Minimum Spanning Trees
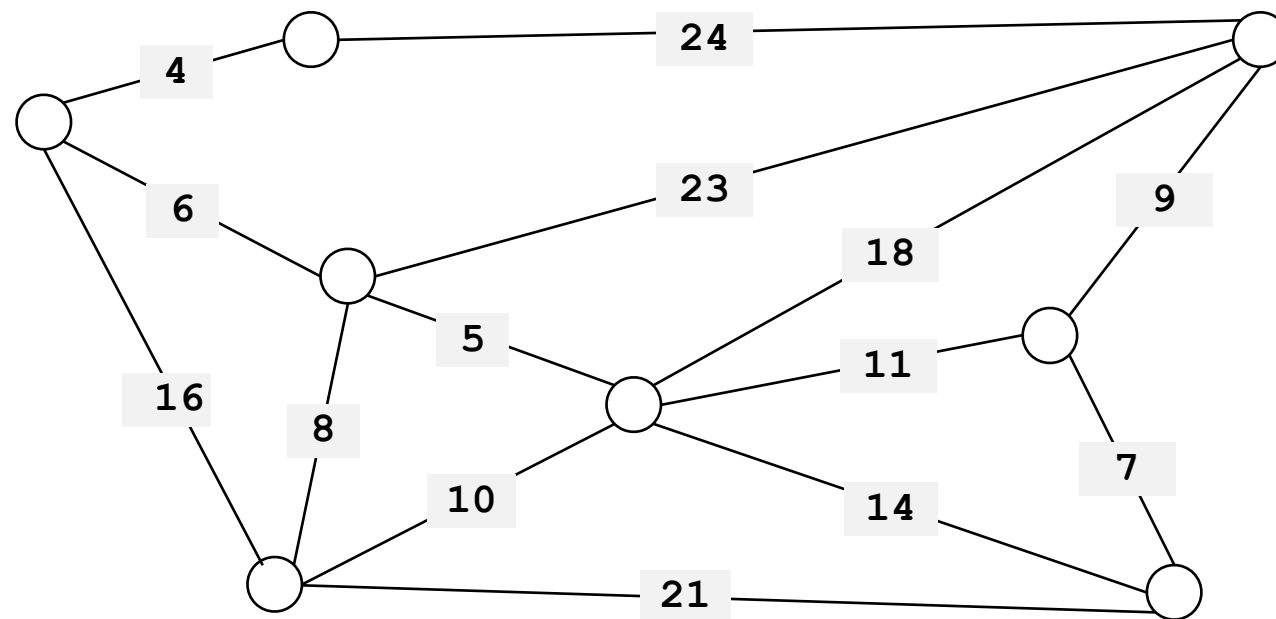
- edge-weighted graph API
- greedy algorithm
- Kruskal's algorithm
- Prim's algorithm
- advanced topics

Minimum spanning tree

Given. Undirected graph G with positive edge weights (connected).

Def. A spanning tree of G is a subgraph T that is connected and acyclic.

Goal. Find a min weight spanning tree.

graph G

Minimum spanning tree

Given. Undirected graph G with positive edge weights (connected).

Def. A spanning tree of G is a subgraph T that is connected and acyclic.
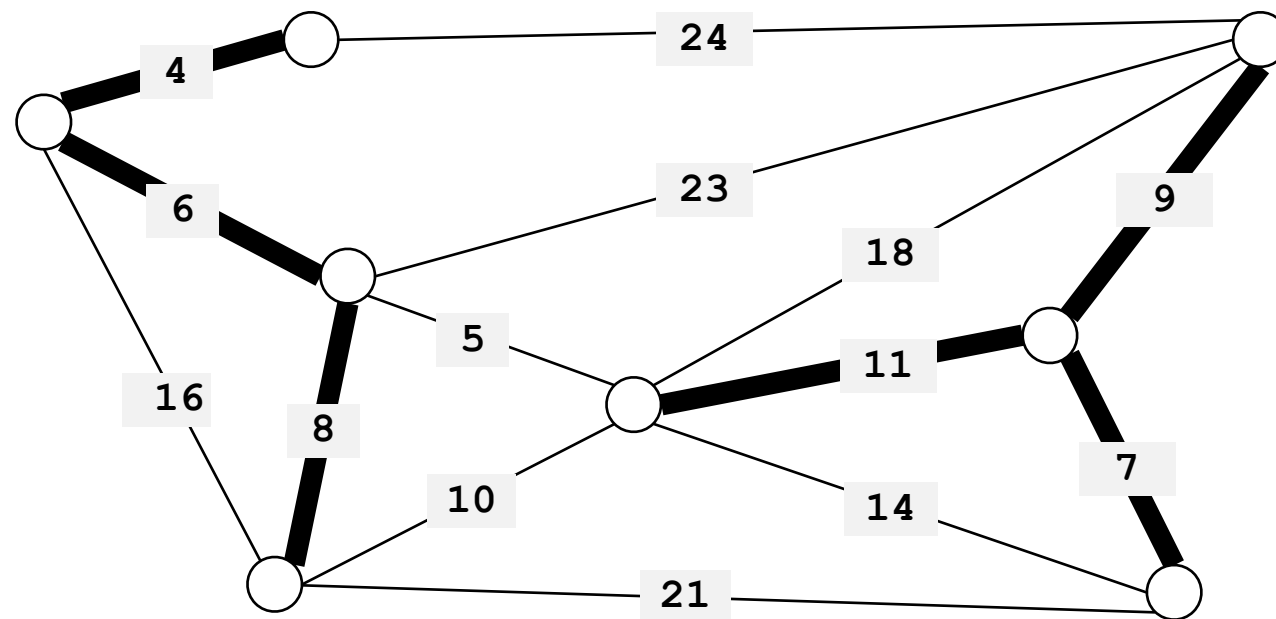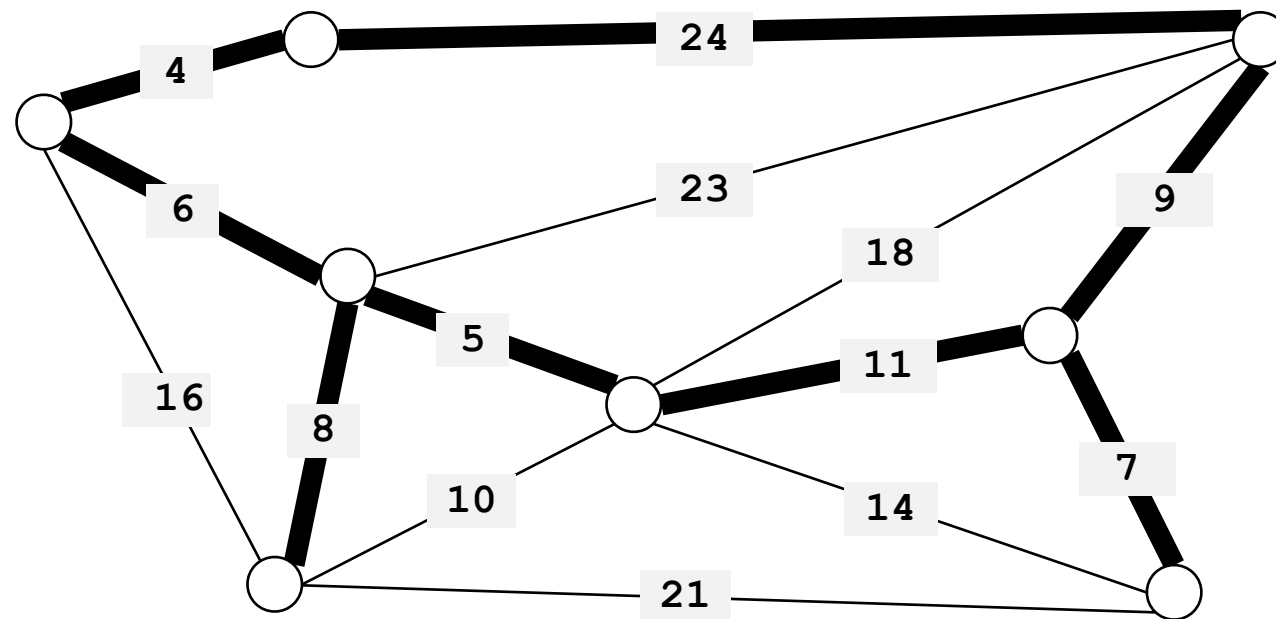
Goal. Find a min weight spanning tree.

not connected

Minimum spanning tree

Given. Undirected graph G with positive edge weights (connected).

Def. A spanning tree of G is a subgraph T that is connected and acyclic.

Goal. Find a min weight spanning tree.

not acyclic

Minimum spanning tree

Given. Undirected graph G with positive edge weights (connected).

Def. A spanning tree of G is a subgraph T that is connected and acyclic.
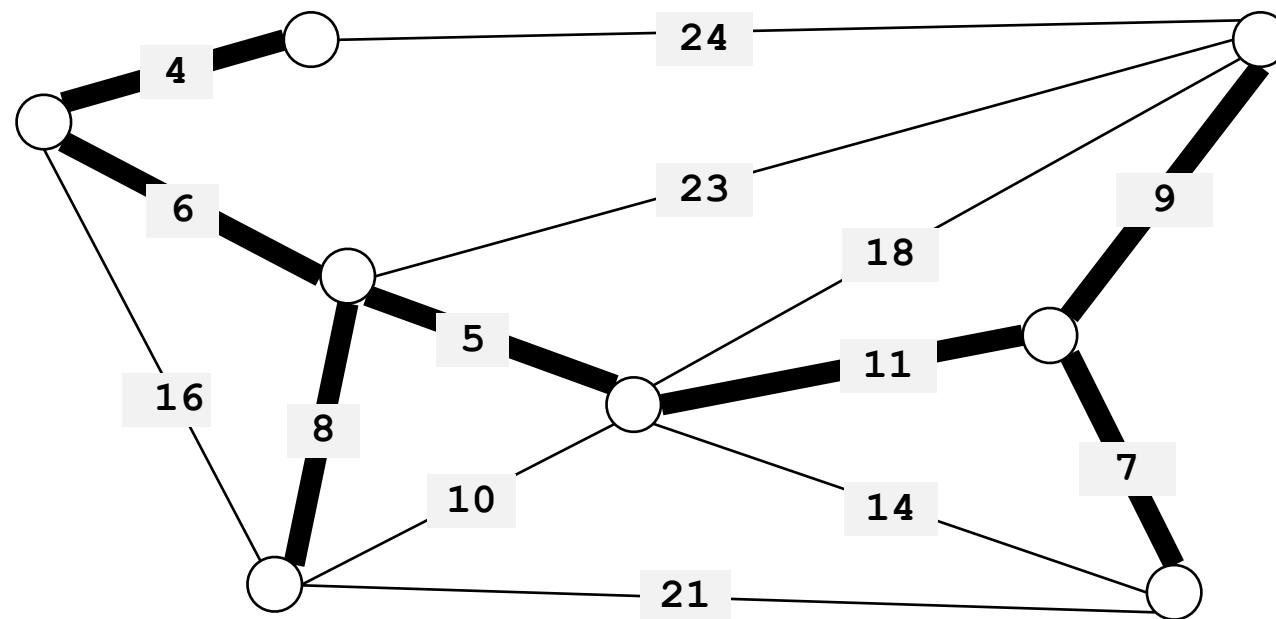
Goal. Find a min weight spanning tree.

spanning tree T: cost = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7

# Brute force. Try all spanning trees?

Applications

MST is fundamental problem with diverse applications.

- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, cable, computer, road).

http://www.ics.uci.edu/~eppstein/gina/mst.html

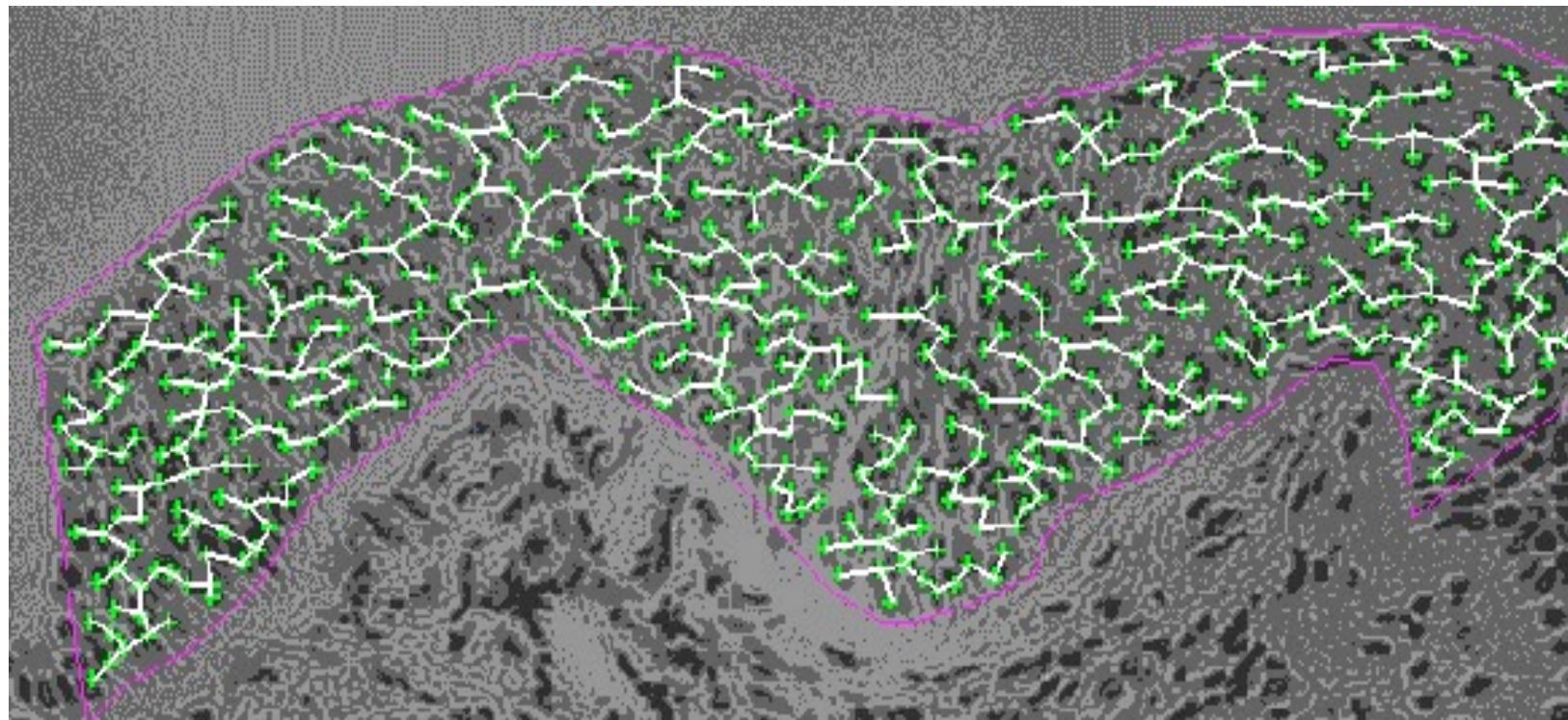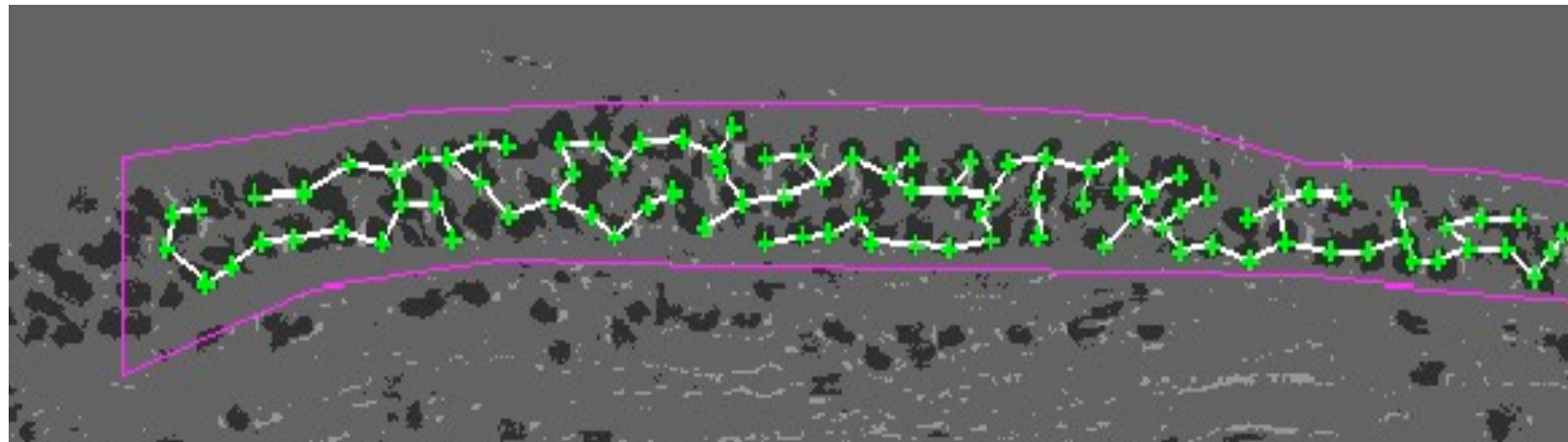# Network design

MST of bicycle routes in North Seattle
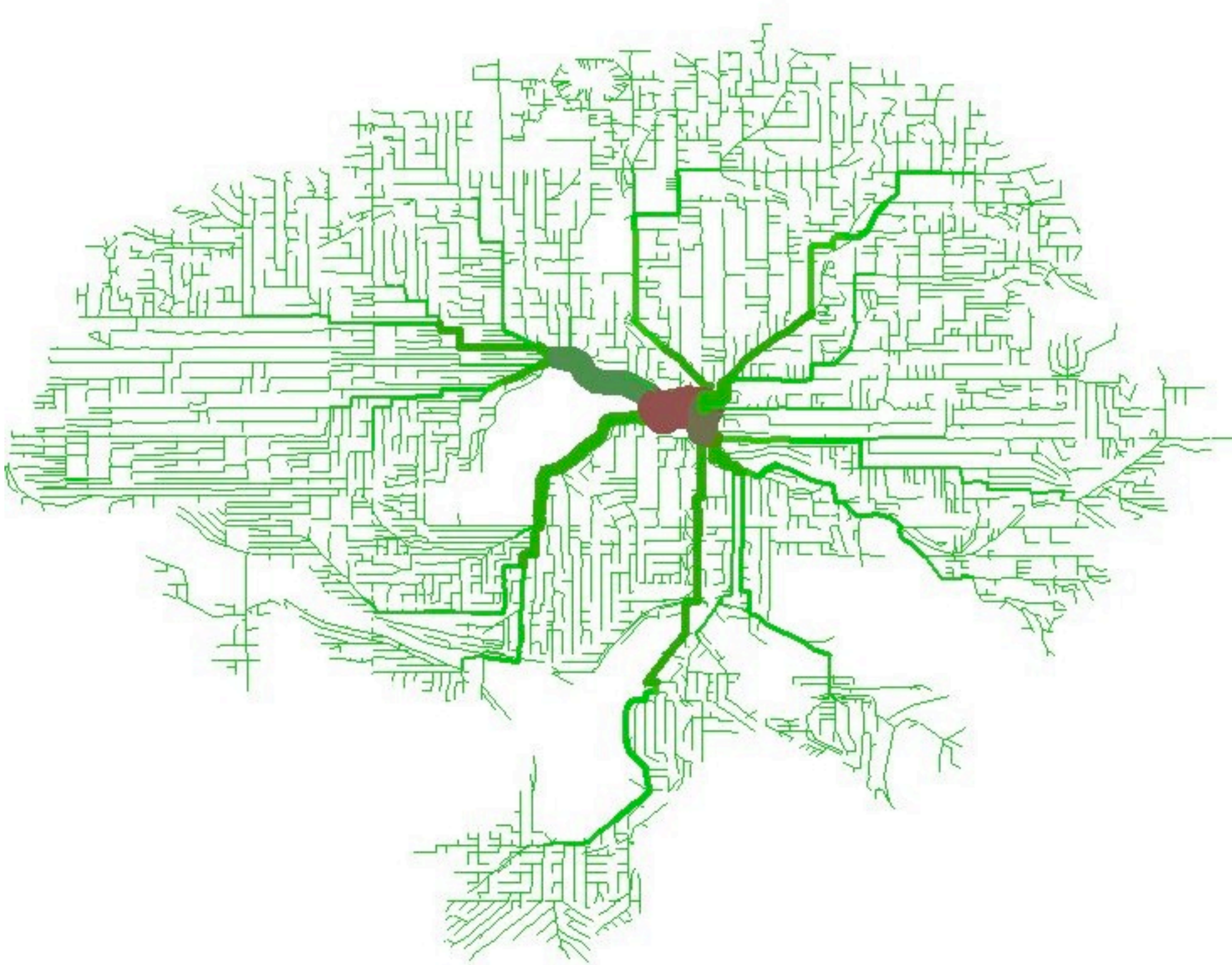
Medical image processing

MST describes arrangement of nuclei in the epithelium for cancer research

`http://www.bccrc.ca/ci/ta01_archlevel.html`

Genetic research

MST of tissue relationships measured by gene expression correlation coefficient
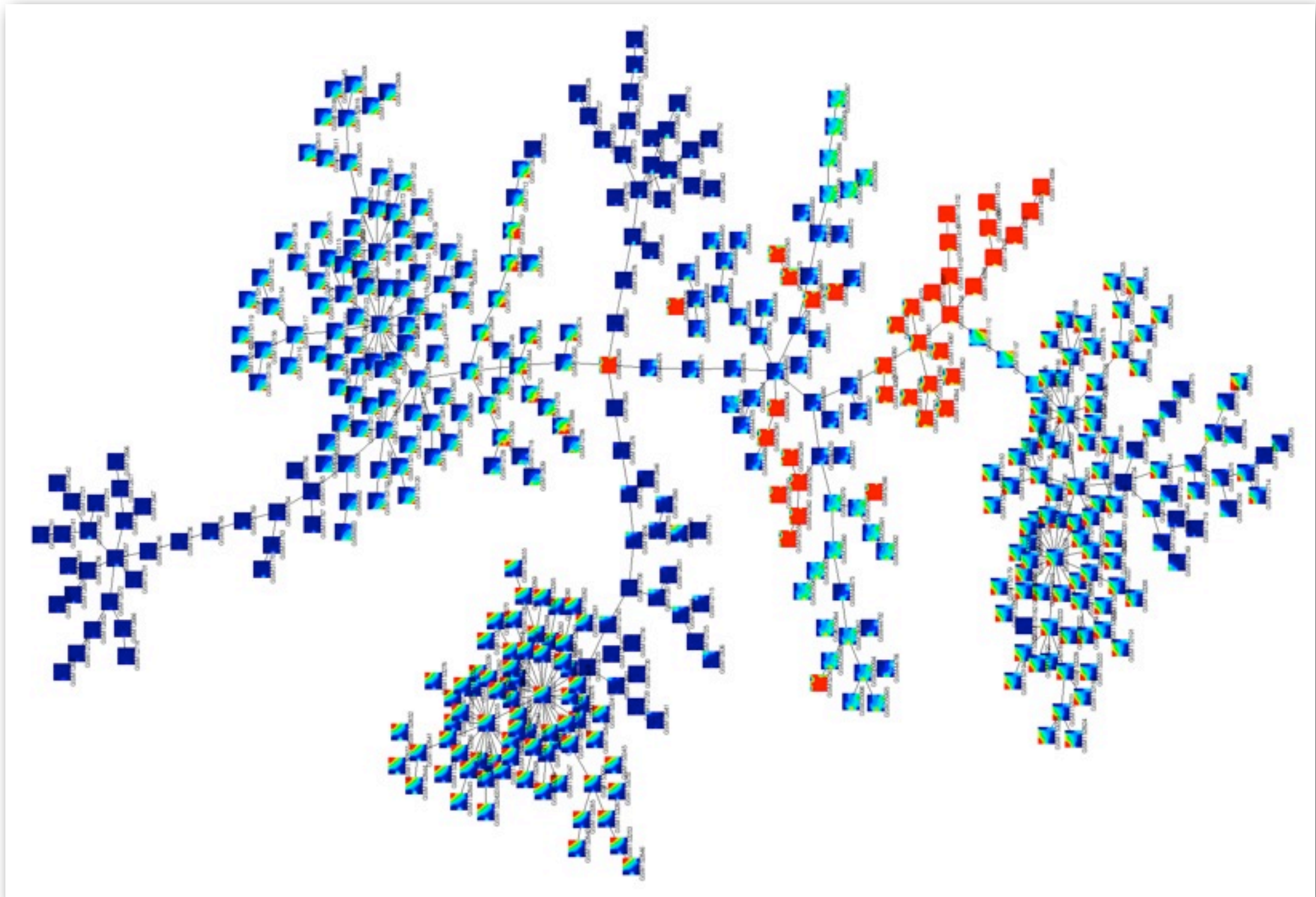
Minimum Spanning Trees

- edge-weighted graph API
- greedy algorithm
- Kruskal's algorithm
- Prim's algorithm
- advanced topics

Weighted edge API

Edge abstraction needed for weighted edges.

public class Edge implements Comparable\<Edge\>

	`Edge(int v, int w, double weight)`	*create a weighted edge v-w*
`int`	`either()`	*either endpoint*
`int`	`other(int v)`	*the endpoint that's not v*
`int`	`compareTo(Edge that)`	*compare this edge to that edge*
`double`	`weight()`	*the weight*
`String`	`toString()`	*string representation*

```
        weight
   v ─────────────── w
```

Idiom for processing an edge **e**: `int v = e.either(), w = e.other(v);`

Weighted edge: Java implementation

```java
public class Edge implements Comparable<Edge>
{
    private final int v, w;
    private final double weight;

    public Edge(int v, int w, double weight)
    {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

    public int either()
    {   return v;   }

    public int other(int vertex)
    {
        if (vertex == v) return w;
        else return v;
    }

    public int compareTo(Edge that)
    {
        if       (this.weight < that.weight) return -1;
        else if  (this.weight > that.weight) return +1;
        else                                 return  0;
    }
}
```

constructor

either endpoint

other endpoint

compare edges by weight

Edge-weighted graph API

```
public class EdgeWeightedGraph
```

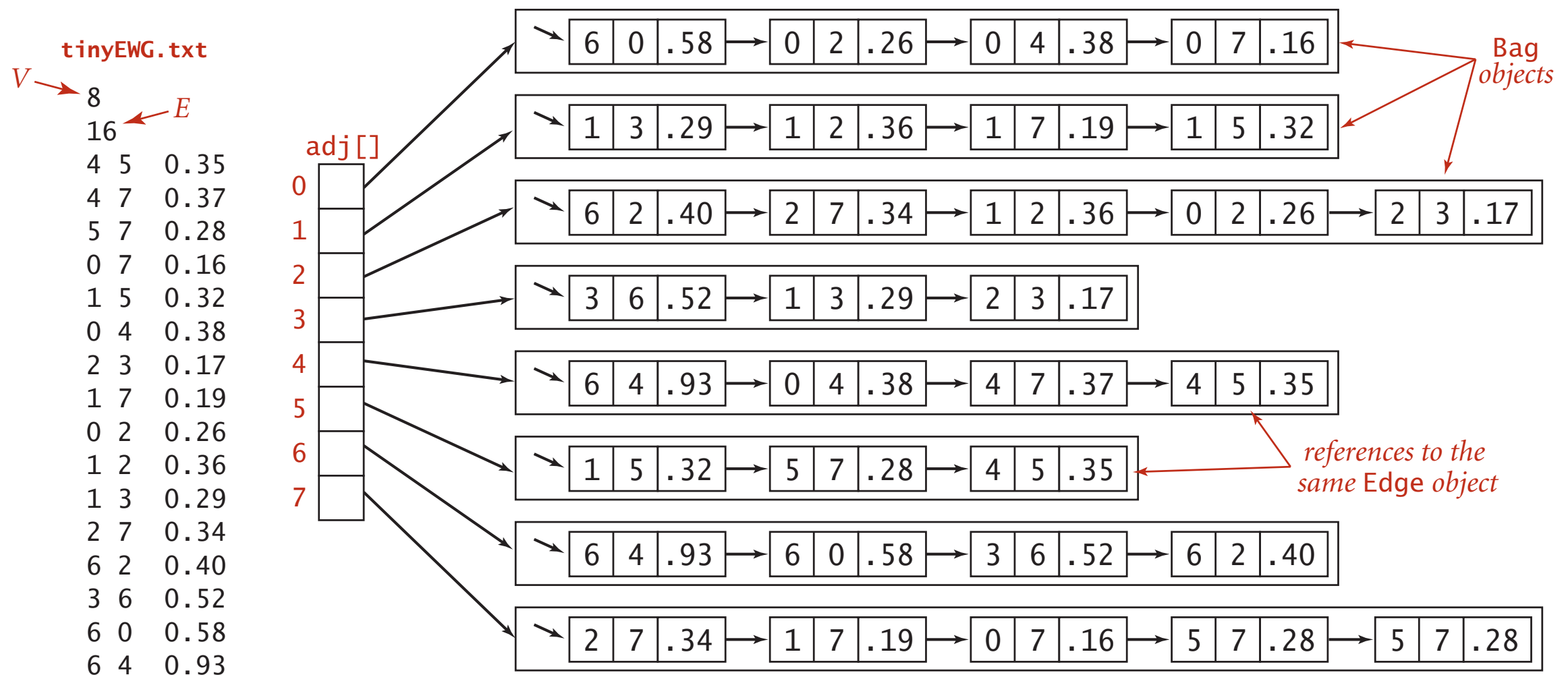|		
	`EdgeWeightedGraph(int V)`	*create an empty graph with V vertices*
	`EdgeWeightedGraph(In in)`	*create a graph from input stream*
`void`	`addEdge(Edge e)`	*add weighted edge e*
`Iterable<Edge>`	`adj(int v)`	*edges incident to v*
`Iterable<Edge>`	`edges()`	*all of this graph's edges*
`int`	`V()`	*return number of vertices*
`int`	`E()`	*return number of edges*
`String`	`toString()`	*string representation*

Conventions. Allow self-loops and parallel edges.

Edge-weighted graph: adjacency-list representation

Maintain vertex-indexed array of **Edge** lists (use **Bag** abstraction).

tinyEWG.txt

V → 8
16 ← E
4 5 0.35
4 7 0.37
5 7 0.28
0 7 0.16
1 5 0.32
0 4 0.38
2 3 0.17
1 7 0.19
0 2 0.26
1 2 0.36
1 3 0.29
2 7 0.34
6 2 0.40
3 6 0.52
6 0 0.58
6 4 0.93

adj[]

0
1
2
3
4
5
6
7

| 6 | 0 | .58 | → | 0 | 2 | .26 | → | 0 | 4 | .38 | → | 0 | 7 | .16 |

| 1 | 3 | .29 | → | 1 | 2 | .36 | → | 1 | 7 | .19 | → | 1 | 5 | .32 |

| 6 | 2 | .40 | → | 2 | 7 | .34 | → | 1 | 2 | .36 | → | 0 | 2 | .26 | → | 2 | 3 | .17 |

| 3 | 6 | .52 | → | 1 | 3 | .29 | → | 2 | 3 | .17 |

| 6 | 4 | .93 | → | 0 | 4 | .38 | → | 4 | 7 | .37 | → | 4 | 5 | .35 |

| 1 | 5 | .32 | → | 5 | 7 | .28 | → | 4 | 5 | .35 |

| 6 | 4 | .93 | → | 6 | 0 | .58 | → | 3 | 6 | .52 | → | 6 | 2 | .40 |

| 2 | 7 | .34 | → | 1 | 7 | .19 | → | 0 | 7 | .16 | → | 5 | 7 | .28 | → | 5 | 7 | .28 |

Bag objects

references to the same Edge object

Edge-weighted graph: adjacency-lists implementation

```
public class EdgeWeightedGraph
{
   private final int V;
   private final Bag<Edge>[] adj;

   public EdgeWeightedGraph(int V)
   {
      this.V = V;
      adj = (Bag<Edge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<Edge>();
   }

   public void addEdge(Edge e)
   {
      int v = e.either(), w = e.other(v);
      adj[v].add(e);
      adj[w].add(e);
   }

   public Iterable<Edge> adj(int v)
   {  return adj[v];   }
}
```

same as `Graph`, but adjacency lists of `Edge`s instead of integers

constructor

add edge to both adjacency lists

Minimum spanning tree API

Q. How to represent the MST?

public class MST

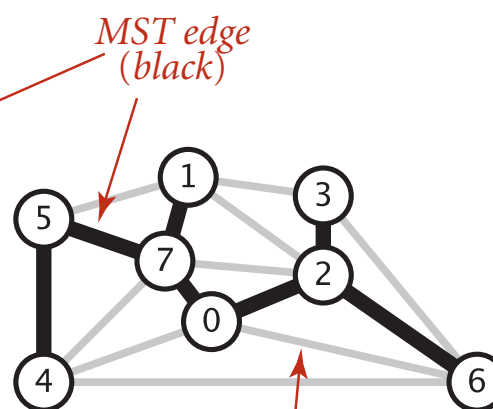|	
MST(EdgeWeightedGraph G)	*constructor*
Iterable<Edge> edges()	*edges in MST*
double weight()	*weight of MST*


```
tinyEWG.txt
V → 8
  16 ← E
   4 5  0.35
   4 7  0.37
   5 7  0.28
   0 7  0.16
   1 5  0.32
   0 4  0.38
   2 3  0.17
   1 7  0.19
   0 2  0.26
   1 2  0.36
   1 3  0.29
   2 7  0.34
   6 2  0.40
   3 6  0.52
   6 0  0.58
   6 4  0.93
```

MST edge (black)

non-MST edge (gray)

```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```

Minimum spanning tree API

Q. How to represent the MST?

public class MST

MST(EdgeWeightedGraph G)	*constructor*
Iterable<Edge> edges()	*edges in MST*
double weight()	*weight of MST*

```java
public static void main(String[] args)
{
    In in = new In(args[0]);
    EdgeWeightedGraph G = new EdgeWeightedGraph(in);
    MST mst = new MST(G);
    for (Edge e : mst.edges())
        StdOut.println(e);
    StdOut.println(mst.weight());
}
```

```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```

Minimum Spanning Trees

- edge-weighted graph API
- **greedy algorithm**
- Kruskal's algorithm
- Prim's algorithm
- advanced topics

Cut property

Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets.
A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.

*crossing edges separating
gray from white vertices
are drawn in red*

e

*minimum-weight crossing edge
must be in the MST*
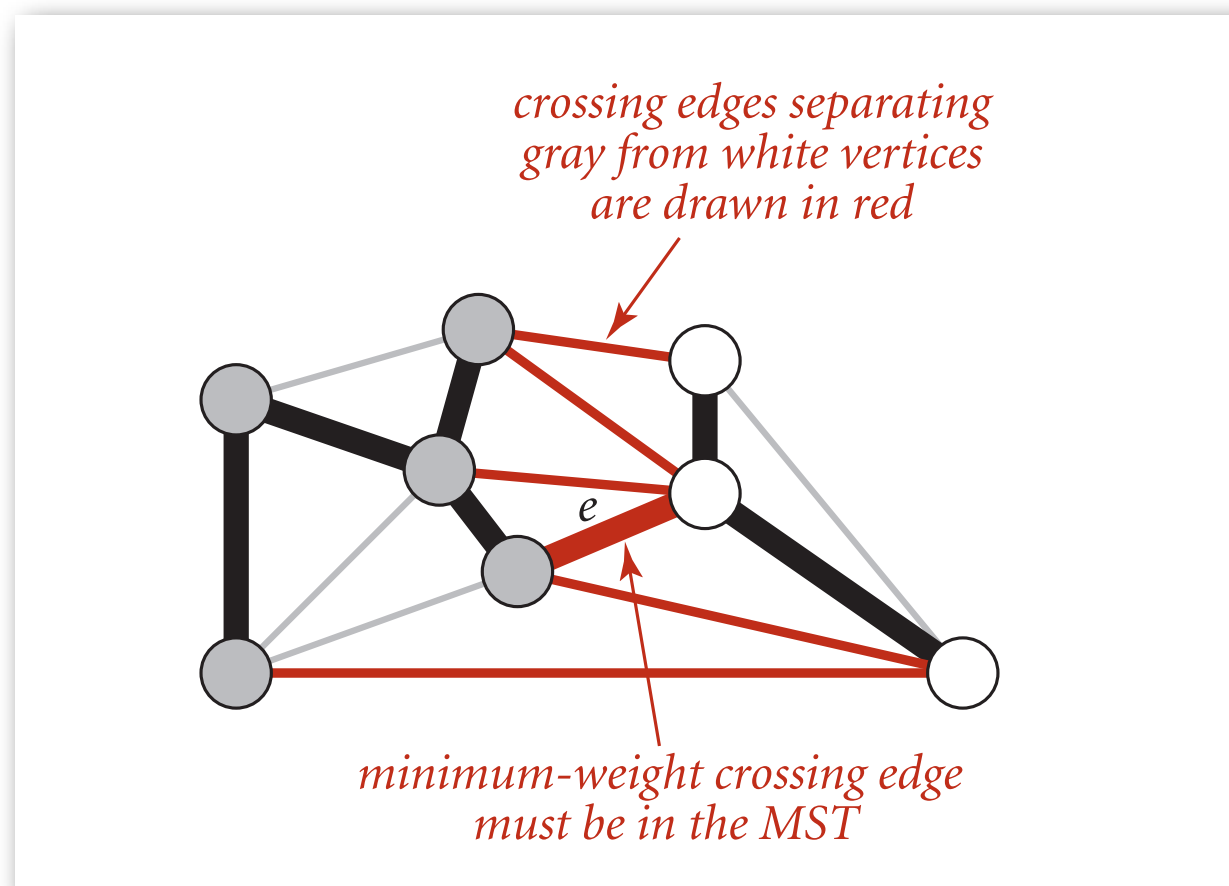
Cut property: correctness proof

Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets.
A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.

Pf. Let e be the min-weight crossing edge in cut.
- Suppose e is not in the MST.
- Adding e to the MST creates a cycle.
- Some other edge f in cycle must be a crossing edge.
- Removing f and adding e is also a spanning tree.
- Since weight of e is less than the weight of f,
  that spanning tree is lower weight.
- Contradiction. ▪
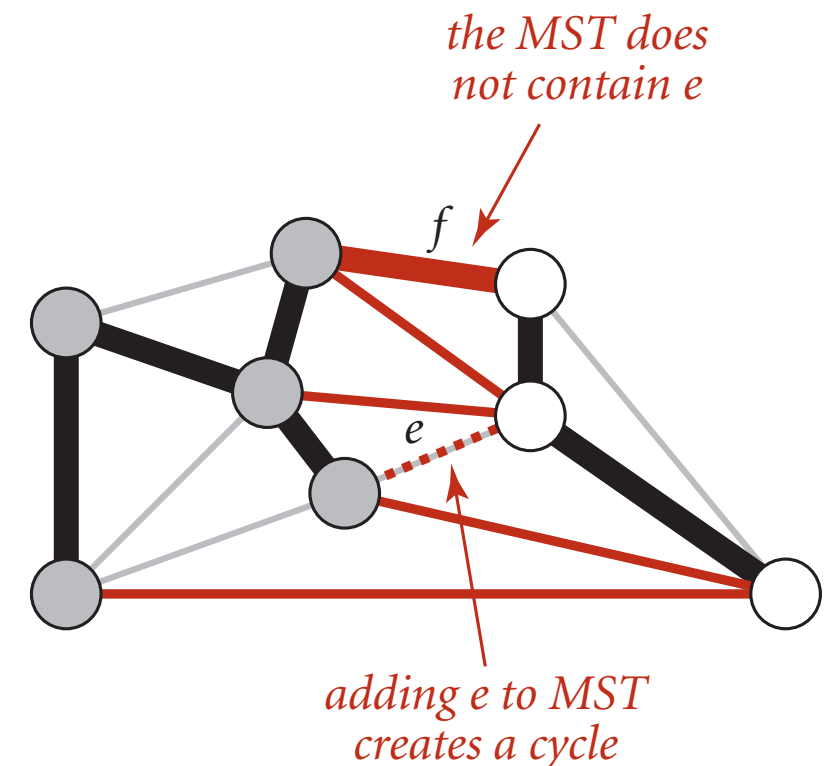
the MST does not contain e

f

e

adding e to MST creates a cycle

Greedy MST algorithm

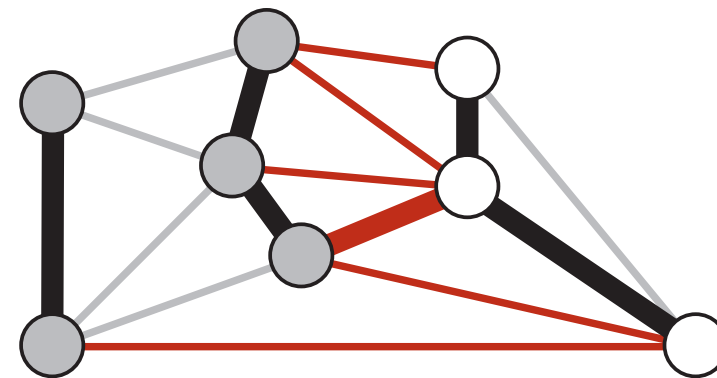**Proposition.** The following algorithm computes the MST:

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Continue until $V - 1$ edges are colored black.
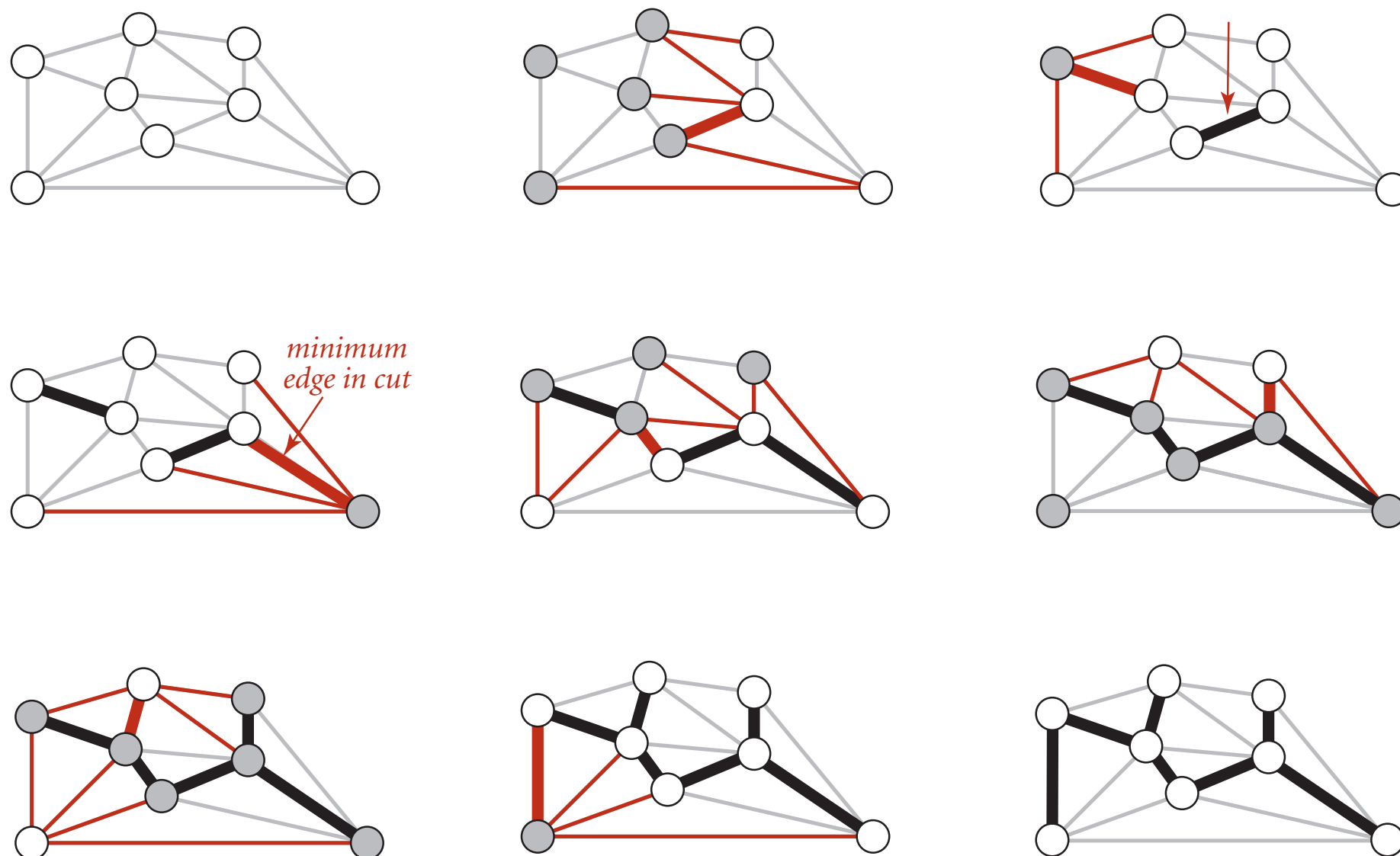
Greedy MST algorithm

Proposition. The following algorithm computes the MST:

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Continue until $V - 1$ edges are colored black.

Pf.

- Any edge colored black is in the MST (via cut property).
- If fewer than $V - 1$ black edges, there exists a cut with no black crossing edges.
  (consider cut whose vertices are one connected component)
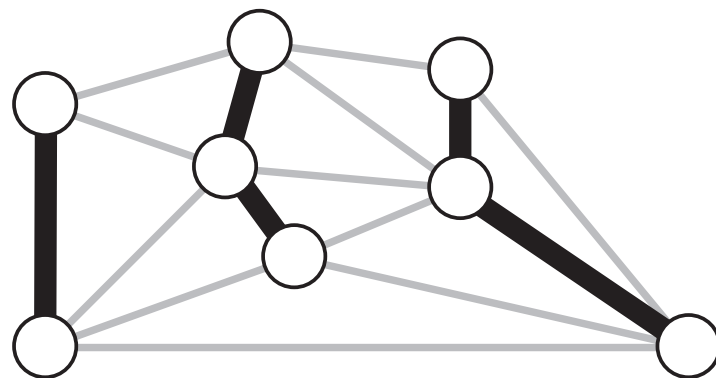
fewer than V-1 edges colored black   *a cut with no black crossing edges*

Greedy MST algorithm

Proposition. The following algorithm computes the MST:

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Continue until $V - 1$ edges are colored black.

Efficient implementations. How to choose cut? How to find min-weight edge?

Ex 1. Kruskal's algorithm. [stay tuned]

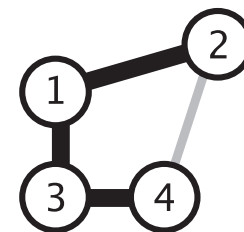Ex 2. Prim's algorithm. [stay tuned]

Ex 3. Borüvka's algorithm.

Removing two simplifying assumptions

Q. What if edge weights are not all distinct?

A. Greedy MST algorithm still correct if equal weights are present!

(our correctness proof fails, but that can be fixed)
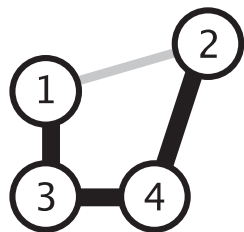

```
1 2   1.00
1 3   0.50
2 4   1.00
3 4   0.50
```

```
1 2   1.00
1 3   0.50
2 4   1.00
3 4   0.50
```

Q. What if graph is not connected?

A. Compute minimum spanning forest = MST of each component.
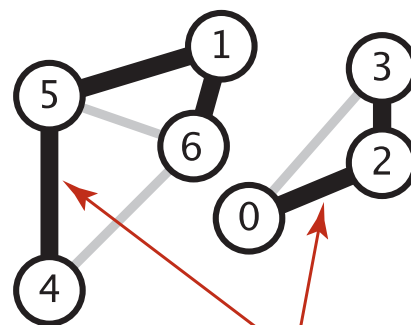

```
4 5   0.61
4 6   0.62
5 6   0.88
1 5   0.11
2 3   0.35
0 3   0.6
1 6   0.10
0 2   0.22
```
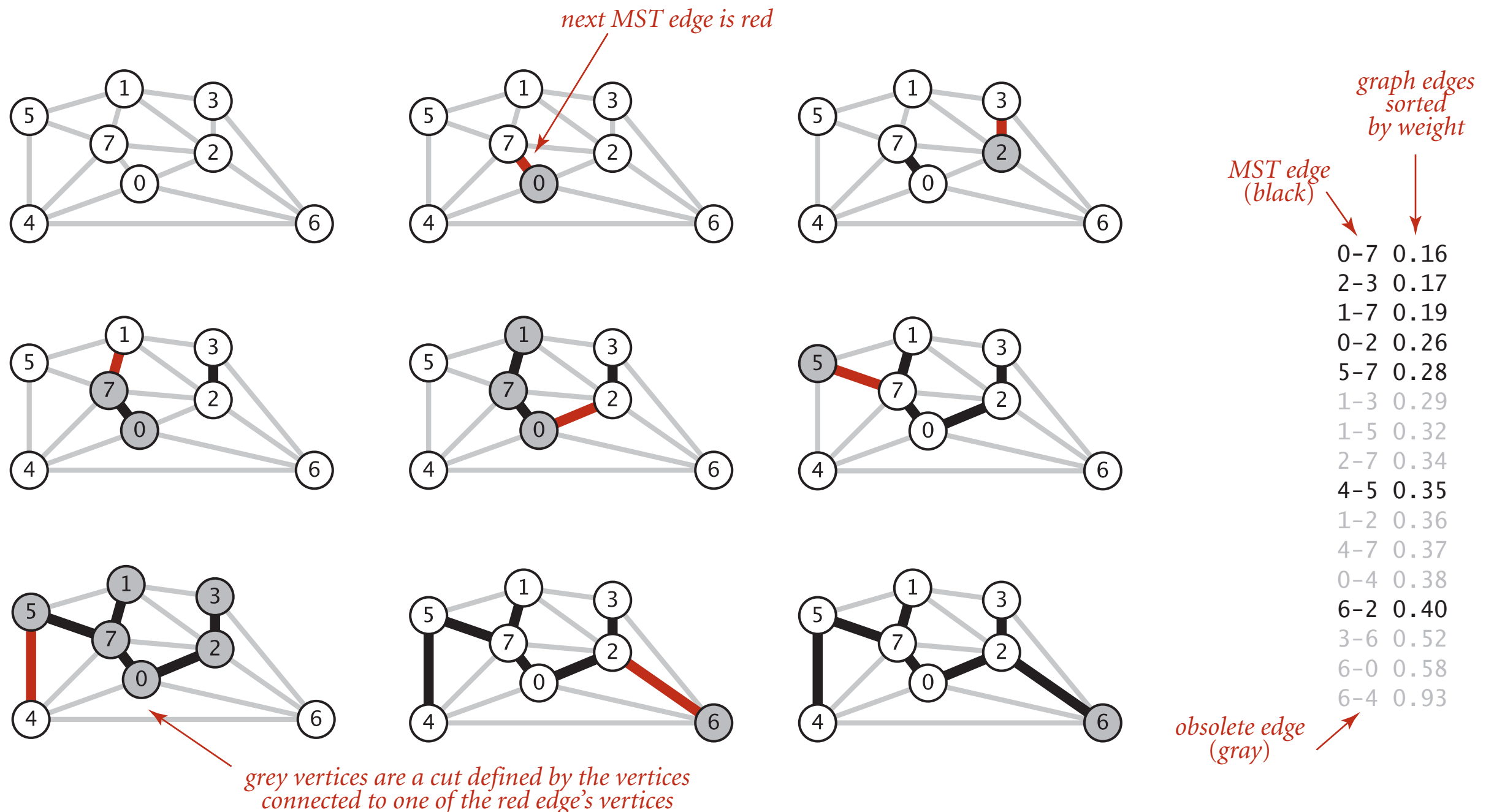
can independently compute
MSTs of components

Minimum Spanning Trees

- edge-weighted graph API
- greedy algorithm
- **Kruskal's algorithm**
- Prim's algorithm
- advanced topics

Kruskal's algorithm

Kruskal's algorithm. [Kruskal 1956] Consider edges in ascending order of weight. Add the next edge to the tree T unless doing so would create a cycle.
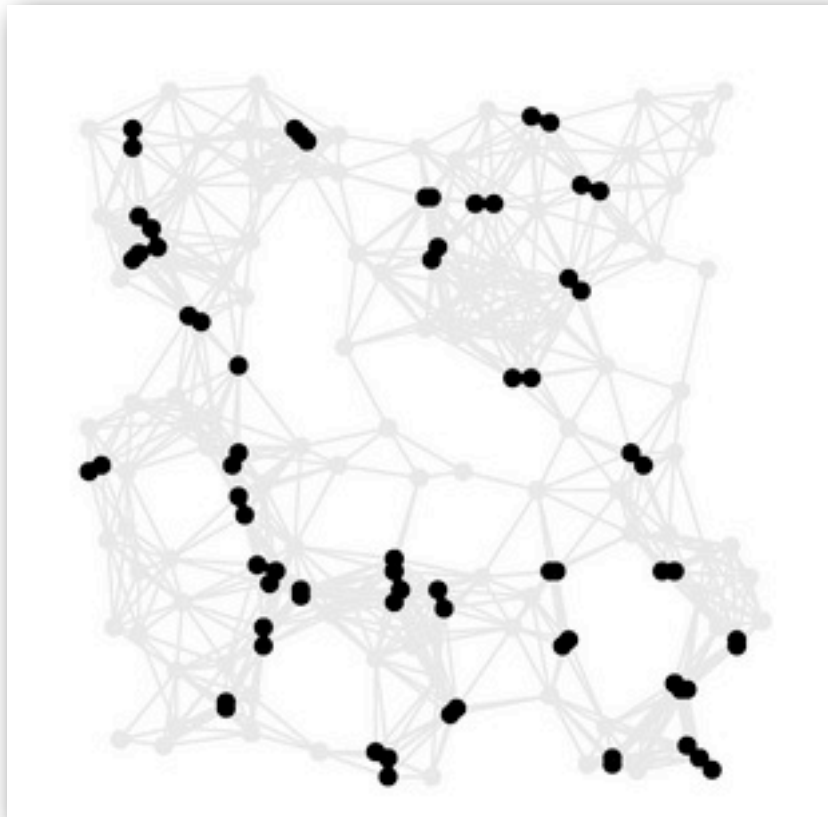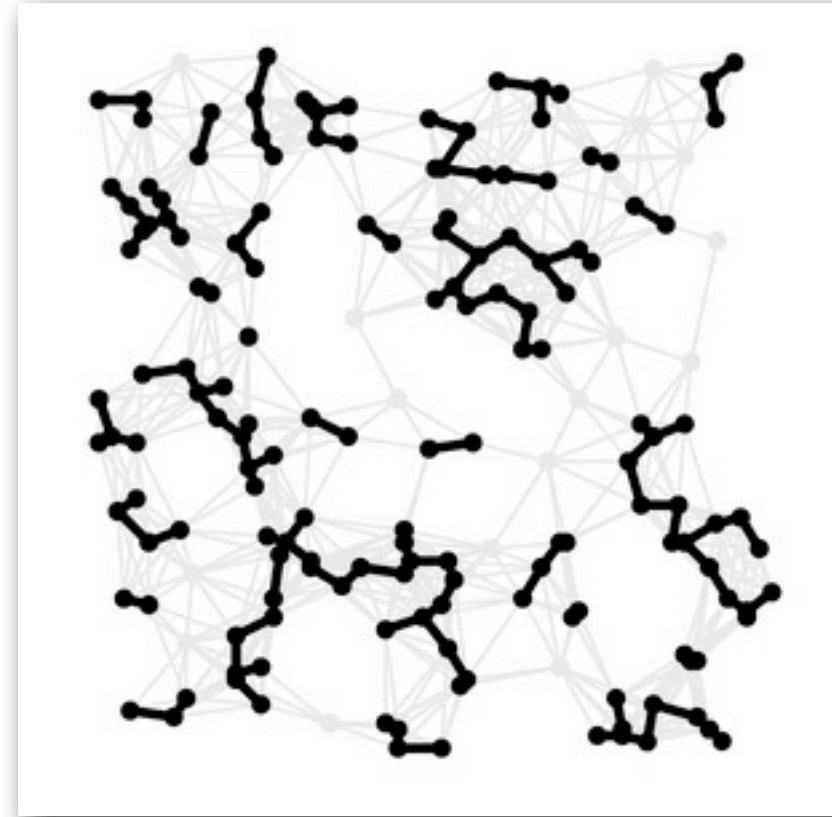

next MST edge is red

graph edges sorted by weight

MST edge (black)

0-7 0.16
2-3 0.17
1-7 0.19
0-2 0.26
5-7 0.28
1-3 0.29
1-5 0.32
2-7 0.34
4-5 0.35
1-2 0.36
4-7 0.37
0-4 0.38
6-2 0.40
3-6 0.52
6-0 0.58
6-4 0.93

obsolete edge (gray)

grey vertices are a cut defined by the vertices connected to one of the red edge's vertices

Kruskal's algorithm visualization

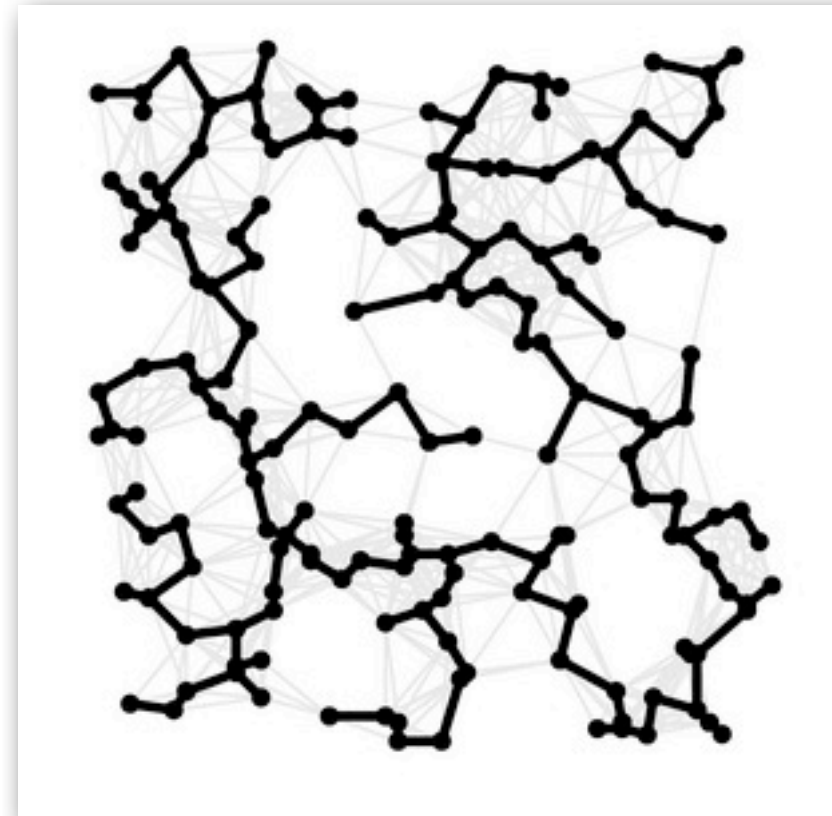# Kruskal's algorithm visualization

25%

75%

50%
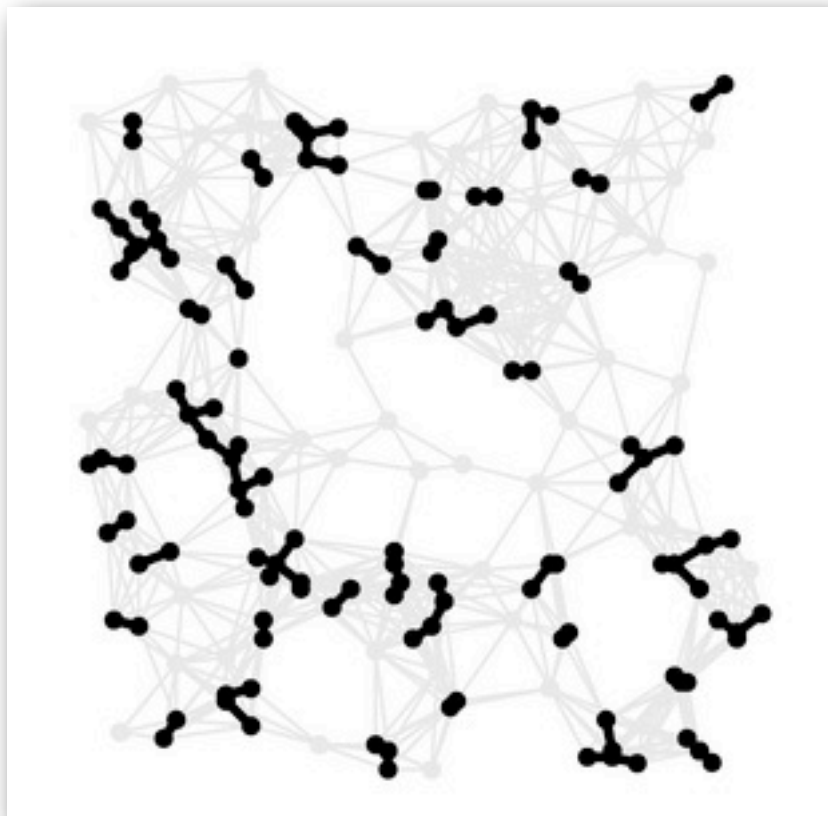
100%

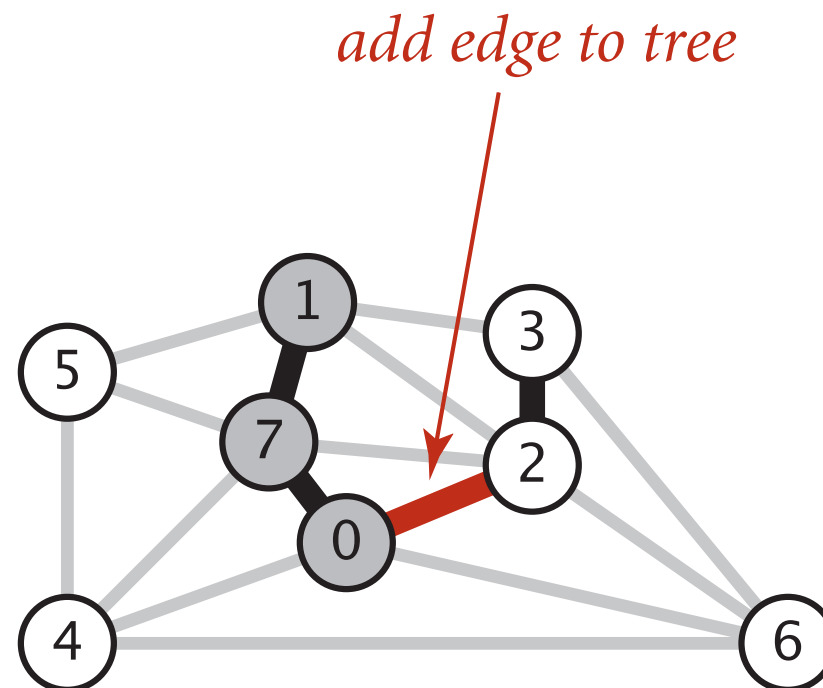# Kruskal's algorithm: proof of correctness

Proposition. Kruskal's algorithm computes the MST.

Pf. Kruskal's algorithm is a special case of the greedy MST algorithm.

- Suppose Kruskal's algorithm colors edge $e = v$–w black.
- Cut = set of vertices connected to v (or to w) in tree T.
- No crossing edge is black.
- No crossing edge has lower weight. Why?
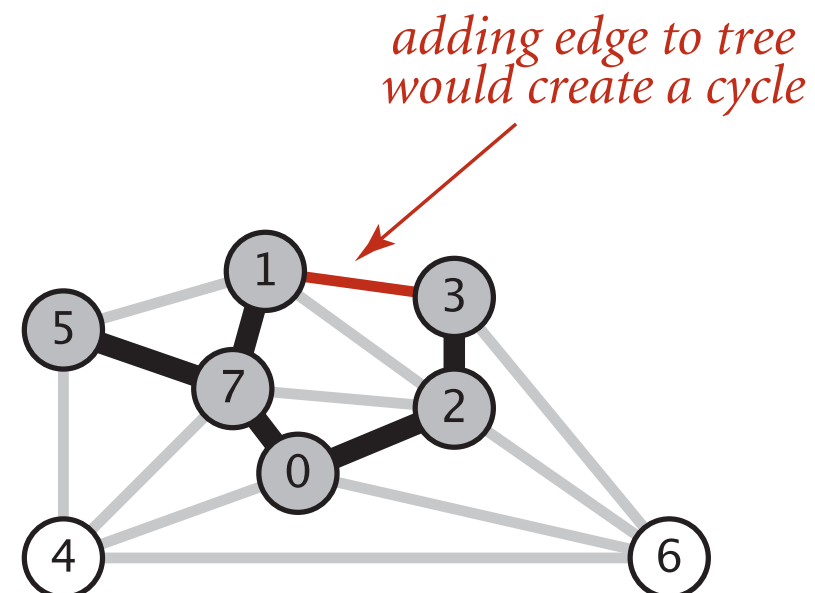
add edge to tree
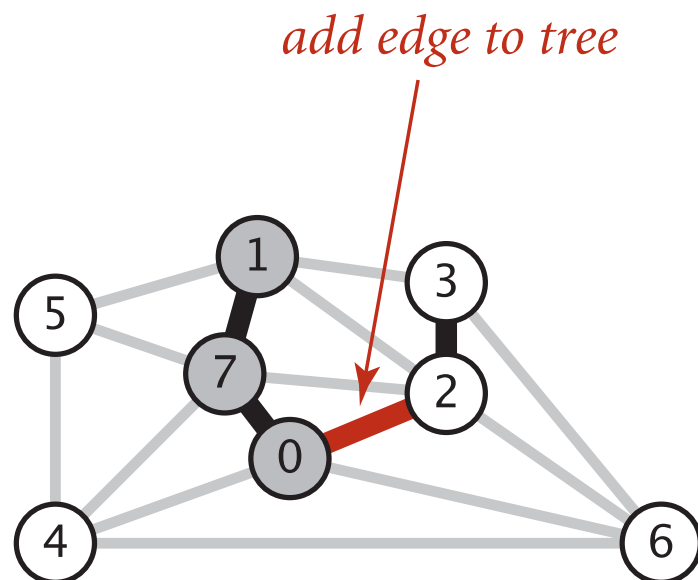
Kruskal's algorithm: implementation challenge

Challenge. Would adding edge v–w to tree T create a cycle? If not, add it.

How difficult?

- O $(E + V)$ time.
- O (V) time.   ⟵   run DFS from v, check if w is reachable
                                   (T has at most V – 1 edges)
- O $(\log V)$ time.
- O $(\log^* V)$ time.   ⟵   use the union-find data structure !
- Constant time.

*add edge to tree*

adding edge to tree would create a cycle

Kruskal's algorithm: implementation challenge
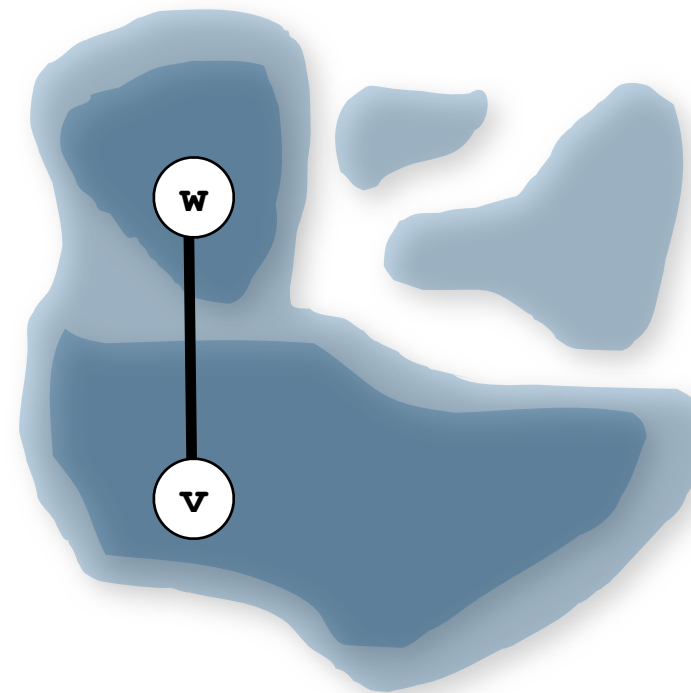
Challenge. Would adding edge $v-w$ to tree T create a cycle? If not, add it.

Efficient solution. Use the union-find data structure.

- Maintain a set for each connected component in T.
- If v and w are in same set, then adding $v-w$ would create a cycle.
- To add $v-w$ to T, merge sets containing v and w.
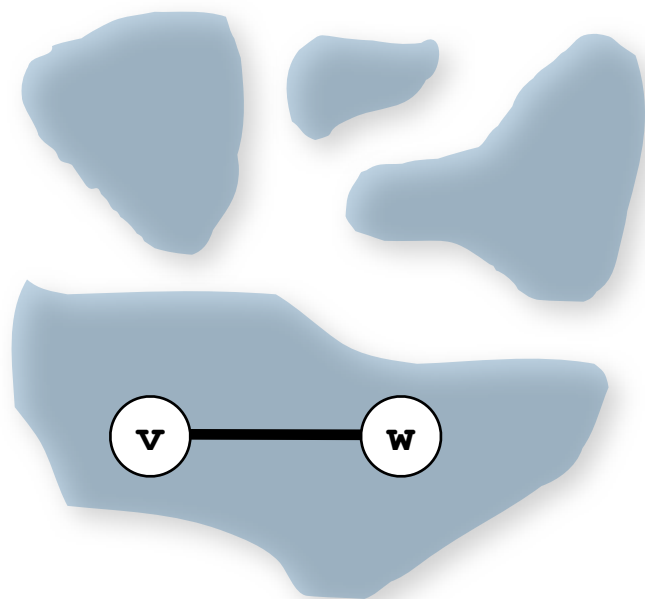
Case 1: adding v–w creates a cycle   **Case 2: add v–w to T and merge sets containing v and w**

Kruskal's algorithm: Java implementation

```java
public class KruskalMST
{
   private Queue<Edge> mst;
   private MinPQ<Edge> pq;

   public KruskalMST(EdgeWeightedGraph G)
   {
      mst = new Queue<Edge>();
      pq = new MinPQ<Edge>(G.edges());
      UnionFind uf = new UnionFind(G.V());
      while (!pq.isEmpty() && mst.size() < G.V()-1)
      {
         Edge e = pq.delMin();
         int v = e.either(), w = e.other(v);
         if (!uf.find(v, w))
         {
            uf.union(v, w);
            mst.enqueue(e);
         }
      }
   }

   public Iterable<Edge> edges()
   {  return mst;  }
}
```

build priority queue

greedily add edges to MST

edge v–w does not create cycle

merge sets

add edge to MST

Kruskal's algorithm running time

Proposition. Kruskal's algorithm computes MST in $O(E \log E)$ time.

Pf.

operation	frequency	time per op
build pq	1	E
del min	E	log E
union	V	log* V †
find	E	log* V †

† amortized bound using weighted quick union with path compression

recall: log* V ≤ 5 in this universe

Remark. If edges are already sorted, order of growth is $E \log^* V$.

Minimum Spanning Trees

- edge-weighted graph API
- greedy algorithm
- Kruskal's algorithm
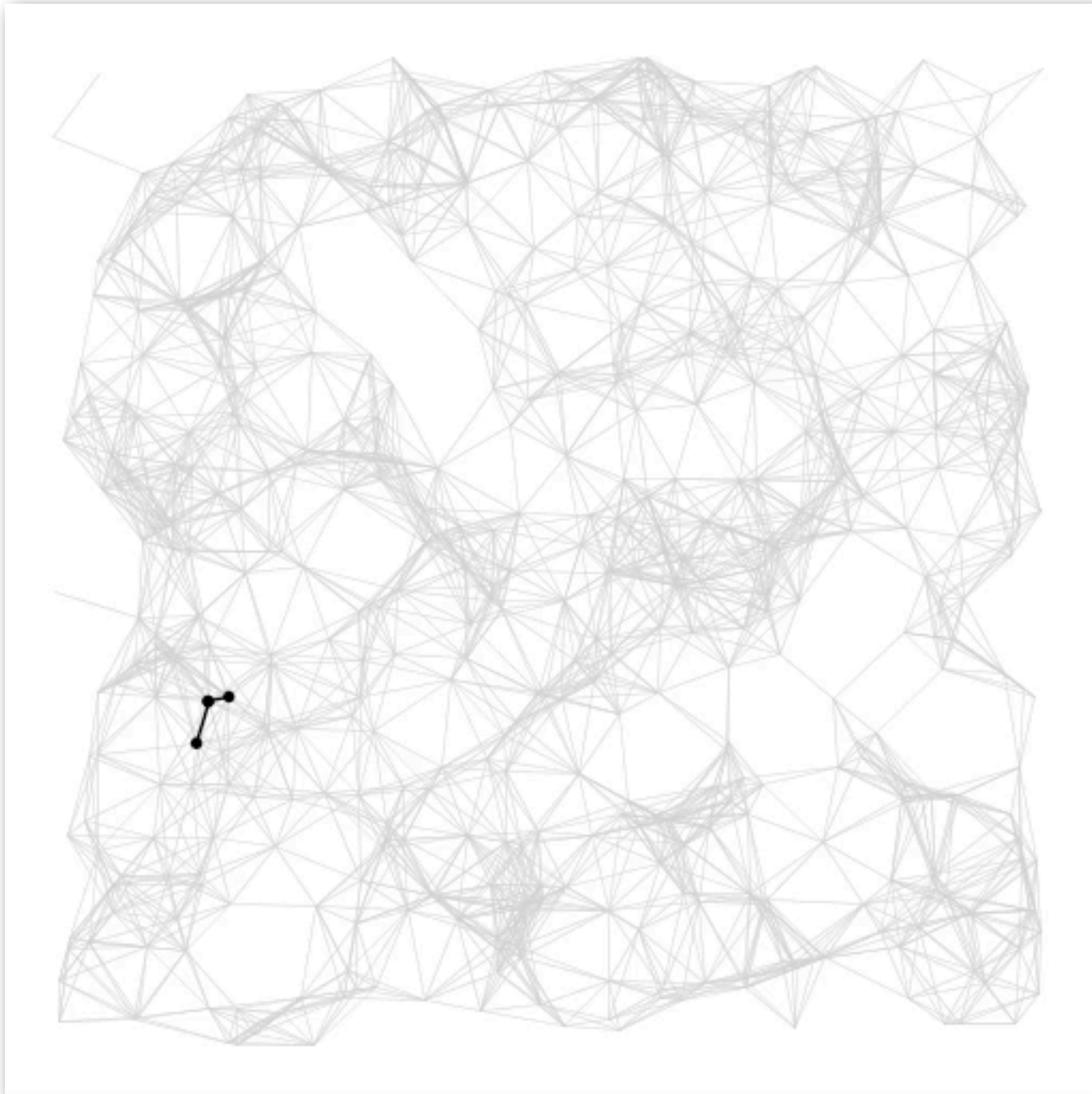- Prim's algorithm
- advanced topics

Prim's algorithm example

Prim's algorithm. [Jarník 1930, Dijkstra 1957, Prim 1959]

Start with vertex 0 and greedily grow tree T. At each step,
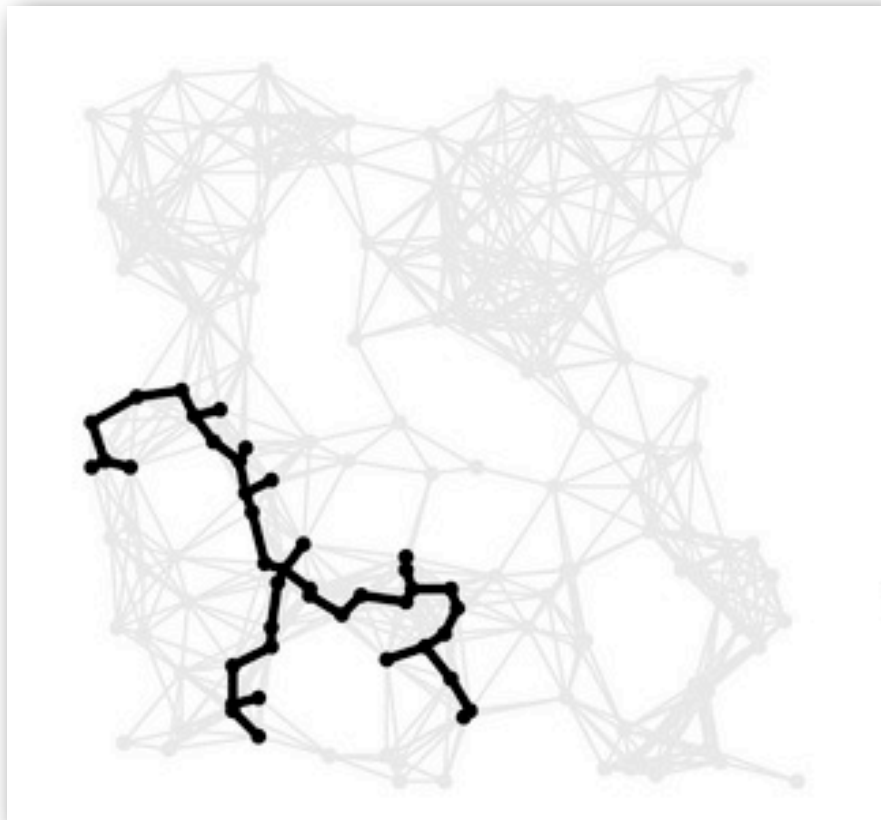add to T the min weight edge with exactly one endpoint in T.
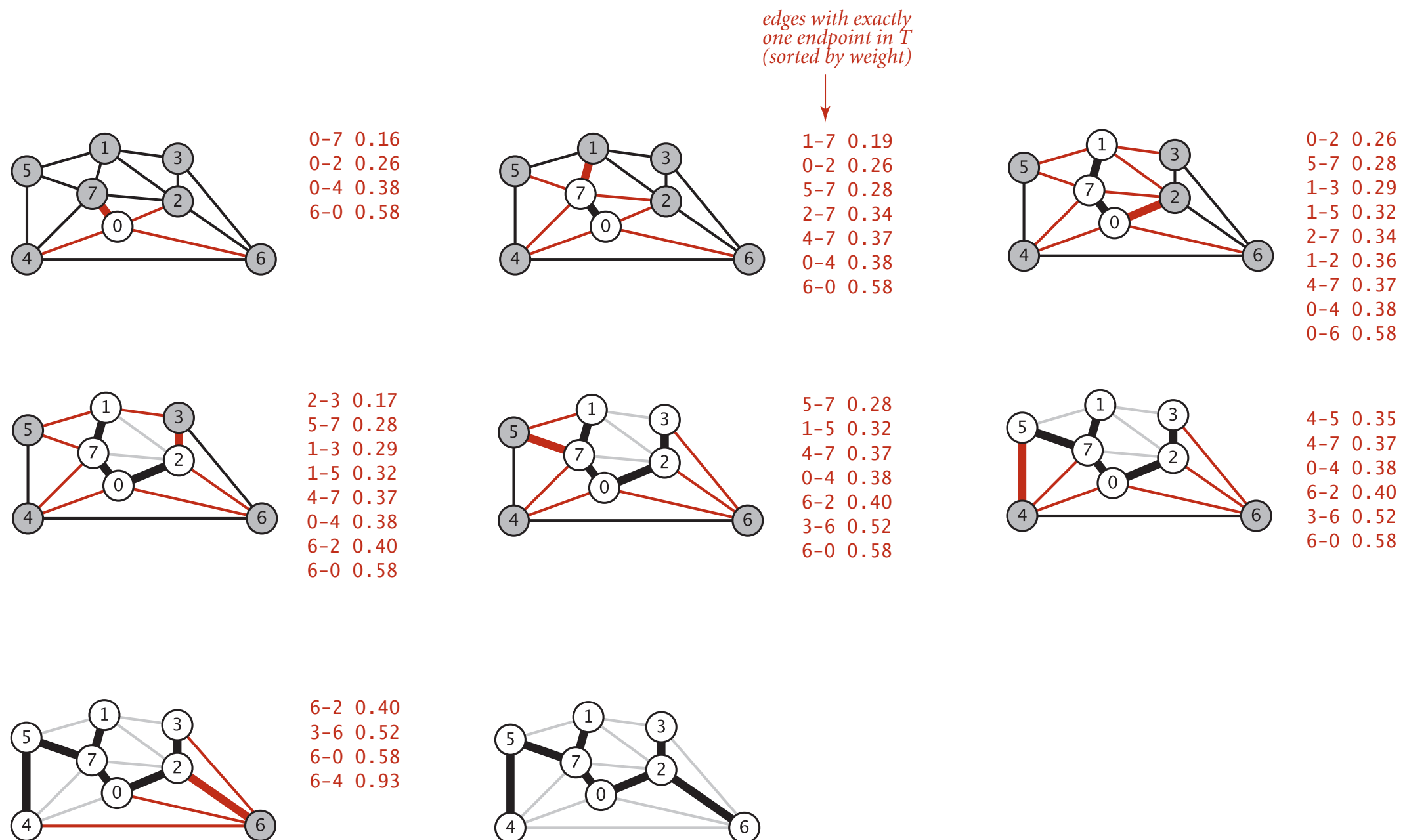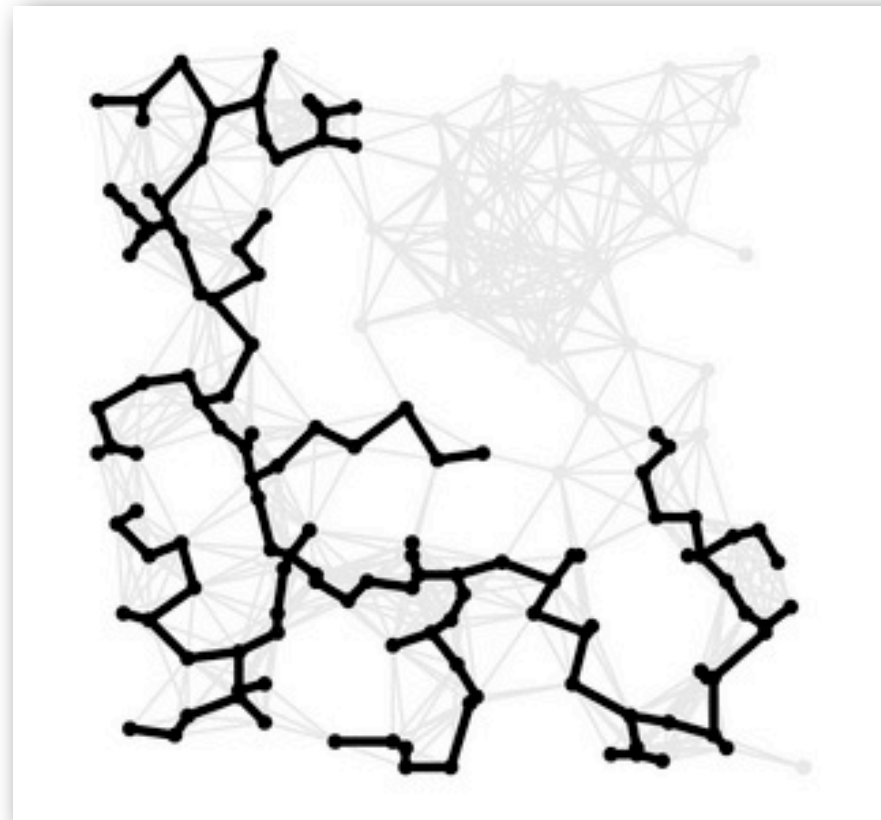
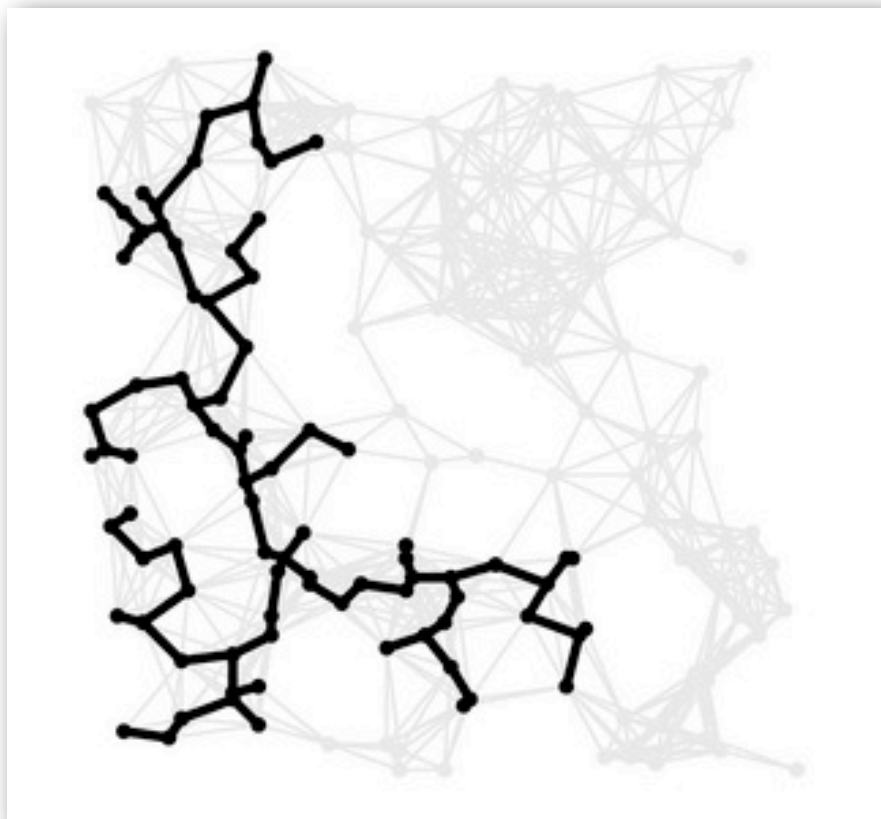# Prim's algorithm: visualization
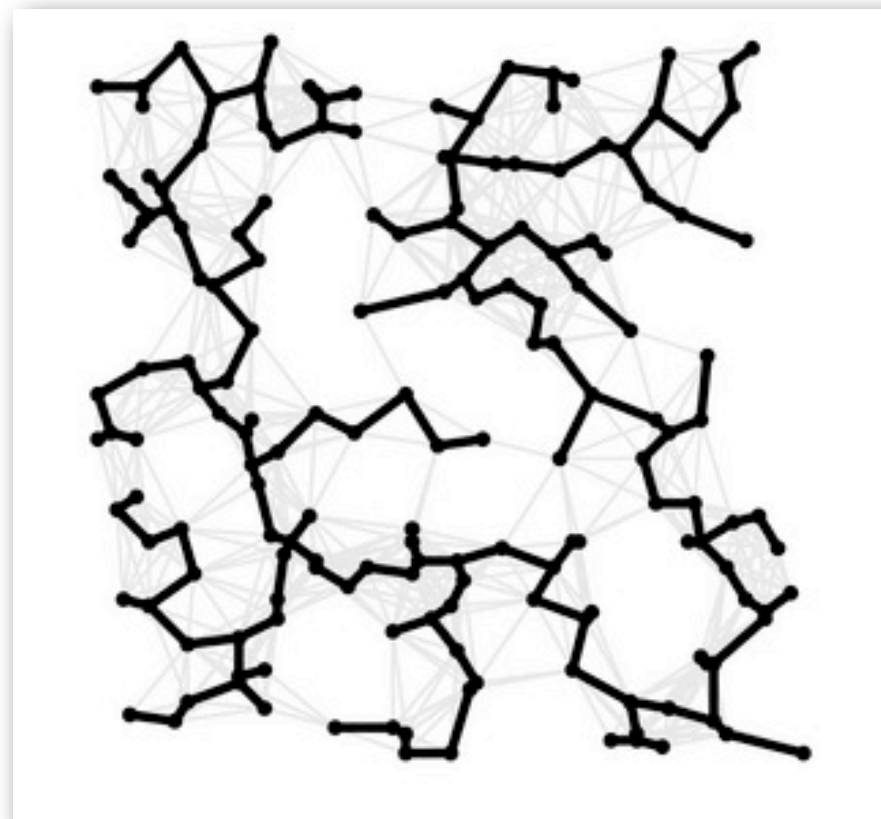
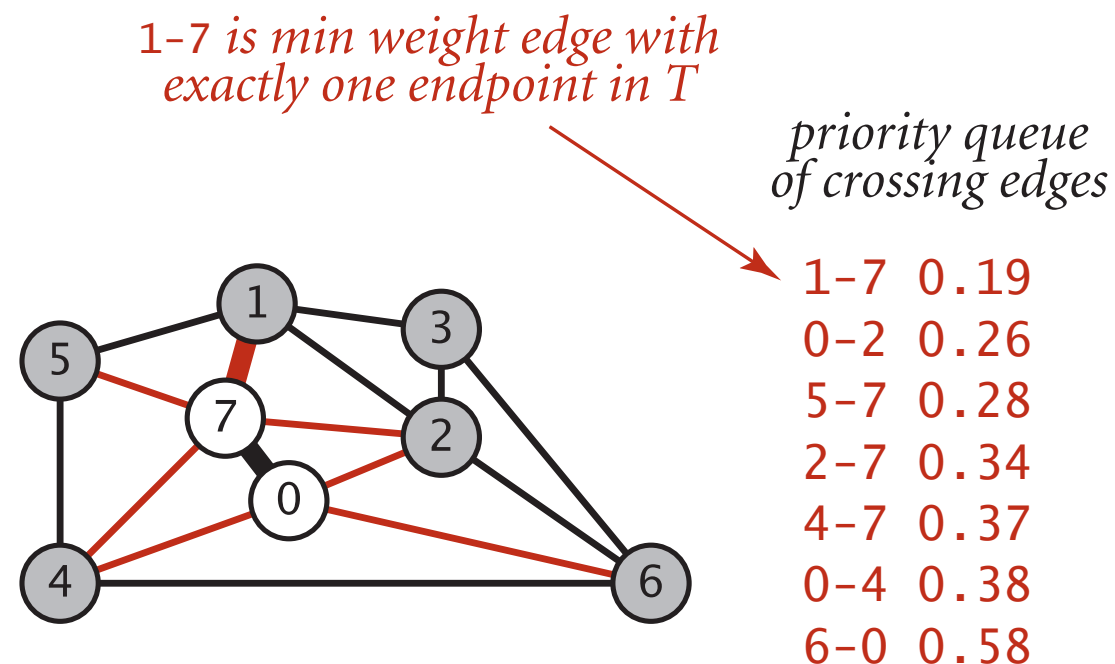# Prim's algorithm: visualization

25%

75%

50%

100%

Prim's algorithm: implementation challenge

Challenge. Find the min weight edge with exactly one endpoint in T.

How difficult?

- $O(E)$ time.   ⟵   try all edges
- $O(V)$ time.
- $O(\log E)$ time.   ⟵   use a priority queue !
- $O(\log^* E)$ time.
- Constant time.

1-7 is min weight edge with
exactly one endpoint in T

priority queue
of crossing edges

```
1-7  0.19
0-2  0.26
5-7  0.28
2-7  0.34
4-7  0.37
0-4  0.38
6-0  0.58
```

Prim's algorithm: proof of correctness

Proposition. Prim's algorithm computes the MST.

Pf. Prim's algorithm is a special case of the greedy MST algorithm.
- Suppose edge e = min weight edge connecting a vertex on the tree to a vertex not on the tree.
- Cut = set of vertices connected on tree.
- No crossing edge is black.
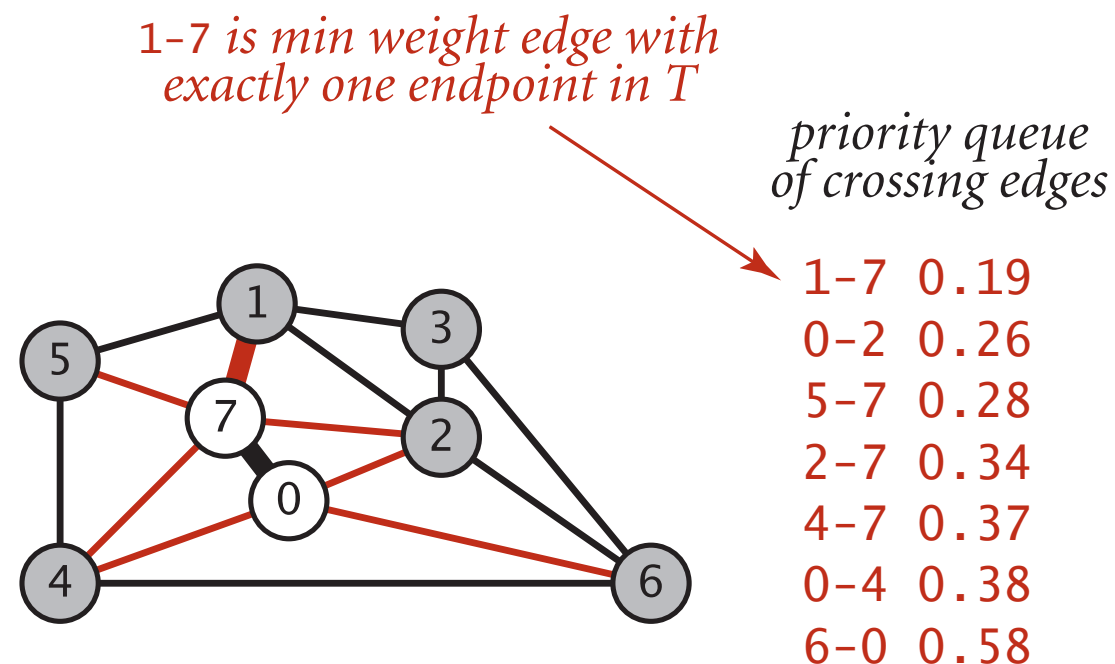- No crossing edge has lower weight.

edge e = 7-5 added to tree
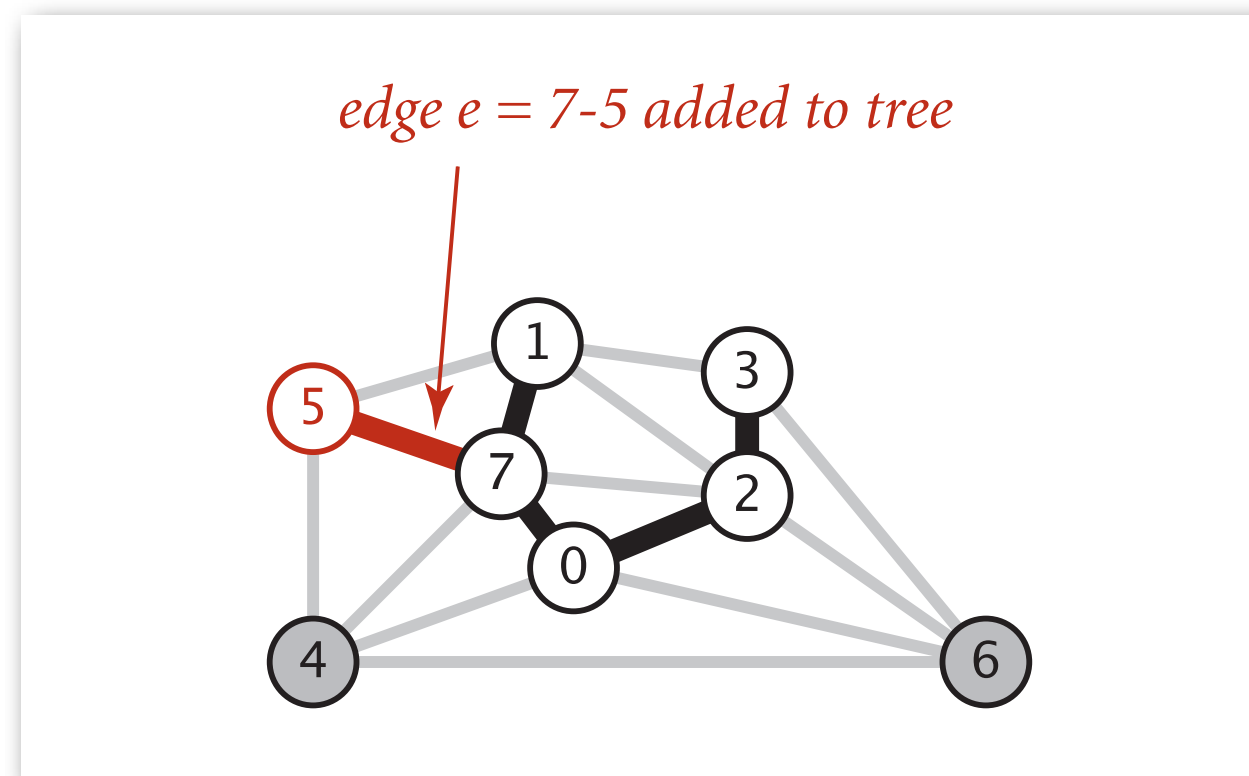
Prim's algorithm: lazy implementation

Challenge. Find the min weight edge with exactly one endpoint in T.

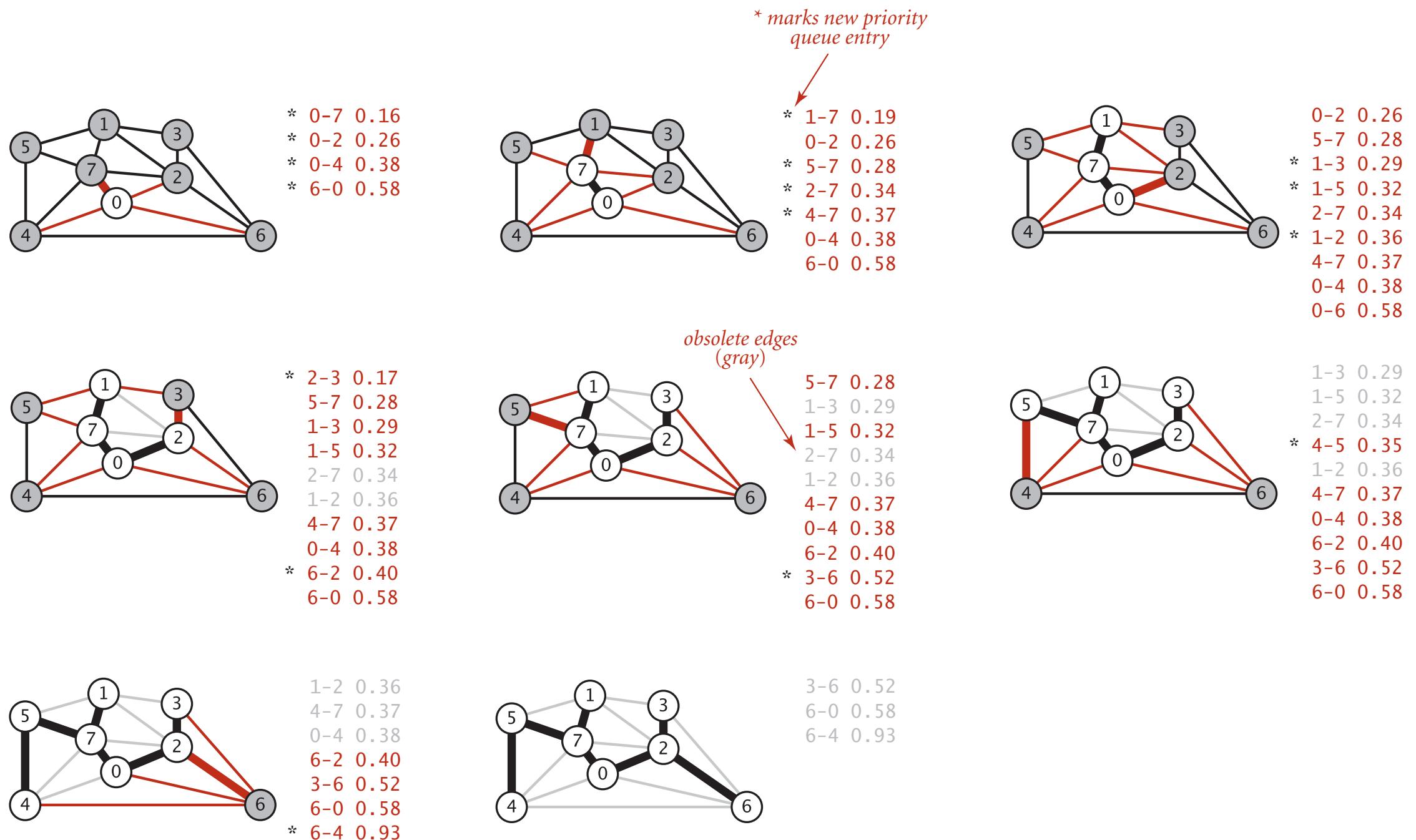Lazy solution. Maintain a PQ of edges with (at least) one endpoint in T.
- Delete min to determine next edge $e = v{-}w$ to add to T.
- Disregard if both endpoints v and w are in T.
- Otherwise, let v be vertex not in T :
  - add to PQ any edge incident to v (assuming other endpoint not in T)
  - add v to T

1-7 is min weight edge with
exactly one endpoint in T

priority queue
of crossing edges

1-7 0.19
0-2 0.26
5-7 0.28
2-7 0.34
4-7 0.37
0-4 0.38
6-0 0.58

Use **MinPQ**: key = edge, prioritized by weight.

(lazy version leaves some obsolete edges on the PQ)

* marks new priority
queue entry

```
* 0-7 0.16
* 0-2 0.26
* 0-4 0.38
* 6-0 0.58
```

```
* 1-7 0.19
  0-2 0.26
* 5-7 0.28
* 2-7 0.34
* 4-7 0.37
  0-4 0.38
  6-0 0.58
```

```
  0-2 0.26
  5-7 0.28
* 1-3 0.29
* 1-5 0.32
  2-7 0.34
* 1-2 0.36
  4-7 0.37
  0-4 0.38
  0-6 0.58
```

```
* 2-3 0.17
  5-7 0.28
  1-3 0.29
  1-5 0.32
  2-7 0.34
  1-2 0.36
  4-7 0.37
  0-4 0.38
* 6-2 0.40
  6-0 0.58
```

obsolete edges
(gray)

```
  5-7 0.28
  1-3 0.29
  1-5 0.32
  2-7 0.34
  1-2 0.36
  4-7 0.37
  0-4 0.38
  6-2 0.40
* 3-6 0.52
  6-0 0.58
```

```
  1-3 0.29
  1-5 0.32
  2-7 0.34
* 4-5 0.35
  1-2 0.36
  4-7 0.37
  0-4 0.38
  6-2 0.40
  3-6 0.52
  6-0 0.58
```

```
  1-2 0.36
  4-7 0.37
  0-4 0.38
  6-2 0.40
  3-6 0.52
  6-0 0.58
* 6-4 0.93
```

```
  3-6 0.52
  6-0 0.58
  6-4 0.93
```

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Prim's algorithm: lazy implementation

```java
public class LazyPrimMST
{
    private boolean[] marked;    // MST vertices
    private Queue<Edge> mst;      // MST edges
    private MinPQ<Edge> pq;       // PQ of edges

     public LazyPrimMST(WeightedGraph G)
      {
          pq = new MinPQ<Edge>();
          mst = new Queue<Edge>();
          marked = new boolean[G.V()];
          visit(G, 0);                              ⟵ assume G is connected

          while (!pq.isEmpty())
          {
              Edge e = pq.delMin();                 ⟵ repeatedly delete the
                                                       min weight edge e = v–w from PQ
              int v = e.either(), w = e.other(v);
              if (marked[v] && marked[w]) continue; ⟵ ignore if both endpoints in T
              mst.enqueue(e);                       ⟵ add edge e to tree
              if (!marked[v]) visit(G, v);
              if (!marked[w]) visit(G, w);          ⟵ add v or w to tree
          }
      }
}
```

Prim's algorithm: lazy implementation

```
private void visit(WeightedGraph G, int v)
{
   marked[v] = true;
   for (Edge e : G.adj(v))
      if (!marked[e.other(v)])
         pq.insert(e);
}

public Iterable<Edge> mst()
{  return mst;  }
```

add v to T

for each edge e = v–w, add to PQ if w not already in T

Prim's algorithm: running time

Proposition. Lazy Prim's algorithm computes the MST in time proportional to $E \log E$ in the worst case.

Pf.

operation	frequency	binary heap
delete min	E	log E
insert	E	log E

Indexed priority queue

Associate an index between 0 and $N - 1$ with each key in a priority queue.

- Client can insert and delete-the-minimum.
- Client can change the key by specifying the index.

```
public class IndexMinPQ<Key extends Comparable<Key>>
```

	`IndexMinPQ(int N)`	*create indexed priority queue with indices 0, 1, ..., N-1*
`void`	`insert(int k, Key key)`	*associate key with index k*
`void`	`decreaseKey(int k, Key key)`	*decrease the key associated with index k*
`boolean`	`contains()`	*is k an index on the priority queue?*
`int`	`delMin()`	*remove a minimal key and return its associated index*
`boolean`	`isEmpty()`	*is the priority queue empty?*
`int`	`size()`	*number of entries in the priority queue*

Prim's algorithm: eager implementation

Challenge. Find min weight edge with exactly one endpoint in T.

Eager solution. Maintain a PQ of vertices connected by an edge to T, where priority of vertex v = weight of shortest edge connecting v to T.
- Delete min vertex v and add its associated edge $e = v\text{–}w$ to T.
- Update PQ by considering all edges $e = v\text{–}x$ incident to v
  - ignore if x is already in T
  - add x to PQ if not already on it
  - decrease priority of x if $v\text{–}x$ becomes shortest edge connecting x to T
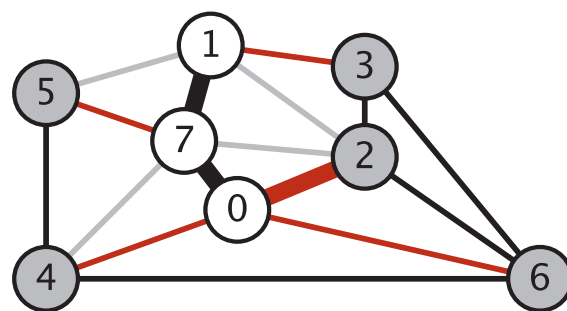

```
0
1 1-7 0.19
2 0-2 0.26      ←——— red: on PQ
3 1-3 0.29
4 0-4 0.38
5 5-7 0.28
6 6-0 0.58
7 0-7 0.16
```

black: on MST

Indexed priority queue implementation

Implementation.

- Start with same code as `MinPQ`.
- Maintain parallel arrays `keys[]`, `pq[]`, and `qp[]` so that:
  - `keys[i]` is the priority of `i`
  - `pq[i]` is the index of the key in heap position `i`
  - `qp[i]` is the heap position of the key with index `i`
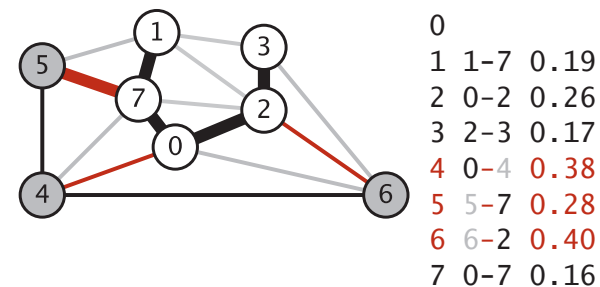- Use `swim(qp[k])` implement `decreaseKey(k, key)`.

i	0	1	2	3	4	5	6	7	8
keys[i]	A	S	O	R	T	I	N	G	–
pq[i]	–	0	6	7	2	1	5	4	3
qp[i]	1	5	4	8	7	6	2	3	–
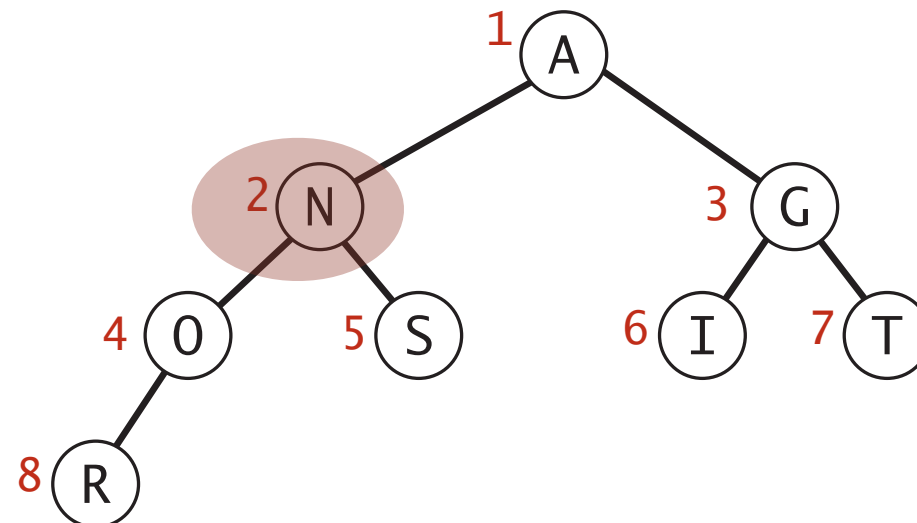
Prim's algorithm example: eager implementation

Use `IndexMinPQ`: key = edge weight, index = vertex.

(eager version has at most one PQ entry per vertex)
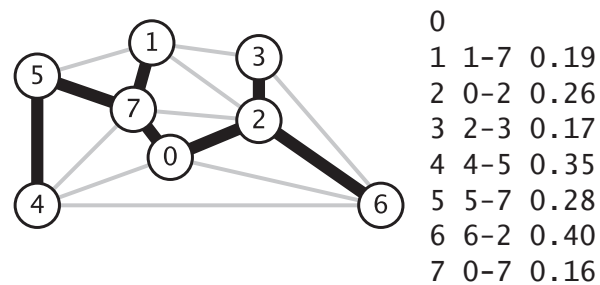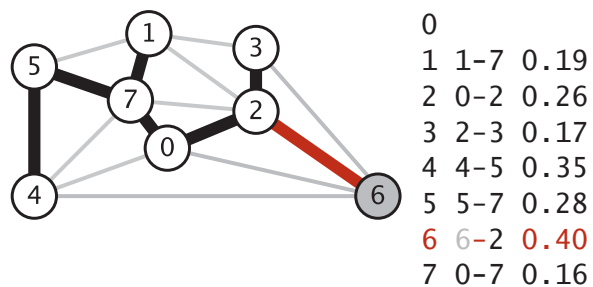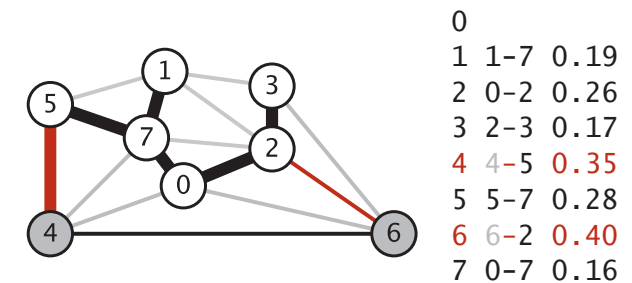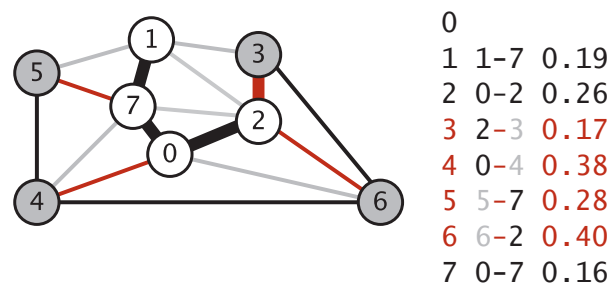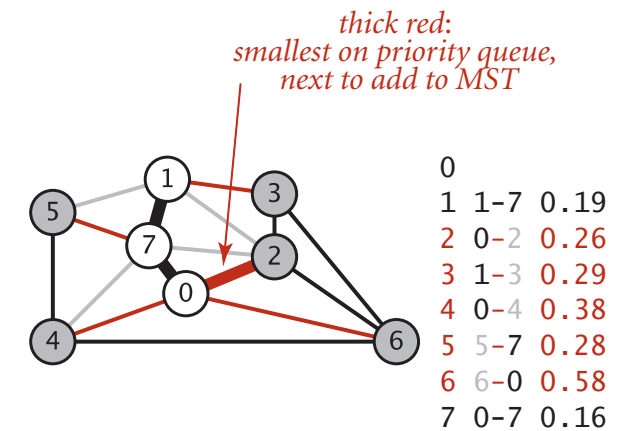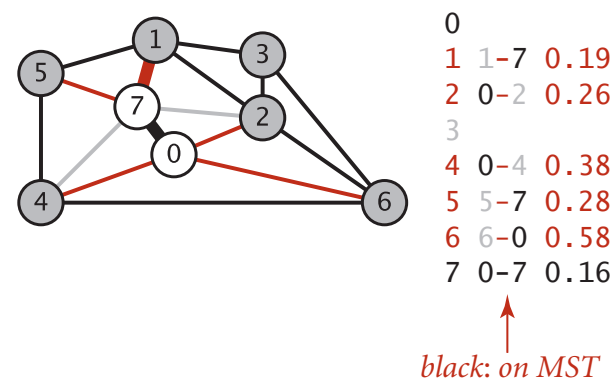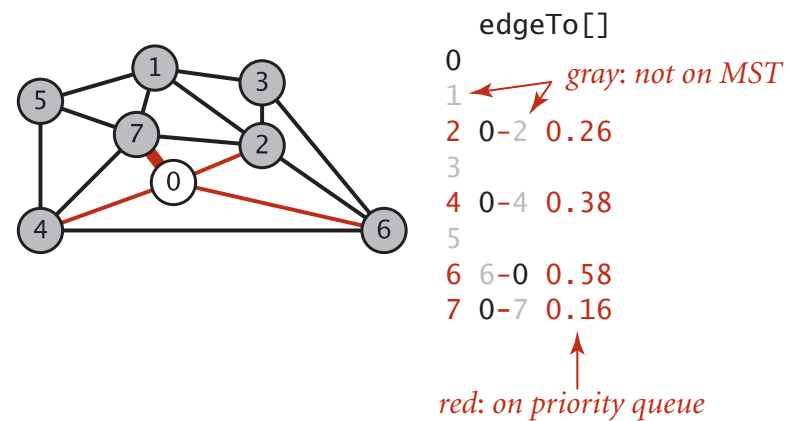
Prim's algorithm: running time

Depends on PQ implementation: V insert, V delete-min, E decrease-key.

PQ implementation	insert	delete-min	decrease-key	total
array	1	V	1	V^2
binary heap	log V	log V	log V	E log V
d-way heap (Johnson 1975)	$d \log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
Fibonacci heap (Fredman-Tarjan 1984)	1 [†]	log V [†]	1 [†]	E + V log V

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

† amortized

Minimum Spanning Trees

- edge-weighted graph API
- greedy algorithm
- Kruskal's algorithm
- Prim's algorithm
- **advanced topics**

# Does a linear-time MST algorithm exist?

deterministic compare-based MST algorithms

year	worst case	discovered by
1975	E log log V	Yao
1976	E log log V	Cheriton-Tarjan
1984	E log* V, E + V log V	Fredman-Tarjan
1986	E log (log* V)	Gabow-Galil-Spencer-Tarjan
1997	E α(V) log α(V)	Chazelle
2000	E α(V)	Chazelle
2002	optimal	Pettie-Ramachandran
20xx	E	???

Remark. Linear-time randomized MST algorithm (Karger-Klein-Tarjan 1995).

Euclidean MST

Given N points in the plane, find MST connecting them, where the distances between point pairs are their Euclidean distances.
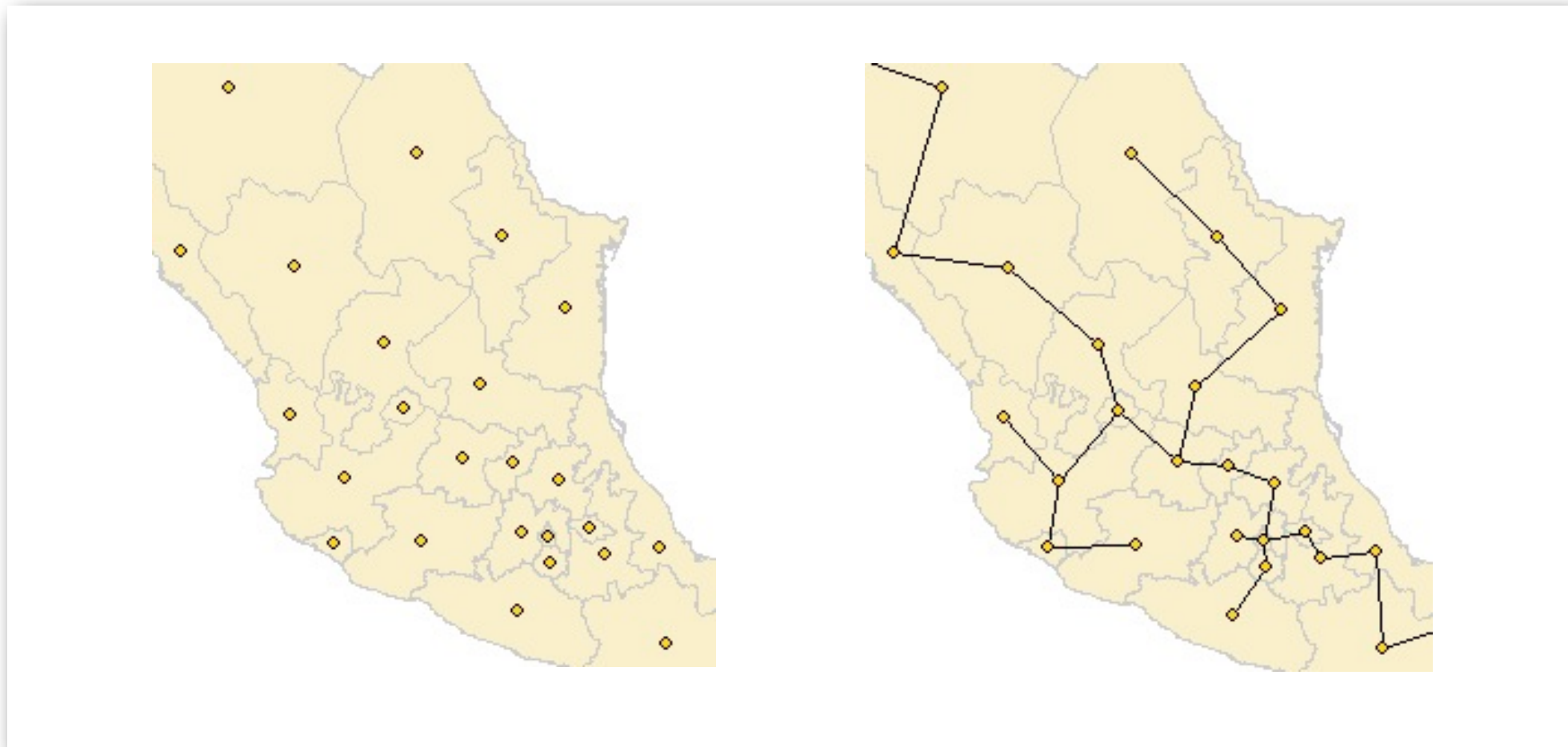
Brute force. Compute $\sim N^2 / 2$ distances and run Prim's algorithm.

**Ingenuity.** Exploit geometry and do it in $\sim c \, N \log N$.

Scientific application: clustering

k-clustering. Divide a set of objects classify into k coherent groups.

Distance function. Numeric value specifying "closeness" of two objects.

Goal. Divide into clusters so that objects in different clusters are far apart.
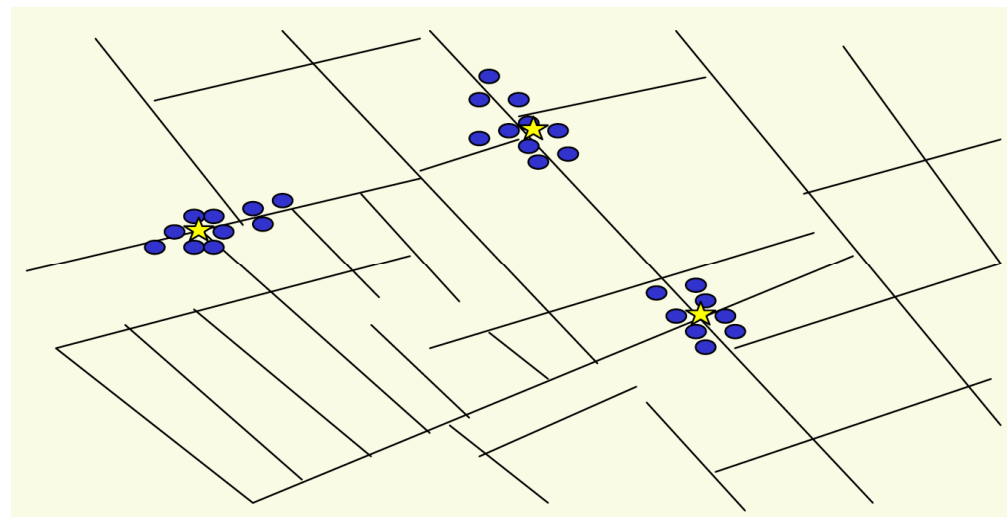
outbreak of cholera deaths in London in 1850s (Nina Mishra)

Applications.
- Routing in mobile ad hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases.
- Skycat: cluster 10^9 sky objects into stars, quasars, galaxies.

Single-link clustering

k-clustering. Divide a set of objects classify into k coherent groups.
Distance function. Numeric value specifying "closeness" of two objects.

Single link. Distance between two clusters equals the distance
between the two closest objects (one in each cluster).

Single-link clustering. Given an integer k, find a k-clustering that maximizes the distance
between two closest clusters.

distance between two clusters

distance between two closest clusters
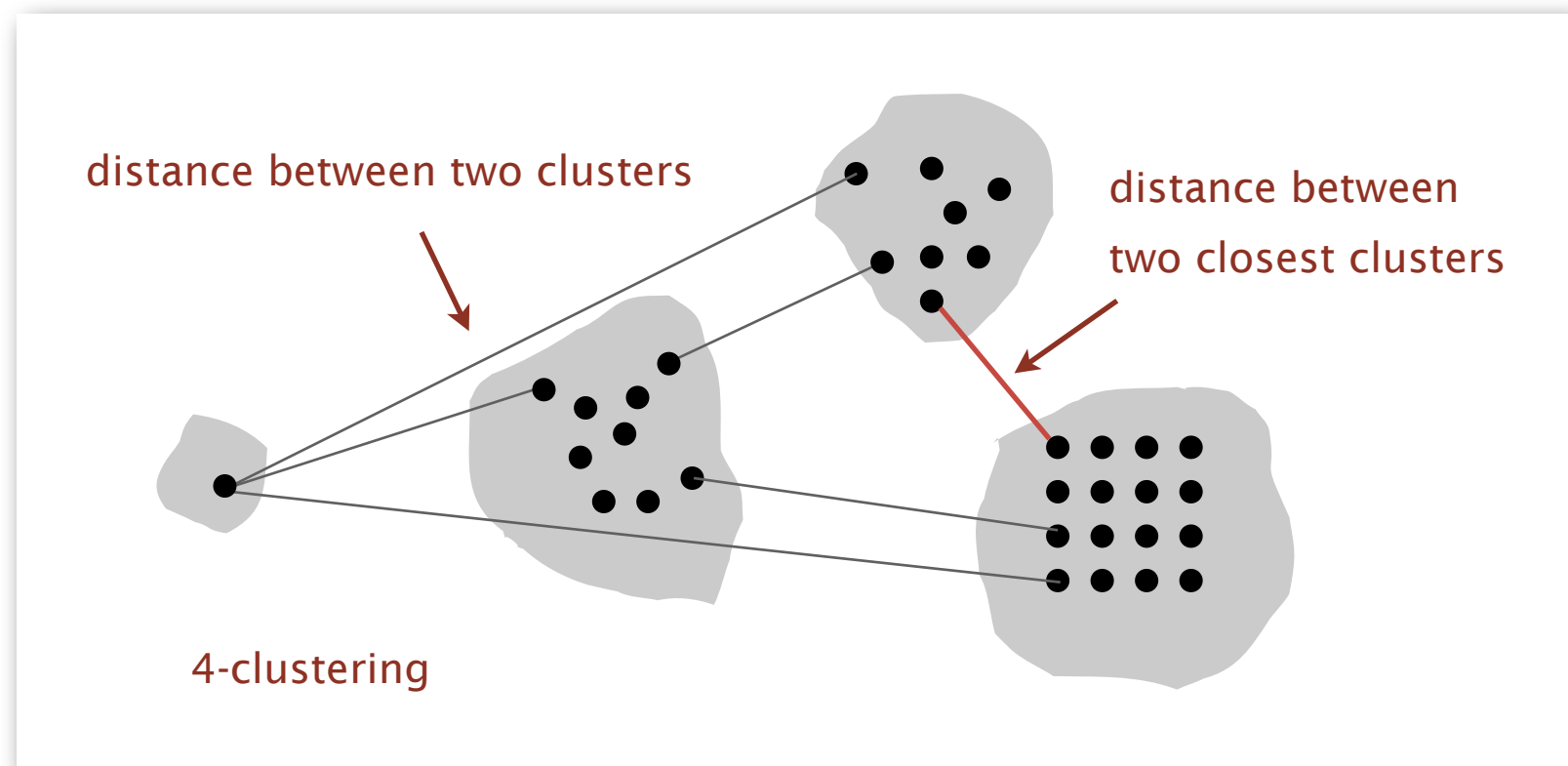
4-clustering

Single-link clustering algorithm

"Well-known" algorithm for single-link clustering:

- Form V clusters of one object each.
- Find the closest pair of objects such that each object is in a different cluster, and merge the two clusters.
- Repeat until there are exactly k clusters.

Observation. This is Kruskal's algorithm
(stop when k connected components).
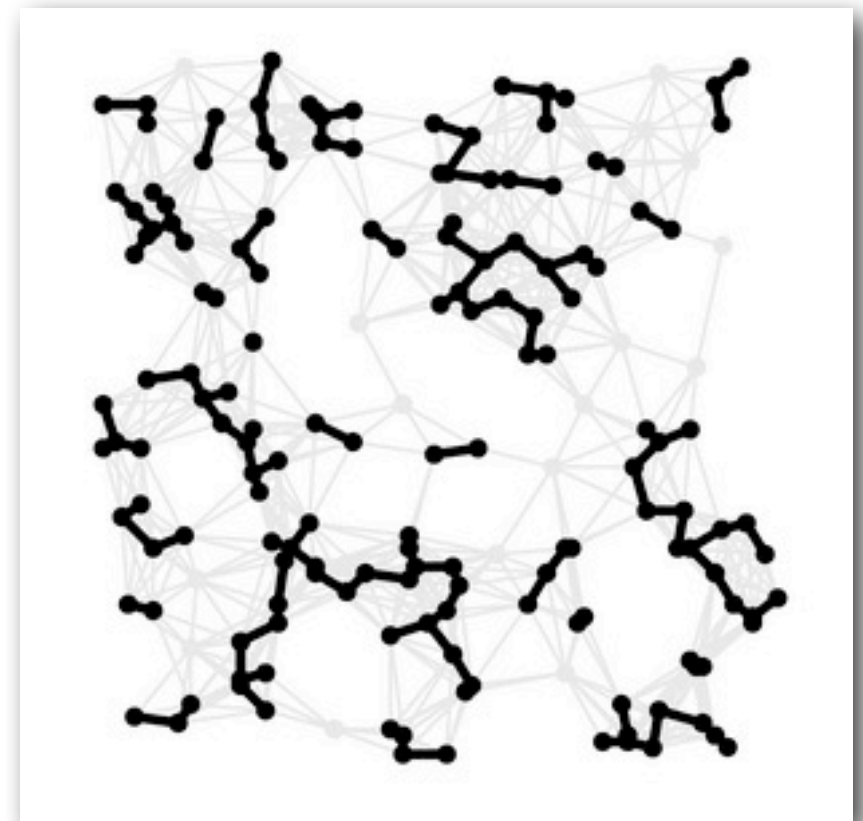
Alternate solution. Run Prim's algorithm and delete k-1 max weight edges.

Dendrogram

Dendrogram. Tree diagram that illustrates arrangement of clusters.

BA   NA   RM   FI   MI   TO

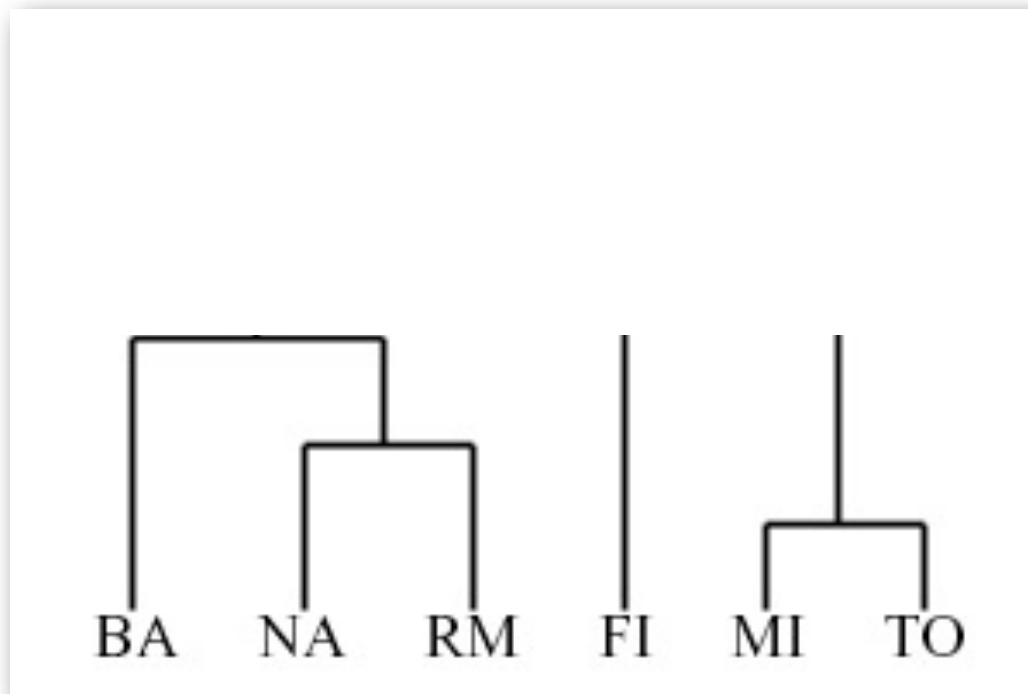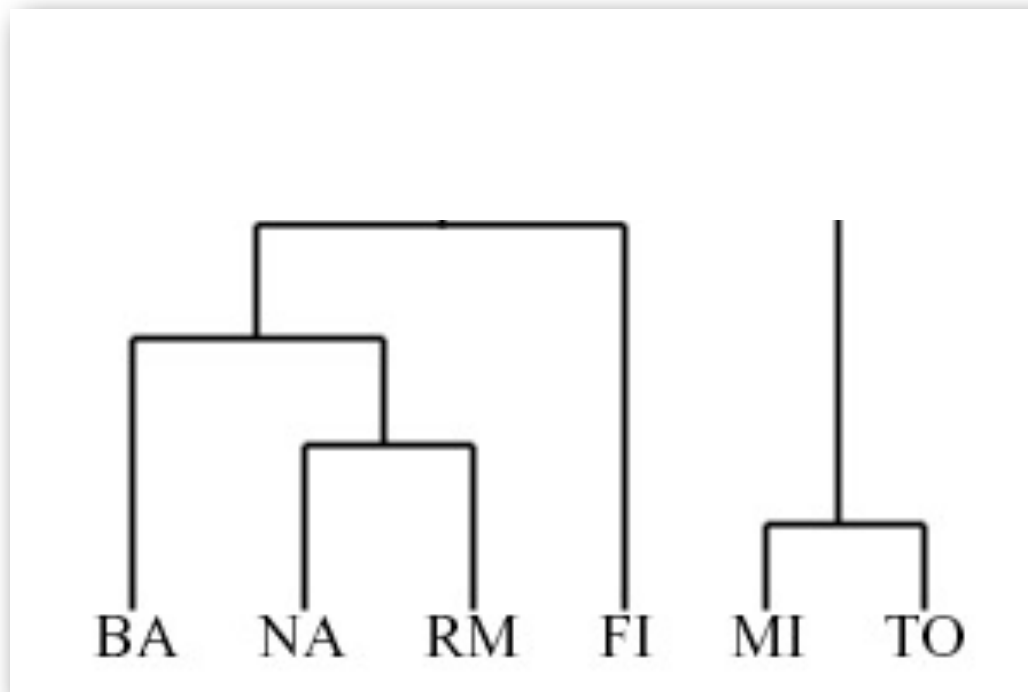http://home.dei.polimi.it/matteucc/Clustering/tutorial_html/hierarchical.html

Dendrogram
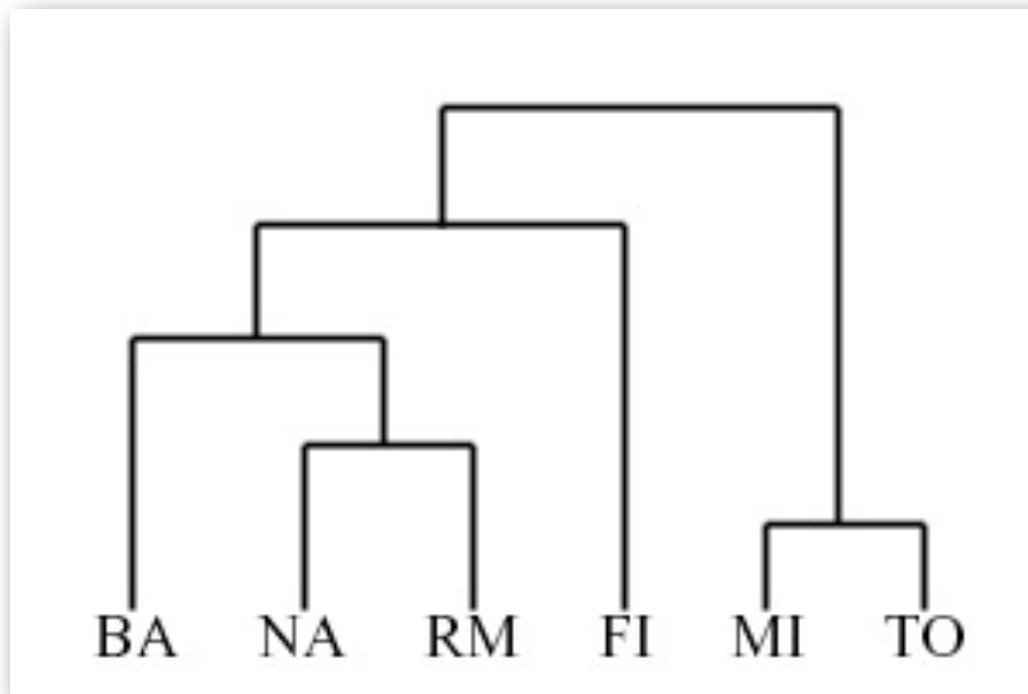
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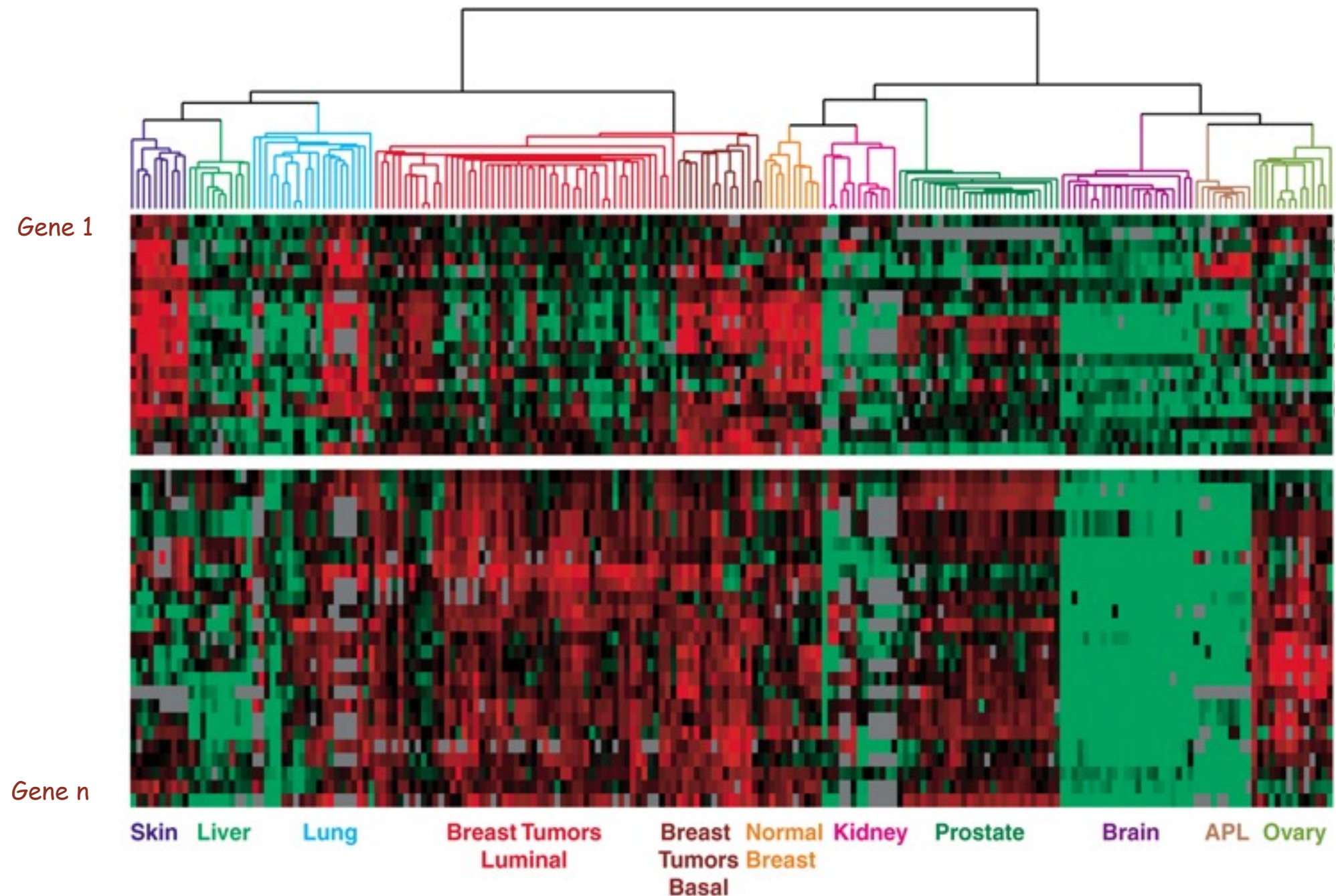
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Dendrogram of cancers in human

Tumors in similar tissues cluster together.

Gene 1

Gene n

Skin Liver Lung Breast Tumors Luminal Breast Tumors Basal Normal Breast Kidney Prostate Brain APL Ovary

Reference: Botstein & Brown group

gene expressed
gene not expressed