Solution

Your Name: ____________________

CS 182
MIDTERM
Spring 2012

Left Neighbor: ____________________ Right Neighbor: ____________________

This exam contains 9 numbered pages. Check your copy and exchange it immediately if it is defective. Print your name and your student id number on the top of this page. Print the name of your left and right neighbors below your name. Good luck!

Problem	Maximum	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Bonus	5	
Total	70	

[10] **PROBLEM 1**: *Logic*

[8] State and prove *modus tollens*, that is, prove that $\begin{array}{c} \neg q \\ p \to q \\ \hline \therefore \neg p. \end{array}$ is a tautology.

$Q = \neg q \wedge (p \to q) \to \neg p$

p	q	Q
0	0	1
0	1	1
1	0	1
1	1	1

[2] Give an example of an application of this rule of inference.

[10] **PROBLEM 2**: *Relations*

[8] Let xRy if x and y are *positive integers* and x divides (evenly) y. Is this relation

- reflexive, YES xRx sin $x = 1 \cdot x$
- symmetric, NOT if $xRy \Rightarrow y = k \cdot x$ $k \in N$, but not $x = l \cdot y$
- antisymmetric, YES
- transitive. YES

if xRy & $yRx \Rightarrow$ $x = k \cdot y$

$y = l \cdot x$

$k \cdot l = 1$ $k, l \geq 1$

$k = 1$ & $l = 1$

Prove your statements.

xRy & $yRz \longrightarrow xRz$

$y = k \cdot x$ $z = l \cdot y \rightarrow z = l \cdot y = l \cdot k \cdot x$ yes

[2] Determine whether R is an equivalence relation or a partial order relation. If it is an equivalence relation, construct the equivalence classes.

$$A\cup(B\cap C)=$$
$$(A\cup B)\cap(A\cup C)$$

[10] **PROBLEM 3**: *Sets*

Without Venn's diagrams prove that

$$(A\cap B)\cup(A\cap\bar{B})=A$$

for any sets A and B.

By identity:

$$(A\cap B)\cup(A\cap\bar{B})=(A\cup A)\cap(A\cup\bar{B})\cap(B\cup A)\cap\overset{=\mathcal{U}}{(B\cup\bar{B})}$$

$$=A\cap\left[A\cup\overset{\emptyset}{(B\cap\bar{B})}\right]$$

$$=A\cap A=A$$

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if $x\in(A\cap B)\cup(A\cap\bar{B})$

$$\Downarrow$$

$x\in A\cap B$ OR $x\in A\cap\bar{B}$

$(x\in A\ \&\ x\in B)$ OR $(x\in A\ \&\ x\in\bar{B})$

identity of proposit.

$$(a\wedge b)\vee(a\wedge\bar{b})$$

$a\equiv x\in A$

$b=x\in B$

[10] **PROBLEM 4**: *Number Theory*

Prove that $n^3 - n$ is divisible by 3 for any integer $n \geq 1$.

$n^3 - n = n(n^2-1) = n(n-1)(n+1)$ ~~mod 3~~

You find with mod 3 argument

[10] **PROBLEM 5**: *Mathematical Induction*

Prove by induction on n that for all natural numbers $n \geq 1$ and every $a \neq 1$ and $0 \leq m \leq n$

$$\sum_{i=m}^{n} a^i = \frac{a^{n+1} - a^m}{a-1}.$$

Base case 2pt

$n \geq m$

$$\sum_{i=m}^{n+1} a^i = \sum_{i=m}^{n} a^i + a^{n+1} =$$

$$= \frac{a^{n+1} - a^m}{a-1} + a^{n+1}$$

$$= \frac{a^{n+1} - a^m + a^{n+1}(a-1)}{a-1} =$$

$$= \frac{\cancel{a^{n+1}} - a^m + a^{n+2} - \cancel{a^{n+1}}}{a-1} = \frac{a^{n+2} - a^m}{a-1} \checkmark$$

[10] **PROBLEM 6**: *Recurrences*

In the class we've discussed the following recurrence:

$$\begin{aligned} T(0) &= 1, \\ T(n) &= 1 + \frac{2}{n}\sum_{j=0}^{n-1} T(j), \quad n \geq 1. \end{aligned}$$

Prove that $T(n) = 2n + 1$ for all $n \geq 0$.

By induction

$T(j) = 2j + 1$

$$T(n) = 1 + \frac{2}{n} \sum_{j=0}^{n-1} (2j+1)$$

$$= 1 + \frac{2}{n}\left(2 \cdot \frac{n(n-1)}{2} + n\right)$$

$$= 1 + \frac{2}{n}\left(n(n-1) + n\right)$$

$$= 1 + 2(n-1) + 2 = 1 + 2(n$$

$1 + 2n - 2 + 2$

[10] **PROBLEM 7**: *Big Oh*

Show that

$$\sum_{i=1}^{n} i^3 \log \overset{i}{\cancel{n}} = \Theta(n^4 \log n).$$

Upper Bound:

$$\sum i^3 \log i \le n^3 \log n \sum^{n} 1 = n^4 \log n$$

Lower Bound

$$\sum_{i=1}^{n} i^3 \log i \geq \sum_{i=n/2}^{n} i^3 \log i \geq$$

$$\geq \log \frac{n}{2} \left(\frac{n}{2}\right)^3 \cdot \frac{n}{2}$$

$$= \frac{n^4}{2^4} \log \frac{n}{2}$$

[5] **BONUS PROBLEM**: *Another Sum*

Prove that for any $0 < p < 1$ the following holds:

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1.$$

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + 1 - p)^n = 1^n = 1$$