a) All azimuthal mappings have the general mapping equations $\alpha = \Lambda$, $r = f(\Delta)$, $\Delta = \frac{\pi}{2} - \Phi$

Radius r in the map can be read from the figure $r = 4R\tan\frac{\Delta}{4} = 4R\tan\left(\frac{\pi}{8} - \frac{\Phi}{4}\right)$.

b) Starting from $\alpha = \Lambda$, $r = 4R\tan\frac{\Delta}{4} = 4R\tan\left(\frac{\pi}{8} - \frac{\Phi}{4}\right)$ the Jacobian matrix is

$\underline{J} = \begin{bmatrix} \frac{\partial\alpha}{\partial\Lambda} & \frac{\partial\alpha}{\partial\Delta} \\ \frac{\partial r}{\partial\Lambda} & \frac{\partial r}{\partial\Delta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{R}{\cos^2\frac{\Delta}{4}} \end{bmatrix}$. With metric matrix $\underline{g} = \begin{bmatrix} r^2 & 0 \\ 0 & 1 \end{bmatrix}$ for the parameters α

and r in the plane, the Cauchy-Green-deformation matrix is $\underline{C} = \underline{J}'\underline{g}\underline{J} = \begin{bmatrix} r^2 & 0 \\ 0 & \frac{R^2}{\cos^4\frac{\Delta}{4}} \end{bmatrix}$.

Because both $\underline{C}$ and the metric matrix for the sphere

$\underline{G} = R^2\begin{bmatrix} \cos^2\Phi & 0 \\ 0 & 1 \end{bmatrix} = R^2\begin{bmatrix} \sin^2\Delta & 0 \\ 0 & 1 \end{bmatrix}$ are diagonal, the principal distortions can be de-

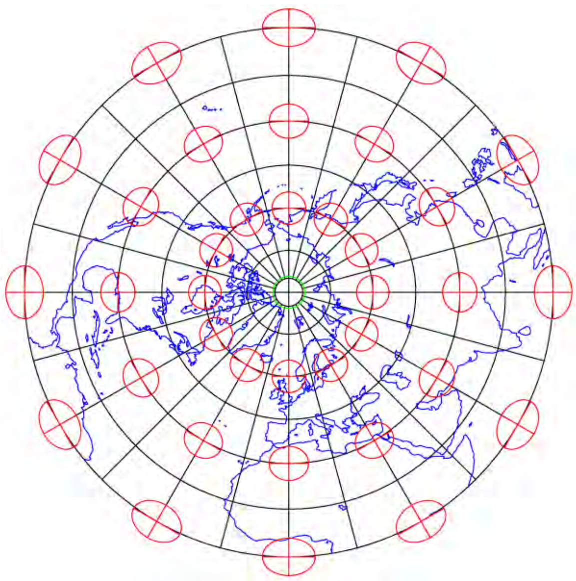
duced from the simplified formulas

$$\Lambda_1 = \sqrt{\frac{C_{11}}{G_{11}}} = \frac{r}{R\sin\Delta} = \frac{4\tan\frac{\Delta}{4}}{\sin\Delta} = \frac{4\sin\frac{\Delta}{4}}{\sin\Delta\cos\frac{\Delta}{4}} = \frac{1}{\cos\frac{\Delta}{2}\cos^2\frac{\Delta}{4}} \geq \Lambda_2 = \sqrt{\frac{C_{22}}{G_{22}}} = \frac{1}{\cos^2\frac{\Delta}{4}}.$$

Thus, the mapping is not conformal. The poles are mapped without distortions $\Lambda_1\big|_{\substack{\Phi=90° \\ \Delta=0°}} =$

$\Lambda_2\big|_{\substack{\Phi=90° \\ \Delta=0°}} = 1$. Going from the poles towards the equator Λ_1 grows stronger than Λ_2.

c)



d) Equidistant parallel circle $\Phi_0 = \text{const} \neq 90°$ means that $r(\Phi_0) = R\cos\Phi_0$ must hold, because the parallel circle Φ_0 has radius $R\cos\Phi_0$. This requirement is met, if r is multiplied by the factor $f(\Phi_0) = \dfrac{\cos\Phi_0}{4\tan\left(\dfrac{\pi}{8} - \dfrac{\Phi_0}{4}\right)}$. The new equation becomes

$r = \dfrac{R\cos\Phi_0}{\tan\left(\dfrac{\pi}{8} - \dfrac{\Phi_0}{4}\right)} \tan\left(\dfrac{\pi}{8} - \dfrac{\Phi}{4}\right)$. As an alternative, the postulate

$\tilde{\Lambda}_1\big|_{\Delta_0} = f(\Delta_0)\,\Lambda_1\big|_{\Delta_0} = \dfrac{f(\Delta_0)}{\cos\dfrac{\Delta_0}{2}\cos^2\dfrac{\Delta_0}{4}} = 1$ ($\tilde{\Lambda}_1$ denotes the distortion Λ_1 for the modified equation(s)) can be used to find how the mapping equation has to be modified for an equidistant mapping of the parallel circle. We get $f(\Delta_0) = \cos\dfrac{\Delta_0}{2}\cos^2\dfrac{\Delta_0}{4}$ which is equivalent to $f(\Phi_0)$ from above.

a) Solid line is the image of a parallel circle, dashed line the image of a meridian

b) No. Parallel circles and meridians are always displayed as curved lines. Only the images of the equator and the reference meridian are straight.

c) Parallel circles and meridians are always orthogonal so are their images in a conformal map.

d) Solid lines are lines of constant northings, dashed lines of constant eastings.

e) Green lines are parameter lines in Cartesian coordinates. These are always straight.

f) Green lines are parameter lines in Cartesian coordinates. These are always orthogonal.

g) No. The set of green lines is different from the set of blue lines by the meridian convergence. The dashed green lines are orthogonal to the solid blue lines only at the equator, and the solid green lines are orthogonal to the dashed blue line only at the reference meridian.

h) The blue dashed line points north because it is the image of a meridian.

i) Dashed circle: The longitude of the dashed blue meridian image. Solid circle: The latitude of the solid blue parallel image.

j) Dashed circle: False easting of the dashed green coordinate line. Solid circle: The northing of the solid green coordinate line.

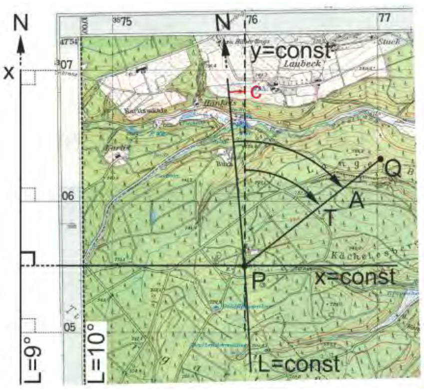
k) The distance is computed from Gauß-Krüger Cartesian coordinate differences using the Pythagorean law.

l) No. A conformal mapping is not length preserving in any arbitrary direction. Equidistance is given only for 2 points on either the reference meridian image or the equator image.

m) Grid bearing is computed from Gauß-Krüger Cartesian coordinate differences using the 'arctan'.

n) North azimuth is found by adding the meridian convergence c in P to T from m). c is computed from a series expansion $c = (01)_c \ell + (11)_c b\ell + \ldots$, $b = B - B_0$, $\ell = L - L_0$, the coefficients of which depend on the geometry of the ellipsoid-of-revolution and the latitude B_0 of the Taylor point (local origin).

o)



a) $\mathbf{G}_1 = \dfrac{\partial \mathbf{X}}{\partial U^1} = \dfrac{\partial \mathbf{X}}{\partial U} = \mathbf{E}_1 + 0\mathbf{E}_2 + \tan U \mathbf{E}_3$, $\mathbf{G}_2 = \dfrac{\partial \mathbf{X}}{\partial U^2} = \dfrac{\partial \mathbf{X}}{\partial V} = 0\mathbf{E}_1 + \mathbf{E}_2 - \tan V \mathbf{E}_3$

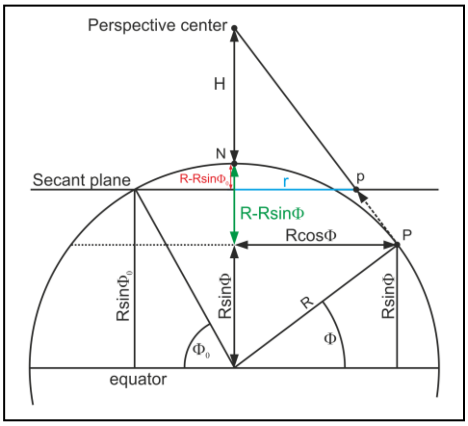
b) $\underline{G} = \begin{bmatrix} 1+\tan^2 U & -\tan U \tan V \\ -\tan U \tan V & 1+\tan^2 V \end{bmatrix}$

c) Because $G_{12} = -\tan U \tan V \neq 0$ this manifold is not orthogonally parameterized.

a) From the intercept theorem (theorem about ratios in similar triangles) we know that

$\frac{r}{R\cos\Phi} = \frac{H+R-R\sin\Phi_0}{H+R-R\sin\Phi}$. As a consequence

$r = R\cos\Phi\frac{H+R-R\sin\Phi_0}{H+R-R\sin\Phi} = R(c-\sin\Phi_0)\frac{\cos\Phi}{c-\sin\Phi}, \; c = \frac{H}{R}+1$. With $\alpha=\Lambda$, the mapping equations are therefore $x = r\cos\Lambda$, $y = r\sin\Lambda$.



b) Using the hint to formulate r as a function of co-latitudes $\Delta = 90° - \Phi, \Delta_0 = 90° - \Phi_0$ we get

$r = R(c-\cos\Delta_0)\frac{\sin\Delta}{c-\cos\Delta}, \; c = \frac{H}{R}+1$. Jacobian and Cauchy-Green matrix are

$$\underline{J} = \begin{bmatrix} -r\sin\Lambda & \frac{\partial r}{\partial \Delta}\cos\Lambda \\ r\cos\Lambda & \frac{\partial r}{\partial \Delta}\sin\Lambda \end{bmatrix}, \underline{C} = \begin{bmatrix} r^2 & 0 \\ 0 & \left(\frac{\partial r}{\partial \Delta}\right)^2 \end{bmatrix} \text{ where}$$

$$\frac{\partial r}{\partial \Delta} = R(c-\cos\Delta_0)\frac{c\cos\Delta - 1}{(c-\cos\Delta)^2} \Rightarrow$$

$$\Lambda_1 = \frac{c-\cos\Delta_0}{c-\cos\Delta} = \frac{c-\sin\Phi_0}{c-\sin\Phi} \quad , \quad \Lambda_2 = \frac{(c-\cos\Delta_0)(c\cos\Delta-1)}{(c-\cos\Delta)^2} = \frac{(c-\sin\Phi_0)(c\sin\Phi-1)}{(c-\sin\Phi)^2}$$

c) Parallel circles are mapped equidistantly if $\Lambda_1 = 1 \;\Rightarrow\; \Phi = \Phi_0$. Only the circle of contact is mapped equidistantly.

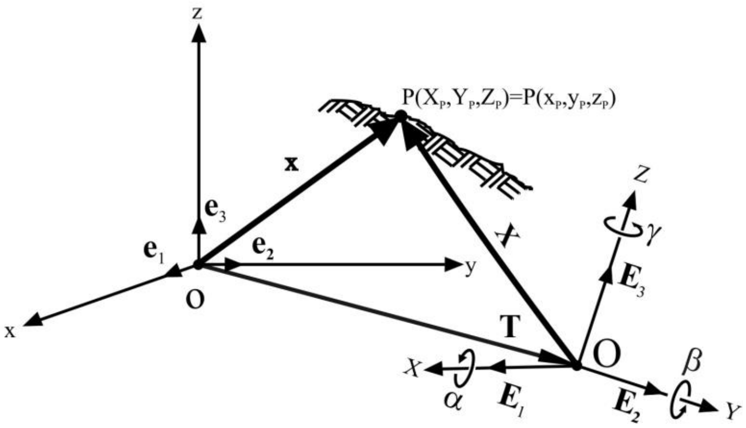
d) Isometry is guaranteed if $\Lambda_1 = \Lambda_2 = 1$ holds. It is seen from

$\Lambda_1 = \Lambda_2 \Leftrightarrow \frac{c-\sin\Phi_0}{c-\sin\Phi} = \frac{(c-\sin\Phi_0)(c\sin\Phi-1)}{(c-\sin\Phi)^2} \Leftrightarrow \sin\Phi = 1 \Leftrightarrow \Phi = 90°$ that only the North Pole is mapped conformally. However, because of $\Lambda_1|_{\Phi=90°} = \frac{c-\sin\Phi_0}{c-1} \neq 1$ the North Pole is not mapped free of distortion. For this reason no point or line is mapped isometrically.

If c) has been solved already, it is sufficient to check if $\Lambda_2|_{\Phi=\Phi_0} = \frac{c\sin\Phi_0 - 1}{c-\sin\Phi_0} = 1$ is satisfied. This is only true for $\Phi_0 = 90°$, however !

a) $\mathbf{X} \mapsto \mathbf{x} = \lambda \underline{R}\mathbf{X} + \mathbf{T},\ \underline{R} := \underline{R}_3(\gamma)\underline{R}_2(\beta)\underline{R}_1(\alpha)$

b)



λ is the scale difference (better: the ratio of the scales in both systems), **T** the translation vector between the two origins of both sets of base vectors, which always points from the origin of the target system (here: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) to the origin of the start system (here: $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$). This is because – in case of λ=1, $\underline{R}=\underline{I}_3$ – the vectorial sum of **X** and **T** builds **x**. $\underline{R}$ is the 3×3 overall rotation matrix, which makes the base vector sets become parallel. It consists of the above sequence of elementary rotation matrices using

$$\underline{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}, \underline{R}_2(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \underline{R}_3(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation angles α,β,γ are reckoned mathematically positive when measured counter-clockwise as seen from the tip of the corresponding axis towards the origin. They are needed in order to make the base vector sets parallel.

c) Transformation close to the identity means that $\lambda \approx 1, \alpha \approx \beta \approx \gamma \approx 0 \Rightarrow \underline{R} = \underline{I}_3 + \delta\underline{R} \approx \underline{I}_3$

a) Point 1:

From "False Easting" E = 631 535.88 m, "False Northing" N = 485 137.03 m and Zone = 18 we can first easily determine the reference meridian $L_0 = 6°(\text{Zone} - 30) - 3° = -75°$. Only 3 points (Bogota, Lima and Washington) are located in the vicinity of this meridian. Second, because E-500 000 m is positive, the point is located east from the meridian. Third the point distance from the equator is roughly only 485 km to the North. So the point must be **Bogota**.

Point 2:

From "False Easting" E = 318 473.70 m, "False Northing" N = 3 772 859.59 m and Zone = 43 we can first easily determine the reference meridian $L_0 = 6°(\text{Zone} - 30) - 3° = 75°$. Only 2 points (Islamabad and New Delhi) are located in the vicinity of this meridian. Second, because E-500 000 m is negative, the point is located west from the meridian. So the point must be **Islamabad**.

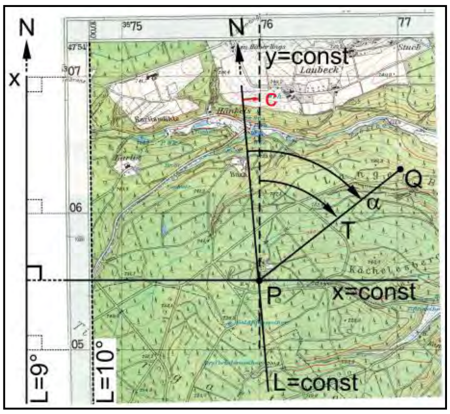
b) The two equations are homogeneous bivariate polynomials of degree n

$$x(L - L_0, Q - Q_0) =: x(\ell, q) = (n,0)_x q^n + (n-1,1)_x q^{n-1}\ell + (n-2,2)_x q^{n-2}\ell^2 + \dots + (0,n)_x \ell^n$$
$$y(L - L_0, Q - Q_0) =: y(\ell, q) = (n,0)_y q^n + (n-1,1)_y q^{n-1}\ell + (n-2,2)_y q^{n-2}\ell^2 + \dots + (0,n)_y \ell^n$$

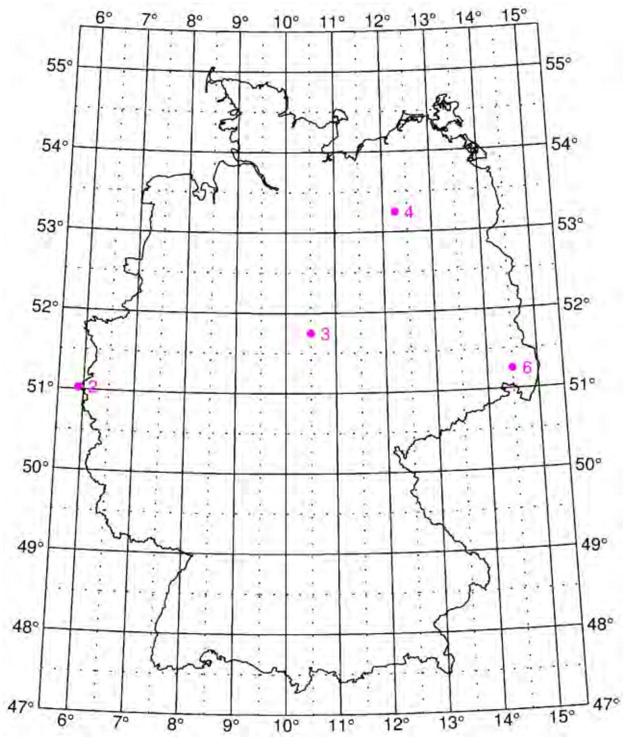
c) Since x and y refer to a local origin P_0 on a reference meridian the following steps are necessary to establish False Northings N and Eastings E:

$y \mapsto E = y + 500\,000\text{ m}, \quad x \mapsto N = X_0 + x$ where X_0 is the length of the meridional arc from the equator to the local origin $P_0(B_0,L_0)$ at latitude B_0. For the reason that L_0 cannot be read from x and y the additional information required for proper georeferencing the point is the zone number $= \dfrac{L_0 + 3°}{6°} + 30$.

d) Meridian convergence c is used in order to compute azimuth α from grid bearing T, i.e. α=T+c. Grid bearing T is computed from map coordinates.



e) From the False Eastings it is seen that points 1 and 7 are definitely not located in Germany (Point 1 is west of the 3°-meridian, point 7 is ~20km west of the 18°-meridian). Point 2 is ~7km west of the 6°-meridian; from its Northing the approximate latitude can be computed using the approximation of a spherical Earth of radius R=6 380 000m: $\Phi\,[°] = 180° \times \text{Northing} / (\pi \times R)$. Since Φ_2=50°.7837 the point is most likely in Germany. Points 3-4 are clearly in Germany because they are close to the 9°- and 12°-meridians and within the north-south window (55° ⇔ 6 124 360m, 47.5° ⇔ 5 289 220m). The latitude of point 5 (Φ_5~45°.7) is clearly south of 47.5°. That is why this point is outside Germany. Finally, point 6 is ~35km west of the 15°-meridian and a bit north of the 51° parallel circle. Therefore, it is also located in Germany.



Points are shown on a conformal conical mapping of the GRS80-ellipsoid (Lambert projection with 2 equidistant parallel circles, Φ_1=48°40', Φ_2=52°40')

The Jacobian matrix of the mapping equations is

$$\underline{J} = \begin{bmatrix} \frac{\partial x}{\partial \Lambda} & \frac{\partial x}{\partial \Theta}\frac{d\Theta}{d\Phi} \\ \frac{\partial y}{\partial \Lambda} & \frac{\partial y}{\partial \Theta}\frac{d\Theta}{d\Phi} \end{bmatrix} = R \begin{bmatrix} \left(e^{\Lambda} + e^{-\Lambda}\right)\cos\Theta & -\left(e^{\Lambda} - e^{-\Lambda}\right)\sin\Theta \frac{d\Theta}{d\Phi} \\ \left(e^{\Lambda} - e^{-\Lambda}\right)\sin\Theta & \left(e^{\Lambda} + e^{-\Lambda}\right)\cos\Theta \frac{d\Theta}{d\Phi} \end{bmatrix} = \begin{bmatrix} y\cot\Theta & -x\tan\Theta \frac{d\Theta}{d\Phi} \\ x\tan\Theta & y\cot\Theta \frac{d\Theta}{d\Phi} \end{bmatrix},$$

and – with $\underline{g} = \underline{I}_2$ – the Cauchy-Green-distortion matrix is derived as

$$\underline{C} = \underline{J}'\underline{J} = \left(x^2\tan^2\Theta + y^2\cot^2\Theta\right)\begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{d\Theta}{d\Phi}\right)^2 \end{bmatrix}.$$

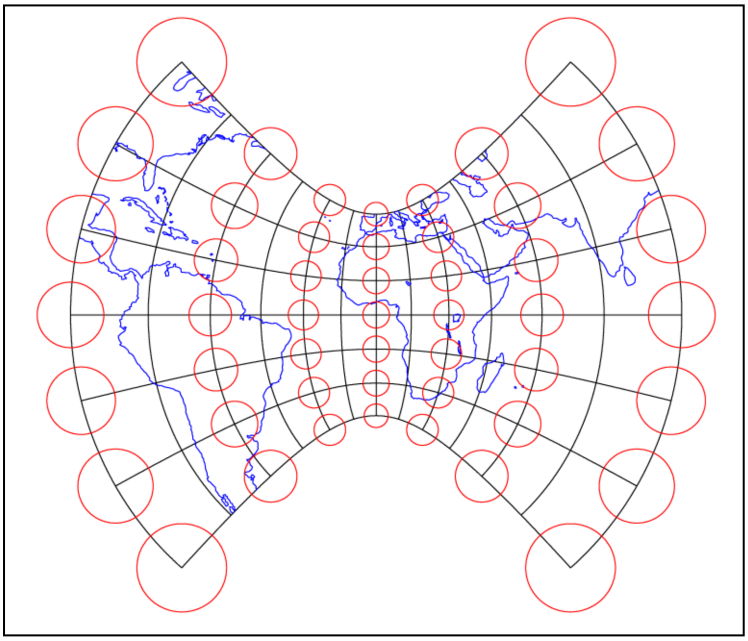
For the reason that $\underline{C}$ is a diagonal matrix the extremal distortions Λ_1 and Λ_2 can be read from the simplified distortion equations $(G_{11} = R^2\cos^2\Phi, G_{12} = G_{21} = 0, G_{22} = R^2)$

$$\Lambda_1 = \sqrt{\frac{C_{11}}{G_{11}}} = \frac{\sqrt{\left(x^2\tan^2\Theta + y^2\cot^2\Theta\right)}}{R\cos\Phi}, \quad \Lambda_2 = \sqrt{\frac{C_{22}}{G_{22}}} = \frac{\sqrt{\left(x^2\tan^2\Theta + y^2\cot^2\Theta\right)}}{R}\frac{d\Theta}{d\Phi}$$

Conformality is given if $\Lambda_1 = \Lambda_2 \sim \frac{1}{\cos\Phi} = \frac{d\Theta}{d\Phi}$ holds. Equality can be proven either by differentiation $\frac{d\Theta}{d\Phi} = \frac{d}{d\Phi}\ln\tan\left(\frac{\pi}{4} + \frac{\Phi}{2}\right) = \frac{1}{\cos\Phi}$ or using the transformation of the spherical arc element, which has been applied in the frame of isometric coordinates (of the ellipsoid-of-revolution)

$$dS^2 = R^2(\cos^2\Phi d\Lambda^2 + d\Phi^2) = R^2\cos^2\Phi(d\Lambda^2 + \frac{d\Phi^2}{\cos^2\Phi}) = R^2\cos^2\Phi(d\Lambda^2 + d\Theta^2).$$

Here we directly see that $\frac{d\Phi^2}{\cos^2\Phi} = d\Theta^2$ and $\frac{1}{\cos^2\Phi} = \frac{d\Theta^2}{d\Phi^2}$, resp. is satisfied.



a) (i) Distortions in direction of the parameter lines ($\Lambda_1 : \Phi = \text{const}$, $\Lambda_2 : \Lambda = \text{const}$),

(ii) equidistant mapping (no distortions) of the parallel circles,

(iii) equidistant mapping (no distortions) of the meridians,

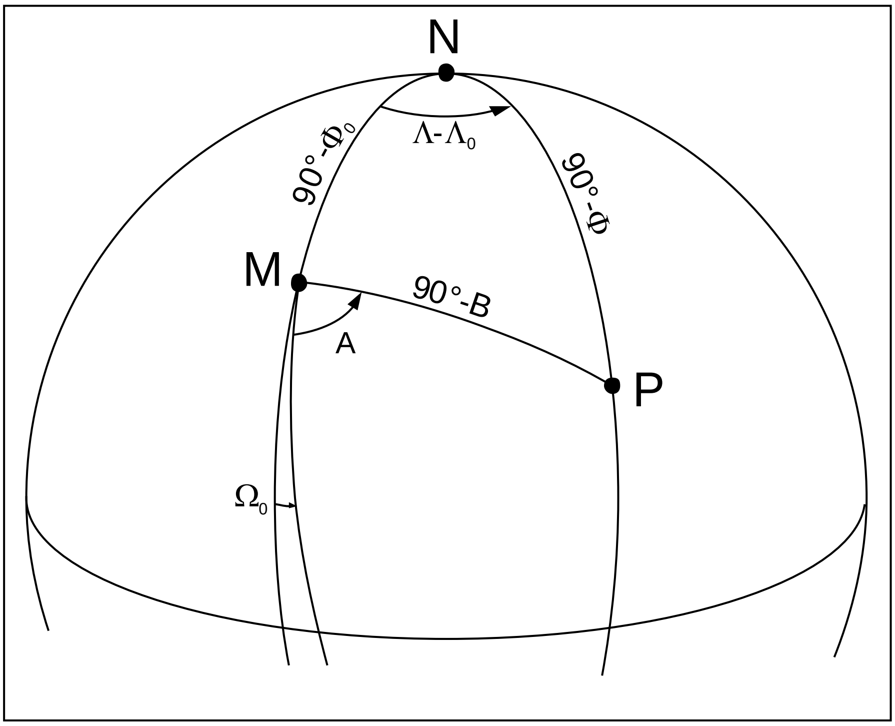
(iv) no distortion at all

b) (i) $\alpha=\Lambda$, $r=Rf(90°-\Phi)$,

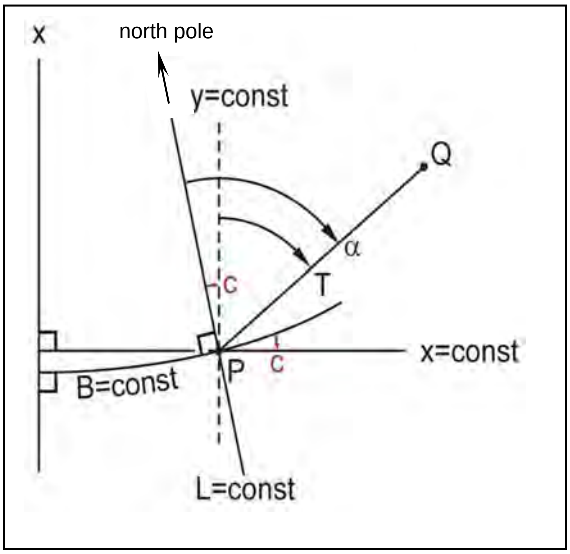
(ii) $x=R\Lambda$, $y=Rf(\Phi)$,

(iii) $\alpha=n\Lambda$, $r=f(90°-\Phi)$, $n=\sin\Phi_0$ "Cone constant"

c) For oblique spherical map projections a so-called meta pole M is chosen, which replaces the standard pole, e.g. the north pole N. The given coordinates of M in the standard system are longitude Λ_0 and latitude Φ_0. Standard pole N, meta pole $M(\Lambda_0,\Phi_0)$ and (given) point $P(\Lambda,\Phi)$ to be mapped in an oblique mapping generate a spherical triangle on the surface of a sphere where the two sides $(90°-\Phi)$ and $(90°-\Phi_0)$ and the angle $\Lambda-\Lambda_0$ in between are known. Unknown is the meta latitude B (or meta co-latitude 90°-B) referring to the meta pole M and meta longitude A to be counted from the meta prime meridian. There exists one degree of freedom Ω_0 in the choice of the meta prime meridian, starting from the meridian Λ_0 passing through N and M. The mapping equations are subsequently formulated in A, B and Ω_0 instead of Λ and Φ.



a) On a map, at a given point, meridian convergence c is the angle measured clockwise between the direction to the north pole and the direction of the northing grid line (grid north), which is parallel to the axis of ordinate. However, the direction to the north pole at the given point somewhere on the map cannot be seen directly. Usage: If two points P and Q on a map are connected by a line, grid bearing T of this line can be computed from map coordinates. The azimuth α of this line, i.e. its angle with respect to the direction to the north pole, cannot be read from the map, usually, because this direction (L=const) is not shown on the map. For this reason the azimuth of the line is derived from the grid bearing using the meridian convergence α=T+c.



b) P_1 is located 80 km west of the meridian 12°E and 5 301 912,12m north of the equator
P_2 is located 12 129,04m east of the meridian 9°E and 5 500 000,00m north of the equator
P_3 is located 34 012,45m east of the meridian 15°E and 4 989 991,34m north of the equator

c) P_1: H = 5 528 844,77m, R = 2 729 741,67m
P_2: N = 5 458 123,55m, E = 371 155,23m, Zone=33
P_3: N = 5 400 000,00m, E = 500 000,00m, Zone=31

In order to prove that the coordinate transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \omega & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \end{bmatrix} \sim \underline{x}' = \underline{J}\,\underline{x} + \underline{T}$$

is not conformal the extremal distortions Λ_1 and Λ_2 with the test $\Lambda_1 \neq \Lambda_2$ are required. Most easily, this is done using the radicand of $\Lambda_{1,2}^2 = \frac{1}{2}\left\{ \mathrm{tr}(\underline{C}\underline{G}^{-1}) \pm \sqrt{[\mathrm{tr}(\underline{C}\underline{G}^{-1})]^2 - 4\det(\underline{C}\underline{G}^{-1})} \right\}$. For the reason that $\underline{G}(x,y)=\underline{g}(x',y')=\underline{I}_2$ the condition $(\mathrm{tr}\,\underline{C})^2 - 4\det\underline{C} = 0 \Leftrightarrow \mathrm{tr}\,\underline{C} = 2\det\underline{J}$ must be satisfied. Because $\mathrm{tr}\,\underline{C} = \mathrm{tr}\begin{bmatrix} 10 & 7 \\ 7 & 41 \end{bmatrix} = 51 \neq 2\det\underline{J} = 38$ the transformation is not conformal.

As an alternative for this proof two orthogonal vectors $\underline{x}_1, \underline{x}_2$ (e.g. $\underline{x}_1 = [1,0]^T, \underline{x}_2 = [0,1]^T$) can be fed in the transformation equations showing that the transformed vectors $\underline{x}_1' = [3,-1]^T, \underline{x}_2' = [4,5]^T$ (which are the columns of $\underline{J}$) are not orthogonal afterwards. Quite the contrary, the dot product of the columns of $\underline{J}$ does not vanish but equals the off-diagonal element of $\underline{C}$, $C_{12}=7$. Even simpler is applying the Cauchy-Riemann differential equations, which are not satisfied here $\frac{\partial x'}{\partial x} = 3 \neq \frac{\partial y'}{\partial y} = 5, \frac{\partial x'}{\partial y} = 4 \neq -\frac{\partial y'}{\partial y} = 1$.

a) Conformality on $\Phi_1 = \pm 30°$ and $\Phi_2 = \pm 75°$ means that $\Lambda_1\big|_{\Phi=\Phi_1} = \Lambda_2\big|_{\Phi=\Phi_1}$ and $\Lambda_1\big|_{\Phi=\Phi_2} = \Lambda_2\big|_{\Phi=\Phi_2}$ must be satisfied. Jacobian matrix $\underline{J}$ and Cauchy-Green-deformation matrix $\underline{C}$ are

$\underline{J} = R\begin{bmatrix} 1 & 0 \\ 0 & a_0 + a_2\Phi^2 \end{bmatrix}$, $\underline{C} = \underline{J}'\underline{g}\underline{J} = \underline{J}'\underline{J} = R^2\begin{bmatrix} 1 & 0 \\ 0 & \left(a_0 + a_2\Phi^2\right)^2 \end{bmatrix}$. For the reason that both $\underline{G}$ and $\underline{C}$ are diagonal matrices eigendirections of Tissot ellipses are identical to the base vectors of the map and simplified rules can be applied to find distortions Λ_1 and Λ_2:

$$\Lambda_1 = \sqrt{\frac{C_{11}}{G_{11}}} = \frac{1}{\cos\Phi} \quad , \quad \Lambda_2 = \sqrt{\frac{C_{22}}{G_{22}}} = a_0 + a_2\Phi^2 .$$

No angular distortion (conformality) leads to the postulate

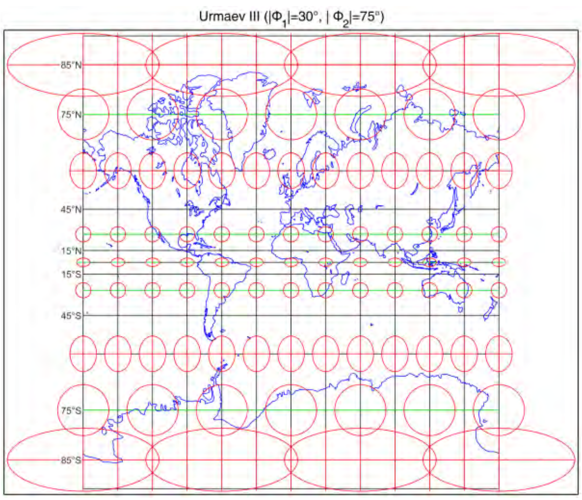
$$\Lambda_1 = \Lambda_2 \Leftrightarrow \frac{1}{\cos\Phi} = a_0 + a_2\Phi^2 \Leftrightarrow 1 = \cos\Phi\, a_0 + \Phi^2\cos\Phi\, a_2 = \begin{bmatrix} \cos\Phi & \Phi^2\cos\Phi \end{bmatrix}\begin{bmatrix} a_0 \\ a_2 \end{bmatrix}.$$

In order to compute unknown coefficients a_0, a_2 two fixed latitudes $\Phi_1 = \pm 30°$ and $\Phi_2 = \pm 75°$ are required (The two different signs do not have an influence on the coefficient matrix $\begin{bmatrix} \cos\Phi & \Phi^2\cos\Phi \end{bmatrix}$). The resulting 2×2 systems of equations

$$\begin{bmatrix} a_0 \\ a_2 \end{bmatrix} = \begin{bmatrix} \cos\Phi_1 & \Phi_1^2\cos\Phi_1 \\ \cos\Phi_2 & \Phi_2^2\cos\Phi_2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is easily solved: $a_0 = 0.6387$, $a_2 = 1.8821$

b) Because $\lim\limits_{|\Phi|\to 90°} \Lambda_2 = \lim\limits_{|\Phi|\to 90°} (a_0 + a_2\Phi^2)$ ("meridional direction") is bounded as compared to $\lim\limits_{|\Phi|\to 90°} \Lambda_1 = \lim\limits_{|\Phi|\to 90°} \dfrac{1}{\cos\Phi}$ ("parallel circle direction") distortion ellipses are oriented as shown.



c) Coefficients a_0, a_2 for an equivalent mapping are computed using the postulate

$$\Lambda_1\Lambda_2 = 1 \Leftrightarrow \cos\Phi = a_0 + a_2\Phi^2 = \begin{bmatrix} 1 & \Phi^2 \end{bmatrix}\begin{bmatrix} a_0 \\ a_2 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} a_0 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & \Phi_1^2 \\ 1 & \Phi_2^2 \end{bmatrix}^{-1}\begin{bmatrix} \cos\Phi_1 \\ \cos\Phi_2 \end{bmatrix} = \frac{1}{\Phi_2^2 - \Phi_1^2}\begin{bmatrix} \Phi_2^2 & -\Phi_1^2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \cos\Phi_1 \\ \cos\Phi_2 \end{bmatrix},$$ i.e.

$$a_0 = \frac{\Phi_2^2\cos\Phi_1 - \Phi_1^2\cos\Phi_2}{\Phi_2^2 - \Phi_1^2} = 0.9817 \, , \, a_2 = \frac{\cos\Phi_2 - \cos\Phi_1}{\Phi_2^2 - \Phi_1^2} = -0.4219$$

Starting from the general mapping equations $x = R\Lambda\cos\Phi_0$, $y = f(\Phi)$ Jacobian matrix $\underline{J}$, Cauchy-Green deformation matrix $\underline{C}$ and distortions are easily computed:

$$\underline{J} = \begin{bmatrix} R\cos\Phi_0 & 0 \\ 0 & f'(\Phi) \end{bmatrix}, \underline{C} = \underline{J}'\underline{g}\underline{J} = \underline{J}'\underline{J} = \begin{bmatrix} R^2\cos^2\Phi_0 & 0 \\ 0 & [f'(\Phi)]^2 \end{bmatrix}$$

and – since $\underline{G}$ is diagonal, too – simplified rules for the distortion ellipse semi axes are allowed:

$$\Lambda_1 = \frac{\cos\Phi_0}{\cos\Phi} \text{ (not required)}, \Lambda_2 = \frac{f'}{R}.$$

For the case of a mapping with equidistant meridians we have to ask for $\Lambda_2 = \dfrac{f'(\Phi)}{R} = 1 \quad \forall\Phi$ so that $f' = R \Rightarrow f = R\Phi + c$. Integration constant c is usually fixed from the additional requirement that the origin of the coordinate system is located on the x-axis, i.e. $y(\Phi = 0) = f(0) = 0 \Leftrightarrow c = 0$. Final mapping equation are therefore $x = R\Lambda\cos\Phi_0$, $y = R\Phi$.

a) $$\begin{bmatrix} X \\ Y \end{bmatrix} = \lambda \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \end{bmatrix}$$

b) $\begin{bmatrix} T_X & T_Y \end{bmatrix}'$ is the translation vector in order to make the origins of both systems coincide, α the rotation angle to align the coordinate axes and λ the ratio of the lengths of both base vector sets. Two control points, i.e. 4 coordinates are necessary in order to estimate the 4 parameters. Although the original model is non-linear in α and λ an alternative model can be set up: Setting $a := \lambda\cos\alpha$ and $b := \lambda\sin\alpha$ makes the model linear in a and b, and permits to rewrite it in the form $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & y & -x \end{bmatrix} \begin{bmatrix} T_X \\ T_Y \\ a \\ b \end{bmatrix}$. For two points therefore the system $\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_1 & y_1 \\ 0 & 1 & y_1 & -x_1 \\ 1 & 0 & x_2 & y_2 \\ 0 & 1 & y_2 & -x_2 \end{bmatrix} \begin{bmatrix} T_X \\ T_Y \\ a \\ b \end{bmatrix}$ with corresponding estimated parameters $\begin{bmatrix} \hat{T}_X & \hat{T}_Y & \hat{a} & \hat{b} \end{bmatrix}'$ is obtained. The estimates $\hat{\alpha}$ and $\hat{\lambda}$ of the original parameters are computed from the estimates $\hat{a}$ and $\hat{b}$ according to $\hat{\alpha} = \arctan\frac{\hat{b}}{\hat{a}}, \hat{\lambda} = \sqrt{\hat{a}^2 + \hat{b}^2}$. However this procedure – using an alternative model – is undesirable for the point of view of adjustment theory. Instead, proper linearization techniques should be used.

c) $$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \end{bmatrix}$$

d) $\begin{bmatrix} T_X & T_Y \end{bmatrix}'$ is the translation vector in order to make the origins of both systems coincide, a, …, d are affine parameters. Three control point, i.e. 6 coordinates are necessary in order to estimate the 6 parameters. As the model can be rewritten in the equivalent form

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 & x & y & 0 & 0 \\ 0 & 1 & 0 & 0 & x & y \end{bmatrix} \begin{bmatrix} T_X & T_Y & a & b & c & d \end{bmatrix}' \text{ or } \begin{bmatrix} X & Y \end{bmatrix} = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} T_X & T_Y \\ a & c \\ b & d \end{bmatrix}$$

the estimates of the six transformation parameters are easily computed from a set of three control points: $\begin{bmatrix} \hat{T}_X & \hat{T}_Y \\ \hat{a} & \hat{c} \\ \hat{b} & \hat{d} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} X_1 & Y_1 \\ X_2 & Y_2 \\ X_3 & Y_3 \end{bmatrix}$. In case of more control points adjustment techniques must be applied.

Solution identical with solution SS 2010, Problem 4

a) The general mapping equations are $\alpha = n\Lambda,\ r = f(\Phi),\ n = \sin\Phi_0$ and $x = r\cos\alpha = r\cos(n\Lambda),\ y = r\sin\alpha = r\sin(n\Lambda)$, respectively.

b) The only unknown quantity in the problem is the polar distance r which can be easily derived from the picture: Point P on the sphere will be mapped onto point p on the chart, i.e. $\sin\Delta = \cos\Phi = \frac{u_P}{R}$ and $\sin\Phi_0 = \frac{u_P}{r}$ so that $r = \frac{R\cos\Phi}{\sin\Phi_0} = \frac{R\cos\Phi}{n}$. Thus the final mapping equations are $x = \frac{R}{n}\cos\Phi\cos(n\Lambda),\ y = \frac{R}{n}\cos\Phi\sin(n\Lambda)$.

c) Metric matrix of the source (sphere): $\underline{G} = R^2\mathrm{diag}(\cos^2\Phi, 1)$

Method 1: Use α,r as parameters of the chart $\rightarrow$ $\underline{g} = \mathrm{diag}(r^2, 1) = \mathrm{diag}(R^2\cos^2\Phi / n^2, 1)$

$$\underline{J} = \begin{bmatrix} \partial\alpha/\partial\Lambda & \partial\alpha/\partial\Phi \\ \partial r/\partial\Lambda & \partial r/\partial\Phi \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & \frac{\partial r}{\partial\Phi} \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & \frac{-R\sin\Phi}{n} \end{bmatrix}$$

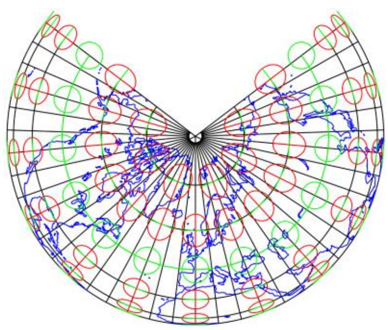
$$\underline{C} = \underline{J}'\underline{g}\underline{J} = \begin{bmatrix} r^2n^2 & 0 \\ 0 & \frac{R^2\sin^2\Phi}{n^2} \end{bmatrix} = R^2\begin{bmatrix} \cos^2\Phi & 0 \\ 0 & \frac{\sin^2\Phi}{n^2} \end{bmatrix} \Rightarrow \Lambda_1^2 = \frac{C_{11}}{G_{11}} = 1,\ \Lambda_2^2 = \frac{C_{22}}{G_{22}} = \frac{\sin^2\Phi}{\sin^2\Phi_0}$$

Method 2: Use x,y as parameters of the chart $\rightarrow$ $\underline{g} = \underline{I}_2$

$$\underline{J} = \begin{bmatrix} \partial x/\partial\Lambda & \partial x/\partial\Phi \\ \partial y/\partial\Lambda & \partial y/\partial\Phi \end{bmatrix} = \begin{bmatrix} -rn\sin(n\Lambda) & \cos\alpha\frac{\partial r}{\partial\Phi} \\ rn\cos(n\Lambda) & \sin\alpha\frac{\partial r}{\partial\Phi} \end{bmatrix} = R\begin{bmatrix} -\cos\Phi\sin(n\Lambda) & -\cos\alpha\frac{\sin\Phi}{n} \\ \cos\Phi\cos(n\Lambda) & -\sin\alpha\frac{\sin\Phi}{n} \end{bmatrix}$$

$$\underline{C} = \underline{J}'\underline{g}\underline{J} = \begin{bmatrix} r^2n^2 & 0 \\ 0 & \frac{R^2\sin^2\Phi}{n^2} \end{bmatrix} = R^2\begin{bmatrix} \cos^2\Phi & 0 \\ 0 & \frac{\sin^2\Phi}{n^2} \end{bmatrix} \Rightarrow \Lambda_1^2 = \frac{C_{11}}{G_{11}} = 1,\ \Lambda_2^2 = \frac{C_{22}}{G_{22}} = \frac{\sin^2\Phi}{\sin^2\Phi_0}$$

d) Semi axes of Tissot distortion ellipses show in direction of parameter line images, because both $\underline{G}$ and $\underline{C}$ are diagonal. Since $\Lambda_1 = 1$ we have an equidistant mapping with undistorted parallel circles: Radii of distortion ellipses equal one in direction of the parallel circles. On the parallel circle $\Phi = \Phi_0$ we have even isometry: $\Lambda_1(\Phi_0) = \Lambda_2(\Phi_0) = 1$. Starting on the circle of contact, $\Phi = \Phi_0$, and going north distortion ellipses experience an elongation while going south they are extremely compressed and become "equatorial tangents" at the equator. Assuming $\Phi_0 = 45°$ then meridional distortion on the pole is $\Lambda_2 = \sqrt{2}$.



a) Step 1: Express the position vector of a point P on surface S_1, $\mathbf{X}_P = X\mathbf{E}_1 + Y\mathbf{E}_2 + Z\mathbf{E}_3$ in terms of its parameters U and V, i.e. $X(U,V)$, $Y(U,V)$, $Z(U,V)$

Step 2: Express the position vector of its image point p on surface S_2, $\mathbf{x}_p = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ in terms of its parameters u and v, i.e. $x(u,v)$, $y(u,v)$, $z(u,v)$

Step 3: Complete the mapping procedure by specifying the mapping equations, i.e. express u and v in terms of U and V: $u = u(U,V)$, $v = v(U,V)$

Step 4: For the deformation analysis, compute the Gauss tangent vectors $\mathbf{G}_U = \partial\mathbf{X}/\partial U$, $\mathbf{G}_V = \partial\mathbf{X}/\partial V$ of surface A and $\mathbf{g}_U = \partial\mathbf{x}/\partial u, \mathbf{g}_V = \partial\mathbf{x}/\partial v$ of surface B, and from this the metric matrices $\underline{G} = \begin{bmatrix} \langle \mathbf{G}_U | \mathbf{G}_U \rangle & \langle \mathbf{G}_U | \mathbf{G}_V \rangle \\ \langle \mathbf{G}_V | \mathbf{G}_U \rangle & \langle \mathbf{G}_V | \mathbf{G}_V \rangle \end{bmatrix}$ and

$\underline{g} = \begin{bmatrix} \langle \mathbf{g}_u | \mathbf{g}_u \rangle & \langle \mathbf{g}_u | \mathbf{g}_v \rangle \\ \langle \mathbf{g}_v | \mathbf{g}_u \rangle & \langle \mathbf{g}_v | \mathbf{g}_v \rangle \end{bmatrix}$

Step 5: Compute the Jacobian matrix $\underline{J} = \begin{bmatrix} \partial u/\partial U & \partial u/\partial V \\ \partial v/\partial U & \partial v/\partial V \end{bmatrix}$ and the Cauchy-Green deformation matrix $\underline{C} = \underline{J}^T \underline{g}\ \underline{J}$

Step 6: The solution of the general eigenvalue problem $(\underline{C} - \Lambda_i^2 \underline{G})\underline{F}_i = 0, i = 1,2$ leads to eigenvalues Λ_1^2, Λ_2^2 and eigenvectors $\underline{F}_1, \underline{F}_2$ which are a measure for the map distortions.

b) In terms of the eigenvalues Λ_1^2, Λ_2^2, conformality means $\Lambda_1^2 = \Lambda_2^2$ (distortion circles), equivalence $\Lambda_1^2 = \Lambda_2^{-2}$ and equidistance $\Lambda_1^2 = 1\ \forall \Lambda_2^2$ or $\Lambda_2^2 = 1\ \forall \Lambda_1^2$

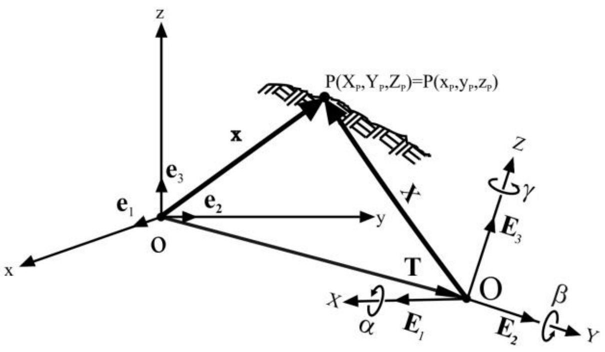
c) The Gauss tangent vectors for the elliptic cone are

$$\mathbf{G}_U = \partial\mathbf{X}/\partial U = -AV\sin U\mathbf{E}_1 + BV\cos U\mathbf{E}_2 + 0\mathbf{E}_3$$
$$\mathbf{G}_V = \partial\mathbf{X}/\partial V = A\cos U\mathbf{E}_1 + B\sin U\mathbf{E}_2 + C\mathbf{E}_3$$

For the reason that the scalar product (dot product) $\langle \mathbf{G}_U | \mathbf{G}_V \rangle = (B^2 - A^2)V\sin U\cos U$ does not vanish, in general, the parameter lines U,V do not intersect by 90°.

a) $\mathbf{X} \mapsto \mathbf{x} = \lambda \underline{R}\mathbf{X} + \mathbf{T}$, $\underline{R} := \underline{R}_3(\gamma)\underline{R}_2(\beta)\underline{R}_1(\alpha)$

b)



λ is the scale difference (better: the ratio of the scales in both systems), $\mathbf{T}$ the translation vector between the two origins of both sets of base vectors, which always points from the origin of the target system (here: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) to the origin of the start system (here: $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$). This is because – in case of λ=1, $\underline{R}=\underline{I}_3$ – the vectorial sum of $\mathbf{X}$ and $\mathbf{T}$ builds $\mathbf{x}$. $\underline{R}$ is the 3×3 overall rotation matrix. It consists of the above sequence of elementary rotation matrices using

$$\underline{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}, \underline{R}_2(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \underline{R}_3(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation angles α,β,γ are reckoned mathematically positive when measured counterclockwise as seen from the tip of the corresponding axis towards the origin. They are needed in order to make the base vector sets parallel.

c) Transformation close to the identity means that $\lambda \approx 1, \alpha \approx \beta \approx \gamma \approx 0 \Rightarrow \underline{R} = \underline{I}_3 + \delta\underline{R} \approx \underline{I}_3$

a) The relevance of Gauß-Krüger-/UTM-coordinates comes from the fact that they are conformal coordinates. Therefore the map distortions are isotropic, i.e. they do not depend on the viewing direction. Moreover the entire globe can be represented using a system of ellipsoidal strips thus limiting the distortions at the strip boundaries.

b) A "Taylor point" $P_0(L_0,B_0)$ is chosen on a meridian with integer longitude ("reference meridian") in the vicinity of the point P(L,B). This guarantees that the chosen conformal approximation $x(L-L_0, B-B_0)$, $y(L-L_0, B-B_0)$ for the conformal bivariate homogeneous polynomials of degree n converge; differences in latitude $B-B_0$ and longitude $L-L_0$ must not exceed a certain limit. For the reason that both longitude L_0 and latitude B_0 can be fixed more or less arbitrarily within the radius of convergence x and y are not unique. Uniqueness is achieved using the following rules: In North-South-direction add to $x(L-L_0, B-B_0)$ the meridional arc length of P_0 thus shifting the intermediate origin of the local coordinate system from P_0 to the intersection of the meridian through P_0 and the equator. In East-West-direction add a constant (usually 500 000 m) to $y(L-L_0, B-B_0)$ in order to avoid negative coordinates for points west of P_0 → False Eastings. Then put the code number $L_0/3°$ in front of the resulting number (or add $L_0/3°\times10^6$ to it).

c) The difference is first in the scale factor of the reference meridian (Gauß-Krüger: 1, UTM: 0,9996). Strip width is different (Gauß-Krüger: typically 3°, UTM: 6°). False Eastings do not have the code number (as mentioned in b)) in front but are equipped with a zone number instead. Zone number is also a function of reference meridian L_0.

a) $\underline{G} = R\begin{bmatrix} \cos^2\Phi & 0 \\ 0 & 1 \end{bmatrix}$

b) $\underline{g} = \underline{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c)

$$\underline{J} = \begin{bmatrix} \frac{\partial x}{\partial \Lambda} & \frac{\partial x}{\partial \Phi} \\ \frac{\partial y}{\partial \Lambda} & \frac{\partial y}{\partial \Phi} \end{bmatrix} = R\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)\frac{1}{\cos^2\frac{\Phi}{2}} \end{bmatrix} \Rightarrow$$

$$\underline{C} = \underline{J}'\underline{g}\underline{J} = \underline{J}'\underline{J} = \frac{R^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)^2\frac{1}{\cos^4\frac{\Phi}{2}} \end{bmatrix}$$

d) The principal distortions (principal stretches) are in direction of the parameter line images because both $\underline{G}$ and $\underline{C}$ are diagonal matrices.

e) The mapping equation do not generate an equal-area mapping because $\Lambda_1 = \sqrt{\frac{C_{11}}{G_{11}}} = \frac{1}{\sqrt{2}\cos\Phi} \neq \frac{1}{\Lambda_2} = \sqrt{\frac{G_{22}}{C_{22}}} = \frac{2\cos^2\frac{\Phi}{2}}{1+\frac{1}{\sqrt{2}}}$ or – using the determinant rules for matrix products – $\det\underline{J} = R^2\frac{1}{2\sqrt{2}}\left(1+\frac{1}{\sqrt{2}}\right)\frac{1}{\cos^2\frac{\Phi}{2}} \neq \sqrt{\det\underline{G}} = R^2\cos\Phi$

f) For distortion-free parameter line images $\Lambda_1 = \Lambda_2 = 1$ is required. This is satisfied for $\Phi = \pm 45°$, only.

a) First, the nonlinear model equation is transformed into a linear equation using new variables a,b,c and d.

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} + \begin{bmatrix} t_u \\ t_v \end{bmatrix},$$

with $a := m_u \cos\beta,\ c := m_u \sin\beta,\ b := -m_v \sin\alpha,\ d := m_v \cos\alpha$. Second, the linear equation is reformulated in order to obtain a matrix-like system of equation. This is possible because the two sets of unknowns (set 1: a,b,t_u, set 2: c,d,t_v) are independent of each other and will reduce the size of the involved matrices

$$[u, v] = [1, U, V] \begin{bmatrix} t_u & t_v \\ a & c \\ b & d \end{bmatrix}.$$

Thus, for three control points the system of equations

$$\underbrace{\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix}}_{3\times 2} = \underbrace{\begin{bmatrix} 1 & U_1 & V_1 \\ 1 & U_2 & V_2 \\ 1 & U_3 & V_3 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} t_u & t_v \\ a & c \\ b & d \end{bmatrix}}_{3\times 2} \sim \underline{\ell} = \underline{A}\underline{\xi}$$

is achieved. From its solution

$$\begin{bmatrix} \hat{t}_u & \hat{t}_v \\ \hat{a} & \hat{c} \\ \hat{b} & \hat{d} \end{bmatrix} = \begin{bmatrix} 1 & U_1 & V_1 \\ 1 & U_2 & V_2 \\ 1 & U_3 & V_3 \end{bmatrix}^{-1} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix}$$

scale factors and rotation angles are determined

$$\hat{\beta} = \arctan\frac{\hat{c}}{\hat{a}},\ \hat{\alpha} = \arctan\frac{-\hat{b}}{\hat{d}},\ \hat{m}_u = \sqrt{\hat{a}^2 + \hat{c}^2},\ \hat{m}_v = \sqrt{\hat{b}^2 + \hat{d}^2}.$$

b) The inverse coefficient matrix $\underline{A}^{-1}$ is easily computed:

$$\begin{bmatrix} 1 & 7 & 5 \\ 1 & 5 & 7 \\ 1 & 4 & 4 \end{bmatrix}^{-1} = \frac{1}{8} \begin{bmatrix} -8 & -8 & 24 \\ 3 & -1 & -2 \\ -1 & 3 & -2 \end{bmatrix}.$$

Finally, multiplication by the matrix $\underline{\ell}$ gives

$$\begin{bmatrix} \hat{t}_u & \hat{t}_v \\ \hat{a} & \hat{c} \\ \hat{b} & \hat{d} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -8 & -8 & 24 \\ 3 & -1 & -2 \\ -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ 5 & 1 \\ 6 & -2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 48 & -48 \\ 4 & 0 \\ -4 & 8 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$$

Therefore, numerical values for scale factors and rotation angles are

$$\hat{\beta} = \arctan\frac{\hat{c}}{\hat{a}} = \arctan(0) = 0°,\ \hat{\alpha} = \arctan\frac{-\hat{b}}{\hat{d}} = \arctan(0.5) = 26°.565\,051\,177\,08$$

$$\hat{m}_u = \sqrt{\hat{a}^2 + \hat{c}^2} = 0.5,\ \hat{m}_v = \sqrt{\hat{b}^2 + \hat{d}^2} = \frac{\sqrt{5}}{2}$$

c) Missing coordinates u_4,v_4 of point 4 are derived from U_4,V_4 using the model equations: $\hat{u}_4 = 6$, $\hat{v}_4 = 0$

a) The "Plate Carrée map" or "isoparametric mapping" is the simplest cylindrical mapping and has the mapping equations $x = R\Lambda$, $y = R\Phi$. Since $\Lambda \in [-\pi, \pi]$, $\Phi \in [-\pi/2, \pi/2]$ the North-South and East-West-extension, respectively, of the map is $\Delta y = R(\Phi_{North} - \Phi_{South}) = \pi R$ and $\Delta x = R(\Lambda_{East} - \Lambda_{West}) = 2\pi R$, respectively. The width-to-height ratio of the map should thus be $\Delta x / \Delta y = 2$ which is not true, however. Therefore the map cannot be a "Plate Carrée map".

b) Using a "Plate Carrée map" for the determination of areas or area ratios is not suitable because it is not an area preserving mapping. This can be seen from the fact that $\det \underline{J} = R^2 \neq \sqrt{\det \underline{G}} = R^2 \cos\Phi$ or $\Lambda_1 = \dfrac{1}{\cos\Phi} \neq \Lambda_2^{-1} = 1$.

c) The general mapping equations for normal cylindrical mappings are given by the formulas $x = R\Lambda$, $y = Rf(\Phi)$. From this, the extremal distortions Λ_1, Λ_2 can be easily derived: $\Lambda_1 = \sqrt{\dfrac{C_{11}}{G_{11}}} = \dfrac{1}{\cos\Phi}, \Lambda_2 = \sqrt{\dfrac{C_{22}}{G_{22}}} = f' = \dfrac{df}{d\Phi}$. By reason of both $\underline{G}$ und $\underline{C}$ being diagonal matrices extremal distortions are along the images of the parameter lines. The equivalence postulate leads to the requirement $\Lambda_1\Lambda_2 = 1 \Leftrightarrow f' = \cos\Phi \Leftrightarrow f = \sin\Phi + c$. In order to achieve y(0)=0 the integration constant c is set to zero, as usual. The final set of mapping equation for our scientific problem of finding the continental-to-ocean area ration is therefore $x = R\Lambda$, $y = R\sin\Phi$. This is nothing else but the cylindrical Lambert equal area mapping of the sphere. Of course, the same result is achieved from the postulate $\det \underline{J} = R^2 f'(\Phi) = \sqrt{\det \underline{G}} = R^2 \cos\Phi \Leftrightarrow f'(\Phi) = \cos\Phi \Leftrightarrow f(\Phi) = \sin\Phi + c$.

a) For the reason that both systems have different origins, different scales and different axes orientation two reasonable proposals are

a_1) the four-parameter (similarity) transformation model (1 scale, 1 rotation, 2 translations)

$$\begin{bmatrix} v \\ u \end{bmatrix} = m \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} + \begin{bmatrix} T_y \\ T_x \end{bmatrix}$$

a_2) the six-parameter (affine) transformation model (4 affine parameters, 2 translations)

$$\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} + \begin{bmatrix} T_y \\ T_x \end{bmatrix}$$

b) $$\underline{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}, \underline{R}_2(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \underline{R}_3(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mathematically positive means that angles are measured/counted counter clockwise as seen from the tip/end of the axis.

c) From "False Easting" E = 729 741.67 m, "False Northing" N = 3 128 844.77 m and Zone = 43, we can first easily determine the reference meridian $L_0 = 6°(\text{Zone} - 30) - 3° = 75°$. Second the distance from the equator is roughly 3 000 km to the North, i.e. 1/3 of a quadrant which makes 30° North. There is only one point in that region: New Delhi.

The exact meaning of the numbers is: The point is located 229 741.67 m East from its reference meridian with longitude $L_0 = 75°$ and 3 128 844.77 m North from the equator.

a) Jacobian matrix $\underline{J}$ and Cauchy-Green-Tensor $\underline{C} = \underline{J}^T \underline{g}\ \underline{J}$, $\underline{g} = \underline{I}_2$

$$\underline{J} = R \begin{bmatrix} \dfrac{\cos\Phi}{\beta+\alpha\cos\Phi} & \dfrac{-\beta\Lambda\sin\Phi}{(\beta+\alpha\cos\Phi)^2} \\ 0 & \beta+\alpha\cos\Phi \end{bmatrix} = \frac{R}{(\beta+\alpha\cos\Phi)^2} \begin{bmatrix} \cos\Phi(\beta+\alpha\cos\Phi) & -\beta\Lambda\sin\Phi \\ 0 & (\beta+\alpha\cos\Phi)^3 \end{bmatrix},$$

$$\underline{C} = \frac{R^2}{(\beta+\alpha\cos\Phi)^4} \begin{bmatrix} \cos^2\Phi(\beta+\alpha\cos\Phi)^2 & -\beta\Lambda\sin\Phi\cos\Phi(\beta+\alpha\cos\Phi) \\ -\beta\Lambda\sin\Phi\cos\Phi(\beta+\alpha\cos\Phi) & \beta^2\Lambda^2\sin^2\Phi+(\beta+\alpha\cos\Phi)^6 \end{bmatrix}$$

so that extremal distortions appear only along the parameter lines if $\beta = 0$ ("Lambert cylindrical mapping"), since $C_{12}=C_{21}=G_{12}=G_{21}=0$ in that case. Alternatively one can use the argument that if $\underline{g}=\underline{I}_2$ and $x = x(\Lambda), y = y(\Phi)$ or $x = x(\Phi), y = y(\Lambda)$ the term

$$C_{12} = C_{21} = J_{11}J_{12} + J_{21}J_{22} = \frac{\partial x}{\partial\Lambda}\frac{\partial x}{\partial\Phi} + \frac{\partial y}{\partial\Lambda}\frac{\partial y}{\partial\Phi} \text{ vanishes.}$$

b) The mapping is of equal-area type if – with given $\underline{G} = R^2 \operatorname{diag}(\cos^2\Phi, 1)$ –

$$\Lambda_1\Lambda_2 = \det(\underline{C}\,\underline{G}^{-1}) = \frac{\det\underline{C}}{\det\underline{G}} = \frac{\det(\underline{J}'\underline{J})}{\det\underline{G}} = \frac{(\det\underline{J})^2}{\det\underline{G}} = 1 \Leftrightarrow \det\underline{J} = \sqrt{\det\underline{G}} = R^2\cos\Phi$$

holds true. Indeed $\det\underline{J} = R^2\cos\Phi$.

c) Conformality is achieved if

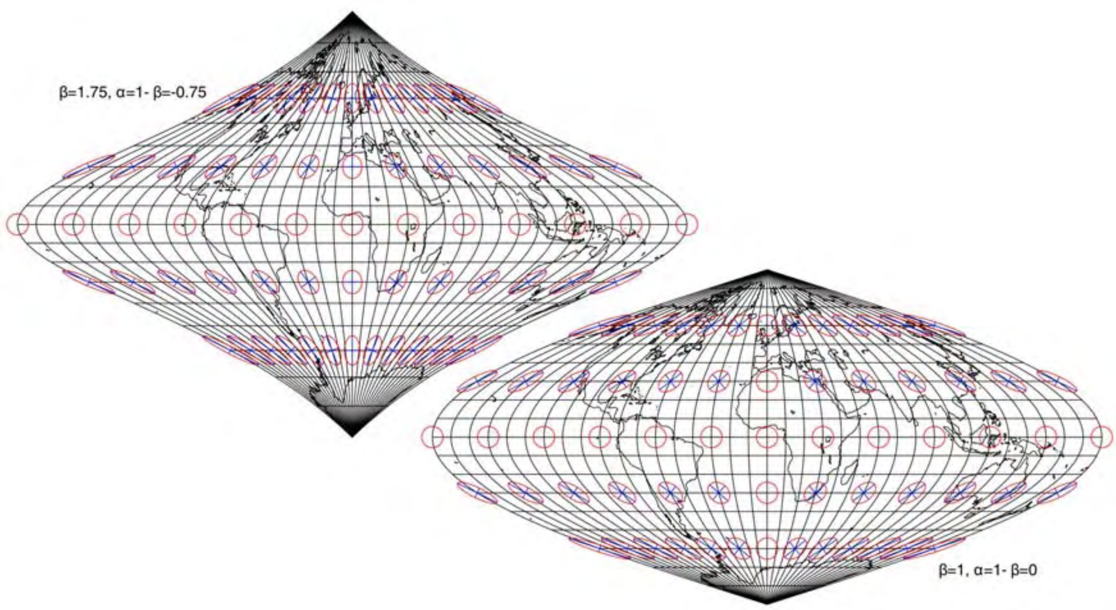
$$\Lambda_1 = \Lambda_2 \Leftrightarrow \left[\operatorname{tr}(\underline{C}\,\underline{G}^{-1})\right]^2 - 4\det(\underline{C}\,\underline{G}^{-1}) = 0$$

$$\overset{\det(\underline{C}\,\underline{G}^{-1})=1}{\Leftrightarrow} \operatorname{tr}(\underline{C}\,\underline{G}^{-1}) = \frac{(\beta+\alpha\cos\Phi)^2+\beta^2\Lambda^2\sin^2\Phi+(\beta+\alpha\cos\Phi)^6}{(\beta+\alpha\cos\Phi)^4} \overset{!}{=} 2$$

is fulfilled.

(c_1) For the case $\alpha = 1-\beta$ this is true on the equator $\Phi = 0, \forall\,\Lambda$ only and

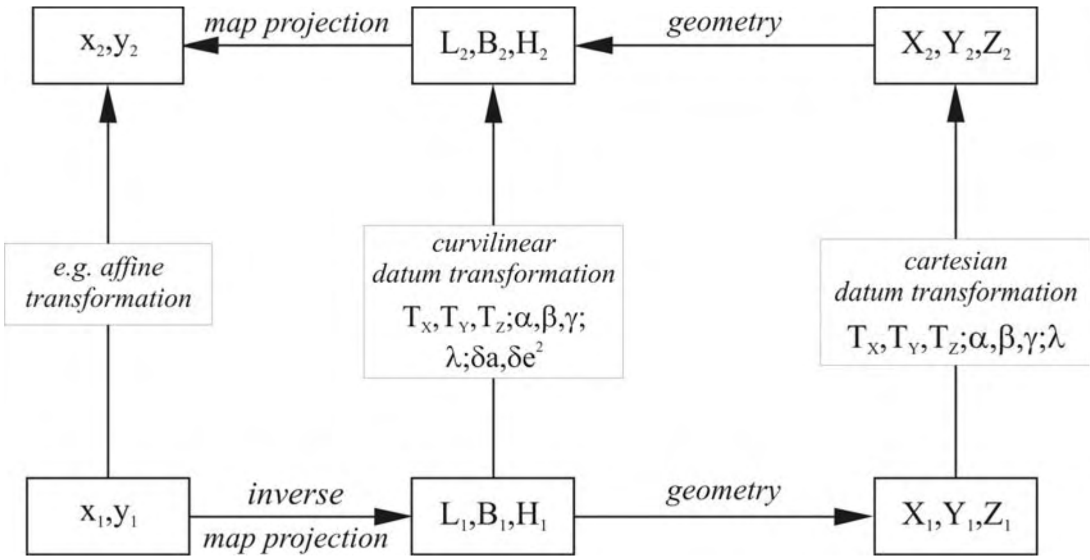
(c_2) for the case $\alpha = 0$, $\beta = 1$ on the central meridian $\Lambda = 0, \forall\,\Phi$, in addition.



a) Conformal coordinates in the sense of Gauß-Krüger/UTM-coordinates (transverse conformal cylindrical mapping) are based on the (necessary) Cauchy-Riemann differential equations $\partial x/\partial L = \partial y/\partial Q$, $\partial x/\partial Q = -\partial y/\partial L$ and the (sufficient) Laplace differential equations (as derivatives of the Cauchy-Riemann differential equations) between two sets (x,y) and (L,Q) of isometric coordinates; (non isometric) ellipsoidal latitude B is transformed to isometric latitude Q beforehand. As "Ansatz" two homogeneous bivariate polynomials $x(L-L_0=\ell, Q-Q_0=q)$ and $y(L-L_0=\ell, Q-Q_0=q)$ of order n are chosen. The point L_0,Q_0 is a – more or less arbitrarily – chosen origin ("Taylor point") to which conformal coordinates x,y refer. It is a point chosen not too far away from the points to be mapped in order to guarantee convergence of the polynomials. In most practical cases L_0 is an integer value. The coefficients of the homogeneous bivariate polynomials are determined from (i) the requirements of conformality ("Cauchy-Riemann differential equations") and (ii) integrability ("Laplace differential equations"), and (iii) the postulate of an equidistant reference meridian. Before the coefficients are evaluated at the (ellipsoidal latitude B_0 of the) Taylor point a back transformation from isometric to ellipsoidal latitude is necessary.
Gauß-Krüger-coordinate "Northing" is then generated from conformal coordinate x by adding the meridional arc length from the equator to the latitude $B_0(Q_0)$ of the Taylor point. Northing describes the metrical distance of a point from the equator of the underlying ellipsoid-of-revolution. Gauß-Krüger-coordinate "False Easting" is computed from conformal coordinate y by (i) adding a constant (mostly 500 km) to avoid negative numbers for points westerly of L_0 and (ii) computing a reference number from L_0 which is put in front of the number. Thus the arbitrariness of the initial origin is removed.
UTM coordinates are generated from conformal coordinates using identical bivariate polynomials except that both are scaled by an (conventional or optimally chosen) scale factor before. Therefore two meridians west and east of the reference meridian are mapped equidistantly. The reference number computed from L_0 is replaced by a zone number which completes the "False Eastings".

b) Point P is located 34,012 45 km east of the reference meridian with longitude $L_0 = 6° \times (\text{zone number} - 30) - 3° = 15°$ and 4989,99134 km north of the equator.

c)



d) Translation parameters T_X, T_Y, T_Z Rotation parameters: α, β, γ Scale parameter: λ

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \lambda \underline{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} T_X \\ T_Y \\ T_Z \end{pmatrix}, \quad \underline{R} = \underline{R}_3(\gamma)\underline{R}_2(\beta)\underline{R}_1(\alpha)$$

$$\underline{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}, \underline{R}_2(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \underline{R}_3(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) The changed mapping equations are

$$\begin{bmatrix} x \\ y \end{bmatrix} = 2Rf(\Phi_0)\tan\left(\frac{\pi}{4}-\frac{\Phi}{2}\right)\begin{bmatrix} \cos\Lambda \\ \sin\Lambda \end{bmatrix},$$

and the unknown function $f(\Phi_0)$ must be derived from the postulate of an equidistant mapping of the parallel circle $\Phi = \Phi_0$. Multiplying the mapping equations with an arbitrary function f results in a multiplication of the given principal distortions with that function: $\Lambda_1 = \Lambda_2 = \dfrac{f}{\cos^2\left(\frac{\pi}{4}-\frac{\Phi}{2}\right)}$. An equidistant mapping of the parallel $\Phi = \Phi_0$ is achieved by finding the function from the postulate

$$\Lambda_1(\Phi = \Phi_0) = \Lambda_2(\Phi = \Phi_0) = \frac{f(\Phi_0)}{\cos^2\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right)} = 1$$

$$\Leftrightarrow \qquad .$$

$$f(\Phi_0) = \cos^2\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right) = \frac{1}{2}(1+\sin\Phi_0) = \frac{\cos\Phi_0}{2\tan\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right)}$$

However, more easily $f(\Phi_0)$ can be found from the fact that in case of an equidistant mapping the image of the parallel $\Phi = \Phi_0$ must have a radius $r = \sqrt{\left(x|_{\Phi=\Phi_0}\right)^2 + \left(y|_{\Phi=\Phi_0}\right)^2} = 2Rf(\Phi_0)\tan\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right) = R\cos\Phi_0$ in the map. We therefore end up with the modified mapping equations

$$\begin{bmatrix} x \\ y \end{bmatrix} = R(1+\sin\Phi_0)\tan\left(\frac{\pi}{4}-\frac{\Phi}{2}\right)\begin{bmatrix} \cos\Lambda \\ \sin\Lambda \end{bmatrix} = 2R\cos^2\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right)\tan\left(\frac{\pi}{4}-\frac{\Phi}{2}\right)\begin{bmatrix} \cos\Lambda \\ \sin\Lambda \end{bmatrix}.$$

b) The plane (which is parallel to the equator) intersects the sphere at $\Phi = \Phi_0$: It is not a tangential plane.

c) The image of the parallel circle $\Phi=\Phi_0$ has clearly a radius

$$r = \sqrt{\left(x|_{\Phi=\Phi_0}\right)^2 + \left(y|_{\Phi=\Phi_0}\right)^2} = 2R\cos^2\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right)\tan\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right) =$$

$$= 2R\sin\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right) = R\sin\left[2\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right)\right] =$$

$$= R\sin\left(\frac{\pi}{2}-\Phi_0\right) = R\cos\Phi_0$$

a) First, the model equations – which are non-linear in the unknowns m_x, m_y, μ, ω, c_1 and c_2 – are transformed into linear equations by introducing new variables a, b, d, e with $a := m_x \cos\mu$, $d := m_x \sin\mu$, $b := -m_y \sin\omega$, $e := m_y \cos\omega$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Next, the equations are re-arranged to

$$[x, y] = [1, X, Y] \begin{bmatrix} c_1 & c_2 \\ a & d \\ b & e \end{bmatrix}.$$

We end up with a matrix equation which can easily be solved by inversion of a 3×3 matrix.

$$\underbrace{\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}}_{3\times 2} = \underbrace{\begin{bmatrix} 1 & X_1 & Y_1 \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} c_1 & c_2 \\ a & d \\ b & e \end{bmatrix}}_{3\times 2} \sim \underline{\ell} = \underline{A}\underline{\xi},$$

The solution is

$$\begin{bmatrix} \hat{c}_1 & \hat{c}_2 \\ \hat{a} & \hat{d} \\ \hat{b} & \hat{e} \end{bmatrix} = \begin{bmatrix} 1 & X_1 & Y_1 \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{bmatrix}^{-1} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} \sim \underline{\hat{\xi}} = \underline{A}^{-1}\underline{\ell}$$

and the original unknown parameters scales and rotation angles are computed from

$$\hat{\mu} = \arctan\frac{\hat{d}}{\hat{a}},\ \hat{\omega} = \arctan\frac{-\hat{b}}{\hat{e}},\ \hat{m}_x = \sqrt{\hat{a}^2 + \hat{d}^2},\ \hat{m}_y = \sqrt{\hat{b}^2 + \hat{e}^2}\ .$$

b) The inverse coefficient matrix $\underline{A}^{-1}$ is easily computed using elementary operations such as the inversion by subdeterminants:

$$\begin{bmatrix} 1 & -1 & 7 \\ 1 & 1 & 5 \\ 1 & -2 & 6 \end{bmatrix}^{-1} = \frac{1}{6-5-7\times 2-(-5\times 2-6+7)} \begin{bmatrix} 6+5\times 2 & -(6-5) & -2-1 \\ -(-6+14) & 6-7 & -(-2+1) \\ -5-7 & -(5-7) & 2 \end{bmatrix}^T =$$

$$= \frac{1}{-13+9} \begin{bmatrix} 16 & -8 & -12 \\ -1 & -1 & 2 \\ -3 & 1 & 2 \end{bmatrix} =$$

$$= -\frac{1}{4} \begin{bmatrix} 16 & -8 & -12 \\ -1 & -1 & 2 \\ -3 & 1 & 2 \end{bmatrix}.$$

Right multiplication with $\underline{\ell}$ results in

$$\begin{bmatrix} \hat{c}_1 & \hat{c}_2 \\ \hat{a} & \hat{d} \\ \hat{b} & \hat{e} \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 16 & -8 & -12 \\ -1 & -1 & 2 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 7 & 5 \\ 4 & 4 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -24 & 24 \\ -4 & -4 \\ 0 & -8 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 1 & 1 \\ 0 & 2 \end{bmatrix},$$

and scale factors and rotations angles can be found to be

$$\hat{\mu} = \arctan\frac{\hat{d}}{\hat{a}} = \arctan(1) = 45°, \hat{\omega} = \arctan\frac{-\hat{b}}{\hat{e}} = \arctan(0) = 0°$$

$$\hat{m}_x = \sqrt{\hat{a}^2 + \hat{d}^2} = \sqrt{2}, \hat{m}_y = \sqrt{\hat{b}^2 + \hat{e}^2} = 2.$$

c) Missing coordinates X_4,Y_4 of point 4 are computed from inversion of the model equations:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1}\begin{bmatrix} x - c_1 \\ y - c_2 \end{bmatrix}$$

$$\begin{bmatrix} \hat{X}_4 \\ \hat{Y}_4 \end{bmatrix} = \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{d} & \hat{e} \end{bmatrix}^{-1}\begin{bmatrix} x_4 - \hat{c}_1 \\ y_4 - \hat{c}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$