Name (print): ____

CS182 **Sample Final Exam**

- This exam is **closed book, closed notes** and is designed to take **2 hours**.
- Understanding a question *is part of the exam*, so please do not ask for additional comments or hints (unless you think the question is ill-formulated, contains typos, inconsistent or ambiguous wording, etc).
- Unless otherwise specified, there is no need to justify the answer you give to a question.
- The questions are not listed in any particular order (they are *not* listed by increasing order of difficulty).
- Use the blank pages as a rough working area (they will *not* be graded, so make sure you write each answer in the space provided immediately near the question).

Question 1. (5+5 points) Simplify the following Boolean expression, where $\overline{p}$ denotes NOT p, $+$ denotes OR, and $\cdot$ denotes AND.

1. $(p+q)\cdot\overline{p}\cdot\overline{q}$

   Answer: ___________________

2. $(p+p\cdot q)\cdot\overline{p\cdot\overline{q}}$

   Answer: ___________________

Question 2. (4 points) 51 students were enrolled in a class whose instructor has given 5 different kinds of grades: A, B, C, D, F (no grades with $+$ or $-$ superscripts). Consider the following statement about the grade distribution:

"There are at least k students who have the same grade as each other"

What is the largest k for which the above statement is always true ?

Answer: ___________________

Question 3. (5 points) The solution to the recurrence $a_n = 6a_{n-1} - 9a_{n-2}$ is of the following form (choose one):

1. $a_n = c_1 3^n + c_2$ where c_1 and c_2 are constants that depend on the boundary conditions.
2. $a_n = c_1 2^n + c_2 3^n$ where c_1 and c_2 are constants that depend on the boundary conditions.
3. $a_n = c_1 3^n + c_2 n 3^n$ where c_1 and c_2 are constants that depend on the boundary conditions.
4. $a_n = c_1 6^n + c_2 9^n$ where c_1 and c_2 are constants that depend on the boundary conditions.

Question 4. (5 points) Let T be a tree (not necessarily binary). Let $Preorder[v]$ denote the preorder number of a node v, and let $Postorder[v]$ denote the postorder number of that node. State how, using only the available $Preorder$ and $Postorder$ information, one can determine in constant time (i.e., in time that does not depend on the height of the tree or its number of nodes) whether two nodes v and w are related by the "ancestor" relationship or not.

Answer: ______________________________________

Question 5. (9 points) Mark by T (= True) or F (= False) each of the following statements.

1. _____ A binary tree always contains a node whose removal from breaks the tree into pieces none of which has more than $n/2$ nodes, but this may not be true if the tree is not binary
2. _____ Let T be an n-node tree (not necessarily binary). For each node v in T, let $Desc[v]$ be the number of descendants of v, i.e., the number of nodes in the subtree of v (counting v as being in its own subtree). For example, if v is the root then $Desc[v] = n$, and if v is a leaf then $Desc[v] = 1$. Then T always contains a node v such that $n/3 \le Desc[v] \le 2n/3$.

3. _____ The following statement is always true: $\exists x(P(x) \to \forall y P(y))$

Question 6. (7 points) Let W be a permutation of the integers $1, 2, \cdots, 50$. In other words W contains every one of the first 50 integers, each exactly once. You are given a sheet of paper on which W was written with a pencil. You are also given an eraser, and you are asked to erase as few integers as possible from W such that the surviving sequence of integers is either increasing or decreasing when read in left-to-right order (either is fine). We assume that erasing a two-digit integer like 37 counts as a single erasure, not two (just the same as erasing a single-digit integer like 9); therefore all of W can be erased using 50 erasure operations.

Using the pigeonhole principle, it is possible to prove that for every such sequence W, you never need to erase more than k integers from W (which is equivalent to proving that there is always a valid surviving sequence of containing $50 - k$ or more integers). What is the smallest value of k for which such a proof can be given?

Answer: $k =$__________________ (When writing down your answer please do not confuse k with $n-k$; k is a number of erasures, not a length of a valid surviving sequence.)

Question 7. (8 points) Simplify the following expression (assuming $n \geq 20$):

$$C(2n, n) + 4C(2n, n-1) + 6C(2n, n-2) + 4C(2n, n-3) + C(2n, n-4)$$

Answer: __________________

Question 8. (6 points) A fair coin is flipped $2n$ times, $n \geq 20$. Let h_k be the number of "head" and t_k be the number of "tail" after the kth coin flip, for $k = 1, 2, \ldots, 2n$ (of course $h_k + t_k = k$). For example, if the first three coin flips come up (head, tail, head) then $h_3 = 2$ and $t_3 = 1$. Let $m < n$. Suppose we want the probability of the following event: $h_n = m$ and $h_{2n} = n$. In other words, we want the probability that the $2n$ coin flips result in n heads and, along the way, the first n coin flips result in m heads. Choose one of the following possible answers (circle only one of the choices).

1. $2^{-2n}C(n, m)C(2n, n)$
2. $2^{-2n}C(n, m)C(n, n-m)$
3. $2^{-n-m}C(n, m)C(2n, n)$
4. $2^{-n-m}C(n, m)C(n, n-m)$
5. None of the above

Question 9. (5 points) Consider the Boolean function $f(x, y) = x \cdot \overline{y} + \overline{x} \cdot y$ where $\overline{x}$ denotes NOT x, $+$ denotes OR, and $\cdot$ denotes AND. (You will recognize that $f(x, y)$ is the "exclusive OR" of x and y.) Simplify $f(f(x, f(x, y)), y)$, i.e., write a simpler expression for it.

Answer: __________________

Question 10. (8+4 points) Let T be a binary tree whose 7 nodes are named a, b, c, d, e, f, g. A preorder traversal of T lists its nodes in the following order:

$$e\ b\ g\ c\ f\ a\ d$$

An inorder traversal of T lists its nodes in the following order:

$$g\ b\ f\ c\ a\ e\ d$$

Draw T in the space below, then give a listing of its nodes according to a postorder traversal.

Question 11. (4+4+6 points) Suppose the alphabet consists of only the five characters a, b, c, d, e and that their respectivie probabilities are $0.12, 0.40, 0.15, 0.08, 0.25$. Consider the following codes.

1. $a = 0,\ b = 10, c = 110\ , d =\ 1110, e = 1111$
2. $a = 10,\ b = 0, c = 110\ , d = 1111\ , e = 1110$
3. $a = 0,\ b = 10, c =\ 11, d =\ 110, e = 111$
4. $a = 110,\ b = 0, c = 1110\ , d = 1111\ , e = 10$

- One of the above codes is not a prefix code (and is therefore not practical). Which one is it?

  Answer: ___________________

- Among the three prefix codes listed above, which one has smallest average code length, and what is that length?

  Answer: ___________________

- The best (i.e., smallest) possible average code length for the given alphabet is not achieved by any of the given three prefix codes. Give an optimal code, and state its average code length.

  Answer: ___________________

Question 12. (5+5+5 points) Write down, for each of the following three finite state machines with no output, a brief description of the binary string that it accepts.

Note by MJA: I seem to have lost the .eps file for this pic – they were simple machines, very similar to the examples covered in class.