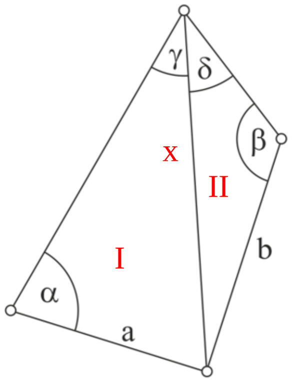
a) In order to determine the redundancy two possibilities exist: (a) Find those observations which can be cancelled without causing the network to collapse. (b) Use A-model: Because of m=6, n=8 (coordinates) and d=3 (2 translations, 1 rotation) the redundancy is $r = m-(n-d) = 1$.

b) All observations have to be processed and all condition equations must be linearly independent.

c) There is only one condition equation to be built from the two triangles with common (but unknown) side x:



Triangle I: $\dfrac{a}{\sin\gamma} = \dfrac{x}{\sin\alpha} \Rightarrow x = \dfrac{a\sin\alpha}{\sin\gamma}$

Triangle II: $\dfrac{b}{\sin\delta} = \dfrac{x}{\sin\beta} \Rightarrow x = \dfrac{b\sin\beta}{\sin\delta}$

Equating both equations and introducing inconsistencies, the final condition equation then reads

$$f(e_a, e_b, e_\alpha, e_\beta, e_\gamma, e_\delta) = (a-e_a)\sin(\alpha-e_\alpha)\sin(\delta-e_\delta) - (b-e_b)\sin(\beta-e_\beta)\sin(\gamma-e_\gamma) = 0.$$

d) If we define the following ordering in the unknown inconsistencies $e = [e_a, e_b, e_\alpha, e_\beta, e_\gamma, e_\delta]'$ and if $e_0 = [e_a^0, e_b^0, e_\alpha^0, e_\beta^0, e_\gamma^0, e_\delta^0]'$ is the vector of approximate values, then the misclosure is $w = f(e_0) + B'e_0$ with matrix B' of coefficients (which has to be evaluated at the Taylor point $e = e_0$)

$$B' = \left[-\frac{\partial f}{\partial e_a} \quad -\frac{\partial f}{\partial e_b} \quad -\frac{\partial f}{\partial e_\alpha} \quad -\frac{\partial f}{\partial e_\beta} \quad -\frac{\partial f}{\partial e_\gamma} \quad -\frac{\partial f}{\partial e_\delta} \right]\Bigg|_{e=e_0}$$

The partial derivatives are

$$\frac{\partial f}{\partial e_a} = -\sin(\alpha-e_\alpha)\sin(\delta-e_\delta) \quad , \quad \frac{\partial f}{\partial e_b} = \sin(\beta-e_\beta)\sin(\gamma-e_\gamma)$$

$$\frac{\partial f}{\partial e_\alpha} = -(a-e_a)\cos(\alpha-e_\alpha)\sin(\delta-e_\delta) \quad , \quad \frac{\partial f}{\partial e_\beta} = (b-e_b)\cos(\beta-e_\beta)\sin(\gamma-e_\gamma)$$

$$\frac{\partial f}{\partial e_\gamma} = (b-e_b)\sin(\beta-e_\beta)\cos(\gamma-e_\gamma) \quad , \quad \frac{\partial f}{\partial e_\delta} = -(a-e_a)\sin(\alpha-e_\alpha)\cos(\delta-e_\delta).$$

e) In the first iteration $e_0 = [0,0,0,0,0,0]'$, while in subsequent iterations the most recent values $e_0 = [\hat{e}_a, \hat{e}_b, \hat{e}_\alpha, \hat{e}_\beta, \hat{e}_\gamma, \hat{e}_\delta]'$ will be used.

f) The normal equations are obtained from minimizing the Lagrange function

$$\mathcal{L}_B(e,\lambda) = \frac{1}{2}e'e + \lambda'(w - B'e) \to \min_{e,\lambda} \quad \Rightarrow \quad \begin{bmatrix} \underset{6\times6}{I} & \underset{6\times1}{-B} \\ \underset{1\times6}{-B'} & \underset{1\times1}{0} \end{bmatrix} \begin{bmatrix} \underset{6\times1}{\hat{e}} \\ \underset{1\times1}{\hat{\lambda}} \end{bmatrix} = \begin{bmatrix} \underset{6\times1}{0} \\ \underset{1\times1}{-w} \end{bmatrix}$$

a)

$$\left.\begin{aligned} d_1 &= s_1 \\ d_2 &= s_2 \\ d_3 &= s_3 \\ d_4 &= s_1 + s_2 \\ d_5 &= s_2 + s_3 \\ d_6 &= s_1 + s_2 + s_3 \end{aligned}\right\} \quad y = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix}, \quad x = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

b)

$$\left.\begin{aligned} d_1 - e_1 + d_2 - e_2 + d_3 - e_3 - d_6 + e_6 &= 0 \\ d_1 - e_1 + d_2 - e_2 - d_4 + e_4 &= 0 \\ d_2 - e_2 + d_3 - e_3 - d_5 + e_5 &= 0 \end{aligned}\right\} \quad y = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}$$

$$B' = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \end{bmatrix}, \quad w = B'y = \begin{bmatrix} 0.07 \\ 0.05 \\ -0.03 \end{bmatrix}$$

c)

$$\hat{e} = B\hat{\lambda} = B(B'B)^{-1}w$$

$$B'B = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}, (B'B)^{-1} = \frac{1}{16}\begin{bmatrix} 8 & -4 & -4 \\ -4 & 8 & 0 \\ -4 & 0 & 8 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}, \hat{\lambda} = \frac{1}{4}\begin{bmatrix} 0.12 \\ 0.03 \\ -0.13 \end{bmatrix}$$

$$\hat{e} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0.07 \\ 0.05 \\ -0.03 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0.12 \\ 0.03 \\ -0.13 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0.15 \\ 0.02 \\ -0.01 \\ -0.03 \\ 0.13 \\ -0.12 \end{bmatrix}$$

d) $A'\hat{e} = 0$ ✓ and/or $\hat{e}'\hat{e} - \hat{\lambda}'w = 0$ ✓ and/or $w - B'\hat{e} = 0$ ✓

This is a typical situation/problem of error propagation. The standard deviation σ_θ follows from the standard deviations σ_ϕ and σ_λ as follows:

$$\theta = \sqrt{(\phi - B)^2 + (\lambda - L)^2 \cos^2 B}$$

$$d\theta = \frac{\partial\theta}{\partial\phi} d\phi + \frac{\partial\theta}{\partial\lambda} d\lambda = \begin{bmatrix} \frac{\partial\theta}{\partial\phi} & \frac{\partial\theta}{\partial\lambda} \end{bmatrix} \begin{bmatrix} d\phi \\ d\lambda \end{bmatrix}$$

$$\sigma_\theta^2 = \begin{bmatrix} \frac{\partial\theta}{\partial\phi} & \frac{\partial\theta}{\partial\lambda} \end{bmatrix} \begin{bmatrix} \sigma_\phi^2 & \\ & \sigma_\lambda^2 \end{bmatrix} \begin{bmatrix} \frac{\partial\theta}{\partial\phi} \\ \frac{\partial\theta}{\partial\lambda} \end{bmatrix} = \left(\frac{\partial\theta}{\partial\phi}\right)^2 \sigma_\phi^2 + \left(\frac{\partial\theta}{\partial\lambda}\right)^2 \sigma_\lambda^2$$

$$\frac{\partial\theta}{\partial\phi} = \frac{\partial\theta}{\partial\xi}\frac{\partial\xi}{\partial\phi} = \frac{\partial\theta}{\partial\xi} = \frac{\phi - B}{\theta} = \frac{\xi}{\theta}$$

$$\frac{\partial\theta}{\partial\lambda} = \frac{\partial\theta}{\partial\eta}\frac{\partial\eta}{\partial\lambda} = \frac{(\lambda - L)\cos^2 B}{\theta} = \frac{\eta\cos B}{\theta}$$

$$\sigma_\theta = \pm\frac{1}{\theta}\sqrt{\xi^2\sigma_\phi^2 + \eta^2\cos^2 B\sigma_\lambda^2}$$

$$\xi = 4,2''$$

$$\eta = -2,34''$$

$$\theta = 4,81''$$

$$\sigma_\theta = \pm 0,34''$$

a) The A-model formulation is $\Delta v = A\Delta x + e$ where

$$\underset{5\times1}{\Delta v} := \begin{bmatrix} v(u_1) - v(\alpha_0, \beta_0, u_1) \\ \vdots \\ v(u_5) - v(\alpha_0, \beta_0, u_5) \end{bmatrix} = v - v_0 \text{ , } v_0 = \frac{e^{\alpha_0 + \beta_0 u}}{1 + e^{\alpha_0 + \beta_0 u}} \qquad \text{"observed-computed"}$$

$$\Delta x = \begin{bmatrix} \alpha - \alpha_0 \\ \beta - \beta_0 \end{bmatrix} = \begin{bmatrix} \Delta\alpha \\ \Delta\beta \end{bmatrix} \qquad \text{"vector of unknowns"}$$

$$\underset{5\times2}{A} = \begin{bmatrix} \frac{\partial v}{\partial \alpha} & \frac{\partial v}{\partial \beta} \end{bmatrix}_{\substack{\alpha=\alpha_0 \\ \beta=\beta_0}} \qquad \text{"design matrix"}$$

$$\underset{5\times1}{e} = \begin{bmatrix} e_{v_1} \\ \vdots \\ e_{v_5} \end{bmatrix} \qquad \text{"inconsistencies"}$$

b) There is no datum problem: d=0

c) The redundancy is $r = m - (n - d)$, where m=5 is the number of observations, n=2 is the number of unknowns and d=0 is the datum defect. Here, we have a redundancy of r=3.

d) $$\Delta x = \begin{bmatrix} \Delta\alpha \\ \Delta\beta \end{bmatrix} = (A'A)^{-1} A'\Delta v \text{ , } \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} + \begin{bmatrix} \Delta\alpha \\ \Delta\beta \end{bmatrix}$$

e)

$$\Delta v = \begin{bmatrix} v(u_1) - v(\alpha_0, \beta_0, u_1) \\ \vdots \\ v(u_5) - v(\alpha_0, \beta_0, u_5) \end{bmatrix} = v - v_0 \text{ , } v_0 = \frac{e^{\alpha_0 + \beta_0 u}}{1 + e^{\alpha_0 + \beta_0 u}} \qquad \text{"observed-computed"}$$

$$\Delta x = \begin{bmatrix} \alpha - \alpha_0 \\ \beta - \beta_0 \end{bmatrix} = \begin{bmatrix} \Delta\alpha \\ \Delta\beta \end{bmatrix} \qquad \text{"vector of unknowns"}$$

$$A = \begin{bmatrix} \frac{e^{\alpha_0 + \beta_0 u_1}}{(1 + e^{\alpha_0 + \beta_0 u_1})^2} & \frac{u_1 e^{\alpha_0 + \beta_0 u_1}}{(1 + e^{\alpha_0 + \beta_0 u_1})^2} \\ \vdots & \vdots \\ \frac{e^{\alpha_0 + \beta_0 u_5}}{(1 + e^{\alpha_0 + \beta_0 u_5})^2} & \frac{u_5 e^{\alpha_0 + \beta_0 u_5}}{(1 + e^{\alpha_0 + \beta_0 u_5})^2} \end{bmatrix} = \begin{bmatrix} \frac{v_1}{1 + e^{\alpha_0 + \beta_0 u_1}} & \frac{u_1 v_1}{1 + e^{\alpha_0 + \beta_0 u_1}} \\ \vdots & \vdots \\ \frac{v_5}{1 + e^{\alpha_0 + \beta_0 u_5}} & \frac{u_5 v_5}{1 + e^{\alpha_0 + \beta_0 u_5}} \end{bmatrix} \qquad \text{"designmatrix"}$$

$$e = \begin{bmatrix} e_{v_1} \\ \vdots \\ e_{v_5} \end{bmatrix} \qquad \text{"inconsistencies"}$$

f) The point constraint has to be linearized as it was done with the observation equations.

Starting from $\tilde{v} = \tilde{v}(\tilde{u}) = \dfrac{e^{\alpha+\beta\tilde{u}}}{1+e^{\alpha+\beta\tilde{u}}}$ we get

$$\tilde{v} = \frac{e^{\alpha_0+\beta_0\tilde{u}}}{1+e^{\alpha_0+\beta_0\tilde{u}}} + \frac{e^{\alpha_0+\beta_0\tilde{u}}}{(1+e^{\alpha_0+\beta_0\tilde{u}})^2}\Delta\alpha + \frac{\tilde{u}e^{\alpha_0+\beta_0\tilde{u}}}{(1+e^{\alpha_0+\beta_0\tilde{u}})^2}\Delta\beta \Rightarrow$$

$$\underbrace{\tilde{v} - \frac{e^{\alpha_0+\beta_0\tilde{u}}}{1+e^{\alpha_0+\beta_0\tilde{u}}}}_{\Delta\tilde{v}} - \underbrace{\frac{e^{\alpha_0+\beta_0\tilde{u}}}{(1+e^{\alpha_0+\beta_0\tilde{u}})^2}\begin{bmatrix}1 & \tilde{u}\end{bmatrix}}_{D'}\underbrace{\begin{bmatrix}\Delta\alpha\\ \Delta\beta\end{bmatrix}}_{\Delta x} = 0$$

This (homogeneous) equation is taken into account in the Lagrange function using a Lagrange multiplier λ

$$\mathcal{L}_A(\Delta x, \lambda) = \frac{1}{2}e'e + \lambda'(D'\Delta x - c) = \frac{1}{2}(\Delta v - A\Delta x)'(\Delta v - A\Delta x) + \lambda'(D'\Delta x - \Delta\tilde{v}) = \min_{\Delta x, \lambda}.$$

The necessary conditions for a minimum of $\mathcal{L}_A$ lead to an (extended) set of normal equation, the solution of which results (after iteration) in the required parameter estimates

$$\left.\begin{aligned}\frac{\partial\mathcal{L}_A}{\partial\Delta x}(\Delta x, \hat{\lambda}) &= A'A\Delta x + D\hat{\lambda} - A'\Delta v \overset{!}{=} 0\\ \frac{\partial\mathcal{L}_A}{\partial\lambda}(\Delta x, \hat{\lambda}) &= D'\Delta x - \Delta\tilde{v} \overset{!}{=} 0\end{aligned}\right\} \Rightarrow \begin{bmatrix}\underset{2\times2}{A'A} & \underset{2\times1}{D}\\ \underset{1\times2}{D'} & \underset{1\times1}{0}\end{bmatrix}\begin{bmatrix}\underset{2\times1}{\Delta x}\\ \underset{1\times1}{\hat{\lambda}}\end{bmatrix} = \begin{bmatrix}\underset{2\times1}{A'\Delta v}\\ \underset{1\times1}{\Delta\tilde{v}}\end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix}\underset{2\times1}{\Delta x}\\ \underset{1\times1}{\hat{\lambda}}\end{bmatrix} = \begin{bmatrix}\underset{2\times2}{A'A} & \underset{2\times1}{D}\\ \underset{1\times2}{D'} & \underset{1\times1}{0}\end{bmatrix}^{-1}\begin{bmatrix}\underset{2\times1}{A'\Delta v}\\ \underset{1\times1}{\Delta\tilde{v}}\end{bmatrix}$$

g) Whenever constraints are added to the problem the square sum of estimated residuals $\hat{e}'\hat{e}$ will increase.

Line and parabola parameters will be estimated from the A-model $y = Ax + e$ including continuity constraints $D'x = 0$. The continuity constraint results from the two requirements $a_1 + b_1 u_A = a_2 + b_2 u_A + c_2 u_A^2$, $a_3 + b_3 u_B = a_2 + b_2 u_B + c_2 u_B^2$, i.e. the function value of the left straight line shall equal the function value of the parabola in point A. The same shall hold for the right straight line and the parabola at point B. The elements of the A-model with constraints are

$$y = \begin{bmatrix} v_1 \\ \vdots \\ v_4 \\ \hdashline v_5 \\ \vdots \\ v_9 \\ \hdashline v_{10} \\ \vdots \\ v_{12} \end{bmatrix}, A = \left[\begin{array}{cc:ccc:cc} 1 & u_1 & & & & & \\ \vdots & \vdots & & & & & \\ 1 & u_4 & & & & & \\ \hdashline & & 1 & u_5 & u_5^2 & & \\ & & \vdots & \vdots & \vdots & & \\ & & 1 & u_9 & u_9^2 & & \\ \hdashline & & & & & 1 & u_{10} \\ & & & & & \vdots & \vdots \\ & & & & & 1 & u_{12} \end{array}\right], x = \begin{bmatrix} a_1 \\ b_1 \\ \hdashline a_2 \\ b_2 \\ c_2 \\ \hdashline a_3 \\ b_3 \end{bmatrix}$$

$$D' = \left[\begin{array}{cc:ccc:cc} 1 & u_A & -1 & -u_A & -u_A^2 & 0 & 0 \\ 0 & 0 & -1 & -u_B & -u_B^2 & 1 & u_B \end{array}\right], c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

a) From the vertical section on the right we find the height $H_T = H_A + i + s\cot z$. The horizontal distance s comes from the triangle $\dfrac{s}{\sin\beta} = \dfrac{b}{\sin(\alpha+\beta)}$ so that

$H_T = H_A + i + \dfrac{b\sin\beta\cot z}{\sin(\alpha+\beta)}$. For the purpose of $\sigma^2_{H_T}$ the law of error propagation must be applied after linearization of H_T. Because H_A and I are taken as error-free they are excluded from error propagation.

General matrix equation for $\sigma^2_{H_T}$

$$dH_T = \frac{\partial H_T}{\partial\alpha}d\alpha + \frac{\partial H_T}{\partial\beta}d\beta + \frac{\partial H_T}{\partial b}db + \frac{\partial H_T}{\partial z}dz$$

$$= \begin{bmatrix} \frac{\partial H_T}{\partial\alpha} & \frac{\partial H_T}{\partial\beta} & \frac{\partial H_T}{\partial b} & \frac{\partial H_T}{\partial z} \end{bmatrix} \begin{bmatrix} d\alpha \\ d\beta \\ db \\ dz \end{bmatrix}$$

$$\sigma^2_{H_T} = \begin{bmatrix} \frac{\partial H_T}{\partial\alpha} & \frac{\partial H_T}{\partial\beta} & \frac{\partial H_T}{\partial b} & \frac{\partial H_T}{\partial z} \end{bmatrix} \underset{4\times4}{\Sigma_{[\alpha,\beta,b,z]}} \begin{bmatrix} \frac{\partial H_T}{\partial\alpha} & \frac{\partial H_T}{\partial\beta} & \frac{\partial H_T}{\partial b} & \frac{\partial H_T}{\partial z} \end{bmatrix}'$$

b)

$$\frac{\partial H_T}{\partial\alpha} = \frac{-b\sin\beta\cot z\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)} \quad , \quad \frac{\partial H_T}{\partial\beta} = \frac{b\cot z\sin\alpha}{\sin^2(\alpha+\beta)}$$

$$\frac{\partial H_T}{\partial b} = \frac{\sin\beta\cot z}{\sin(\alpha+\beta)} \quad , \quad \frac{\partial H_T}{\partial z} = \frac{-b\sin\beta}{\sin(\alpha+\beta)\sin^2 z}$$

c) Because $\alpha+\beta = 100\text{ gon}$ and $z = 50\text{ gon}$ we get

$$\frac{\partial H_T}{\partial\alpha} = 0 \quad , \quad \frac{\partial H_T}{\partial\beta} = b\sin\alpha \quad , \quad \frac{\partial H_T}{\partial b} = \sin\beta \quad , \quad \frac{\partial H_T}{\partial z} = -2b\sin\beta$$

The assumption of no correlation gives $\Sigma_{[\alpha,\beta,b,z]} = \begin{bmatrix} \sigma^2_\alpha & 0 & 0 & 0 \\ 0 & \sigma^2_\beta & 0 & 0 \\ 0 & 0 & \sigma^2_b & 0 \\ 0 & 0 & 0 & \sigma^2_z \end{bmatrix}$ so that

$$\sigma^2_{H_T} = \sigma^2_\beta b^2\sin^2\alpha + \sigma^2_b\sin^2\beta + 4\sigma^2_z b^2\sin^2\beta$$

$$= b^2\left(\frac{0,005\text{ gon}}{200\text{ gon}/\pi}\right)^2\left(\sin^2\alpha + 4\sin^2\beta\right) + \sigma^2_b\sin^2\beta$$

$$\sigma_{H_T} = 1,16\text{ cm}$$

a) $f_i(a,b,e_{u_i},e_{v_i}) = v_i - [a + b(u_i - e_{u_i}) + e_{v_i}] = 0$

b)

$$
\begin{aligned}
& f_i(a,b,e_{u_i},e_{v_i}) = \\
&= f_i(a_0,b_0,e^0_{u_i},e^0_{v_i}) + \left.\frac{\partial f_i}{\partial a}\right|_{TP}(a-a_0) + \left.\frac{\partial f_i}{\partial b}\right|_{TP}(b-b_0) + \left.\frac{\partial f_i}{\partial e_{u_i}}\right|_{TP}(e_{u_i}-e^0_{u_i}) + \left.\frac{\partial f_i}{\partial e_{v_i}}\right|_{TP}(e_{v_i}-e^0_{v_i}) \\
&= f_i(a_0,b_0,e^0_{u_i},e^0_{v_i}) - \left.\frac{\partial f_i}{\partial e_{u_i}}\right|_{TP} e^0_{u_i} - \left.\frac{\partial f_i}{\partial e_{v_i}}\right|_{TP} e^0_{v_i} + \left.\frac{\partial f_i}{\partial a}\right|_{TP}\Delta a + \left.\frac{\partial f_i}{\partial b}\right|_{TP}\Delta b + \left.\frac{\partial f_i}{\partial e_{u_i}}\right|_{TP} e_{u_i} + \left.\frac{\partial f_i}{\partial e_{v_i}}\right|_{TP} e_{v_i} \\
&= v_i - (a_0 + b_0 u_i) + \begin{bmatrix} \left.\frac{\partial f_i}{\partial a}\right|_{TP} & \left.\frac{\partial f_i}{\partial b}\right|_{TP} \end{bmatrix} \begin{bmatrix} \Delta a \\ \Delta b \end{bmatrix} + \begin{bmatrix} \left.\frac{\partial f_i}{\partial e_{u_i}}\right|_{TP} & \left.\frac{\partial f_i}{\partial e_{v_i}}\right|_{TP} \end{bmatrix} \begin{bmatrix} e_{u_i} \\ e_{v_i} \end{bmatrix} \\
&= \quad w_i \quad + \begin{bmatrix} -1 & -(u_i - e^0_{u_i}) \end{bmatrix} \begin{bmatrix} \Delta a \\ \Delta b \end{bmatrix} + \begin{bmatrix} b_0 & -1 \end{bmatrix} \begin{bmatrix} e_{u_i} \\ e_{v_i} \end{bmatrix} \\
&= \quad w_i \quad + \quad A_i \quad \begin{bmatrix} \Delta a \\ \Delta b \end{bmatrix} + \quad B'_i \quad \begin{bmatrix} e_{u_i} \\ e_{v_i} \end{bmatrix} \\
&= \quad w_i \quad + \quad A_i \quad \Delta\xi \quad + \quad B'_i \quad e_i
\end{aligned}
$$

with $\Delta a := a - a_0,\ \Delta b := b - b_0$. For the reason that a and e_{v_i} appear linear there is no necessity to expand them in the Taylor series. In this case misclosure and parameter vector consist of the elements $w_i = v_i - b_0 u_i,\quad \Delta\xi = [a \quad \Delta b]'$, instead.

c) The Taylor point of approximation is $"TP" := \{a_0, b_0, e^0_{u_i}, e^0_{v_i}\}$ or $"TP" := \{b_0, e^0_{u_i}\}$, respectively. In the first iteration $e^0_{u_i} = e^0_{v_i} = 0$ or $e^0_{u_i} = 0$, respectively, is set.

d) $\underset{m\times 1}{w} + \underset{m\times 2}{A}\,\underset{2\times 1}{\Delta\xi} + \underset{m\times 2m}{B'}\,\underset{2m\times 1}{e} = \underset{m\times 1}{0}$

e) $$\underset{m\times 2}{A} = \begin{bmatrix} -1 & -(u_1 - e^0_{u_1}) \\ \vdots & \vdots \\ -1 & -(u_m - e^0_{u_m}) \end{bmatrix}, \quad \underset{m\times 2m}{B'} = \begin{bmatrix} \underbrace{\begin{matrix} b_0 & 0 & \cdots & 0 \\ 0 & b_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_0 \end{matrix}}_{m\times m} & \underbrace{\begin{matrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{matrix}}_{m\times m} \end{bmatrix}$$

a) $\mathcal{L}_A(x,\lambda) = \frac{1}{2}\underset{1\times 1}{e'e} + \underset{1\times d}{\lambda'}(\underset{d\times n}{D'}\underset{n\times 1}{x} - \underset{d\times 1}{c}) = \frac{1}{2}(y-Ax)'(y-Ax) + \lambda'(D'x-c) \rightarrow \min_{x,\lambda}$

b)

$$\frac{\partial \mathcal{L}_A(x)}{\partial x}(\hat{x},\hat{\lambda}) = 0$$
$$\frac{\partial \mathcal{L}_A(x)}{\partial \lambda}(\hat{x},\hat{\lambda}) = 0$$

c)

$$\left.\begin{array}{lll} -A'y + A'A\hat{x} + D\hat{\lambda} = 0 \Rightarrow A'A\hat{x} + D\hat{\lambda} = A'y & (1) \\ D'\hat{x} - c = 0 \Rightarrow D'\hat{x} = c & (2) \end{array}\right\} \Rightarrow \begin{bmatrix} A'A & D \\ D' & 0 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} A'y \\ c \end{bmatrix}$$

d) If A has full column rank then the normal equation matrix $N = A'A$ is invertible, i.e. $N^{-1} = (A'A)^{-1}$ exists. Thus equation (1) can be left multiplied by $-D'N^{-1} \Rightarrow -D'\hat{x} - D'N^{-1}D\hat{\lambda} = -D'N^{-1}A'y$, added to equation (2) $\Rightarrow -D'N^{-1}D\hat{\lambda} = -D'N^{-1}A'y + c$ and solved for $\hat{\lambda} \Rightarrow \hat{\lambda} = (D'N^{-1}D)^{-1}(D'N^{-1}A'y - c)$.

e) The estimate for $\hat{x}$ is obtained if $\hat{\lambda}$ is substituted in (1: $\hat{x} = N^{-1}A'y - N^{-1}D\hat{\lambda}$), which is then solved for $\hat{x}$ $\Rightarrow \hat{x} = N^{-1}[D(D'N^{-1}D)^{-1}(D'N^{-1}A'y - c) - A'y]$

a) $P_A = A(A^TWA)^{-1}A^TW$ and $P_A^\perp = I - P_A$

b) From $P_A y$ the vector $\hat{y} = P_A y$ of estimated (reduced) observations is computed, while $P_A^\perp y$ generates the vector $\hat{e} = P_A^\perp y$ of estimated inconsistencies.

c) Two (2 translations)

d) Zero

e) Point error ellipses and point error ellipsoids

f) $P = Q_y^{-1}$

g) $\Sigma_{\hat{x}} \equiv D\{\underline{\hat{x}}\} = (A^TPA)^{-1}$ or $\Sigma_{\hat{x}} = \sigma^2(A^TPA)^{-1}$ (σ^2 known)

$Q_{\hat{y}} \equiv D\{\underline{\hat{y}}\} = AQ_{\hat{x}}A^T = A(A^TPA)^{-1}A^T$ or $Q_{\hat{y}} = \sigma^2 A(A^TPA)^{-1}A^T$ (σ^2 known)

$\hat{\Sigma}_{\hat{x}} \equiv \hat{D}\{\underline{\hat{x}}\} = \hat{\sigma}^2(A^TPA)^{-1}$, $\hat{\sigma}^2 = \dfrac{\hat{e}^TP\hat{e}}{m - rk(A)}$ (σ^2 unknown)

h) Type-I-error probability is the probability α to reject a true null hypothesis (or to accept a wrong alternative hypothesis). Type-II-error probability is the probability β to reject a true alternative hypothesis (or to accept a wrong null hypothesis). For a picture see lecture notes and powerpoint viewgraphs.

i) $P(a \le \underline{x} \le b) = \int_a^b f(x)dx = F(b) - F(a)$. f(x) denotes the probability density function, F(x) is the cumulative distribution function. For a picture see lecture notes and powerpoint viewgraphs.

j) Since $\underline{x}$ is not standard normally distributed the critical value k_x for testing a sample value x cannot be read from the attached table. However the quantity $\underline{y} = \dfrac{\underline{x} - \mu}{\sigma} = \dfrac{\underline{x} - 3}{0.5}$ is standard normally distributed and – for the given significance level α=0.01 – a value k_y=2.325 can be found, instead. Finally, k_x is computed from $k_x = \sigma k_y + \mu = 4.1625$.

k) $\underline{\hat{e}}^TP\underline{\hat{e}} \sim \chi^2_{m-n}$ (χ^2 distribution with m-n degrees of freedom). For a sketch, see lecture notes and powerpoint viewgraphs.

Problem a)

i) As 6 distances have been measured connecting 4 points, m=6, n=8. Because planar distance networks exhibit a datum defect of d=3 the redundancy is r=m-(n-d)=1. For example, distance observation s_{24} is redundant.

ii) The reason for d=3 is that the network can be shifted along two orthogonal directions and rotated in the plane without affecting the distance observations (2 translational and 1 rotational degree of freedom). Therefore, the $m\times n$ design matrix A will have 3 linear dependent columns and so the normal equation matrix N has; N is not invertible.

iii) The two ways are: Fix coordinates of one point and one coordinate of a second point or fix coordinates of one point and one azimuth of a line.

iv) Reduction of solution space: Remove the coordinates (coordinate corrections) mentioned in (iii) from the vector of unknowns and remove corresponding columns from the design matrix. These terms go into the vector of (reduced) observations. Augmentation of solution space: Set up additional datum constraints on the vector of unknowns and use the technique of constrained Lagrangian to solve the system.

Problem b)

i) Starting from $\underline{\hat{u}}_2 = \dfrac{s_{13}^2 - \underline{s}_{23}^2 + \underline{s}_{12}^2}{2s_{13}}$ first the total derivative $d\underline{\hat{u}}_2$ of $\underline{\hat{u}}_2$ has to be computed with respect to all involved stochastic quantities, i.e. with respect to $\underline{s}_{23}$ and $\underline{s}_{12}$.

$$d\underline{\hat{u}}_2 = \frac{\partial \underline{\hat{u}}_2}{\partial \underline{s}_{12}} d\underline{s}_{12} + \frac{\partial \underline{\hat{u}}_2}{\partial \underline{s}_{23}} d\underline{s}_{23} = \begin{bmatrix} \dfrac{\partial \underline{\hat{u}}_2}{\partial \underline{s}_{12}} & \dfrac{\partial \underline{\hat{u}}_2}{\partial \underline{s}_{23}} \end{bmatrix} \begin{bmatrix} d\underline{s}_{12} \\ d\underline{s}_{23} \end{bmatrix} = \begin{bmatrix} \dfrac{\underline{s}_{12}}{s_{13}} & -\dfrac{\underline{s}_{23}}{s_{13}} \end{bmatrix} \begin{bmatrix} d\underline{s}_{12} \\ d\underline{s}_{23} \end{bmatrix}$$

Then, compute the variance

$$\sigma_{\hat{u}_2}^2 = \begin{bmatrix} \dfrac{\underline{s}_{12}}{s_{13}} & -\dfrac{\underline{s}_{23}}{s_{13}} \end{bmatrix} Q_{[s_{12}, s_{23}]} \begin{bmatrix} \dfrac{\underline{s}_{12}}{s_{13}} & -\dfrac{\underline{s}_{23}}{s_{13}} \end{bmatrix}^T =$$

$$= \frac{1}{s_{13}^2} \begin{bmatrix} \underline{s}_{12} & -\underline{s}_{23} \end{bmatrix} \begin{bmatrix} \sigma_{s_{12}}^2 & 0 \\ 0 & \sigma_{s_{23}}^2 \end{bmatrix} \begin{bmatrix} \underline{s}_{12} \\ -\underline{s}_{23} \end{bmatrix} = \frac{\underline{s}_{12}^2 \sigma_{s_{12}}^2 + \underline{s}_{23}^2 \sigma_{s_{23}}^2}{s_{13}^2}$$

ii) The variance $\sigma_{\hat{F}}^2$ of the area $\hat{F}$ is computed using the same rule as before

$$\underline{\hat{F}} = \frac{s_{13}\underline{\hat{v}}_2}{2} \Rightarrow d\underline{\hat{F}} = \frac{s_{13}}{2} d\underline{\hat{v}}_2 \Rightarrow \sigma_{\hat{F}}^2 = \frac{s_{13}^2}{4} \sigma_{\hat{v}_2}^2$$

a) One equation is necessary and sufficient because the redundancy is one. The condition equation follows (for error-free observations) from simple planar trigonometry $\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}$ or $a\sin\beta-b\sin\alpha=0$. In adjustment problems inconsistencies must be taken into account, i.e. the final condition equation is $(a-e_a)\sin(\beta-e_\beta)-(b-e_b)\sin(\alpha-e_\alpha)=0$.

b) The condition equation is non-linear in the e's and has to be linearized: Let $f(e_a,e_b,e_\alpha,e_\beta)=(a-e_a)\sin(\beta-e_\beta)-(b-e_b)\sin(\alpha-e_\alpha)=0$ then its Taylor expansion with respect to the unknown inconsistencies leads to

$$f(e_a,e_b,e_\alpha,e_\beta)=\ f(e_a^0,e_b^0,e_\alpha^0,e_\beta^0)+\left.\frac{\partial f}{\partial a}\right|_{e_a=e_a^0,\ldots,e_\beta=e_\beta^0}(e_a-e_a^0)+\ldots+\left.\frac{\partial f}{\partial \beta}\right|_{e_a=e_a^0,\ldots,e_\beta=e_\beta^0}(e_\beta-e_\beta^0)=0$$

where

$$f(e_a^0,e_b^0,e_\alpha^0,e_\beta^0)=(a-e_a^0)\sin(\beta-e_\beta^0)-(b-e_b^0)\sin(\alpha-e_\alpha^0)=$$
$$=a\sin(\beta-e_\beta^0)-b\sin(\alpha-e_\alpha^0)-\sin(\beta-e_\beta^0)e_a^0+\sin(\alpha-e_\alpha^0)e_b^0$$

$$\left.\frac{\partial f}{\partial e_a}\right|_{e_a=e_a^0,\ldots,e_\beta=e_\beta^0}(e_a-e_a^0)=-\sin(\beta-e_\beta^0)(e_a-e_a^0)=-\sin(\beta-e_\beta^0)e_a+\sin(\beta-e_\beta^0)e_a^0$$

$$\left.\frac{\partial f}{\partial e_b}\right|_{e_a=e_a^0,\ldots,e_\beta=e_\beta^0}(e_b-e_b^0)=\sin(\alpha-e_\alpha^0)(e_b-e_b^0)=\sin(\alpha-e_\alpha^0)e_b-\sin(\alpha-e_\alpha^0)e_b^0$$

$$\left.\frac{\partial f}{\partial e_\alpha}\right|_{e_a=e_a^0,\ldots,e_\beta=e_\beta^0}(e_\alpha-e_\alpha^0)=(b-e_b^0)\cos(\alpha-e_\alpha^0)(e_\alpha-e_\alpha^0)=$$
$$=(b-e_b^0)\cos(\alpha-e_\alpha^0)e_\alpha-(b-e_b^0)\cos(\alpha-e_\alpha^0)e_\alpha^0$$

$$\left.\frac{\partial f}{\partial e_\beta}\right|_{e_a=e_a^0,\ldots,e_\beta=e_\beta^0}(e_\beta-e_\beta^0)=-(a-e_a^0)\cos(\beta-e_\beta^0)(e_\beta-e_\beta^0)=$$
$$=-(a-e_a^0)\cos(\beta-e_\beta^0)e_\beta+(a-e_a^0)\cos(\beta-e_\beta^0)e_\beta^0.$$

The terms marked in red and blue cancel after summation. Then, all terms containing individual e's are collected in $-B'e$, the remaining quantities (marked in magenta) go into the misclosure w. If the order $e=[e_a,e_b,e_\alpha,e_\beta]'$ is chosen we obtain:

$$B'=\left[\sin(\beta-e_\beta^0),-\sin(\alpha-e_\alpha^0),-(b-e_b^0)\cos(\alpha-e_\alpha^0),(a-e_a^0)\cos(\beta-e_\beta^0)\right]$$
$$w=a\sin(\beta-e_\beta^0)-b\sin(\alpha-e_\alpha^0)-(b-e_b^0)\cos(\alpha-e_\alpha^0)e_\alpha^0+(a-e_a^0)\cos(\beta-e_\beta^0)e_\beta^0$$

Clearly, in the first iteration of the adjustment algorithm initial approximate values $e_a^0=e_b^0=e_\alpha^0=e_\beta^0=0$ are chosen.

c) There is no datum problem in the B-model because here we only search for estimates of the inconsistencies; no unknown parameters such as coordinates are present.

d)

$$\mathcal{L}_B(e,\lambda) = \frac{1}{2} e'e + \lambda'(w - B'e) \Rightarrow$$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}_B(e,\lambda)}{\partial e}(\hat{e},\hat{\lambda}) &= \hat{e} - B\hat{\lambda} = 0 \\ \frac{\partial \mathcal{L}_B(e,\lambda)}{\partial \lambda}(\hat{e},\hat{\lambda}) &= -B'\hat{e} = -w \end{aligned}\right\} \Rightarrow \begin{bmatrix} I & -B \\ -B' & 0 \end{bmatrix} \begin{bmatrix} \hat{e} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ -w \end{bmatrix}$$

e) If the first equation is multiplied by B' on the left and then the second equation is added, one can solve for $\hat{\lambda} = (B'B)^{-1} w$. This result is inserted into first equation which is then solved for $\hat{e} = B(B'B)^{-1} w$.

a) $P_A = A(A^TWA)^{-1}A^TW$ and $P_A^\perp = I - P_A$

b) From $P_A y$ the vector $\hat{y} = P_A y$ of estimated (reduced) observations is computed, while $P_A^\perp y$ generates the vector $\hat{e} = P_A^\perp y$ of estimated inconsistencies.

c) Two (2 translations)

d) Zero

e) Point error ellipses and point error ellipsoids

f) $P = Q_y^{-1}$

g) $\Sigma_{\hat{x}} \equiv D\{\underline{\hat{x}}\} = (A^TPA)^{-1}$ or $\Sigma_{\hat{x}} = \sigma^2(A^TPA)^{-1}$ $\quad(\sigma^2$ known)

$\Sigma_{\hat{y}} \equiv D\{\underline{\hat{y}}\} = AQ_{\hat{x}}A^T = A(A^TPA)^{-1}A^T$ or $\Sigma_{\hat{y}} = \sigma^2 A(A^TPA)^{-1}A^T$ $\quad(\sigma^2$ known)

$\hat{\Sigma}_{\hat{x}} \equiv \hat{D}\{\underline{\hat{x}}\} = \hat{\sigma}^2(A^TPA)^{-1},\ \hat{\sigma}^2 = \dfrac{\hat{e}^TP\hat{e}}{m - rk(A)}$ $\quad(\sigma^2$ unknown)

h) Type-I-error probability is the probability α to reject a true null hypothesis (or to accept a wrong alternative hypothesis). Type-II-error probability is the probability β to reject a true alternative hypothesis (or to accept a wrong null hypothesis). For a picture see lecture notes and powerpoint viewgraphs.

i) $P(a \le \underline{x} \le b) = \int_a^b f(x)dx = F(b) - F(a)$. f(x) denotes the probability density function, F(x) is the cumulative distribution function. For a picture see lecture notes and powerpoint viewgraphs.

j) Since $\underline{x}$ is not standard normally distributed the critical value k_x for testing a sample value x cannot be read from the attached table. However the quantity $\underline{y} = \dfrac{\underline{x} - \mu}{\sigma} = \dfrac{\underline{x} - 3}{0.5}$ is standard normally distributed and – for the given significance level α=0.01 – a value k_y=2.325 can be found, instead. Finally, k_x is computed from $k_x = \sigma k_y + \mu = 4.1625$.

k) $\underline{\hat{e}}^TP\underline{\hat{e}} \sim \chi^2_{m-n}$ (χ^2 distribution with m-n degrees of freedom). For a sketch, see lecture notes and powerpoint viewgraphs.

In order to compute the variance $\sigma^2_{x_0}$ from the linear law of error propagation the total differential of the function $\underline{x}_0 = \underline{s}\cos\underline{\alpha}\sin\underline{\beta}/\sin(\underline{\alpha}+\underline{\beta})$ is needed with respect to all involved stochastic quantities, i.e. coordinates $\underline{x}_\beta$, $\underline{y}_\beta$ and observations $\underline{\alpha}$, $\underline{\beta}$.

$$
\begin{aligned}
dx_0 &= \frac{\partial x_0}{\partial x_\beta}dx_\beta + \frac{\partial x_0}{\partial y_\beta}dy_\beta + \frac{\partial x_0}{\partial \alpha}d\alpha + \frac{\partial x_0}{\partial \beta}d\beta \\
&= \frac{\partial x_0}{\partial s}\frac{\partial s}{\partial x_\beta}dx_\beta + \frac{\partial x_0}{\partial s}\frac{\partial s}{\partial y_\beta}dy_\beta + \frac{\partial x_0}{\partial \alpha}d\alpha + \frac{\partial x_0}{\partial \beta}d\beta \\
&= \frac{\partial x_0}{\partial s}\left(\frac{\partial s}{\partial x_\beta}dx_\beta + \frac{\partial s}{\partial y_\beta}dy_\beta\right) + \frac{\partial x_0}{\partial \alpha}d\alpha + \frac{\partial x_0}{\partial \beta}d\beta \\
&= \frac{\cos\underline{\alpha}\sin\underline{\beta}}{\sin(\underline{\alpha}+\underline{\beta})}\left(-\frac{x_\alpha - x_\beta}{s}dx_\beta - \frac{y_\alpha - y_\beta}{s}dy_\beta\right) - \frac{\underline{s}\sin\underline{\beta}\cos\underline{\beta}}{\sin^2(\underline{\alpha}+\underline{\beta})}d\alpha + \frac{\underline{s}\sin\underline{\alpha}\cos\underline{\alpha}}{\sin^2(\underline{\alpha}+\underline{\beta})}d\beta \\
&= \frac{\cos\underline{\alpha}\sin\underline{\beta}}{\sin(\underline{\alpha}+\underline{\beta})}dx_\beta + 0dy_\beta - \frac{\underline{s}\sin\underline{\beta}\cos\underline{\beta}}{\sin^2(\underline{\alpha}+\underline{\beta})}d\alpha + \frac{\underline{s}\sin\underline{\alpha}\cos\underline{\alpha}}{\sin^2(\underline{\alpha}+\underline{\beta})}d\beta \\
&= \cos\underline{\alpha}\sin\underline{\beta}dx_\beta + 0dy_\beta - s\sin\underline{\beta}\cos\underline{\beta}d\alpha + s\sin\underline{\alpha}\cos\underline{\alpha}d\beta \\
&= \left[\cos\underline{\alpha}\sin\underline{\beta},\ 0,\ -\underline{s}\sin\underline{\beta}\cos\underline{\beta},\ \underline{s}\sin\underline{\alpha}\cos\underline{\alpha}\right]\begin{bmatrix} dx_\beta \\ dy_\beta \\ d\alpha \\ d\beta \end{bmatrix} \\
&= \qquad\qquad\qquad\qquad A \qquad\qquad\qquad\qquad\qquad z
\end{aligned}
$$

because $y_\alpha = \underline{y}_\beta,\ x_\alpha - \underline{x}_\beta = -\underline{s}$ (see the graphics) and $\underline{\alpha}+\underline{\beta} = 90°$.

Linear law of error propagations: $\sigma^2_{x_0} = AQ_zA' = AQ_{[x_\beta, y_\beta, \alpha, \beta]}A'$

$$
\text{with} \quad Q_{[x_\beta, y_\beta, \alpha, \beta]} = \left[\begin{array}{cc|cc} \sigma^2_{x_\beta} & \sigma_{x_\beta y_\beta} & & \\ \sigma_{x_\beta y_\beta} & \sigma^2_{y_\beta} & & \\ \hline & & \sigma^2_\alpha & \\ & & & \sigma^2_\beta \end{array}\right]
$$

$$
\Rightarrow \sigma^2_{x_0} = \cos^2\underline{\alpha}\sin^2\underline{\beta}\sigma^2_{x_\beta} + \underline{s}^2(\sin^2\underline{\beta}\cos^2\underline{\beta}\sigma^2_\alpha + \sin^2\underline{\alpha}\cos^2\underline{\alpha}\sigma^2_\beta).
$$

Remark: Since $\underline{\alpha}+\underline{\beta} = 90°$ the terms $\sin\underline{\alpha}, \cos\underline{\alpha}, \sin\underline{\beta}$ and $\cos\underline{\beta}$ can be replaced with $\cos\underline{\beta}, \sin\underline{\beta}, \cos\underline{\alpha}$ and $\sin\underline{\alpha}$.

a) m=8, n=4, d=0, r=m-(n-d)=4

b)

$$\underline{\ell} = [\underline{x}_1, \underline{y}_1, \ldots, \underline{x}_4, \underline{y}_4]' \qquad \xi = [t_x, t_y, a, b]' \qquad A = \begin{bmatrix} 1 & 0 & u_1 & v_1 \\ 0 & 1 & v_1 & -u_1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & u_4 & v_4 \\ 0 & 1 & v_4 & -u_4 \end{bmatrix}$$

c)

$$\underline{\hat{\lambda}} = \sqrt{\underline{\hat{a}}^2 + \underline{\hat{b}}^2} = \sqrt{\underline{\hat{\xi}}_3^2 + \underline{\hat{\xi}}_4^2}, \quad \underline{\hat{\alpha}} = \arctan\frac{\underline{\hat{b}}}{\underline{\hat{a}}} = \arctan\frac{\underline{\hat{\xi}}_4}{\underline{\hat{\xi}}_3}, \quad \underline{\hat{t}}_x = \underline{\hat{\xi}}_1, \quad \underline{\hat{t}}_y = \underline{\hat{\xi}}_2$$

d)

$$\left.\begin{array}{r} d\lambda = \dfrac{a}{\sqrt{a^2+b^2}}da + \dfrac{b}{\sqrt{a^2+b^2}}db \\ \dfrac{d\alpha}{\cos^2\alpha} = -\dfrac{b}{a^2}da + \dfrac{a}{a^2}db \Rightarrow d\alpha = \dfrac{-b}{a^2+b^2}da + \dfrac{a}{a^2+b^2}db \end{array}\right\} \Rightarrow$$

$$\begin{bmatrix} d\lambda \\ d\alpha \end{bmatrix} = \frac{1}{a^2+b^2}\begin{bmatrix} a\sqrt{a^2+b^2} & b\sqrt{a^2+b^2} \\ -b & a \end{bmatrix}\begin{bmatrix} da \\ db \end{bmatrix} \sim \begin{bmatrix} d\lambda \\ d\alpha \end{bmatrix} = A\begin{bmatrix} da \\ db \end{bmatrix} \Rightarrow$$

$$Q_{[\hat{\lambda},\hat{\alpha}]} = AQ_{[\hat{a},\hat{b}]}A'$$

where

$$Q_{[\hat{a},\hat{b}]} = \begin{bmatrix} \sigma_{\hat{a}}^2 & \sigma_{\hat{a}\hat{b}} \\ \sigma_{\hat{a}\hat{b}} & \sigma_{\hat{b}}^2 \end{bmatrix} = \begin{bmatrix} \sigma_{\hat{\xi}_3}^2 & \sigma_{\hat{\xi}_3\hat{\xi}_4} \\ \sigma_{\hat{\xi}_3\hat{\xi}_4} & \sigma_{\hat{\xi}_4}^2 \end{bmatrix}$$

a) $\underline{\hat{\sigma}}_0^2 = \dfrac{\underline{\hat{e}}^T I \underline{\hat{e}}}{m-n} = \dfrac{\underline{\hat{e}}^T \underline{\hat{e}}}{11-2} = 1.52845$

b) Null hypothesis H_0: $\underline{\hat{\sigma}}_0^2 = \sigma_0^2$ $\leftrightarrow$ Alternative hypothesis H_a: $\underline{\hat{\sigma}}_0^2 \neq \sigma_0^2$

Test quantity $\underline{T} = \underline{\hat{\sigma}}_0^2 / \sigma_0^2 = 1.52845 \sim F(1-\frac{\alpha}{2}, f_1, f_2)$

Critical value $k_\alpha = F(0.975, 9, \infty) = 2.12$

As $-k_\alpha < \underline{T} < k_\alpha$: deviation is not significant, hence accept H_0

c) Test quantity $\underline{T} = \underline{\hat{e}}^T I \underline{\hat{e}} = 13.756 \sim \chi^2(m-n, \lambda)$

Critical value ($\alpha = 5\%$) $k_\alpha = \chi^2(9,0) = 16.92$

As $-k_\alpha < \underline{T} < k_\alpha$: global test detects no outlier

Critical value ($\alpha = 25\%$) $k_\alpha = \chi^2(9,0) = 11.39$

As $\underline{T} > k_\alpha$: global test detects outlier

For a fixed configuration (problem), outlier detection exclusively depends on the choice of the level of significance.

d) The global test is an overall model test. It detects general model errors. It is not possible to attribute errors to individual observations. Thus, the global test can only provide general information on whether or not there exists a model error. Detection of concrete gross errors needs a testing strategy that checks each individual observation (referred to as individual test).

e) Successively screening the whole observation vector (each individual observation) for a gross error in terms of an individual test. Generally, the observation with the largest gross error is rejected. If there are further gross errors in the data after the rejection, repeat the procedure in an iterative way until all gross errors are eliminated.

f) Range of local redundancy number: $0 \leq \underline{r}_i \leq 1$

The smaller $\underline{r}_i$, the worse is the possibility to control the appropriate observation ("badly controlled network").

The greater $\underline{r}_i$, the better is the possibility to control the appropriate observation ("well controlled network").

Thus, the smaller the ratio $\underline{\sigma}_{\hat{y}_i}^2 / \sigma_{y_i}^2$ the better network control (variance of the adjusted observation much smaller than the variance of the real observation).

On the other hand, the greater the ratio $\underline{\sigma}_{\hat{y}_i}^2 / \sigma_{y_i}^2$ the worse network control (variance of the adjusted observation equal to the variance of the real observation)

a)

$$\left.\begin{array}{l}\underline{w}_1 = \alpha + e_{w_1} \\ \underline{w}_2 = \beta + e_{w_2} \\ \underline{w}_3 = \gamma + e_{w_3} \\ \underline{w}_4 = -\alpha + \beta + e_{w_4} \\ \underline{w}_5 = -\alpha + \gamma + e_{w_5} \\ \underline{w}_6 = -\beta + \gamma + e_{w_6}\end{array}\right\} \Rightarrow \underline{y} = \begin{bmatrix}\underline{w}_1 \\ \underline{w}_2 \\ \underline{w}_3 \\ \underline{w}_4 \\ \underline{w}_5 \\ \underline{w}_6\end{bmatrix}, x = \begin{bmatrix}\alpha \\ \beta \\ \gamma\end{bmatrix}, A = \begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1\end{bmatrix}$$

b)

$$N = A'A = \begin{bmatrix}3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3\end{bmatrix}, N^{-1} = \frac{1}{4}\begin{bmatrix}2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2\end{bmatrix}, A'\underline{y} = \begin{bmatrix}\underline{w}_1 - \underline{w}_4 - \underline{w}_5 \\ \underline{w}_2 + \underline{w}_4 - \underline{w}_6 \\ \underline{w}_3 + \underline{w}_5 + \underline{w}_6\end{bmatrix},$$

$$\begin{bmatrix}\hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2\end{bmatrix}\begin{bmatrix}\underline{w}_1 - \underline{w}_4 - \underline{w}_5 \\ \underline{w}_2 + \underline{w}_4 - \underline{w}_6 \\ \underline{w}_3 + \underline{w}_5 + \underline{w}_6\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2\underline{w}_1 + \underline{w}_2 + \underline{w}_3 - \underline{w}_4 - \underline{w}_5 \\ \underline{w}_1 + 2\underline{w}_2 + \underline{w}_3 + \underline{w}_4 - \underline{w}_6 \\ \underline{w}_1 + \underline{w}_2 + 2\underline{w}_3 + \underline{w}_5 + \underline{w}_6\end{bmatrix}$$

c) Redundancy $r = m - (n - d)$ with m=6 observations, n=3 unknowns and d=0 datum defects, i.e. $r = 3$. As an example (other combinations possible) we have:

$$\left.\begin{array}{l}-\underline{w}_1 + \underline{w}_2 = \underline{w}_4 \\ -\underline{w}_1 + \underline{w}_3 = \underline{w}_5 \\ -\underline{w}_2 + \underline{w}_3 = \underline{w}_6\end{array}\right\} \Rightarrow B = \begin{bmatrix}-1 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1\end{bmatrix}, k = \begin{bmatrix}-\underline{w}_1 + \underline{w}_2 - \underline{w}_4 \\ -\underline{w}_1 + \underline{w}_3 - \underline{w}_5 \\ -\underline{w}_2 + \underline{w}_3 - \underline{w}_6\end{bmatrix}.$$

d) Transition A- to B-model is based on the orthogonality of A and B, i.e. $BA = 0$. This is satisfied.

e) Lagrange function $L = \frac{1}{2}\underline{e}'\underline{e} + \lambda'(B\underline{e} - \underline{k}) \to \min_{\underline{e},\lambda}$, necessary conditions and normal equation system

$$\left.\begin{array}{l}\frac{\partial L}{\partial \underline{e}}(\hat{\underline{e}}) = \hat{\underline{e}} + B'\hat{\lambda} = 0 \\ \frac{\partial L}{\partial \lambda}(\hat{\lambda}) = B\hat{\underline{e}} = \underline{k}\end{array}\right\} \Rightarrow \begin{bmatrix}I_6 & B' \\ B & 0\end{bmatrix}\begin{bmatrix}\hat{\underline{e}} \\ \hat{\lambda}\end{bmatrix} = \begin{bmatrix}0 \\ \underline{k}\end{bmatrix}$$

f)

$$\hat{\underline{e}} = -B'\hat{\lambda} \Rightarrow B\hat{\underline{e}} = -BB'\hat{\lambda} \Rightarrow \hat{\lambda} = -\left(BB'\right)^{-1} k$$

$$\Rightarrow \hat{\underline{e}} = B'\left(BB'\right)^{-1} k = B'\left(BB'\right)^{-1}\begin{bmatrix}-\underline{w}_1 + \underline{w}_2 - \underline{w}_4 \\ -\underline{w}_1 + \underline{w}_3 - \underline{w}_5 \\ -\underline{w}_2 + \underline{w}_3 - \underline{w}_6\end{bmatrix}$$

a) The law of linear error propagation is $\sigma_c^2 = AQA'$ where Q is the variance-covariance matrix of given quantities a, b and γ. Matrix A contains the partial derivatives of the explicit function $c = \arccos(\cos a \cos b + \sin a \sin b \cos\gamma)$ with respect to the quantities a, b and γ. However, much more easily is to apply the implicit function formulation together with the total differential of the implicit function.

b) $$Q=\begin{bmatrix} \sigma_a^2 & \sigma_{ab} & \sigma_{a\gamma} \\ \sigma_{ab} & \sigma_b^2 & \sigma_{b\gamma} \\ \sigma_{a\gamma} & \sigma_{b\gamma} & \sigma_\gamma^2 \end{bmatrix}$$

$$\begin{aligned}
\frac{d\cos c}{dc}dc &= \frac{\partial(\cos a\cos b+\sin a\sin b\cos\gamma)}{\partial a}da + \frac{\partial(\cos a\cos b+\sin a\sin b\cos\gamma)}{\partial b}db + \\
&\quad + \frac{\partial(\cos a\cos b+\sin a\sin b\cos\gamma)}{\partial\gamma}d\gamma \\
-\sin c\,dc &= (-\sin a\cos b+\cos a\sin b\cos\gamma)da + (-\cos a\sin b+\sin a\cos b\cos\gamma)db + \\
&\quad -\sin a\sin b\sin\gamma d\gamma \\
dc &= \frac{\sin a\cos b-\cos a\sin b\cos\gamma}{\sin c}da + \frac{\cos a\sin b-\sin a\cos b\cos\gamma}{\sin c}db + \\
&\quad + \frac{\sin a\sin b\sin\gamma}{\sin c}d\gamma \\
&= A\begin{bmatrix} da \\ db \\ d\gamma \end{bmatrix}
\end{aligned}$$

where $A = \frac{1}{\sin c}\left[\sin a\cos b-\cos a\sin b\cos\gamma \quad \cos a\sin b-\sin a\cos b\cos\gamma \quad \sin a\sin b\sin\gamma\right]$

a) The estimated parameters and their functions are independent on σ_0^2 while the variance-covariance information is not.

b) The variance ratio test is based on a test quantity built from the ratio of two independent variances σ_1^2 (with f_1 degrees of freedom) and σ_2^2 (with f_2 degrees of freedom). The larger variance has to be in the numerator. The test quantity is Fisher-distributed $F_{1-\alpha,\text{ degree of freedom of numerator, degree of freedom of denominator}}$.

c) The relation $\lambda_0 = \lambda(\alpha_L, q = 1, \gamma_0 = 1-\beta_0) = \lambda(\alpha_G, q = m-n > 1, \gamma_0 = 1-\beta_0)$ establishes the matching between significance levels α_L (local test, individual test) and α_G (global test). For given power of test $\gamma_0 = 1-\beta_0$ and significance level α_L the non-centrality parameter λ_0 is computed iteratively. λ_0 is used later fort the computation of α_G with given degrees of freedom $q = m-n$ of the global test.

d) The global test (or "overall model test") uses the χ^2-test of the weighted square sum of residuals $\underline{T} = \underline{\hat{e}}_0' Q_y^{-1} \underline{\hat{e}}_0 \sim \chi^2(m-n)$ or the F-test of the estimated variance component $\frac{\underline{T}}{m-n} = \frac{\underline{\hat{e}}_0' Q_y^{-1} \underline{\hat{e}}_0}{m-n} = \underline{\hat{\sigma}}_0^2 \sim F(m-n, \infty)$. The global test is applied in order to perfom a general test. The local test (individual test), however, is performed – using a simplified model for Q_y – with intent to locate a blunder in the data. It is based on a test quantity which is built from the individual estimated residual normalized by its variance $\frac{\underline{\hat{e}}_i}{\sigma_{\hat{e}_i}} \sim N(0,1)$.

e) $\frac{\underline{\hat{e}}_i}{\sigma_{\hat{e}_i}} \sim N(0,1)$

f) Minimal detectable bias is a term of internal reliability and expresses how big an error in an observation must be in order to be detectable by a statistical test. It depends on non-centrality parameter λ_0, given (a priori) variance $\sigma_{y_i}^2$ and computed (a posteriori) variance $\sigma_{\hat{y}_i}^2$ of the i-th observation. The larger the precision of the observation (as expressed by $\sigma_{y_i}^2$) the smaller possible errors can be detected.

g) Local redundancy r_i is the share of observation y_i in the overall redundancy $r = m-n$. It is $0 \le r_i = 1 - \frac{\sigma_{\hat{y}_i}^2}{\sigma_{y_i}^2} \le 1$.

h) The quantity $EV_i = 100\% \times r_i = 70\%$ provides information on the fact that observation y_i is being check by other observation to an extent of 70%. This is a good measure.

i) If in a leveling network a point is tied to the network only by a single leveling line, its EV_i is close to zero. The corresponding height difference observation cannot be checked, an error in the observation will not be detectable. Another example is a point in a horizontal control network which is connected to the other points only by a single distance and a single direction observation.

a) The model of the null hypothesis H_0 is the usual A-model $E\{\underline{y}\} = Ax, D\{\underline{y}\} = Q_y$ with estimated square sum of residuals $\underline{\hat{e}}_0^T Q_y^{-1} \underline{\hat{e}}_0$. The vector $\underline{y}$ contains m observations, vector x comprises n unknown parameters.

b) The model of the alternative hypothesis H_a is the usual A-model augmented by a part $C\nabla$, $E\{\underline{y}\} = Ax + C\nabla = \begin{bmatrix} A & C \end{bmatrix} \begin{bmatrix} x \\ \nabla \end{bmatrix}, = D\{\underline{y}\} = Q_y$. Vector ∇ contains q additional parameters which refine the model of the null hypothesis with the intention to model outliers. Its estimated square sum of residuals is $\underline{\hat{e}}_a^T Q_y^{-1} \underline{\hat{e}}_a$.

c) For the reason that in both models m observations will be processed, it is possible to estimated $1 \le q \le m - n$ additional parameters in the augmented model. For the case q=m-n redundancy is lost and $\underline{\hat{e}}_a = \underline{0}$.

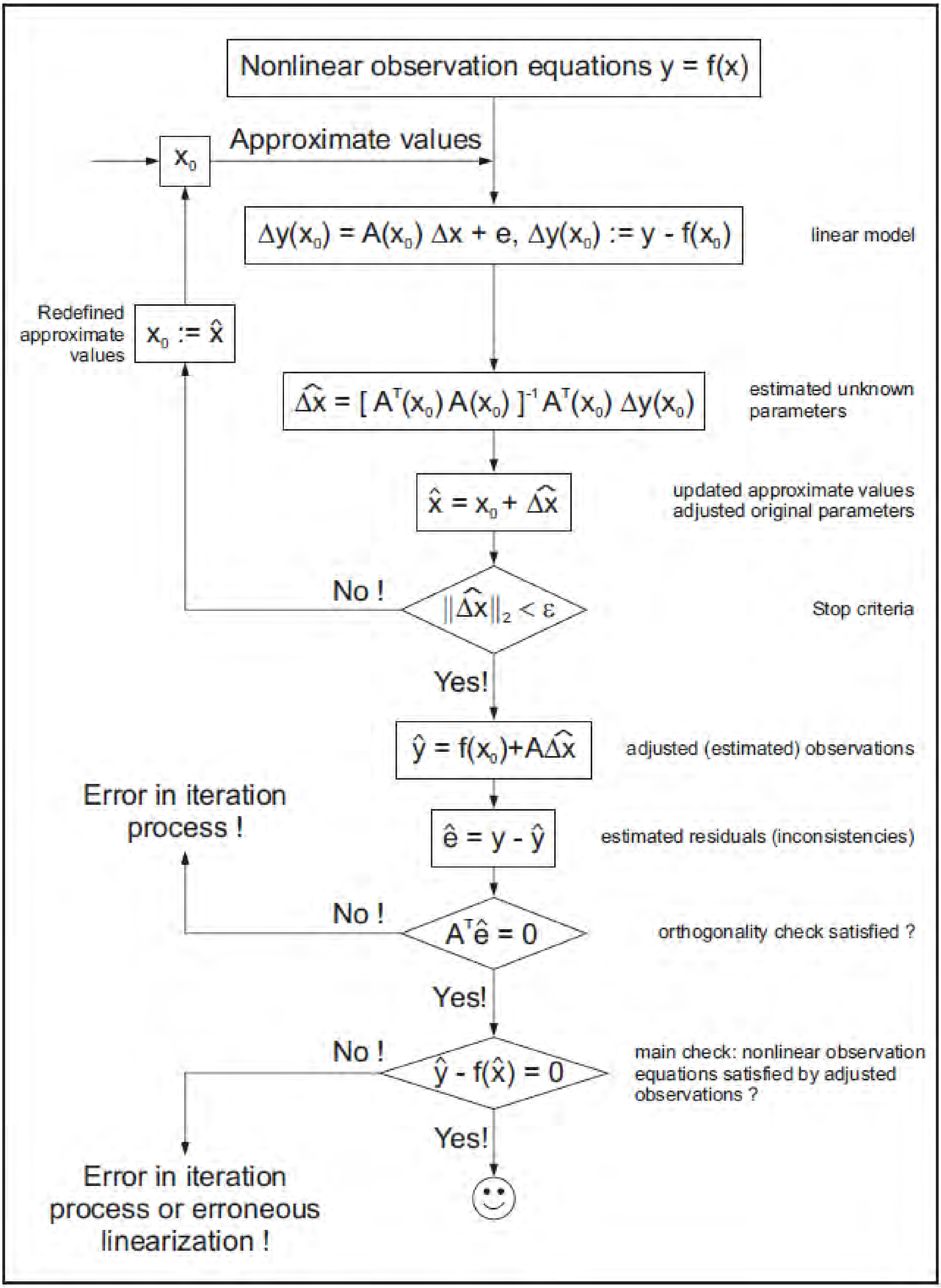
d) The general test quantity is $\underline{T} = \underline{\hat{e}}_0^T Q_y^{-1} \underline{\hat{e}}_0 - \underline{\hat{e}}_a^T Q_y^{-1} \underline{\hat{e}}_a \sim \chi^2_{q,\lambda}$. It is χ^2-distributed with q degrees of freedom and non-centrality parameter λ. In case that the null hypothesis is true $\lambda = 0$ holds true otherwise $\lambda > 0$.

In case of the biggest possible number of additional parameters ("global test") $\underline{T} = \underline{\hat{e}}_0^T Q_y^{-1} \underline{\hat{e}}_0 \sim \chi^2_{m-n,\lambda=0}$ holds true, for q=1 ("individual test") $T = \dfrac{\hat{\nabla}}{\sigma_{\hat{\nabla}}} \sim N(0,1)$ and under the assumption of a simplified stochastic model $\underline{T} = \dfrac{\underline{\hat{e}}_i}{\sigma_{\hat{e}_i}} \sim N(0,1)$.

e) The DIA-principle is first to check the existence of any model errors (D=Detection), second their identification and possible causes (I=Identification), third the reaction on them (A=Adaption).

f) Problems are encountered with global and individual test if both lead to contradicting results/conclusions. The reason for this situation is in the test parameters of both tests which are performed independently of each other with individual probabilities for type-I- and type-II-errors.

g) In order to avoid these problems test parameters are balanced. Starting from a constant power of test in both tests, the type-I-error probability of the global test is (iteratively) derived from the type-I-error probability of the local test.



Nonlinear observation equations y = f(x)
Approximate values
x0
Δy(x0) = A(x0) Δx + e, Δy(x0) := y - f(x0)
linear model
Redefined approximate values
x0 := x̂
Δx̂ = [A^T(x0) A(x0)]^-1 A^T(x0) Δy(x0)
estimated unknown parameters
x̂ = x0 + Δx̂
updated approximate values adjusted original parameters
No !
||Δx̂||2 < ε
Stop criteria
Yes!
ŷ = f(x0)+AΔx̂
adjusted (estimated) observations
Error in iteration process !
ê = y - ŷ
estimated residuals (inconsistencies)
No !
A^Tê = 0
orthogonality check satisfied ?
Yes!
No !
ŷ - f(x̂) = 0
main check: nonlinear observation equations satisfied by adjusted observations ?
Yes!
Error in iteration process or erroneous linearization !

a) The reduced observation vector is $\ell = d_{ik} - d_{ik}^0$ where d_{ik} is the observed distance and $d_{ik}^0 = (1+m_0)\sqrt{\left(x_i^0 - x_k^0\right)^2 + \left(y_i^0 - y_k^0\right)^2} + c_0$ is the approximate distance as computed from approximate values $m_0, c_0, x_i^0, x_k^0, y_i^0, y_k^0$ (=Taylor point)

b) The design matrix consists of the partial derivatives $\left.\frac{\partial d_{ik}}{\partial m}\right|_{TP}, \left.\frac{\partial d_{ik}}{\partial c}\right|_{TP}, \left.\frac{\partial d_{ik}}{\partial x_i}\right|_{TP}, \ldots, \left.\frac{\partial d_{ik}}{\partial y_k}\right|_{TP}$, evaluated at the Taylor point.

$$\left.\frac{\partial d_{ik}}{\partial m}\right|_{TP} = s_{ik}^0 := \sqrt{\left(x_i^0 - x_k^0\right)^2 + \left(y_i^0 - y_k^0\right)^2}$$

$$\left.\frac{\partial d_{ik}}{\partial c}\right|_{TP} = 1$$

$$\left.\frac{\partial d_{ik}}{\partial x_i}\right|_{TP} = (1+m_0)\frac{x_i^0 - x_k^0}{s_{ik}^0}$$

$$\left.\frac{\partial d_{ik}}{\partial x_k}\right|_{TP} = -\left.\frac{\partial d_{ik}}{\partial x_i}\right|_{TP}$$

$$\left.\frac{\partial d_{ik}}{\partial y_i}\right|_{TP} = (1+m_0)\frac{y_i^0 - y_k^0}{s_{ik}^0}$$

$$\left.\frac{\partial d_{ik}}{\partial y_k}\right|_{TP} = -\left.\frac{\partial d_{ik}}{\partial y_i}\right|_{TP}$$

$$\xi = [\Delta x_i, \ldots, \Delta c]^T$$

This is a variance ratio test where two independent variances are compared. The test quantity $\underline{T}$ is the ratio of both variances and the larger variance has be in the numerator of $\underline{T}$. $\underline{T}$ is F-distributed with f_1 and f_2 degrees of freedom: $\underline{T} \sim F_{1-\alpha,f_1,f_2}$. f_1 are the degrees of freedom of the numerator (f_1=58) and f_2 are those of the denominator (f_2=41). Because of the setting-up of the alternative hypothesis $H_a : \sigma^2_{\text{Teilnetz 1}} > \sigma^2_{\text{Teilnetz 2}}$ we deal with a one-sided test, here. The critical value from the table results to $1.589 < k_\alpha < 1.66$ meaning that due to $\underline{T} = \underline{\hat{\sigma}}^2_{\text{Teilnetz 1}} / \underline{\hat{\sigma}}^2_{\text{Teilnetz 2}} = 1.877 > k_\alpha$ there is no reason to accept the null hypothesis. The variances of both subnets are different.

A1) Version 1: $\tan z_{ik} = \dfrac{s_{ik}}{h_k - h_i} = \dfrac{s_{ik}}{h_{ik}}$ Version 2: $\sin z_{ik} = \dfrac{s_{ik}}{d_{ik}}$

Version 3: $\cos z_{ik} = \dfrac{h_k - h_i}{d_{ik}} = \dfrac{h_{ik}}{d_{ik}}$

with $s_{ik}^2 = (u_k - u_i)^2 + (v_k - v_i)^2 = u_{ik}^2 + v_{ik}^2 \quad , \quad s_{ik} = \sqrt{u_{ik}^2 + v_{ik}^2}$

$$d_{ik}^2 = s_{ik}^2 + (h_k - h_i)^2 = s_{ik}^2 + h_{ik}^2 \quad , \quad d_{ik} = \sqrt{s_{ik}^2 + h_{ik}^2} = \sqrt{u_{ik}^2 + v_{ik}^2 + h_{ik}^2}$$

$$u_k = u_k^0 + \Delta u_k \quad , \quad v_k = v_k^0 + \Delta v_k \quad , \quad h_k = h_k^0 + \Delta h_k$$

A2) Implicit differentiation based on version 1

$$\frac{1}{\cos^2 z_{ik}} dz_{ik} = \frac{1}{h_{ik}} ds_{ik} - \frac{s_{ik}}{h_{ik}^2} dh_k$$

$$dz_{ik} = \frac{\cos^2 z_{ik}}{h_{ik}} ds_{ik} - \frac{s_{ik} \cos^2 z_{ik}}{h_{ik}^2} dh_k = \frac{\cos^2 z_{ik}}{h_{ik}} ds_{ik} - \frac{\tan z_{ik} \cos^2 z_{ik}}{h_{ik}} dh_k$$

$$= \frac{\cos z_{ik}}{h_{ik}} \left(\cos z_{ik} ds_{ik} - \sin z_{ik} dh_k \right) = \frac{1}{d_{ik}^2} \left(h_{ik} ds_{ik} - s_{ik} dh_k \right)$$

$$2 s_{ik} ds_{ik} = 2 u_{ik} du_k + 2 v_{ik} dv_k$$

$$ds_{ik} = \frac{u_{ik}}{s_{ik}} du_k + \frac{v_{ik}}{s_{ik}} dv_k \Rightarrow dz_{ik} = \frac{1}{d_{ik}^2} \left(\frac{h_{ik} u_{ik}}{s_{ik}} du_k + \frac{h_{ik} v_{ik}}{s_{ik}} dv_k - s_{ik} dh_k \right)$$

Transit to finite quantities and insert approximate values

$$\Delta z_{ik} = z_{ik} - z_{ik}^0 = \frac{1}{\left(d_{ik}^0\right)^2} \left(\frac{(h_k^0 - h_i)(u_k^0 - u_i)}{s_{ik}^0} \Delta u_k + \frac{(h_k^0 - h_i)(v_k^0 - v_i)}{s_{ik}^0} \Delta v_k - s_{ik}^0 \Delta h_k \right)$$

$$= \frac{1}{d_{ik}^0} \left(\cos z_{ik}^0 \sin t_{ik}^0 \Delta u_k + \cos z_{ik}^0 \cos t_{ik}^0 \Delta v_k - \sin z_{ik}^0 \Delta h_k \right)$$

with

$$s_{ik}^0 = \sqrt{(u_k^0 - u_i)^2 + (v_k^0 - v_i)^2}$$

$$d_{ik}^0 = \sqrt{(u_k^0 - u_i)^2 + (v_k^0 - v_i)^2 + (h_k^0 - h_i)^2} = \sqrt{(s_{ik}^0)^2 + (h_k^0 - h_i)^2}$$

A3) The part $\Delta z_{ik} = z_{ik} - z_{ik}^0$ ("Observed" - "computed") enters the vector **y,** the coefficients $\dfrac{\cos z_{ik}^0 \sin t_{ik}^0}{d_{ik}^0} \quad , \quad \dfrac{\cos z_{ik}^0 \cos t_{ik}^0}{d_{ik}^0} \quad , \quad -\dfrac{\sin z_{ik}^0}{d_{ik}^0}$ go into the design matrix **A.**

A4) The measurement z_{ik} has to be converted to radian [rad] because z_{ik}^0 as computed from coordinates is in radian. **A** is in units of [m^{-1}] while **x** is in [m].

A1: $\sigma_{\hat{a}} = 3.83 \cdot 10^{-3}$, $\sigma_{\hat{b}} = 0.027$, $\sigma_{\hat{c}} = 0.042$

A2: Null hypothesis H_0: $a = 0$ $\leftrightarrow$ Alternative hypothesis H_a: $a \neq 0$

Test quantity $T = \dfrac{|\hat{a}|}{\sigma_{\hat{a}}} = 1.4 \sim t(q = m - n = 3)$ (two-sided test)

Critical values $k_\alpha = t_{\frac{\alpha}{2}}(3) = 3.182$

As $-k_\alpha < T < k_\alpha$: accept H_0 (deviation is not significant)

A3: Null hypothesis H_0: $b = 0.05$ $\leftrightarrow$ Alternative hypothesis H_a: $b > 0.05$

Test quantity $T = \dfrac{|\hat{b}\text{-}0.05|}{\sigma_{\hat{b}}} = 2.8 \sim t(q = m - n = 3)$ (one-sided test)

Critical value $k_\alpha = t_\alpha(3) = 2.35$

As $T > k_\alpha$: reject H_0 (deviation is significant)

A4: The data is properly represented by a regression line. The quadratic term is not significant, hence can be neglected. The slope of the regression line has to be taken from the adjustment. The value of 0.05 is statistically insignificant.

A5: $\hat{\sigma}_0^2 = \dfrac{\hat{e}^T\hat{e}}{m-n} = \dfrac{\hat{e}^T\hat{e}}{6-3} = 5.47 \cdot 10^{-4}$

A6: Null hypothesis H_0: $\hat{\sigma}_0^2 = \sigma_0^2$ $\leftrightarrow$ Alternative hypothesis H_a: $\hat{\sigma}_0^2 \neq \sigma_0^2$

Test quantity $T = \dfrac{\hat{\sigma}_0^2}{\sigma_0^2} = 5.47 \cdot 10^{-4} \sim F(q_1{=}m\text{-}n{=}3, q_2{=}\infty, 0)$

Critical values $k_{1,\alpha} = F_{1-\frac{\alpha}{2}}(3, \infty, 0) = 3.116$

$$k_{2,\alpha} = F_{\frac{\alpha}{2}}(3, \infty, 0) = F_{0.025}(3, \infty, 0) = \frac{1}{F_{1-\frac{\alpha}{2}}(\infty, 3, 0)} = 0.072$$

As $T \notin [k_{2,\alpha}, k_{1,\alpha}]$: reject H_0 (the values significantly deviate from each other)

a)

$$
\begin{aligned}
v &\doteq v\big|_{a_0,b_0} + \frac{\partial v}{\partial a}\bigg|_{a_0,b_0}\Delta a + \frac{\partial v}{\partial b}\bigg|_{a_0,b_0}\Delta b \\
&= a_0 e^{b_0 u} + e^{b_0 u}\Delta a + a_0 u e^{b_0 u}\Delta b \\
&= v_0 + \begin{bmatrix} e^{b_0 u} & a_0 u e^{b_0 u} \end{bmatrix}\begin{bmatrix}\Delta a \\ \Delta b\end{bmatrix} \\
&= v_0 + Ax
\end{aligned}
$$

$$\Rightarrow y := v - v_0 = Ax$$

b) $\tilde{v} = a_0 e^{b_0\tilde{u}} + \begin{bmatrix} e^{b_0\tilde{u}} & a_0\tilde{u}e^{b_0\tilde{u}} \end{bmatrix}\begin{bmatrix}\Delta a \\ \Delta b\end{bmatrix} = \tilde{v}_0 + Bx \Rightarrow \tilde{v} - \tilde{v}_0 =: c = Bx$

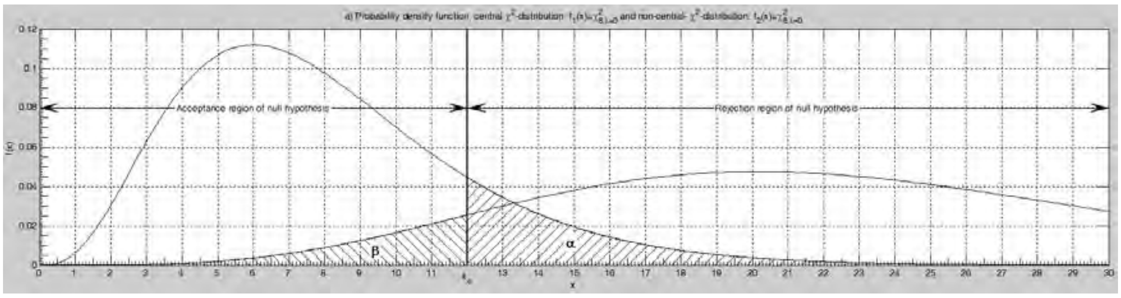
c) $\mathcal{L}(x,\lambda) = \frac{1}{2}(y - Ax)^T(y - Ax) + \lambda^T(Bx - c) = \min\limits_{x,\lambda}$

d)

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x}(\hat{x},\hat{\lambda}) &= -A^T y + A^T A\hat{x} + B^T\hat{\lambda} = 0 \\
\frac{\partial\mathcal{L}}{\partial \lambda}(\hat{x},\hat{\lambda}) &= B\hat{x} - c \qquad\qquad\qquad = 0
\end{aligned}
$$

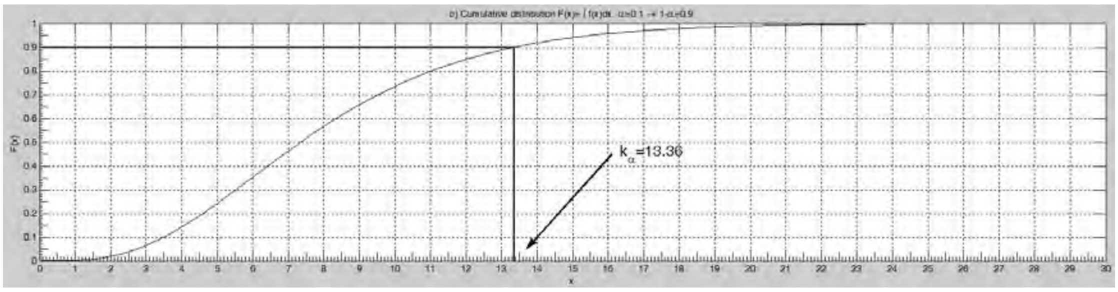
e) $\begin{bmatrix} \underbrace{A^T A}_{2\times 2} & \underbrace{B^T}_{2\times 1} \\ \underbrace{B}_{1\times 2} & \underbrace{0}_{1\times 1} \end{bmatrix}\begin{bmatrix} \underbrace{\hat{x}}_{2\times 1} \\ \underbrace{\hat{\lambda}}_{1\times 1}\end{bmatrix} = \begin{bmatrix} \underbrace{A^T y}_{2\times 1} \\ \underbrace{c}_{1\times 1}\end{bmatrix} \sim N\hat{\xi} = n$

a)



b) The critical value k_α is most important for the decision whether to accept or reject the null hypothesis: For a sample value x (in this case) right from k_α, the null hypothesis will be rejected otherwise accepted.

c)



The given probability of a type-I-error was α=10%=0.1. From the graphics above the critical value can be found: $1-\alpha=0.9 \Rightarrow k_\alpha \approx 13.3$ (13.4).

d) Using the critical value $k_\alpha \approx 13.3$ (13.4) in the table with 8 degrees of freedom α is found to be in the interval α=0.9 and α=0.8. Linear interpolation gives the exact value α=0.875 (0.8716) $\Rightarrow$ $\beta=1-\alpha$=0.125 (0.1284)=12.5% (12.84%)

A1) Number of observations = m, number of unknowns n = 3 (a,b,ω), redundancy = m-n

A2)

$$Z(t) = Z(t)\Big|_{\substack{a=a_0\\b=b_0\\\omega=\omega_0}} + \frac{\partial Z(t)}{\partial a}\Big|_{\substack{a=a_0\\b=b_0\\\omega=\omega_0}} da + \frac{\partial Z(t)}{\partial b}\Big|_{\substack{a=a_0\\b=b_0\\\omega=\omega_0}} db + \frac{\partial Z(t)}{\partial \omega}\Big|_{\substack{a=a_0\\b=b_0\\\omega=\omega_0}} d\omega$$

$$= a_0 \sin\omega_0 t + b_0 \cos\omega_0 t + da \sin\omega_0 t + db \cos\omega_0 t + t(a_0 \cos\omega_0 t - b_0 \sin\omega_0 t)d\omega$$

$$= a_0 \sin\omega_0 t + b_0 \cos\omega_0 t + [\sin\omega_0 t,\ \cos\omega_0 t,\ t(a_0 \cos\omega_0 t - b_0 \sin\omega_0 t)][da\ db\ d\omega]^T$$

A3)

$$\underset{m\times 1}{\mathbf{y}} = \begin{bmatrix} Z(t_1) - (a_0 \sin\omega_0 t_1 + b_0 \cos\omega_0 t_1) \\ \vdots \\ Z(t_m) - (a_0 \sin\omega_0 t_m + b_0 \cos\omega_0 t_m) \end{bmatrix}$$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} \sin\omega_0 t_1 & \cos\omega_0 t_1 & t_1(a_0 \cos\omega_0 t_1 - b_0 \sin\omega_0 t_1) \\ \vdots & \vdots & \vdots \\ \sin\omega_0 t_m & \cos\omega_0 t_m & t_m(a_0 \cos\omega_0 t_m - b_0 \sin\omega_0 t_m) \end{bmatrix}$$

$$\underset{n\times 1}{\mathbf{x}} = [da\ db\ d\omega]^T$$

A4) The unknowns are estimable because the columns of **A** are linear independent.

A5) $\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}$ or $\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P}\mathbf{y}$ and

$\mathbf{D}(\hat{\mathbf{x}}) = (\mathbf{A}^T\mathbf{A})^{-1}$ or $\mathbf{D}(\hat{\mathbf{x}}) = (\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}$

A6)

$$f = \sqrt{\hat{a}^2 + \hat{b}^2} = (\hat{a}^2 + \hat{b}^2)^{1/2} \Rightarrow$$

$$df = \frac{1}{2}(\hat{a}^2 + \hat{b}^2)^{-1/2}(2\hat{a}\ d\hat{a} + 2\hat{b}\ d\hat{b}) = (\hat{a}^2 + \hat{b}^2)^{-1/2}[\hat{a},\hat{b}][d\hat{a}\ d\hat{b}]^T$$

$$D(f) = (\hat{a}^2 + \hat{b}^2)^{-1/2}[\hat{a},\hat{b}]D(\hat{a},\hat{b})\left[(\hat{a}^2 + \hat{b}^2)^{-1/2}[\hat{a},\hat{b}]\right]^T$$

$$= (\hat{a}^2 + \hat{b}^2)^{-1}[\hat{a},\hat{b}]D(\hat{a},\hat{b})[\hat{a},\hat{b}]^T$$

$$= (\hat{a}^2 + \hat{b}^2)^{-1}[\hat{a},\hat{b}]\begin{bmatrix} \sigma_{\hat{a}}^2 & \sigma_{\hat{a}\hat{b}} = 0 \\ \sigma_{\hat{a}\hat{b}} = 0 & \sigma_{\hat{b}}^2 \end{bmatrix}\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}$$

$$= \frac{\hat{a}^2\sigma_{\hat{a}}^2 + \hat{b}^2\sigma_{\hat{b}}^2}{\hat{a}^2 + \hat{b}^2}$$

A1) $\hat{\sigma}_0^2 = \dfrac{\mathbf{e}^T\mathbf{I}\mathbf{e}}{m-n} = \dfrac{\mathbf{e}^T\mathbf{e}}{11-2} = 1.52845$

A2) Null hypothesis H_0: $\hat{\sigma}_0^2 = \sigma_0^2$ $\leftrightarrow$ Alternative hypothesis H_a: $\hat{\sigma}_0^2 \neq \sigma_0^2$

Test quantity $T = \hat{\sigma}_0^2 / \sigma_0^2 = 1.52845 \sim F(1\text{-}\alpha, f_1, f_2)$

Critical value $k_\alpha = F(1-0.95, 9, \infty) = 1.880$ (linear interpolation yields 1.8845)

As $-k_\alpha < T < k_\alpha$: deviation is not significant, hence accept H_0

A3) Test quantity $T = \mathbf{e}^T\mathbf{I}\mathbf{e} = 13.756 \sim \chi^2(m\text{-}n, \lambda)$

Critical value ($\alpha = 5\%$) $k_\alpha = \chi^2(9,0) = 16.92$

As $-k_\alpha < T < k_\alpha$: global test detects no outlier

Critical value ($\alpha = 25\%$) $k_\alpha = \chi^2(9,0) = 11.39$

As $T > k_\alpha$: global test detects outlier

For a fixed configuration (problem), outlier detection exclusively depends on the choice of the level of significance.

A4) The global test is an overall model test. It detects general model errors. It is not possible to attribute errors to individual observations. Thus, the global test can only provide general information on whether or not there exists a model error. Detection of concrete gross errors needs a testing strategy that checks each individual observation (referred to as individual test).

A5) Successively screening the whole observation vector (each individual observation) for a gross error in terms of an individual test. Generally, the observation with the largest gross error is rejected. If there are further gross errors in the data after the rejection, repeat the procedure in an iterative way until all gross errors are eliminated.

A6) Range of local redundancy number: $0 \leq r_i \leq 1$

The smaller r_i, the worse is the possibility to control the appropriate observation ("badly controlled network").

The greater r_i, the better is the possibility to control the appropriate observation ("well controlled network").

Thus, the smaller the ratio $\sigma_{\hat{y}_i}^2 / \sigma_{y_i}^2$ the better network control (variance of the adjusted observation much smaller than the variance of the real observation).

On the other hand, the greater the ratio $\sigma_{\hat{y}_i}^2 / \sigma_{y_i}^2$ the worse network control (variance of the adjusted observation equal to the variance of the real observation