MA 266 Semester PRACTICE PROBLEMS FOR FINAL EXAM

Caution. Watch this page for updates. This is a generic review sheet and may contain some material that will not appear on the real final. For Chapter 7 (systems of diff. eqns.) practice problems: see Review7.pdf. Consult your instructor for further information about the coverage of the final. The form of the questions may differ.

The approximate format of the Fall 2000 Final is about fifteen short answer questions, about five medium length problems and about four longer problems, for a total of 200 points.
You should also review using the Review sheets for each section, particularly Review 6 and Review 7.

Table 6.2.1 Elementary Laplace Transforms will be included with the final exam.
1. Which of $y_1(t) = t,\ y_2(t) = t^2,\ y_3(t) = t^3$ are solutions of the differential equation $t^2y'' - 3ty' + 3y = 0$?
2. Use the given direction field to sketch the solution of the corresponding initial value problem (figure 1) with $y(0) = 1$. Determine (approximately) the interval in which the solution is valid.

3. What is the largest interval for which a unique solution of the initial value problem $ty' + \frac{1}{t+1}y = \frac{1}{t-3},\ y(1) = 0$, is guaranteed?

4. Which of the following initial value problem are guaranteed to have a unique solution? **A.** $y' = (ty)^{1/3},\ y(0) = 0$,
B. $y' = (ty)^{1/3},\ y(0) = 1$,
C. $y' = (ty)^{1/3},\ y(1) = 0$,
D. $y' = (ty)^{1/3},\ y(1) = 1$.

5. Consider the differential equation $dy/dt = -(y-1)(y-4)^2/10,\ t \geq 0,\ y \geq 0$, where the graph of $F(y) = -(y-1)(y-4)^2/10$ is indicated below in figure 4.
(a) What are the equlibrium solutions?
(b) Which equilibrium solutions are stable?
(c) For which open intervals of y is the graph of $y(t)$ increasing?
(d) Sketch the graph of the solution of the differential equation for $t \geq 0$

with each of the initial values $y(0) = 0,\ y(0) = 1,\ y(0) = 2,\ y(0) = 3,\ y(0) = 4,\ y(0) = 5$.

6. Determine whether each of the following differential equations is separable, homogeneous, linear, or exact:

(a) $x + 2y + (2x + y)dy/dx = 0$

(b) $2y + 1 + (x + 2)dy/dx = 0$

7. Find the explicit form of the general solution of the differential equation $y' = y^2 - 1$.

8. Find the explicit form of the solution of the initial value problem $y' = y^3,\ y(0) = 1$. What is the largest open interval on which the solution is valid?

9. Find the general solution of the differential equation $y' + (1 + \frac{1}{t})y = \frac{1}{t}$.

10. Find the solution of the initial value problem $y' = \frac{2y+t^2}{t}, \quad y(1) = 2$.

11.(a) Solve the initial value problem $y' - y = 2e^{-t} - 2,\ y(0) = a$.

(b) For which initial value(s) a will the solution approach infinity as t approaches infinity?

(c) For which initial value(s) a will the solution approach negative infinity as t approaches infinity?

(d) For which initial value(s) a will the solution remain bounded as t approaches infinity?

12. Find an implicit form of the general solution of the differential equation $y^2 + 1 + (2xy + 1)dy/dx = 0$.

13. Consider the initial value problem $y' = xy + y^2,\ y(3) = -1$.

(a) Is the solution increasing or decreasing near $(x, y) = (3, -1)$?

(b) Is the solution concave upward or downward near $(x, y) = (3, -1)$?

(c) Are the Euler tangent line method approximations of the solution near $(x, y) = (3, -1)$ greater than or less than the solution?

14. Determine approximate values at $t = 0.5$ of the solution of the initial value problem $y' = 3t + y,\ y(0) = 1$ by using the Euler tangent line method with $h = 0.25$.

15. Use the formula $y = tv$ to express the differential equation $dy/dt =$

$(t+y)/(t-y)$ in terms of t, v, and dv/dt.
16. Suppose y' is proportional to y, $y(0)=2$, and $y(1)=8$.
(a) Find y in terms of t.
(b) For what value of t does $y(t)=20$?

17. Find the general solution of these differential equations.
(a) $y''-4y'+3y=0$, (b) $y''-4y'+4y=0$, (c) $y''-4y'+5y=0$.

18. Find the solution of the initial value problem
$y'''-2y''+y'=0,\ y(0)=2,\ y'(0)=0,\ y''(0)=1.$

19 Find the general solution of the differential equation $y^{(5)}+By^{(4)}+Cy'''+Dy''+Ey'+Fy=0$, if its corresponding characteristic equation is $(r+1)(r^2-2r+5)^2=0$.

In Problems 20–22 find the general solution of the homogeneous differential equation in (a) and use the method of undetermined coefficients to find the **form** of a particular solution of the nonhomogeneous equations in (b) and (c).

20.(a) $y''+9y=0$, (b) $y''+9y=te^{-3t}$, (c) $y''+9y=\cos(3t)$
21.(a) $y''+6y'+5y=0$, (b) $y''+6y'+5y=t^2e^{-t}$, (c) $y''+6y'+5y=e^t+e^{-t}$

22.(a) $y''+6y'+9y=0$, (b) $y''+6y'+9y=te^{-3t}$, (c) $y''+6y'+9y=\cos(3t)+t$

23.(a) Find the general solution of the homogeneous differential equation $y^v+2y'''+y'=0$, which has characteristic equation
$r^5+2r^3+r=0$.

(b) Use the method of undetermined coefficients to find the **form** of a particular solution of the nonhomogeneous equation
$y^v+2y'''+y'=t+t\cos t$. You do not need to solve for the values of the coefficients.

24. Find the solution of the initial value problem
$y''+5y'+6y=24e^t,\ y(0)=0,\ y'(0)=0.$

25. Find the general solution of the differential equation

$y'' - y' = 4t.$

26. The differential equation $t^2y'' - ty' + y = 0$ has solution $y_1(t) = t$. Use the method of reduction of order to find a solution $y_2 = tv(t)$ of the equation that is not a constant multiple of y_1.

27. The differential equation $t^2y'' - 2ty' + 2y = 0$ has solutions $y_1 = t$ and $y_2 = t^2$. Use the method of variation of parameters to find a solution of $t^2y'' - 2ty' + 2y = 2t^2$.

28. The functions $y_1 = t^2$ and $y_2 = t^{-2}$ are solutions of the differential equation
$t^2y'' + ty' - 4y = 0$.

(a) Evaluate the Wronskian $W(t^2, t^{-2})$.

(b) Find the solution of the initial value problem $t^2y'' + ty' - 4y = 0,\ y(1) = 1,\ y'(1) = 6$.

29. (a) For what values of ω will resonance occur, so the solution of the initial value problem
$u'' + 4u = 4\cos(\omega t),\ u(0) = 0,\ u'(0) = 0$, becomes unbounded as $t \to \infty$?

(b) For what value of ω does the solution of the initial value problem
$u'' + 4u' + 4u = 4\cos(\omega t),\ u(0) = 0,\ u'(0) = 0$, become unbounded as $t \to \infty$?

(c) For what positive value(s) of m will the solution of the initial value problem
$mu'' + 2u' + u = 0,\ u(0) = 0,\ u'(0) = 1$, oscillate, so $u(t)$ will periodically have value zero?

30. At time $t = 0$ a tank contains 40 ounces of salt mixed in 100 gallons of water. A solution that contains 4 oz of salt per gallon of solution is then pumped into the tank at a rate of 5 gal/min. The well-stirred mixture flows out of the tank at the rate of 3 gal/min. Set up and solve an initial value problem that gives the amount of salt in the tank after t minutes.

31. Use the definition of the Laplace transform as an improper integral to evaluate $\mathcal{L}\{u_1(t)e^{-t}\}$.

32. Find the Laplace transforms of the functions $f(t)$.

(a) $f(t) = \sin(3t) + \cos(3t)$,

(b) $f(t) = e^t(1 + \cos(2t))$,
(c) $f(t) = t^2 - u_1(t)(t^2 - 1)$.

33. Find the inverse Laplace transforms of the functions $F(s)$.
(a) $F(s) = s/(s-1)^2$, (b) $F(s) = s/(s^2 - 2s - 3)$, (c) $F(s) = (se^{-s})/(s^2 + 2s + 5)$.

34.(a) $\mathcal{L}\{\int_0^t \sin 2(t-\tau) \cdot \cos(3\tau)\, d\tau\} =$

(b) $\mathcal{L}^{-1}\{\frac{6}{s^4} \cdot \frac{s}{s^2+4}\} = \int_0^t$

35 Find the Laplace transform of the function $f(t)$, where the graph of f is given below. (see figure 3)

36. Find the solution of the initial value problem
$y'' + y = F(t),\ y(0) = 0,\ y'(0) = 0$, where $F(t) = t$, if $0 \le t < \pi$ and $= \pi$, if $t \ge \pi$.

37. Find the solution of the initial value problem
$y'' + y = \delta(t - \pi),\ y(0) = 0,\ y'(0) = 1$.

38. Express the initial value problem $y'' + 2y' + (\sin t)y = \cos t,\ y(0) = 1,\ y'(0) = -3$,
as a system of first order differential equations with initial conditions.

39. Tank 1 initially holds 50 gallons of brine with concentration 2 oz/gal and Tank 2 initially holds 25 gallons of pure water. Brine with concentration 4 oz/gal flows into Tank 1 at a rate of 6 gal/min and brine with concentration 3 oz/gal flows into Tank 2 at a rate of 4 gal/min. The well–mixed solution in Tank 1 flows out of the system at a rate of 3 gal/min and the solution in Tank 2 flows out of the system at a rate of 7 gal/min. Also, the solution in Tank 1 flows into Tank 2 at a rate of 5 gal/min while the solution in Tank 2 flows into Tank 1 at a rate of 2 gal/min. SET UP an initial value problem that gives the amounts of salt in Tank 1 and Tank 2 at time t. DO NOT SOLVE THE INITIAL VALUE PROBLEM!

Problems 40-up: See the review sheet #7.
ANSWERS:
1. $y_1(t) = t$ and $y_3(t) = t^3$

2. The solution is valid for approximately $-1 < t < 1$.

3. $0 < t < 3$

4. B and D.
(The function $f(t, y) = (ty)^{1/3}$ is continuous for all (t, y); $\frac{\partial f}{\partial y}$ is continuous for $y = 0$.)

5.(a) $y = 1$ and $y = 4$ (d)

(b) $y = 1$

(c) $0 < y < 1$

6.(a) homogeneous and exact, (b) separable and linear

7. $y = \frac{1+Ce^{2t}}{1-Ce^{2t}}$, $y = \pm 1$

8. $y = \frac{1}{\sqrt{1-2t}}$, $t < \frac{1}{2}$

9. $y = \frac{1}{t} + \frac{C}{te^t}$

10. $y = t^2 \ln t + 2t^2$

11. (a) $y = 2 - e^{-t} + (a-1)e^t$, (b) $a > 1$, (c) $a < 1$, (d) $a = 1$.

12. $xy^2 + x + y = C$

13. (a) decreasing, (b) concave downward, (c) The Euler tangent line approximations are greater than the solution near $(x, y) = (3, -1)$.
14. 1.75
15. $t dv/dt = (1 + v^2)/(1 - v)$

16. (a) $y = 2e^{(\ln 4)t} = 2 \cdot 4^t$, (b) $t = \ln 10/\ln 4$

17.(a) $y = C_1 e^t + C_2 e^{3t}$

(b) $y = C_1 e^{2t} + C_2 t e^{2t}$

(c) $y = C_1 e^{2t}\cos t + C_2 e^{2t}\sin t$

18. $y = 3 - e^t + te^t$

19 $y = C_1 e^{-t} + C_2 e^t\cos(2t) + C_3 e^t\sin(2t) + C_4 te^t\cos(2t) + C_5 te^t\sin(2t)$
20.(a) $y = C_1\cos(3t) + C_2\sin(3t)$, (b) $y = (At+B)e^{-3t}$, (c) $y = t(A\cos(3t) + B\sin(3t))$

21.(a) $y = C_1 e^{-t} + C_2 e^{-5t}$, (b) $y = t(At^2+Bt+C)e^{-t}$, (c) $y = Ae^t + Bte^{-t}$

22.(a) $y = C_1 e^{-3t} + C_2 te^{-3t}$, (b) $y = t^2(At+B)e^{-3t}$, (c) $y = A\cos(3t) + B\sin(3t) + Ct + D$

23.(a) $y = C_1 + C_2\cos t + C_3\sin t + C_4 t\cos t + C_5 t\sin t$,
(b) $y = t(At+B) + t^2((Ct+D)\cos t + (Et+F)\sin t)$

24. $y = -8e^{-2t} + 6e^{-3t} + 2e^t$

25. $y = C_1 + C_2 e^t - 2t^2 - 4t$

26. $y_2 = t\ln|t|$
(Any $y_2 = At\ln|t| + Bt$, where A and B are fixed numbers and $A \neq 0$ is correct.)

27. $y = C_1 t + C_2 t^2 + 2t^2\ln|t|$, C_1 and C_2 any real numbers

28.(a) $W(t^2, t^{-2}) = -4t^{-1}$, (b) $y = 2t^2 - t^{-2}$

29.(a) $\omega = \pm 2$, (b) none, (c) $m > 1$

30. $dQ/dt = (4)(5) - (Q/(100+2t))(3)$, $Q(0) = 40$; $Q(t) = 8(t+50) - \frac{360(50)^{3/2}}{(t+50)^{3/2}}$

31. $\mathcal{L}\{u_1(t)e^{-t}\} = \int_0^\infty u_1(t)e^{-t}e^{-st}\,dt$

$= \int_1^\infty e^{-(s+1)t}\, dt$
$= \lim\limits_{A\to\infty} \int_1^A e^{-(s+1)t}\, dt = \lim\limits_{A\to\infty} [\frac{e^{-(s+1)t}}{-(s+1)}]_1^A$
$= \lim\limits_{A\to\infty} (\frac{e^{-(s+1)}}{s+1} - \frac{e^{-(s+1)A}}{s+1})$
$= e^{-(s+1)}/(s+1)$
32.(a) $\frac{3}{s^2+9} + \frac{s}{s^2+9}$, (b) $\frac{1}{s-1} + \frac{s-1}{(s-1)^2+4}$, (c) $\frac{2}{s^3} - e^{-s}(\frac{2}{s^3} + \frac{2}{s^2})$

33. (a) $f(t) = e^t + te^t$, (b) $f(t) = \frac{3}{4}e^{3t} + \frac{1}{4}e^{-t}$
(c) $f(t) = u_1(t)(e^{-t+1}\cos(2t-2) - \frac{1}{2}e^{-t+1}\sin(2t-2))$

34.(a) $\mathcal{L}\{\int_0^t \sin 2(t-\tau)\cdot\cos(3\tau)\, d\tau\} = \frac{2}{s^2+4}\cdot\frac{s}{s^2+9}$

(b) $\mathcal{L}^{-1}\{\frac{6pt}{s^4}\cdot\frac{s}{s^2+4}\} = \int_0^t (t-\tau)^3 \cos 2\tau\, d\tau = \int_0^t \tau^3 \cos 2(t-\tau)\, d\tau$

35 $10\frac{e^{-2s}}{s} - 10\frac{e^{-4s}}{s} + 5\frac{e^{-6s}}{s} - 5\frac{e^{-8s}}{s}$
36. $y = t - \sin t - u_\pi(t)(t - \pi - \sin(t-\pi)) = t - \sin t$ if $0 < t < \pi$ and $= \pi - 2\sin t$, if $t \geq \pi$.
37. $y = \sin t + u_\pi(t)(\sin(t-\pi)) = \sin t$, if $0 < t < \pi$, and $= 0$, if $t \geq \pi$.
38. $x_1' = x_2,\ x_2' = \cos t - 2x_2 - (\sin t)x_1,\ x_1(0) = 1,\ x_2(0) = -3$
39.
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{pmatrix} 24 - 8x_1/50 + 2x_2/25 \\ 12 + 5x_1/50 - 9x_2/25 \end{pmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}.$$

Don't forget Review 7.