PROBLEM 1

a. (2 pts) Evaluate $-17 \bmod 5$.

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b. (2 pts) What is the prime factorization of 12! (factorial of 12)?

$12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$

$2 \times 3 \times 2 \times 2 \times 5 \times 2 \times 3 \times 7 \times 2 \times 2 \times 2 \times 3 \times 3 \times 2 \times 5 \times 11 \times$
$2 \times 2 \times 3$

$2^{10} \times 3^5 \times 5^2 \times 7 \times 11$

c. (3 pts) What is the LCM (least common multiple) of the following integers $2^2 \cdot 3 \cdot 5$ and $2^3 \cdot 7$?

$2^3 \times 3 \times 5 \times 7$

$= 8 \times 15 \times 7 = 120 \times 7 = 840$

d. (3 pts) What is the GCD (greatest common divisor) of the following integers $2^3 \cdot 3^2 \cdot 5 \cdot 13$ and $2^2 \cdot 3^3 \cdot 7 \cdot 11$?

$2^2 \times 3^2 = 4 \times 9 = 36$

PROBLEM 2

Give the big-O estimate for each of the following functions. Provide a simple function $g(x)$ of the smallest order.

a. (2 pts) $f(x) = x^2 \log(x^3 - 1) + x^{1.5}$.

$x^2 \lg x$

b. (2 pts) $f(x) = \lfloor (x^2 + 3)/2 \rfloor$.

x^2

c. (2 pts) $f(x) = 2^x + x^6$.

2^x

d. (4 pts) Show that $f(x) = 3x^2 + 4x$ is $O(x^2)$ by providing constants C and x_0 as evidence.

PROBLEM 3

a. (5 pts) Compute the following:

$$\sum_{i=1}^{n}(2i-1).$$

$$\sum_{i=1}^{n} 2i - \sum_{i=1}^{n} 1$$

$$2\sum_{i=1}^{n} i - n$$

$$= \cancel{2} \cdot \frac{n(n+1)}{\cancel{2}} - n = n^2.$$

b. (2 + 3 pts) Determine if the following functions are bijections from $\mathbb{R}$ to $\mathbb{R}$.

i. $f(x) = 2x + 1$.

yes.

ii. $f(x) = 2x^2 - 1$.

no.

PROBLEM 4

a. (6 pts) Show **using** set identities that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

$$\overline{A \cup (B \cap C)}$$

$$= \overline{A} \cap \overline{(B \cap C)}$$

$$= \overline{A} \cap (\overline{B} \cup \overline{C})$$

$$= (\overline{B} \cup \overline{C}) \cap \overline{A}$$

$$= (\overline{C} \cup \overline{B}) \cap A.$$

b. (4 pts) What is the Cartesian product $A \times B$ for sets $A = \{\text{sunny}, \text{rainy}\}$ and $B = \{\text{nights}, \text{days}\}$.

$A \times B = \{$(sunny, nights), (sunny, days),
(rainy, nights), (rainy, dys)$\}$.

PROBLEM 5

a. (4 pts) Use a direct proof to show that the sum of two odd integers is even.

Suppose we have 2 odd integers o_1 and o_2.

then $o_1 = 2m+1$ for some integer m

$o_2 = 2n+1$ for some integer n

Thus. $o_1 + o_2 = 2m+1 + 2n+1 = 2(m+n) + 2$, which is even.

b. (6 pts) Show **using** rules of inference that the premise "Everyone who exercises has a sore muscle" and "Jimmy exercises" imply that "Jimmy has a sore muscle." Use $E(x)$ to denote "x exercises" and $S(x)$ to denote that "x has a sore muscle."

$[\forall x\ (E(x) \to S(x))] \Rightarrow [E(\text{Jimmy}) \to S(\text{Jimmy})]$

$E(\text{Jimmy})$

$\wedge\ E(\text{Jimmy}) \to S(\text{Jimmy})$

$\therefore\ S(\text{Jimmy})$.

PROBLEM 6

Let $S(x)$ denote "x is a student," $F(x)$ denote "x is on the faculty," and $A(x, y)$ denote "x asked y a question." Let x be drawn from the universe of all people in the world (i.e., x need not necessarily be a student or faculty). Translate the following into logical statements.

a. (4 pts) Every student has asked Professor Smith a question.

$$\forall x \left(S(x) \rightarrow A(x, \text{Smith}) \right)$$

b. (6 pts) Some student has not asked any faculty member a question.

$$\exists x, \forall y \left([S(x) \wedge F(y)] \rightarrow \neg A(x, y) \right).$$

PROBLEM 7

(5 pts) Show that $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.

$[(p \to q) \wedge (q \to r)] \to (p \to r)$

$\equiv [(\neg p \vee q) \wedge (\neg q \vee r)] \to (\neg p \vee r)$

$\equiv \neg[(\neg p \vee q) \wedge (\neg q \vee r)] \vee (\neg p \vee r)$

$\equiv \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee (\neg p \vee r)$

$\equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r)$

$\equiv \neg p \vee (p \wedge \neg q) \vee (q \wedge \neg r) \vee r$

$\equiv (\neg p \vee p) \wedge (\neg p \vee \neg q) \vee (q \wedge \neg r) \vee r$

$\equiv T \wedge (\neg p \vee \neg q) \vee [(q \wedge \neg r) \vee r]$

$\equiv (\neg p \vee \neg q) \vee [(q \vee r) \wedge (\neg r \vee r)]$

$\equiv (\neg p \vee \neg q) \vee [(q \vee r) \wedge T]$

$\equiv (\neg p \vee \neg q) \vee q \vee r$

$\equiv \neg p \vee (\neg q \vee q) \vee r$

$\equiv \neg p \vee T \vee r$

$\equiv T.$ $\square$

(5 pts) Show using identities that $\neg(p \vee (\neg p \wedge q))$ and $(\neg p \wedge \neg q)$ are logically equivalent.

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge [\neg(\neg p \wedge q)]$

$\equiv \neg p \wedge [p \vee \neg q]$

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$

$\equiv F \vee (\neg p \wedge \neg q)$

$\equiv \neg q \vee \neg q.$ $\square$