数值积分——复化 Gauss 求积公式

湘潭大学, 数学与计算科学学院

一、复化 Gauss 求积公式

Gauss 积分:

具有 n 个积分节点，代数精度为 $2n-1$ 阶的数值积分，称为 Gauss 积分，记为 $G_n(f)$ 在 $[-1,1]$ 上的 Gauss 积分公式为

$$G_n(f)=\sum_{i=1}^{n}\alpha_i^{(n)}f(x_i^{(n)})$$

在 $[-1,1]$Gauss 积分的积分节点 $\{x_i^{(n)}\}$ 及积分系数 $\{\alpha_i^{(n)}\}$ 是已知的（文末表格）

对于一般区间的积分 $I(f)=\int_a^b f(x)\mathrm{d}x$, 作线性变量代换, 得到 Gauss 积分公式

$$G_n(f)=\frac{b-a}{2}\sum_{i=1}^{n}\alpha_i^{(n)}f\left(\frac{(a+b)+(b-a)x_i^{(n)}}{2}\right)$$

复化 Gauss 求积: 将 $[a,b]$ 划分成 m 个小区间，得到 $m+1$ 个节点，记 $k=0,1,2,\cdots,m$，$h=\frac{b-a}{m}, X_i=a+kh.$

$$I(f)=\int_a^b f(x)\mathrm{d}x=\sum_{i=0}^{m-1}\int_{X_i}^{X_{i+1}}f(x)\mathrm{d}x\approx\sum_{i=0}^{m-1}\text{每个小区间下的 GS 积分}$$

二、算法

♡ **Gauss 求积公式：** `[output] = GS求积(a,b,n,f)`

1. 输入
   - $[a,b]$：积分区间
   - n: 节点个数
   - f: 支持向量运算的被积函数
2. 类似于 Newton-Cotes 求积公式的实现

   已知它积分区间为 $[-1,1]$ 对应的权重和节点坐标，代入到下方公式中

$$G_n(f)=\frac{b-a}{2}\sum_{i=1}^{n}\alpha_i^{(n)}f\left(\frac{(a+b)+(b-a)x_i^{(n)}}{2}\right)$$

3. 输出

- output = $G_n(f)$

♡ **复化 Gauss 求积公式：** `[GS] = 复化GS求积(a,b,n1,n2,f)`

1. 输入

- $[a,b]$：积分区间
- $n1$: GS 求积时的节点个数
- $n2$: $[a,b]$ 分成 $n2$ 个小区间
- f: 支持向量运算的被积函数

2. 复化

- 计算步长 $h = \frac{b-a}{n2}$
- k = [0 1 …n], 区间节点 Xi = a+k*h
- [Xi(i),Xi(i+1)] 上的积分近似值 = `GS求积(Xi(i),Xi(i+1),n1,f)`
- 对每个小区间都这样做，求和得到

$$GS = \sum_{i=0}^{n2-1} \text{每个小区间下的 GS 积分}$$

3. 输出

- GS：通过复化 Gauss 公式得到的积分近似值

三、北太天元源程序

Gauss 积分节点及权重的数据处理

```
% 对导入数据进行处理
% 导入表格 从(4,3)取到(251,4) 长方形，勾选 忽略
format long;
jiedian = cell(1,20);%创建空的元胞数组
quanzhong = cell(1,20);
d = 3:1:21;
j = 0;
quanzhong{1} = main(1,1);
jiedian{1} = main(1,2);
for i=2:1:20
    j=j+d(i-1);
    quanzhong{i} = main(j+1:j+i,1);
    jiedian{i}= main(j+1:j+i,2);
end
save('gs.mat','jiedian','quanzhong');
```

将上述代码保存为 `GS数据处理.m` 文件。

Gauss 求积

```
function output = GS求积(a,b,n,f)
    % [a,b]
  % n : 积分节点个数
  % f: 支持向量运算的函数
load('gs.mat');
xi = jiedian{n}; % 是列向量
Xi = ((a+b)+(b-a)*xi)./2;
Yi = f(Xi);
alpha = quanzhong{n}; % 是列向量
A = alpha .* Yi;
Sum = sum(A);
output = (b-a)/2 * Sum;
end
```

将上述代码保存为 GS求积.m 文件。

复化 Gauss 求积

```
function GS = 复化GS求积(a,b,n1,n2,f)
    % [a,b]
  % n1 : 一个小区间内选取的积分节点数目
  % n2 : 将[a,b]分划成 n2 个小区间
  % f: 支持向量运算的函数
S = zeros(1,n2);
h = (b-a)/n2;
k = 0:1:n2;
jd = a + k * h;
for i =1:1:n2
   S(i) = GS求积(jd(i),jd(i+1),n1,f);
end
  GS = sum(S);
end
```

将上述代码保存为 复化GS求积.m 文件。

四、数值算例

例 1 用 *Gauss* 求积公式计算

$$\int_{-4}^{4} \frac{\mathrm{d}x}{1+x^2} = 2\arctan(4) \approx 2.65163533$$

分别取 $n = 3, 4, \ldots, 20$, 计算出对应的积分近似值，并观察随 n 增加它与 2.65163533 的误差变化

GS 求积例子 1

```
% GS求积例子1
clc,clear all,format long;
f1 = @(x)1./(1+x.^2);
a = -4; b = 4;
zhenshi = 2.65163533; % 真实值取8位小数的值
gs = zeros(1,18);
delta = zeros(1,18);
i=1;
% 进行GS求积
for n=3:1:20
gs(i) = GS求积(a,b,n,f1);
delta(i) = abs(gs(i)-zhenshi);
i++;
end
n = 3:1:20;
zhenshi1 = ones(1,18)*zhenshi;
figure(1);
   plot(n,gs,'-*r');
   hold on
   plot(n,zhenshi1,'-b');
figure(2);
   plot(n,delta,'-*r');
disp('delta',delta);

% 进行复化GS求积 对比看看
% 变n1, 不变n2
GS1 = zeros(1,18);
delta1 = zeros(1,18);
i=1;
for n = 3:1:20
   GS1(i) = 复化GS求积(a,b,n,10,f1);
   delta1(i) = abs(GS1(i)-zhenshi);
   i++;
end
n = 3:1:20;
figure(3);
   plot(n,GS1,'-*r');
   hold on
   plot(n,zhenshi1,'-b');
figure(4);
   plot(n,delta1,'-*r');
disp('GS1',GS1);
disp('delta1',delta1);
% 不变n1, 变n2
GS2 = zeros(1,18);
delta2 = zeros(1,18);
```

```
i=1;
for n = 3:1:20
    GS2(i) = 复化GS求积(a,b,10,n,f1);
    delta2(i) = abs(GS2(i)-zhenshi);
    i++;
end
n = 3:1:20;
figure(5);
    plot(n,GS2,'-*r');
    hold on
    plot(n,zhenshi1,'-b');
figure(6);
    plot(n,delta2,'-*r');
disp('GS2',GS2);
disp('delta2',delta2);
```

将上述代码保存为 GS求积例子1.m

运行后得到

可以发现，Gauss 求积公式，不仅精度高、还非常稳定

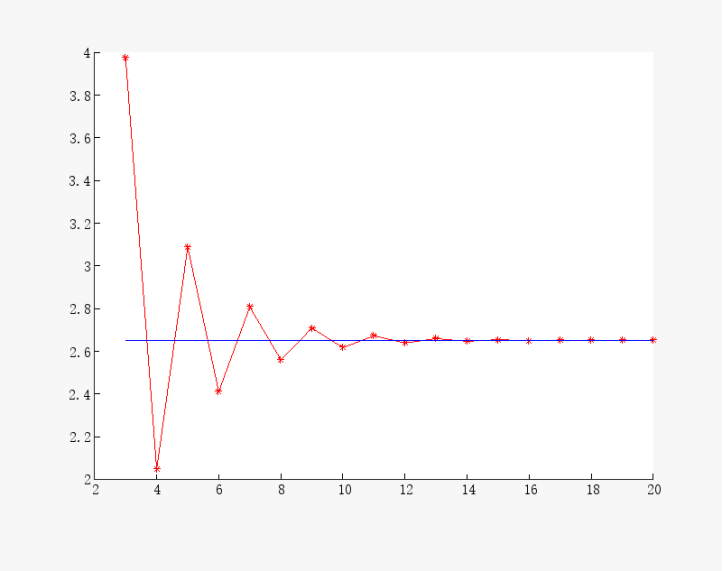

图 **1** **GS** 求积下的值

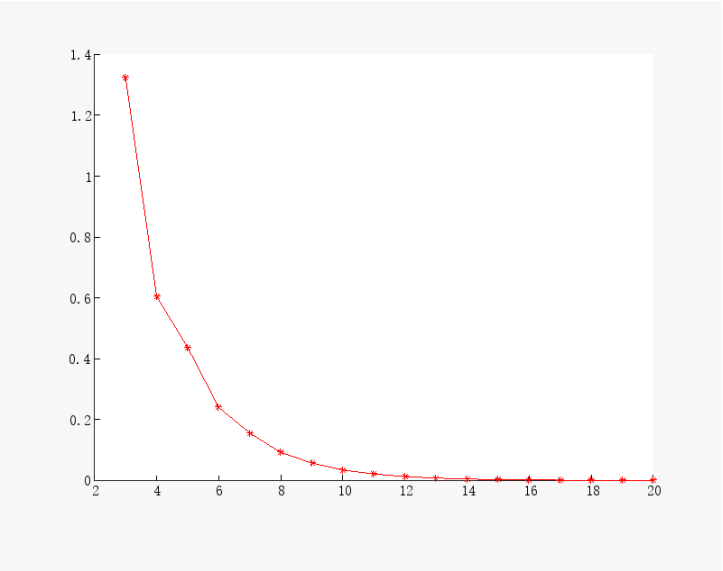

图 **2**　**GS** 求积下的误差

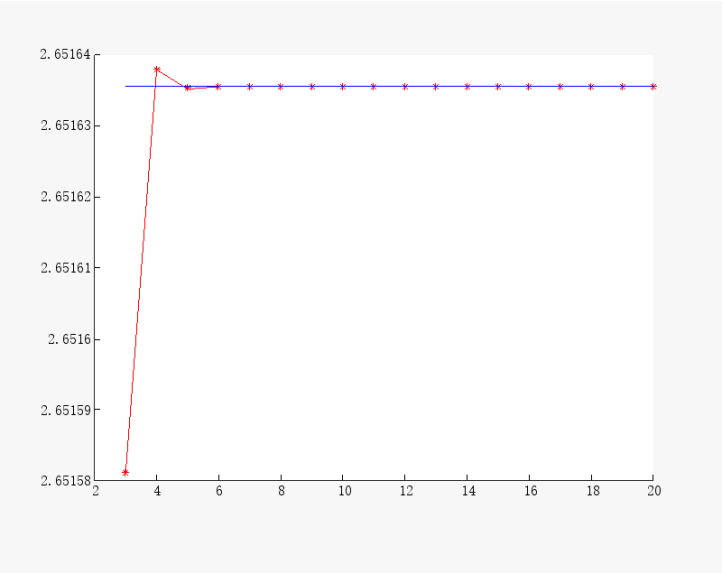

图 **3**　变 **n1**，不变 **n2**

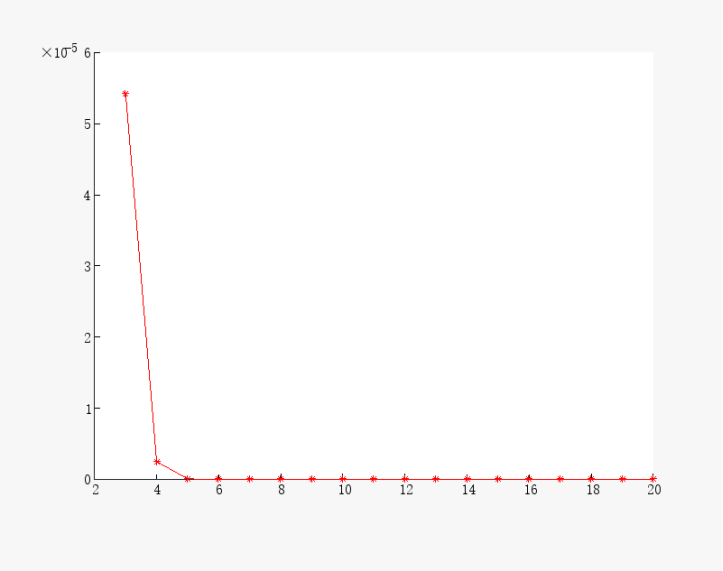

图 **4**　误差

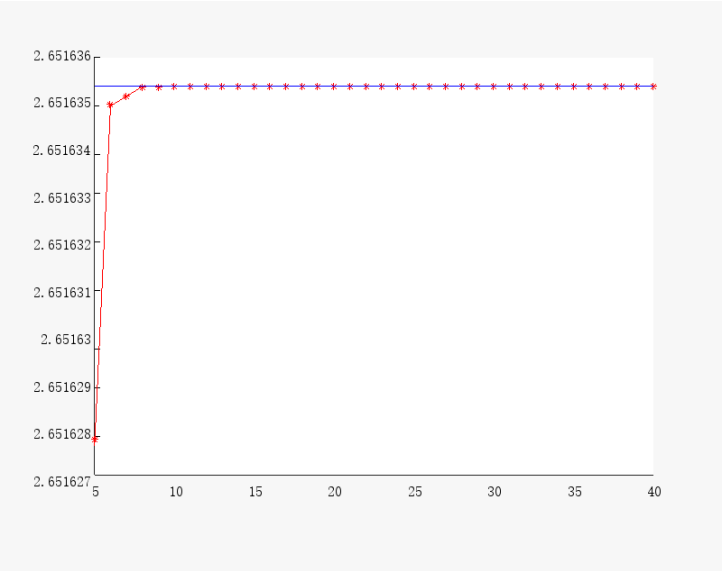

图 **5**　变 **n2**，不变 **n1**

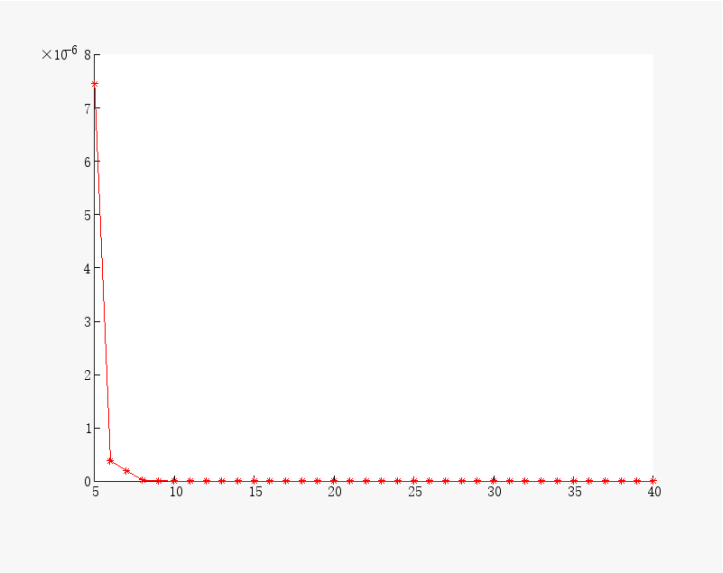

图 **6**　误差

例 2 用 *Gauss* 求积公式计算

$$\pi = 4\int_0^1 \frac{1}{1+x^2}\mathrm{d}x$$

计算积分值与 π 的误差并作图；

GS 求积例子 2

```
% 复化Gauss求积例子2 pi
clc;clear all;format long;
f = @(x) 4./(1+x.^2);

N1 = 3:1:20; l1 =length(N1);
N2 = 5:1:100; l2 = length(N2);
GS1 = zeros(1,l1);
GS2 = zeros(1,l2);
delta1 = zeros(1,l1);
delta2 = zeros(1,l2);
% n1变，n2不变
k = 1;
for n1 = N1
    GS1(k) = 复化GS求积(0,1,n1,5,f);
    delta1(k) = abs(pi - GS1(k));
    k++;
end
P1 = ones(1,l1)*pi;
figure(1);
     plot(N1,GS1,'-b');
    set(gca, 'YLim', [3.14159, 3.141595]);
    hold on
       plot(N1,P1,'r');
figure(2);
     plot(N1,delta1,'r')
disp('GS1',GS1);
disp(pi);disp('delta1',delta1);

% n1不变，n2变
k=1;
for n = N2
    GS2(k) = 复化GS求积(0,1,3,n,f);
    delta2(k) = abs(pi - GS2(k));
    k++;
end
P2 = ones(1,l2)* pi;
figure(3);
    plot(N2,GS2,'r');
    set(gca, 'YLim', [3.14159, 3.141595]);
figure(4);
     plot(N2,delta2,'b');
```

```
disp('GS2',GS2);
disp(pi);disp('delta2',delta2);
```

将上述代码保存为 GS求积例子2.m

运行后得到

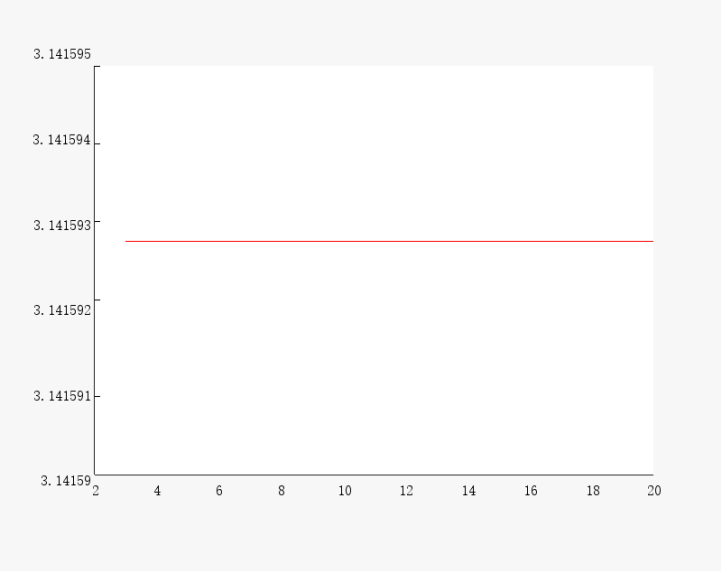

图 7　变 **n1**，不变 **n2**

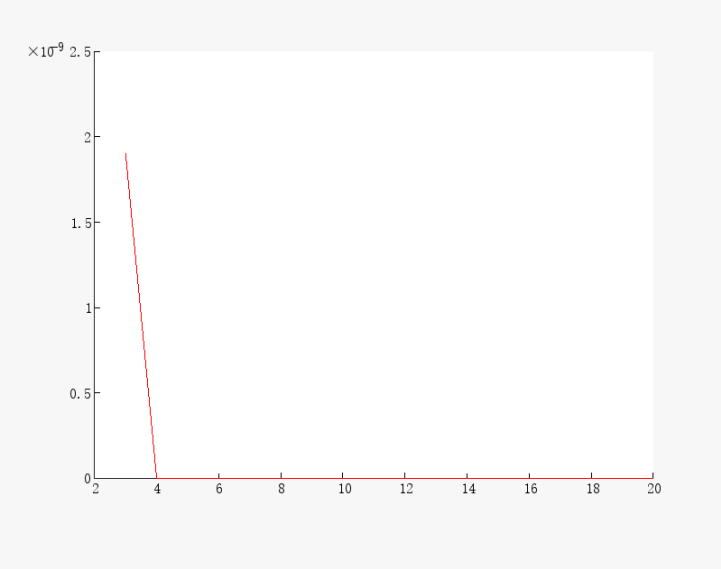

图 **8**　误差

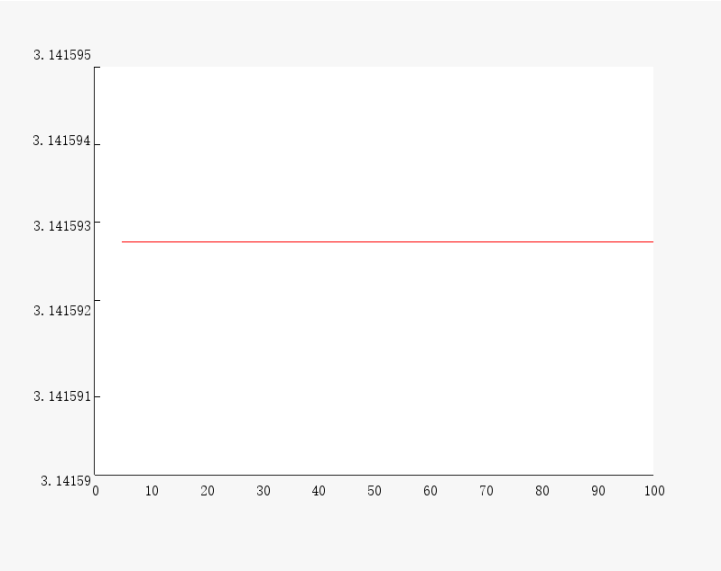

图 **9**　不变 **n1**，变 **n2**

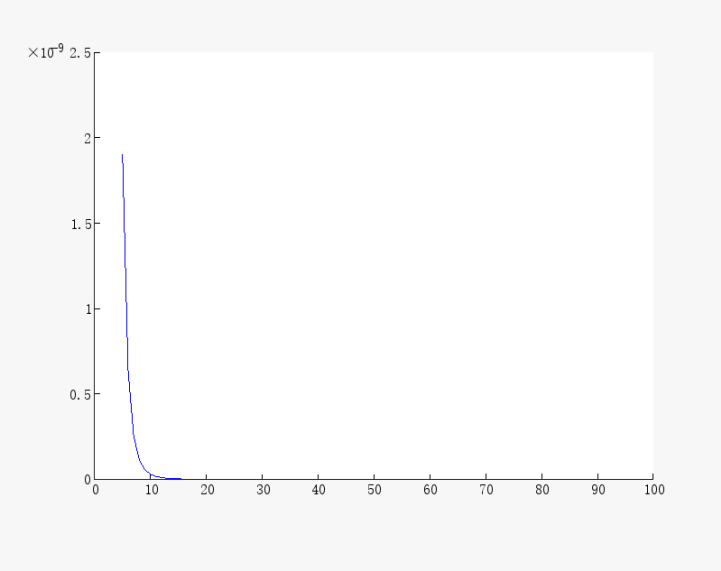

图 **10**　误差

积分值与 π 的误差非常非常接近于 0，导致图像看上去是一条直线

```
列 71 -- 75
 0.000000000000000   0.000000000000001   0.000000000000001   0.000000000000000   0.000000000000001
列 76 -- 80
 0.000000000000000   0.000000000000001   0.000000000000001   0.000000000000001   0.000000000000000
列 81 -- 85
 0.000000000000001   0.000000000000001   0.000000000000000   0.000000000000000   0.000000000000000
列 86 -- 90
 0.000000000000000   0.000000000000000   0.000000000000000   0.000000000000001   0.000000000000000
列 91 -- 95
 0.000000000000000   0.000000000000000   0.000000000000000   0.000000000000000   0.000000000000000
列 96
 0.000000000000000
```

图 **11**　误差的部分截取

Gauss积分节点及权重

n = 1

i	权重	坐标
1	2.0000000000000000	0.0000000000000000

n = 2

i	权重	坐标
1	1.0000000000000000	-0.5773502691896257
2	1.0000000000000000	0.5773502691896257

n = 3

i	权重	坐标
1	0.8888888888888888	0.0000000000000000
2	0.5555555555555556	-0.7745966692414834
3	0.5555555555555556	0.7745966692414834

n = 4

i	权重	坐标
1	0.6521451548625461	-0.3399810435848563
2	0.6521451548625461	0.3399810435848563
3	0.3478548451374538	-0.8611363115940526
4	0.3478548451374538	0.8611363115940526

n = 5

i	权重	坐标
1	0.5688888888888889	0.0000000000000000
2	0.4786286704993665	-0.5384693101056831
3	0.4786286704993665	0.5384693101056831
4	0.2369268850561891	-0.9061798459386640
5	0.2369268850561891	0.9061798459386640

n = 6

i	权重	坐标
1	0.3607615730481386	0.6612093864662645
2	0.3607615730481386	-0.6612093864662645
3	0.4679139345726910	-0.2386191860831969
4	0.4679139345726910	0.2386191860831969
5	0.1713244923791704	-0.9324695142031521
6	0.1713244923791704	0.9324695142031521

n = 7

i	权重	坐标
1	0.4179591836734694	0.0000000000000000
2	0.3818300505051189	0.4058451513773972
3	0.3818300505051189	-0.4058451513773972
4	0.2797053914892766	-0.7415311855993945
5	0.2797053914892766	0.7415311855993945
6	0.1294849661688697	-0.9491079123427585
7	0.1294849661688697	0.9491079123427585

n = 8

i	权重	坐标
1	0.3626837833783620	-0.1834346424956498
2	0.3626837833783620	0.1834346424956498
3	0.3137066458778873	-0.5255324099163290
4	0.3137066458778873	0.5255324099163290
5	0.2223810344533745	-0.7966664774136267
6	0.2223810344533745	0.7966664774136267
7	0.1012285362903763	-0.9602898564975363
8	0.1012285362903763	0.9602898564975363

n = 9

i	权重	坐标
1	0.3302393550012598	0.0000000000000000
2	0.1806481606948574	-0.8360311073266358

3	0.1806481606948574	0.8360311073266358
4	0.0812743883615744	-0.9681602395076261
5	0.0812743883615744	0.9681602395076261
6	0.3123470770400029	-0.3242534234038089
7	0.3123470770400029	0.3242534234038089
8	0.2606106964029354	-0.6133714327005904
9	0.2606106964029354	0.6133714327005904

n = 10

i	权重	坐标
1	0.2955242247147529	-0.1488743389816312
2	0.2955242247147529	0.1488743389816312
3	0.2692667193099963	-0.4333953941292472
4	0.2692667193099963	0.4333953941292472
5	0.2190863625159820	-0.6794095682990244
6	0.2190863625159820	0.6794095682990244
7	0.1494513491505806	-0.8650633666889845
8	0.1494513491505806	0.8650633666889845
9	0.0666713443086881	-0.9739065285171717
10	0.0666713443086881	0.9739065285171717

n = 11

i	权重	坐标
1	0.2729250867779006	0.0000000000000000
2	0.2628045445102467	-0.2695431559523450
3	0.2628045445102467	0.2695431559523450
4	0.2331937645919905	-0.5190961292068118
5	0.2331937645919905	0.5190961292068118
6	0.1862902109277343	-0.7301520055740494
7	0.1862902109277343	0.7301520055740494
8	0.1255803694649046	-0.8870625997680953
9	0.1255803694649046	0.8870625997680953
10	0.0556685671161737	-0.9782286581460570
11	0.0556685671161737	0.9782286581460570

n = 12

i	权重	坐标
1	0.2491470458134028	-0.1252334085114689
2	0.2491470458134028	0.1252334085114689
3	0.2334925365383548	-0.3678314989981802
4	0.2334925365383548	0.3678314989981802
5	0.2031674267230659	-0.5873179542866175
6	0.2031674267230659	0.5873179542866175
7	0.1600783285433462	-0.7699026741943047
8	0.1600783285433462	0.7699026741943047
9	0.1069393259953184	-0.9041172563704749
10	0.1069393259953184	0.9041172563704749
11	0.0471753363865118	-0.9815606342467192
12	0.0471753363865118	0.9815606342467192

n = 13

i	权重	坐标
1	0.2325515532308739	0.0000000000000000
2	0.2262831802628972	-0.2304583159551348
3	0.2262831802628972	0.2304583159551348
4	0.2078160475368885	-0.4484927510364469
5	0.2078160475368885	0.4484927510364469
6	0.1781459807619457	-0.6423493394403402
7	0.1781459807619457	0.6423493394403402
8	0.1388735102197872	-0.8015780907333099
9	0.1388735102197872	0.8015780907333099
10	0.0921214998377285	-0.9175983992229779

11	0.0921214998377285	0.9175983992229779
12	0.0404840047653159	-0.9841830547185881
13	0.0404840047653159	0.9841830547185881

n = 14

i	权重	坐标
1	0.215263853463157	-0.108054948707343
2	0.2152638534631578	0.1080549487073437
3	0.2051984637212956	-0.3191123689278897
4	0.2051984637212956	0.3191123689278897
5	0.1855383974779378	-0.5152486363581541
6	0.1855383974779378	0.5152486363581541
7	0.1572031671581935	-0.6872929048116855
8	0.1572031671581935	0.6872929048116855
9	0.1215185706879032	-0.8272013150697650
10	0.1215185706879032	0.8272013150697650
11	0.0801580871597602	-0.9284348836635735
12	0.0801580871597602	0.9284348836635735
13	0.0351194603317519	-0.9862838086968123
14	0.0351194603317519	0.9862838086968123

n = 15

i	权重	坐标
1	0.2025782419255613	0.0000000000000000
2	0.1984314853271116	-0.2011940939974345
3	0.1984314853271116	0.2011940939974345
4	0.1861610000155622	-0.3941513470775634
5	0.1861610000155622	0.3941513470775634
6	0.1662692058169939	-0.5709721726085388
7	0.1662692058169939	0.5709721726085388
8	0.1395706779261543	-0.7244177313601701
9	0.1395706779261543	0.7244177313601701
10	0.1071592204671719	-0.8482065834104272
11	0.1071592204671719	0.8482065834104272
12	0.0703660474881081	-0.9372733924007060
13	0.0703660474881081	0.9372733924007060
14	0.0307532419961173	-0.9879925180204854
15	0.0307532419961173	0.9879925180204854

n = 16

i	权重	坐标
1	0.1894506104550685	-0.0950125098376374
2	0.1894506104550685	0.0950125098376374
3	0.1826034150449236	-0.2816035507792589
4	0.1826034150449236	0.2816035507792589
5	0.1691565193950025	-0.4580167776572274
6	0.1691565193950025	0.4580167776572274
7	0.1495959888165767	-0.6178762444026438
8	0.1495959888165767	0.6178762444026438
9	0.1246289712555339	-0.7554044083550030
10	0.1246289712555339	0.7554044083550030
11	0.0951585116824928	-0.8656312023878318
12	0.0951585116824928	0.8656312023878318
13	0.0622535239386479	-0.9445750230732326
14	0.0622535239386479	0.9445750230732326
15	0.0271524594117541	-0.9894009349916499
16	0.0271524594117541	0.9894009349916499

n = 17

i	权重	坐标
1	0.1794464703562065	0.0000000000000000
2	0.1765627053669926	-0.1784841814958479

3	0.1765627053669926	0.1784841814958479
4	0.1680041021564500	-0.3512317634538763
5	0.1680041021564500	0.3512317634538763
6	0.1540457610768103	-0.5126905370864769
7	0.1540457610768103	0.5126905370864769
8	0.1351363684685255	-0.6576711592166907
9	0.1351363684685255	0.6576711592166907
10	0.1118838471934040	-0.7815140038968014
11	0.1118838471934040	0.7815140038968014
12	0.0850361483171792	-0.8802391537269859
13	0.0850361483171792	0.8802391537269859
14	0.0554595293739872	-0.9506755217687678
15	0.0554595293739872	0.9506755217687678
16	0.0241483028685479	-0.9905754753144174
17	0.0241483028685479	0.9905754753144174

n = 18

i	权重	坐标
1	0.1691423829631436	-0.0847750130417353
2	0.1691423829631436	0.0847750130417353
3	0.1642764837458327	-0.2518862256915055
4	0.1642764837458327	0.2518862256915055
5	0.1546846751262652	-0.4117511614628426
6	0.1546846751262652	0.4117511614628426
7	0.1406429146706507	-0.5597708310739475
8	0.1406429146706507	0.5597708310739475
9	0.1225552067114785	-0.6916870430603532
10	0.1225552067114785	0.6916870430603532
11	0.1009420441062872	-0.8037049589725231
12	0.1009420441062872	0.8037049589725231
13	0.0764257302548891	-0.8926024664975557
14	0.0764257302548891	0.8926024664975557
15	0.0497145488949698	-0.9558239495713977
16	0.0497145488949698	0.9558239495713977
17	0.0216160135264833	-0.9915651684209309
18	0.0216160135264833	0.9915651684209309

n = 19

i	权重	坐标
1	0.1610544498487837	0.0000000000000000
2	0.1589688433939543	-0.1603586456402254
3	0.1589688433939543	0.1603586456402254
4	0.1527660420658597	-0.3165640999636298
5	0.1527660420658597	0.3165640999636298
6	0.1426067021736066	-0.4645707413759609
7	0.1426067021736066	0.4645707413759609
8	0.1287539625393362	-0.6005453046616810
9	0.1287539625393362	0.6005453046616810
10	0.1115666455473340	-0.7209661773352294
11	0.1115666455473340	0.7209661773352294
12	0.0914900216224500	-0.8227146565371428
13	0.0914900216224500	0.8227146565371428
14	0.0690445427376412	-0.9031559036148179
15	0.0690445427376412	0.9031559036148179
16	0.0448142267656996	-0.9602081521348300
17	0.0448142267656996	0.9602081521348300
18	0.0194617882297265	-0.9924068438435844
19	0.0194617882297265	0.9924068438435844

n = 20

i	权重	坐标

1	0.1527533871307258	-0.0765265211334973
2	0.1527533871307258	0.0765265211334973
3	0.1491729864726037	-0.2277858511416451
4	0.1491729864726037	0.2277858511416451
5	0.1420961093183820	-0.3737060887154195
6	0.1420961093183820	0.3737060887154195
7	0.1316886384491766	-0.5108670019508271
8	0.1316886384491766	0.5108670019508271
9	0.1181945319615184	-0.6360536807265150
10	0.1181945319615184	0.6360536807265150
11	0.1019301198172404	-0.7463319064601508
12	0.1019301198172404	0.7463319064601508
13	0.0832767415767048	-0.8391169718222188
14	0.0832767415767048	0.8391169718222188
15	0.0626720483341091	-0.9122344282513259
16	0.0626720483341091	0.9122344282513259
17	0.0406014298003869	-0.9639719272779138
18	0.0406014298003869	0.9639719272779138
19	0.0176140071391521	-0.9931285991850949
20	0.0176140071391521	0.9931285991850949