$$\omega \times b = \Omega \cdot b = -B \cdot \omega$$

matrix representation:

$$\omega^s_{ts} = \begin{pmatrix} \omega^s_{ts1} \\ \omega^s_{ts2} \\ \omega^s_{ts3} \end{pmatrix} \Rightarrow \Omega^s_{ts} = \begin{pmatrix} 0 & -\omega^s_{ts3} & \omega^s_{ts2} \\ \omega^s_{ts3} & 0 & -\omega^s_{ts1} \\ -\omega^s_{ts2} & \omega^s_{ts1} & 0 \end{pmatrix}$$

$$\Omega^s_{st} = -\Omega^s_{ts}$$

$$= \quad \Omega^s_{ts} = C^s_t \cdot \Omega^t_{ts} \cdot C^t_s = -C^s_t \cdot \Omega^t_{st} \cdot C^t_s$$

- Derivative of DCM:

the changes of the s-system w.r.t. t-system in the time interval Δt

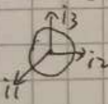
$$\dot{C}^s_t(t) = \lim_{\Delta t \to 0} \frac{\Delta C^s_t(t, t+\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{C^s_t(t+\Delta t) - C^s_t(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta C^s_t(t, t+\Delta t) \cdot C^s_t(t) - C^s_t(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(I - \Delta A) \cdot C^s_t(t) - C^s_t(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{-\Delta A}{\Delta t} \cdot C^s_t = -\Omega^s_{ts} \cdot C^s_t \quad ?$$

$$\Rightarrow \dot{C}^s_t = C^s_t \cdot \Omega^t_{st}$$

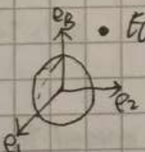
4. Coordinate Systems

- Coordinate Systems
  - Quasi-inertial coordinate system (i-system) — no rotation of axes w.r.t. inertial space

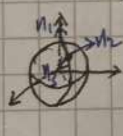


    - i_1 — Aries (♈) 白羊座
    - i_3 — rotation axis of the earth
    - i_2 — complements right-handed orthogonal system

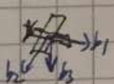
    quasi → mis-modelling of 10^{-8} g
  - Earth-centered earth-fixed coordinate system (e-system) — "earth-surface"



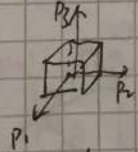
    - e_1 — meridian of Greenwich (0° Lon)
    - e_3 — rotation axis of the earth
    - e_2 — complements right-handed orthogonal system
  - Local level coordinate system (n-system)



    origin in a point of interest near the earth surface
    - n_1 — towards north (on the ellipsoid)
    - n_2 — towards east
    - n_3 — downwards ellipsoidal normal
  - Body coordinate system (b-system)



    origin in the center of gravity of the vehicle
    - b_1 — roll, pointing forward
    - b_2 — pitch, to the right
    - b_3 — heading (yaw), downward
  - Platform coordinate system (p-system)



    rigidly attached to the IMU
    - p_1 — x-accelerometer / gyro
    - p_2 — y-accelerometer / gyro
    - p_3 — z-accelerometer / gyro

- transformation
  - Quasi-inertial to ECEF: rotation about e_3/i_3 with $\omega_E \cdot \Delta t$

$$\omega^e_{ie} = \omega^i_{ie} = (0 \; 0 \; \omega_E)^T \Rightarrow C^e_i = \begin{pmatrix} \cos(\omega_E \cdot \Delta t) & \sin(\omega_E \cdot \Delta t) & 0 \\ -\sin(\omega_E \cdot \Delta t) & \cos(\omega_E \cdot \Delta t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Omega^e_{ie} = \begin{pmatrix} 0 & -\omega_E & 0 \\ \omega_E & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Local level to ECEF: rotation about n_2 with $\phi+\frac{\pi}{2}$
  rotation about n_3 with $-\lambda$

$$C_n^e = C(3,-\lambda)\cdot C(2,\phi+\tfrac{\pi}{2}) = \begin{pmatrix} -\sin\phi\cos\lambda & -\sin\lambda & -\cos\phi\cos\lambda \\ -\sin\phi\sin\lambda & \cos\lambda & -\cos\phi\sin\lambda \\ \cos\phi & 0 & -\sin\phi \end{pmatrix}$$

$$\dot{C}_n^e = C_n^e\cdot\Omega_{en}^n \Rightarrow C_e^n\cdot\dot{C}_n^e = \Omega_{en}^n = \begin{pmatrix} 0 & \sin\phi\cdot\dot\lambda & -\dot\phi \\ -\sin\phi\cdot\dot\lambda & 0 & -\cos\phi\cdot\dot\lambda \\ \dot\phi & \cos\phi\cdot\dot\lambda & 0 \end{pmatrix}$$

- Body to local: rotation about b_1 with $-R$ (roll)
  b_2 with $-P$ (pitch)
  b_3 with $-Y$ (yaw)

$$C_b^n = C(3,-Y)\cdot C(2,-P)\cdot C(1,-R) = \begin{pmatrix} \quad \end{pmatrix}$$

$$\dot{C}_b^n = C_b^n\cdot\Omega_{nb}^b \Rightarrow \Omega_{nb}^b = C_n^b\cdot\dot{C}_b^n$$

- platform to body

$$C_p^b \approx \begin{pmatrix} 1 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 1 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 1 \end{pmatrix}$$

- Inertial Measurement Unit (IMU): accelerometer & gyroscope

  measurement at time t_k:

  $\begin{cases} \text{velocity increment}: \Delta v^p(t_k) = \int_{t_{k-1}}^{t_k} a^p(\tau)\,d\tau & 10Hz - 200Hz \\ \text{angular increment}: \Delta\alpha^p(t_k) = \int_{t_{k-1}}^{t_k} \omega_{ip}^p(\tau)\,d\tau \end{cases}$

  - non-linear platform motion within readout interval can not be recovered from readout data

5. Measurements of inertial sensors

assumption: $b \Leftrightarrow p$

- stationary accelerometer triad:

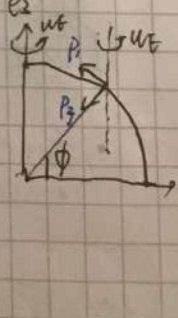
  $\begin{cases} \text{leveled}: a_1^p = 0,\ a_2^p = 0,\ a_3^p = g \\ \text{mis-leveled}: a^p = a^b = C_n^b a^n \Rightarrow \begin{cases} a_1^p = -g\sin P \\ a_2^p = g\sin R\cos P \\ a_3^p = g\cos R\cos P \end{cases} \end{cases}$

- stationary gyro triad:

  assumption: $Y = P = R = 0$

$$\omega_{ip}^p = \begin{pmatrix} \omega_E\cos\phi \\ 0 \\ -\omega_E\sin\phi \end{pmatrix}$$

- $p_2 \perp$ e_3 & p_1 & p_3
- used for calibration & alignment



- moving gyro triad:

  assumption: $Y = R = P = 0$ (b & p assigned with n)

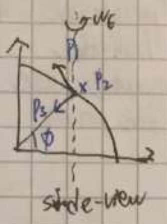
$$\underset{\omega_{in}^n}{\omega_{ip}^p} = \begin{pmatrix} \omega_E\cos\phi + \dot\lambda\cos\phi \\ -\dot\phi \\ -\underset{\omega_{ie}^n}{\omega_E\sin\phi} - \underset{\omega_{en}^n}{\dot\lambda\sin\phi} \end{pmatrix} = \begin{pmatrix} \omega_E\cos\phi + \frac{v_E}{N+h} \\ -\frac{v_N}{M+h} \\ \cancel{-\omega_E\sin\phi}\ \frac{v_E}{N+h}\tan\phi \end{pmatrix}$$

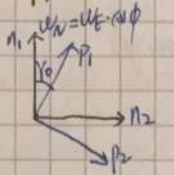
- **Alignment**: relationship between b-system and n-system ⇒ initial attitude R_0, P_0, Y_0 ⇒ DCM.
  - accelerometer levelling: align accelerometer to n-system ⇒ $a_2 = a_1 = 0$

$$\begin{cases} \sin P_0 = -\dfrac{a_1^p}{g} \\ \sin R_0 = \dfrac{a_2^p}{g\cos P_0} \end{cases}$$

  - Gyro compassing: maximum of gyro in a leveled plane ⟶ north direction
    0 ⟶ east

$$\left.\begin{aligned} \omega_{ip_2}^p &= -\omega_N \cdot \sin Y_0 = -\omega_E \cdot \cos\phi \cdot \sin Y_0 \\ \omega_{ip_1}^p &= \omega_N \cdot \cos Y_0 = \omega_E \cdot \cos\phi \cdot \cos Y_0 \end{aligned}\right] \Rightarrow \tan Y_0 = -\frac{\omega_{ip_2}^p}{\omega_{ip_1}^p}$$





$\omega_{in_2}^n = 0$

assignment 03 T2?

捷联式

6. Differential Equations for a Strap Down IMU

\ 3 accelerometers & 3 gyros are fixed to the platform carrying IMU

accelerometer measurement: $\vec{a} = \ddot{\vec{x}} - g$

- Differential equations in the e-system

$$x^i = C_e^i \cdot x^e \xrightarrow[\frac{\partial^2}{\partial t^2}]{\frac{\partial}{\partial t}} \begin{aligned} &\bullet\ \dot{x}^e = v^e \\ &\bullet\ \dot{v}^e = C_p^e \cdot a^p - 2\Omega_{ie}^e \cdot v^e - \Omega_{ie}^e \cdot \Omega_{ie}^e \cdot x^e + g^e \end{aligned}$$

$\Downarrow$

$$C_p^i = C_e^i \cdot C_p^e \Rightarrow \bullet\ \dot{C}_p^e = C_p^e \cdot (\Omega_{ip}^p - \Omega_{ie}^p)$$

with $C_p^e = C_n^e \cdot C_b^n \cdot C_p^b$

complete system.

- Differential equations in the n-system

why choosing n-system?
  - directly obtain attitude angle Y, R, P from the mechanization equations
  - directly obtain geographic coordinate differences $\Delta\phi, \Delta\lambda, \Delta h$
  - simple representation of gravity vector g
  - Schuler effect ⇒ computational error in navigation parameters on the horizontal plane are bounded.

$M = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}}$

M: meridian radius of curvature

N: prime vertical radius of curvature

$N = \frac{a}{\sqrt{1-e^2\sin^2\phi}}$

$$\begin{pmatrix} x_1^e \\ x_2^e \\ x_3^e \end{pmatrix} = \begin{pmatrix} (N+h)\cos\phi\cos\lambda \\ (N+h)\cos\phi\sin\lambda \\ [N(1-e^2)+h]\sin\phi \end{pmatrix} \xrightarrow{\frac{\partial}{\partial t}}$$

$$v^n = C_e^n \cdot \dot{x}^e$$

$$\Rightarrow \begin{pmatrix} v_N \\ v_E \\ v_D \end{pmatrix} = \begin{pmatrix} \dot{\phi}(M+h) \\ \dot{\lambda}(N+h)\cos\phi \\ -\dot{h} \end{pmatrix} \Rightarrow \bullet \begin{pmatrix} \dot{\phi} \\ \dot{\lambda} \\ \dot{h} \end{pmatrix} = \begin{pmatrix} \frac{v_N}{M+h} \\ \frac{v_E}{(N+h)\cos\phi} \\ -v_D \end{pmatrix}$$

$$\dot{v}^e = \frac{d}{dt}(\dot{x}^e) = \frac{d}{dt}(C_n^e \cdot v^n) = C_n^e\left(\frac{d}{dt}v^n + \Omega_{en}^n v^n\right) \Rightarrow$$

$$\bullet\ \dot{v}^n = C_p^n \cdot a^p - (2\Omega_{ie}^n + \Omega_{en}^n) \cdot v^n - C_e^n \Omega_{ie}^e \Omega_{ie}^e x^e(\phi, \lambda, h) + g^n$$

with C

$$\bullet\ \dot{C}_p^n = C_p^n(\Omega_{ip}^p - \Omega_{in}^p)$$

complete system

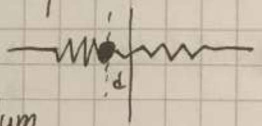
$$(\omega_{ip}^p - \omega_{ie}^p = \omega_{ip}^p + \omega_{ei}^p = \omega_{ep}^p)$$

- DCM → Quaternions

$$\begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \omega_{st_1}^t & \omega_{st_2}^t & \omega_{st_3}^t \\ -\omega_{st_1}^t & 0 & \omega_{st_3}^t & -\omega_{st_2}^t \\ -\omega_{st_2}^t & -\omega_{st_3}^t & 0 & \omega_{st_1}^t \\ -\omega_{st_3}^t & \omega_{st_2}^t & -\omega_{st_1}^t & 0 \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

7. Sensors

- Accelerometers
  - spring suspended mass



$$k \cdot d = m \cdot a_i^p$$

Common types:

capacitive	电容	capacity
piezoelectric	压电—电压	voltage
piezoresistive	电阻	resistance
hall effect	电磁变化	magnetic fields
magnetoresistive	磁抗	
heat transfer	温度	

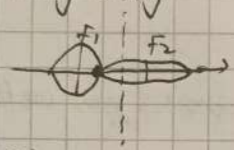
  - Pendulum

$$\frac{d}{L} \approx \alpha = \frac{m}{k} \cdot a_i^p$$

  - closed loop pendulum (remove non-linearity)

$$T = m \cdot L \cdot a_i^p$$

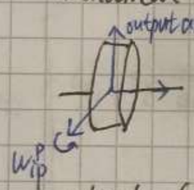
  - Vibrating string



$$f = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

$$f_1^2 - f_2^2 = \frac{m}{2\mu L^2} a_i^p$$

- Gyroscopes
  - mechanical (rotating mass)

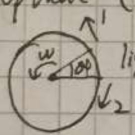


rotational velocity about input axis ⟶ force F ⟶ torque M = change in angular momentum H ($H = I \cdot \omega$)

↓ keep constant (torque generator)

torque generated ∝ rotational velocity

  - optical (laser, fibre optics)



$$\Delta t_1 = \frac{2\pi r + r \cdot \Delta\varphi}{c}, \quad \Delta t_2 = \frac{2\pi r - r\Delta\varphi}{c} \Rightarrow \omega_{ip}^p = \frac{c^2(\Delta t_1 - \Delta t_2)}{4\pi r^2}$$

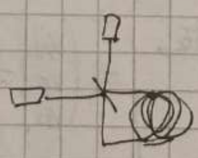
time difference ∝ rotational velocity; area enclosed by light path

    - laser gyro

insensitive to accelerations; large working range

"lock effect" ⟵ min small high frequency movements

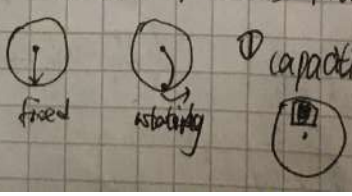
    - fibre optic gyro



diff of interferometry phase

  - vibrating mass (MEMS)

work on the basis of coriolis force: $F_c = 2 \cdot M \cdot (v \times \omega)$



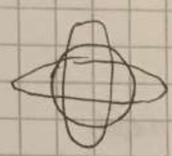
fixed rotating

① capacitive sensing element attached to the resonator

- immunity to shocks & vibrations

② Tuning Fork
- two masses driven to oscillate with equal amplitude in opposite directions
- rotation generates coriolis force → orthogonal vibration results
- capacity is a measure for the angular rate (amplitude & $\Delta\varphi$)

! very sensitive to shocks and vibrations



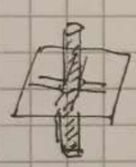
共鸣器 半球
③ Wine glass resonator / Hemispherical resonator
- a vibrating ring is driven to resonance
- rotation displaces nodal points → measure of angular rate

! non-sensitive to shocks and vibrations

④ vibrating wheel
- wheel is driven to vibrate around its axis of symmetry
- rotation about either in plane axis results in the wheel tilting
- changes detected with capacitive electrodes under the wheel
- degree of angular rate

! sensitive in 2 directions

⑤ Foucault pendulum gyroscope
- vibrating rod 棒
- rotation around the rod leads to an orthogonal oscillation
- amplitude and phase are a measure for the rate of rotation



! relative large metal rod necessary
no planar realization possible

	mechanical	optical	MEMS
drift (°/h)	0.1-0.0001	FOG 0.1-100 RLG 0.0001-1	1-500
price (eur)	>10000	100-50000	10-100

8. Integration of the attitude equation in the e-system $\left(\omega_{ep}^p = \omega_{ip}^p - \omega_{ie}^p = \omega_{ip}^p - C_e^p \cdot \omega_{ie}^e\right)$

measurement: $\Delta\alpha_{ip}^p(t)$

① $\Delta\beta_{ep}^p(t_{k-1}) \approx \Delta\alpha_{ip}^p(t_{k-1}) - C_e^p(t_{k-2})\,\omega_{ie}^e\,\Delta t$

$\Delta\beta_{ep}^p(t_k) \approx \Delta\alpha_{ip}^p(t_k) - C_e^p(t_{k-1})\cdot\omega_{ie}^e\,\Delta t$

② $\hat{\omega}_{ep}^p(t_{k-2}) = \dfrac{3\Delta\beta^p(t_{k-1}) - \Delta\beta^p(t_k)}{2\Delta t}$

$\hat{\omega}_{ep}^p(t_{k-1}) = \dfrac{\Delta\beta^p(t_{k-1}) + \Delta\beta^p(t_k)}{2\Delta t}$

$\hat{\omega}_{ep}^p(t_k) = \dfrac{3\Delta\beta^p(t_k) - \Delta\beta^p(t_{k-1})}{2\Delta t}$

③ $\dot{q}(t) = \frac{1}{2}A(t)\,q(t) = f(t,q) \Rightarrow \hat{q}(t_k) = \hat{q}(t_{k-2}) + \frac{\delta t}{6}(k_1 + 4k_2 + k_3)$ $\begin{cases} k_1 = f(t_{k-2}, q_{k-2}) \\ k_2 = f(t_{k-1}, q_{k-2} + k_1 \cdot \frac{\delta t}{2}) \\ k_3 = f(t_k, q_{k-2} - k_1 \cdot \delta t + k_2 \cdot 2\delta t) \end{cases}$

④ normalise the quaternion

⑤ compute DCM from quaternion

assignment 05_2019 T2?

9. Integration of the velocity and position equations in the e-system

① $$\hat{a}^p(t_{k-2}) = \frac{3\Delta v^p(t_{k-1}) - \Delta v^p(t_k)}{2\delta t}$$

$$\hat{a}^p(t_{k-1}) = \frac{\Delta v^p(t_{k-1}) + \Delta v^p(t_k)}{2\delta t} \qquad \delta t = 2\delta t$$

$$\hat{a}^p(t_k) = \frac{3\Delta v^p(t_k) - \Delta v^p(t_{k-1})}{2\delta t}$$

② from Chapter 6 $\Rightarrow \dot{v}^e = C_p^e a^p - \underbrace{2\Omega_{ie}^e v^e - \Omega_{ie}^e \Omega_{ie}^e x^e + g^e}_{\text{see as constant}}$

$$\Rightarrow \hat{v}^e(t_k) = \int_{t_{k-1}}^{t_k} C_p^e a^p - \left(2\Omega_{ie}^e v^e + \Omega_{ie}^e \Omega_{ie}^e x^e - g^e\right)_{t_{k-1}} \cdot \delta t$$

$$\xrightarrow{\text{Simpson's Rule}} \int_{t_{k-1}}^{t_k} C_p^e a^p \, d\tau = \frac{\delta t}{6}\left((C_p^e a^p)(t_{k-2}) + 4 \cdot C_p^e a^p(t_{k-1}) + C_p^e a^p(t_k)\right)$$

③ $$\hat{x}^e(t_k) = \hat{x}^e(t_{k-1}) + \hat{v}^e(t_{k-1}) \cdot \delta t$$

10. Linearized error equations in the e-system

- errors: $\begin{cases} \text{initial conditions for integration} \\ \text{measurements for accelerometer \& gyroscope} \\ \text{gravity field} \end{cases}$

$$\frac{d}{dt}\begin{pmatrix} \psi^e \\ \delta\dot{x}^e \\ \delta x^e \end{pmatrix} = \begin{pmatrix} -\Omega_{ie}^e & 0 & 0 \\ a^e \times & -2\Omega_{ie}^e & -\Omega_{ie}^e \cdot \Omega_{ie}^e - \Gamma^e \\ 0 & I & 0 \end{pmatrix}_{9\times 9} \cdot \begin{pmatrix} \psi^e \\ \delta\dot{x}^e \\ \delta x^e \end{pmatrix}_{9\times 1} + \begin{pmatrix} -C_p^e & 0 & 0 \\ 0 & C_p^e & I \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \delta\omega_{ip}^p \\ \delta a^p \\ \delta g^e \end{pmatrix}$$

ψ^e: $\delta C_p^e = (I - \Psi^e) \cdot C_p^e - C_p^e$ difference between true DCM and a DCM accounting for a small addition rotation

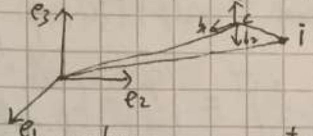
$$\Psi^e \cdot k = \psi^e \times k = -k \times \psi^e \qquad \Psi^e = \begin{pmatrix} 0 & -\delta\alpha_3 & -\delta\alpha_2 \\ \delta\alpha_3 & 0 & \delta\alpha_1 \\ \delta\alpha_2 & -\delta\alpha_1 & 0 \end{pmatrix} \qquad \dot{\psi}^e = -C_p^e \, \delta\omega_{ep}^p$$

Γ^e: the gradient of the gravitational acceleration

Integrated Positioning and Navigation

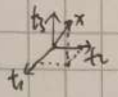
1. Vectors and Coordinates

- Inertial navigation system
  — sense the 6 parameters describing the motion of a body.



$$x_i^e(t) = x_c^e(t) + C_b^e(t)\cdot x_i^b(t)$$

- vector



$$\vec{x}^t = \begin{bmatrix} x_1^t \\ x_2^t \\ x_3^t \end{bmatrix}$$ — coordinates if a vector.

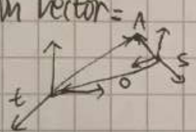
$$\underline{t} = (\vec{t}_1\ \vec{t}_2\ \vec{t}_3)$$ — base vectors

$$\Rightarrow\ x = \underline{t}\cdot\vec{x}^t = (\vec{t}_1\vec{t}_2\vec{t}_3)\cdot\begin{pmatrix} x_1^t \\ x_2^t \\ x_3^t \end{pmatrix} = x_1^t\vec{t}_1 + x_2^t\vec{t}_2 + x_3^t\vec{t}_3$$

- vector $\neq$ position vector (vector attached to the origin)

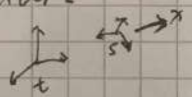
- transformation between coordinate systems

position vector:



$$x^t(A) = t\cdot x^t \qquad x^s(A) = s\cdot x^s \quad\Rightarrow\quad s\cdot x^s = s\cdot O^s + t\cdot x^t = s\cdot O^s + s\cdot C_t^s x^t$$

vector:



$$\vec{x} = t\cdot x^t \qquad \vec{x} = s\cdot x^s \quad\Rightarrow\quad s\cdot x^s = t\cdot x^t = s\cdot C_t^s\cdot x^t$$

DCM: "direction cosine matrix"

Direction Cosine Matrix:

$$x^s = C_t^s x^t \quad \& \quad \underline{t} = \underline{s}\cdot C_t^s$$

$$(C_t^s)^{-1} = C_s^t = (C_t^s)^T$$

2. Parameterization of the DCM

- fully-described DCM

$$C_t^s(1,\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \qquad C_t^s(2,\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \qquad C_t^s(3,\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Euler angles
  two coordinate systems can be transformed into each other by 3 subsequent single axis rotations:
  ① $C_t^s = C(3,\gamma)\cdot C(2,\beta)\cdot C(3,\alpha)$ 6 possible sequences
  ② $C_t^s = C(1,\alpha)\cdot C(2,\beta)\cdot C(3,\gamma)$ 6 possible sequences

$$C_t^s = \begin{pmatrix} \cos\beta\cos\gamma & \cos\beta\sin\gamma & -\sin\beta \\ -\sin\gamma\cos\alpha + \sin\beta\cos\gamma\sin\alpha & \cos\gamma\cos\alpha + \sin\beta\sin\gamma\sin\alpha & \cos\beta\sin\alpha \\ \sin\gamma\sin\alpha + \sin\beta\cos\gamma\cos\alpha & -\cos\gamma\sin\alpha + \sin\beta\sin\gamma\cos\alpha & \cos\beta\cos\alpha \end{pmatrix} \simeq \begin{pmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha = \arctan\left(\dfrac{C_t^s(2,3)}{C_t^s(3,3)}\right) \\ \beta = \arcsin\left(-C_t^s(1,3)\right) \\ \gamma = \arctan\left(\dfrac{C_t^s(1,2)}{C_t^s(1,1)}\right) \end{cases}$$

- not unique $\Rightarrow -90^\circ < \beta \le 90^\circ$
- if $\beta = \pm 90^\circ \Rightarrow$ unable to get α & γ

$\det(C_s^t) = 1$
$(C_t^s)^{-1} = (C_t^s)^T = C_s^t$

(if $\det(C_t^s) = -1$ → mirroring
others → rotation & dilation

- Euler Symmetric Parameters

two coordinate systems can be transformed into each other by a single rotation about a unit vec

$\vec{f} = t \cdot f^t = s \cdot f^s \Rightarrow f^t = f^s$

$$\Rightarrow C_t^s = \begin{pmatrix} \cos\phi + f_1^2(1-\cos\phi) & f_1f_2(1-\cos\phi)+f_3\sin\phi & f_1f_3(1-\cos\phi)-f_2\sin\phi \\ f_1f_2(1-\cos\phi)-f_3\sin\phi & \cos\phi+f_2^2(1-\cos\phi) & f_2f_3(1-\cos\phi)+f_1\sin\phi \\ f_1f_3(1-\cos\phi)+f_2\sin\phi & f_2f_3(1-\cos\phi)-f_1\sin\phi & \cos\phi+f_3^2(1-\cos\phi) \end{pmatrix} \approx \begin{pmatrix} 1 & \phi f_3 & -\phi f_2 \\ -\phi f_3 & 1 & \phi f_1 \\ \phi f_2 & -\phi f_1 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \vec{q} = \left(\cos\frac{\phi}{2}, f_1\sin\frac{\phi}{2}, f_2\sin\frac{\phi}{2}, f_3\sin\frac{\phi}{2}\right)^T = \begin{pmatrix} q_0 \\ q \end{pmatrix} = \begin{pmatrix} \cos\frac{\phi}{2} \\ \vec{f}\cdot\sin\frac{\phi}{2} \end{pmatrix}$$

"Quaternion"

$$\Rightarrow C_t^s = \begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_3q_0) & 2(q_1q_3-q_2q_0) \\ 2(q_1q_2-q_3q_0) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_1q_0) \\ 2(q_1q_3+q_2q_0) & 2(q_2q_3-q_1q_0) & q_0^2-q_1^2-q_2^2+q_3^2 \end{pmatrix} \approx \begin{pmatrix} 1 & 2q_3 & -2q_2 \\ -2q_3 & 1 & 2q_1 \\ 2q_2 & -2q_1 & 1 \end{pmatrix}$$

$$q_0^2+q_1^2+q_2^2+q_3^2 = 1$$

$$\Rightarrow \begin{cases} q_0 = \frac{1}{2}\sqrt{C_t^s(1,1)+C_t^s(2,2)+C_t^s(3,3)+1} \\ q_1 = \dfrac{C_t^s(2,3)-C_t^s(3,2)}{4q_0} \\ q_2 = \dfrac{C_t^s(3,1)-C_t^s(1,3)}{4q_0} \\ q_3 = \dfrac{C_t^s(1,2)-C_t^s(2,1)}{4q_0} \end{cases}$$

- Orthogonality (important due to rounding errors)

Euler Angles: scalar product of different rows $= 0$ (the same for column)
scalar product of each row itself $= 1$

Quaternion: normalize $q = \dfrac{q}{\sqrt{q^T q}}$

- Quaternion algebra
  - conjugate: $\bar{q} = [q_0 \ -q_1 \ -q_2 \ -q_3]^T$
  - scalar product: $p \cdot q = p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3$
  - multiplication: $p \circ q = \begin{pmatrix} p_0 \\ \vec{p} \end{pmatrix} \circ \begin{pmatrix} q_0 \\ \vec{q} \end{pmatrix} = \begin{pmatrix} p_0q_0 - \vec{p}\cdot\vec{q} \\ p_0\vec{q} + q_0\vec{p} + \vec{p}\times\vec{q} \end{pmatrix}$
  - transformation with quaternions:

$$x^t = q \circ x^s \circ \bar{q}$$
$$x^s = \bar{q} \circ x^t \circ q \qquad x = (0 \ x_1 \ x_2 \ x_3)^T$$

3. Rotational Motion

- angular velocity vector

$$\vec{w}_{ts}^s = (w_{ts1}^s, \ w_{ts2}^s \ w_{ts3}^s)^T$$

the angular velocity vector (in the s-system) of the s-system w.r.t. t-system

$$\vec{w}_{st}^s = -\vec{w}_{ts}^s \qquad \vec{w}_{ts}^s = C_t^s \cdot w_{ts}^t = -C_t^s \cdot w_{st}^t$$