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ECE 20200 : Linear Circuit Analysis II
School of ECE, Purdue University

LECTURE 4

- Properties of Laplace Transform (continued)
- Examples

Reference: Decarlo/Lin pp 575-584

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Properties of Laplace Transform (continued)

Covered in Lecture 2.

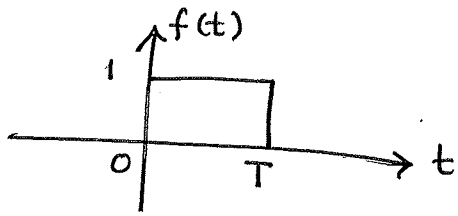
1) Linearity $\mathcal{L}\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)$

2) Time-shift $\mathcal{L}\left[f(t-T)u(t-T)\right] = e^{-sT} F(s) \quad T > 0$

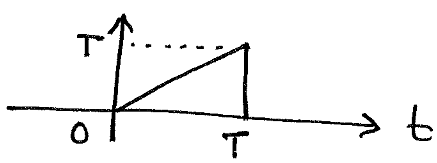
3) Time-multiplication $\mathcal{L}\left[t f(t)\right] = -\frac{d}{ds} F(s)$

Example:

$$f(t) = u(t)u(T-t) = u(t) - u(t-T)$$



$$g(t) = t f(t)$$



$$F(s) = \mathcal{L}\left[f(t)\right] = \frac{1}{s} - \frac{e^{-sT}}{s}$$

$$G(s) = -\frac{d}{ds} F(s) = \frac{1}{s^2} - \frac{e^{-sT}}{s^2} - \frac{Te^{-sT}}{s}$$

Remark:

$$\mathcal{L}\left[f(t-T)\right] = e^{-sT} F(s), \quad \text{then}$$

$$\mathcal{L}\left[t f(t-T)\right] = -\frac{d}{ds}\left[e^{-sT} F(s)\right]$$

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4) Multiplication by t^n

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F(s)}{ds^n}$$

5) Frequency Shift

$$\mathcal{L}\left[f(t)\right] = F(s), \quad \text{then}$$

$$\mathcal{L}\left[e^{-at} f(t)\right] = F(s+a)$$

Proof:
$$\mathcal{L}\left[e^{-at} f(t)\right] = \int_{0^-}^{\infty} f(t)\, e^{-at}\, e^{-st}\, dt$$
$$= \int_{0^-}^{\infty} f(t)\, e^{-\overbrace{(s+a)}^{\hat{s}}t}\, dt$$
$$= F(\overbrace{s+a}^{\hat{s}})$$

Example:

$$f(t) = \cos(\omega t)\, u(t) \Rightarrow F(s) = \frac{s}{s^2+\omega^2}$$

$$g(t) = e^{-at} f(t) = e^{-at}\cos(\omega t)\, u(t)$$

$$\Rightarrow G(s) = F(\hat{s})\Big|_{\hat{s}=s+a} = \frac{s+a}{(s+a)^2+\omega^2} \leftarrow$$

Similarly,

$$\mathcal{L}\left[e^{-at}\sin(\omega t)\, u(t)\right] = \frac{\omega}{\hat{s}^2+\omega^2}\Bigg|_{\hat{s}=s+a}$$
$$= \frac{\omega}{(s+a)^2+\omega^2} \leftarrow$$

6) Time or frequency scaling

Let $a > 0$ and $F(s) = \mathcal{L}[f(t)]$. Then

$$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Example:

$$\mathcal{L}[\delta(at)] = \frac{1}{a} \cdot 1 = \frac{1}{a}$$

Example:

$$\mathcal{L}[\sin(t)u(t)] = \frac{1}{s^2+1} \text{. Then}$$

$$\mathcal{L}[\sin(\omega t)] = \frac{1}{\omega} \cdot \frac{1}{\left(\frac{s}{\omega}\right)^2+1} = \frac{\omega}{s^2+\omega^2}$$

7) Time differentiation

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0^-)$$

$$\mathcal{L}\left[\frac{d^2}{dt^2} f(t)\right] = s^2F(s) - sf(0^-) - \dot{f}(0^-)$$

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\dot{f}(0^-) - \cdots - f^{(n-1)}(0^-)$$

Interpretation 1: Differentiation in time domain means multiplication by 's' in frequency domain (s-domain).

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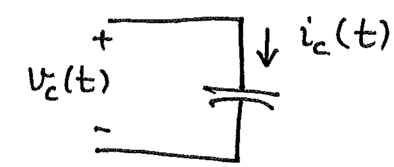
Example:

$$\mathcal{L}[\delta(t)] = \mathcal{L}\left[\frac{d}{dt}u(t)\right]$$
$$= s \cdot \frac{1}{s} - u(0^-)$$
$$= 1 - 0$$
$$= 1 \leftarrow$$

Interpretation 2:

Example: A NEW s-domain equivalent circuit of a charged capacitor

time-domain



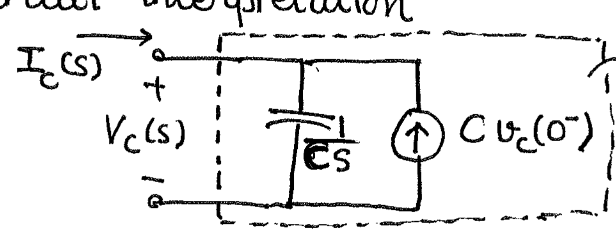
$$i_c(t) = C\frac{dv_c(t)}{dt}$$
$$i_c(t) = C\,\dot{v}_c(t)$$

s-domain

$$\mathcal{L}[i_c(t)] = \mathcal{L}[C\,\dot{v}_c(t)]$$
$$I_c(s) = C\,\mathcal{L}[\dot{v}_c(t)]$$
$$I_c(s) = C\left(s V_c(s) - v_c(0^-)\right)$$
$$I_c(s) = C s V_c(s) - C v_c(0^-)$$

currents

circuit interpretation



model of charged capacitor in s-domain

Example: Find solution to differential equation

$$f''(t) = 2u(t) \text{ when } f'(0^-) \neq 0 \text{ and}$$

$$f(0^-) \neq 0$$

Solution: Take Laplace Transform on both sides

$$s^2F(s) - sf(0^-) - f'(0^-) = \frac{2}{s}$$

$$s^2F(s) = \frac{2}{s} + sf(0^-) + f'(0^-)$$

$$F(s) = \frac{2}{s^3} + \frac{f(0^-)}{s} + \frac{f'(0^-)}{s^2}$$

$$\therefore f(t) = 2\frac{t^2}{2}u(t) + f(0^-)u(t) + f'(0^-)tu(t)$$

8) Time Integration

$$\mathcal{L}\left[\int_{-\infty}^{t} f(q)\,dq\right] = \frac{F(s)}{s} + \frac{\int_{-\infty}^{0^-} f(q)\,dq}{s}$$

$$\mathcal{L}\left[\int_{0^-}^{t} f(q)\,dq\right] = \frac{F(s)}{s}$$

Example: Find $f(t)$ when

$$\int_{0^-}^{t} f(q)\,dq + \frac{d}{dt}f(t) = 2u(t)$$

Assume $f(0^-) \neq 0$ and $\int_{-\infty}^{0^-} f(\tau)\,d\tau = 0$

$$\int_{0^-}^{t} f(q)\,dq + \frac{d}{dt} f(t) = 2u(t)$$

Take Laplace transform on both sides

$$\frac{F(s)}{s} + sF(s) - f(0^-) = \frac{2}{s}$$

$$\frac{F(s)}{s} + sF(s) = \frac{2}{s} + f(0^-)$$

$$F(s) + s^2 F(s) = 2 + s f(0^-)$$

$$(s^2+1)\,F(s) = 2 + s f(0^-)$$

$$F(s) = \frac{2}{s^2+1} + \frac{f(0^-)\,s}{s^2+1}$$

$$\therefore\ f(t) = 2\sin(t)\,u(t) + f(0^-)\cos(t)\,u(t)$$