Question 1. (30 points) Let $P = p_1 p_2 \cdots p_m$ be a string and let f be the function (as defined in class) such that $f(i)$ is the length of the longest proper prefix of $p_1 p_2 \cdots p_i$ that is also a suffix of $p_1 p_2 \cdots p_i$. For example, if $P = abracadabra$ then $f(4) = 1$ and $f(11) = 4$. Suppose that m is a power of 2, i.e., it is 2^q for some integer q (assume $q \geq 8$). Answer each of the following questions.

1. Assuming $f(m) = m - 2$, write down a simple equation for $f(j)$ as a function of j for $2 \leq j < m$. Briefly justify your answer.
2. Assuming $f(m/4) = 0$ and $f(m) = 3m/4$, what are the values of $f(m/2)$ and $f(3m/4)$?

Question 2. (30 points) Suppose the vertices of directed graph G are given already partitioned into L sets $S_1, \ldots, S_L$, such that every edge goes from a vertex in S_i to a vertex in S_{i+1} ($1 \leq i \leq L - 1$); in other words, if (v, w) is an edge and v is in S_i, then w is in S_{i+1}. Moreover, we assume that for every vertex v in S_i and every vertex w in S_{i+1}, there *is* an edge (v, w) in G. This implies that the total number of edges in G is

$$|S_1| * |S_2| + |S_2| * |S_3| + \cdots + |S_L - 1| * |S_L|.$$

A network flow problem has the above graph G as input, with all edge capacities equal to 1, with $S_1 = \{s\}$ and $S_L = \{t\}$. Let β be the value of a maximum flow from s to t. Write down β as a function of the $|S_i|$'s.

Question 3. (40 points) Suppose you are given a sorted set of n distinct red points on the x axis, whose x coordinates are $r_1 < r_2 < \cdots < r_n$. You are also given a sorted set of n distinct blue points on the x axis, whose x coordinates are $b_1 < b_2 < \cdots < b_n$. No red point coincides with a blue one.

1. Design an $O(n)$ time algorithm that computes a matching (i.e., a "marriage") of red to blue points in such a way that the sum of the n distances between the n matched pairs is minimized. More formally, the algorithm must compute a permutation π of the integers $\{1, \ldots, n\}$ that minimizes the summation

$$\sum_{i=1}^{n} |r_i - b_{\pi(i)}|$$

   Prove that your algorithm is correct, i.e., that the π it produces indeed minimizes the above summation.

2. Repeat the above for the problem of minimizing the longest distance between matched pairs. In other words, instead of minimizing $\sum_{i=1}^{n} |r_i - b_{\pi(i)}|$, we now want a π that minimizes $\max\{|r_1 - b_{\pi(1)}|, |r_2 - b_{\pi(2)}|, \ldots, |r_n - b_{\pi(n)}|\}$.

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