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LECTURE 2

- Laplace Transform
- Properties of Laplace Transform
  - Linearity
  - Time-shift
  - Time multiplication

Reference: Decarlo/Lin pp 554-564

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Laplace Transform

- One-sided or unilateral Laplace Transform

$$\mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt = F(s)$$

  - 's' is a complex variable, $s = \sigma + j\omega$
  - 's' is called complex frequency
  - The range of 'σ' is chosen so that the integral converges.
  - 'ω' usually refers to the sinusoidal frequency.

Remarks:

1) ROC (Region of Convergence) is the set of all '$\sigma + j\omega$' for which the integral exists.
2) $F(s) = \mathcal{L}[f(t)]$ uniquely represents $f(t)$ only when $f(t) = 0$ for $t < 0$.
   If $f_1(t) = f_2(t)$ for $t \geq 0$ but
   $f_1(t) \neq f_2(t)$ for $t < 0$,
   then $F_1(s) = F_2(s)$.
3) For some $f(t)$, $F(s)$ does not exist.

Examples

1. Find $\mathcal{L}[\delta(t)]$

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t) e^{-st}\, dt = e^{-s(0)} = 1 \leftarrow \text{ROC}: \forall s$$

2. Find $\mathcal{L}[u(t)]$

$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} u(t) e^{-st}\, dt = \left. \frac{e^{-st}}{-s} \right|_{0}^{\infty}$$

$$= \frac{1}{s} \leftarrow \text{if } \operatorname{Re}\{s\} > 0 \quad \text{ROC}$$

3. Find $\mathcal{L}[e^{-at}u(t)]$

$$\mathcal{L}[e^{-at}u(t)] = \int_{0^-}^{\infty} e^{-at} u(t) e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} e^{-(a+s)t}\, dt$$

$$= \left. \frac{e^{-(s+a)t}}{-(s+a)} \right|_{0^-}^{\infty}$$

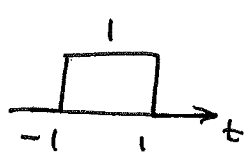
$$= \frac{1}{s+a} \quad \text{if } \sigma + a > 0$$

$$\sigma > -a \quad \text{ROC}$$

$$\sigma = \operatorname{Re}\{s\}$$

4. Find $\mathcal{L}[u(t+1) - u(t-1)]$

$$\mathcal{L}[u(t+1) - u(t-1)] = \int_{0^-}^{\infty} [u(t+1) - u(t-1)] e^{-st}\, dt$$



$$= \int_{0^-}^{1} e^{-st}\, dt = \left. \frac{e^{-st}}{-s} \right|_{0^-}^{1} = \frac{1 - e^{-s}}{s} \leftarrow$$

$$\text{ROC}: \forall s$$

5. Find $F(s)$ if $f(t) = (10 + 5e^{-7t})u(t)$

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}\left[(10+5e^{-7t})\,u(t)\right]$$

$$= \int_{0^-}^{\infty} (10+5e^{-7t})\,u(t)\,e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} (10+5e^{-7t})\,e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} 10e^{-st}\,dt + \int_{0^-}^{\infty} 5e^{-7t}e^{-st}\,dt$$

$$= \frac{10}{s} + \frac{5}{s+7}$$

Properties of Laplace Transform

1) Linearity

$$\mathcal{L}\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)$$

by virtue of the linearity of the Laplace Transform integral.

2) Time-shift (Right shift only)

$$\mathcal{L}\left[f(t-T)\,u(t-T)\right] = e^{-sT}F(s) \qquad , T > 0$$

Proof:

$$\mathcal{L}\left[f(t-T)\,u(t-T)\right] = \int_{0^-}^{\infty} f(t-T)\,u(t-T)\,e^{-st}\,dt$$

$$= \int_{T^-}^{\infty} f(t-T)\,e^{-st}\,dt$$

Let $q = t - T \Rightarrow t = q + T$

$\frac{dq}{dt} = 1 \Rightarrow dq = dt$

$t = T^-$ $\quad$ $t = \infty$

$q = T^- - T = 0^-$ $\quad$ $q = \infty$

Therefore

$$\int_{-T}^{\infty} f(t-T)\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} f(q)\, e^{-s(q+T)}\, dq$$

$$= \int_{0^-}^{\infty} f(q)\, e^{-sq}\, e^{-sT}\, dq$$

$$= e^{-sT} \int_{0^-}^{\infty} f(q) e^{-sq}\, dq$$

$$= e^{-sT} F(s)$$

3) Time-multiplication

$F(s) = \mathcal{L}[f(t)]$, then

$$\mathcal{L}[t f(t)] = -\frac{d}{ds} F(s)$$

Proof: $\mathcal{L}[t f(t)] = \int_{0^-}^{\infty} t f(t) e^{-st}\, dt$

$$= \int_{0^-}^{\infty} f(t)\left(-\frac{d}{ds} e^{-st}\right) dt$$

$$= -\frac{d}{ds} \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

$$= -\frac{d}{ds} \left(F(s)\right)$$

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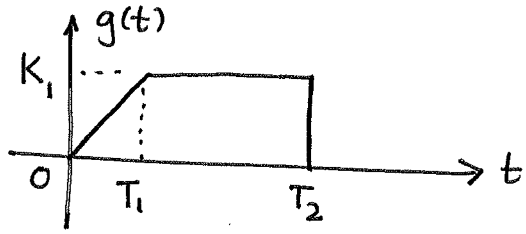
Examples: Find $\mathcal{L}[r(t)]$

$$\mathcal{L}[r(t)] = \mathcal{L}[t\,u(t)]$$
$$= -\frac{d}{ds}\left[\mathcal{L}[u(t)]\right]$$
$$= -\frac{d}{ds}\left(\frac{1}{s}\right)$$
$$= \frac{1}{s^2} \leftarrow$$

- Find $\mathcal{L}[g(t)]$



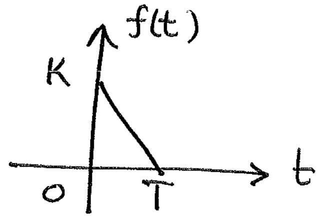
$$g(t) = \frac{K_1}{T_1} r(t) - \frac{K_1}{T_1} r(t-T_1) - K_1 u(t-T_2)$$

$$\mathcal{L}[g(t)] = G(s) = \frac{K_1}{T_1}\frac{1}{s^2} - \frac{K_1}{T_1}\frac{1}{s^2}e^{-sT_1} - K_1 \cdot \frac{1}{s} \cdot e^{-sT_2}$$

- Find $F(s)$ when $f(t) = te^{-at}u(t)$

$$F(s) = -\frac{d}{ds}\mathcal{L}\left[e^{-at}u(t)\right]$$
$$= -\frac{d}{ds}\left(\frac{1}{s+a}\right)$$
$$= \frac{1}{(s+a)^2}$$

- Find $F(s)$ when $f(t)$ is given by



$$f(t) = Ku(t) - \frac{K}{T} r(t) + \frac{K}{T} r(t-T)$$

$$\therefore F(s) = \frac{K}{s} - \frac{K}{T} \cdot \frac{1}{s^2} + \frac{K}{T} \frac{e^{-sT}}{s^2} \leftarrow$$

- Find $F(s)$ when $f(t) = \sin(\omega t)\, u(t)$

$$f(t) = \sin(\omega t) u(t) = \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) u(t)$$

$$= \frac{e^{j\omega t}}{2j} u(t) - \frac{e^{-j\omega t}}{2j} u(t)$$

$$F(s) = \frac{1}{2j} \left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right]$$

$$= \frac{1}{2j} \left[\frac{s + j\omega - (s - j\omega)}{(s - j\omega)(s + j\omega)} \right]$$

$$= \frac{1}{2j} \quad \frac{2j\omega}{s^2 + \omega^2}$$

$$= \frac{\omega}{s^2 + \omega^2} \leftarrow$$

- Find $F(s)$ when $f(t)$ is $u(t+1) + 5\,\delta(t-2) \cos(15.2\pi t) u(t)$

$$f(t) = u(t+1) + 5\,\delta(t-2) \cos(15.2\pi(2)) u(2)$$

$$= u(t+1) + 5\cos(30.4\pi)\ \delta(t-2)$$

$$F(s) = \frac{1}{s} + 5\cos(30.4\pi)\, e^{-2s} \leftarrow$$