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Summer 2012

ECE 20200 : Linear Circuit Analysis II
School of ECE, Purdue University

LECTURE 8

- Transfer Functions $H(s)$
- Impulse / Step Responses
- Initial / Final Value Theorems

Reference: Decarlo / Lin PP 626-634
PP 726-729

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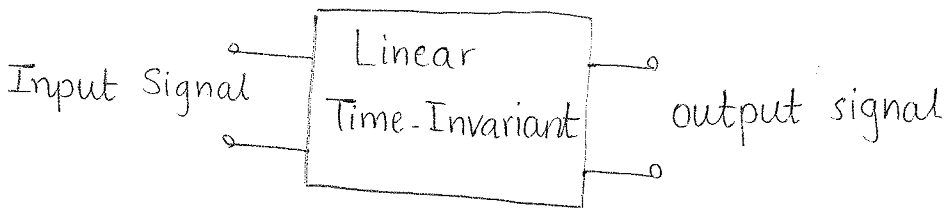
Transfer Functions



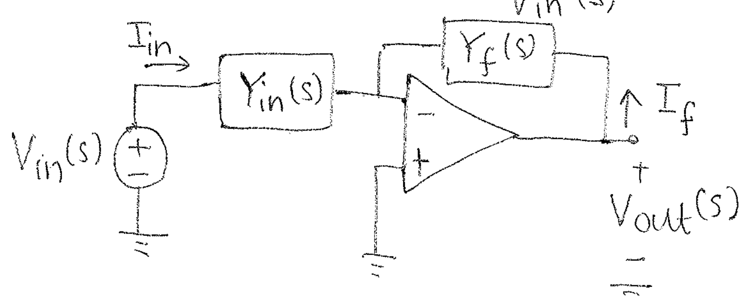
Transfer Function $H(s) = \dfrac{\mathcal{L}[\text{output signal}]}{\mathcal{L}[\text{input signal}]}$

Interpretation: $H(s)$ denotes the Laplace transform of the circuit or more generally the Laplace transform of some linear, time-invariant physical process having input stimuli and observable responses.

Remark: $Z(s) = \dfrac{V_{out}(s)}{I_{in}(s)}$ and $Y(s) = \dfrac{I_{out}(s)}{V_{in}(s)}$

are special types of transfer functions.

Example 1: Show that $\dfrac{V_{out}(s)}{V_{in}(s)} = H(s) = \dfrac{-Y_{in}(s)}{Y_f(s)}$



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Check!
$$H(s) = -\frac{Z_f(s)}{Z_{in}(s)}$$

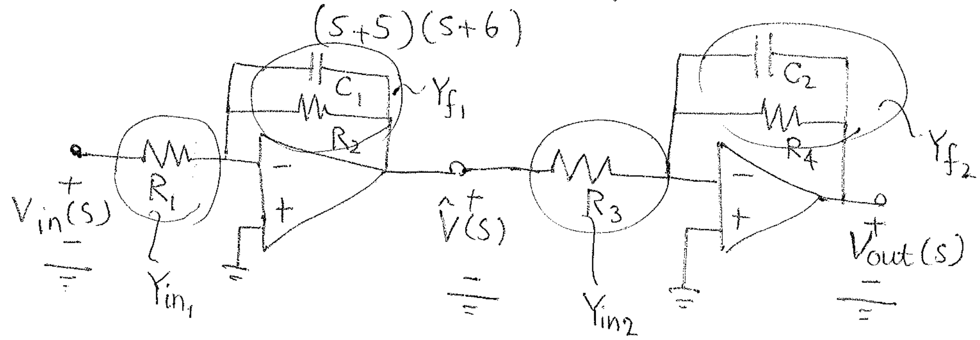
$$I_{in}(s) = -I_f(s)$$

$$Y_{in}(s)\,V_{in}(s) = -Y_f(s)\,V_{out}(s)$$

$$\therefore \quad \frac{V_{out}(s)}{V_{in}(s)} = -\frac{Y_{in}(s)}{Y_f(s)} = H(s)$$

Example 2: Find R_1, R_2, C_1 and R_3, R_4, C_2 so that

$$H(s) = \frac{30}{(s+5)(s+6)}$$ (Note: The solution is not unique)



$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{V_{out}(s)}{\hat{V}(s)} \cdot \frac{\hat{V}(s)}{V_{in}(s)} = \frac{30}{(s+5)(s+6)}$$

$$H_1(s) \cdot H_2(s) = \frac{30}{(s+5)(s+6)}$$

Using the result from example 1,

$$\left(-\frac{Y_{in1}(s)}{Y_{f_1}(s)}\right)\left(-\frac{Y_{in2}(s)}{Y_{f_2}(s)}\right) = \frac{30}{(s+5)(s+6)}$$

$$= \left(\frac{-5}{s+5}\right)\left(\frac{-6}{s+6}\right)$$

Admittance formulas

$$Y_{in_1}(s) = \frac{1}{R_1} = G_1 \qquad Y_{in_2}(s) = \frac{1}{R_3} = G_3$$

$$Y_{f_1}(s) = C_1 s + G_2 \qquad Y_{f_2}(s) = C_2 s + G_4$$

$$\therefore H_1(s) = \frac{-Y_{in_1}(s)}{Y_{f_1}(s)} = -\frac{G_1}{C_1 s + G_2} = \frac{-5}{s+5}$$

$$H_2(s) = \frac{-Y_{in_2}(s)}{Y_{f_2}(s)} = \frac{-G_3}{C_2 s + G_4} = \frac{-6}{s+6}$$

$$\therefore G_1 = G_2 = 5\ S \qquad G_3 = G_4 = 6\ S$$

$$C_1 = 1\ F \qquad C_2 = 1\ F$$

$$R_1 = R_2 = 0.2\ \Omega \qquad R_3 = R_4 = \frac{1}{6}\ \Omega$$

Impulse and Step Responses

Some Definitions

1. Impulse Response : Response of circuit to an impulse.

$$\delta(t) \longrightarrow \boxed{H(s)} \longrightarrow ?$$

output $= h(t) = \mathcal{L}^{-1}\left[H(s) \cdot \mathcal{L}[\delta(t)]\right] = \mathcal{L}^{-1}[H(s)]$

2. Step Response : Response of circuit to a step input

$$u(t) \longrightarrow \boxed{H(s)} \longrightarrow ?$$

output $= s(t) = \mathcal{L}^{-1}\left[H(s) \cdot \mathcal{L}[u(t)]\right] = \mathcal{L}^{-1}\left[H(s) \cdot \frac{1}{s}\right]$

3. Relationship between impulse response and step response

$$h(t) = \mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}\left[s \frac{H(s)}{s}\right] = \frac{d}{dt}[s(t)]$$

$\therefore$ impulse response $= \frac{d}{dt}$ [step response]

Example 2 - continued.

Recall $H(s) = \dfrac{30}{(s+5)(s+6)}$

(a) Find the impulse response.

$$H(s) = \frac{30}{(s+5)(s+6)} = \frac{30}{s+5} - \frac{30}{s+6}$$

$$\therefore h(t) = \mathcal{L}^{-1}[H(s)] = 30e^{-5t}u(t) - 30e^{-6t}u(t)$$

(b) Find the step response.

$$\frac{H(s)}{s} = \frac{30}{s(s+5)(s+6)} = \frac{1}{s} - \frac{6}{s+5} + \frac{5}{s+6}$$

$$\therefore s(t) = \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right] = u(t) - 6e^{-5t}u(t) + 5e^{-6t}u(t)$$

(c) Check the relationship.

$$\frac{d}{dt}(s(t)) = \frac{d}{dt}\left[(1-6e^{-5t}+5e^{-6t})u(t)\right]$$

$$= (1-6e^{-5t}+5e^{-6t})\frac{d}{dt}(u(t)) + \frac{d}{dt}(1-6e^{-5t}+5e^{-6t}) \cdot u(t)$$

$$= (1-6e^{-5t}+5e^{-6t})\delta(t) + (30e^{-5t} - 30e^{-6t})u(t)$$

$$= (1-6+5)\delta(t) + 30e^{-5t}u(t) - 30e^{-6t}u(t)$$

$$= 30e^{-5t}u(t) - 30e^{-6t}u(t) = h(t) \quad \checkmark$$

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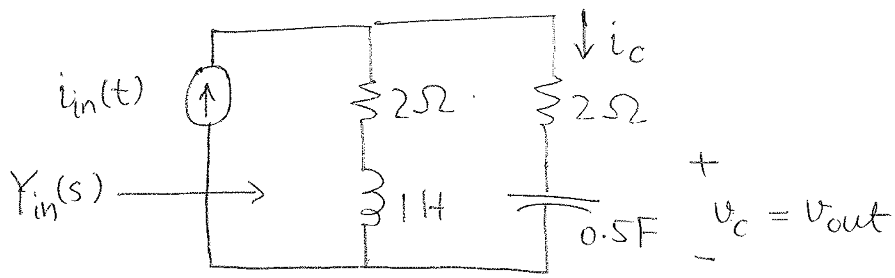
Example 3.



(a) Find $H(s) = \dfrac{V_{out}(s)}{I_{in}(s)}$

Compute $Y_{in}(s)$ first.

$$Y_{in}(s) = \frac{1}{s+2} + \frac{1}{2+\frac{2}{s}} = \frac{1}{s+2} + \frac{0.5s}{s+1}$$

Then get $I_c(s)$ using current division

$$I_c(s) = \frac{\frac{0.5s}{s+1}}{\frac{0.5s}{s+1} + \frac{1}{s+2}} I_{in}(s) = \frac{2s(s+2)}{s^2+4s+2} I_{in}(s)$$

$$\therefore V_c(s) = \frac{2}{s} \cdot I_c(s) = \frac{4(s+2)}{s^2+4s+2} I_{in}(s) = V_{out}(s)$$

$$\therefore \frac{V_{out}(s)}{I_{in}(s)} = \frac{4(s+2)}{s^2+4s+2} = H(s)$$

(b) Find the impulse response $h(t)$.

$$h(t) = \mathcal{L}^{-1}[H(s)] = \left(2e^{-3.4142t} + 2e^{-0.58579t}\right)u(t)$$

(c) Find the step response $s(t)$

$$s(t) = \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right] = \left(4 - 0.58579e^{-3.4142t} - 3.4142e^{-0.58579t}\right) u(t)$$

* solutions to parts (b) and (c) are obtained using MATLAB

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Initial/Final Value Theorems

Final Value Theorem

Suppose $F(s) = \mathcal{L}[f(t)]$ only has poles in OLHCP except possibly a single pole at the origin (i.e., $s=0$). Then

$$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t) = f(\infty)$$

Example 1. Find $f(\infty)$ when $F(s) = \frac{(s+2)(s-1)}{s(s+1)(s+3)}$

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{\cancel{s}\,(s+2)(s-1)}{\cancel{s}\,(s+1)(s+3)} = \frac{2(-1)}{(1)(3)} = -\frac{2}{3}$$

Example 2. Find $f(\infty)$ when $F(s) = \frac{1}{s^2+1}$

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\frac{1}{s^2+1} = 0 \quad ?$$

$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin(t)u(t)$ which does not have a limit

$\therefore$ Final value Theorem does not work here! ($\because s^2+1=0 \Rightarrow s=\pm j$ poles)

Initial Value Theorem

$\mathcal{L}[f(t) = F(s) = \frac{n(s)}{d(s)}$ with $\deg(n(s)) < \deg(d(s))$. Then

$$\lim_{s\to\infty} sF(s) = f(0^+)$$

Example: Find $f(1^+)$ when $F(s) = \frac{e^{-s}}{s}\left[\frac{2.7183s+1}{s+3}\right]$

$f(1^+) = \hat{f}(0^+)$ where $\hat{F}(s) = e^s F(s)$

$$\hat{f}(0^+) = \lim_{s\to\infty} s\hat{F}(s) = \lim_{s\to\infty} s\,e^s\,\frac{e^{-s}}{s}\left[\frac{2.7183s+1}{s+3}\right]$$

$$= \lim_{s\to\infty}\left[\frac{2.7183s+1}{s+3}\right]$$

$$= 2.7183$$