University of Stuttgart
Institute of Geodesy

Physical Geodesy

Spherical harmonic expansion of Earth’s gravitational potential

GIS

Spherical harmonic expansion

Legendre functions

Task 1: Prepare figures for fully normalized zonal, tesseral and sectorial Legendre functions $\bar{P}_{lm}(\cos\theta)$ and spherical harmonics $\bar{Y}_{lm}(\theta,\lambda) = \bar{P}_{lm}(\cos\theta)\cos m\lambda$ of degree $l = 10$ within $\theta \in [0° \; 180°]$ using both Rodrigues-Ferrers and recursive formulas. How many zero crossings do the fully normalized Legendre functions $\bar{P}_{lm}(\cos\theta)$ contain dependent on degree l and order m? Compare results from the two aforementioned formulas. How many zero crossings do the fully normalized spherical harmonics $Y_{lm}(\theta,\lambda)$ contain in North-South direction and East-West direction dependent on degree l and order m?

- Rodrigues-Ferrers

$$P_l(t) = \frac{1}{2^l l!}\frac{\mathrm{d}^l\left(t^2-1\right)^l}{\mathrm{d}t^l} \qquad t = \cos\theta = \sin\phi$$

$$P_{l,m}(t) = \left(1-t^2\right)^{\frac{m}{2}}\frac{\mathrm{d}^m P_l(t)}{\mathrm{d}t^m}$$

- Recursive
  - ✓ Initials

$$P_{0,0}(t) = 1 \quad P_{1,0}(t) = t \quad P_{1,1}(t) = \sqrt{1-t^2}$$

  - ✓ Sectorial

$$P_{l,l}(t) = (2l-1)\sqrt{1-t^2}P_{l-1,l-1}(t)$$

  - ✓ Other

$$P_{l,m}(t) = \frac{1}{l-m}\left\{(2l-1)tP_{l-1,m}(t) - (l-1+m)P_{l-2,m}(t)\right\}$$

Spherical harmonic expansion

Normalized Legendre functions

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- Normalization

$$\bar{P}_{l,m}(t) = \begin{cases} \sqrt{2l+1}\,P_{l,m}(t) & m = 0 \\ \sqrt{2(2l+1)\dfrac{(l-m)!}{(l+m)!}}\,P_{l,m}(t) & m > 0 \end{cases}$$

- Recursive
  - ✓ Initials

$$\bar{P}_{0,0}(t) = 1 \quad \bar{P}_{1,1}(t) = \sqrt{3(1-t^2)}$$

  - ✓ Sectorial

$$\bar{P}_{l,l}(t) = \sqrt{\frac{2l+1}{2l}}\sqrt{1-t^2}\,\bar{P}_{l-1,l-1}(t)$$

  - ✓ Other

$$\bar{P}_{l,m}(t) = \sqrt{\frac{2l+1}{(l+m)(l-m)}}\left\{\sqrt{2l-1}\,t\,\bar{P}_{l-1,m}(t) - \sqrt{\frac{(l-1+m)(l-1-m)}{2l-3}}\,\bar{P}_{l-2,m}(t)\right\}$$

Spherical harmonic expansion

Normalized Legendre functions (examples)

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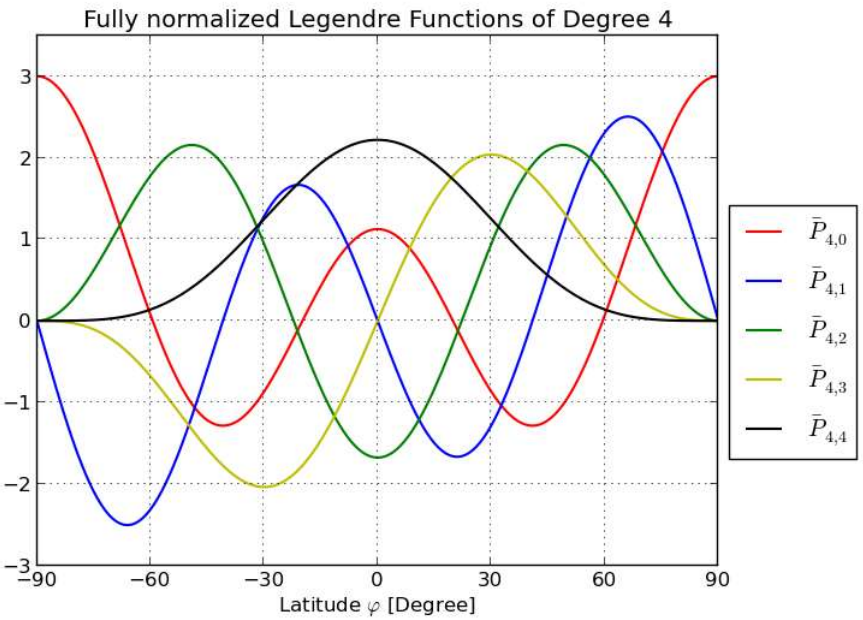
$$\bar{P}_{4,m}(\cos\theta) \quad m = 0,\dots,4$$



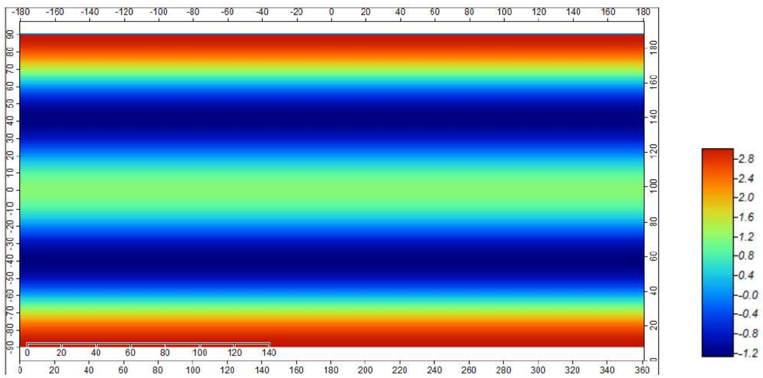
Spherical harmonic expansion

Normalized surface spherical harmonics (examples)

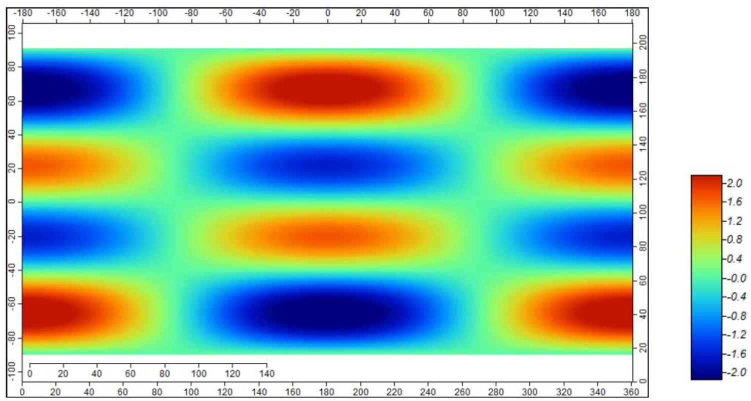
Task 1: Prepare figures for fully normalized zonal, tesseral and sectorial Legendre functions $\bar{P}_{lm}(\cos\theta)$ and spherical harmonics $\bar{Y}_{lm}(\theta,\lambda) = \bar{P}_{lm}(\cos\theta)\cos m\lambda$ of degree $l = 10$ within $\theta \in [0^\circ\ 180^\circ]$ using both Rodrigues-Ferrers and recursive formulas. How many zero crossings do the fully normalized Legendre functions $\bar{P}_{lm}(\cos\theta)$ contain dependent on degree l and order m? Compare results from the two aforementioned formulas. How many zero crossings do the fully normalized spherical harmonics $Y_{lm}(\theta,\lambda)$ contain in North-South direction and East-West direction dependent on degree l and order m?

$$\bar{Y}_{l,m}(\theta,\lambda) = \bar{P}_{lm}(\cos\theta)\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}$$

$\bar{Y}_{4,0}(\theta,\lambda)$



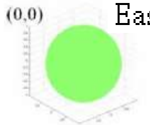
$\bar{Y}_{4,1}(\theta,\lambda)$

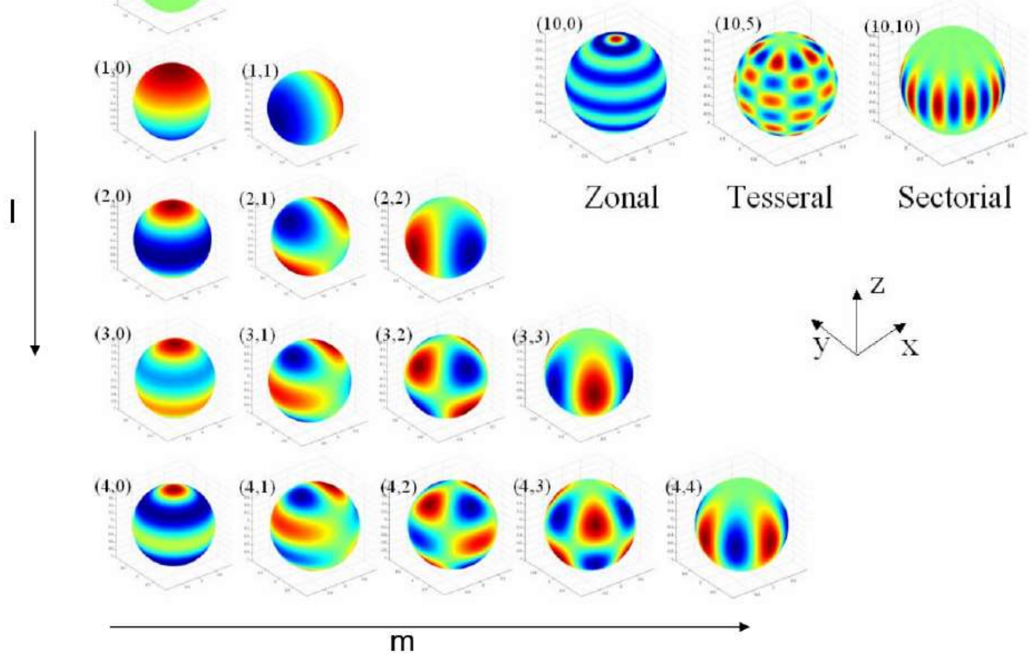


Spherical harmonic expansion

Normalized surface spherical harmonics: examples

Task 1: Prepare figures for fully normalized zonal, tesseral and sectorial Legendre functions $\bar{P}_{lm}(\cos\theta)$ and spherical harmonics $\bar{Y}_{lm}(\theta,\lambda) = \bar{P}_{lm}(\cos\theta)\cos m\lambda$ of degree $l = 10$ within $\theta \in [0^\circ\ 180^\circ]$ using both Rodrigues-Ferrers and recursive formulas. How many zero crossings do the fully normalized Legendre functions $\bar{P}_{lm}(\cos\theta)$ contain dependent on degree l and order m? Compare results from the two aforementioned formulas. How many zero crossings do the fully normalized spherical harmonics $Y_{lm}(\theta,\lambda)$ contain in North-South direction and East-West direction dependent on degree l and order m?





$$\bar{Y}_{l,m}(\theta,\lambda) = \bar{P}_{lm}(\cos\theta)\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}$$

Spherical harmonic expansion

Addition theorem

Task 2: Consider a Legendre polynomial in $\cos\psi_{PQ}$, in which the ψ_{PQ} is the spherical distance between point P and Q. The addition separates the composite angle argument into contributions from the point P and Q individually

$$P_l(\cos\psi_{PQ}) = \frac{1}{2l+1}\sum_{m=0}^{l}\bar{P}_{lm}(\cos\theta_P)\,\bar{P}_{lm}(\cos\theta_Q)\{\cos m\lambda_P \cos m\lambda_Q + \sin m\lambda_P \sin m\lambda_Q\}$$

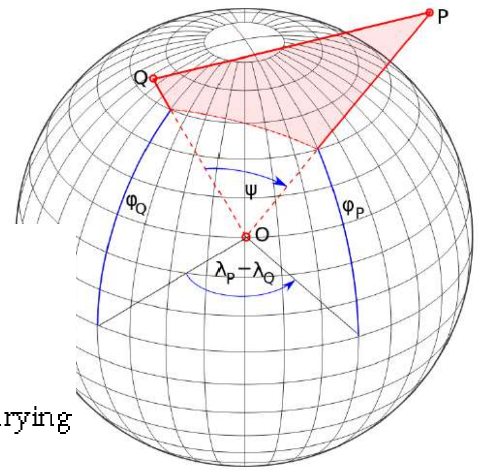
For all P and Q in a same meridian ($\lambda_P = \lambda_Q$), we have

$$P_l(\cos\psi_{PQ}) = \frac{1}{2l+1}\sum_{m=0}^{l}\bar{P}_{lm}(\cos\theta_P)\bar{P}_{lm}(\cos\theta_Q)$$

For $\theta_P = 90^o$ and $\theta_Q \in [0^o\ 90^o]$ display the difference between the right and left hand side of above equation for different ψ and for different degree l varying from 0 to 100.

Spherical distance

$$\cos\psi = \sin\varphi_P \sin\varphi_Q + \cos\varphi_P \cos\varphi_Q \cos(\lambda_P - \lambda_Q)$$



Task 3: When $\theta_P = \theta_Q = \theta$ then we have

$$P_l(1) = 1 = \frac{1}{2l+1}\sum_{m=0}^{l}\bar{P}^2_{lm}(\cos\theta)$$

For $\theta = [0^o\ 180^o]$ display the right hand side of above equation for different degree l varying from 0 to 100. Do you get 1 for all degree and θ?

Spherical harmonic expansion

Spherical harmonic synthesis

The gravitational potential V in the exterior (mass-free) domain is determined by means of a spherical harmonics series expansion as

$$V(\lambda,\theta,r) = \frac{GM}{R}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^{l+1}\sum_{m=0}^{l}\bar{P}_{l,m}(\cos\theta)(\bar{c}_{l,m}\cos m\lambda + \bar{s}_{l,m}\sin m\lambda)$$

Various models with coefficients $\bar{c}_{lm}$ and $\bar{s}_{lm}$ exist, which have been estimated for instance from the analysis of terrestrial or satellite gravity data. One of these models is the EGM96 (Earth Gravity Model 1996) of the NASA.

Task 4: Determine the gravity and gravitational potential W and V at a point P with the following spherical coordinates by applying the EGM96 (available at ILIAS)

$$\begin{aligned} \lambda &= (10+k)^\circ \\ \theta &= (42+k)^\circ \\ r &= 6379245.458\ [\mathrm{m}] \end{aligned}$$

- Gravity

$$W = V + V_c$$

- Gravitation

$$V(r,\theta,\lambda) = \frac{GM}{R}\sum_{l=0}^{L}\left(\frac{R}{r}\right)^{l+1}\sum_{m=0}^{l}\bar{c}_{lm}\bar{Y}_{lm}^{c}(\theta,\lambda) + \bar{s}_{lm}\bar{Y}_{lm}^{s}(\theta,\lambda)$$

- Centrifugal

$$V_c(r,\theta,\lambda) = \frac{1}{2}\omega^2 r^2 \sin^2\theta$$

Spherical harmonic expansion

Spherical harmonic synthesis

$$V(r,\theta,\lambda) = \frac{GM}{R}\sum_{l=0}^{L}\left(\frac{R}{r}\right)^{l+1}\sum_{m=0}^{l}\bar{c}_{lm}\bar{Y}_{lm}^{c}(\theta,\lambda) + \bar{s}_{lm}\bar{Y}_{lm}^{s}(\theta,\lambda)$$

EGM96

0	0	1.000000e+000	0
1	0	0	0
1	1	0	0
2	0	-4.84165371736e-004	0
2	1	-1.86987635955e-010	1.19528012031e-009
2	2	2.43914352397e-006	-1.40016683653e-006
3	0	9.57254173791e-007	0
3	1	2.02998882184e-006	2.48513158715e-007
3	2	9.04627768604e-007	-6.19025944204e-007
3	3	7.21072657057e-007	1.41435626957e-006
4	0	5.39873863789e-007	0
4	1	-5.36321616971e-007	-4.73440265853e-007
4	2	3.50694105785e-007	6.62671572540e-007
4	3	9.90771803829e-007	-2.00928369177e-007
4	4	-1.88560802734e-007	3.08853169333e-007
5	0	6.85323475630e-008	0
5	1	-6.21012128527e-008	-9.44226127525e-008
5	2	6.52438297612e-007	-3.23349612668e-007
5	3	-4.51955406070e-007	-2.14847190624e-007
5	4	-2.95301647654e-007	4.96658876768e-008
5	5	1.74971983203e-007	-6.69384278218e-007
6	0	-1.49957994713e-007	0
...	...	...	...

Spherical harmonic expansion

Spherical harmonic synthesis

Task 5: Using the spherical harmonic series expression, for an airborne gravimetry campaign discuss the question whether the airplane should fly as high or rather as low as possible to capture a better representation of gravity field.