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Dynamic System Estimation

Linear Dynamic Systems

3

Linear Dynamic Systems

State space description of a Linear Dynamic System

$$\boxed{\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(t)\boldsymbol{x}(t) + \boldsymbol{G}(t)\boldsymbol{w}(t) + \boldsymbol{L}(t)\boldsymbol{s}(t)} \tag{3.1}$$

$\boldsymbol{x}(t)$	Set of random variables describing the linear system (the state vector)
$\boldsymbol{w}(t)$	Random forcing function
$\boldsymbol{s}(t)$	Deterministic control input
$\boldsymbol{F}(t)$	square matrix
$\boldsymbol{G}(t)$,$\boldsymbol{L}(t)$	matrices (not necessarily square!)

Here we will consider only linear dynamic models without control input:

$$\boxed{\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(t)\boldsymbol{x}(t) + \boldsymbol{G}(t)\boldsymbol{w}(t)} \tag{3.2}$$

Linear Dynamic Systems - cont'd

For given initial conditions $\boldsymbol{x}_0(t)$, the general solution to equ.(3.2) can be written:

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t, t')\boldsymbol{G}(t')\boldsymbol{w}(t')dt' \qquad (3.3)$$

The general solution is the sum of the the solution of the homogeneous equation and a particular solution of the non-homogeneous equation!

$\boldsymbol{\Phi}(t, t_0)$ is called the state transition matrix. The following relations hold for this matrix:

$$\begin{aligned} \frac{d}{dt}\boldsymbol{\Phi}(t, t_0) &= \boldsymbol{F}(t)\boldsymbol{\Phi}(t, t_0) \\ \boldsymbol{\Phi}(t_2, t_0) &= \boldsymbol{\Phi}(t_2, t_1)\boldsymbol{\Phi}(t_1, t_0) \\ \boldsymbol{\Phi}(t, t) &= \boldsymbol{\Phi}(t, t_0)\boldsymbol{\Phi}(t_0, t) = \boldsymbol{I} \Rightarrow \boldsymbol{\Phi}^{-1}(t, t_0) = \boldsymbol{\Phi}(t_0, t) \end{aligned} \qquad (3.4)$$

Up to now $\boldsymbol{\Phi}(t, t_0)$ is still unknown!

Linear Dynamic Systems - cont'd

Transition matrix for stationary systems: in stationary systems, the matrix $\boldsymbol{F}$ in equs. (3.1) and (3.2) is time-invariant. Stationary systems can often be used to replace approximatively more complex systems over short time periods. A general Taylor expansion gives:

$$\boxed{\boldsymbol{x}(t) = \boldsymbol{x}(t_0) + \dot{\boldsymbol{x}}(t_0)(t - t_0) + \frac{1}{2!}\ddot{\boldsymbol{x}}(t_0)(t - t_0)^2 + \ldots} \tag{3.5}$$

From the homogeneous part of equ. (3.2) we can replace

$$\begin{aligned} \dot{\boldsymbol{x}}(t_0) &= \boldsymbol{F}\boldsymbol{x}(t_0) \\ \ddot{\boldsymbol{x}}(t_0) &= \dot{\boldsymbol{F}}\boldsymbol{x}(t_0) + \boldsymbol{F}\dot{\boldsymbol{x}}(t_0) = \boldsymbol{F}\boldsymbol{F}\boldsymbol{x}(t_0) = \boldsymbol{F}^2\boldsymbol{x}(t_0) \\ &\vdots \\ \boldsymbol{x}^{(n)}(t_0) &= \boldsymbol{F}^n\boldsymbol{x}(t_0) \end{aligned} \tag{3.6}$$

Linear Dynamic Systems - cont'd

Substituting equ. (3.6) into (3.5):

$$\boldsymbol{x}(t) = \boldsymbol{x}(t_0) + \boldsymbol{F}\boldsymbol{x}(t_0)(t - t_0) + \frac{\boldsymbol{F}^2}{2!}\boldsymbol{x}(t_0)(t - t_0)^2 + \ldots$$

$$\boxed{\boldsymbol{x}(t) = \left[\boldsymbol{I} + \boldsymbol{F}(t - t_0) + \frac{\boldsymbol{F}^2}{2!}(t - t_0)^2 + \ldots\right]\boldsymbol{x}(t_0)} \qquad (3.7)$$

The term in square brackets is by definition the matrix exponential

$$\begin{aligned} e^{\boldsymbol{A}} &= \boldsymbol{I} + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \frac{\boldsymbol{A}^3}{3!} + \ldots \\ e^{\boldsymbol{F}(t-t_0)} &= \boldsymbol{I} + \boldsymbol{F}(\boldsymbol{t} - \boldsymbol{t_0}) + \frac{\boldsymbol{F}^2(t - t_0)^2}{2!} + \ldots \end{aligned} \qquad (3.8)$$

For stationary systems, the state transition matrix depends only on the time interval $(t - t0)$ and the matrix $\boldsymbol{F}$

$$\boxed{\boldsymbol{\Phi}(t, t_0) = e^{\boldsymbol{F}(t-t_0)}} \qquad (3.9)$$

Linear Dynamic Systems - cont'd

We are now in a position to discretize the continuous system of equ. (3.2):

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(t)\boldsymbol{x}(t) + \boldsymbol{G}(t)\boldsymbol{w}(t)$$

$$\Downarrow$$

$$\boxed{\begin{aligned} \boldsymbol{x}(t_n) &= \boldsymbol{\Phi}(t_n, t_{n-1})\boldsymbol{x}(t_{n-1}) + \boldsymbol{u}(t_n) \\ &\text{or} \\ \boldsymbol{x}_n &= \boldsymbol{\Phi}(t_n, t_{n-1})\boldsymbol{x}_{n-1} + \boldsymbol{u}_n \end{aligned}} \tag{3.10}$$

with

$$\boldsymbol{u}_n = \int_{t_{n-1}}^{t_n} \boldsymbol{\Phi}(t, t')\boldsymbol{G}(t')\boldsymbol{w}(t')dt' \tag{3.11}$$

For stationary systems, the state transition matrix is computed from equ. (3.9). The discretization of equ. (3.10) holds also for non-stationary systems; but then the state transition matrix cannot be computed from equ. (3.9)!

Linear Dynamic Systems - cont'd

You will find the examples discussed in this lecture as Jupyter notebook under

`https://github.com/spacegeodesy/ParameterEstimationDynamicSystems/blob/master/example03.ipynb`