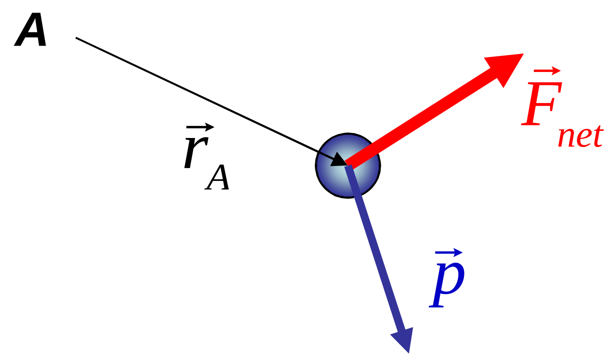
PHYS 172: Modern Mechanics **Spring 2012**

Lecture 20 – Angular momentum Read 11.4 – 11.7

The angular momentum principle

$$\frac{d\vec{p}}{dt} = \vec{F}_{net}$$



$$\frac{d\vec{L}_A}{dt} = ? \Rightarrow \frac{d\left(\vec{r}_A \times \vec{p}\right)}{dt} = \underbrace{\frac{d\vec{r}_A}{dt} \times \vec{p}}_{= \vec{v} \times \gamma m \vec{v} = 0} + \vec{r}_A \times \underbrace{\frac{d\vec{p}}{dt}}_{= \vec{F}_{net}}$$

torque:

$$\vec{\tau}_A \equiv \vec{r}_A \times \vec{F}_{net}$$

The angular momentum principle for a point particle

$$\frac{d\vec{L}_A}{dt} = \vec{r}_A \times \vec{F}_{net} = \vec{\tau}_A$$

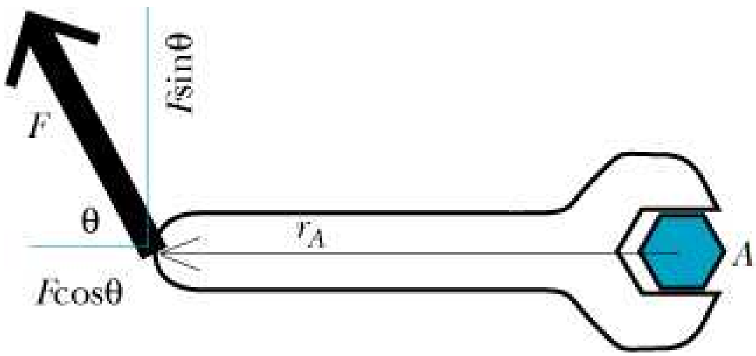
$$\Delta\vec{L}_A = \left(\vec{r}_A \times \vec{F}_{net}\right)\Delta t = \vec{\tau}_A \Delta t$$

Note:
The angular momentum principle is derived from the momentum principle

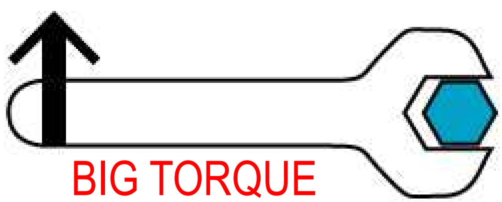
Torque

$$\vec{\tau}_A \equiv \vec{r}_A \times \vec{F}_{net}$$

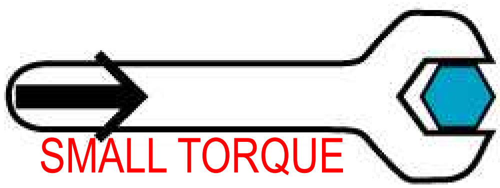
$$\left|\vec{\tau}_A\right| \equiv \left|\vec{r}_A\right| \cdot \left|\vec{F}_{net}\right| \cdot \sin\theta$$

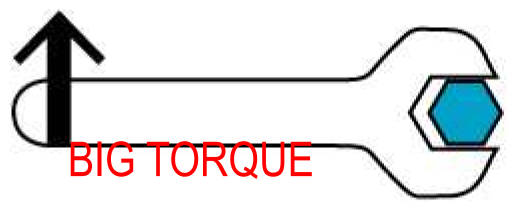


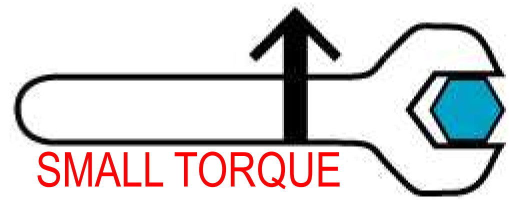
θ=90°



θ=0°







Example:
momentum and angular momentum principles

Use the momentum principle: $\boxed{\dfrac{d\vec{p}}{dt} = \vec{F}_{net}}$

$$\frac{d(mv)}{dt} = mg \qquad \frac{dv}{dt} = g$$

Use the angular momentum principle:

$$\boxed{\frac{d\vec{L}_A}{dt} = \vec{r}_A \times \vec{F}_{net} = \vec{\tau}_A}$$

$$\tau_A = xmg$$

$$\vec{L}_A = \vec{r}_A \times \vec{p}$$

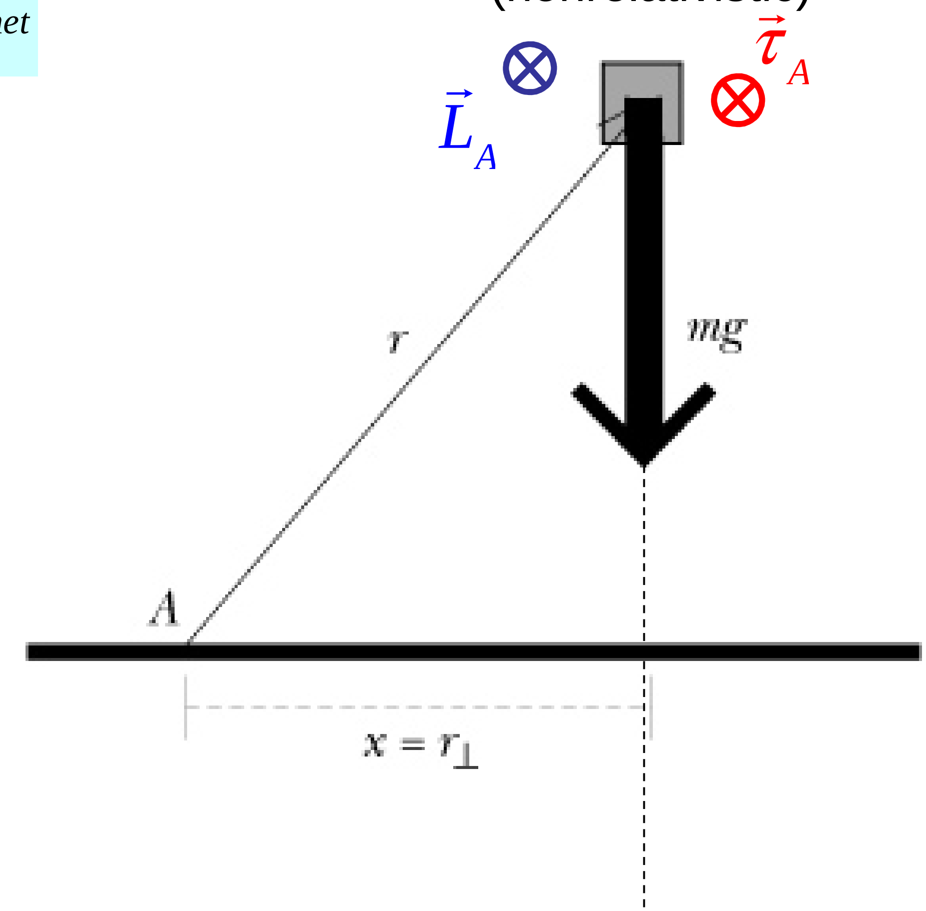
$$L_A = r_\perp p = xmv$$

$$\frac{d(xmv)}{dt} = xm\frac{dv}{dt}$$

$$\frac{dv}{dt} = g$$

Falling object

(nonrelativistic)

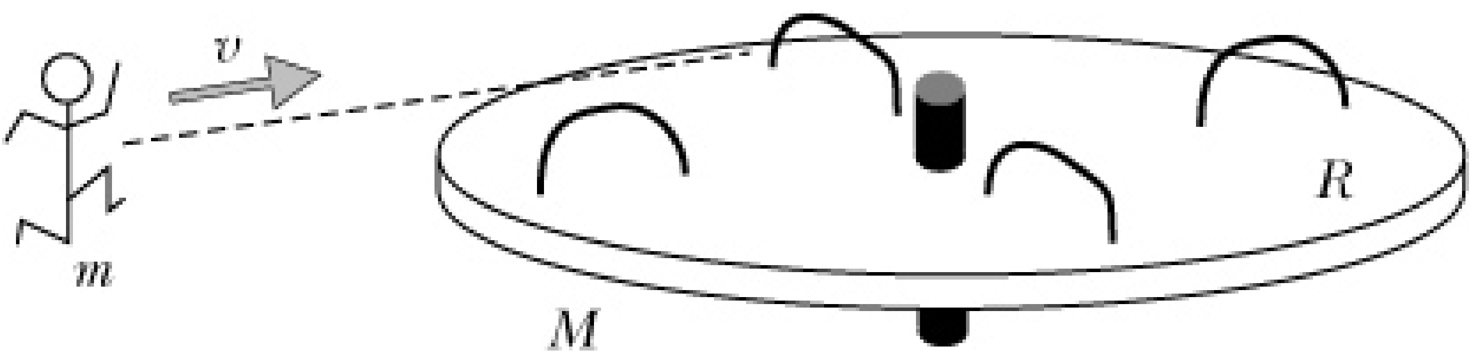


Conservation of angular momentum

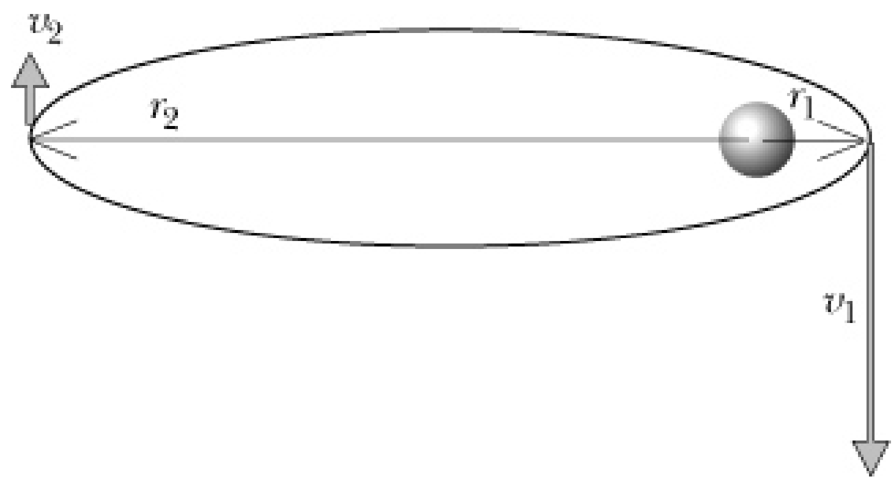
$$\boxed{\Delta\vec{L}_{A,system} + \Delta\vec{L}_{A,surroundings} = 0}$$

Important: both *L*'s must be about the same point (axis)

Example:



A comet



$$\frac{d\vec{L}_A}{dt} = \vec{r}_A \times \vec{F}_{net} = \vec{\tau}_A$$

CLICKER: What is the direction of the torque on the comet in point B about the star due to gravitational pull?

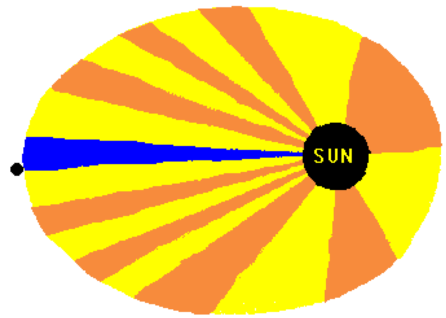
A) Into the page
B) Out of the page
C) It is zero

$$\vec{L}_A = \vec{r}_A \times \vec{p} \longrightarrow r_1 m v_1 = r_2 m v_2 \longrightarrow r_1 v_1 = r_2 v_2$$

(nonrelativistic)

Example: Kepler and elliptical orbits

Kepler, 1609: “a radius vector joining any planet to the Sun sweeps out equal areas in equal lengths of time”



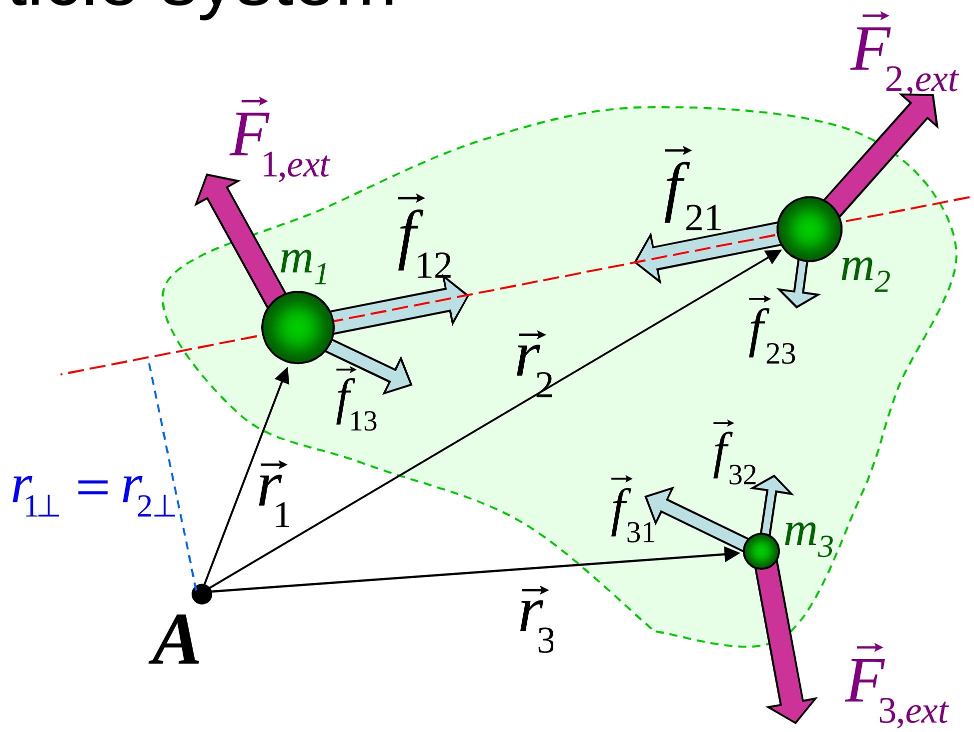
Can be easily proven using conservation of angular momentum
See book p. 430 (11.4)

Multiparticle system

$$\frac{d\vec{L}_1}{dt} = \vec{r}_1 \times \vec{F}_{1,ext} + \boxed{\vec{r}_1 \times \vec{f}_{12}} + \vec{r}_1 \times \vec{f}_{13}$$

$$\frac{d\vec{L}_2}{dt} = \vec{r}_2 \times \vec{F}_{2,ext} + \boxed{\vec{r}_2 \times \vec{f}_{21}} + \vec{r}_2 \times \vec{f}_{23}$$

$$\frac{d\vec{L}_3}{dt} = \vec{r}_3 \times \vec{F}_{3,ext} + \vec{r}_3 \times \vec{f}_{31} + \vec{r}_3 \times \vec{f}_{32}$$



Internal forces produce no torque!

$$\frac{d\overbrace{\left(\vec{L}_1 + \vec{L}_2 + \vec{L}_3\right)}^{\vec{L}_{tot,A}}}{dt} = \frac{d\vec{L}_1}{dt} + \frac{d\vec{L}_2}{dt} + \frac{d\vec{L}_3}{dt} = \overbrace{\vec{r}_1 \times \vec{F}_{1,ext} + \vec{r}_2 \times \vec{F}_{2,ext} + \vec{r}_3 \times \vec{F}_{3,ext}}^{\vec{\tau}_{net,ext,A}}$$

The angular momentum principle for a multiparticle system

$$\boxed{\begin{aligned} \frac{d\vec{L}_{tot,A}}{dt} &= \vec{\tau}_{net,ext,A} \\ \Delta\vec{L}_{tot,A} &= \vec{\tau}_{net,ext,A}\Delta t \end{aligned}}$$

The angular momentum principle relative to the center of mass:

$$\frac{d\vec{L}_{cm}}{dt} = \frac{d}{dt}\left[\left(\vec{r}_{cm,cm} \times \vec{P}_{tot}\right) + \vec{L}_{rot}\right] = \frac{d\vec{L}_{rot}}{dt}$$

$$\boxed{\begin{aligned} \frac{d\vec{L}_{rot}}{dt} &= \vec{\tau}_{net,cm} \\ \Delta\vec{L}_{rot} &= \vec{\tau}_{net,cm}\Delta t \end{aligned}}$$

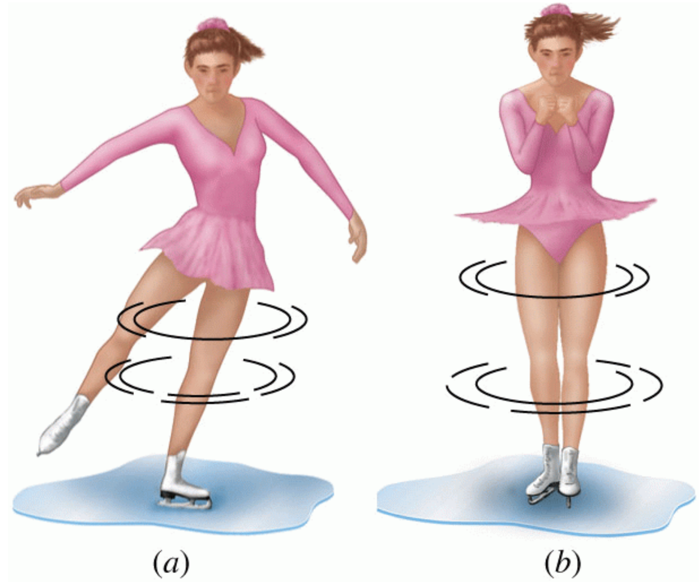
The Three Fundamental Principles

Momentum	Angular Momentum	Energy
$\frac{d\vec{P}}{dt} = \vec{F}_{net,ext}$	$\frac{d\vec{L}_A}{dt} = \vec{\tau}_{net,ext,A}$	$\Delta E = W + Q$
If external forces: momentum changes.	If external torques: angular momentum changes	If work is done: energy of system changes
If no external forces: Momentum of system is constant	If no external torques: Angular momentum of system is constant	If no work done: Energy of system is constant

NOTE: Emmy Noether showed that the three conservation laws are deeply connected with symmetries in our physical laws. It's a fascinating insight that we don't have time to explore!

Angular momentum: a system with no torque

Dorothy Hamill , 1985



$$\frac{d\vec{L}_{rot}}{dt} = \vec{\tau}_{net,cm} = \vec{0}$$

$$\vec{L}_{rot} = I\vec{\omega}$$

$$I_i > I_f$$

$$I = m_1 r_{\perp 1}^2 + m_2 r_{\perp 2}^2 + \ldots$$

$$I_f \omega_f = I_i \omega_i$$

$$\omega_f = \frac{I_i}{I_f}\omega_i$$

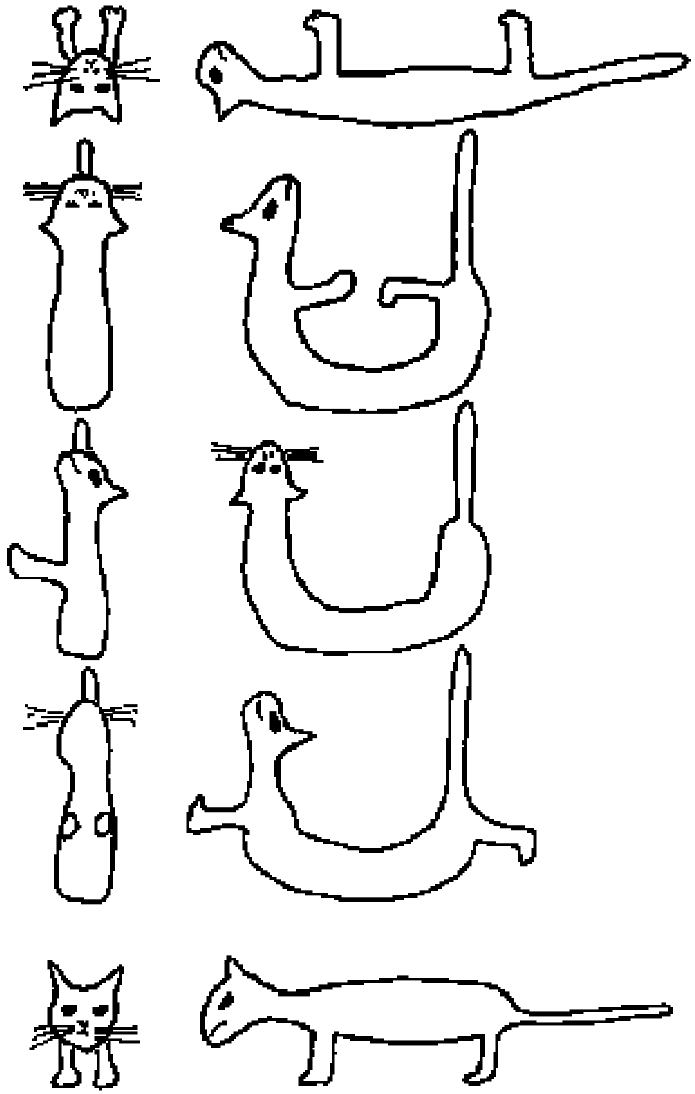
http://www.hep.phys.soton.ac.uk/courses/phys2006/

Angular momentum: a system with no torque

Cat always lands on its feet

http://www.youtube.com/watch?v=RHhXbOhK_hs

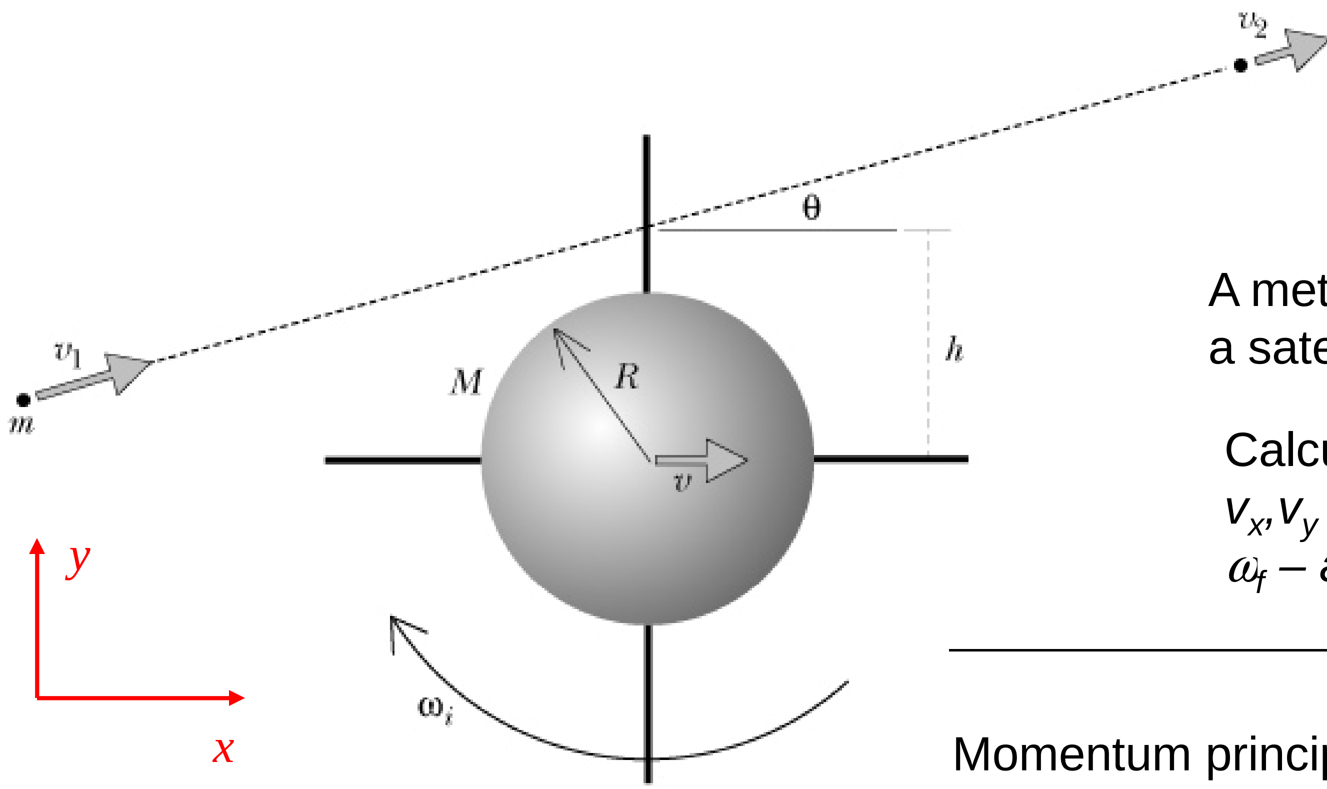
Angular momentum: application



A free-falling cat cannot alter its total angular momentum. Nonetheless, by swinging its tail and twisting its body to alter its moment of inertia, the cat can manage to alter its orientation

See also book example: High dive page 437

Angular momentum: application



A meteor rips through
a satellite with solar panels.

Calculate:
v_x,v_y of center of mass
ω_f – angular velocity

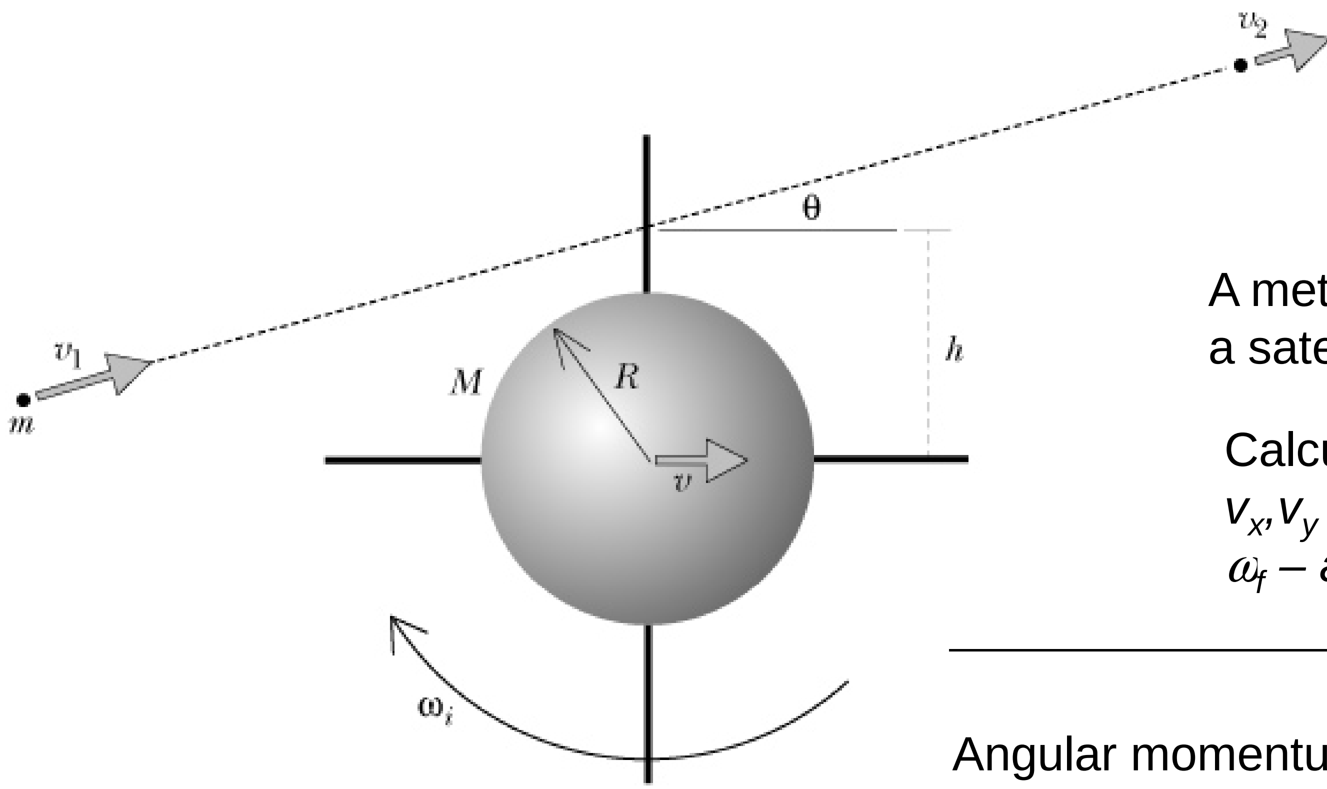
Momentum principle:

$$\left\langle \left(Mv + mv_1\cos\theta \right), mv_1\sin\theta, 0 \right\rangle = \left\langle \left(Mv_x + mv_2\cos\theta \right), \left(Mv_y + mv_2\sin\theta \right), 0 \right\rangle$$

$$\boxed{v_x = v + \frac{m}{M}\left(v_1 - v_2\right)\cos\theta}$$

$$\boxed{v_y = \frac{m}{M}\left(v_1 - v_2\right)\sin\theta}$$

Angular momentum: application



A meteor rips through
a satellite with solar panels.

Calculate:
v_x,v_y of center of mass
ω_f – angular velocity

Angular momentum principle:

$$I\omega_i + mv_1 h\cos\theta = I\omega_f + mv_2 h\cos\theta$$

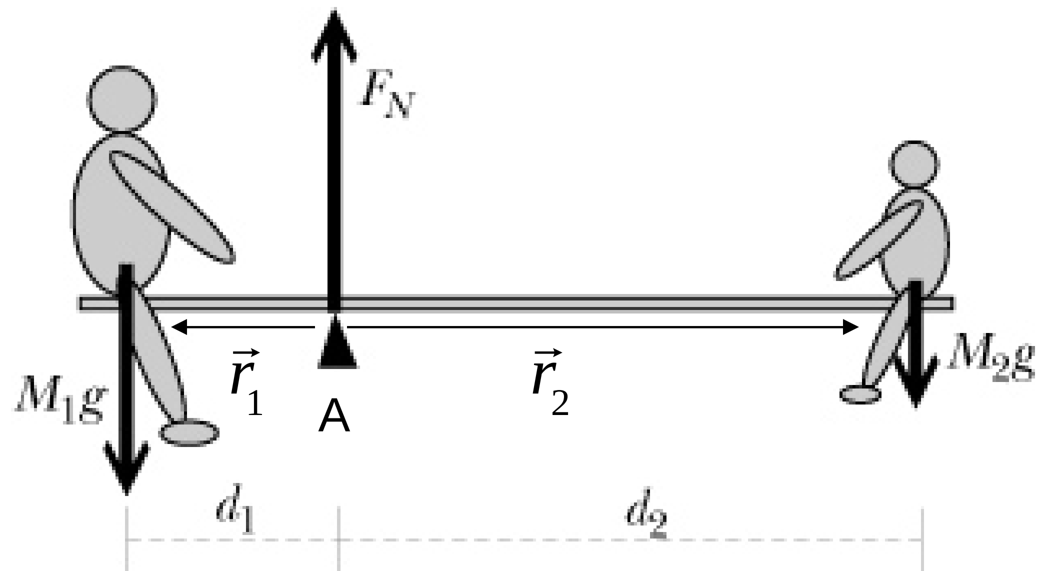
For sphere:

$$I = \tfrac{2}{5}MR^2$$

$$\boxed{\omega_f = \omega_i + \frac{hm}{I}\left(v_1 - v_2\right)\cos\theta}$$

Direction?

Static equilibrium: seesaw



$$\frac{d\vec{L}_A}{dt} = \vec{\tau}_{net,ext,A} = 0$$

$$\tau_{net,ext,A} = \vec{r}_1 \times \left(M_1 \vec{g}\right) + \vec{r}_2 \times \left(M_2 \vec{g}\right) = 0$$

$$\left(M_1 \vec{r}_1 + M_2 \vec{r}_2\right) \times \vec{g} = 0$$

$$-M_1 \vec{r}_1 = M_2 \vec{r}_2$$

$$-M_1 \left\langle -d_1, 0, 0 \right\rangle = M_2 \left\langle d_2, 0, 0 \right\rangle$$

$$M_1 d_1 = M_2 d_2$$

$$\vec{r}_{cm} = \frac{M_1 \vec{r}_1 + M_2 \vec{r}_2}{M_1 + M_2} = \vec{0}$$