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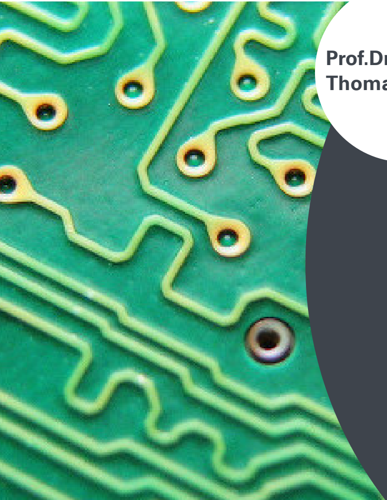

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Integrated Positioning and Navigation

8

Integration of the attitude equation in the e-system

Integration of the attitude equation in the e-system

$$\dot{\boldsymbol{q}}_p^e = \frac{1}{2}\boldsymbol{A}\boldsymbol{q}_p^e \qquad \text{with } \boldsymbol{A} = \begin{bmatrix} 0 & \omega_{ep1}^p & \omega_{ep2}^p & \omega_{ep3}^p \\ -\omega_{ep1}^p & 0 & \omega_{ep3}^p & -\omega_{ep2}^p \\ -\omega_{ep2}^p & -\omega_{ep3}^p & 0 & \omega_{ep1}^p \\ -\omega_{ep3}^p & \omega_{ep2}^p & -\omega_{ep1}^p & 0 \end{bmatrix} \tag{8.1}$$

$$\dot{\boldsymbol{q}}(t) = \frac{1}{2}\boldsymbol{A}(t)\boldsymbol{q}(t) = \boldsymbol{f}(t,\boldsymbol{q}) \text{ (subscript and superscript omitted)}$$

Use a numerical procedure for integration.

Example Runge-Kutta integrator of 3rd order*

$$\begin{aligned} \hat{\boldsymbol{q}}(t_k) &= \hat{\boldsymbol{q}}(t_{k-2}) + \frac{\delta t}{6}\left(\boldsymbol{k}_1 + 4\boldsymbol{k}_2 + \boldsymbol{k}_3\right) \\ \boldsymbol{k}_1 &= \boldsymbol{f}(t_{k-2}, \boldsymbol{q}_{k-2}) \\ \boldsymbol{k}_2 &= \boldsymbol{f}(t_{k-1}, \boldsymbol{q}_{k-2} + \boldsymbol{k}_1 \cdot \tfrac{\delta t}{2}) \\ \boldsymbol{k}_3 &= \boldsymbol{f}(t_k, \boldsymbol{q}_{k-2} - \boldsymbol{k}_1 \cdot \delta t + \boldsymbol{k}_2 \cdot 2\delta t) \end{aligned} \tag{8.2}$$

(* see lecture notes for *Parameter Estimation in Dynamic Systems*!)

Integration of the attitude equation in the e-system - cont't

The elements of the matrix $\boldsymbol{A}$

$$\boldsymbol{\omega}_{ep}^{p} = \boldsymbol{\omega}_{ip}^{p} - \boldsymbol{C}_{e}^{p}\boldsymbol{\omega}_{ie}^{e} \tag{8.3}$$

Gyro output (last equation of Module 7):

$$\Delta\boldsymbol{\alpha}_{ip}^{p}(t_k) = \int_{t_{k-1}}^{t_k} \boldsymbol{\omega}_{ip}^{p}(\tau)d\tau \tag{8.4}$$

Formal integration of equation (8.3):

$$\Delta\boldsymbol{\beta}_{ep}^{p}(t_k) = \int_{t_{k-1}}^{t_k} \boldsymbol{\omega}_{ep}^{p}(\tau)d\tau = \int_{t_{k-1}}^{t_k} \left[\boldsymbol{\omega}_{ip}^{p}(\tau) - \boldsymbol{C}_{e}^{p}(\tau)\boldsymbol{\omega}_{ie}^{e}\right] d\tau \tag{8.5}$$

From now on use the following abbreviations:

$$\Delta t = t_k - t_{k-1},\ \ \boldsymbol{\omega}^{p} = \boldsymbol{\omega}_{ep}^{p},\ \ \Delta\boldsymbol{\alpha}^{p} = \Delta\boldsymbol{\alpha}_{ip}^{p},\ \ \Delta\boldsymbol{\beta}^{p} = \Delta\boldsymbol{\beta}_{ep}^{p},\ \ \boldsymbol{q} = \boldsymbol{q}_{p}^{e} \tag{8.6}$$

Integration of the attitude equation in the e-system - cont't

From equation (8.5) we obtain:

$$\begin{aligned} \Delta\boldsymbol{\beta}^p(t_{k-1}) &= \int\limits_{t_{k-2}}^{t_{k-1}} \boldsymbol{\omega}_{ep}^p(\tau)d\tau \\ &= \Delta\boldsymbol{\alpha}^p(t_{k-1}) - \int\limits_{t_{k-2}}^{t_{k-1}} \boldsymbol{C}_e^p(\tau)\boldsymbol{\omega}_{ie}^e d\tau \end{aligned} \tag{8.7}$$

$$\text{For small } \Delta t: \quad \boldsymbol{C}_e^p(\tau) \approx \boldsymbol{C}_e^p(t_{k-1}) \approx \boldsymbol{C}_e^p(t_{k-2}) \tag{8.8}$$

From equations (8.7) and (8.8) :

$$\boxed{\begin{aligned} \Delta\boldsymbol{\beta}^p(t_{k-1}) &= \Delta\boldsymbol{\alpha}^p(t_{k-1}) - \boldsymbol{C}_e^p(t_{k-2})\boldsymbol{\omega}_{ie}^e\Delta t + \ldots \\ \Delta\boldsymbol{\beta}^p(t_k) &= \Delta\boldsymbol{\alpha}^p(t_k) - \boldsymbol{C}_e^p(t_{k-1})\boldsymbol{\omega}_{ie}^e\Delta t + \ldots \end{aligned}} \tag{8.9}$$

$\Delta\boldsymbol{\beta}$ is a function of gyro output and known quantities!

Integration of the attitude equation in the e-system - cont't

In general, we can express $\boldsymbol{\omega}^p$ in a Taylor-expansion in the interval $[t_{k-2}, t_k]$:

$$\boldsymbol{\omega}^p(t) = \boldsymbol{\omega}^p(t_{k-2}) + \dot{\boldsymbol{\omega}}^p(t_{k-2}) \cdot (t - t_{k-2}) + O(\delta t^2), \quad t - t_{k-2} \leq \delta t$$

1) Find expressions for: $\boldsymbol{\omega}^p(t_{k-2}), \dot{\boldsymbol{\omega}}^p(t_{k-2})$ (8.10)

2) determine $\boldsymbol{\omega}^p(t)$ for $t = k-1, k$

From eqn. (8.5)

$$\Delta\boldsymbol{\beta}^p(t_{k-1}) = \int_{t_{k-2}}^{t_{k-1}} \boldsymbol{\omega}^p(\tau)d\tau = \boldsymbol{\omega}^p(t_{k-2})\Delta t + \frac{1}{2}\dot{\boldsymbol{\omega}}^p(t_{k-2})\Delta t^2 + \ldots \tag{8.11}$$

$$\Delta\boldsymbol{\beta}^p(t_k) = \int_{t_{k-1}}^{t_k} \boldsymbol{\omega}^p(\tau)d\tau = \boldsymbol{\omega}^p(t_{k-2})\Delta t + \frac{3}{2}\dot{\boldsymbol{\omega}}^p(t_{k-2})\Delta t^2 + \ldots \tag{8.12}$$

Integration of the attitude equation in the e-system - cont't

From equations (8.11) and (8.12):

$$\boxed{\begin{aligned} \boldsymbol{\omega}^p(t_{k-2}) &= \frac{1}{2\Delta t}\left(3\Delta\boldsymbol{\beta}^p(t_{k-1}) - \Delta\boldsymbol{\beta}^p(t_k)\right) + \ldots \\ \dot{\boldsymbol{\omega}}^p(t_{k-2}) &= \frac{1}{\Delta t^2}\left(\Delta\boldsymbol{\beta}^p(t_k) - \Delta\boldsymbol{\beta}^p(t_{k-1})\right) + \ldots \end{aligned}} \quad (8.13)$$

Insert (8.13) in equation (8.10)

$$\begin{aligned} \boldsymbol{\omega}^p(t_{k-2}) &= \frac{1}{2\Delta t}\left(3\Delta\boldsymbol{\beta}^p(t_{k-1}) - \Delta\boldsymbol{\beta}^p(t_k)\right) + \ldots \\ \boldsymbol{\omega}^p(t_{k-1}) &= \frac{1}{2\Delta t}\left(3\Delta\boldsymbol{\beta}^p(t_{k-1}) - \Delta\boldsymbol{\beta}^p(t_k)\right) + \frac{1}{\Delta t}\left(\Delta\boldsymbol{\beta}^p(t_k) - \Delta\boldsymbol{\beta}^p(t_{k-1})\right) + \ldots \\ \boldsymbol{\omega}^p(t_k) &= \frac{1}{2\Delta t}\left(3\Delta\boldsymbol{\beta}^p(t_{k-1}) - \Delta\boldsymbol{\beta}^p(t_k)\right) + \frac{2}{\Delta t}\left(\Delta\boldsymbol{\beta}^p(t_k) - \Delta\boldsymbol{\beta}^p(t_{k-1})\right) + \ldots \end{aligned} \quad (8.14)$$

Integration of the attitude equation in the e-system - cont't

Set $\delta t = 2\Delta t$ and indicate approximation by $\hat{}$ (remove higher order terms)

$$
\begin{aligned}
\hat{\boldsymbol{\omega}}^p(t_{k-2}) &= \frac{3\Delta\boldsymbol{\beta}^p(t_{k-1}) - \Delta\boldsymbol{\beta}^p(t_k)}{\delta t} \\
\hat{\boldsymbol{\omega}}^p(t_{k-1}) &= \frac{\Delta\boldsymbol{\beta}^p(t_{k-1}) + \Delta\boldsymbol{\beta}^p(t_k)}{\delta t} \\
\hat{\boldsymbol{\omega}}^p(t_k) &= \frac{3\Delta\boldsymbol{\beta}^p(t_k) - \Delta\boldsymbol{\beta}^p(t_{k-1})}{\delta t}
\end{aligned}
\tag{8.15}
$$

$\boldsymbol{\omega}^p$ is a function of gyro output and known quantities!

Integration of the attitude equation in the e-system - cont't

Computation of $\boldsymbol{k}_1$ (cf. Equ. (8.2)):

$$\boldsymbol{k}_1 = \boldsymbol{f}(t_{k-2}, \boldsymbol{q}_{k-2}) = \frac{1}{2}\boldsymbol{A}(t_{k-2})\hat{\boldsymbol{q}}_{k-2}$$

$$\boldsymbol{A}(t_{k-2}) = \begin{bmatrix} 0 & \omega_1^p(t_{k-2}) & \omega_2^p(t_{k-2}) & \omega_3^p(t_{k-2}) \\ -\omega_1^p(t_{k-2}) & 0 & \omega_3^p(t_{k-2}) & -\omega_2^p(t_{k-2}) \\ -\omega_2^p(t_{k-2}) & -\omega_3^p(t_{k-2}) & 0 & \omega_1^p(t_{k-2}) \\ -\omega_3^p(t_{k-2}) & \omega_2^p(t_{k-2}) & -\omega_1^p(t_{k-2}) & 0 \end{bmatrix} \tag{8.16}$$

where we can use the values computed according to equation (8.15)

Similarly for $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$.

$$\begin{aligned} &\Rightarrow\ \hat{\boldsymbol{q}}(t_k) = \boldsymbol{D}\cdot\hat{\boldsymbol{q}}(t_{k-2}) = \\ &\Rightarrow\ \boldsymbol{D} = \boldsymbol{f}(\hat{\boldsymbol{\omega}}^p(t_{k-2}), \hat{\boldsymbol{\omega}}^p(t_{k-1}), \hat{\boldsymbol{\omega}}^p(t_k)) \end{aligned} \tag{8.17}$$

Integration of the attitude equation in the e-system - cont't

Summary of integration procedure:

1. Evaluate equation (8.9) $\Rightarrow \Delta\boldsymbol{\beta}^p(t_{k-1}), \Delta\boldsymbol{\beta}^p(t_k)$
2. Evaluate equation (8.15) $\Rightarrow \hat{\boldsymbol{\omega}}^p(t_{k-2}), \hat{\boldsymbol{\omega}}^p(t_{k-1}), \hat{\boldsymbol{\omega}}^p(t_k)$
3. Evaluate equation (8.2) $\Rightarrow \hat{\boldsymbol{q}}(t_k)$
4. Normalise the quaternion: equation (2.13)
5. Compute DCM $\boldsymbol{C}$ from normalised quaternion $\boldsymbol{q}$: equation (2.12)