CS182 **Homework 7** **Spring 2012**

In all of the following questions, show the details of your work (it is not enough to just give the answer).

Question 1. (10 points) Let ℓ, m, n be positive integers such that $\ell > m + n$. Let $f(\ell, m, n)$ be the number of distinct ℓ-bit binary strings whose leftmost m bits are all 1, or whose last n bits are all 0 (or both, i.e., the "or" is not exclusive). White down (in closed form) the expression for $f(\ell, m, n)$.

Question 2. (10 points) Let n and k be positive integers with $k < n$. Write down, as a function of n, the number of distinct solutions to the equation

$$x_1 + x_2 + \cdots + x_k = n$$

where $x_1, x_2, \ldots, x_k$ are *positive* integers.

Question 3. (10 points) Suppose that you start at the origin of the coordinate system in the plane, i.e., at position $(0, 0)$. Then you make a total of n unit steps each of which is either horizontal or vertical, i.e., if you are at position (x, y) then after a horizontal step you become at position $(x + 1, y)$, whereas a vertical step would have taken you to position $(x, y + 1)$. Note that you cannot move down or to the left, only up and to the right.

1. Write down, as a function of n, the number of different possible paths after n steps.
2. Let m be a positive integer with $m \leq n$. Write down, as a function of m and n, the number of different paths that start at position $(0, 0)$ and end at position $(m, n - m)$.

Question 4. (10 points) Let S be a sequence of $N = n^2 + 1$ distinct integers:

$$S = (x_1, x_2, \ldots, x_N)$$

A *subsequence* of S is a sequence of the form $(x_{i_1}, x_{i_2}, \ldots, x_{i_t})$ where $1 \leq i_1 < i_2 < \cdots < i_t \leq N$; the *length* of that subsequence is t (= the number of elements in it). For example, if $S = (8, 6, 5, 33, 10, 67, 2, 39)$ then $(6, 33, 10, 2)$ is a subsequence of S but $(6, 67, 33)$ not a subsequence of S. A subsequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_t})$ of S is said to be *monotone* if either $x_{i_1} < x_{i_2} < \ldots < x_{i_t}$) or $x_{i_1} > x_{i_2} > \ldots > x_{i_t}$); in the former case the subsequence is said to be *increasing*, and in the latter case it is said to be *decreasing*. For example, if $S = (8, 6, 5, 33, 10, 67, 2, 39)$ then $(6, 33, 67)$ is an increasing subsequence of S, and $(8, 5, 2)$ is a decreasing subsequence of S.
Prove that there is a subsequence of S that is monotone and has length $n + 1$.

Hint: Start of proof. Let ℓ_i be the length of a longest increasing subsequence of S whose rightmost symbol is x_i, $1 \leq i \leq N$. If some $\ell_i \geq n + 1$ then we are done because there is an increasing subsequence of lenth $\geq n+1$. So suppose all ℓ_i are smaller than $n+1$. Put x_i in "pigeonhole" number ℓ_i, $1 \leq i \leq N$. [You need to continue the proof.]

Date due: April 5, 2012