MA 266: Midterm 2 — Thursday Mar 21

Name: **Section number:**

Time: 70 minutes.

This exam has 8 questions. Answer *all* questions. All questions are of equal credit.

Show *detailed* working.

No calculators.

Put away books, notes, calculators, cell phones, and other electronic devices. No discussion during the exam.

Question 1. (a): Find the solution of the initial value problem

$$y'' + 4y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 0.$$

(b): How does the solution behave as $t \to +\infty$?

Answer. (a): The characteristic equation is

$$r^2 + 4r + 5 = 0.$$

Its roots are

$$r = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm \mathrm{i}.$$

Hence the general solution is

$$y(t) = c_1 \mathrm{e}^{-2t} \cos t + c_2 \mathrm{e}^{-2t} \sin t.$$

Since

$$y'(t) = c_1 \left(-2\mathrm{e}^{-2t} \cos t - \mathrm{e}^{-2t} \sin t \right) + c_2 \left(-2\mathrm{e}^{-2t} \sin t + \mathrm{e}^{-2t} \cos t \right),$$

we find that

$$1 = y(0) = c_1, \quad 0 = -2c_1 + c_2.$$

Hence $c_1 = 1$ and $c_2 = 2$ and we get

$$y(t) = \mathrm{e}^{-2t} \cos t + 2\mathrm{e}^{-2t} \sin t.$$

(b): $y(t) \to 0$ as $t \to \infty$.

Question 2. Consider the differential equation.

$$x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0.$$

(a): Show that $y_1(x) = x$ and $y_2(x) = x\mathrm{e}^x$ are solutions of the differential equation.

(b): Determine the Wronskian $W(y_1, y_2)(x)$ of y_1 and y_2.

(c): Do $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions? Explain your answer.

Answer. (a): Note that

$$x^2y_1'' - x(x+2)y_1' + (x+2)y_1 = -x(x+2) + (x+2)x = 0,$$

and also that

$$\begin{aligned} &x^2y_2'' - x(x+2)y_2' + (x+2)y_2 \\ &\quad = x^2\left(2\mathrm{e}^x + x\mathrm{e}^x\right) - x(x+2)\left(\mathrm{e}^x + x\mathrm{e}^x\right) + (x+2)x\mathrm{e}^x \\ &\quad = \left(2x^2 + x^3 - x^2 - 2x - x^3 - 2x^2 + x^2 + 2x\right)\mathrm{e}^x \\ &\quad = 0. \end{aligned}$$

Hence y_1 and y_2 are solutions.

(b): We have

$$W(y_1, y_2)(x) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} x & x\mathrm{e}^x \\ 1 & \mathrm{e}^x + x\mathrm{e}^x \end{pmatrix} = x^2\mathrm{e}^x.$$

(c): Since $x^2\mathrm{e}^x \neq 0$ for $x \neq 0$ the Wronskian $W(y_1, y_2)(x)$ is nonzero for at least one x, and therefore y_1 and y_2 are fundamental.

Question 3. A mass weighing 8lb stretches a spring 0.5ft. The mass is pulled down 1ft from the equilibrium position and then set in motion with an upward velocity of 2ft/sec. Assume that there is no damping force and that the downward direction is the positive direction. The gravity constant g is 32ft/sec^2.

(a): Show that the function $u(t)$ describing the displacement of the mass from the equilibrium position as a function of time t satisfies the differential equation:

$$u'' + 64u = 0.$$

(b): Determine the initial conditions for $u(t)$.

(c): By solving the corresponding initial value problem, find $u(t)$ explicitly in terms of t.

Answer.

For an undamped spring with no external force,

$$mu''(t) + ku(t) = 0.$$

In this case

$$m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4}\frac{\text{lb sec}^2}{\text{ft}}$$

and

$$k = \frac{8\text{lb}}{0.5\text{ft}} = 16\frac{\text{lb}}{\text{ft}}.$$

Hence

$$\frac{1}{4}u'' + 16u = 0 \quad \Leftrightarrow \quad u'' + 64u = 0.$$

(b): The initial conditions are $u(0) = 1\text{ft}$ and $u'(0) = -2\text{ft/sec}$.

(c): The general solution is

$$u(t) = c_1 \cos 8t + c_2 \sin 8t.$$

The initial conditions give

$$c_1 = 1, \quad 8c_2 = -2.$$

Hence

$$u(t) = \cos 8t - \frac{1}{4}\sin 8t.$$

Question 4. Consider the differential equation

$$t^2y'' - 2y = 3t^2 - 1, \quad t > 0.$$

(a): Show that the functions $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are solutions of the corresponding homogeneous equation.

(b): Show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions for the homogeneous equation.

(c): It is known that the particular solution of the differential equation takes the form

$$A + t^2 \log(t),$$

for some constant A. By determining the value for the constant A, find the general solution $y(t)$ of the differential equation.

Answer. (a): We have

$$t^2y_1'' - 2y_1 = 2t^2 - 2t^2 = 0, \quad t^2y_2'' - 2y_2 = 2t^{-1} - 2t^{-1} = 0,$$

hence y_1 and y_2 are solutions of the homogeneous DE.

(b): The Wronskian is

$$W(y_1, y_2)(t) = \det \begin{pmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{pmatrix} = -1 - 2 = -3 \neq 0.$$

Hence y_1 and y_2 are fundamental.

(c): Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^2 \log t) = 2t \log t + t, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}(t^2 \log t) = 2 \log t + 3$$

Hence

$$\begin{aligned} t^2y''(t) - 2y(t) &= t^2 (2 \log t + 3) - 2 \left(A + t^2 \log t\right) \\ &= 3t^2 - 2A \end{aligned}$$

Thus $A = \frac{1}{2}$, and therefore

$$y_p(t) = \frac{1}{2} + t^2 \log(t).$$

The general solution is consequently

$$y(t) = c_1t^2 + c_2t^{-1} + \frac{1}{2} + t^2 \log(t).$$

Question 5. Find the general solution of the differential equation

$$y'' + y' + 4y = 2\sinh t.$$

Hint: write $\sinh t = \left(\mathrm{e}^{t} - \mathrm{e}^{-t}\right)/2$.

Answer. The characteristic equation is

$$r^2 + r + 4 = 0.$$

Hence

$$r = \frac{-1 \pm \sqrt{1-16}}{2} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2}\mathrm{i},$$

and therefore the complementary solution is

$$y_c(t) = c_1\mathrm{e}^{-t/2}\cos\left(\sqrt{15}t/2\right) + c_2\mathrm{e}^{-t/2}\sin\left(\sqrt{15}t/2\right).$$

Since $r = 1$ and $r = -1$ are not roots of the characteristic equation, the particular solution takes the form

$$y_p(t) = A\mathrm{e}^{t} + B\mathrm{e}^{-t},$$

for constants A and B. We have

$$y_p'' + y_p' + 4y_p = 6A\mathrm{e}^{t} + 4B\mathrm{e}^{-t}.$$

Therefore, $A = \frac{1}{6}$ and $B = -\frac{1}{4}$, and we get

$$y_p(t) = \frac{1}{6}\mathrm{e}^{t} - \frac{1}{4}\mathrm{e}^{-t}.$$

Thus the general solution is

$$y(t) = c_1\mathrm{e}^{-t/2}\cos\left(\sqrt{15}t/2\right) + c_2\mathrm{e}^{-t/2}\sin\left(\sqrt{15}t/2\right) + \frac{1}{6}\mathrm{e}^{t} - \frac{1}{4}\mathrm{e}^{-t}$$

Question 6. Find the solution of the initial value problem

$$y''' - y'' + y' - y = 0, \qquad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = -2.$$

Answer.

The characteristic equation is

$$r^3 - r^2 + r - 1 = 0.$$

Note that $r = 1$ is a root. Hence

$$(r-1)(r^2+1) = 0.$$

Thus the roots are $r = 1, \mathrm{i}, -\mathrm{i}$ and the general solution is

$$y(t) = c_1 \mathrm{e}^t + c_2 \cos t + c_3 \sin t.$$

Substituting the initial conditions, we get

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 + c_3 &= -1 \\ c_1 - c_2 &= -2. \end{aligned}$$

Solving for c_1 and c_2 (using equations 1 and 3), we obtain

$$c_1 = 0, \quad c_2 = 2.$$

The second equation now gives $c_3 = -1$. Hence

$$y(t) = 2\cos t - \sin t.$$

Question 7. (a): Find the general solution of the differential equation

$$y'' + 8y' + 16y = 0. \tag{1}$$

(b): Find the general solution of the differential equation

$$y^{(4)} + 8y^{(2)} + 16y = 0. \tag{2}$$

(c): Is the following statement true or false? *All solutions of (1) tend to zero as* $t \to \infty$, *whereas no nonzero solution of (2) does.*

Answer. (a): The characteristic equation is

$$r^2 + 8r + 16 = 0,$$

and this is equivalent to

$$(r+4)^2 = 0.$$

Hence $r = -4$ is a double root of the equation. Therefore the general solution is

$$y(t) = c_1 \mathrm{e}^{-4t} + c_2 t \mathrm{e}^{-4t}$$

(b): The characteristic equation is

$$r^4 + 8r^2 + 16 = 0.$$

Hence

$$(r^2 + 4)^2 = 0.$$

Thus the equation has double roots at $r = \pm 2\mathrm{i}$. Thus the general solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + c_3 t \cos 2t + c_4 t \sin 2t.$$

(c): True.

Question 8. Consider the differential equation

$$t(t-1)y^{(3)} + \mathrm{e}^t y'' + 4t^2 y = 0.$$

(a): Is the differential equation linear or nonlinear? Explain your answer.

(b): What is the order of the differential equation?

(c): If the initial conditions

$$y(1/2) = 0, \quad y'(1/2) = 1, \quad y''(1/2) = -4,$$

are given, determine an interval I in which the solution of the corresponding initial value problem is guaranteed to exist uniquely.

Answer.

(a): Linear, since it is of the form $P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \ldots + P_{n-1}(t)y' + P_n(t)y = G(t)$.

(b): Third order.

(c): Dividing by $t(t-1)$, we note that the DE has the form

$$y^{(3)} + \frac{\mathrm{e}^t}{t(t-1)}y'' + \frac{4t^2}{t(t-1)}y = 0.$$

Thus all terms are continuous except possibly when $t = 0$ and $t = 1$. Hence the possible choices for I are

$$(-\infty, 0),\ (0, 1),\ (1, \infty).$$

Since the solution must satisfy the initial conditions at $t = 1/2$, we must have

$$I = (0, 1).$$