Question 1. (10 points) The number of strings that satisfy the first condition is $2^{\ell-m}$, the number of strings that satisfy the second condition is $2^{\ell-n}$, and the number of strings that satisfy both conditions is $2^{\ell-m-n}$. Therefore the number of strings that satisfy the first or the second or both is

$$2^{\ell-m} + 2^{\ell-n} - 2^{\ell-m-n}$$

where we used the fact that $|A \cup B| = |A| + |B| - |A \cap B|$.

Question 2. (10 points) The number we seek is the same as the number of distinct solutions to the equation

$$(y_1 + 1) + (y_2 + 1) + \cdots + (y_k + 1) = n$$

where $y_1, y_2, \ldots, y_k$ are *non-negative* integers. Subtracting k from both sides, we seek the number of distinct solutions to the equation

$$y_1 + y_2 + \cdots + y_k = n - k$$

where $y_1, y_2, \ldots, y_k$ are non-negative integers. This case was covered in class and is in the notes in Module 6 (as Example 15), therefore the answer is:

$$C(n - k + (k - 1), k - 1) = C(n - 1, k - 1)$$

Question 3. (10 points)

1. 2^n because at each step there are 2 possible moves (right or up).
2. Each choice of which m steps are the horizontal ones (with the remaining $n-m$ being vertical) corresponds to a distinct path from $(0, 0)$ to $(m, n-m)$. There are $C(n, m)$ different choices for which m of the n steps are horizontal.

Question 4. (10 points) Let ℓ_i be the length of a longest increasing subsequence of S whose rightmost symbol is x_i, $1 \leq i \leq N$. If some $\ell_i \geq n+1$ then we are done because there is an increasing subsequence of lenth $\geq n+1$. So suppose all ℓ_i are smaller than $n+1$. Put x_i in "pigeonhole" number ℓ_i, for all $1 \leq i \leq N$.

Claim: Let x_i and x_j be in the same pigeonhole (i.e., $\ell_i = \ell_j$). If $i < j$ then $x_i > x_j$.

Proof. By contradiction: Suppose that $x_i < x_j$. Then a longest increasing subsequence of length ℓ_i that ends at x_i can be extended by appending x_j at its end, thereby obtaining a longer (by 1) increasing subsequence ending at x_j, which would imply that $\ell_j \geq \ell_i + 1$ (a contradiction to $\ell_i = \ell_j$).

An immediate consequence of the above claim is that the x_i's that are in the same pigeonhole form a decreasing subsequence of S. Therefore the proof would be complete if we could show that some pigeonhole contains $n + 1$ (or more) pigeons. This is proved by contradiction: There are no more than n pigeonholes (because all ℓ_i's are smaller than $n + 1$), and if each pigeonhole contained less than $n + 1$ pigeons then the total number of pigeons would be $\leq n^2$, contradicting the fact that there are $n^2 + 1$ pigeons.