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Dynamic System Estimation

Random Processes

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Random Processes

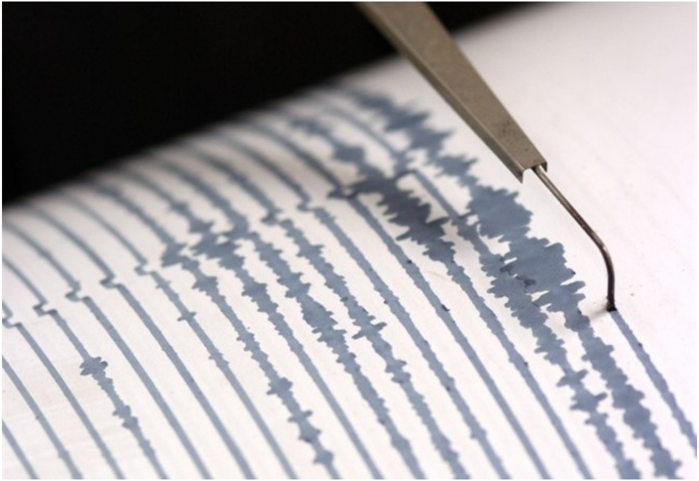
Review:
A realisation of a random variable (RV) is the outcome of an experiment. Performing the experiment once gives one realisation of the RV. Measurement errors are typically continuous RVs that can take any value according to an associated probability density distribution.
Example: the Gaussian bell-shaped probability density distribution.
RVs can be scalar or more-dimensional.

A realisation of a random process (RP, sometimes also called stochastic process) is obtained if the outcome of the experiment is a "function" of an independent variable, usually time. A RP is called continuous, if its argument is continuous time, $x(t)$. A RP is called discrete, if its argument is a discrete variable, $x(t_i), i = 1, 2, 3, \ldots$ Performing the experiment once gives one realisation of the RP. RPs can be scalar or more-dimensional.

Can you think of examples for continuous and discrete random processes?

Random Processes - cont'd

Examples for continuous and discrete random processes?



Random Processes - cont'd

Example (after A. Gelb (ed.), 1984. Applied Optimal Estimation, MIT Press). Shown are 4 realisations of a scalar RP. There may be infinitely many "realisations" constituting the ensemble of the RP.

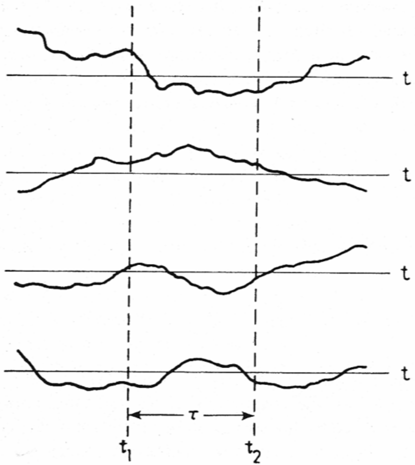


Figure 4.1: Example for scalar random process

Random Processes

If $\boldsymbol{y}(t)$ is a RP and $p(\boldsymbol{y}(t))$ is its probability density distribution, than the expected value of the RP at time t is its mean value:

$$\boldsymbol{\mu}(t) = E(\boldsymbol{y}(t)) = \int_{-\infty}^{\infty} \boldsymbol{y}(t) p(\boldsymbol{y}(t)) d\boldsymbol{y}(t) \tag{4.1}$$

The auto-covariance and the auto-correlation of the RP are measures for the self-similarity at two epochs in time.

$$\begin{aligned} \mathrm{Cov}_{yy}(t_1, t_2) &= E((\boldsymbol{y}(t_1) - \boldsymbol{\mu}(t_1))(\boldsymbol{y}(t_2) - \boldsymbol{\mu}(t_2))^T) \\ \mathrm{Cor}_{yy}(t_1, t_2) &= E(\boldsymbol{y}(t_1)\boldsymbol{y}(t_2))^T) \end{aligned} \tag{4.2}$$

The cross-covariance of two RP is a measure for the similarity of the two RP at two epochs in time.

$$\mathrm{Cov}_{xy}(t_1, t_2) = E((\boldsymbol{x}(t_1) - \boldsymbol{\mu}_{\boldsymbol{x}}(t_1))(\boldsymbol{y}(t_2) - \boldsymbol{\mu}_{\boldsymbol{y}}(t_2))^T) \tag{4.3}$$

Random Processes

Stationarity: A RP is said to be stationary, if its probability density distribution is independent of time, i.e.:

$$p(\boldsymbol{y}(t)) = p(\boldsymbol{y}(t+\Delta t)) \tag{4.4}$$

For stationary RPs, the mean value $\boldsymbol{\mu}$ is independent of time, and the auto-covariance and auto-correlation functions depend only on the time interval:

$$\begin{aligned} \mathrm{Cov}_{yy}(\Delta t) &= E((\boldsymbol{y}(t)-\boldsymbol{\mu})(\boldsymbol{y}(t+\Delta t)-\boldsymbol{\mu})^T) \\ \mathrm{Cor}_{yy}(\Delta t) &= E(\boldsymbol{y}(t)\boldsymbol{y}(t+\Delta t)^T) \end{aligned} \tag{4.5}$$

For stationary RPs, the auto-covariance is an even function, which attains its maximum at zero lag; for zero lag, the auto-covariance function value is the variance of the stationary RP:

$$\mathrm{Cov}_{yy}(\Delta t) = \mathrm{Cov}_{yy}(-\Delta t), \quad |\mathrm{Cov}_{yy}(\Delta t)| \leq \mathrm{Cov}_{yy}(0) \tag{4.6}$$

Random Processes

For stationary RPs also the cross-covariance function depends only on the time interval:

$$\mathrm{Cov}_{xy}(\Delta t) = E((\boldsymbol{x}(t) - \boldsymbol{\mu}_{\boldsymbol{x}})(\boldsymbol{y}(t+\Delta t) - \boldsymbol{\mu}_{\boldsymbol{y}})^T) \tag{4.7}$$

Equ. (4.6) does not hold for the cross-covariance function.

The Fourier transform of the auto-correlation function is called the power spectral density (psd) of the RP.

$$\boxed{\Phi_y(f) = \int_{-\infty}^{\infty} \mathrm{Cor}_{yy}(\tau) e^{-i2\pi f \tau} d\tau} \tag{4.8}$$

Similarly, the auto-correlation function of the RP is the inverse Fourier transform of the power spectral density of the RP.

$$\boxed{\mathrm{Cor}_{yy}(\tau) = \int_{-\infty}^{\infty} \Phi_y(f) e^{i2\pi f \tau} df} \tag{4.9}$$

Random Processes

Ergodicity: A stationary RP is called ergodic, if the statistics of the RP (mean, variance, etc.) can be derived from a single realisation of the RP by operations in the time domain. The time average of a single realisation of the RP is:

$$\boldsymbol{m} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+t/2} \boldsymbol{y}(t)dt \tag{4.10}$$

Ergodicity means, that this time average is equal to the ensemble average (eqn. (4.1)). Similarly, the the auto- and cross-correlation functions for ergodic RPs can be computed from:

$$\begin{aligned} \mathrm{Cov}_{yy}(\Delta t) &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+t/2} (\boldsymbol{y}(t) - \boldsymbol{m})(\boldsymbol{y}(t+\Delta t) - \boldsymbol{m})^T dt \\ \mathrm{Cov}_{xy}(\Delta t) &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+t/2} (\boldsymbol{x}(t) - \boldsymbol{m}_x)(\boldsymbol{y}(t+\Delta t) - \boldsymbol{m}_y)^T dt \end{aligned} \tag{4.11}$$

Random Processes

White noise: A stationary RP is called a white noise process, if its auto-covariance function is zero for non-zero lag (here: zero mean white noise):

$$\mathrm{Cov}_{ww}(\Delta t) = \sigma^2 \delta(t - \Delta t)$$
$$\delta(t - \Delta t) = 0 \text{ for } t \neq 0, \qquad \int_{-\infty}^{+\infty} g(t')\delta(t')dt = g(t) \tag{4.12}$$

The psd of a white noise process is obtained from eqn. (4.8):

$$\Phi_w(f) = \int_{-\infty}^{+\infty} \sigma^2 \delta(\tau) e^{-i2\pi\tau} d\tau = \sigma^2 \tag{4.13}$$

The psd of a white noise process is a constant. Its spectral power is evenly distributed; hence the name “white noise”. Inserting eqn. (4.13) into (4.9), we obtain a definition for the Dirac function in terms of the integral of a complex exponential function.

$$\int_{-\infty}^{+\infty} \sigma^2 e^{i2\pi\tau} ds = \sigma^2 \delta(\tau) \tag{4.14}$$

Random Processes

Random constants: A random constant is a stationary RP that take on a constant value for all times, but the constant value is a random variable that changes for each realisation. Examples??

The random constant can be described by its differential equation:

$$\boxed{\dot{R}(t) = 0, \quad R(t_0) = R_0} \tag{4.15}$$

The random constant is not an ergodic RP. The auto-covariance function for a random constant is:

$$\boxed{\mathrm{Cov}_{RR}(\Delta t) = \sigma_R^2} \tag{4.16}$$

The RP is fully correlated. The psd of the random constant is obtained from eqn. (4.8):

$$\boxed{\Phi_R(f) = \int_{-\infty}^{+\infty} \sigma_R^2 e^{-i2\pi\tau} d\tau = \sigma_R^2 \delta(f)} \tag{4.17}$$

Random Processes

Random walk: A random walk is a RP described by the following differential equation:

$$\boxed{\begin{aligned} &\dot{R}(t) = W(t), \quad R(t_0) = 0 \\ &\Rightarrow R(t) = \int\limits_{t_0}^{t} W(t)dt \end{aligned}} \tag{4.18}$$

The random walk process is not a stationary RP. Its auto-covariance function is given by

$$\boxed{\mathrm{Cov}_{RR}(t_1, t_2) = \begin{cases} \sigma^2(t_2 - t_1) & \text{if } t_2 \geq t_1 \\ \sigma^2(t_1 - t_2) & \text{if } t_1 > t_2 \end{cases}} \tag{4.19}$$

σ^2 is the amplitude of the psd of the white noise RP $W(t)$.

Examples??

Random Processes

Gaussian white noise random process: A white noise RP is called a Gaussian white noise RP, if the probability density distribution of the underlying random variable is the Gaussian distribution (bell-shaped curve).

If the white noise RP underlying the random walk RP is a Gaussian white noise RP, then it is called a Wiener process; the Wiener process is the integral of the Gaussian white noise RP.

Gauss-Markov-process of first order: The differential equation

$$\boxed{\dot{X}(t) = -\beta X(t) + W(t), \quad \beta > 0} \tag{4.20}$$

where $W(t)$ is a zero-mean Gaussian white noise RP with psd amplitude equal to $2\sigma^2\beta$ describes a Gauss-Markov-process of first order. Its auto-covariance is given by:

$$\boxed{\mathrm{Cov}_{XX}(t_1, t_2) = \mathrm{Cov}_{XX}(\Delta t) = \sigma^2 e^{-\beta|\Delta t|}} \tag{4.21}$$

σ^2 is the variance of the RP; the RP is stationary.

Random Processes

The psd of a Gauss-Markov-process of first order is given by

$$\boxed{\Phi_X(f) = \frac{2\sigma^2\beta}{4\pi^2 f^2 + \beta^2}} \tag{4.22}$$

β is the parameter describing the correlation length of the RP.

Relation to Random Constant RP?

Relation to Gaussian white noise RP?

Shape of auto-covariance function? Shape of psd?

Random Processes

Gauss-Markov-process of second order: The differential equation

$$\ddot{X}(t) + 2\beta\dot{X}(t) + \beta^2 X(t) = W(t), \quad \beta \geq 0 \tag{4.23}$$

where $W(t)$ is a zero-mean Gaussian white noise RP with psd amplitude equal to $4\sigma^2\beta^3$ describes a Gauss-Markov-process of second order. Its auto-covariance and its psd are is given by:

$$\mathrm{Cov}_{XX}(t_1, t_2) = \sigma^2 e^{-\beta|\Delta t|}(1 + \beta\Delta t), \quad \Delta t = |t_2 - t_1| \tag{4.24}$$

$$\Phi_X(f) = \frac{2\sigma^2\beta^3}{(4\pi^2 f^2 + \beta^2)^2} \tag{4.25}$$

Shape of auto-covariance function and psd?

Random Processes

Gauss-Markov-process of third order: The differential equation

$$\boxed{\dddot{X}(t) + 3\beta\ddot{X}(t) + 3\beta^2\dot{X}(t) + \beta^3 X(t) = W(t), \quad \beta \geq 0} \tag{4.26}$$

where $W(t)$ is a zero-mean Gaussian white noise RP with psd amplitude equal to $16/3\sigma^2\beta^5$ describes a Gauss-Markov-process of third order. Its auto-covariance and its psd are is given by:

$$\boxed{\mathrm{Cov}_{XX}(t_1, t_2) = \sigma^2 e^{-\beta|\Delta t|}(1 + \beta\Delta t + \frac{1}{3}\beta^2\Delta t^2), \;\; \Delta t = |t_2 - t_1|} \tag{4.27}$$

$$\boxed{\Phi_X(f) = \frac{\frac{16}{3}\sigma^2\beta^5}{(4\pi^2 f^2 + \beta^2)^3}} \tag{4.28}$$

Shape of auto-covariance function and psd?

Gauss-Markov-processes of higher order defined by the appropriate differential equations. Relation between Gauss-Markov-processes of different order?

Random Processes

Discrete random processes: The continuous RPs "random constant", "random walk" and "first order Markov process" can be described by the differential equation

$$\dot{X}(t) = -\beta X(t) + \alpha W(t), \quad \alpha, \beta \geq 0 \tag{4.29}$$

where $W(t)$ is a zero-mean Gaussian white noise RP. For the "random constant" both α and β are zero. For the "random walk" β is zero and α is unity. For the "first order Markov process" β is non-zero positive and α is unity. For the discrete counterparts of these continuous RPs the differential equation (4.29) is replaced by the difference equation

$$X_{n+1} = b_n X_n + a_{n+1} W_{n+1} \tag{4.30}$$

where W_n is a zero-mean Gaussian white noise random sequence with variance σ^2.

- For the discrete "random constant" $b_n = 1$ and $a_n = 0$.
- For the discrete "random walk", $b_n = 1$ and $a_n = 1$.
- For the discrete "first order Markov process", $b_n \neq 0$ and $a_n = 1$.

Random Processes

Variance propagation: (see basic lectures on adjustment theory).

Random constant:

$$\boxed{\begin{aligned} X_{n+1} &= X_n \\ \sigma^2_{X,n+1} &= \sigma^2_{X,n} \end{aligned}} \tag{4.31}$$

Random walk:

$$\boxed{\begin{aligned} X_{n+1} &= X_n + W_{n+1} \\ \sigma^2_{X,n+1} &= \sigma^2_{X,n} + \sigma^2_{W,n+1} \end{aligned}} \tag{4.32}$$

First order Markov process (exponential decay in autocorrelation):

$$\boxed{\begin{aligned} X_{n+1} &= b_n X_n + a_{n+1} W_{n+1} \\ \sigma^2_{X,n+1} &= b_n^2 \sigma^2_{X,n} + a^2_{n+1} \sigma^2_{W,n+1} \end{aligned}} \tag{4.33}$$