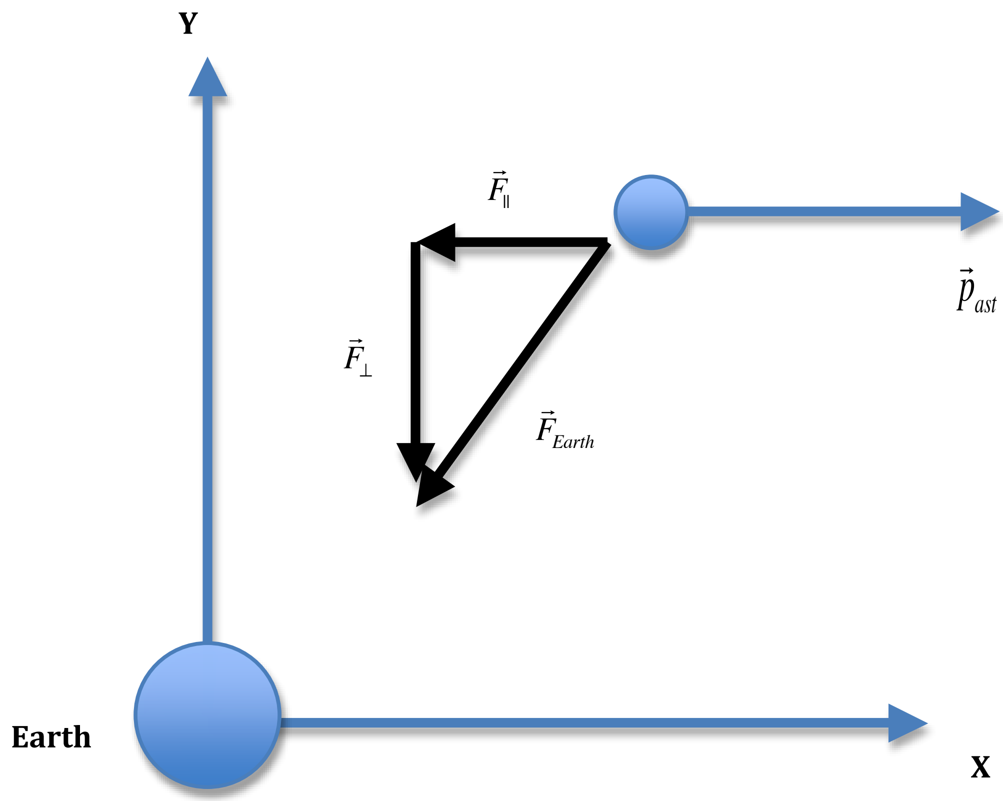
Solution to Problem of the Week #3

a, b, and c shown on diagram below



d. $\vec{F}_{\perp}$ is responsible for changing the direction of the asteroid's momentum.

e. The magnitude of the asteroid's momentum is decreasing because the direction in which $\vec{F}_{\parallel}$ acts is opposite to that of $\vec{p}_{ast}$.

f. In order to calculate the gravitational force on the asteroid due to the Earth, we must determine the distance between the two bodies and the appropriate unit vector – a unit vector that points from the asteroid towards the Earth. Our intuition informs us that because gravity is an attractive force, the force on the asteroid due to the Earth must point from the asteroid towards the Earth. Recall that in general:

$\vec{F}_{grav\, on\, 2\, by\, 1} = -G\dfrac{m_1 m_2}{|\vec{r}|^2}\hat{r}$,where $\vec{r} = \vec{r}_2 - \vec{r}_1$ extends from the center of object 1 to the center of object 2. In our case, object 1 is Earth and object 2 is the asteroid. The negative sign in the above equation assures that the force attracts body 2 towards body 1.

Therefore,

$$\vec{r} = \vec{r}_{ast} - \vec{r}_{Earth} = \langle 1.7\times10^8, 1.1\times10^8, 0\rangle - \langle 0,0,0\rangle = \langle 1.7\times10^8, 1.1\times10^8, 0\rangle\ m$$

The unit vector is given by:

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle 1.7\times10^8, 1.1\times10^8, 0\rangle}{\sqrt{(1.7\times10^8)^2 + (1.1\times10^8)^2}} = \frac{\langle 1.7\times10^8, 1.1\times10^8, 0\rangle}{2.02\times10^8} = \langle 0.84, .54, 0\rangle$$

Check that this is a vector of unit length:

$(0.84)^2 + (0.54)^2 \sim 0.997$ - very close to 1.0

$$\vec{F}_{grav\, on\, 2\, by\, 1} = -G\frac{m_1 m_2}{|\vec{r}|^2}\hat{r} = -6.7\times10^{-11}\frac{(6\times10^{24})(2\times10^{13})}{4.1\times10^{16}}\langle 0.84, 0.54, 0\rangle\ N$$

$$\vec{F}_{grav\, on\, 2\, by\, 1} = -\langle 1.65\times10^{11}, 1.06\times10^{11}, 0\rangle\ N$$

Notice that the direction of the force due to the Earth on the asteroid has components in the negative-x and negative-y directions as indicated on our diagram.

g. We will use the Momentum Principle in its update form:

$$\vec{p}_{final} = \vec{p}_{initial} + \vec{F}_{grav\, on\, 2\, by\, 1}\Delta t$$

$$\vec{p}_{final} = \langle 2.4\times10^{16}, 0, 0\rangle + \langle -1.65\times10^{11}, -1.06\times10^{11}, 0\rangle \cdot 3.6\times10^4\ kg\cdot m/s$$
$$= \langle 1.81\times10^{16}, -3.82\times10^{15}, 0\rangle\ kg\cdot m/s$$

h. What is the new position of the asteroid 10 hours later? Show all steps in your work and express your answer as a three-component vector.

We will use the position update equation:

$$\Delta\vec{r} = \vec{v}_{average}\Delta t$$

$$\vec{r}_{final} = \vec{r}_{initial} + \frac{(\vec{p}_{initial} + \vec{p}_{final})}{2m_{asteroid}}\Delta t = \vec{r}_{initial} + \vec{v}_{initial}\cdot\Delta t + \frac{\vec{F}}{2m_{asteroid}}(\Delta t)^2 = \vec{r}_{initial} + \vec{v}_{initial}\cdot\Delta t + \frac{1}{2}\vec{a}(\Delta t)^2$$

$$\vec{r}_{final} = \left\langle 1.7\times10^{8}, 1.1\times10^{8}, 0\right\rangle + \frac{(\left\langle 2.4\times10^{16}, 0, 0\right\rangle + \left\langle 1.8\times10^{16}, -3.8\times10^{15}, 0\right\rangle)\cdot 3.6\times10^{4}}{2\times2\times10^{13}}\ m$$
$$= \left\langle 2.1\times10^{8}, 1.1\times10^{8}, 0\right\rangle\ m$$

We have assumed that the force on the asteroid due to the Earth did not change significantly during the time interval. Let's check that that was a reasonable assumption.

We can do this by comparing the force on the asteroid at the beginning and end of the time interval. We need only to calculate the quantity

$$\frac{\vec{F}_{final}}{\vec{F}_{initial}} = \frac{\vec{r}^{\,2}_{initial}}{\vec{r}^{\,2}_{final}} = \frac{4.1\times10^{16}}{5.62\times10^{16}} = .72$$

So we see that the force exerted by the Earth on the asteroid at the final position is 72% of that experienced by the asteroid at its intial position. While our assumption of a constant force over the interval is not strictly correct, it is also not too horribly wrong.