Universität Stuttgart

Prof.Dr.
Thomas Hobiger

Dynamic System Estimation

5

State and covariance prediction

State and covariance prediction

Recall that we have been able to express the continuous linear dynamic system (3.2)

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(t)\boldsymbol{x}(t) + \boldsymbol{G}(t)\boldsymbol{w}(t)$$

in a discretized form

$$\boldsymbol{x}_n = \boldsymbol{\Phi}(t_n, t_{n-1})\boldsymbol{x}_{n-1} + \underbrace{\int\limits_{t_{n-1}}^{t_n} \boldsymbol{\Phi}(t_n, t')\boldsymbol{G}(t')\boldsymbol{w}(t')dt'}_{\boldsymbol{u}_n}$$

where we succeeded in finding a way to obtain the state transition matrix $\boldsymbol{\Phi}$ by

$$\boldsymbol{\Phi}(t, t_0) = e^{\boldsymbol{F}(t-t_0)}$$

What are the implications of the term $\boldsymbol{u}_n$???

State and covariance prediction - cont'd

The random forcing function $\boldsymbol{w}(t)$ is assumed to be uncorrelated in time (white noise) and not correlated with the state $\boldsymbol{x}(t)$. Moreover, we assume that random variables have zero ensemble average values, i.e. $E(\boldsymbol{w}(t)) = 0$. This implies that if the state $\boldsymbol{x}(t)$ at some time t_0 is unbiased the state will remain unbiased as there are no biasing contributions from the random forcing function.

However, the situation for the covariance propagation is different. We obtain

$$\mathrm{Cov}_{x_n,x_n} = \boldsymbol{\Phi}(t_n, t_{n-1})\mathrm{Cov}_{x_{n-1},x_{n-1}}\boldsymbol{\Phi}^T(t_n, t_{n-1}) + \boldsymbol{Q} \tag{5.1}$$

with

$$\begin{aligned}\boldsymbol{Q} &= E\left(\int\limits_{t_{n-1}}^{t_n}\int\limits_{t_{n-1}}^{t_n} \boldsymbol{\Phi}(t_n,\tau)\boldsymbol{G}(\tau)\boldsymbol{w}(\tau)\boldsymbol{w}^T(\alpha)\boldsymbol{G}^T(\alpha)\boldsymbol{\Phi}^T(t_n,\alpha)d\tau d\alpha\right) \\ &= \int\limits_{t_{n-1}}^{t_n}\int\limits_{t_{n-1}}^{t_n} \boldsymbol{\Phi}(t_n,\tau)\boldsymbol{G}(\tau)E\left(\boldsymbol{w}(\tau)\boldsymbol{w}^T(\alpha)\right)\boldsymbol{G}^T(\alpha)\boldsymbol{\Phi}^T(t_n,\alpha)d\tau d\alpha\end{aligned} \tag{5.2}$$

State and covariance prediction - cont'd

Considering the characteristics of white noise (cf. (4.12))

$$E\left(\boldsymbol{w}(\tau)\boldsymbol{w}^T(\alpha)\right) = \delta(\tau - \alpha)$$

equation (5.1) simplifies to

$$\boldsymbol{Q} = \int_{t_{n-1}}^{t_n} \boldsymbol{\Phi}(t_n, \tau)\boldsymbol{G}(\tau)\boldsymbol{\Lambda}(\tau)\boldsymbol{G}^T(\tau)\boldsymbol{\Phi}^T(t_n, \tau)d\tau \qquad (5.3)$$

where $\boldsymbol{\Lambda}(\tau)$ is the so-called spectral density matrix.

Equation (5.3) can be evaluated analytically in some special cases as discussed in this lecture's Jupyter notebook `https://github.com/spacegeodesy/ParameterEstimationDynamicSystems/blob/master/example05.ipynb`

State and covariance prediction - cont'd

In many cases the analytical evaluation of integral (5.3) resp. (5.1) is either very complex or not possible at all. Moreover, for practical reasons numerical methods are preferable when an algorithm is implemented. Based on the work by [Loan, 1978] (not discussed here) one can find a "cookbook" for computing $\boldsymbol{\Phi}$ and $\boldsymbol{Q}$ numerically. The steps are

- STEP 1: Form a $2n \times 2n$ matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{bmatrix} -\boldsymbol{F} & \boldsymbol{G}\boldsymbol{W}\boldsymbol{G}^T \\ \boldsymbol{0} & \boldsymbol{F}^T \end{bmatrix} \Delta t \tag{5.4}$$

- STEP 2: Compute the matrix exponential of $\boldsymbol{A}$, i.e.

$$\boldsymbol{B} = \exp(\boldsymbol{A}) \tag{5.5}$$

  which contains implicit information about $\boldsymbol{\Phi}$ and $\boldsymbol{Q}$ as the matrix $\boldsymbol{B}$ is build up by

$$\boldsymbol{B} = \left[\begin{array}{c:c} \ldots & \boldsymbol{\Phi}^{-1}\boldsymbol{Q} \\ \hdashline \boldsymbol{0} & \boldsymbol{\Phi}^T \end{array}\right] = \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix} \tag{5.6}$$

  where the submatrices $\boldsymbol{B}_{ij}$ are of dimension $n \times n$.

State and covariance prediction - cont'd

- STEP 3: The state transition matrix can be found from (5.6) as

$$\boldsymbol{\Phi} = \boldsymbol{B}_{22}^{T} \tag{5.7}$$

- STEP 4: Finally, the covariance matrix $\boldsymbol{Q}$ can also be deduced from (5.6) as

$$\boldsymbol{Q} = \boldsymbol{\Phi}\boldsymbol{B}_{12} \tag{5.8}$$

Numerical evaluation of Eqs. (5.4) - (5.7) is preferable in many cases and easily programmable when $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{W}$ and Δt are given.

A corresponding numerical example can be found in the 2nd part of this lecture's Jupyter notebook `https://github.com/spacegeodesy/ParameterEstimationDynamicSystems/blob/master/example05.ipynb`

References

[Loan, 1978] Loan, C. V. (1978).
Computing integrals involving the matrix exponential.
IEEE Transactions on Automatic Control, 23(3):395–404.