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Summer 2012

LECTURE 3

- Inverse Laplace Transform
- Partial Fraction Expansion

Reference: Decarlo / Lin pp 565-575

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Inverse Laplace Transform

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{\Gamma} F(s) e^{+st} \, ds$$

Bad news: This integral requires solid background in complex variables and knowledge of residue theorem of complex variables.

Good news: We are not going to use this integral to find inverse Laplace Transform.

What are we going to use?

- Table 12.1 (p-564) and partial-fraction expansion.
- Table 12.2 (p-584) may be useful as well.

Example 1. Find $f(t)$ if $F(s) = \dfrac{20s^2 + 30s + 20}{s(s+2)}$

Step 1: Write down the form of partial fraction expansion of $F(s)$.

$$F(s) = K + \frac{A}{s} + \frac{B}{s+2}$$

Step 2: Compute the coefficients

(a) $K = \lim_{s \to \infty} F(s) = 20$

(b) $A = \left. \dfrac{20s^2 + 30s + 20}{s+2} \right|_{s=0} = \dfrac{20}{2} = 10$

(c) $B = \left. \dfrac{20s^2 + 30s + 20}{s} \right|_{s=-2} = \dfrac{80 - 60 + 20}{-2} = -20$

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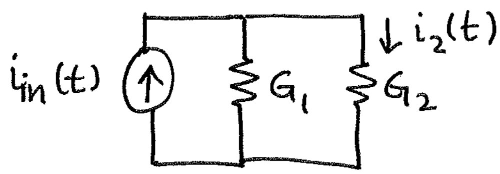
Step 3 : Take inverse Laplace transform with the help of Table 12.1

$$F(s) = 20 + \frac{10}{s} + \frac{-20}{s+2}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = 20\,\delta(t) + 10u(t) - 20e^{-2t}u(t)$$

Example 2. Find $i_{in}(t)$ when $I_2(s) = \dfrac{2s^4+2s^3+3s^2+3s+2}{s^2\,(s+1)^2}$



$G_1 = 0.125\,℧$

$G_2 = 0.5\,℧$

Step 1: $i_{in}(t)\ \boxed{?}\ i_2(t)$

$$i_2(t) = \frac{G_2}{G_1+G_2}\, i_{in}(t)$$

$$i_{in}(t) = \frac{G_1+G_2}{G_2}\, i_2(t) = \frac{0.5+0.125}{0.5}\, i_2(t)$$

$$\therefore\ i_{in}(t) = \frac{5}{4}\, i_2(t)$$

Step 2: Find $i_2(t)$ by finding inverse Laplace transform of $I_2(s)$

step 2.1. Write down the form of partial fraction expansion for $I_2(s)$

$$I_2(s) = K + \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$$

step 2.2 Compute the coefficients

(a) $K = \lim_{s \to \infty} I_2(s) = 2$

(b) $$B = \left.\frac{2s^4 + 2s^3 + 3s^2 + 3s + 2}{(s+1)^2}\right|_{s=0} = 2$$

(c) $$D = \left.\frac{2s^4 + 2s^3 + 3s^2 + 3s + 2}{s^2}\right|_{s=-1} = 2$$

(d) It remains to find 'A' & 'C'. The approach we take is to generate two equations in two unknowns 'A' and 'C'. At this point, we have

$$I_2(s) = 2 + \frac{A}{s} + \frac{2}{s^2} + \frac{C}{s+1} + \frac{2}{(s+1)^2}$$

(i) $$I_2(1) = \frac{12}{4} = [\text{Expression of } I_2(s)]_{s=1}$$

$$= 3 = 2 + \frac{A}{1} + \frac{2}{1} + \frac{C}{2} + \frac{2}{4}$$

which reduces to $A + 0.5C = -1.5$ —(1)

(ii) $$I_2(-2) = \frac{2(16) - 2(8) + 3(4) - 6 + 2}{4}$$

$$= 6 = 2 + \frac{A}{(-2)} + \frac{2}{(-2)^2} + \frac{C}{(-2+1)} + \frac{2}{(-2+1)^2}$$

which reduces to

$-2A - 4C = 6$ —(2)

Solving (1) & (2), we get

$A = -1$

$C = -1$

step 2.3 Take inverse Laplace transform with the help of the table.

$$I_2(s) = 2 + \frac{-1}{s} + \frac{2}{s^2} + \frac{-1}{s+1} + \frac{2}{(s+1)^2}$$

$$i_2(t) = \mathcal{L}^{-1}[I_2(s)]$$

$$= 2\,\delta(t) - u(t) + 2r(t) - e^{-t}u(t) + 2te^{-t}u(t)$$

Step 3: Since $i_{in}(t) = \frac{5}{4}\, i_2(t)$ (from step 1)

$$i_{in}(t) = 2.5\,\delta(t) - 1.25\,u(t) + 2.5\,r(t) - 1.25e^{-t}u(t) + 2.5te^{-t}u(t) \leftarrow$$

Example 3: Find $f(t)$ when $F(s) = \dfrac{As+B}{s^2+\omega^2}$

Note that, from the table, we have

(i) $\mathcal{L}[\sin(\omega t)u(t)] = \dfrac{\omega}{s^2+\omega^2}$

(ii) $\mathcal{L}[\cos(\omega t)u(t)] = \dfrac{s}{s^2+\omega^2}$

$$F(s) = \frac{As+B}{s^2+\omega^2} = A\frac{s}{s^2+\omega^2} + \frac{B}{\omega}\frac{\omega}{s^2+\omega^2}$$

$$\therefore f(t) = \mathcal{L}^{-1}[F(s)] = A\cos(\omega t)u(t) + \frac{B}{\omega}\sin(\omega t)u(t) \leftarrow$$

Example 4. (complex poles)

Find $h(t)$ when $H(s) = \dfrac{10s^2 - 8s + 16}{s^3 + s^2 + 16s + 16}$

$$= \frac{10s^2 - 8s + 16}{(s+1)(s^2+16)}$$

Step 1: Write down the form of partial fraction expansion

$$H(s) = \frac{As + B}{s^2 + 16} + \frac{C}{s+1}$$

Step 2: Compute A, B and C

(a) $C = \left.\dfrac{10s^2 - 8s + 16}{s^2 + 16}\right|_{s=-1}$

$= 2 \leftarrow$

(b) To find A and B, use two equations, two unknown approach.

(i) $H(0) = \dfrac{16}{16} = 1 = \dfrac{2}{1} + \dfrac{0(A) + B}{16}$

$B = -16$

(ii) $H(1) = \dfrac{10 - 8 + 16}{2(1+16)} = \dfrac{9}{17} = \dfrac{2}{1+1} + \dfrac{(1)(A) + (-16)}{1+16}$

$A = 8$

Step 3: Find inverse Laplace transform with the help of table

$$H(s) = \frac{8s + (-16)}{s^2 + 16} + \frac{2}{s+1}$$

$$H(s) = 8\frac{s}{s^2+16} - 4\frac{4}{s^2+16} + \frac{2}{s+1}$$

$$\therefore h(t) = \mathcal{L}^{-1}[H(s)] = 8\cos(4t)u(t) - 4\sin(4t)u(t) + 2e^{-t}u(t) \leftarrow$$

Example 5. Find $f(t)$ if $F(s) = \dfrac{15}{s(s^2+4s+5)}$

Step 1: Write down the form of partial fraction expansion

$$F(s) = \frac{A}{s} + \frac{Bs+C}{(s+2)^2+1}$$

Step 2: Compute A, B and C

(a) $A = \left.\dfrac{15}{s^2+4s+5}\right|_{s=0} = 3$

(b) $F(1) = \dfrac{3}{1} + \dfrac{B+C}{10} = \dfrac{15}{10}$

$$\therefore B+C = -15$$

$F(-1) = \dfrac{3}{(-1)} + \dfrac{B(-1)+C}{2} = \dfrac{15}{(-1)(2)}$

$$-B+C = -9$$

$$\therefore C = -12, \quad B = -3$$

Step 3: Find inverse Laplace transform with the help of table.

$$F(s) = \frac{3}{s} + \frac{-3s-12}{(s+2)^2+1}$$

$$= \frac{3}{s} - 3\frac{s+2}{(s+2)^2+1} - 6\frac{1}{(s+2)^2+1}$$

$$\therefore f(t) = 3u(t) - 3e^{-2t}\cos(t)u(t) - 6e^{-2t}\sin(t)u(t) \leftarrow$$