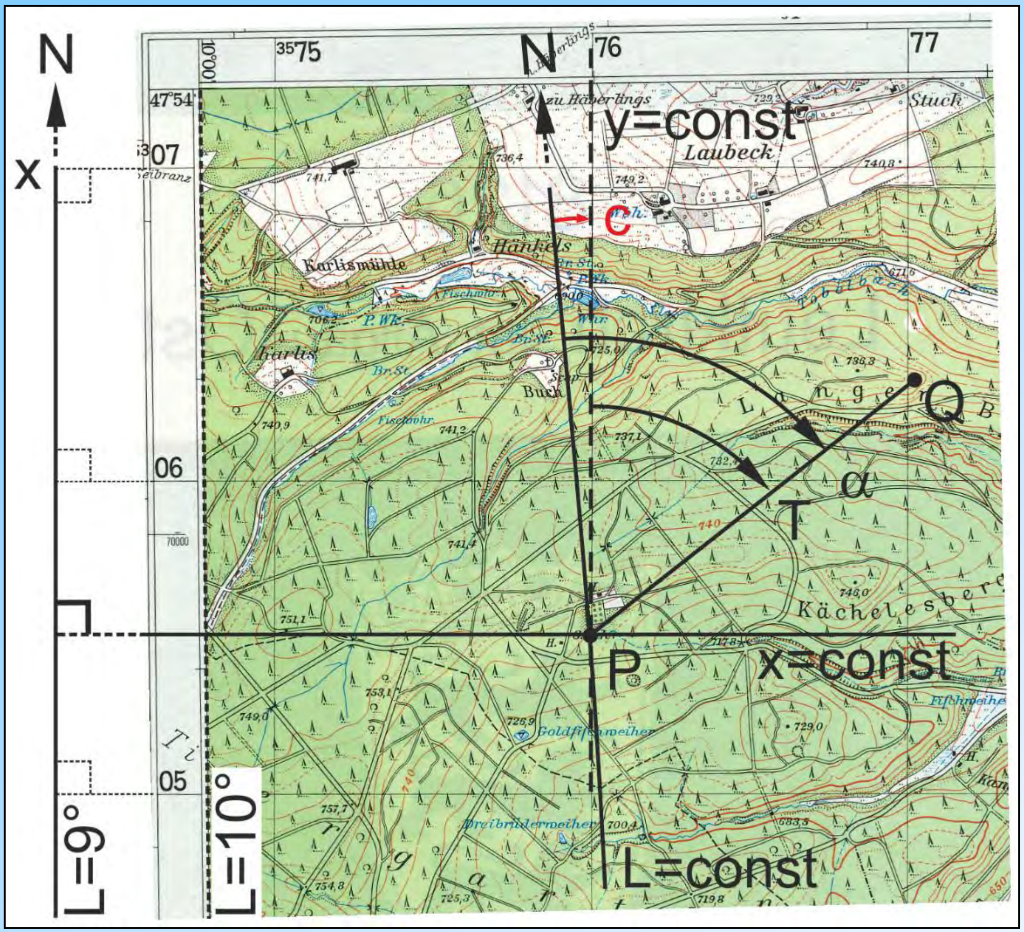
Conformal coordinates: Meridian convergence

Meridian convergence c (Azimuth α, grid bearing T: α =T+c)



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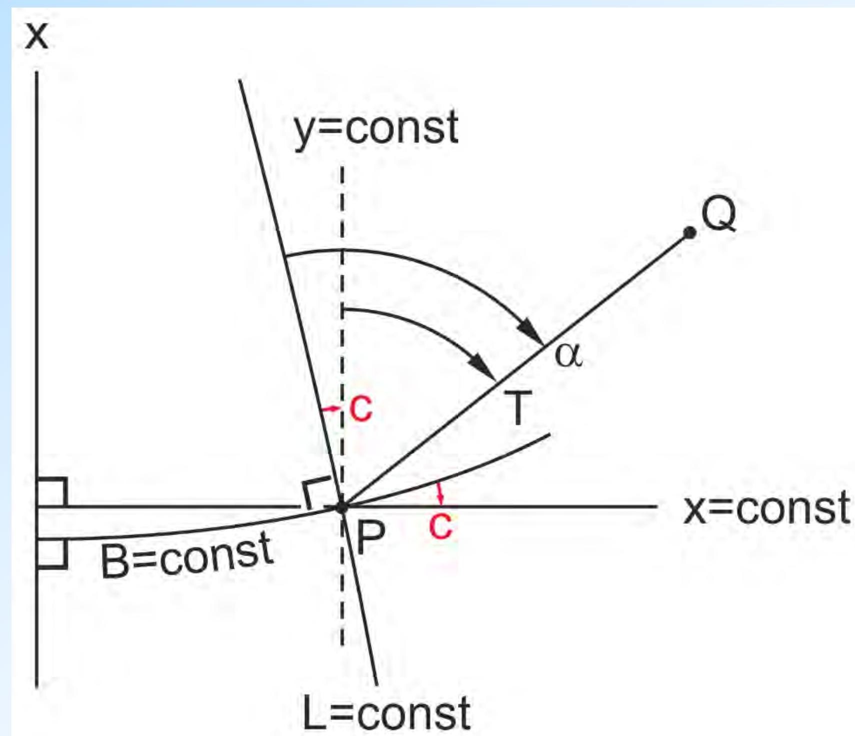
$$\tan c = \frac{x_\ell}{y_\ell} = \frac{\partial x / \partial \ell}{\partial y / \partial \ell}$$



→ Conformal coordinates: Meridian convergence and distortion

Conformal coordinates: Meridian convergence and distortion

$$c = (01)_c \ell + (11)_c b\ell + (21)_c b^2\ell + (03)_c \ell^3 + (31)_c b^3\ell + (13)_c b\ell^3 + (23)_c b^2\ell^3 + (05)_c \ell^5$$

$$(01)_c = \sin B,\ (11)_c = \cos B,\ (21)_c = -\frac{1}{2}\sin B,\ (03)_c = \frac{1}{3}\cos^3 B t(1+3\eta^2),\ (31)_c = -\frac{1}{6}\cos B$$

$$(13)_c = \frac{1}{3}\cos^3 B[1-2t^2+\eta^2(3-12t^2)],\ (23)_c = \frac{1}{6}\cos^3 B t(-7+2t^2),\ (05)_c = \frac{1}{15}\cos^5 B t(2-t^2)$$

$$\Lambda = m_0 + (02)_\Lambda \ell^2 + (12)_\Lambda b\ell^2 + (22)_\Lambda b^2\ell^2 + (04)_\Lambda \ell^4 + (32)_\Lambda b^3\ell^2 + (14)_\Lambda b\ell^4 + (06)_\Lambda \ell^6$$

$$(02)_\Lambda = \frac{m_0}{2}\cos^2 B(1+\eta^2),\quad (12)_\Lambda = -m_0\cos^2 B t(1+2\eta^2),\quad (32)_\Lambda = \frac{2m_0}{3}\cos^2 B t$$

$$(22)_\Lambda = \frac{m_0}{2}\cos^2 B[-1+t^2+\eta^2(-2+6t^2)],\quad (06)_\Lambda = \frac{m_0}{720}\cos^6 B(61-148t^2+16t^4)$$

$$(14)_\Lambda = \frac{m_0}{6}\cos^4 B t(-7+2t^2),\quad (04)_\Lambda = \frac{m_0}{24}\cos^4 B[5-4t^2+\eta^2(14-28t^2)]$$

all coefficients (ij) have to be evaluated at latitude B_0 of the local origin P_0 !

→ Conformal coordinates: Inverse series

Conformal coordinates: Inverse series

$$L_P = L_0 + \ell(x,y),\ B_P = B_0\left(\frac{X_0}{m_0}\right) + b(x,y)$$

$$\ell := L_P - L_0,\ y := y_P,\ x := X_P - X_0$$

$$|x| < 100\ \text{km},\ |y| < \frac{\ell_{max}}{180° / \pi} R_0 \cos B_0,\ R_0 = 6380\text{km}$$

$$\begin{aligned}
b = {} & [10]\, x \\
& + [20]\, x^2 + [02]\, y^2 \\
& + [30]\, x^3 + [12]\, x\, y^2 \\
& + [40]\, x^4 + [22]\, x^2 y^2 + [04]\, y^4 \\
& + [32]\, x^3 y^2 + [14]\, x\, y^4 \\
& + [42]\, x^4 y^2 + [24]\, x^2 y^4 + [06]\, y^6 + [34]\, x^3 y^4 + [16]\, x y^6
\end{aligned}$$

all coefficients [ij] have to be evaluated at latitude B_0 of the local origin P_0 !

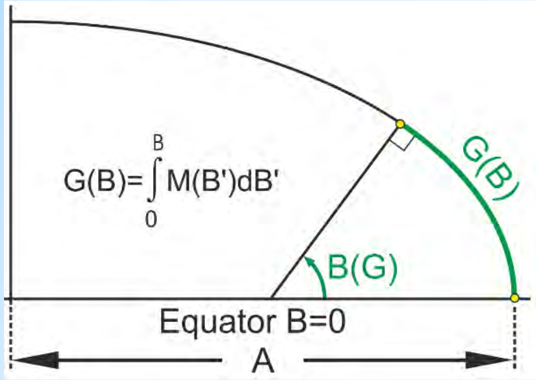
$$\begin{aligned}
\ell = {} & [01]\, y \\
& + [11]\, x\, y \\
& + [21]\, x^2 y + [03]\, y^3 \\
& + [31]\, x^3 y + [13]\, x\, y^3 \\
& + [41]\, x^4 y + [23]\, x^2 y^3 + [05]\, y^5 \\
& + [33]\, x^3 y^3 + [15]\, x\, y^5 + [25]\, x^2 y^5 + [07]\, y^7
\end{aligned}$$

physical units !

$\rightarrow$ Meridional arc length $\rightarrow$ Latitude

Meridional arc length G → Latitude B [rad]

$$B(G) = \frac{G}{e_0} + F_2 \sin\left(2\frac{G}{e_0}\right) + F_4 \sin\left(4\frac{G}{e_0}\right) + F_6 \sin\left(6\frac{G}{e_0}\right) + F_8 \sin\left(8\frac{G}{e_0}\right)$$



$$e_0 = A\left(1 - \frac{1}{4}E^2 - \frac{3}{64}E^4 - \frac{5}{256}E^6 - \frac{175}{16384}E^8 - \frac{411}{65536}E^{10}\right)$$

$$F_2 = \frac{3}{8}E^2 + \frac{3}{16}E^4 + \frac{213}{2048}E^6 + \frac{255}{4096}E^8 + \frac{166479}{655360}E^{10}$$

$$F_4 = \frac{21}{256}E^4 + \frac{21}{256}E^6 + \frac{533}{8192}E^8 - \frac{120563}{327680}E^{10}$$

$$F_6 = \frac{151}{6144}E^6 + \frac{147}{4096}E^8 + \frac{2732071}{9175040}E^{10}$$

$$F_8 = \frac{1097}{131072}E^8 - \frac{273697}{4587520}E^{10}$$

$$e_0 = A\left(1 - \frac{1}{4}E'^2 + \frac{13}{64}E'^4 - \frac{45}{256}E'^6 + \frac{2577}{16384}E'^8 - \frac{9417}{65536}E'^{10}\right)$$

$$F_2 = \frac{3}{8}E'^2 - \frac{3}{16}E'^4 + \frac{213}{2048}E'^6 - \frac{255}{4096}E'^8 + \frac{166479}{655360}E'^{10}$$

$$F_4 = \frac{21}{256}E'^4 - \frac{21}{256}E'^6 + \frac{533}{8192}E'^8 - \frac{152083}{327680}E'^{10}$$

$$F_6 = \frac{151}{6144}E'^6 - \frac{155}{4096}E'^8 + \frac{2767911}{9175040}E'^{10}$$

$$F_8 = \frac{1097}{131072}E'^8 - \frac{427277}{4587520}E'^{10}$$

→ Conformal coordinates: Inverse series

Conformal coordinates: Inverse series

Meridian convergence

$$
\begin{aligned}
c = {} & [01]_c\, y \\
& + [11]_c\, x\, y \\
& + [21]_c\, x^2 y + [03]_c\, y^3 \\
& + [31]_c\, x^3 y + [13]_c\, x\, y^3 \\
& + [41]_c\, x^4 y + [23]_c\, x^2 y^3 + [05]_c\, y^5 \\
& \qquad + [33]_c\, x^3 y^3 + [15]_c\, x\, y^5 + [25]_c\, x^2 y^5
\end{aligned}
$$

Scale factor, distortion

$$
\begin{aligned}
\Lambda = m_0 & + [02]_\Lambda\, y^2 \\
& + [12]_\Lambda\, x\, y^2 \\
& + [22]_\Lambda\, x^2 y^2 + [04]_\Lambda\, y^4
\end{aligned}
$$

all coefficients [ij] have to be evaluated at latitude B_0 of the local origin P_0 !

→ Conformal coordinates: Inverse series, series coefficients

Conformal coordinates: Inverse series, series coefficients

$$[10] = \frac{1+\eta^2}{m_0 N}, \quad [12] = \frac{-1-t^2+\eta^2(-2+2t^2)+\eta^4(-1+3t^2)}{2m_0^3 N^3}, \quad [20] = -\frac{3t(\eta^2+\eta^4)}{2m_0^2 N^2}$$

$$[30] = \frac{\eta^2(-1+t^2)+\eta^4(-2+6t^2)}{2m_0^3 N^3}, \quad [40] = \frac{t\eta^2}{2m_0^4 N^4}, \quad [22] = \frac{t[-2-2t^2+\eta^2(9+t^2)]}{4m_0^4 N^4}$$

$$[02] = -\frac{t(1+\eta^2)}{2m_0^2 N^2}, \quad [32] = \frac{-2-8t^2-6t^4+\eta^2(7-6t^2+3t^4)}{12m_0^5 N^5}, \quad [24] = \frac{t(7+16t^2+9t^4)}{12m_0^6 N^6}$$

$$[14] = \frac{5+14t^2+9t^4+\eta^2(11-30t^2-9t^4)}{24m_0^5 N^5}, \quad [04] = \frac{t[5+3t^2+\eta^2(6-6t^2)]}{24m_0^4 N^4}$$

$$[06] = -\frac{t(61+90t^2+45t^4)}{720m_0^6 N^6}, \quad [34] = \frac{7+55t^2+93t^4+45t^6}{36m_0^7 N^7}$$

$$[16] = -\frac{61+331t^2+495t^4+225t^6}{720m_0^7 N^7}, \quad [42] = -\frac{t(2+5t^2+3t^4)}{6m_0^6 N^6}$$

all coefficients [ij] have to be evaluated at latitude B_0 of the local origin P_0 !

→ Conformal coordinates: Inverse series, series coefficients

Conformal coordinates: Inverse series, series coefficients

$$[01]=\frac{1}{m_0 N\cos B},\quad [11]=\frac{t}{m_0^2 N^2\cos B},\quad [21]=\frac{1+2t^2+\eta^2}{2m_0^3 N^3\cos B},\quad [03]=-\frac{1+2t^2+\eta^2}{6m_0^3 N^3\cos B}$$

$$[31]=\frac{t(5+6t^2+\eta^2)}{6m_0^4 N^4\cos B},\quad [13]=-\frac{t(5+6t^2+\eta^2)}{6m_0^4 N^4\cos B},\quad [41]=\frac{5+28t^2+24t^4}{24m_0^5 N^5\cos B}$$

$$[23]=-\frac{5+28t^2+24t^4+\eta^2(6+8t^2)}{12m_0^5 N^5\cos B},\quad [05]=\frac{5+28t^2+24t^4+\eta^2(6+8t^2)}{120m_0^5 N^5\cos B}$$

$$[33]=-\frac{t(61+180t^2+120t^4)}{36m_0^6 N^6\cos B},\quad [15]=\frac{t(61+180t^2+120t^4)}{120m_0^6 N^6\cos B}$$

$$[25]=\frac{61+662t^2+1320t^4+720t^6}{240m_0^7 N^7\cos B},\quad [07]=-\frac{61+662t^2+1320t^4+720t^6}{5040m_0^7 N^7\cos B}$$

all coefficients [ij] have to be evaluated at latitude B_0 of the local origin P_0 !

see also: http://www.gis.uni-stuttgart.de/lehre/campus-docs/geo2gk.pdf

Conformal coordinates: Inverse series, series coefficients

$$[01]_c = \frac{t}{m_0 N}, \quad [11]_c = \frac{1+t^2+\eta^2}{m_0^2 N^2}, \quad [21]_c = \frac{t(1+t^2-\eta^2)}{m_0^3 N^3}, \quad [03]_c = -\frac{t(1+t^2-\eta^2)}{3m_0^3 N^3}$$

$$[31]_c = \frac{1+4t^2+3t^4}{3m_0^4 N^4}, \quad [13]_c = -\frac{1+4t^2+3t^4}{3m_0^4 N^4}, \quad [33]_c = -\frac{2(2+17t^2+30t^4+15t^6)}{9m_0^6 N^6}$$

$$[23]_c = -\frac{2t(2+5t^2+3t^4)}{3m_0^5 N^5}, \quad [05]_c = \frac{t(2+5t^2+3t^4)}{15m_0^5 N^5}, \quad [41]_c = \frac{t(2+5t^2+3t^4)}{3m_0^5 N^5}$$

$$[15]_c = \frac{2+17t^2+30t^4+15t^6}{15m_0^6 N^6}, \quad [25]_c = \frac{t(17+77t^2+105t^4+45t^6)}{15m_0^7 N^7}$$

$$[02]_\Lambda = \frac{1+\eta^2}{2m_0 N^2}, \quad [12]_\Lambda = -\frac{2t\eta^2}{m_0^2 N^3}, \quad [22]_\Lambda = \frac{\eta^2(-1+t^2)}{m_0^3 N^4}, \quad [04]_\Lambda = \frac{1+6\eta^2}{24m_0^3 N^4}$$

all coefficients [ij] have to be evaluated at latitude B_0 of the local origin P_0 !

→ Gauß-Krüger coordinates: Summary

Gauß-Krüger coordinates: Summary

Ellipsoidal coordinates L, B → Gauß-Krüger coordinates R, H (m_0=1)

choose L_0, B_0 so that $|\ell|=|L_P-L_0| < \ell_{max}= 2°$, $|b|=|B_P-B_0| < 1°$, L_0... multiple of 3°

$$\Rightarrow \text{ False Easting } R = y(\ell, b) + 10^6 \frac{L_0}{3°} + 5\times 10^5, \quad \text{Northing } H = X_0(B_0) + x(\ell, b)$$

(X_0... Meridional arc length from the equator to latitude B_0; B_0=B_P is admissible)

Gauß-Krüger coordinates R, H (m_0=1) → Ellipsoidal coordinates L, B

$$\text{choose } X_0 \text{ so that } |x| = |H - X_0| < 100 \text{ km}, |y| = \left|R - 10^6 Kz - 5\times 10^5\right| < \frac{\ell_{max}}{\rho} R_0 \cos B_0,$$

$$\rho=\frac{180°}{\pi}, R_0 = 6380\text{km} \Rightarrow \text{Longitude } L_P = L_0 + \ell(x, y), \quad \text{latitude } B_P = B_0 + b(x, y)$$

(B_0... Latitude corresponding to meridional arc length X_0, L_0=3°×Kz, Kz... 1^{st} digit of False Easting, X_0=H is admissible)

→ UTM coordinates: Summary

UTM coordinates: Summary

Ellipsoidal coordinates L, B → UTM coordinates False Easting, False Northing, Zone (m_0= 0,999 6)

choose L_0, B_0 so that $|\ell|=|L_P-L_0| < \ell_{max}= 3.5°$, $|b|=|B_P-B_0| < 1°$, $L_0+3°$=multiple of 6°

$$\Rightarrow \text{False Easting} = y(\ell,b) + 5\times 10^5,\ \text{False Northing} = X_0 + x(\ell,b) + \begin{cases} 0 & X_0 + x(\ell,b) > 0 \\ 10^7 & X_0 + x(\ell,b) < 0 \end{cases},$$

$$\text{Zone} = (L_0 + 3°)/6° + 30$$

(X_0... Meridional arc length from the equator to latitude B_0; B_0=B is admissible)

UTM coordinates False Easting, False Northing, Zone → Ellipsoidal coordinates L, B (m_0= 0,999 6)

$$\text{choose } X_0 \text{ so that } |x| = |\text{False Northing} - X_0| < 100 \text{ km},$$

$$|y| = |\text{False Easting} - 5\times 10^5| < \frac{\ell_{max}}{\rho} R_0 \cos B_0$$

$$\Rightarrow \text{Longitude } L_P = (\text{Zone} - 30)\times 6° - 3° + \ell(x,y), \quad \text{Latitude } B_P = B_0 + b(x,y)$$

(B_0... Latitude corresponding to meridional arc length X_0/m_0; X_0=False Northing is admissible)

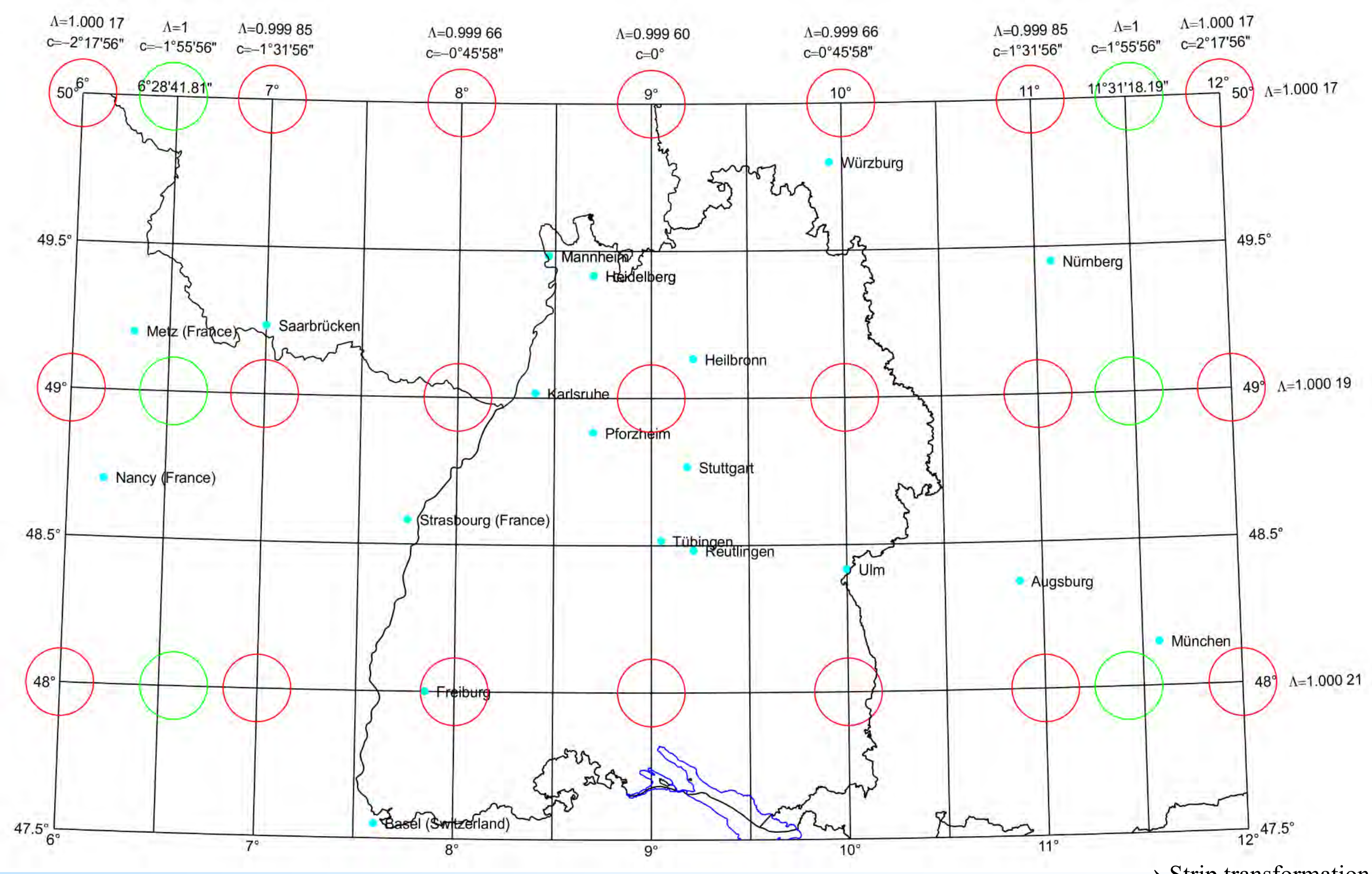
→ Comparison Gauß-Krüger ↔ UTM coordinates

Comparison Gauß-Krüger ↔ UTM coordinates

	Gauß-Krüger	UTM
Strip width	3°	6°
Strip Overlap	0.5°	0.5°
Strip extension in longitude at B=50° (incl. overlap)	~ 215 km (~ 286 km)	~ 430 km (~ 502 km)
Scale of reference meridian	1	0,999 6
Scale on strip boundary at B=50° (incl. overlap)	1,000 14 (1,000 25)	1,000 17 (1,000 37)
Max. length distortion	< 14 cm/km	< 40 cm/km
No distortion (B=50°) at	$L=L_0 \Leftrightarrow \ell=0°$	$L \approx L_0 \pm 2.518° \Leftrightarrow \lvert\ell\rvert \approx 2.518°$
Interpretation	"transverse tangent cylinder"	"transverse secant cylinder"
Coordinate labeling	Rechtswert, Hochwert False Easting, Northing	False Easting False Northing

→ UTM-Map Southern Germany

UTM Map Southern Germany (GRS80)

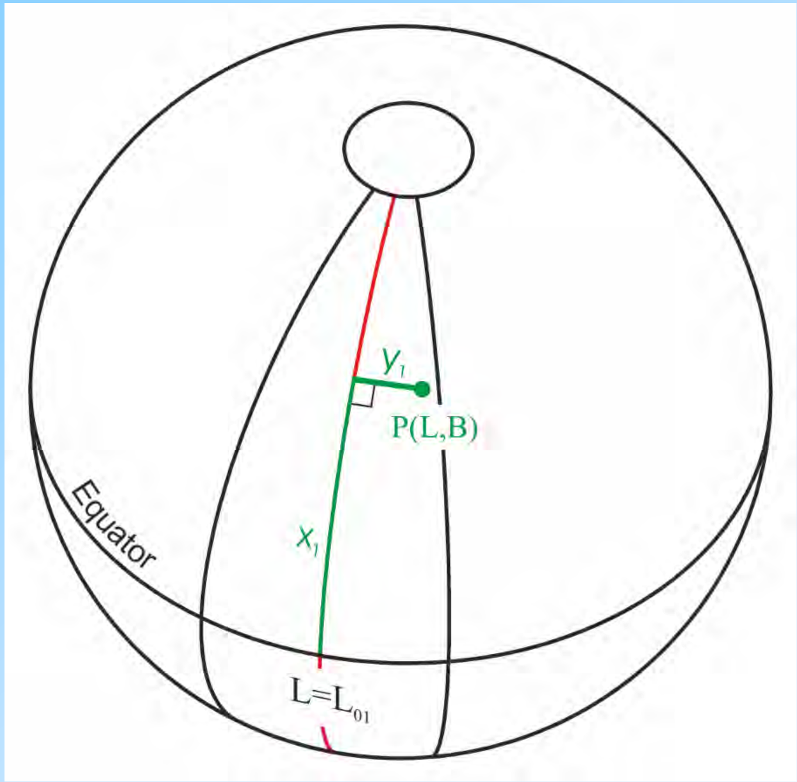


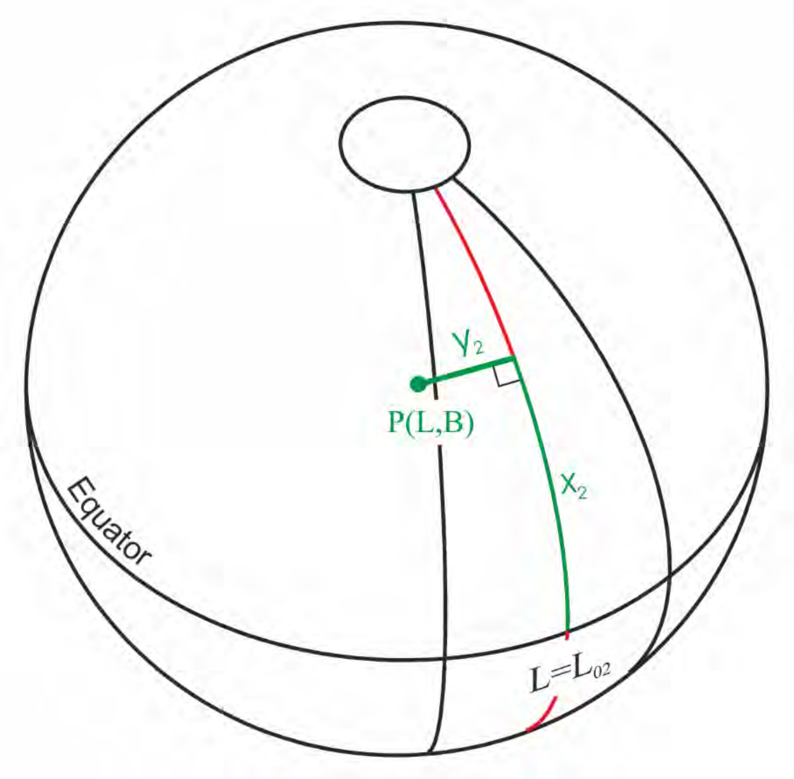
→ Strip transformation

Strip transformation

Given: Gauß-Krüger/UTM-coordinates of a point P with respect to reference meridian $L=L_{01}$

Wanted: Gauß-Krüger/UTM-coordinates of the same point P but with respect to the next reference meridian $L=L_{02}=L_{01} \pm 2(\ell_{max}-0.5°)$

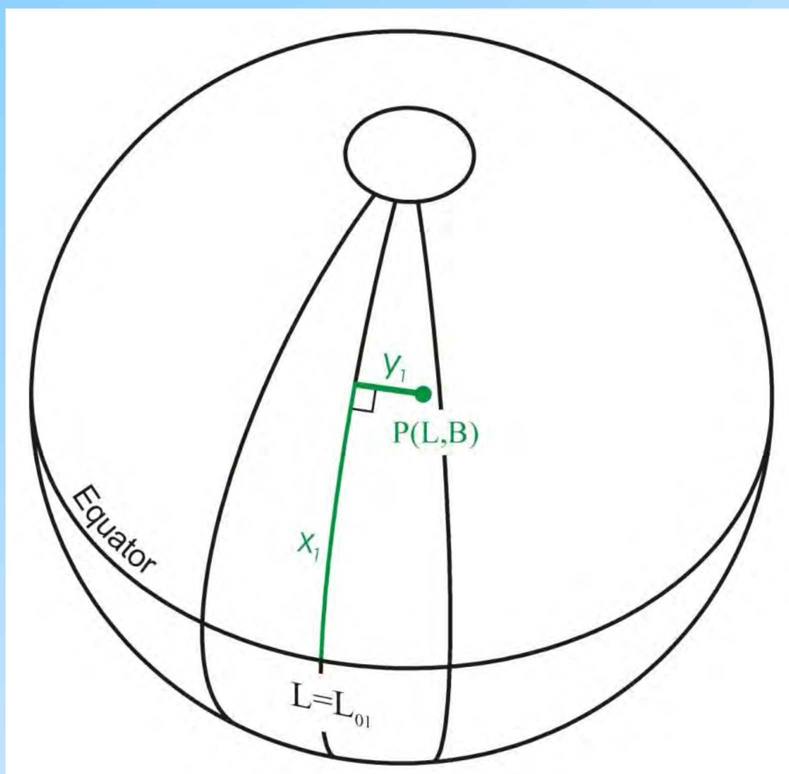




→ Strip transformation

Strip transformation

Answer: (a) Transform X_1, y_1 (L_{01}) into the invariants L, B
(b) Change the reference meridian $L_{01} \rightarrow L_{02}$
(c) Compute X_2, y_2 (L_{02})



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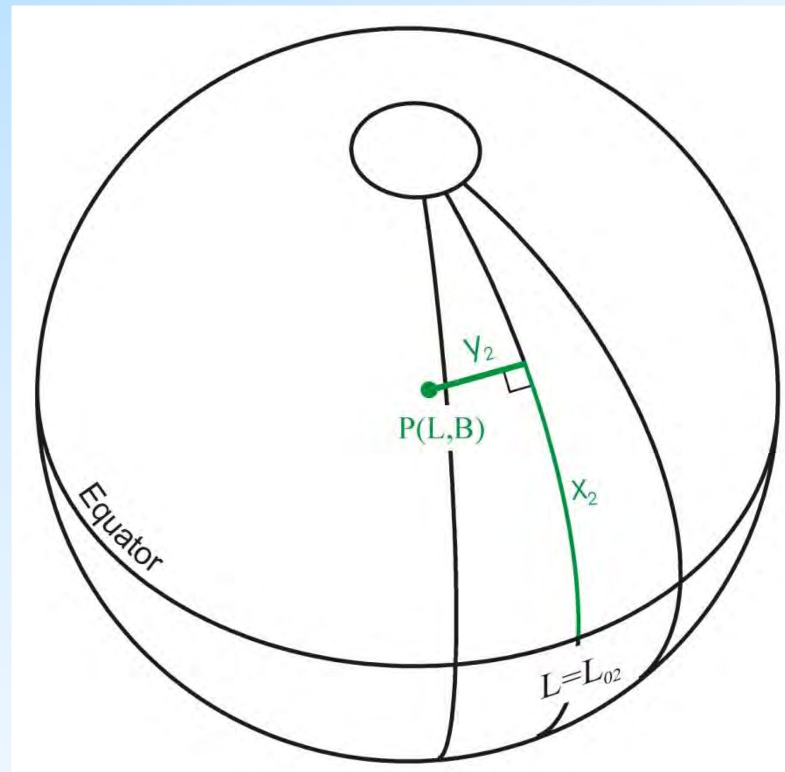
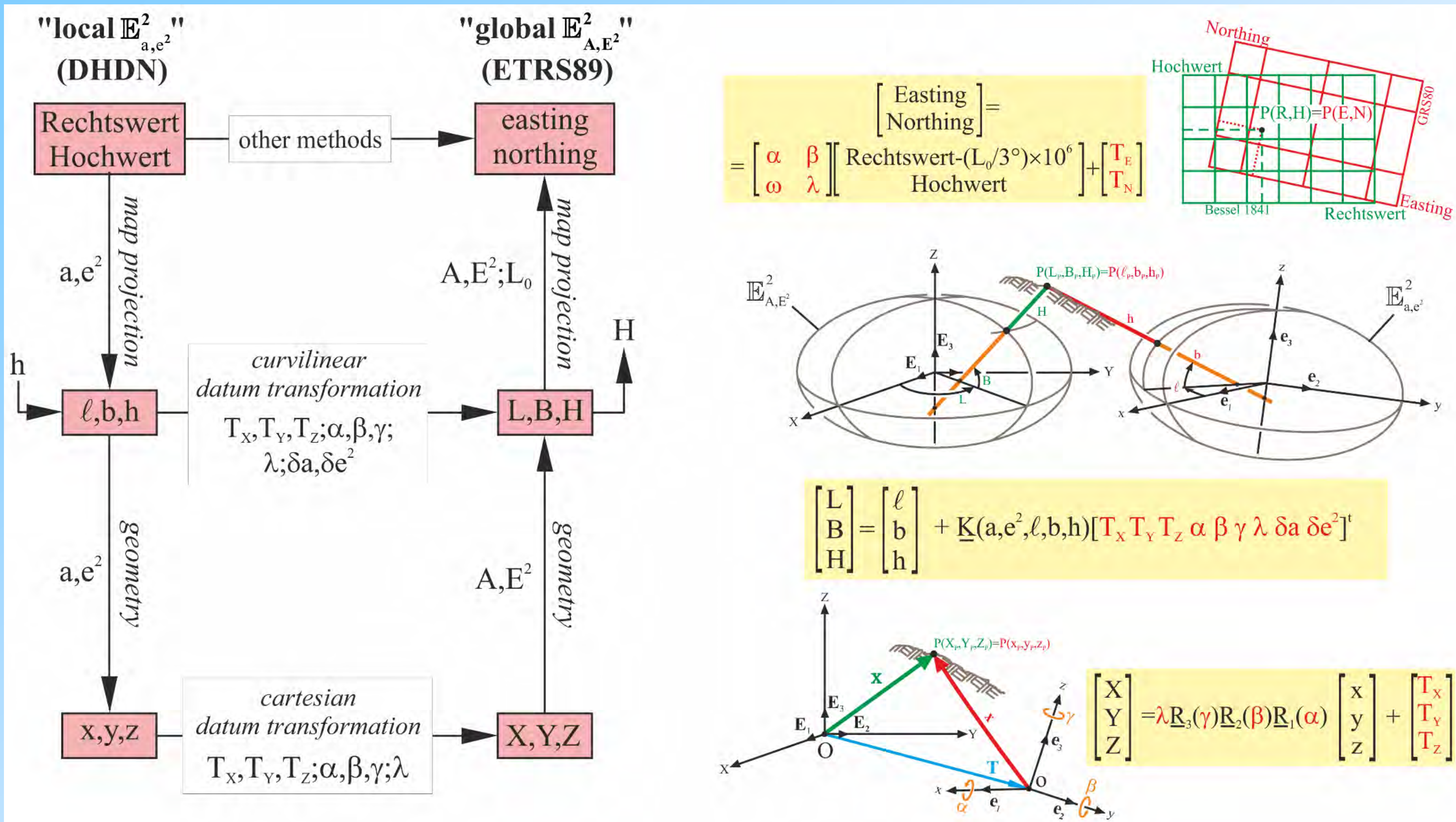


→ Datum transformations: Overview

Datum transformations: Overview



→ "Geometry transformation" (X,Y,Z) ↔ (L,B,H)

"Geometry transformation" (X,Y,Z) ↔ (L,B,H)

P(L,B,H) is close to one of the poles

$$\cos B_{(0)} = \frac{\sqrt{X^2+Y^2}}{A\sqrt{1+E'^2}} = \frac{\sqrt{X^2+Y^2}}{A}\sqrt{1-E^2}$$

$$i \Leftarrow i+1$$

$$N_{(i)} \Leftarrow \frac{A\sqrt{1+E'^2}}{\sqrt{1+E'^2\cos^2 B_{(i-1)}}} = \frac{A}{\sqrt{1-E^2\sin^2 B_{(i-1)}}}$$

$$H_{(i)} \Leftarrow \left|\frac{Z}{\sin B_{(i-1)}}\right| - \frac{N_{(i)}}{1+E'^2} = \left|\frac{Z}{\sin B_{(i-1)}}\right| - N_{(i)}(1-E^2)$$

$$\cos B_{(i)} \Leftarrow \frac{\sqrt{X^2+Y^2}}{N_{(i)}+H_{(i)}}$$

$$L = \arctan\frac{Y}{X}$$

P(L,B,H) is distant from one of the poles

$$\tan B_{(0)} = \frac{Z(1+E'^2)}{\sqrt{X^2+Y^2}} = \frac{Z}{(1-E^2)\sqrt{X^2+Y^2}}$$

$$i \Leftarrow i+1$$

$$N_{(i)} \Leftarrow \frac{A\sqrt{1+E'^2}}{\sqrt{1+E'^2\cos^2 B_{(i-1)}}} = \frac{A}{\sqrt{1-E^2\sin^2 B_{(i-1)}}}$$

$$H_{(i)} \Leftarrow \frac{\sqrt{X^2+Y^2}}{\cos B_{(i-1)}} - N_{(i)}$$

$$\tan B_{(i)} \Leftarrow \frac{Z(1+E'^2)}{\sqrt{X^2+Y^2}}\left(1+E'^2\frac{H_{(i)}}{N_{(i)}+H_{(i)}}\right)^{-1} = \frac{Z}{\sqrt{X^2+Y^2}}\,\frac{N_{(i)}+H_{(i)}}{(1-E^2)N_{(i)}+H_{(i)}}$$

$$\cos^2 B_{(i)} \Leftarrow (1+\tan^2 B_{(i)})^{-1} \quad , \quad \sin^2 B_{(i)} \Leftarrow \tan^2 B_{(i)}(1+\tan^2 B_{(i)})^{-1}$$

Simplified algorithm

$$B_{(0)} = \arctan\frac{Z}{\sqrt{X^2+Y^2}}$$

$$i \Leftarrow i+1$$

$$N_{(i)} \Leftarrow \frac{A\sqrt{1+E'^2}}{\sqrt{1+E'^2\cos^2 B_{(i-1)}}} = \frac{A}{\sqrt{1-E^2\sin^2 B_{(i-1)}}}$$

$$B_{(i)} \Leftarrow \arctan\frac{Z\left(1+E'^2\right)+N_{(i)}E'^2\sin B_{(i-1)}}{\left(1+E'^2\right)\sqrt{X^2+Y^2}} = \arctan\frac{Z+N_{(i)}E^2\sin B_{(i-1)}}{\sqrt{X^2+Y^2}}$$

$$H = \frac{\sqrt{X^2+Y^2}}{\cos B} - N$$

Iterate until (e.g. $\varepsilon \le 1\text{mm}$)

$$\left|H_{(i)} - H_{(i-1)}\right| < \varepsilon \quad \text{and} \quad \left|B_{(i)} - B_{(i-1)}\right| < \frac{\varepsilon}{A}$$

or

$$\left|\tan B_{(i)} - \tan B_{(i-1)}\right| < \frac{\varepsilon}{A}\left(1+\tan^2 B_{(i)}\right)$$

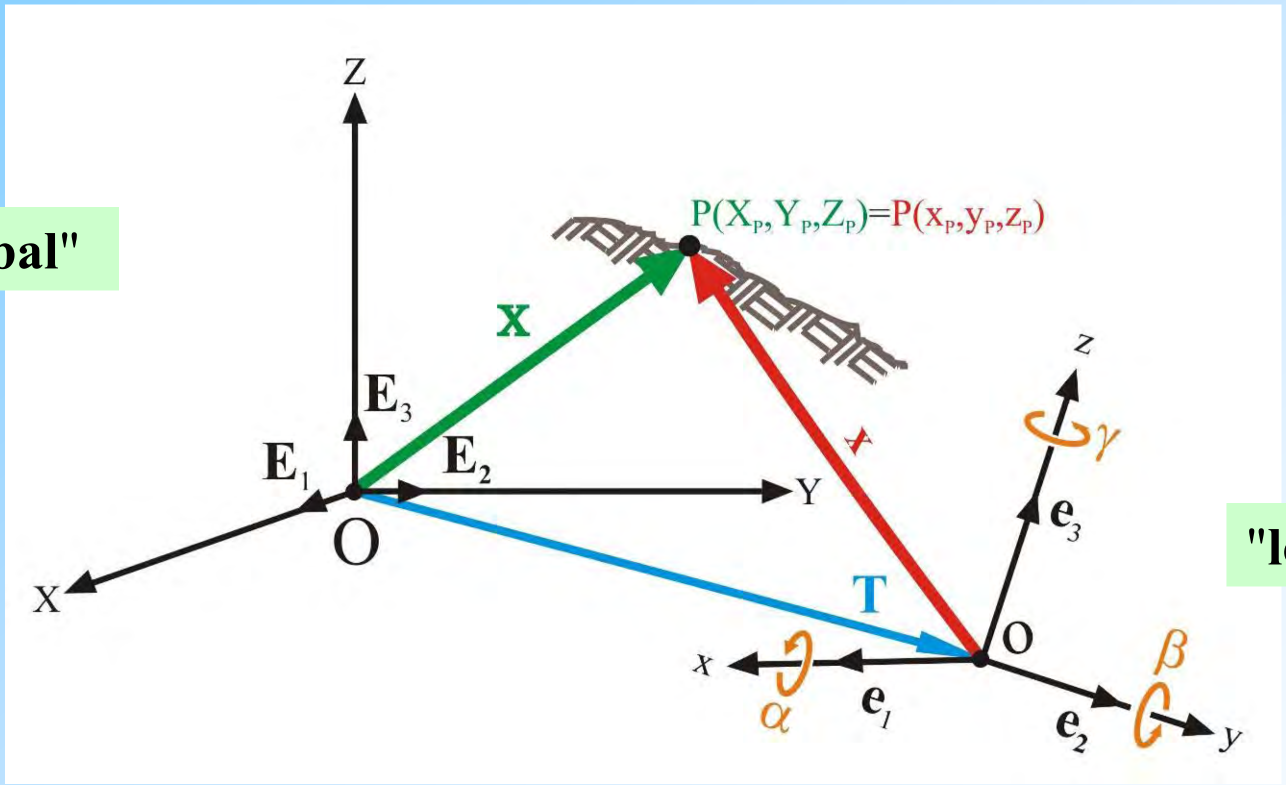
or

$$\left|\cos B_{(i)} - \cos B_{(i-1)}\right| < \frac{\varepsilon}{A}\left|\sin B_{(i)}\right|$$

→ Datum transformation: 7-Parameter-transformation

Datum transformation: 7-parameter-transformation

"global"



"local"

$\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ ... Base vectors system 1 (e.g. global system)

$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ ... Base vectors system 2 (e.g. local system)

α, β, γ ... Rotation angles to make $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ parallel

T_X, T_Y, T_Z ... Coordinates of the vector of translations (shifts) $O \rightarrow o$

→ Datum transformation: 7-Parameter-transformation

Datum transformation: 7-parameter-transformation

Transformation model for base vectors and coordinates

$$\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix} = \lambda \underline{R}_3(\gamma)\underline{R}_2(\beta)\underline{R}_1(\alpha) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \qquad \lambda \text{ ... Scale factor}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \lambda \underline{R}_3(\gamma)\underline{R}_2(\beta)\underline{R}_1(\alpha) \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} T_X \\ T_Y \\ T_Z \end{pmatrix} \sim \underline{X} = \lambda \underline{R}(\alpha,\beta,\gamma)\underline{x} + \underline{T}$$

$$\underline{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}, \underline{R}_2(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \underline{R}_3(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(mathematically positive: ccw)

→ Datum transformation: 7-Parameter-transformation

Datum transformation: 7-parameter-transformation

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \lambda \underline{R}_3(\gamma)\underline{R}_2(\beta)\underline{R}_1(\alpha)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} T_X \\ T_Y \\ T_Z \end{pmatrix}$$

Target system — Start system

Determination of 7 parameters using coordinates of points with coordinates in both systems (identical points, homologous points, control points): **Analysis**

? how many do we need ?

Problems: (1) Equation is not linear in the unknown parameters

(2) more points (coordinates) available than necessary to uniquely determine the parameters

Solution possibilities for the **Analysis**:

(1) Linearize the equation and use adjustment procedures to estimate the parameters

(2) Other procedures

→ 7-parameter-transformation: Analysis

7-parameter-transformation: Analysis

Approximate values (Taylor point)

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \lambda \underline{R}_3(\gamma)\underline{R}_2(\beta)\underline{R}_1(\alpha)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} T_X \\ T_Y \\ T_Z \end{pmatrix}$$

$$\lambda = \lambda_0 + \delta\lambda,\ \alpha = \alpha_0 + \delta\alpha$$
$$\beta = \beta_0 + \delta\beta,\ \gamma = \gamma_0 + \delta\gamma$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \lambda_0 \underline{R}_3(\gamma_0)\underline{R}_2(\beta_0)\underline{R}_1(\alpha_0)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} T_X \\ T_Y \\ T_Z \end{pmatrix} +$$

$$+[\ \underline{R}_3(\gamma_0)\underline{R}_2(\beta_0)\underline{R}_1(\alpha_0)\delta\lambda + \lambda_0\underline{R}_3(\gamma_0)\underline{R}_2(\beta_0)\underline{R}_1'(\alpha_0)\delta\alpha +$$

$$\lambda_0\underline{R}_3(\gamma_0)\underline{R}_2'(\beta_0)\underline{R}_1(\alpha_0)\delta\beta + \lambda_0\underline{R}_3'(\gamma_0)\underline{R}_2(\beta_0)\underline{R}_1(\alpha_0)\delta\gamma\ ]\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

→ 7-parameter-transformation: Analysis

7-parameter-transformation: Analysis

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_i - \lambda_0 \underline{R}(\alpha_0, \beta_0, \gamma_0) \begin{pmatrix} x \\ y \\ z \end{pmatrix}_i = \underline{R}(\alpha_0, \beta_0, \gamma_0) \begin{pmatrix} x \\ y \\ z \end{pmatrix}_i \delta\lambda + \lambda_0 \underline{R}_3(\gamma_0) \underline{R}_2(\beta_0) \underline{R}_1'(\alpha_0) \begin{pmatrix} x \\ y \\ z \end{pmatrix}_i \delta\alpha +$$

$$+ \lambda_0 \underline{R}_3(\gamma_0) \underline{R}_2'(\beta_0) \underline{R}_1(\alpha_0) \begin{pmatrix} x \\ y \\ z \end{pmatrix}_i \delta\beta +$$

$$+ \lambda_0 \underline{R}_3'(\gamma_0) \underline{R}_2(\beta_0) \underline{R}_1(\alpha_0) \begin{pmatrix} x \\ y \\ z \end{pmatrix}_i \delta\gamma + \begin{pmatrix} T_X \\ T_Y \\ T_Z \end{pmatrix} \qquad i = 1, \ldots, n$$

$$\underline{y}_i \qquad = \qquad \underline{A}_i \quad \underline{\xi}$$

→ 7-parameter-transformation: Analysis

7-parameter-transformation: Analysis

$$\text{with}\quad \underset{3\times1}{\underline{y}_i} := \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_i - \lambda_0 \underline{R}(\alpha_0,\beta_0,\gamma_0)\begin{pmatrix} x \\ y \\ z \end{pmatrix}_i \quad , \quad \underset{3\times7}{\underline{A}_i} := \left[\underset{3\times4}{\underline{A}_i^1} \quad \underset{3\times3}{\underline{A}_i^2} \right]$$

$$\underset{3\times4}{\underline{A}_i^1} := \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\underline{R}(\alpha_0,\beta_0,\gamma_0)\begin{pmatrix} x \\ y \\ z \end{pmatrix}_i}_{\delta\lambda} \\ 0 & 0 & 1 & \\ T_X & T_Y & T_Z & \end{bmatrix}$$

$$\underset{3\times3}{\underline{A}_i^2} := \left[\underbrace{\lambda_0 \underline{R}_3(\gamma_0)\underline{R}_2(\beta_0)\underline{R}_1'(\alpha_0)\begin{pmatrix} x \\ y \\ z \end{pmatrix}_i}_{\delta\alpha} \quad \underbrace{\lambda_0 \underline{R}_3(\gamma_0)\underline{R}_2'(\beta_0)\underline{R}_1(\alpha_0)\begin{pmatrix} x \\ y \\ z \end{pmatrix}_i}_{\delta\beta} \quad \underbrace{\lambda_0 \underline{R}_3'(\gamma_0)\underline{R}_2(\beta_0)\underline{R}_1(\alpha_0)\begin{pmatrix} x \\ y \\ z \end{pmatrix}_i}_{\delta\gamma} \right]$$

$$\underset{7\times1}{\underline{\xi}} := [T_X \quad T_Y \quad T_Z \quad \delta\lambda \quad \delta\alpha \quad \delta\beta \quad \delta\gamma]^t$$

→ 7-parameter-transformation: Analysis

7-parameter-transformation: Analysis

Usual magnitude of the unknown parameters:

Rotations α,β,γ: few arc seconds (–3" ... +3")

Scale λ: $\approx 1 \Rightarrow \delta\lambda \approx 10^{-5} \ldots 10^{-7}$

(CTI-transformation)

Translations T_X, T_Y, T_Z : up to several 100m (–600m ... +600m)

$$\Rightarrow \lambda_0 = 1, \alpha_0 = \beta_0 = \gamma_0 = 0 \Rightarrow \underline{R}(\alpha,\beta,\gamma) \doteq \begin{bmatrix} 1 & \delta\gamma & -\delta\beta \\ -\delta\gamma & 1 & \delta\alpha \\ \delta\beta & -\delta\alpha & 1 \end{bmatrix} = \underline{I}_3 + \delta\underline{R},\ \underline{R}^T\underline{R} \approx \underline{I}_3$$

$$\underbrace{\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_i - \begin{pmatrix} x \\ y \\ z \end{pmatrix}_i}_{\underline{y}_i} = \underbrace{\left[\begin{array}{ccc:c:ccc} 1 & 0 & 0 & x & 0 & -z & y \\ 0 & 1 & 0 & y & z & 0 & -x \\ 0 & 0 & 1 & z & -y & x & 0 \end{array}\right]_i}_{\underline{A}_i} \underbrace{\left[\begin{array}{c} T_X \\ T_Y \\ T_Z \\ \hdashline \delta\lambda \\ \hdashline \delta\alpha \\ \delta\beta \\ \delta\gamma \end{array}\right]}_{\underline{\xi}}$$

Note:

- Possible stochasticity of design matrix $\underline{A}$
- Iteration needed
- Quality control
- Outlier detection

→ 7-parameter-transformation: Analysis

GIS

7-parameter-transformation: Analysis

<u>Given:</u> Transformation parameters from analysis model "local" → "global"

<u>Wanted:</u> Synthesis in the model "global" → "local"

$$\underline{X} = \lambda \underline{R}\underline{x} + \underline{T}$$

$$\Rightarrow \underline{x} = \lambda^{-1}\underline{R}^{-1}(\underline{X} - \underline{T}) = \lambda^{-1}\underline{R}^{T}(\underline{X} - \underline{T}) \quad \text{since } \underline{R} \text{ is orthogonal}$$

$$\left.\begin{aligned} &= \lambda^{-1}\underline{R}^{T}\underline{X} - \lambda^{-1}\underline{R}^{T}\underline{T} \\ &= \tilde{\lambda}\ \underline{\tilde{R}}\ \underline{X} + \ \underline{\tilde{T}} \end{aligned}\right\} \Rightarrow \tilde{\lambda} = \lambda^{-1}, \quad \underline{\tilde{R}} = \underline{R}^{T}, \quad \underline{\tilde{T}} = -\lambda^{-1}\underline{R}^{T}\underline{T}$$

$$\text{CTI}: \lambda = 1 + \delta\lambda, \quad \underline{R} = \underline{I}_3 + \delta\underline{R} \Rightarrow$$

$$\tilde{\lambda} = (1+\delta\lambda)^{-1} \doteq 1 - \delta\lambda, \quad \underline{\tilde{R}} = \underline{R}^{T} = \underline{R}^{-1} = \underline{I}_3 - \delta\underline{R}$$

$$\underline{X} \doteq \underline{x} + (+\delta\lambda\underline{I}_3 + \delta\underline{R})\underline{x} + \underline{T}$$

$$\underline{x} \doteq \underline{X} + (-\delta\lambda\underline{I}_3 - \delta\underline{R})\underline{X} - \underline{T} \underbrace{+(\delta\lambda\underline{I}_3 + \delta\underline{R})\underline{T}}_{\approx 0}$$

→ 7-parameter-transformation: Synthesis

7-parameter-transformation: Synthesis

<u>Analysis:</u> Determine the unknown parameters from a certain number of homologous points in both systems

<u>Synthesis:</u> Transform points from system 1 to system 2 using the estimated parameters (Start system → target system)

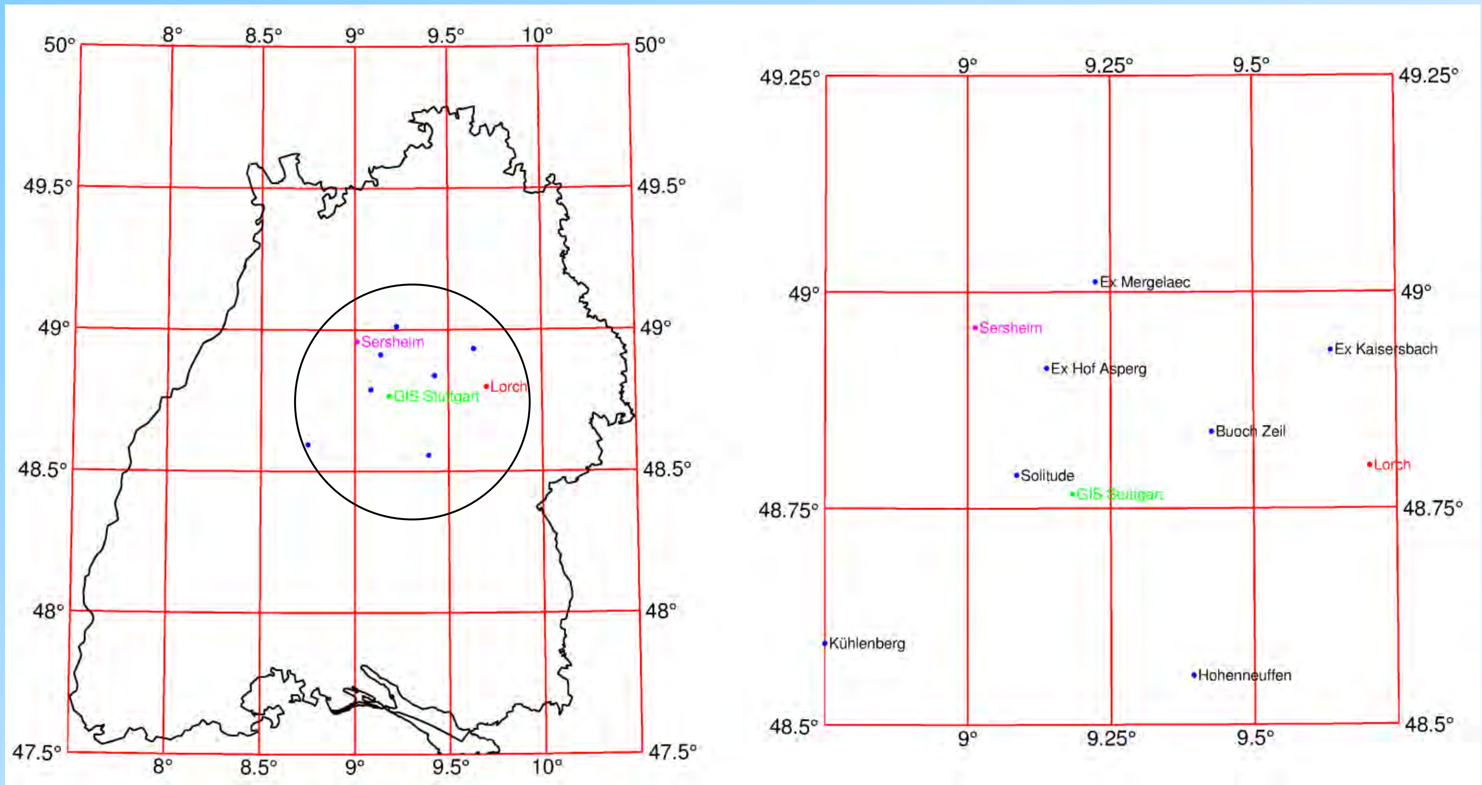
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \hat{\lambda}\underline{R}_3(\hat{\gamma})\underline{R}_2(\hat{\beta})\underline{R}_1(\hat{\alpha})\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \hat{T}_X \\ \hat{T}_Y \\ \hat{T}_Z \end{pmatrix}$$

Attention: Stating und using numerical values for transformation parameters without specifying the model from which there were computed is meaningless and can lead to wrong results !
Reason:

Model A: $\underline{X} = \lambda\underline{R}(\alpha,\beta,\gamma)\underline{x} + \underline{T}$ $\neq$ Model B: $\underline{x} = \lambda\underline{R}(\alpha,\beta,\gamma)\underline{X} + \underline{T}$

→ 7-parameter-transformation: Example

7-parameter-transformation: Example



Given: Transformation parameters from analysis model "local" → "global", i.e.

$$\underline{X}_i = \lambda \underline{R}\,\underline{x}_i + \underline{T}\,,\; i = 1,\ldots,7 \quad \Rightarrow \quad [\hat{T}_X, \hat{T}_Y, \hat{T}_Z] = [640.933\mathrm{m}, 71.927\mathrm{m}, 414.787\mathrm{m}]$$

$$[\hat{\alpha}, \hat{\beta}, \hat{\gamma}] = [-1''.7074, 0''.9070, 1''.0815]$$

$$\hat{\lambda} = 1 + \widehat{\delta\lambda} = 1 + 0.0000058$$

→ 7-parameter-transformation: Example

7-parameter-transformation: Example

Synthesis in the **local-to-global** model with local-to-global parameters

Given: Coordinates x,y,z of point "Lorch" in the local system

$$x = 4149\,297.818\text{m}\,,\; y = 709\,461.957\text{m}\,,\; z = 4\,776\,101.269\text{m}$$

Wanted: Coordinates X,Y,Z of point "Lorch" in the global system

$$\underline{\hat{X}}_{\text{Lorch}} = \hat{\lambda}\underline{\hat{R}}\underline{x}_{\text{Lorch}} + \underline{\hat{T}} \quad \text{with} \quad \underline{\hat{R}} = \underline{R}(\hat{\alpha},\hat{\beta},\hat{\gamma})$$

$$\hat{X} = 4\,149\,945.535\text{m}\,,\; \hat{Y} = 709\,476.708\text{m}\,,\; \hat{Z} = 4\,776\,567.876\text{m}$$

Synthesis in the **global-to-local** model with local-to-global parameters

Given: Coordinates X,Y,Z of point "Sersheim" in the global system

$$X = 4144\,220.260\text{m}\,,\; Y = 657\,329.504\text{m}\,,\; Z = 4\,787\,730.742\text{m}$$

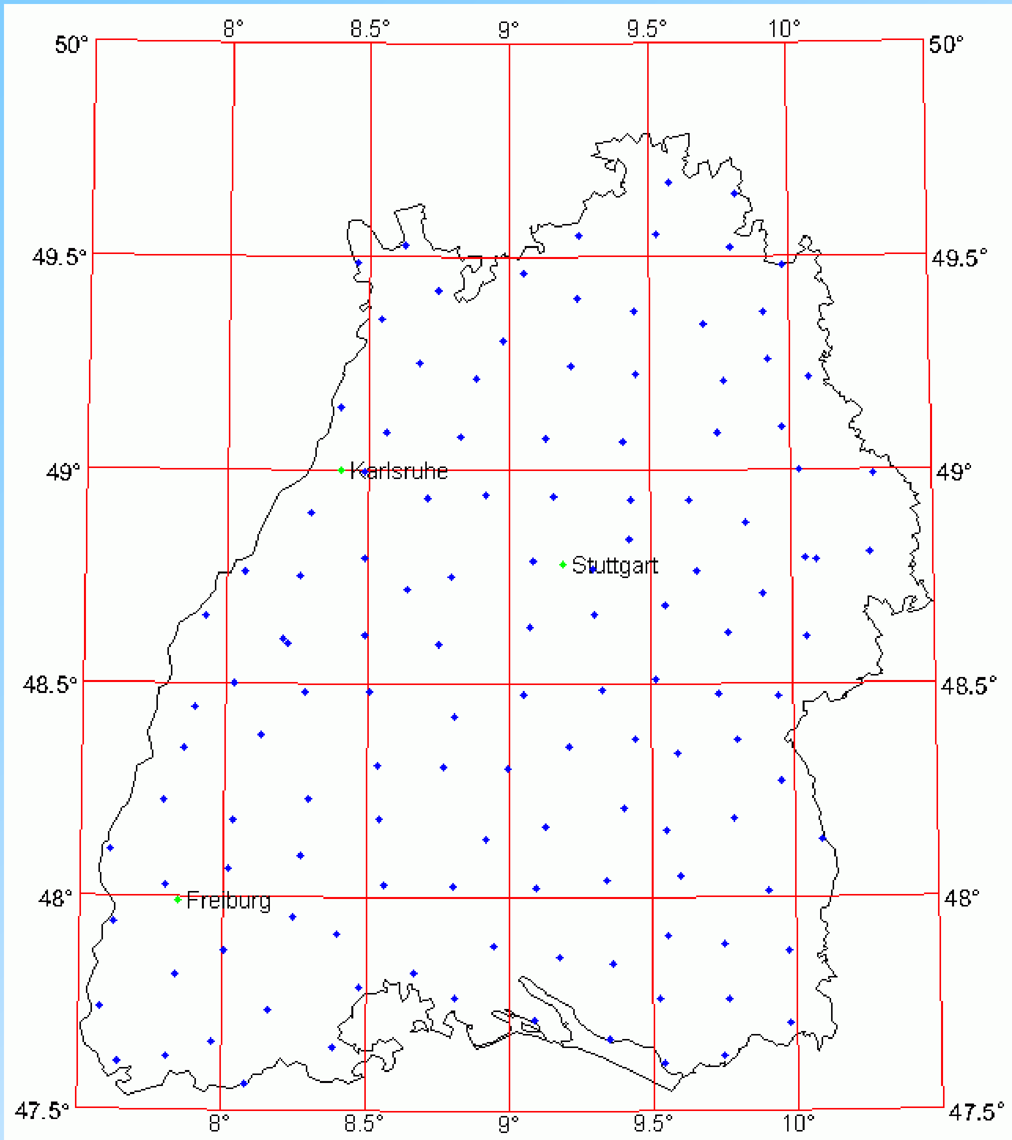
Wanted: Coordinates x,y,z of point "Sersheim" in the local system

$$\underline{\hat{x}}_{\text{Sersheim}} = \lambda\underline{R}\underline{X}_{\text{Sersheim}} + \underline{T} \quad \text{with} \quad \lambda = 1 - \widehat{\delta\lambda},\; \underline{R} = \underline{R}(-\hat{\alpha},-\hat{\beta},-\hat{\gamma})$$

$$\hat{x} = 4\,143\,572.899\text{m}\,,\; \hat{y} = 657\,315.118\text{m}\,,\; \hat{z} = 4\,787\,264.528\text{m}$$

→ 7-parameter-transformation: Example

7-parameter-transformation: Example



131 homologous points of the BWREF-network: UTM-coordinates given with respect to the ETRS89 ("global") and Gauß-Krüger-coordinates given with respect to the German DHDN ("local")

Analysis:

$$\hat{T}_X = 583\text{m}$$

$$\hat{T}_Y = 112\text{m}$$

$$\hat{T}_Z = 406\text{m}$$

$$\hat{\alpha} = -2.3''$$

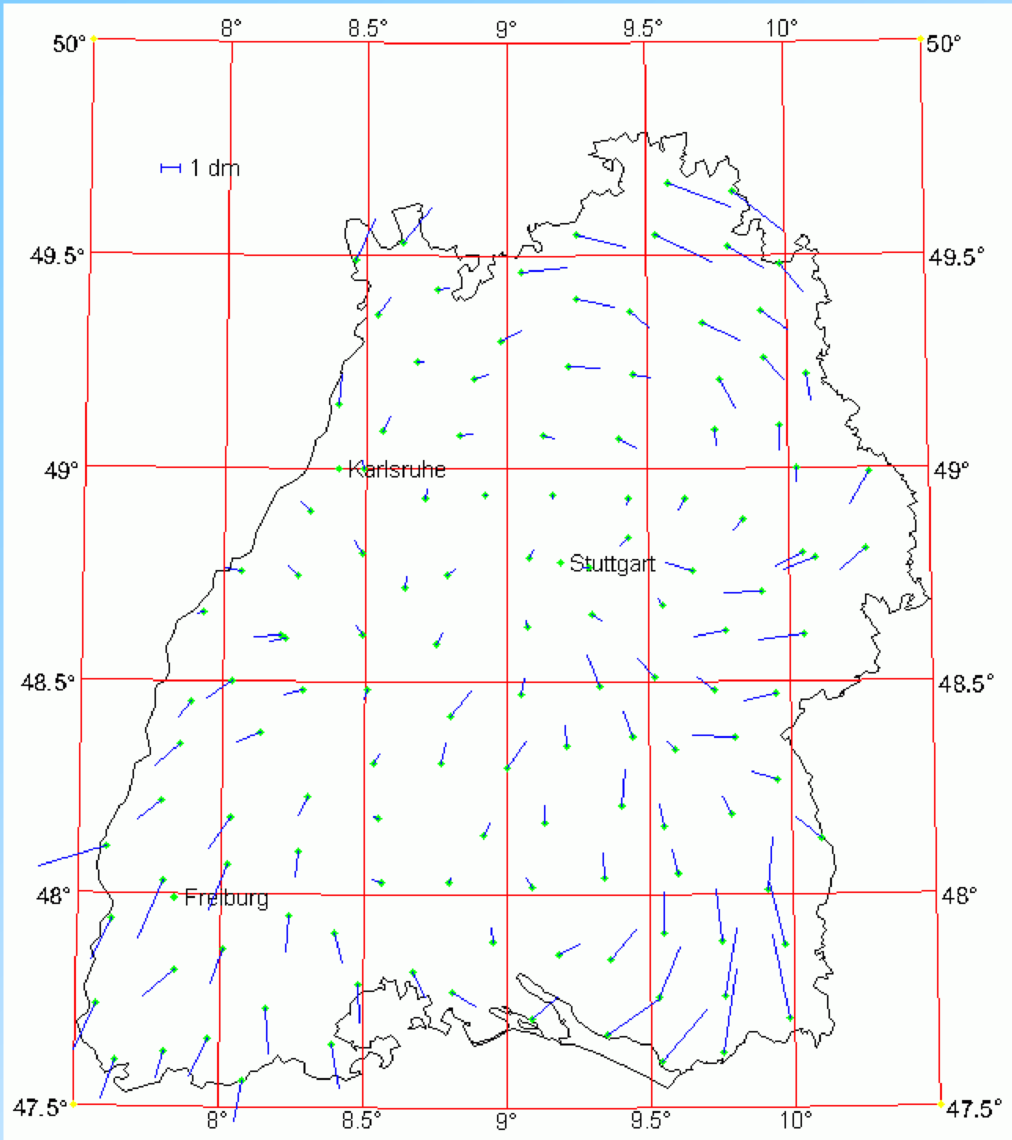
$$\hat{\beta} = -0.3''$$

$$\hat{\gamma} = 2.1''$$

$$\hat{\lambda} = 1.000\,009$$

→ 7-parameter-transformation: Example

7-parameter-transformation: Example



Synthesis of the homologous points:

Using the parameter estimates from the analysis step, control points can be transformed from the start to the target system, $\underline{\hat{X}}$. These values will not coincide with the given coordinates $\underline{X}$, giving rise to residual estimates $\underline{X} - \underline{\hat{X}}$. In most cases, however, control points should not be affected by the transformation, i.e. residuals should be disseminated over newly transformed points, only. This is done using a true-neighborhood-post-transformation.

→ True-neighborhood-post-transformation correction

True-neighborhood-post-transformation correction

One possibility for this post-transformation correction is based on an inverse distance weighting of the d×m matrix of residuals $\underline{\hat{E}} = \underline{X} - \underline{\hat{X}}$ of m control points P in d-dimensional space.

Let $\underline{s}_{PQ}$ be the m×n matrix of distances between the m control points P and n transformation points Q, as computed in the starting system

$$\underline{s}_{PQ} = \begin{bmatrix} s_{P_1Q_1} & s_{P_1Q_2} & \cdots & s_{P_1Q_n} \\ s_{P_2Q_1} & s_{P_2Q_2} & \cdots & s_{P_2Q_n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{P_mQ_1} & s_{P_mQ_2} & \cdots & s_{P_mQ_n} \end{bmatrix}.$$

The d×n matrix of coordinates of Q transformed into the target system, $\underline{\hat{X}}_Q$, by using the estimates of the transformation parameters from the analysis step are then "corrected" on the basis of the following post-transformation:

$$\underset{d\times n}{\underline{\tilde{X}}_Q} = \underline{\hat{X}}_Q + d\underline{\hat{X}} \quad , \quad \underset{d\times n}{d\underline{\hat{X}}} = \underset{d\times m}{\underline{\hat{E}}} \frac{\underset{m\times n}{\underline{w}}}{\underline{1}_m \underline{1}_m^T \underline{w}} \quad , \quad \underset{m\times n}{\underline{w}} = \frac{\underline{1}_m \underline{1}_n^T}{\underset{m\times n}{\underline{s}_{PQ}^{\alpha}} + \beta \underline{1}_m \underline{1}_n^T}$$

→ True-neighborhood-post-transformation correction

True-neighborhood-post-transformation correction

$\underline{1}_r$ is the r×1 summation array, $\underline{1}_r^T = [1, 1, \ldots, 1]$, α and β are constants (smoothing factors), usually set to $\alpha = 2, \beta = 0$. The (dyadic) product $\underline{1}_q \underline{1}_r^T$ generates an q×r matrix of ones and $\underline{s}_{PQ}^{\alpha}$ means elementwise exponentiation of the entries of $\underline{s}_{PQ}$. Matrix divisions in the computation of $d\hat{\underline{X}}$ and $\underline{w}$ are understood as elementwise divisions of corresponding matrix entries.

If β=0 and Q is a point from the set of control points (Q∈P), say $Q \equiv P_2$, then

$$\frac{\underset{m\times n}{\underline{w}}}{\underline{1}_m \underline{1}_m^T \underline{w}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T}_{m\times n}$$

is an m×1 unit vector with a 1 at the position of Q=P and

$$\tilde{\underline{X}}_{Q=P} = \hat{\underline{X}}_{Q=P} + d\hat{\underline{X}} = \hat{\underline{X}}_{Q=P} + \hat{\underline{E}}_2 = \hat{\underline{X}}_{Q=P} + (\underline{X}_{Q=P} - \hat{\underline{X}}_{Q=P}) = \underline{X}_{Q=P}$$

The coordinates of point Q∈P after the post-transformation equal the given coordinates of point P in the target system and remain unchanged.

A numerical example is given later for the case of the 4-parameter transformation.

→ 2D-models: 6-parameter affine transformation

2D-models: 6-parameter affine transformation

Given: A planar geodetic network, the points of which are equipped with different kinds of coordinates [e.g. Gauß-Krüger coordinates with respect to a "local" ellipsoid (a, e^2) and UTM coordinates with respect to a "global" ellipsoid (A, E^2)]

Unknown: Translation between and mutual orientation of the ellipsoids

Wanted: Transformation parameters from the model "6-parameter affine transformation (2 rotations, 2 translations, 2 scale factors)

Model equation:

$$\begin{bmatrix} \text{Easting} \\ \text{Northing} \end{bmatrix}_{\text{UTM}} = \begin{bmatrix} \alpha & \beta \\ \omega & \lambda \end{bmatrix} \begin{bmatrix} \text{Easting} \\ \text{Northing} \end{bmatrix}_{\text{GK}} + \begin{bmatrix} T_E \\ T_N \end{bmatrix} \sim$$

$$\sim \underline{X} = \underline{J}\,\underline{x} + \underline{T}$$

affine parameters

→ 2D-models: 6-parameter affine transformation

2D-models: 6-parameter affine transformation

1st representation/interpretation

$\underline{T}$ ... coordinate array of translation vector
$\underline{J}$ ... deformation matrix (Jacobian matrix)

$$\underline{J} = \begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{bmatrix} \overset{\text{Polar decomposition}}{=} \underline{R}\,\underline{S} \quad \begin{cases} \underline{R} \text{ ... rotation matrix, } \underline{R}^T\underline{R} = \underline{I}_2 \\ \underline{S} \text{ ... stretch matrix, symmetric } \underline{S} = \underline{S}^T\text{, p.d.} \\ \text{(in general full matrix)} \end{cases}$$

$$\underline{J} = \underline{U}\underline{\Sigma}\underline{V}^T, \quad \underline{R} = \underline{U}\underline{V}^T, \quad \underline{S} = \underline{V}\underline{\Sigma}\underline{V}^T$$

singular value decomposition

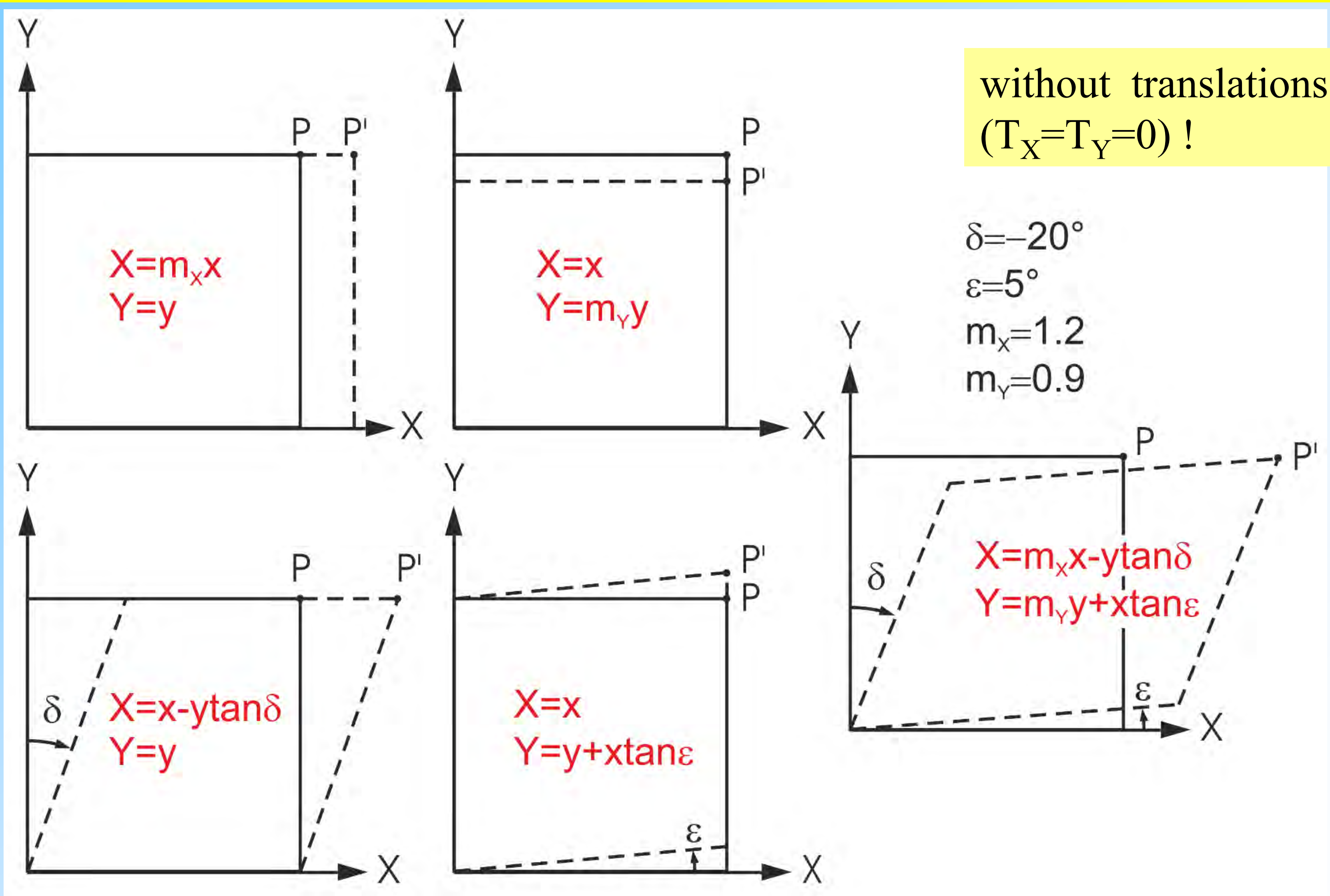
$\frac{\partial X}{\partial x}, \frac{\partial Y}{\partial y}$ ... scale factors m_X und m_Y

$\frac{\partial X}{\partial y}, \frac{\partial Y}{\partial x}$ ... shear strain ($\tan\delta$ and $\tan\varepsilon$)

→ 2D-models: 6-parameter affine transformation

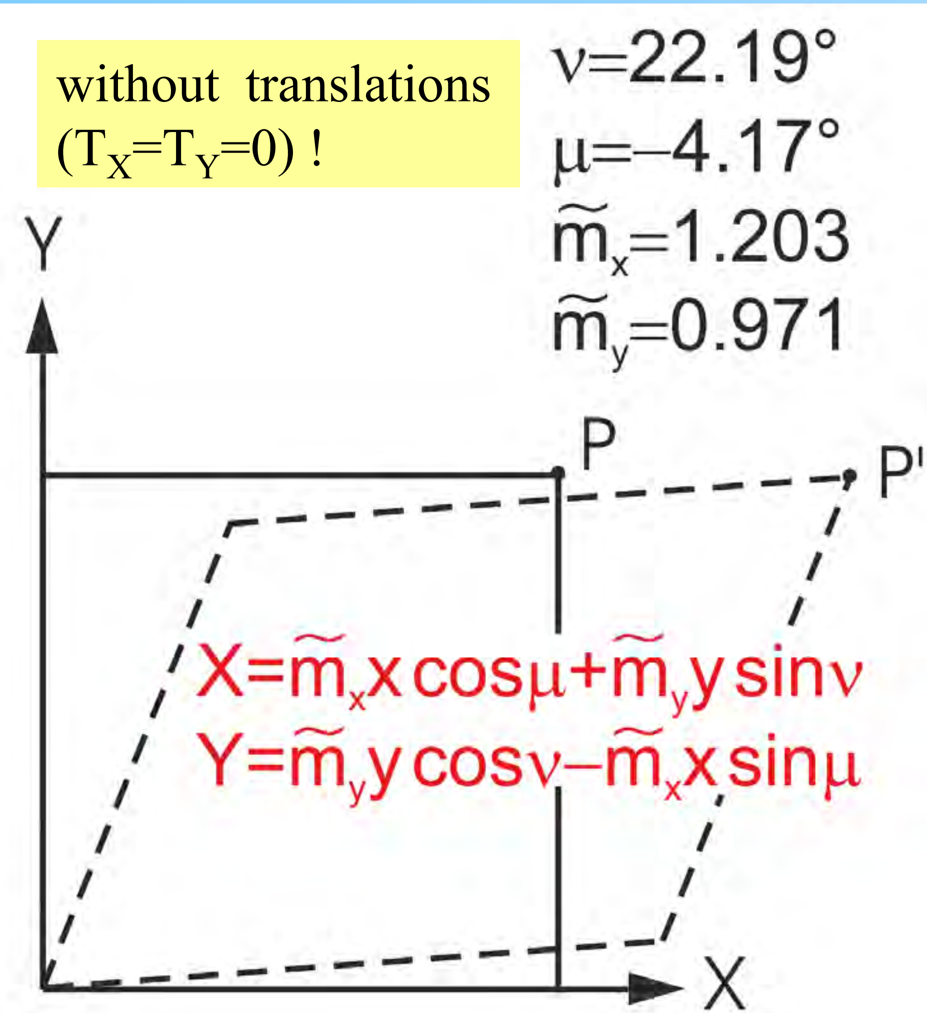
2D-models: 6-parameter affine transformation

without translations ($T_X = T_Y = 0$) !



→ 2D-models: 6-parameter affine transformation

2D-models: 6-parameter affine transformation



2nd representation/interpretation

$\underline{T}$ ... coordinate array of translation vector

$$\underline{J} = \begin{bmatrix} \cos\mu & \sin\nu \\ -\sin\mu & \cos\nu \end{bmatrix} \begin{bmatrix} \tilde{m}_X & 0 \\ 0 & \tilde{m}_Y \end{bmatrix}$$

$$\Downarrow$$

$$X = \tilde{m}_X x\cos\mu + \tilde{m}_Y y\sin\nu + T_X$$

$$Y = \tilde{m}_Y y\cos\nu - \tilde{m}_X x\sin\mu + T_Y$$

→ 2D-models: 6-parameter affine transformation

2D-models: 6-parameter affine transformation

Evaluation of the model equation:

$$\begin{bmatrix} E \\ N \end{bmatrix}_{UTM} := \begin{bmatrix} \text{Easting} \\ \text{Northing} \end{bmatrix}_{UTM} = \begin{bmatrix} \alpha & \beta \\ \omega & \lambda \end{bmatrix} \begin{bmatrix} \text{Easting } E_{GK} \\ \text{Northing } N_{GK} \end{bmatrix} + \begin{bmatrix} T_E \\ T_N \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} E \\ N \end{bmatrix}_{UTM} = \underbrace{\begin{bmatrix} 1 & 0 & E & N & 0 & 0 \\ 0 & 1 & 0 & 0 & E & N \end{bmatrix}_{GK}}_{2\times 6} \begin{bmatrix} T_E \\ T_N \\ \alpha \\ \beta \\ \omega \\ \lambda \end{bmatrix}$$

6 unknown parameters $\Rightarrow$ coordinates of at least 3 stations required !

2D-models: 6-parameter affine transformation

"Vector valued" "Matrix valued"

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}_{UTM} = \begin{bmatrix} 1 & 0 & E_1 & N_1 & 0 & 0 \\ 1 & 0 & E_2 & N_2 & 0 & 0 \\ 1 & 0 & E_3 & N_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & E_1 & N_1 \\ 0 & 1 & 0 & 0 & E_2 & N_2 \\ 0 & 1 & 0 & 0 & E_3 & N_3 \end{bmatrix}_{GK} \begin{bmatrix} T_E \\ T_N \\ \alpha \\ \beta \\ \omega \\ \lambda \end{bmatrix} \sim \begin{bmatrix} E_1 & N_1 \\ E_2 & N_2 \\ E_3 & N_3 \end{bmatrix}_{UTM} = \begin{bmatrix} 1 & E_1 & N_1 \\ 1 & E_2 & N_2 \\ 1 & E_3 & N_3 \end{bmatrix}_{GK} \begin{bmatrix} T_E & T_N \\ \alpha & \omega \\ \beta & \lambda \end{bmatrix}$$

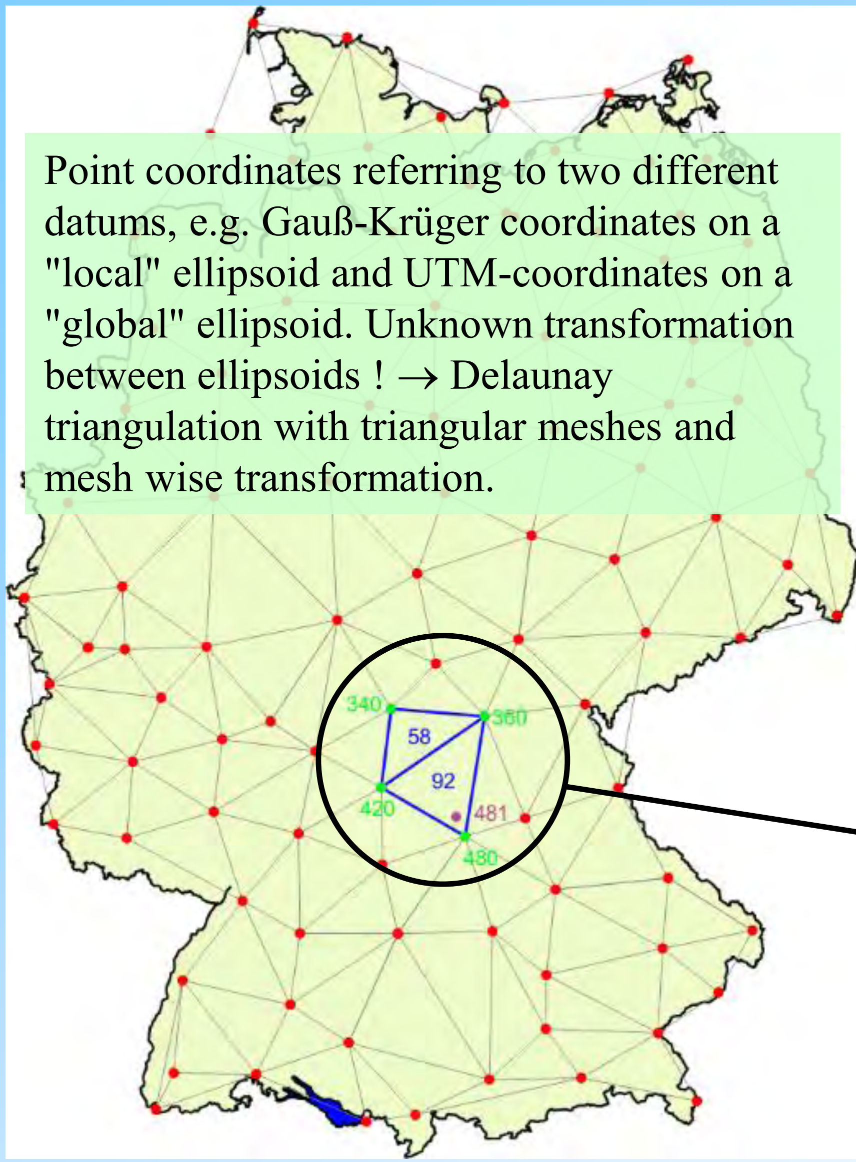
$$\Rightarrow \begin{bmatrix} T_E & T_N \\ \alpha & \omega \\ \beta & \lambda \end{bmatrix} = \begin{bmatrix} 1 & E_1 & N_1 \\ 1 & E_2 & N_2 \\ 1 & E_3 & N_3 \end{bmatrix}_{GK}^{-1} \begin{bmatrix} E_1 & N_1 \\ E_2 & N_2 \\ E_3 & N_3 \end{bmatrix}_{UTM}$$

homologous points must refer to the same meridian strip and zone, respectively, otherwise strip transformation has to be applied beforehand !

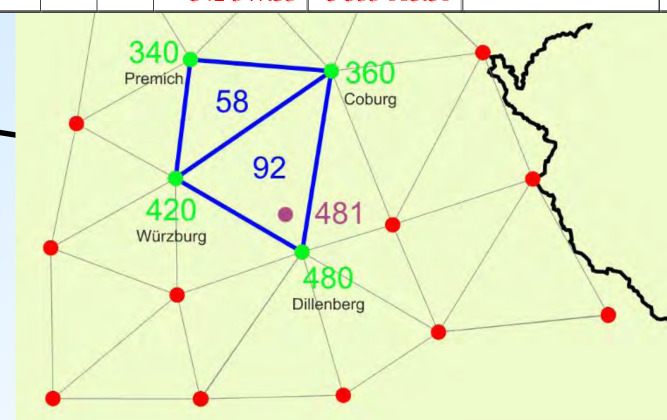
→ 6-parameter affine transformation: Example

6-parameter affine transformation : Example

Point coordinates referring to two different datums, e.g. Gauß-Krüger coordinates on a "local" ellipsoid and UTM-coordinates on a "global" ellipsoid. Unknown transformation between ellipsoids ! → Delaunay triangulation with triangular meshes and mesh wise transformation.



Analysis							
Point ID	Point name	Mesh ID	UTM Zone	Datum 1 ("local" ellipsoid) Gauß-Krüger coordinates False Easting / Datum 2 ("global" ellipsoid) UTM coordinates False Easting	Northing / Northing	Parameter T_E / α / β	T_N / ω / λ
340	Premich	58	32	3 571 616.15 571 718.14	5 576 621.39 5 575 032.32	125.501 4	631.467 4
360	Coburg		32	3 642 085.67 642 159.72	5 572 145.41 5 570 558.01	0.999 603 441 8	−0.000 001 665 5
420	Würzburg		32	3 565 170.11 565 274.61	5 517 037.60 5 515 472.28	0.000 000 877 2	0.999 601 835 8
360	Coburg	92	32	3 642 085.67 642 159.51	5 572 145.41 5 570 558.92	127.889 3	620.807 9
480	Dillenberg		32	3 629 220.02 629 298.92	5 481 538.96 5 479 988.40	0.999 603 978 0	−0.000 004 059 1
420	Würzburg		32	3 565 170.11 565 274.39	5 517 037.60 5 515 473.24	0.000 000 398 7	0.999 603 971 9
Synthesis							
481	N.N.	92	32	3 542 234.16 542 347.53	5 535 256.98 5 533 685.50		



 GIS

6-parameter affine transformation : Discussion

Discussion:

- For a mesh wise, non over determined transformation equations are exactly satisfied (+)
- In the vicinity of a line connecting two points the transformation parameters of the neighboring mesh can be used without loss of precision (+)
- Transition between neighboring mesh is continuous (+)
- Outlier search, check of results, information on precision is not possible (–) → should not be used as a stand alone solution
- can be extended to a transformation for more than only one mesh (+) → over determined transformation → adjustment procedures

→ Other 2D-models

2D-models: 5-parameter transformations

5-parameter transformation

Model equation 1: 2 translations, 1 rotation, 2 scale factors

$$\begin{bmatrix} E_{UTM} \\ N_{UTM} \end{bmatrix} = \begin{bmatrix} m_1 \cos\delta & -m_2 \sin\delta \\ m_1 \sin\delta & m_2 \cos\delta \end{bmatrix} \begin{bmatrix} E_{GK} \\ N_{GK} \end{bmatrix} + \begin{bmatrix} T_E \\ T_N \end{bmatrix} =$$

$$= \begin{bmatrix} \alpha & \beta \\ \omega & \lambda \end{bmatrix} \begin{bmatrix} E_{GK} \\ N_{GK} \end{bmatrix} + \begin{bmatrix} T_E \\ T_N \end{bmatrix} \Rightarrow$$

Attention !

$$\Rightarrow m_1 = \sqrt{\alpha^2 + \omega^2}, \quad m_2 = \sqrt{\beta^2 + \lambda^2}, \quad \tan\delta = \frac{\omega}{\alpha} = \frac{-\beta}{\lambda}$$

Model equation 2: 2 translations, 2 rotations, 1 scale factor

$$\begin{bmatrix} E_{UTM} \\ N_{UTM} \end{bmatrix} = m \begin{bmatrix} \cos\varepsilon & -\sin\delta \\ \sin\varepsilon & \cos\delta \end{bmatrix} \begin{bmatrix} E_{GK} \\ N_{GK} \end{bmatrix} + \begin{bmatrix} T_E \\ T_N \end{bmatrix} \Rightarrow$$

Attention !

$$\Rightarrow m = \sqrt{\alpha^2 + \omega^2} = \sqrt{\beta^2 + \lambda^2}, \quad \tan\varepsilon = \frac{\omega}{\alpha}, \quad \tan\delta = \frac{-\beta}{\lambda}$$

→ Other 2D-models

2D-models: 3- and 4-parameter transformation

<u>4-parameter transformation:</u> 2 translations, 1 rotation, 1 scale factor

$$\begin{bmatrix} E_{UTM} \\ N_{UTM} \end{bmatrix} = m \begin{bmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} E_{GK} \\ N_{GK} \end{bmatrix} + \begin{bmatrix} T_E \\ T_N \end{bmatrix} =$$

$$= \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} E_{GK} \\ N_{GK} \end{bmatrix} + \begin{bmatrix} T_E \\ T_N \end{bmatrix} \Rightarrow$$

$$\Rightarrow m = \sqrt{\alpha^2 + \beta^2}, \quad \tan\delta = \frac{\beta}{\alpha}$$

Attention !

<u>3-parameter transformation:</u> 2 translations, 1 rotation

$$\begin{bmatrix} E_{UTM} \\ N_{UTM} \end{bmatrix} = \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} E_{GK} \\ N_{GK} \end{bmatrix} + \begin{bmatrix} T_E \\ T_N \end{bmatrix}$$

→ True-neighborhood-post-transformation correction: Example

True-neighborhood-post-transformation correction: Example

4-parameter transformation with $(x,y) = (E,N)_{GK}$, $(X,Y) = (E,N)_{UTM}$

m=4 Control points P		1	2	3	4
Start system	x_P	1	3	5	3
	y_P	1	0.5	1	5
Target system	X_P	–1.19	1.09	3.59	1.6
	Y_P	4.73	3.90	4.25	9.27

$\hat{T}_E = -2.520$

$\hat{T}_N = 3.632$

$\hat{\delta} = 5°19'48.5003''$

$\hat{m} = 1.2$

Target system	$\hat{X}_P$	–1.214	1.119	3.564	1.621
	$\hat{Y}_P$	4.715	3.895	4.269	9.270
Resid-uals $\underline{\hat{E}}$	$X_P - \hat{X}_P$	0.024	–0.029	0.026	–0.021
	$Y_P - \hat{Y}_P$	0.015	0.005	–0.019	0.000

n=2 Transformation points Q		A	B
Start system	x_Q	2	4
	y_Q	2	3

$\hat{T}_E$, $\hat{T}_N$, $\hat{\delta}$, $\hat{m}$

Target system	$\hat{X}_Q$	0.092	2.592
	$\hat{Y}_Q$	5.798	6.770
	$\tilde{X}_Q$	0.0954	2.5905
	$\tilde{Y}_Q$	5.8052	6.7664

→ True-neighborhood-post-transformation correction: Example

True-neighborhood-post-transformation correction: Example

Distance matrix $\underline{s}$ and weight matrix $\underline{w}$ for the choice $\alpha = 2$, $\beta = 0$:

$$\underline{s}_{PQ}^{\alpha=2} = \begin{bmatrix} 2 & 13 \\ 3.25 & 7.25 \\ 10 & 5 \\ 10 & 5 \end{bmatrix} \overset{\beta=0}{\Rightarrow} \underset{m\times n}{\underline{w}} = \begin{bmatrix} 1/2 & 1/13 \\ 1/3.25 & 1/7.25 \\ 1/10 & 1/5 \\ 1/10 & 1/5 \end{bmatrix} \Rightarrow \frac{\underset{m\times n}{\underline{w}}}{\underline{1}_m \underline{1}_m^T \underline{w}} = \begin{bmatrix} 0.496 & 0.125 \\ 0.305 & 0.224 \\ 0.099 & 0.325 \\ 0.099 & 0.325 \end{bmatrix}$$

Post-transformation correction matrix $d\hat{\underline{X}}$

$$\underset{2\times n}{d\hat{\underline{X}}} = \underset{2\times m}{\hat{\underline{E}}} \frac{\underset{m\times n}{\underline{w}}}{\underline{1}_m \underline{1}_m^T \underline{w}} = \begin{bmatrix} 0.0035 & -0.0019 \\ 0.0069 & -0.0035 \end{bmatrix} = \begin{bmatrix} d\hat{X}_{Q_A} & d\hat{X}_{Q_B} \\ d\hat{Y}_{Q_A} & d\hat{Y}_{Q_B} \end{bmatrix}$$

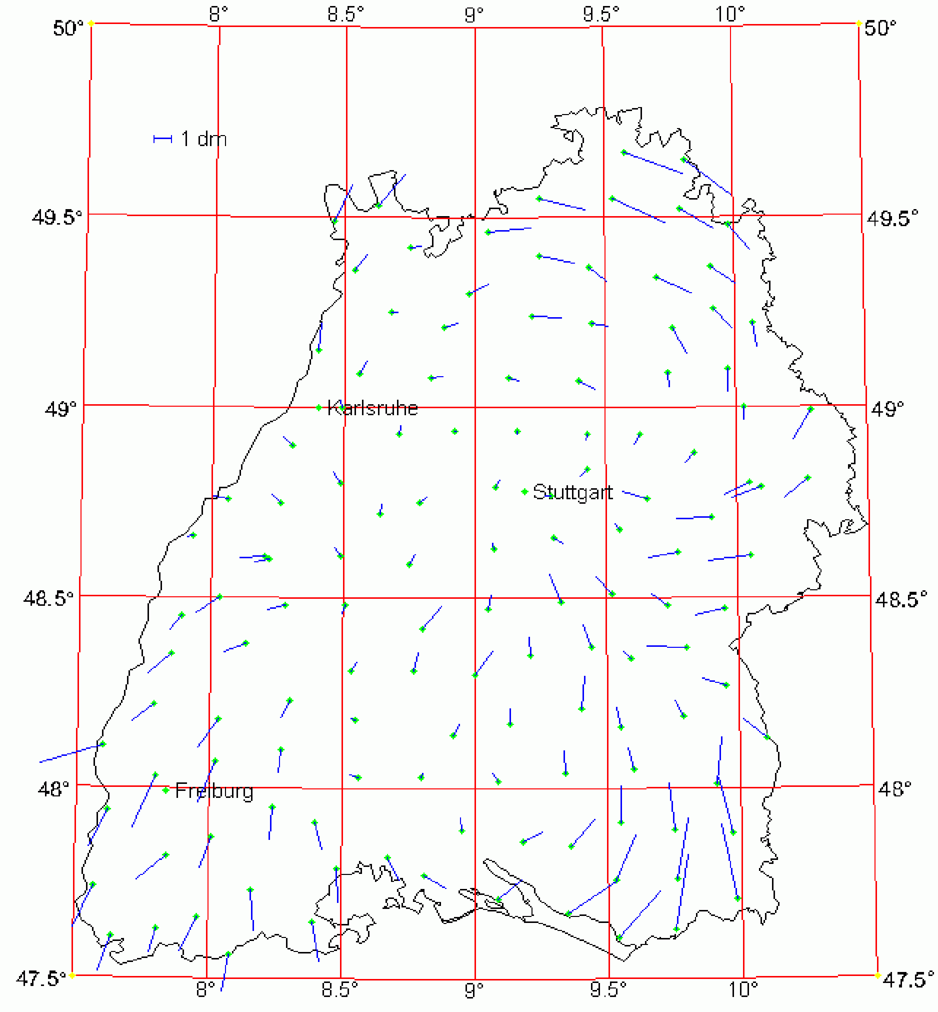
Post-transformation corrected transformation points

$$\tilde{\underline{X}}_Q = \hat{\underline{X}}_Q + d\hat{\underline{X}} = \begin{bmatrix} \tilde{X}_{Q_A} & \tilde{X}_{Q_B} \\ \tilde{Y}_{Q_A} & \tilde{Y}_{Q_B} \end{bmatrix} = \begin{bmatrix} 0.0954 & 2.5905 \\ 5.8052 & 6.7664 \end{bmatrix}$$

→ Comparison 3D-2D: Baden-Württemberg

Comparison 3D-2D: Baden-Württemberg

7-parameter transformation (3D)

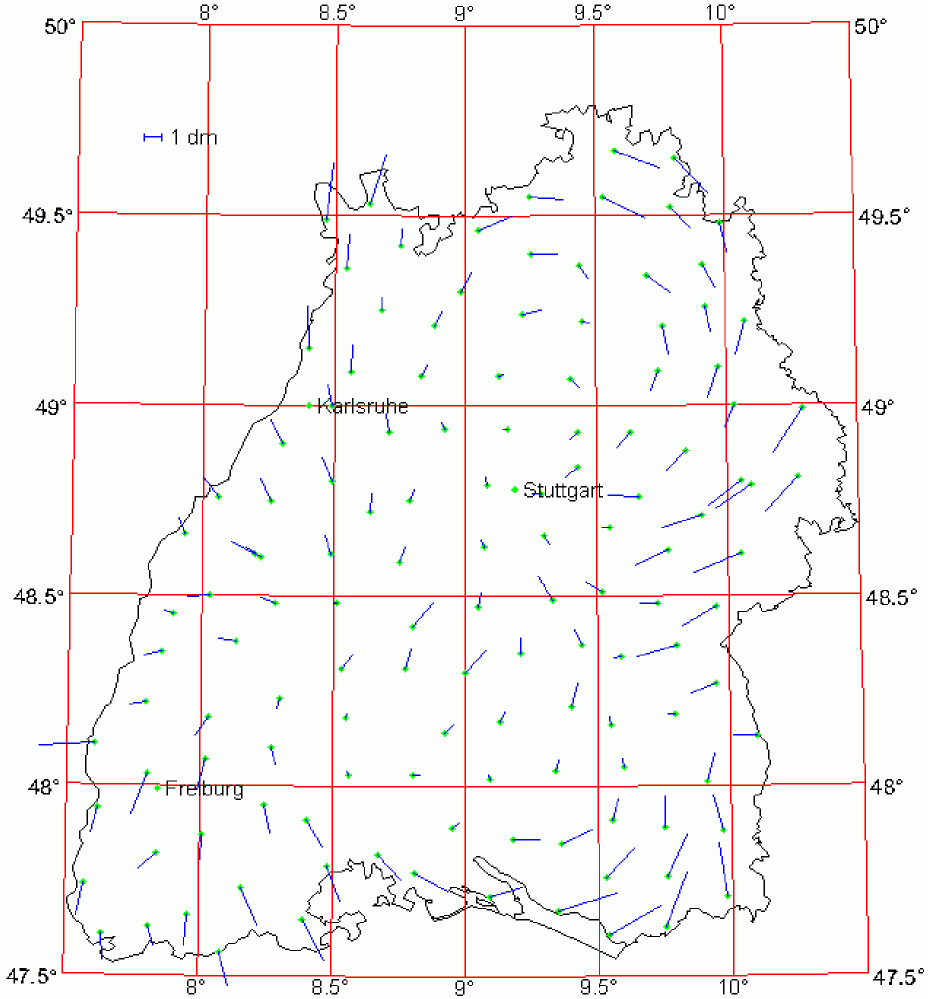


$$\hat{T}_X = 582{,}902\text{m} \pm 0{,}802\text{m}$$
$$\hat{T}_Y = 112{,}168\text{m} \pm 1{,}150\text{m}$$
$$\hat{T}_Z = 405{,}603\text{m} \pm 0{,}808\text{m}$$
$$\hat{\alpha} = -2{,}255'' \pm 0{,}032''$$
$$\hat{\beta} = -0{,}335'' \pm 0{,}028''$$
$$\hat{\gamma} = 2{,}068'' \pm 0{,}029''$$
$$\delta\lambda = 9{,}117 \times 10^{-6} \pm 0{,}108 \times 10^{-6}$$
$$\hat{\sigma} = 0{,}103\text{m}$$
$$\text{MSE} = 0{,}124\text{m}$$

→ Comparison 3D-2D: Baden-Württemberg

Comparison 3D-2D: Baden-Württemberg

6-parameter transformation (2D)



$$\hat{T}_E = 437{,}195\text{m} \pm 0{,}900\text{m}$$

$$\hat{T}_N = 119{,}757\text{m} \pm 0{,}900\text{m}$$

$$\hat{\varepsilon} = 0{,}196'' \pm 0{,}043''$$

$$\hat{\delta} = -0.165'' \pm 0{,}036''$$

$$\hat{m}_X - 1 = -3{,}997 \times 10^{-4} \pm 0{,}002 \times 10^{-7}$$

$$\hat{m}_Y - 1 = -3{,}988 \times 10^{-4} \pm 0{,}002 \times 10^{-7}$$

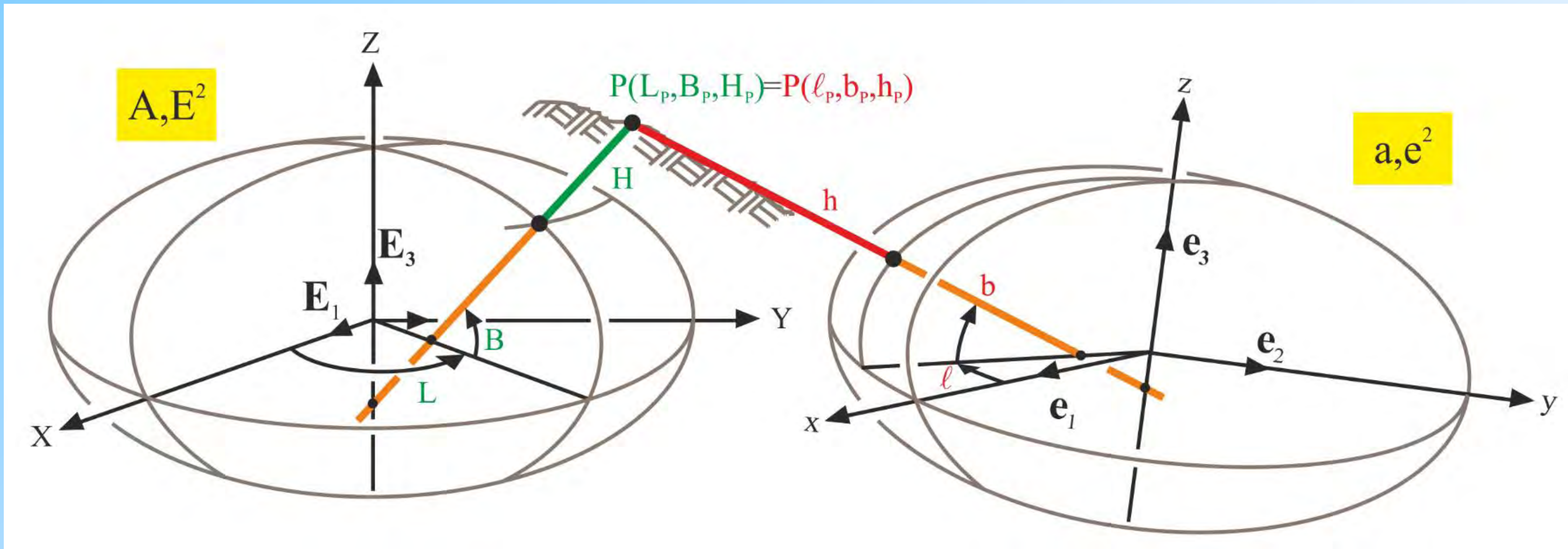
$$\hat{\sigma} = 0{,}120\text{m}$$

$$\text{MSE} = 0{,}118\text{m}$$

→ Curvilinear datum transformation

Curvilinear datum transformation

"Homologous points are equipped with ellipsoidal coordinates"



$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} [N(B)+H]\cos B \cos L \\ [N(B)+H]\cos B \sin L \\ [(1-E^2)N(B)+H]\sin B \end{bmatrix} - \begin{bmatrix} [n(b)+h]\cos b \cos \ell \\ [n(b)+h]\cos b \sin \ell \\ [(1-e^2)n(b)+h]\sin b \end{bmatrix}$$

"global"-"local"

→ Curvilinear datum transformation

Curvilinear datum transformation

Both coordinate system differ from each other just a little $\Rightarrow$ differences between ellipsoidal coordinates (B,L,H) and (b,ℓ,h) are small, as long as the geometries (A,a,E^2,e^2) of the ellipsoids are similar.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} [N(B)+H]\cos B \cos L \\ [N(B)+H]\cos B \sin L \\ [(1-E^2)N(B)+H]\sin B \end{bmatrix}$$

$\rightarrow$ Taylor series expansion !

Case 1: Geometry of both ellipsoids is known $\Rightarrow$ Taylor point is (ℓ,b,h)

$L=\ell+\delta\ell$, $B=b+\delta b$, $H=h+\delta h$

Case 2: Geometry of one (here: global) ellipsoid is unknown $\Rightarrow$ Taylor point is (ℓ,b,h,a,e^2)

$L=\ell+\delta\ell$, $B=b+\delta b$, $H=h+\delta h$,
$A=a+\delta a$, $E^2=e^2+\delta e^2$

$\rightarrow$ Curvilinear datum transformation: Analysis case 1

Curvilinear datum transformation : Analysis case 1

Case 1: Geometry of both ellipsoids is known $\Rightarrow$ Taylor point is (ℓ, b, h)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} [N(B)+H]\cos B \cos L \\ [N(B)+H]\cos B \sin L \\ [(1-E^2)N(B)+H]\sin B \end{bmatrix}\Bigg|_{\substack{L=\ell \\ B=b \\ H=h}} + \begin{bmatrix} \frac{\partial X}{\partial L} & \frac{\partial X}{\partial B} & \frac{\partial X}{\partial H} \\ \frac{\partial Y}{\partial L} & \frac{\partial Y}{\partial B} & \frac{\partial Y}{\partial H} \\ \frac{\partial Z}{\partial L} & \frac{\partial Z}{\partial B} & \frac{\partial Z}{\partial H} \end{bmatrix}\Bigg|_{\substack{L=\ell \\ B=b \\ H=h}} \begin{bmatrix} \delta\ell \\ \delta b \\ \delta h \end{bmatrix} =$$

$$= \begin{bmatrix} [N(b)+h]\cos b \cos \ell \\ [N(b)+h]\cos b \sin \ell \\ [(1-E^2)N(b)+h]\sin b \end{bmatrix} + \begin{bmatrix} \frac{\partial X}{\partial L} & \frac{\partial X}{\partial B} & \frac{\partial X}{\partial H} \\ \frac{\partial Y}{\partial L} & \frac{\partial Y}{\partial B} & \frac{\partial Y}{\partial H} \\ \frac{\partial Z}{\partial L} & \frac{\partial Z}{\partial B} & \frac{\partial Z}{\partial H} \end{bmatrix}\Bigg|_{\substack{L=\ell \\ B=b \\ H=h}} \begin{bmatrix} \delta\ell \\ \delta b \\ \delta h \end{bmatrix}$$

$$N(B) = \frac{A}{\sqrt{1-E^2 \sin^2 B}}$$

$$N(b) = \frac{A}{\sqrt{1-E^2 \sin^2 b}}$$

$\rightarrow$ Curvilinear datum transformation: Analysis case 1

Curvilinear datum transformation : Analysis case 1

$$\Rightarrow \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} [N(b)-n(b)]\cos b \cos \ell \\ [N(b)-n(b)]\cos b \sin \ell \\ [(1-E^2)N(b)-(1-e^2)n(b)]\sin b \end{bmatrix} +$$

$$+ \begin{bmatrix} -[N(b)+h]\cos b \sin \ell & -[M(b)+h]\sin b \cos \ell & \cos b \cos \ell \\ [N(b)+h]\cos b \cos \ell & -[M(b)+h]\sin b \sin \ell & \cos b \sin \ell \\ 0 & [M(b)+h]\cos b & \sin b \end{bmatrix} \begin{bmatrix} \delta\ell \\ \delta b \\ \delta h \end{bmatrix}$$

$$M(b) = \frac{A(1-E^2)}{(1-E^2\sin^2 b)^{3/2}}$$

$$n(b) = \frac{a}{\sqrt{1-e^2\sin^2 b}}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \begin{bmatrix} x \\ y \\ z \end{bmatrix} \overset{CTI}{=} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & -z & y & x \\ 0 & 1 & 0 & z & 0 & -x & y \\ 0 & 0 & 1 & -y & x & 0 & z \end{bmatrix}}_{\underline{P}} \underbrace{\begin{bmatrix} T_X \\ T_Y \\ T_Z \\ \delta\alpha \\ \delta\beta \\ \delta\gamma \\ \delta\lambda \end{bmatrix}}_{\underline{\xi}}$$

Equate both equations and solve for δL, δb, δh !

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} [n(b)+h]\cos b \cos \ell \\ [n(b)+h]\cos b \sin \ell \\ [(1-e^2)n(b)+h]\sin b \end{bmatrix}$$

→ Curvilinear datum transformation: Analysis case 1

Curvilinear datum transformation : Analysis case 1

$$\begin{bmatrix}\delta\ell\\ \delta b\\ \delta h\end{bmatrix} = \underbrace{\begin{bmatrix} -[N(b)+h]\cos b\sin\ell & -[M(b)+h]\sin b\cos\ell & \cos b\cos\ell \\ [N(b)+h]\cos b\cos\ell & -[M(b)+h]\sin b\sin\ell & \cos b\sin\ell \\ 0 & [M(b)+h]\cos b & \sin b\end{bmatrix}^{-1}\begin{bmatrix}1&0&0&0&-z&y&x\\0&1&0&z&0&-x&y\\0&0&1&-y&x&0&z\end{bmatrix}}_{\underline{A}} \begin{bmatrix}T_X\\T_Y\\T_Z\\\delta\alpha\\\delta\beta\\\delta\gamma\\\delta\lambda\end{bmatrix} +$$

$$\overset{!}{=} \underbrace{\begin{bmatrix} -[N(b)+h]\cos b\sin\ell & -[M(b)+h]\sin b\cos\ell & \cos b\cos\ell \\ [N(b)+h]\cos b\cos\ell & -[M(b)+h]\sin b\sin\ell & \cos b\sin\ell \\ 0 & [M(b)+h]\cos b & \sin b\end{bmatrix}^{-1}\begin{bmatrix}[N(b)-n(b)]\cos b\cos\ell\\ [N(b)-n(b)]\cos b\sin\ell \\ [(1-E^2)N(b)-(1-e^2)n(b)]\sin b\end{bmatrix}}_{\underline{r}}$$

$$\begin{bmatrix}L-\ell\\ B-b\\ H-h\end{bmatrix}_i =: \begin{bmatrix}\delta\ell\\ \delta b\\ \delta h\end{bmatrix}_i = \underset{3\times 7}{\underline{A}_i}\begin{bmatrix}T_X\\T_Y\\T_Z\\\delta\alpha\\\delta\beta\\\delta\gamma\\\delta\lambda\end{bmatrix} + \underset{3\times 1}{\underline{r}_i} = \underset{3\times 7}{\underline{A}_i}\,\underset{7\times 1}{\underline{\xi}} + \underset{3\times 1}{\underline{r}_i} \qquad i = 1,\ldots,n$$

→ Curvilinear datum transformation: Analysis case 1

Curvilinear datum transformation : Analysis case 1

Columns 1-3 of A_i (belonging to T_X, T_Y, T_Z) and array r_i

$$\underline{A}_i(1:3) = \begin{bmatrix} \dfrac{-\sin\ell}{[N(b)+h]\cos b} & \dfrac{\cos\ell}{[N(b)+h]\cos b} & 0 \\ \dfrac{-\sin b\cos\ell}{M(b)+h} & \dfrac{-\sin b\sin\ell}{M(b)+h} & \dfrac{\cos b}{M(b)+h} \\ \cos b\cos\ell & \cos b\sin\ell & \sin b \end{bmatrix} \quad \underline{r}_i = \begin{bmatrix} 0 \\ \dfrac{E^2N(b)-e^2n(b)}{M(b)+h}\sin b\cos b \\ \dfrac{a^2}{n(b)} - \dfrac{A^2}{N(b)} \end{bmatrix}$$

Columns 4-7 of A_i (belonging to $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\lambda$)

$$\underline{A}_i(4:7) = \begin{bmatrix} \dfrac{(1-e^2)n(b)+h}{[N(b)+h]\cos b}\sin b\cos\ell & \dfrac{(1-e^2)n(b)+h}{[N(b)+h]\cos b}\sin b\sin\ell & -\dfrac{n(b)+h}{N(b)+h} & 0 \\ -\left(\dfrac{a^2}{n(b)}+h\right)\dfrac{\sin\ell}{M(b)+h} & \left(\dfrac{a^2}{n(b)}+h\right)\dfrac{\cos\ell}{M(b)+h} & 0 & -\dfrac{n(b)e^2\sin b\cos b}{M(b)+h} \\ -n(b)e^2\sin b\cos b\sin\ell & n(b)e^2\sin b\cos b\cos\ell & 0 & \dfrac{a^2}{n(b)}+h \end{bmatrix}$$

→ Curvilinear datum transformation: Analysis case 2

Curvilinear datum transformation : Analysis case 2

Case 2: Geometry of one (here: global) ellipsoid is unknown ⇒ Taylor point is (ℓ, b, h, a, e^2)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \left.\begin{bmatrix} [N(B)+H]\cos B\cos L \\ [N(B)+H]\cos B\sin L \\ [(1-E^2)N(B)+H]\sin B \end{bmatrix}\right|_{\substack{L=\ell \\ B=b \\ H=h \\ A=a \\ E^2=e^2}} + \left.\begin{bmatrix} \frac{\partial X}{\partial L} & \frac{\partial X}{\partial B} & \frac{\partial X}{\partial H} \\ \frac{\partial Y}{\partial L} & \frac{\partial Y}{\partial B} & \frac{\partial Y}{\partial H} \\ \frac{\partial Z}{\partial L} & \frac{\partial Z}{\partial B} & \frac{\partial Z}{\partial H} \end{bmatrix}\right|_{\substack{L=\ell \\ B=b \\ H=h \\ A=a \\ E^2=e^2}} \begin{bmatrix} \delta\ell \\ \delta b \\ \delta h \end{bmatrix} + \left.\begin{bmatrix} \frac{\partial X}{\partial A} & \frac{\partial X}{\partial E^2} \\ \frac{\partial Y}{\partial A} & \frac{\partial Y}{\partial E^2} \\ \frac{\partial Z}{\partial A} & \frac{\partial Z}{\partial E^2} \end{bmatrix}\right|_{\substack{L=\ell \\ B=b \\ H=h \\ A=a \\ E^2=e^2}} \begin{bmatrix} \delta a \\ \delta e^2 \end{bmatrix} =$$

→ Curvilinear datum transformation: Analysis case 2

Curvilinear datum transformation : Analysis case 2

$$
= \begin{bmatrix} [n(b)+h]\cos b\cos\ell \\ [n(b)+h]\cos b\sin\ell \\ [(1-e^2)n(b)+h]\sin b \end{bmatrix} + \begin{bmatrix} -[n(b)+h]\cos b\sin\ell & -[m(b)+h]\sin b\cos\ell & \cos b\cos\ell \\ [n(b)+h]\cos b\cos\ell & -[m(b)+h]\sin b\sin\ell & \cos b\sin\ell \\ 0 & [m(b)+h]\cos b & \sin b \end{bmatrix} \begin{bmatrix} \delta\ell \\ \delta b \\ \delta h \end{bmatrix} +
$$

$$
+ \begin{bmatrix} \dfrac{n(b)\cos b\cos\ell}{a} & \dfrac{m(b)\cos b\sin^2 b\cos\ell}{2(1-e^2)} \\ \dfrac{n(b)\cos b\sin\ell}{a} & \dfrac{m(b)\cos b\sin^2 b\sin\ell}{2(1-e^2)} \\ \dfrac{n(b)(1-e^2)\sin b}{a} & \dfrac{[m(b)\sin^2 b-2n(b)]\sin b}{2} \end{bmatrix} \begin{bmatrix} \delta a \\ \delta e^2 \end{bmatrix}
$$

$$
\left(m(b) = \frac{a(1-e^2)}{(1-e^2\sin^2 b)^{3/2}} \right)
$$

Curvilinear datum transformation : Analysis case 2

$$\Rightarrow \begin{bmatrix} X\\ Y\\ Z\end{bmatrix} - \begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} -[n(b)+h]\cos b\sin\ell & -[m(b)+h]\sin b\cos\ell & \cos b\cos\ell \\ [n(b)+h]\cos b\cos\ell & -[m(b)+h]\sin b\sin\ell & \cos b\sin\ell \\ 0 & [m(b)+h]\cos b & \sin b \end{bmatrix} \begin{bmatrix} \delta\ell\\ \delta b\\ \delta h\end{bmatrix} +$$

$$+\begin{bmatrix} \dfrac{n(b)\cos b\cos\ell}{a} & \dfrac{m(b)\cos b\sin^2 b\cos\ell}{2(1-e^2)} \\ \dfrac{n(b)\cos b\sin\ell}{a} & \dfrac{m(b)\cos b\sin^2 b\sin\ell}{2(1-e^2)} \\ \dfrac{n(b)(1-e^2)\sin b}{a} & \dfrac{[m(b)\sin^2 b-2n(b)]\sin b}{2} \end{bmatrix} \begin{bmatrix} \delta a\\ \delta e^2\end{bmatrix}$$

$$\begin{bmatrix} X\\ Y\\ Z\end{bmatrix} - \begin{bmatrix} x\\ y\\ z\end{bmatrix} \overset{CTI}{=} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & -z & y & x \\ 0 & 1 & 0 & z & 0 & -x & y \\ 0 & 0 & 1 & -y & x & 0 & z \end{bmatrix}}_{\underline{P}} \begin{bmatrix} T_X\\ T_Y\\ T_Z\\ \delta\alpha\\ \delta\beta\\ \delta\gamma\\ \delta\lambda\end{bmatrix}$$

Equate both equations, augment the array of unknowns by δa, δe^2 and solve for $\delta\ell$, δb, δh !

→ Curvilinear datum transformation: Analysis case 2

Curvilinear datum transformation : Analysis case 2

$$\begin{bmatrix} \delta\ell \\ \delta b \\ \delta h \end{bmatrix} = \begin{bmatrix} -[n(b)+h]\cos b \sin\ell & -[m(b)+h]\sin b \cos\ell & \cos b \cos\ell \\ [n(b)+h]\cos b \cos\ell & -[m(b)+h]\sin b \sin\ell & \cos b \sin\ell \\ 0 & [m(b)+h]\cos b & \sin b \end{bmatrix}^{-1} *$$

$$* \begin{bmatrix} 1 & 0 & 0 & 0 & -z & y & x & -\dfrac{n(b)\cos b\cos\ell}{a} & -\dfrac{m(b)\cos b\sin^2 b\cos\ell}{2(1-e^2)} \\ 0 & 1 & 0 & z & 0 & -x & y & -\dfrac{n(b)\cos b\sin\ell}{a} & -\dfrac{m(b)\cos b\sin^2 b\sin\ell}{2(1-e^2)} \\ 0 & 0 & 1 & -y & x & 0 & z & -\dfrac{n(b)(1-e^2)\sin b}{a} & -\dfrac{[m(b)\sin^2 b - 2n(b)]\sin b}{2} \end{bmatrix} \underbrace{\begin{bmatrix} T_X \\ T_Y \\ T_Z \\ \delta\alpha \\ \delta\beta \\ \delta\gamma \\ \delta\lambda \\ \delta a \\ \delta e^2 \end{bmatrix}}_{\underline{\xi}}$$

→ Curvilinear datum transformation: Analysis case 2

Curvilinear datum transformation : Analysis case 2

$$\begin{bmatrix} L-\ell \\ B-b \\ H-h \end{bmatrix}_i =: \begin{bmatrix} \delta\ell \\ \delta b \\ \delta h \end{bmatrix}_i = \underset{3\times 9}{\underline{A}_i} \begin{bmatrix} T_X \\ T_Y \\ T_Z \\ \delta\alpha \\ \delta\beta \\ \delta\gamma \\ \delta\lambda \\ \delta a \\ \delta e^2 \end{bmatrix} = \underset{3\times 9}{\underline{A}_i}\ \underset{9\times 1}{\underline{\xi}} \qquad i = 1,\ldots,n$$

Columns 1-3 of A_i (belonging to T_X, T_Y, T_Z)

$$\underline{A}_i(1:3) = \begin{bmatrix} \dfrac{-\sin\ell}{[n(b)+h]\cos b} & \dfrac{\cos\ell}{[n(b)+h]\cos b} & 0 \\ \dfrac{-\sin b\cos\ell}{m(b)+h} & \dfrac{-\sin b\sin\ell}{m(b)+h} & \dfrac{\cos b}{m(b)+h} \\ \cos b\cos\ell & \cos b\sin\ell & \sin b \end{bmatrix}$$

→ Curvilinear datum transformation: Analysis case 2

Curvilinear datum transformation : Analysis case 2

Columns 4-7 of A_i (belonging to $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\lambda$)

$$\underline{A}_i(4:7)=\begin{bmatrix} \frac{(1-e^2)n(b)+h}{[n(b)+h]\cos b}\sin b\cos\ell & \frac{(1-e^2)n(b)+h}{[n(b)+h]\cos b}\sin b\sin\ell & -1 & 0 \\ -\left(\frac{a^2}{n(b)}+h\right)\frac{\sin\ell}{m(b)+h} & \left(\frac{a^2}{n(b)}+h\right)\frac{\cos\ell}{m(b)+h} & 0 & -\frac{n(b)e^2\sin b\cos b}{m(b)+h} \\ -n(b)e^2\sin b\cos b\sin\ell & n(b)e^2\sin b\cos b\cos\ell & 0 & \frac{a^2}{n(b)}+h \end{bmatrix}$$

Columns 8-9 of A_i (belonging to δa, δe^2)

$$\underline{A}_i(8:9)=\begin{bmatrix} 0 & 0 \\ \frac{n(b)e^2\sin b\cos b}{a[m(b)+h]} & \frac{m(b)e^2\sin^2 b+2(1-e^2)n(b)}{2[m(b)+h](1-e^2)}\sin b\cos b \\ -\frac{a}{n(b)} & -\frac{m(b)e^2\cos^2 b-[2n(b)-m(b)](1-e^2)}{2(1-e^2)}\sin^2 b \end{bmatrix}$$

→ Curvilinear datum transformation: Synthesis

Curvilinear datum transformation : Synthesis

Case 2 only (case 1 accordingly)

a) $$\begin{bmatrix} L \\ B \\ H \end{bmatrix} - \begin{bmatrix} \ell \\ b \\ h \end{bmatrix} = \underline{A}(\ell, b, h, a, e^2)\underline{\xi} \quad \Rightarrow \begin{bmatrix} \hat{L} \\ \hat{B} \\ \hat{H} \end{bmatrix} = \begin{bmatrix} \ell \\ b \\ h \end{bmatrix} + \underset{3\times 9}{\underline{A}}(\ell, b, h, a, e^2)\, \underset{9\times 1}{\underline{\hat{\xi}}}\ \text{"local"} \rightarrow \text{"global"}$$

b) $$\begin{bmatrix} \ell \\ b \\ h \end{bmatrix} - \begin{bmatrix} L \\ B \\ H \end{bmatrix} = \underline{A}(L, B, H, A, E^2)\underline{\xi} \Rightarrow \begin{bmatrix} \hat{\ell} \\ \hat{b} \\ \hat{h} \end{bmatrix} = \begin{bmatrix} L \\ B \\ H \end{bmatrix} + \underset{3\times 9}{\underline{A}}(L, B, H, A, E^2)\, \underset{9\times 1}{\underline{\hat{\xi}}}\ \text{"global"} \rightarrow \text{"local"}$$

c) $$\begin{bmatrix} L \\ B \\ H \end{bmatrix} - \begin{bmatrix} \ell \\ b \\ h \end{bmatrix} = \underline{A}(\ell, b, h, a, e^2)\underline{\xi} \quad \Rightarrow \begin{bmatrix} \hat{L} \\ \hat{B} \\ \hat{H} \end{bmatrix} = \begin{bmatrix} \ell \\ b \\ h \end{bmatrix} + \underset{3\times 9}{\underline{A}}(\ell, b, h, a, e^2)(-\underset{9\times 1}{\underline{\hat{\xi}}}\ \text{"global"} \xrightarrow{\text{CTI}} \text{"local"})$$

d) $$\begin{bmatrix} \ell \\ b \\ h \end{bmatrix} - \begin{bmatrix} L \\ B \\ H \end{bmatrix} = \underline{A}(L, B, H, A, E^2)\underline{\xi} \Rightarrow \begin{bmatrix} \hat{\ell} \\ \hat{b} \\ \hat{h} \end{bmatrix} = \begin{bmatrix} L \\ B \\ H \end{bmatrix} + \underset{3\times 9}{\underline{A}}(L, B, H, A, E^2)(-\underset{9\times 1}{\underline{\hat{\xi}}}\ \text{"local"} \xrightarrow{\text{CTI}} \text{"global"})$$

→ Curvilinear datum transformation: Summary

Curvilinear datum transformation : Summary

Model with known ellipsoid geometries (Case 1):

$$\begin{bmatrix} L-\ell \\ B-b \\ H-h \end{bmatrix}_i =: \begin{bmatrix} \delta\ell \\ \delta b \\ \delta h \end{bmatrix}_i = \underset{3\times 7}{\underline{A}_i}\ \underset{7\times 1}{\underline{\xi}} + \underset{3\times 1}{\underline{r}_i} \qquad i = 1,\dots,n$$

$\underline{A}$ and $\underline{r}$ depend only on local ellipsoidal coordinates and on both ellipsoid geometries

- In order to determine uniquely the transformation parameters at least 3 homologous points are required
- the more homologous points are available and the larger the geodetic network is, the better the unknown transformation parameters can be determined
- without the knowledge of local ellipsoidal heights, equation for δh cannot be set up (⇒ Consequence ?)
- dependency of design matrix $\underline{A}$ on local ellipsoidal height is weak

→ Curvilinear datum transformation: Summary

Curvilinear datum transformation : Summary

Model with one unknown (here: global) ellipsoid geometry (Case 2):

$$\begin{bmatrix} L-\ell \\ B-b \\ H-h \end{bmatrix}_i =: \begin{bmatrix} \delta\ell \\ \delta b \\ \delta h \end{bmatrix}_i = \underset{3\times 9}{\underline{A}_i}\ \underset{9\times 1}{\underline{\xi}} \qquad i = 1,\ldots,n$$

$\underline{A}$ depends only on local ellipsoidal coordinates and on the geometry of local ellipsoid

- In order to determine uniquely the transformation parameters at least 3 homologous points are required
- the more homologous points are available and the larger the geodetic network is, the better the unknown transformation parameters can be determined
- without the knowledge of local ellipsoidal heights, equation for δh must be omitted
- without the height equation, $\delta\lambda$ and δa cannot be separated/estimated (why ?)
- the dependency of design matrix $\underline{A}$ on local ellipsoidal height is weak
- analysis is numerically very sensitive due to weak condition of the normal equation matrix
- neglecting $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\lambda$ $\rightarrow$ "Standard-Molodensky formulae" (Synthesis)

→ Curvilinear datum transformation: Molodensky formulae

Curvilinear datum transformation : Molodensky formulae

$$L = \ell + \frac{T_Y \cos\ell - T_X \sin\ell}{[n(b)+h]\cos b}$$

$$B = b + \frac{1}{m(b)+h}[-T_X \sin b \cos\ell - T_Y \sin b \sin\ell + T_Z \cos b + \delta a \frac{n(b)e^2 \sin b \cos b}{a} +$$

$$+\delta f \frac{m(b)+n(b)(1-e^2)}{\sqrt{1-e^2}} \sin b \cos b]$$

$\delta a = A - a$ difference of major semi axes

$\delta f = F - f$ difference in flattenings

$F = 1 - \sqrt{1-E^2}$ flattening global ellipsoid (~1:300)

$f = 1 - \sqrt{1-e^2}$ flattening local ellipsoid (~1:300)

$$n(b) = \frac{a}{\sqrt{1-e^2\sin^2 b}}, \quad m(b) = \frac{a(1-e^2)}{(1-e^2\sin^2 b)^{3/2}}$$

→ Curvilinear datum transformation: Summary

Curvilinear datum transformation : Summary

Analysis

- Given $(L,B,H,\ell,b,h)_i$, $i=1,\dots,n$, transformation parameters can be determined (e.g. using adjustment methods) → coordinates and terms "local" and "global" are interchangeable
- If in case 2 (9-parameter model), equation for H-h must be omitted due to lack on knowledge about H, $\delta\lambda$ und δa cannot be separated
- If, in addition, (also) h is unknown, a secondary error is introduced
- a systematic error in h should not exceed 100m ("Accuracy"), the precision of h should be better than 5m
- homologous points should always be spread over a wide area

Synthesis

- If h is unknown, H cannot be determined; additionally a secondary error is generated in L,B
- From the equation for H–h the height difference of both ellipsoids can be assessed

In general

- missing or inaccurate h: Impact on transformation parameters and their precision is strong; impact on synthesis coordinates (and possibly) map coordinates is highly reduced

→ Datum transformation: Polynomial models

Curvilinear datum transformation: Polynomial models

$$L = \ell + \delta\ell = \ell + \sum_{i=0}^{i_{max}} \sum_{j=0}^{j_{max}} [ij]_\ell \, U^i V^j =$$

$$B = b + \delta b = b + \sum_{i=0}^{i_{max}} \sum_{j=0}^{j_{max}} [ij]_b \, U^i V^j \qquad U := \ell - \ell_0 \, , \, V := b - b_0$$

$$H = h + \delta h = h + \sum_{i=0}^{i_{max}} \sum_{j=0}^{j_{max}} [ij]_h \, U^i V^j$$

$$\begin{aligned}\delta b\ [''] = \ & 0.16984 - 0.76173\,U + 0.09585\,V + 1.09919\,U^2 - 4.57801\,U^3 - 1.13239\,U^2V + 0.49831\,V^3 \\ & - 0.98399\,U^3V + 0.12415\,UV^3 + 0.11450\,V^4 + 27.05396\,U^5 + 2.03449\,U^4V + 0.73357\,U^2V^3 \\ & - 0.37548\,V^5 - 0.14197\,V^6 - 59.96555\,U^7 + 0.07439\,V^7 - 4.76082\,U^8 + 0.03385\,V^8 \\ & + 49.04320\,U^9 - 1.30575\,U^6V^3 - 0.07653\,U^3V^9 + 0.08646\,U^4V^9\end{aligned}$$

$$\begin{aligned}\delta\ell\ [''] = \ & - 0.88437 + 2.05061\,V + 0.26361\,U^2 - 0.76804\,UV + 0.13374\,V^2 - 1.31974\,U^3 - 0.52162\,U^2V \\ & - 1.05853\,UV^2 - 0.49211\,U^2V^2 + 2.17204\,UV^3 - 0.06004\,V^4 + 0.30139\,U^4V + 1.88585\,UV^4 \\ & - 0.81162\,UV^5 - 0.05183\,V^6 - 0.96723\,UV^6 - 0.12948\,U^3V^5 + 3.41827\,U^9 - 0.44507\,U^8V \\ & + 0.18882\,UV^8 - 0.01444\,V^9 + 0.04794\,UV^9 - 0.59013\,U^9V^3\end{aligned}$$

$$\begin{aligned}\delta h\ [m] = \ & - 36.526 + 3.900\,U - 4.723\,V - 21.553\,U^2 + 7.294\,UV + 8.886\,V^2 - 8.440\,U^2V - 2.930\,UV^2 \\ & + 56.937\,U^4 - 58.756\,U^3V - 4.061\,V^4 + 4.447\,U^4V + 4.903\,U^2V^3 - 55.873\,U^6 + 212.005\,U^5V \\ & + 3.081\,V^6 - 254.511\,U^7V - 0.756\,V^8 + 30.654\,U^8V - 0.122\,UV^9\end{aligned}$$

This is the end !