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Summer 2012

LECTURE 9

- Nodal Analysis in s-domain
- Mesh Analysis in s-domain

Reference: Decarlo/Lin pp 634-640

- Construct an s-domain equivalent circuit
- To find $Z(s)$, $Y(s)$, $H(s)$, ignore initial conditions
- To find $V(s)$, $I(s)$, do not ignore initial conditions
- Write a nodal equation for each non-reference node or a loop equation for each loop
- A super node for an ungrounded voltage source or a super mesh for a current source.
- Use Cramer's Rule or Inverse matrix method to determine $V(s)$ (or) $I(s)$ and then $H(s)$ or $Z(s)$ or $Y(s)$.
- Do partial fraction and inverse Laplace transform.
- Review Chapter 3.

Cramer's Rule

Given
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

Then
$$x = \frac{\begin{vmatrix} r & b & c \\ s & e & f \\ t & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & r & c \\ d & s & f \\ g & t & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a & b & r \\ d & e & s \\ g & h & t \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

where
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= aei - afh - bdi + bfg + cdh - ceg$$

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Matrix Inversion

Given $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$ $[A][x] = [R]$

Then $[x] = [A]^{-1}[R]$ where $[A]^{-1}$ is the inverse of $[A]$

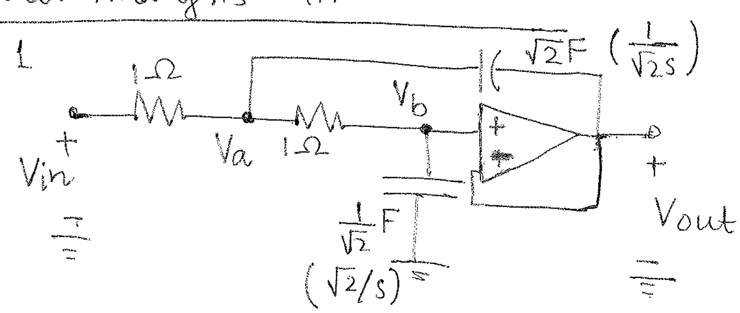
Inverse of 2x2 matrix and 3x3 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} c & b \\ i & h \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ \begin{vmatrix} f & d \\ i & g \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & \begin{vmatrix} c & a \\ f & d \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \begin{vmatrix} b & a \\ h & g \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

Nodal-Analysis in s-domain

Example 1



Find $H(s) = \dfrac{V_{out}}{V_{in}}$

1) $V_b = V_{out}$

2) Nodal equation at node V_a:

$$\frac{V_a - V_{in}}{1} + \frac{V_a - V_{out}}{\frac{1}{\sqrt{2}s}} + \frac{V_a - V_b}{1} = 0$$

$$(2+\sqrt{2}s)V_a - (1+\sqrt{2}s)V_{out} = V_{in}$$

3) Nodal equation at node V_b

$$\frac{V_b - V_a}{1} + \frac{V_b - 0}{\sqrt{2}/s} = 0$$

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$$-V_a + \left(1+\frac{s}{\sqrt{2}}\right)V_b = 0$$

$$-V_a + \left(1+\frac{s}{\sqrt{2}}\right)V_{out} = 0$$

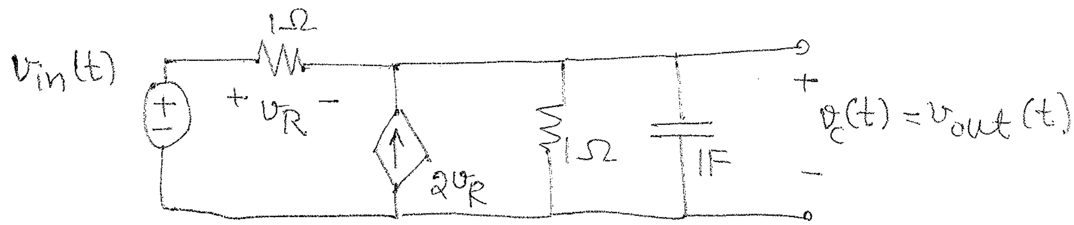
$$V_a = \left(1+\frac{s}{\sqrt{2}}\right)V_{out}$$

Plug in step 2,

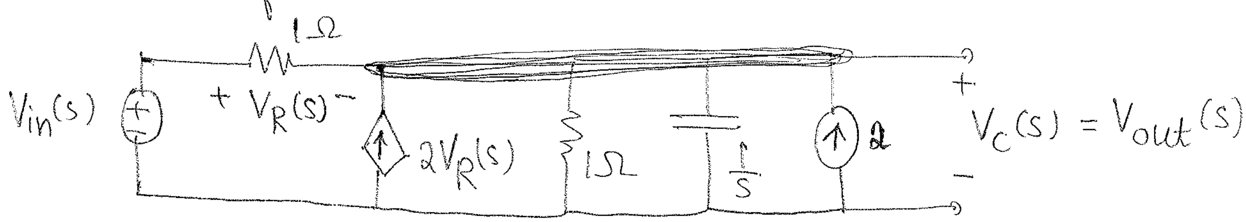
$$(2+\sqrt{2}s)\left(1+\frac{s}{\sqrt{2}}\right)V_{out} - (1+\sqrt{2}s)V_{out} = V_{in}$$

$$\frac{V_{out}}{V_{in}} = \frac{1}{s^2+\sqrt{2}s+1} = H(s)$$

Example 2: Find $v_{out}(t)$ when $v_{in}(t) = 8u(t)$ V and $v_c(0^-)=2$V



s-domain equivalent circuit.



Nodal equation

$$\frac{V_{out}-V_{in}}{1} + (+2)(V_{out}-V_{in}) + \frac{V_{out}}{1} + \frac{V_{out}}{1/s} - 2 = 0$$

$$(s+4)V_{out} = \underbrace{3V_{in}}_{\text{due to input}} + \underbrace{2}_{\text{due to IC}}$$

$$V_{out} = \frac{3}{(s+4)}V_{in} + \frac{2}{(s+4)}$$

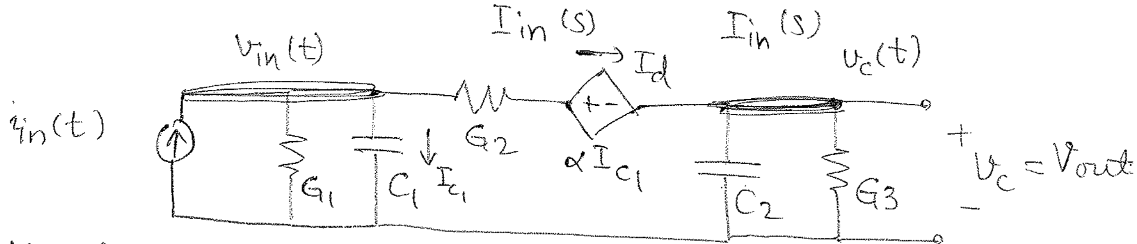
$$V_{out} = \frac{3}{s+4} \quad \frac{8}{s} + \frac{2}{s+4}$$

$$V_{out} = \frac{-6}{s+4} + \frac{6}{s} + \frac{2}{s+4}$$

$$v_{out}(t) = \underbrace{-6e^{-4t}u(t) + 6u(t)}_{\text{zero-state response}} + \underbrace{2e^{-4t}u(t)}_{\text{zero-input response}}$$

$$v_{out}(t) = -4e^{-4t}u(t) + 6u(t)$$

Example 3. Find $H(s) = \dfrac{V_{out}(s)}{I_{in}(s)} = \dfrac{V_c(s)}{I_{in}(s)}$



Modified Nodal Analysis

Define I_d through floating V-source. Then write 3 (modified) nodal equations.

1) $I_{in}(s) = (C_1 s + G_1) V_{in}(s) + I_d(s)$

2) $0 = (C_2 s + G_3) V_c(s) - I_d(s)$

3) Constraint Equation:

$$V_{in}(s) - V_c(s) = R_2 I_d + \alpha I_{c_1}$$

$$I_{c_1} = C_1 s V_{in}(s) \Rightarrow \alpha I_{c_1} = \alpha C_1 s V_{in}(s)$$

$$\therefore \quad V_{in}(s) - \alpha C_1 s V_{in}(s) - V_c(s) - R_2 I_d(s) = 0$$

$$(1 - \alpha C_1 s) V_{in}(s) - V_c(s) - R_2 I_d(s) = 0$$

$$(\alpha C_1 s - 1) V_{in}(s) + V_c(s) + R_2 I_d(s) = 0$$

Put into matrix form

$$\begin{bmatrix} C_1 s + G_1 & 0 & 1 \\ 0 & C_2 s + G_3 & -1 \\ \alpha C_1 s - 1 & 1 & R_2 \end{bmatrix} \begin{bmatrix} V_{in} \\ V_{out} \\ I_d \end{bmatrix} = \begin{bmatrix} I_{in}(s) \\ 0 \\ 0 \end{bmatrix}$$

If $\alpha = 0.5$ and all other parameters $= 1$, then

$$\begin{bmatrix} s+1 & 0 & 1 \\ 0 & s+1 & -1 \\ 0.5s-1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_{in} \\ V_{out} \\ I_d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} I_{in}$$

Find $H(s)$ using Cramer's Rule

$$H(s) = \frac{V_{out}(s)}{I_{in}(s)} = \frac{\begin{vmatrix} s+1 & 1 & 1 \\ 0 & 0 & -1 \\ 0.5s-1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} s+1 & 0 & 1 \\ 0 & s+1 & -1 \\ 0.5s-1 & 1 & 1 \end{vmatrix}} = \frac{-(0.5s-1)}{0.5s^2+3.5s+3} = \frac{2-s}{(s+1)(s+6)}$$

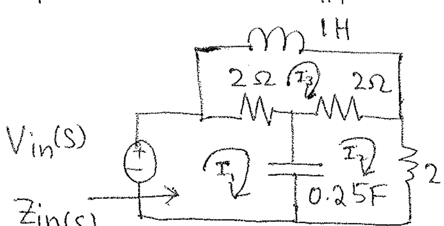
Find the step response

$$s(t) = \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right] = \mathcal{L}^{-1}\left[\frac{1/3}{s} - \frac{3/5}{s+1} + \frac{4/15}{s+6}\right]$$

$$= \left(\frac{1}{3} - \frac{3}{5}e^{-t} + \frac{4}{15}e^{-6t}\right)u(t)$$

Mesh Analysis in s-domain

Example 1: Find $Z_{in}(s)$ by mesh analysis.



Loop 1 : $2(I_1 - I_3) + \frac{4}{s}(I_1 - I_2) = V_{in}$

Loop 2 : $2(I_2 - I_3) + 2I_2 + \frac{4}{s}(I_2 - I_1) = 0$

Loop 3 : $sI_3 + 2(I_3 - I_2) + 2(I_3 - I_1) = 0$

In matrix form

$$\begin{bmatrix} 2+\frac{4}{s} & -\frac{4}{s} & -2 \\ -\frac{4}{s} & 4+\frac{4}{s} & -2 \\ -2 & -2 & s+4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_{in} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} V_{in}$$

Using Cramer's Rule,

$$\frac{I_1(s)}{V_{in}(s)} = \frac{\begin{vmatrix} 1 & -4/s & -2 \\ 0 & 4+4/s & -2 \\ 0 & -2 & s+4 \end{vmatrix}}{\begin{vmatrix} 2+\frac{4}{s} & -\frac{4}{s} & -2 \\ -\frac{4}{s} & 4+\frac{4}{s} & -2 \\ -2 & -2 & s+4 \end{vmatrix}} = 0.5 \qquad \therefore Z_{in}(s) = \frac{V_{in}(s)}{I_1(s)} = 2\,\Omega$$

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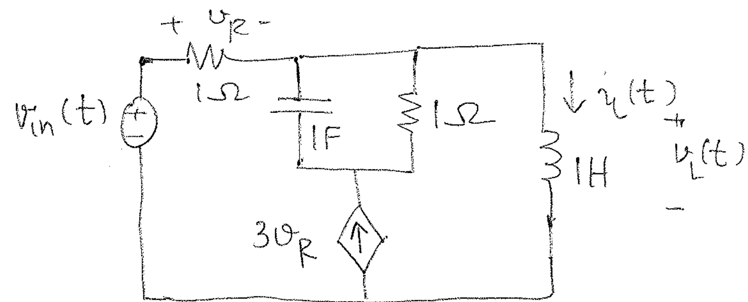
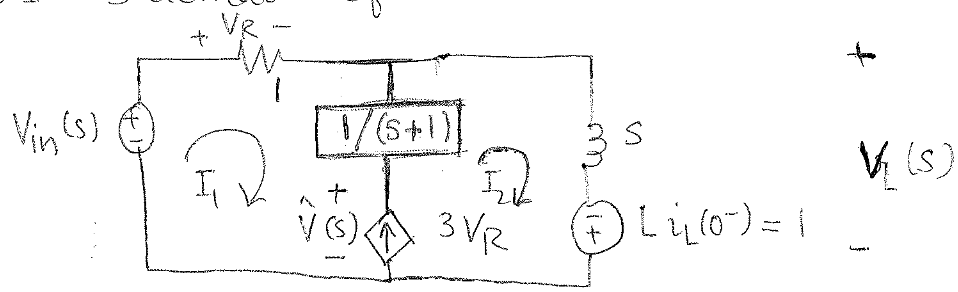
Example 2: Write and solve loop equations for the circuit below assuming $v_c(0^-) = 0$ and $i_L(0^-) = 1A$



Step 1: S-domain equivalent circuit



Step 2: Write loop equations

1) $V_{in}(s) = \left(1 + \frac{1}{s+1}\right) I_1(s) - \frac{1}{s+1} I_2(s) + \hat{V}(s)$

2) $1 = -\frac{1}{s+1} I_1(s) + \left(1 + \frac{1}{s+1} + s\right) I_2(s) - \hat{V}(s)$

3) $-3V_R = I_1 - I_2 \Rightarrow -3I_1 = I_1 - I_2$

$4I_1 - I_2 = 0$

Step 3: Put into Matrix form

$$\begin{bmatrix} \frac{s+2}{s+1} & -\frac{1}{s+1} & 1 \\ -\frac{1}{s+1} & \frac{s^2+s+1}{s+1} & -1 \\ 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \hat{V} \end{bmatrix} = \begin{bmatrix} V_{in} \\ 1 \\ 0 \end{bmatrix}$$

Step 4: Find $I_1(s)$ and $I_2(s)$ when $V_{in}(s) = \frac{1}{s}$. Then find $i_1(t)$ & $i_2(t)$

Use Cramer's Rule, Matrix inversion or MATLAB to do step 4.