Universität Stuttgart

Prof.Dr.
Thomas Hobiger

Dynamic System Estimation

Comparison between Kalman filter and Sequential Least Squares estimation

10

Kalman filter vs. sequential LSQ

The table below compares the two algorithms

Kalman filter	sequential LSQ
state estimation • prediction • correction/update	Est. of parameters, constant in time
$\hat{\boldsymbol{x}}_{n\|n} = \hat{\boldsymbol{x}}_{n\|n-1} + \boldsymbol{K}_n \left(\boldsymbol{z}_n - \boldsymbol{H}_n \hat{\boldsymbol{x}}_{n\|n-1}\right)$ $\boldsymbol{K}_n = \boldsymbol{P}_{n\|n-1} \boldsymbol{H}_n^T \left(\boldsymbol{H}_n \boldsymbol{P}_{n\|n-1} \boldsymbol{H}_n^T + \boldsymbol{R}_n\right)^{-1}$ $\boldsymbol{P}_{n\|n} = \left(\boldsymbol{I} - \boldsymbol{K}_n \boldsymbol{H}_n\right) \boldsymbol{P}_{n\|n-1}$	$\hat{\boldsymbol{x}}_n = \hat{\boldsymbol{x}}_{n-1} + \left[\hat{\sigma}^2_{0n-1} \boldsymbol{\Sigma}(\hat{\boldsymbol{x}}_{n-1})^{-1} + \boldsymbol{A}_{\boldsymbol{n}}^T \boldsymbol{P}_{\boldsymbol{n}} \boldsymbol{A}_{\boldsymbol{n}}\right]^{-1} \cdot \boldsymbol{A}_{\boldsymbol{n}}^T \boldsymbol{P}_{\boldsymbol{n}} \left[\boldsymbol{y}_{\boldsymbol{n}} - \boldsymbol{A}_{\boldsymbol{n}} \hat{\boldsymbol{x}}_{n-1}\right]$ $\boldsymbol{\Sigma}(\hat{\boldsymbol{x}}_n) = \hat{\sigma}^2_{0n} \left[\hat{\sigma}^2_{0n-1} \boldsymbol{\Sigma}(\hat{\boldsymbol{x}}_{n-1})^{-1} + \boldsymbol{A}_{\boldsymbol{n}}^T \boldsymbol{P}_{\boldsymbol{n}} \boldsymbol{A}_{\boldsymbol{n}}\right]^{-1}$
$\hat{\boldsymbol{x}}_{n\|n-1} = \boldsymbol{\Phi}_{n-1\|n-1} \cdot \hat{\boldsymbol{x}}_{n-1\|n-1}$ $\boldsymbol{P}_{n\|n-1} = \boldsymbol{\Phi}_{n-1\|n-1} \boldsymbol{P}_{n-1\|n-1} \boldsymbol{\Phi}_{n-1\|n-1}^T + \boldsymbol{Q}$	

Question: Can we proof that the Kalman filter algorithm can be related to the sequential LSQ in case of time-invariant parameters?

Kalman filter vs. sequential LSQ - cont'd

The Kalman filer degrades to the sequential LSQ estimation, if

- the state transition matrix $\boldsymbol{\Phi}_{n-1|n-1}$ is the identity matrix
- the process noise covariance matrix $\boldsymbol{Q}$ is zero

$$\boldsymbol{\Phi}_{n-1|n-1} = \boldsymbol{I} \quad \Rightarrow \quad \hat{\boldsymbol{x}}_{n|n-1} = \hat{\boldsymbol{x}}_{n-1|n-1}$$

$$\boldsymbol{Q} = \boldsymbol{0},\, \boldsymbol{\Phi}_{n-1|n-1} = \boldsymbol{I} \quad \Rightarrow \quad \boldsymbol{P}_{n|n-1} = \boldsymbol{P}_{n-1|n-1}$$

In other word the prediction step is not effective. Thus in the following we drop the notation $n|n-1$ and just write

$$\boldsymbol{P}_{n|n} = (\boldsymbol{I} - \boldsymbol{K}_n \boldsymbol{H}_n)\, \boldsymbol{P}_{n|n-1} \quad \Rightarrow \quad \boldsymbol{P}_n = (\boldsymbol{I} - \boldsymbol{K}_n \boldsymbol{H}_n)\, \boldsymbol{P}_{n-1}$$

$$\boldsymbol{K}_n = \boldsymbol{P}_{n|n-1} \boldsymbol{H}_n^T \left(\boldsymbol{H}_n \boldsymbol{P}_{n|n-1} \boldsymbol{H}_n^T + \boldsymbol{R}_n \right)^{-1}$$
$$\Rightarrow \quad \boldsymbol{K}_n = \boldsymbol{P}_{n-1} \boldsymbol{H}_n^T \left(\boldsymbol{H}_n \boldsymbol{P}_{n-1} \boldsymbol{H}_n^T + \boldsymbol{R}_n \right)^{-1}$$

$$\boldsymbol{P}_n = \boldsymbol{P}_{n-1} - \boldsymbol{K}_n \boldsymbol{H}_n \boldsymbol{P}_{n-1}$$
$$\Rightarrow \quad \boldsymbol{P}_n = \boldsymbol{P}_{n-1} - \boldsymbol{P}_{n-1} \boldsymbol{H}_n^T \left(\boldsymbol{H}_n \boldsymbol{P}_{n-1} \boldsymbol{H}_n^T + \boldsymbol{R}_n \right)^{-1} \boldsymbol{H}_n \boldsymbol{P}_{n-1}$$

Matrix inversion Lemma, for $\boldsymbol{B}$, $\boldsymbol{D}$ being positive definite matrices

$$\left(\boldsymbol{B}^{-1} - \boldsymbol{C} \boldsymbol{D}^{-1} \boldsymbol{C}^T \right)^{-1} = \boldsymbol{B} - \boldsymbol{B} \boldsymbol{C} \left(\boldsymbol{D} + \boldsymbol{C}^T \boldsymbol{B} \boldsymbol{C} \right)^{-1} \boldsymbol{C}^T \boldsymbol{B}$$

Kalman filter vs. sequential LSQ - cont'd

We identify

$$\boldsymbol{B} = \boldsymbol{P}_{n-1},\ \boldsymbol{C} = \boldsymbol{H}_n^T,\ \boldsymbol{D} = \boldsymbol{R}_n$$

and are able to write

$$\boldsymbol{P}_n = \left(\boldsymbol{P}_{n-1}^{-1} + \boldsymbol{H}_n^T \boldsymbol{R}_n \boldsymbol{H}_n\right)^{-1}$$

which turns out to be equivalent to the covariance update for the sequential LSQ.

A similar derivation can be made for $\hat{\boldsymbol{x}}_n$.