Exercise on 14.05.2019

Task 1 (3 points)

The differential equation of a first order Gauss-Markov process is defined as

$$\dot{x} = -\beta x + W(t).$$

Find the transition matrix $\Phi(\Delta t)$ in order to rewrite the differential equation system from above as discrete difference equations like

$$x_{n+1} = \Phi \cdot x_n + w_{n+1}.$$

Perform 30 realizations with 100 time steps each (i.e. $t = 1 : 100, \Delta t = 1$) of the first order Gauss-Markov process using $\beta = 0.1$ and an initial value $x_0 = 0$ for each realization. The noise contribution w_{n+1} for each step is obtained from a normally distributed random number generator (mean 0, variance 1). Plot your results and the variance of all realizations for each respective time step. Interprete and discuss your results.

Carry out a second simulation of the first order Gauss-Markov process where β is changed to 0.9. Interprete the differences to your first simulation. What happens for the cases $\beta \to 0.0$ and $\beta \to 1.0$?

Task 2 (2 points)

Determine the transition matrix $\mathbf{\Phi}$ for $\Delta t = 1$ of a stochastic process, which is defined by

$$\ddot{x} = 0 + W(t).$$

Use your transition matrix in order to set up the following difference equations:

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix}_{n+1} = \mathbf{\Phi} \cdot \begin{bmatrix} x \\ \dot{x} \end{bmatrix}_n + \begin{bmatrix} 0 \\ w_{n+1} \end{bmatrix}$$

Compute the $\boldsymbol{x}_n$ for $n = 1 \ldots 20$ where the initial value is $\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The random values w_{n+1} can be found in the text file `random02.txt`. Each column of the text file can be used for one realization of the stochastic process. Repeat the stochastic process 100 times (i.e. 100 realization with 20 time steps each). In addition, compute the mean value of all realizations for each time step and its variance. Plot and interprete your results.

Task 3 (2 points)

The following differential equation system is given

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & 0 \\ -\omega_2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

with the initial values $\boldsymbol{x}_0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2.3 \\ 1.7 \\ 6.4 \end{bmatrix}$ as well as $\omega_2 = 0.7$ and $\omega_3 = 0.3$. Determine the transition matrix $\mathbf{\Phi}$ for $\Delta t = 0.25$ and with that compute the solution of x_1, x_2 and x_3 for 40 time steps (Δt). Plot your results.

Task 4 (3 points)

The second order Gauss-Markov process is defined as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2f\omega_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$$

Compute empirically $\mathbf{\Phi}(\Delta t = 1)$ for the case $f = 0.25$, $\omega_0 = 0.03$. With $\mathbf{\Phi}$ you can rewrite the differential equation system from above as discrete difference equations like

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{n+1} = \mathbf{\Phi} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n + \begin{bmatrix} 0 \\ w_{n+1} \end{bmatrix}$$

Perform 100,000 time steps using $\Delta t = 1$ and the initial values $x_1 = 0$ and $x_2 = 0$. The noise contribution w_{n+1} for each step is obtained by a normal distributed random number generator (mean 0, variance 1). Compute the empiric covariance of x_1 for $\tau = \pm 1000$ using e.g the Matlab function `xcov`. In addition calculate the theoretic covariance for x_1 which is obtained by

$$Cov_{x_1,x_1}(\tau) = \sigma_{x_1}^2 \exp^{(-f \cdot \omega_0 \cdot |\tau|)} \cdot \left(\cos(\beta\,|\tau|) + f\frac{\omega_0}{\beta} \cdot \sin(\beta\,|\tau|) \right)$$

where $\sigma_{x_1}^2$ is the variance of x_1. The parameter β is obtained by

$$\beta = \omega_0 \cdot \sqrt{1 - f^2}$$

meaning, your Matlab code can look like

```
% x1 ... your stochastic GM-process
tau=-1000:1000; % +/- 1000 "lags"
th= ... your theoretical covariance
plot(tau,xcov(x1,x1,1000,'biased'),tau,th);
```

How does your result correspond to the theoretic results?
What happens for the case $f \to 0.0$ and $f \to 1.0$?