longitude Λ and spherical latitude Φ, but generalized coordinates, i.e. meta-longitude A (capital α !) and meta-latitude B (capital β !). The North Pole as that point where, e.g., the tangential plane is attached to the sphere is replaced by an arbitrary point called Meta North Pole M. For the very introductory example of

→ Aspect of Projection

Aspect of Projection

the isoparametric mapping the generalized mapping equations are x = RA, y = RB with meta coordinates A,B referring to Meta North Pole M of (ordinary) spherical longitude Λ_0 and latitude Φ_0 given with respect to the North Pole N. The old base vector $\mathbf{E}_3$ is simply replaced by $\mathbf{E}_3^*$ now pointing towards the direction of M instead of N. The "old" equator as the plane orthogonal to $\mathbf{E}_3$ is changed to a meta-equator the normal of which is $\mathbf{E}_3^*$. The freedom to choose coordinates Λ_0, Φ_0 of the Meta North Pole M gives rise to all possible aspects: For $\Phi_0 = \pi/2$ the normal aspect is obtained, for $\Phi_0 = 0$ the transverse aspect is established. All other values for Φ are connected with the oblique aspect.

<u>First task</u>: Given coordinates of Meta North Pole $M(\Lambda_0,\Phi_0,\Omega_0)$ and (standard) coordinates Λ,Φ of a point P referring to (traditional) North Pole N, compute its meta coordinates A,B ($\Delta\Lambda := \Lambda - \Lambda_0$)

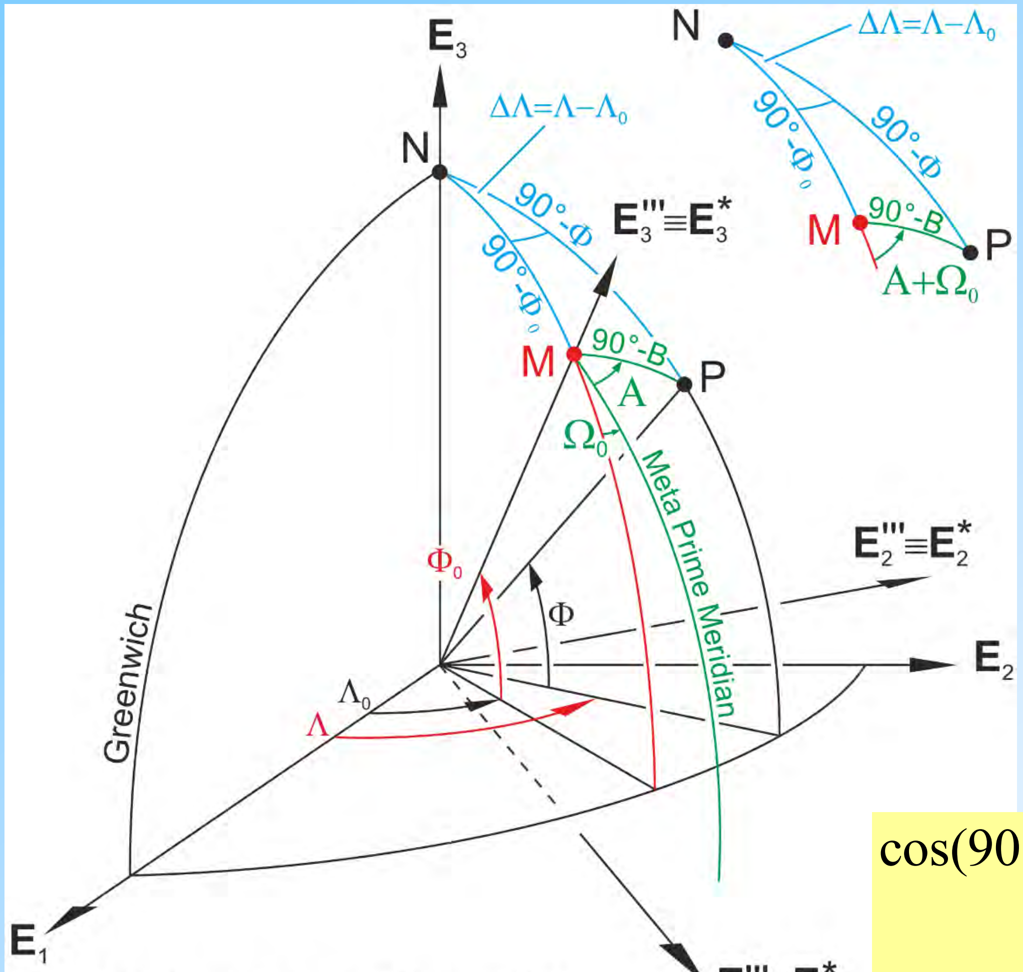
$$B = \arcsin(\sin\Phi\sin\Phi_0 + \cos\Phi\cos\Phi_0\cos\Delta\Lambda) \qquad -\pi/2 \le B \le \pi/2$$

$$A = \arctan\frac{\sin\Delta\Lambda}{\sin\Phi_0\cos\Delta\Lambda - \cos\Phi_0\tan\Phi} - \Omega_0 \qquad 0 \le A \le 2\pi$$

Both formulas are unique as long as the rule of quadrants is considered while computing A.

→ Transformation to Meta North Pole

Transformation to Meta North Pole $M(\Lambda_0,\Phi_0,\Omega_0)$



Spherical Law of Sines:

$$\sin(A+\Omega_0)\cos B = \cos\Phi \sin\Delta\Lambda$$

Spherical Sine-Cosine Law:

$$\cos(A+\Omega_0)\cos B = \cos\Phi\sin\Phi_0\cos\Delta\Lambda - \sin\Phi\cos\Phi_0$$

$$\tan(A+\Omega_0) = \frac{\sin(A+\Omega_0)\cos B}{\cos(A+\Omega_0)\cos B} = \ldots$$

Spherical Cosine Rule for Sides:

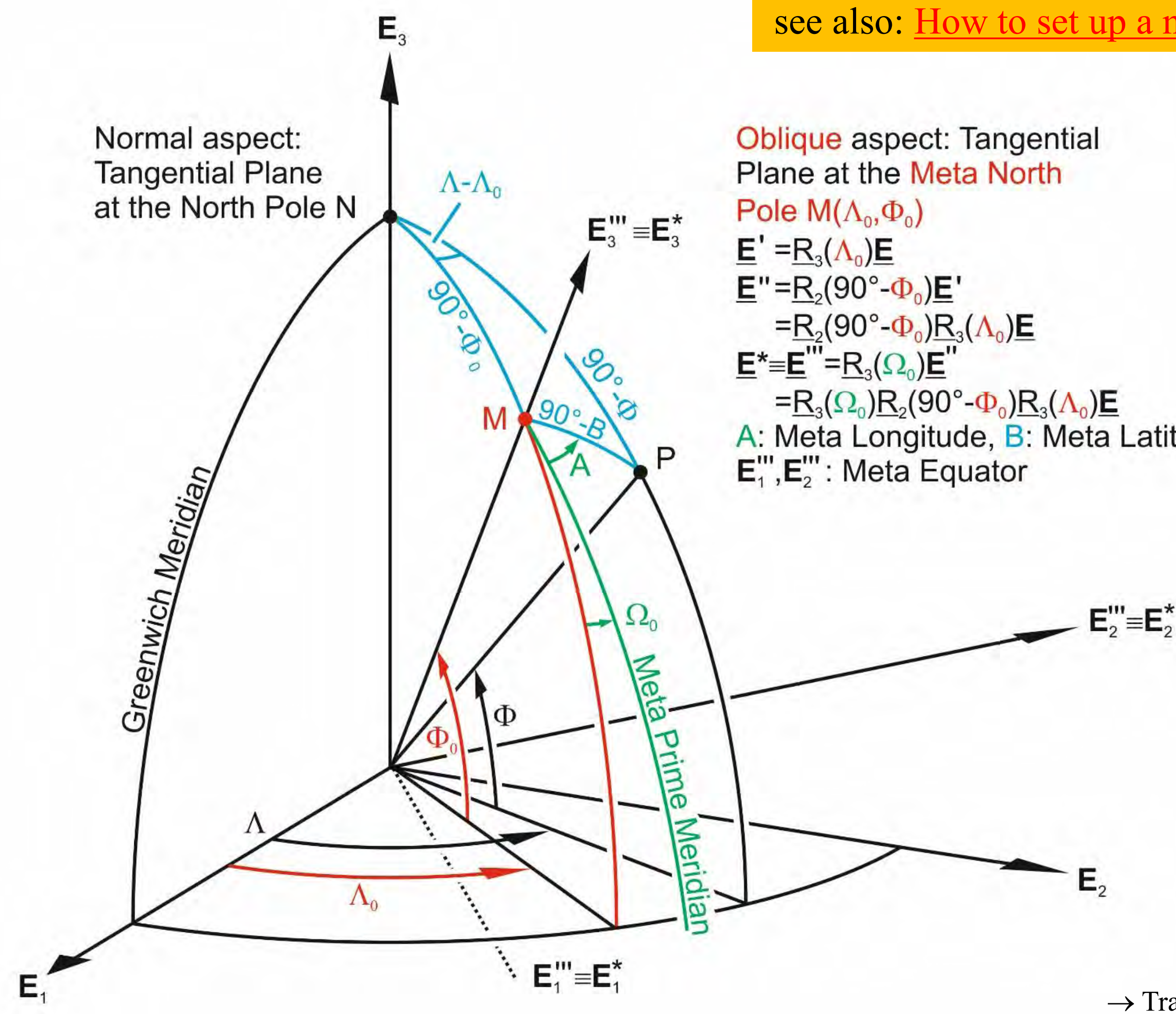
$$\cos(90-B) = \cos(90-\Phi)\cos(90-\Phi_0) + \sin(90-\Phi)\sin(90-\Phi_0)\cos\Delta\Lambda$$

$$\Rightarrow \sin B = \sin\Phi\sin\Phi_0 + \cos\Phi\cos\Phi_0\cos\Delta\Lambda$$

→ Transformation using rotation matrices

Transformation to Meta North Pole using rotation matrices

see also: How to set up a meta pole system by rotations



Oblique aspect: Tangential Plane at the Meta North Pole M(Λ_0,Φ_0)

$\underline{\mathbf{E}}' = \underline{R}_3(\Lambda_0)\underline{\mathbf{E}}$

$\underline{\mathbf{E}}'' = \underline{R}_2(90°-\Phi_0)\underline{\mathbf{E}}'$

$\quad = \underline{R}_2(90°-\Phi_0)\underline{R}_3(\Lambda_0)\underline{\mathbf{E}}$

$\underline{\mathbf{E}}^* \equiv \underline{\mathbf{E}}''' = \underline{R}_3(\Omega_0)\underline{\mathbf{E}}''$

$\quad = \underline{R}_3(\Omega_0)\underline{R}_2(90°-\Phi_0)\underline{R}_3(\Lambda_0)\underline{\mathbf{E}}$

A: Meta Longitude, B: Meta Latitude

$\mathbf{E}_1'''$,$\mathbf{E}_2'''$: Meta Equator

$$\underline{R}_1(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\xi & \sin\xi \\ 0 & -\sin\xi & \cos\xi \end{bmatrix}$$

$$\underline{R}_2(\xi) = \begin{bmatrix} \cos\xi & 0 & -\sin\xi \\ 0 & 1 & 0 \\ \sin\xi & 0 & \cos\xi \end{bmatrix}$$

$$\underline{R}_3(\xi) = \begin{bmatrix} \cos\xi & \sin\xi & 0 \\ -\sin\xi & \cos\xi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Positive rotation:</u>

Counterclockwise (ccw) as seen from the arrowhead towards the origin

→ Transformation using rotation matrices