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Summer 2012

ECE 20200 : Linear Circuit Analysis II
School of ECE, Purdue University

LECTURE 7

- Equivalent circuits of L with initial conditions
- Equivalent circuits of C with initial conditions

Reference: Decarlo/Lin pp 618-625

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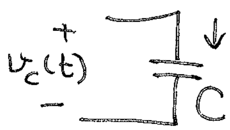
Equivalent s-domain circuits for 'L' and 'C' with initial conditions

Integration Property of Laplace Transform

$$\mathcal{L}\left[\int_{-\infty}^{t} f(q)\,dq\right] = \frac{F(s)}{s} + \frac{\int_{-\infty}^{0^-} f(q)\,dq}{s}$$

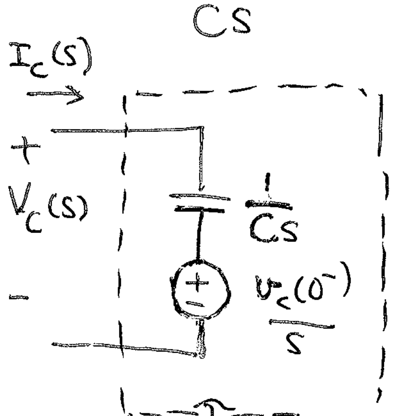
Capacitor



$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(q)\,dq$$

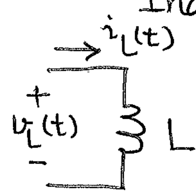
Laplace Transform

$$V_C(s) = \frac{I_C(s)}{Cs} + \frac{v_C(0^-)}{s}$$



equivalent s-domain circuit for 'C' with initial condition

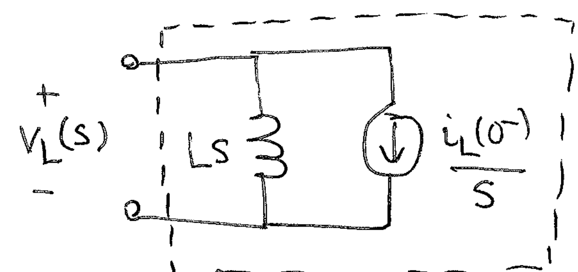
Inductor



$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(q)\,dq$$

Laplace Transform

$$I_L(s) = \frac{V_L(s)}{Ls} + \frac{i_L(0^-)}{s}$$

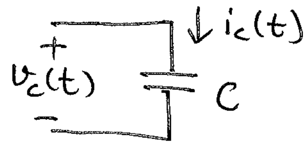


equivalent s-domain circuit for 'L' with initial condition

Differentiation Property of Laplace Transform

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^-)$$

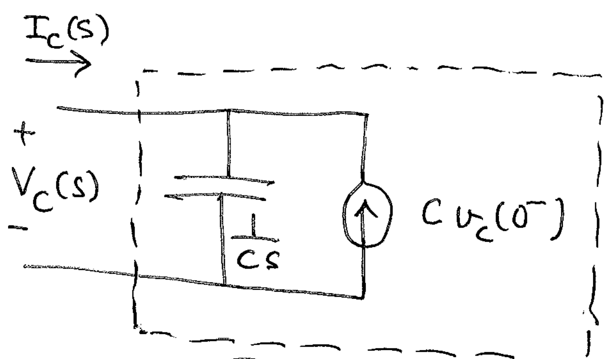
Capacitor



$$i_c(t) = C\frac{dv_c(t)}{dt}$$

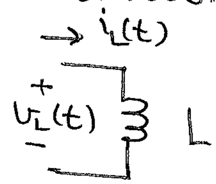
Laplace Transform

$$I_C(s) = Cs\,V_C(s) - C\,v_c(0^-)$$



equivalent s-domain circuit for 'C' with initial condition

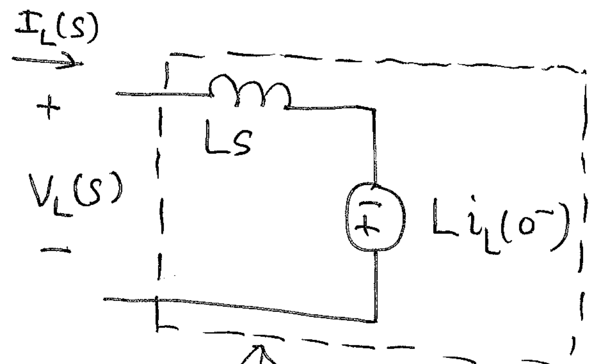
Inductor



$$v_L(t) = L\frac{di(t)}{dt}$$

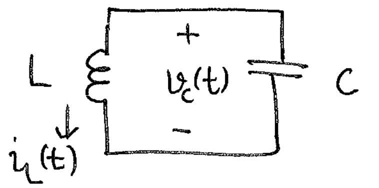
Laplace Transform

$$V_L(s) = Ls\,I_L(s) - L\,i_L(0^-)$$



equivalent s-domain circuit for 'L' with initial condition

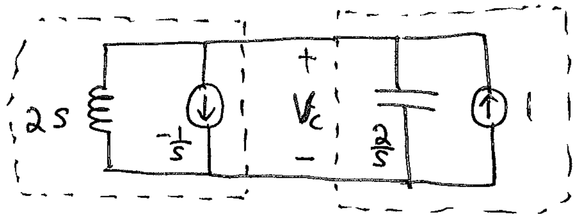
Example 1:



$L = 2H$, $C = 0.5\,F$

$i_L(0^-) = -1\,A$, $v_c(0^-) = 2\,V$

Find $v_c(t)$ for $t > 0$



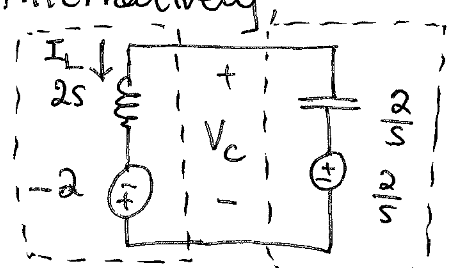
$$2s \,//\, \frac{2}{s} = \frac{2\cancel{s}\left(\frac{2}{\cancel{s}}\right)}{2s + \frac{2}{s}} = \frac{4}{\frac{2s^2+2}{s}}$$

$$Z = \frac{2s}{s^2+1}$$

$$V_c = \left(1 + \frac{1}{s}\right)\cdot\frac{2s}{s^2+1} = \frac{2(s+1)}{s^2+1} = \frac{2s}{s^2+1} + \frac{2}{s^2+1}$$

$\therefore\ v_c(t) = [2\cos(t)\,u(t) + 2\sin(t)\,u(t)]\ V \leftarrow$

Alternatively,



$$V_c = 2s\left(I_L\right) - (-2)$$

$$= 2s\left(\frac{\frac{2}{s} - 2}{\frac{2}{s} + 2s}\right) + 2$$

$$= 2s\left(\frac{2 - 2s}{2 + 2s^2}\right) + 2$$

$$= \frac{2s(1-s)}{s^2+1} + 2$$

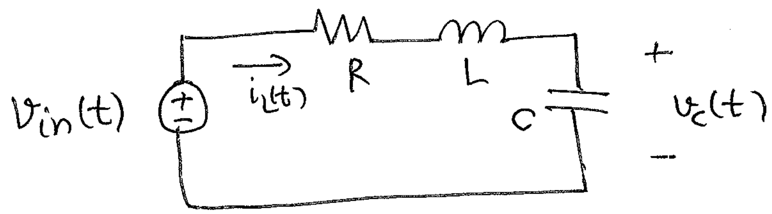
$$= \frac{2s \cancel{- 2s^2} \cancel{+ 2s^2} + 2}{s^2+1}$$

$$= \frac{2s+2}{s^2+1} = \frac{2s}{s^2+1} + \frac{2}{s^2+1}$$

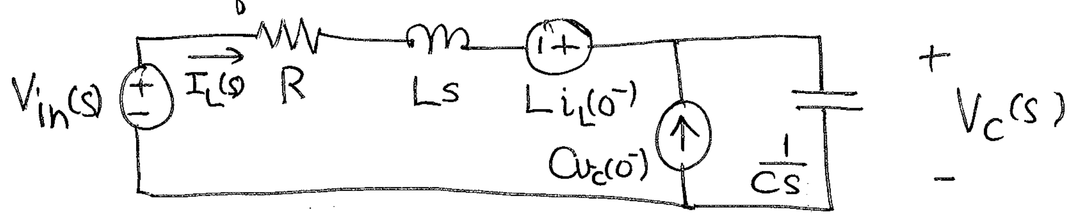
$\therefore\ v_c(t) = [2\cos(t)\,u(t) + 2\sin(t)\,u(t)]\ V \leftarrow$

Example 2

(a) Draw s-domain equivalent circuit and solve for $V_C(s)$.

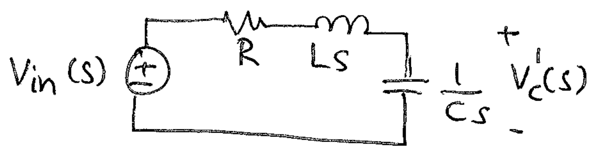


s-domain equivalent circuit



solve for $V_C(s)$ using superposition

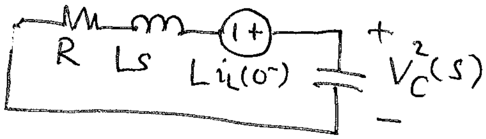
(a) Input source:



$$V_C^1(s) = \frac{\frac{1}{Cs}}{\frac{1}{Cs} + R + Ls} V_{in}(s)$$

$$\boxed{V_C^1(s) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} V_{in}(s)}$$

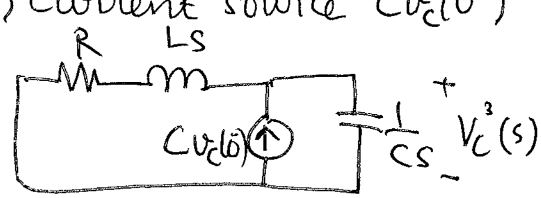
(b) V-source $Li_L(0^-)$:



$$V_C^2(s) = \frac{\frac{1}{\cancel{L}C}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \cancel{L} i_L(0^-)$$

$$\boxed{V_C^2(s) = \frac{\frac{1}{C}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} i_L(0^-)}$$

(c) Current source $Cv_c(0^-)$



$$V_C^3(s) = \frac{\frac{1}{Cs} \cdot (R + Ls)}{\frac{1}{Cs} + R + Ls} C v_c(0^-)$$

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$$\boxed{V_c^3(s) = \frac{s + \frac{R}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} v_c(0^-)}$$

$$\therefore V_C(s) = V_c^1(s) + V_c^2(s) + V_c^3(s)$$

$$= \frac{\frac{1}{LC} V_{in}(s)}{s^2 + \frac{R}{L}s + \frac{1}{LC}} + \frac{\frac{1}{C} i_L(0^-)}{s^2 + \frac{R}{L}s + \frac{1}{LC}} + \frac{s + \frac{R}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} v_c(0^-)$$

(b) If $R = 2\Omega$, $C = 0.25\,F$, $L = 0.25\,H$, $v_{in}(t) = (1 - e^{-4t})u(t)\,V$ then find $V_c(s)$ and then $v_c(t)$.

$$V_{in}(s) = \frac{1}{s} - \frac{1}{s+4}$$

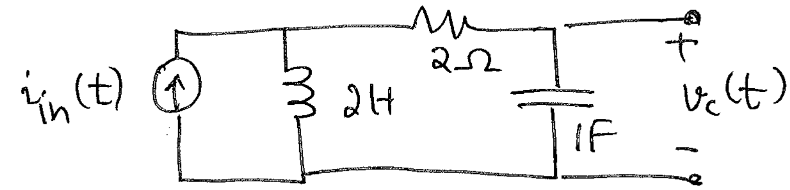
Plugging into $V_c(s)$, we get

$$V_c(s) = \left[\frac{16}{s(s+4)^2} - \frac{16}{(s+4)^3}\right] + \left[\frac{4\, i_L(0^-)}{(s+4)^2}\right] + \left[\frac{s+8}{(s+4)^2} v_c(0^-)\right]$$

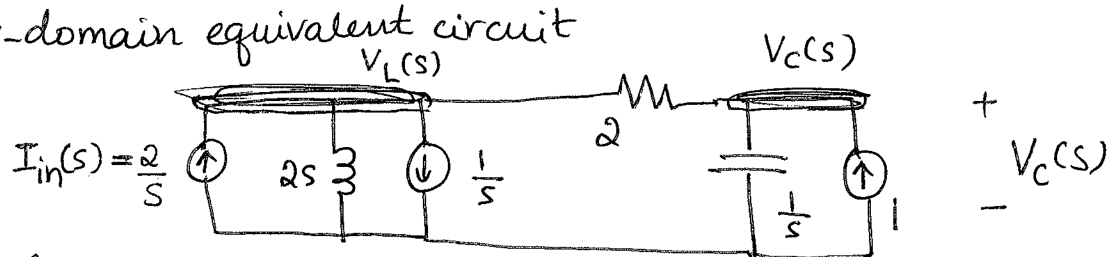
$$V_c(s) = \left(\frac{1}{s} - \frac{1}{s+4} - \frac{4}{(s+4)^2} - \frac{16}{(s+4)^3}\right) + \left(\frac{4\, i_L(0^-)}{(s+4)^2}\right) +$$
$$\left[\left(\frac{1}{s+4} + \frac{4}{(s+4)^2}\right) v_c(0^-)\right]$$

$$v_c(t) = \left(1 - e^{-4t} - 4t\,e^{-4t} - 8t^2 e^{-4t}\right)u(t) + 4 i_L(0^-)\, t e^{-4t} u(t)$$
$$+ v_c(0^-)\,(1 + 4t)\, e^{-4t} u(t) \leftarrow$$

Example 3. Find $v_c(t)$ when $v_c(0^-) = 1\,V$, $i_L(0^-) = 1\,A$ and $i_{in}(t) = 2u(t)\,A$



s-domain equivalent circuit



Writing down node equations at $V_L(s)$ and $V_C(s)$

At $V_L(s)$ node: $$\frac{2}{s} - \frac{1}{s} - \frac{V_L(s)}{2s} - \frac{V_L(s) - V_C(s)}{2} = 0$$

$$V_L(s)\left(\frac{1}{2s} + \frac{1}{2}\right) - \frac{V_C(s)}{2} = \frac{1}{s}$$

At $V_C(s)$ node: $$\frac{V_C(s) - V_L(s)}{2} + \frac{V_C(s)}{1/s} - 1 = 0$$

$$-\frac{1}{2}V_L(s) + \left(s + \frac{1}{2}\right)V_C(s) = 1$$

Putting into matrix form,

$$\begin{bmatrix} \frac{1}{2s} + \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & s + \frac{1}{2} \end{bmatrix} \begin{bmatrix} V_L \\ V_C \end{bmatrix} = \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} V_L \\ V_C \end{bmatrix} = \begin{bmatrix} \frac{1}{2s} + \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & s + \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix} = \frac{2s}{s^2 + s + 0.5} \begin{bmatrix} s + 0.5 & 0.5 \\ 0.5 & \frac{s+1}{2s} \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix}$$

$$\therefore V_L = \frac{3s + 1}{s^2 + s + 0.5} \quad \text{and} \quad V_C = \frac{s + 2}{s^2 + s + 0.5} = \frac{s + 2}{(s + 0.5)^2 + (0.5)^2}$$

By MATLAB,

$$v_c(t) = e^{-0.5t}\left[\cos(0.5)t\,u(t) + 3\sin(0.5t)\,u(t)\right]$$