Last Time

Faraday's Law

Today

Maxwell Equations – complete!
Wave solutions

Maxwell's Equations (so far)

$$\oint \vec{E} \cdot \hat{n} dA = \frac{1}{\epsilon_o} \sum Q_{\text{enclosed}}$$ **GAUSS' LAW**

$$\oint \vec{B} \cdot \hat{n} dA = 0$$ **GAUSS' LAW (Magnetism)**

$$\oint \vec{E} \cdot d\vec{l} = \cancel{0}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \left[\sum I_{\text{enclosed}} + 0 \right]$$

Maxwell's Equations (incomplete)

$$\oint \vec{E} \cdot \hat{n} dA = \frac{1}{\epsilon_o} \sum Q_{\text{enclosed}}$$

GAUSS' LAW

$$\oint \vec{B} \cdot \hat{n} dA = 0$$

GAUSS' LAW (Magnetism)

Chapter 23 (This week)

$$\oint \vec{E} \cdot d\vec{l} = \boxed{-\frac{d}{dt} \int \vec{B} \cdot \hat{n} dA}$$

FARADAY'S LAW

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \left[\sum I_{\text{enclosed}} + \qquad 0 \qquad \right]$$

Maxwell's Equations (incomplete)

$$\oint \vec{E} \cdot \hat{n} dA = \frac{1}{\epsilon_o} \sum Q_{\text{enclosed}}$$

GAUSS' LAW

$$\oint \vec{B} \cdot \hat{n} dA = 0$$

GAUSS' LAW (Magnetism)

Chapter 23 (Last week)

$$\oint \vec{E} \cdot d\vec{l} = \boxed{-\frac{d}{dt} \int \vec{B} \cdot \hat{n} dA}$$

FARADAY'S LAW

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \left[\sum I_{\text{enclosed}} + \cancel{0} \right]$$

Maxwell's Equations – The Full Story

$$\oint \vec{E} \cdot \hat{n} dA = \frac{1}{\epsilon_o} \sum Q_{\text{enclosed}}$$

GAUSS' LAW

$$\oint \vec{B} \cdot \hat{n} dA = 0$$

GAUSS' LAW (Magnetism)

Chapter 23 (Last week)

$$\oint \vec{E} \cdot d\vec{l} = \boxed{-\frac{d}{dt} \int \vec{B} \cdot \hat{n} dA}$$

FARADAY'S LAW

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \left[\sum I_{\text{enclosed}} + \boxed{\epsilon_o \frac{d}{dt} \int \vec{E} \cdot \hat{n} dA} \right]$$

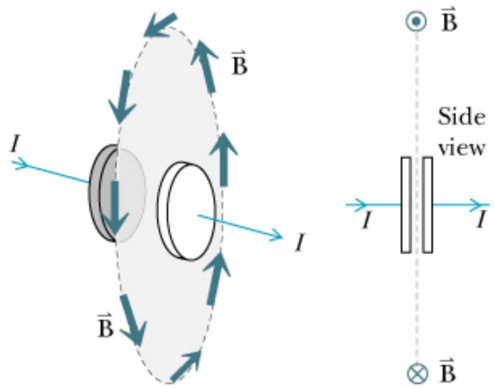
AMPERE-MAXWELL LAW

Chapter 24 (This week)

Adding time to Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \sum I_{\text{enclosed}}$$ Ampere's Law (incomplete)

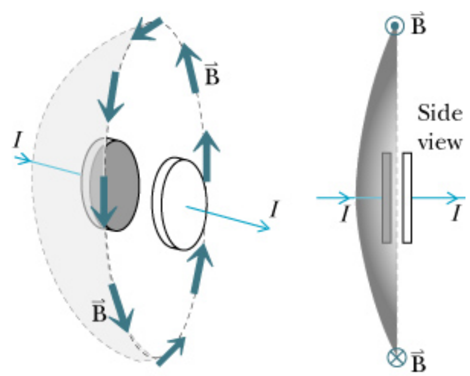
The law stated above gives this contradiction:



Here,

$$\mu_o \sum I_{\text{enclosed}} = 0 \quad \Rightarrow \quad \oint \vec{B} \cdot d\vec{l} = 0$$

capacitor



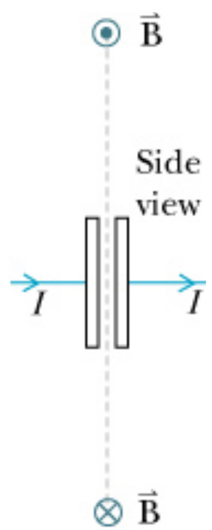
But moving the soap film gives this:

$$\mu_o \sum I_{\text{enclosed}} \neq 0 \quad \Rightarrow \quad \oint \vec{B} \cdot d\vec{l} = \neq 0$$

Something is amiss!

Adding time to Ampere's Law

$$|E| = \frac{1}{\epsilon_o}\frac{Q}{A}$$ Inside a capacitor



Electric flux through this surface:

$$\int \vec{E}\cdot\hat{n}dA = |E|A = \frac{1}{\epsilon_o}\frac{Q}{A}A = \frac{Q}{\epsilon_o}$$

Charge is on the plate, not on our surface

Find the time derivative:

$$\frac{d}{dt}\int \vec{E}\cdot\hat{n}dA = \frac{d}{dt}\frac{Q}{\epsilon_o} \equiv \frac{1}{\epsilon_o}I$$

Current in the wire, not on our surface

So the flux acts like a "current" inside the capacitor:

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_o\left[\sum I_{\text{enclosed}} + \epsilon_o\frac{d}{dt}\int \vec{E}\cdot\hat{n}dA\right]}$$

AMPERE'S LAW
(Full form)

This term contributes outside the capacitor

This term contributes inside the capacitor

Maxwell's Equations – The Full Story

$$\oint \vec{E} \cdot \hat{n} dA = \frac{1}{\epsilon_o} \sum Q_{\mathrm{enclosed}}$$ **GAUSS' LAW**

$$\oint \vec{B} \cdot \hat{n} dA = 0$$ **GAUSS' LAW (Magnetism)**

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot \hat{n} dA$$ **FARADAY'S LAW**

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \left[\sum I_{\mathrm{enclosed}} + \epsilon_o \frac{d}{dt} \int \vec{E} \cdot \hat{n} dA \right]$$ **AMPERE-MAXWELL LAW**

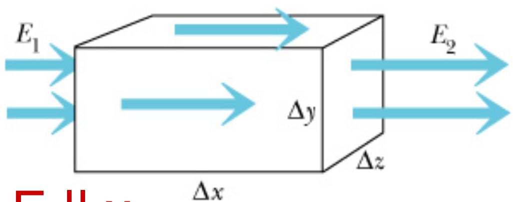
Everything there is to know about electricity & magnetism is contained in these four laws plus the force law:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

This is IT!

Differential Form of Gauss' Law (Sec. 22.8)

$$\oint \vec{E} \cdot \hat{n} dA = \frac{1}{\epsilon_o} \sum Q_{\text{enclosed}}$$ **GAUSS' LAW**



E || x

Think about a region of space, enclosed by a box. Divide Gauss' law by the volume of the box:

$$\lim_{\Delta V \to 0} \frac{\int \vec{E} \cdot \hat{n} dA}{\Delta V} = \lim_{\Delta V \to 0} \frac{1}{\epsilon_o} \frac{\sum Q_{\text{enclosed}}}{\Delta V} \equiv \frac{1}{\epsilon_o} \rho$$

Take the limit of a small box

Work on the left hand side of the equation:

$$\lim_{\Delta V \to 0} \frac{\int \vec{E} \cdot \hat{n} dA}{\Delta V} = \lim_{\Delta V \to 0} \frac{(E_2 - E_1) \cancel{\Delta y \Delta z}}{\Delta x \cancel{\Delta y \Delta z}} = \lim_{\Delta x \to 0} \frac{(E_2 - E_1)}{\Delta x} \equiv \frac{\partial E_x}{\partial x} = \frac{1}{\epsilon_o} \rho$$

For a general case where **E** can point in any direction:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \equiv \boxed{\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_o} \rho}$$

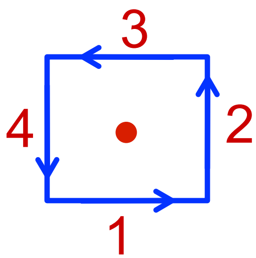
GAUSS' LAW
Differential Form

"Parallel Derivative"

where $\vec{\nabla} \equiv \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

Differential Form of Ampere's Law (Sec. 22.9)

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \left[\sum I_{\text{enclosed}} + \epsilon_o \frac{d}{dt} \int \vec{E} \cdot \hat{n} dA \right]$$ Ampere's Law



Current I
out of the board

Write I in terms of current density J: $I = \vec{J} \cdot \hat{n} \Delta A$

Divide Ampere's Law by a very small ΔA:

$$\lim_{\Delta A \to 0} \frac{\oint \vec{B} \cdot d\vec{l}}{\Delta A} = \lim_{\Delta A \to 0} \frac{\mu_o \vec{J} \cdot \hat{n} \Delta A}{\Delta A} + \lim_{\Delta A \to 0} \epsilon_o \frac{d}{dt} \frac{\int \vec{E} \cdot \hat{n} dA}{\Delta A}$$

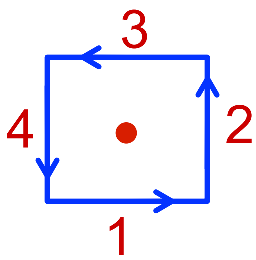
In our geometry, n = z

$$= \lim_{\Delta A \to 0} \frac{\mu_o J_z \cancel{\Delta A}}{\cancel{\Delta A}} + \lim_{\Delta A \to 0} \epsilon_o \frac{d}{dt} \frac{E_z \cancel{\Delta A}}{\cancel{\Delta A}}$$

$$\boxed{\lim_{\Delta A \to 0} \frac{\oint \vec{B} \cdot d\vec{l}}{\Delta A} = \mu_o J_z + \epsilon_o \frac{dE_z}{dt}}$$

Differential Form of Ampere's Law (Sec. 22.9)

$$\oint \vec{B}\cdot d\vec{l} = \mu_o \left[\sum I_{\text{enclosed}} + \epsilon_o \frac{d}{dt}\int \vec{E}\cdot \hat{n}\, dA \right]$$ Ampere's Law



Current I
out of the board

We divided Ampere's Law by a very small ΔA, and got this:

$$\lim_{\Delta A \to 0} \frac{\oint \vec{B}\cdot d\vec{l}}{\Delta A} = \mu_o J_z + \epsilon_o \frac{dE_z}{dt}$$

Now work on the left hand side:

$$\lim_{\Delta A \to 0} \frac{\oint \vec{B}\cdot d\vec{l}}{\Delta A} = \lim_{\Delta A \to 0} \frac{(B_{1,x} - B_{3,x})\Delta x + (B_{2,y} - B_{4,y})\Delta y}{\Delta x \Delta y}$$

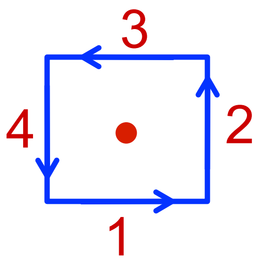
Definition of derivative!

$$= \lim_{\Delta y \to 0} \frac{(B_{1,x} - B_{3,x})}{\Delta y} + \lim_{\Delta x \to 0} \frac{(B_{2,y} - B_{4,y})}{\Delta x} = -\frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x}$$

"Crossed derivative"

Differential Form of Ampere's Law (Sec. 22.9)

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \left[\sum I_{\text{enclosed}} + \epsilon_o \frac{d}{dt} \int \vec{E} \cdot \hat{n} dA \right]$$ Ampere's Law



Current I
out of the board

We divided Ampere's Law by a very small ΔA, and got this:

$$\lim_{\Delta A \to 0} \frac{\oint \vec{B} \cdot d\vec{l}}{\Delta A} = \mu_o J_z + \epsilon_o \frac{dE_z}{dt}$$

$$= -\frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x}$$

For a loop in any direction, this can be re-expressed as:

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_o \left(\vec{J} + \epsilon_o \frac{\partial \vec{E}}{\partial t} \right)}$$

AMPERE'S LAW
Differential Form

Curl: Here's the Math

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} \underbrace{\begin{matrix} \hat{x} & \hat{y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ B_x & B_y \end{matrix}}_{\boxed{\text{copy 1st two colums}}} = \underbrace{\begin{matrix} +(\quad\quad - \quad\quad)\,\hat{x} \\ +(\quad\quad - \quad\quad)\,\hat{y} \\ +(\quad\quad - \quad\quad)\,\hat{z} \end{matrix}}_{\boxed{\text{set up the answer}}}$$

Blast from the Past!
Lecture 12

Curl: Here's the Math

$$\vec{\nabla} \times \vec{B} = \left| \begin{array}{ccc|cc} \hat{x} & \hat{y} & \hat{z} & \hat{x} & \hat{y} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ B_x & B_y & B_z & B_x & B_y \end{array} \right.$$

$-$

$+$

$$= +(\ B_z\ -\ B_y\ )\ \hat{x}$$

$$+(\ B_x\ -\ B_z\ )\ \hat{y}$$

$$+(\ B_y\ -\ B_x\ )\ \hat{z}$$

Blast from the Past!
Lecture 12

Maxwell's Equations – The Full Story

Divergence

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_o}\rho$$

$$\oint \vec{E} \cdot \hat{n} dA = \frac{1}{\epsilon_o} \sum Q_{\text{enclosed}}$$

GAUSS' LAW

Flux

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\oint \vec{B} \cdot \hat{n} dA = 0$$

GAUSS' LAW (Magnetism)

Curl

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot \hat{n} dA$$

FARADAY'S LAW

Circulation

$$\vec{\nabla} \times \vec{B} = \mu_o \left(\vec{J} + \epsilon_o \frac{\partial \vec{E}}{\partial t} \right)$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \left[\sum I_{\text{enclosed}} + \epsilon_o \frac{d}{dt} \int \vec{E} \cdot \hat{n} dA \right]$$

AMPERE'S LAW

Maxwell's Equations – No Charges

In the ABSENCE of "sources" = charges, currents:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_o}\rho$$

GAUSS' LAW

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

GAUSS' LAW (Magnetism)

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

FARADAY'S LAW

$$\vec{\nabla} \times \vec{B} = \mu_o \left(\vec{J} + \epsilon_o \frac{\partial \vec{E}}{\partial t} \right)$$

AMPERE'S LAW

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t}$$

This says once a wave starts, it keeps going!

Maxwell's Equations – No Charges

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t}$$

What happens if we feed one equation into the other?

$$\vec{\nabla} \times \left\{ \vec{\nabla} \times \vec{B} = \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t} \right\}$$

$$\mu_o \epsilon_o = \frac{1}{c^2}$$ Use This

$$\Rightarrow \quad \vec{\nabla} \times \vec{\nabla} \times \vec{B} = \mu_o \epsilon_o \frac{\partial}{\partial t} \left(\vec{\nabla} \times \vec{E} \right) = \mu_o \epsilon_o \frac{\partial}{\partial t} \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left(\vec{B} \right)$$

Maxwell's Equations – No Charges

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t}$$

What happens if we feed one equation into the other?

$$\vec{\nabla} \times \left\{ \vec{\nabla} \times \vec{B} = \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t} \right\}$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{\nabla} \times \vec{B} = \mu_o \epsilon_o \frac{\partial}{\partial t} \left(\vec{\nabla} \times \vec{E} \right) = \mu_o \epsilon_o \frac{\partial}{\partial t} \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left(\vec{B} \right)$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{B}$$

("Vector identity" -- see Wolfram alpha)

$$\Rightarrow \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{B} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{B}$$

Maxwell's Equations – No Charges

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{B} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B}$$

How do you solve a Differential Equation?
Know the answer! (Ask Wolfram Alpha)

→ This is a WAVE EQUATION, with speed c

Using similar ideas, you can show that **E** obeys the same equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$$

Maxwell's Equations – No Charges

These two equations together

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t}$$

... lead to these two equations:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{B} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B}$$

B is waving

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$$

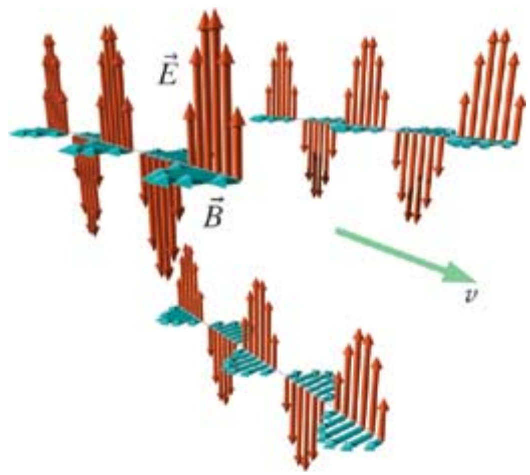
E is waving

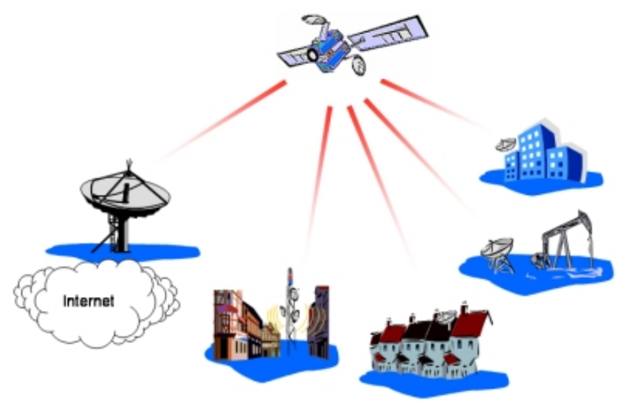
E and B wave solutions

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{B}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{E}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$$

These waves travel at the speed of light!



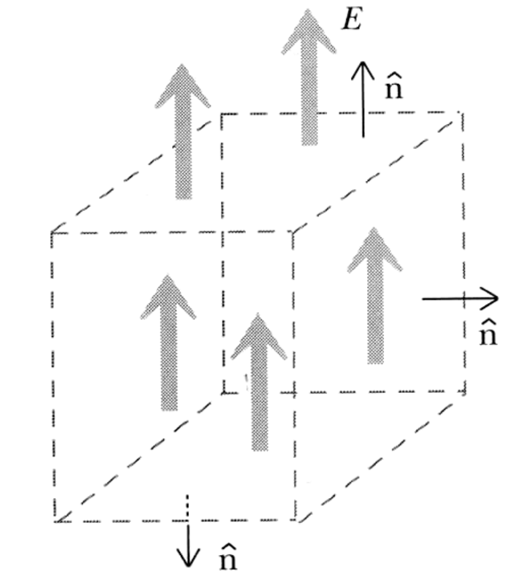


A Pulse and Gauss's Laws

$$\oint \vec{E} \cdot \hat{n} dA = \frac{\sum q_{inside}}{\varepsilon_0}$$

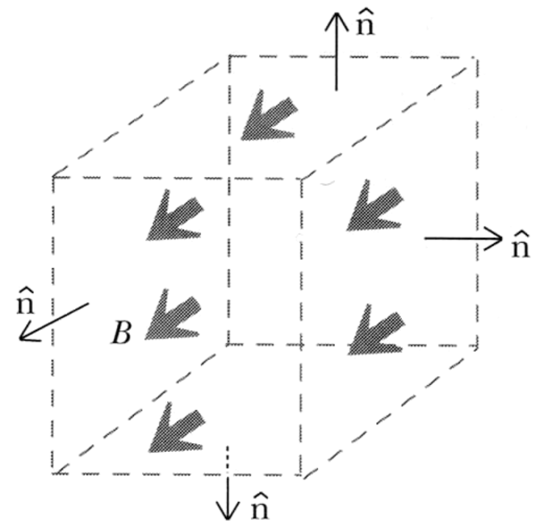
$$\oint \vec{E} \cdot \hat{n} dA = 0$$

Pulse is consistent with Gauss's law



$$\oint \vec{B} \cdot \hat{n} \Delta A = 0$$

Pulse is consistent with Gauss's law for magnetism



A Pulse and Faraday's Law

$$\boxed{emf = -\frac{d\Phi_{mag}}{dt}}$$

Since pulse is 'moving', B depends on time and thus causes E

$$\Delta\Phi_{mag} = Bhv\Delta t$$

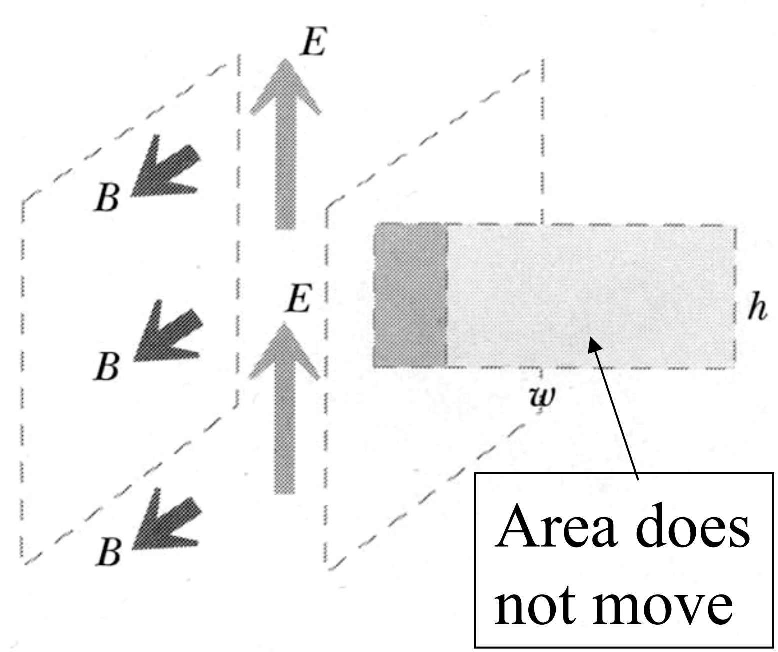
$$\frac{\Delta\Phi_{mag}}{\Delta t} = \frac{d\Phi_{mag}}{dt} = Bhv$$

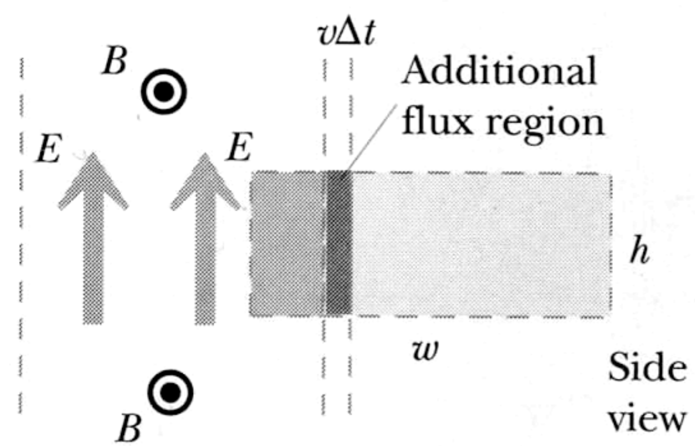
emf

$$\boxed{emf = \left|\oint \vec{E}\cdot d\vec{l}\right| = Eh}$$

$$\boxed{E = Bv}$$

Is direction right?





A Pulse and Ampere-Maxwell Law

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0\left[\sum \underbrace{I_{inside\_path}}_{=0} + \varepsilon_0 \frac{d\Phi_{elec}}{dt}\right]}$$

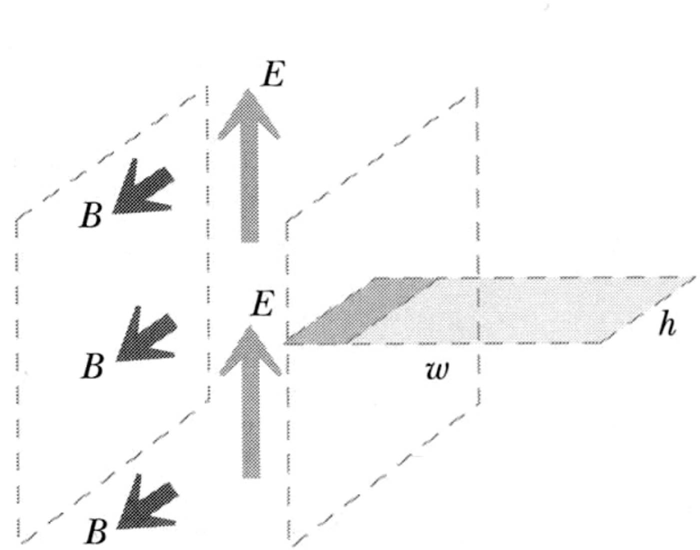
$$\Delta\Phi_{elec} = Ehv\Delta t$$

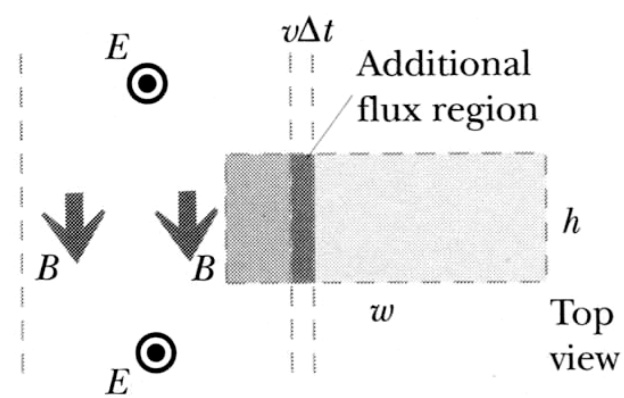
$$\frac{\Delta\Phi_{elec}}{\Delta t} = \frac{d\Phi_{elec}}{dt} = Ehv$$

$$\left|\oint \vec{B}\cdot d\vec{l}\right| = Bh$$

$$Bh = \mu_0\varepsilon_0 Evh$$

$$\boxed{B = \mu_0\varepsilon_0 vE}$$





A Pulse: Speed of Propagation

$$\boxed{B = \mu_0 \varepsilon_0 v E} \qquad \boxed{E = Bv}$$

$$B = \mu_0 \varepsilon_0 v B v$$

$$\downarrow$$

$$1 = \mu_0 \varepsilon_0 v^2$$

$$\downarrow$$

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s} \qquad \boxed{E = cB}$$

Based on Maxwell's equations, pulse must propagate at speed of light

Today

Maxwell Equations – complete!
Wave solutions