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Dynamic System Estimation

Ordinary Differential Equations

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Ordinary Differential Equations

Ordinary Differential Equations express relations between

- derivatives of a function $(\boldsymbol{y}', \boldsymbol{y}'', \boldsymbol{y}''', \ldots, \boldsymbol{y}^{(m)})$
- the function itself $(\boldsymbol{y})$
- and the independent variable (t)

$$\boxed{\boldsymbol{F}(t, \boldsymbol{y}', \boldsymbol{y}'', \boldsymbol{y}''', \ldots, \boldsymbol{y}^{(m)}) = 0} \tag{2.1}$$

The prime $(')$ denotes the derivative with respect to the independent variable t. The function $\boldsymbol{y}$ depends on only one independent variable; therefore Eqn. 2.1 describes an Ordinary Differential Equation (ODE) in contrast to Partial Differential Equations involving multiple independent variables.

Eqn. 2.1 is an ODE of order m; it is solved by integration (m times).

Each of the m integrations requires specification of initial values for some value t_0 of the independent variable t.

$$\boxed{\boldsymbol{y}^{(i-1)}(t_0) = \boldsymbol{y}_0^{(i-1)} \quad i = 1, \ldots, m} \tag{2.2}$$

Ordinary Differential Equations - cont'd

If the relation $\boldsymbol{F}$ in Eqn. 2.1 is linear in the function $\boldsymbol{y}$ and its derivatives, it is called a Linear Differential Equation

$$\boldsymbol{y}^{(m)}(t) + \boldsymbol{A}_1(t)\boldsymbol{y}^{(m-1)}(t) + \cdots + \boldsymbol{A}_{m-1}(t)\boldsymbol{y}'(t) + \boldsymbol{A}_m(t)\boldsymbol{y}(t) = 0 \tag{2.3}$$

The $\boldsymbol{A}_i(t)$ are square matrices with elements that are functions of t (but do not depend on $\boldsymbol{y}$ and its derivatives). Since $\boldsymbol{y}$ in Eqn. 2.3 is a vector-valued function, Eqn. 2.3 represents a System of Linear Differential Equations of m^{th}-order. For a scalar-valued function y:

$$y^{(m)}(t) + a_1(t)y^{(m-1)}(t) + \cdots + a_{m-1}(t)y'(t) + a_m(t)y(t) = 0 \tag{2.4}$$

The $a_i(t)$ are scalar coefficients that are functions of t (but do not depend on y and its derivatives). Eqn. 2.4 represents a scalar Linear Differential Equation of m^{th}-order. The corresponding initial values are:

$$y^{(i-1)}(t_0) = y_0^{(i-1)} \; i = 1, \ldots, m \tag{2.5}$$

Ordinary Differential Equations - cont'd

A scalar Linear Differential Equation of m^{th}-order can be transformed into a System of m Linear Differential Equations of 1^{st}-order through substitution:

$$
\begin{aligned}
y_1(t) &= y(t) \\
y_2(t) &= y'(t) \\
&\vdots \\
y_{(i)}(t) &= y^{(i-1)}(t) \\
&\vdots \\
y_{(m)}(t) &= y^{(m-1)}(t)
\end{aligned}
\tag{2.6}
$$

$$
\begin{gathered}
\frac{d}{dt} y_i(t) = y^{(i)} \quad i = 1, \ldots, m-1 \\
\frac{d}{dt} y_{(m)}(t) + a_1(t) y_{(m)}(t) + a_2(t) y_{(m-1)}(t) + \ldots a_m(t) y_1(t) = b(t)
\end{gathered}
\tag{2.7}
$$

Ordinary Differential Equations - cont'd

Eqn. 2.7 can be rearranged to read:

$$\frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ \vdots \\ y_m(t) \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_m(t) & a_{m-1}(t) & a_{m-2}(t) & \dots & a_1(t) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ \vdots \\ y_m(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b(t) \end{bmatrix} \tag{2.8}$$

Accordingly the initial values (Eqn. (2.5)) must be transformed:

$$\begin{bmatrix} y(t_0) \\ y'(t_0) \\ y''(t_0) \\ \vdots \\ y^{(m-1)}(t_0) \end{bmatrix} = \begin{bmatrix} y_1(t_0) \\ y_2(t_0) \\ y_3(t_0) \\ \vdots \\ y_m(t_0) \end{bmatrix} \tag{2.9}$$

Ordinary Differential Equations - cont'd

Because any higher order Linear Differential Equation can be transformed into a System of Linear Differential Equations of 1^{st} order, it is sufficient to study the solutions of the latter type. In short we can re-write Eqns. (2.8) and (2.9) as

$$\boxed{\begin{aligned} &\boldsymbol{y}'(t) + \boldsymbol{A}(t)\boldsymbol{y}(t) = \boldsymbol{b}(t) \\ &\boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{aligned}} \tag{2.10}$$

$$\begin{aligned} &\boldsymbol{y}'(t) = \left[y_1'(t),\ y_2'(t),\ \dots\ , y_m'(t)\right]^T \\ &\boldsymbol{y}_0 = \left[y_1(t_0),\ y_2(t_0),\ \dots\ , y_m(t_0)\right]^T \\ &\boldsymbol{b}(t) = \left[0,\ 0,\ \dots\ , b(t)\right]^T \end{aligned}$$

Analytical solutions to Eqn. (2.10) can be obtained in many (but not all) cases as the sum of the solution of the corresponding homogeneous equation

$$\boxed{\boldsymbol{y}'(t) + \boldsymbol{A}(t)\boldsymbol{y}(t) = 0} \tag{2.11}$$

and a particular solution of the original equation (Eqn. 2.10). Otherwise a numerical solution can be obtained.

Ordinary Differential Equations - cont'd

Example for a method of numerical solution: Runge-Kutta-Method. The method is first explained for a scalar first-order Differential Equation.

$$\boxed{\begin{array}{l} y'(t) + a(t)y(t) = b(t) \quad \Rightarrow \quad y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{array}} \tag{2.12}$$

If the solution is available for t_n, than, in principle, the solution for t_{n+1} could be obtained through a Taylor expansion

$$y(t_{n+1}) = y(t_n) + y'(t_n)(t_{n+1} - t_n) + \frac{1}{2!} y''(t_n)(t_{n+1} - t_n)^2 + \frac{1}{3!} y''(t_n)(t_{n+1} - t_n)^3 + \ldots$$

with

$$\begin{array}{l} y'(t_n) = f \\ y''(t_n) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f \\ \text{etc.} \end{array}$$

Ordinary Differential Equations - cont'd

In the Runge-Kutta-Method of numerical integration, the higher derivatives (y'', y''', etc.) are replaced by first order derivatives. The order of the algorithm indicates up to which term of the Taylor expansion the algorithm is "correct".
Notation:

$$\begin{aligned} y(t_{n+1}) &= y_{n+1} \\ (t_{n+1} - t_n) &= h \end{aligned}$$

First order algorithm:

$$\boxed{y_{n+1} = y_n + h \cdot f(y_n, t_n)} \tag{2.13}$$

Second order algorithm:

$$\boxed{\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2) \\ k_1 &= f(y_n, t_n) \\ k_2 &= f(y_n + hk_1, t_n + h) \end{aligned}} \tag{2.14}$$

Ordinary Differential Equations - cont'd

Third order algorithm:

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 4k_2 + k_3) \\
k_1 &= f(y_n, t_n) \\
k_2 &= f(y_n + \frac{h}{2}k_1, t_n + \frac{h}{2}) \\
k_3 &= f(y_n - hk_1 + 2hk_2, t_n + h)
\end{aligned}
\tag{2.15}
$$

Forth order algorithm:

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
k_1 &= f(y_n, t_n) \\
k_2 &= f(y_n + \frac{h}{2}k_1, t_n + \frac{h}{2}) \\
k_3 &= f(y_n + \frac{h}{2}k_2, t_n + \frac{h}{2}) \\
k_4 &= f(y_n + hk_3, t_n + h)
\end{aligned}
\tag{2.16}
$$

Ordinary Differential Equations - cont'd

The Runge-Kutta-Method can also be applied to the numerical integration of Systems of 1^{st}-order Linear Differential Equations. Example for third order algorithm:

$$
\boxed{
\begin{aligned}
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \frac{h}{6}(\boldsymbol{k}_1 + 4\boldsymbol{k}_2 + \boldsymbol{k}_3) \\
\boldsymbol{k}_1 &= \boldsymbol{f}(\boldsymbol{y}_n, t_n) \\
\boldsymbol{k}_2 &= \boldsymbol{f}(\boldsymbol{y}_n + \frac{h}{2}\boldsymbol{k}_1, t_n + \frac{h}{2}) \\
\boldsymbol{k}_3 &= \boldsymbol{f}(\boldsymbol{y}_n - h\boldsymbol{k}_1 + 2h\boldsymbol{k}_2, t_n + h)
\end{aligned}
}
\tag{2.17}
$$

You will find the examples discussed in this lecture as Jupyter notebook under

`https://github.com/spacegeodesy/ParameterEstimationDynamicSystems/blob/master/example02.ipynb`