

Quicksort

- quicksort
- selection
- duplicate keys
- system sorts

Two classic sorting algorithms

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.

Mergesort.

← last lecture

- Java sort for objects.
- Perl, C++ stable sort, Python stable sort, Firefox JavaScript, ...

Quicksort.

← this lecture

- Java sort for primitive types.
- C qsort, Unix, Visual C++, Python, Matlab, Chrome JavaScript, ...

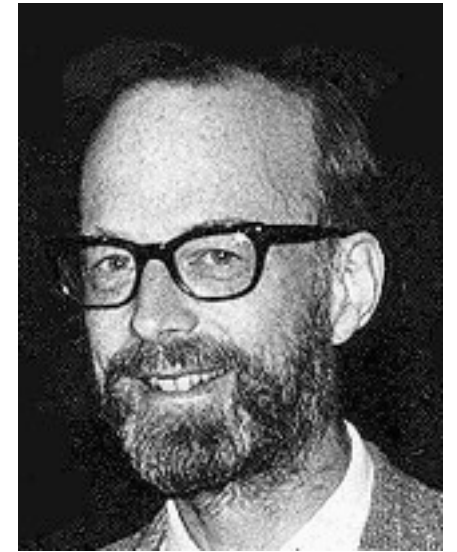
Quicksort

- quicksort
- selection
- duplicate keys
- system sorts

Quicksort

Basic plan.

- **Shuffle** the array.
- **Partition** so that, for some j
 - element $a[j]$ is in place
 - no larger element to the left of j
 - no smaller element to the right of j
- **Sort** each piece recursively.



Sir Charles Antony Richard Hoare
1980 Turing Award

input	Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
shuffle	K	← R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
partition	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
sort left	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
sort right	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

Quicksort partitioning

Basic plan.

- Scan i from left for an item that belongs on the right.
- Scan j from right for item item that belongs on the left.
- Exchange $a[i]$ and $a[j]$.
- Repeat until pointers cross.

	i	j	$a[i]$															
			0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values	0	16	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
scan left, scan right	1	12	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
exchange	1	12	K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
scan left, scan right	3	9	K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
exchange	3	9	K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
scan left, scan right	5	6	K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
exchange	5	6	K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
scan left, scan right	6	5	K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
final exchange	6	5	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
result	6	5	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S

Partitioning trace (array contents before and after each exchange)

Quicksort: Java code for partitioning

```
private static int partition(Comparable[] a, int lo, int hi)
{
    int i = lo, j = hi+1;
    while (true)
    {
        while (less(a[++i], a[lo]))           find item on left to swap
            if (i == hi) break;

        while (less(a[lo], a[--j]))           find item on right to swap
            if (j == lo) break;

        if (i >= j) break;                     check if pointers cross
        exch(a, i, j);                         swap

        exch(a, lo, j);                       swap with partitioning item
        return j;                             return index of item now known to be in place
    }
}
```

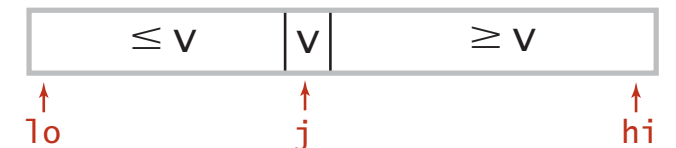
before



during



after



Quicksort: Java implementation

```
public class Quick
{
    private static int partition(Comparable[] a, int lo, int hi)
    { /* see previous slide */ }

    public static void sort(Comparable[] a)
    {
        StdRandom.shuffle(a);
        sort(a, 0, a.length - 1);
    }

    private static void sort(Comparable[] a, int lo, int hi)
    {
        if (hi <= lo) return;
        int j = partition(a, lo, hi);
        sort(a, lo, j-1);
        sort(a, j+1, hi);
    }
}
```

← shuffle needed for
performance guarantee
(stay tuned)

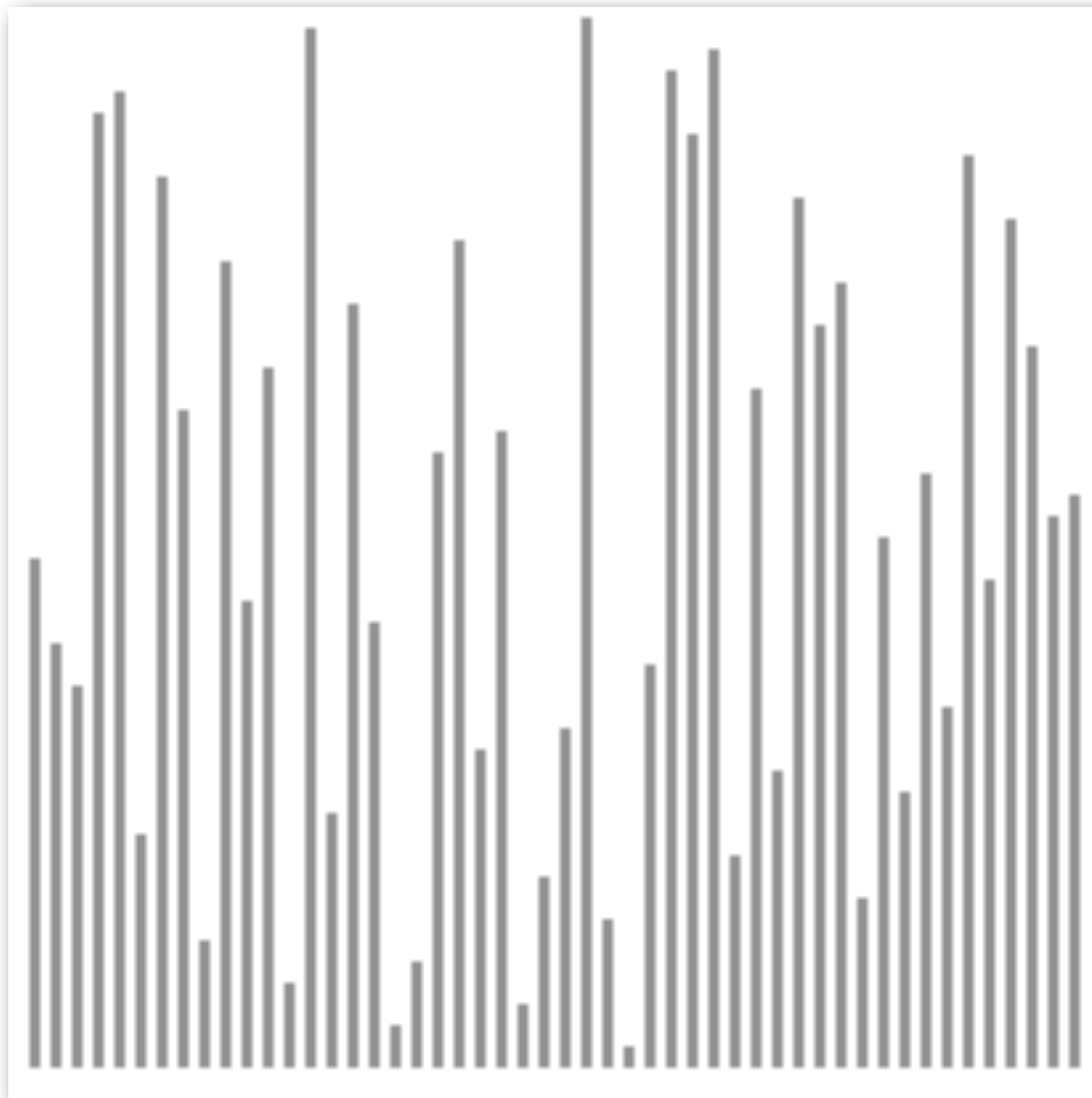
Quicksort trace

	lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values				Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
random shuffle				K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
	0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
	0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	1		1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	4		4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	7	7	8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	8		8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	10	13	15	A	C	E	E	I	K	L	M	O	P	S	Q	R	T	U	X
	10	12	12	A	C	E	E	I	K	L	M	O	P	R	Q	S	T	U	X
	10	11	11	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	10		10	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	14	14	15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	15		15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result				A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

Quicksort trace (array contents after each partition)

Quicksort animation

50 random elements



<http://www.sorting-algorithms.com/quick-sort>

- ▲ algorithm position
- █ in order
- █ current subarray
- █ not in order

Quicksort: implementation details

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but is not worth the cost.

Terminating the loop. Testing whether the pointers cross is a bit trickier than it might seem.

Staying in bounds. The `(j == lo)` test is redundant (why?), but the `(i == hi)` test is not.

Preserving randomness. Shuffling is needed for performance guarantee.

Equal keys. When duplicates are present, it is (counter-intuitively) better to stop on elements equal to the partitioning element.

Quicksort: empirical analysis

Running time estimates:

- Home PC executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort ($N \log N$)			quicksort ($N \log N$)		
computer	thousand	million	billion	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min	instant	0.6 sec	12 min
super	instant	1 second	1 week	instant	instant	instant	instant	instant	instant

Lesson 1. Good algorithms are better than supercomputers.

Lesson 2. Great algorithms are better than good ones.

Quicksort: best-case analysis

Best case. Number of compares is $\sim N \lg N$.

			a[]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0		0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
2		2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4		4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
6		6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8		8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
10		10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12		12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: worst-case analysis

Worst case. Number of compares is $\sim \frac{1}{2} N^2$.


			a[]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: average-case analysis

Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf I. C_N satisfies the recurrence $C_0 = C_1 = 0$ and for $N \geq 2$:

$$C_N = (N+1) + \frac{C_0 + C_1 + \dots + C_{N-1}}{N} + \frac{C_{N-1} + C_{N-2} + \dots + C_0}{N}$$



- Multiply both sides by N and collect terms:

$$NC_N = N(N+1) + 2(C_0 + C_1 + \dots + C_{N-1})$$

- Subtract this from the same equation for $N-1$:

$$NC_N - (N-1)C_{N-1} = 2N + 2C_{N-1}$$

- Rearrange terms and divide by $N(N+1)$:

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}$$

Quicksort: average-case analysis

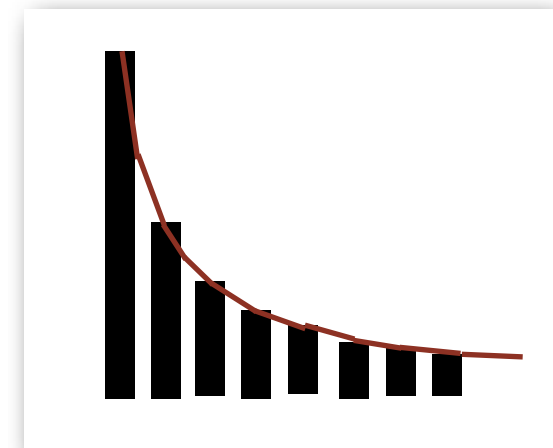
- Repeatedly apply above equation:

$$\begin{aligned}\frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1} \\ &= \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\ &= \frac{C_{N-3}}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\ &= \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \dots + \frac{2}{N+1}\end{aligned}$$

previous equation

- Approximate sum by an integral:

$$\begin{aligned}C_N &= 2(N+1) \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{N+1} \right) \\ &\sim 2(N+1) \int_3^{N+1} \frac{1}{x} dx\end{aligned}$$



- Finally, the desired result:

$$C_N \sim 2(N+1) \ln N \approx 1.39N \lg N$$

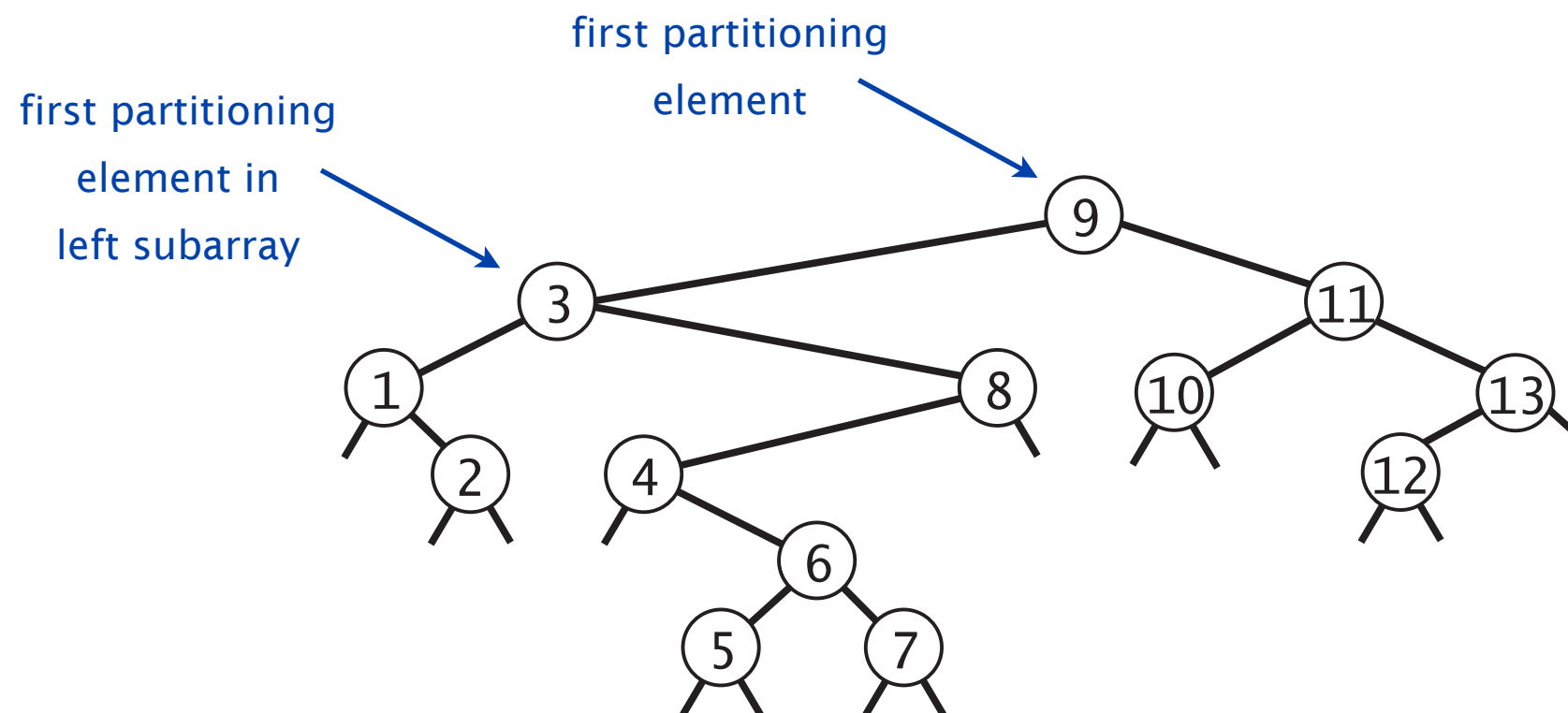
Quicksort: average-case analysis

Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf 2. Consider BST representation of keys 1 to N .

shuffle

9	10	2	5	8	7	6	1	11	12	13	3	4
---	----	---	---	---	---	---	---	----	----	----	---	---

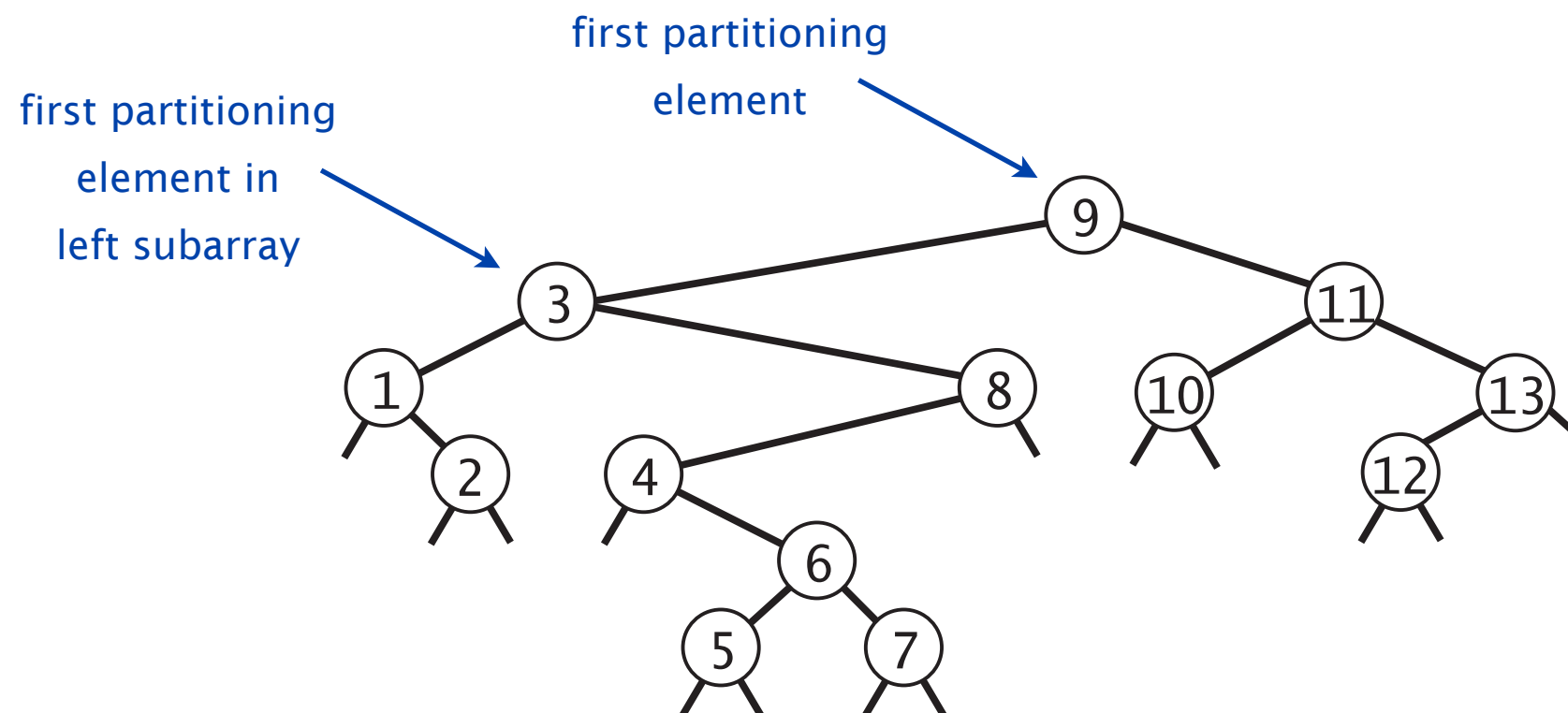


Quicksort: average-case analysis

Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf 2. Consider BST representation of keys 1 to N .

- A key is compared only with its ancestors and descendants.
- Probability i and j are compared equals $2 / |j - i + 1|$.



3 and 6 are compared
(when 3 is partition)

1 and 6 are not compared
(because 3 is partition)


Quicksort: average-case analysis

Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf 2. Consider BST representation of keys 1 to N .

- A key is compared only with its ancestors and descendants.
- Probability i and j are compared equals $2 / |j - i + 1|$.

- Expected number of compares =
$$\sum_{i=1}^N \sum_{j=i+1}^N \frac{2}{j - i + 1} = 2 \sum_{i=1}^N \sum_{j=2}^{N-i+1} \frac{1}{j}$$


all pairs i and j

$$\leq 2N \sum_{j=1}^N \frac{1}{j}$$
$$\sim 2N \int_{x=1}^N \frac{1}{x} dx$$
$$= 2N \ln N$$

Quicksort: summary of performance characteristics

Worst case. Number of compares is quadratic.

- $N + (N - 1) + (N - 2) + \dots + 1 \sim \frac{1}{2} N^2$.
- More likely that your computer is struck by lightning bolt.

Average case. Number of compares is $\sim 1.39 N \lg N$.

- 39% more compares than mergesort.
- **But** faster than mergesort in practice because of less data movement.

Random shuffle.

- Probabilistic guarantee against worst case.
- Basis for math model that can be validated with experiments.

Caveat emptor. Many textbook implementations go **quadratic** if array

- Is sorted or reverse sorted.
- Has many duplicates (even if randomized!)

Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Can delay insertion sort until end.

```
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo + CUTOFF - 1)
    {
        Insertion.sort(a, lo, hi);
        return;
    }
    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```

Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Can delay insertion sort until end.

Median of sample.

- Best choice of pivot element = median.
- Estimate true median by taking median of sample.

```
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo) return;

    int m = medianOf3(a, lo, lo + (hi - lo)/2, hi);
    swap(a, lo, m);

    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```

Quicksort: practical improvements

Insertion sort small subarrays.

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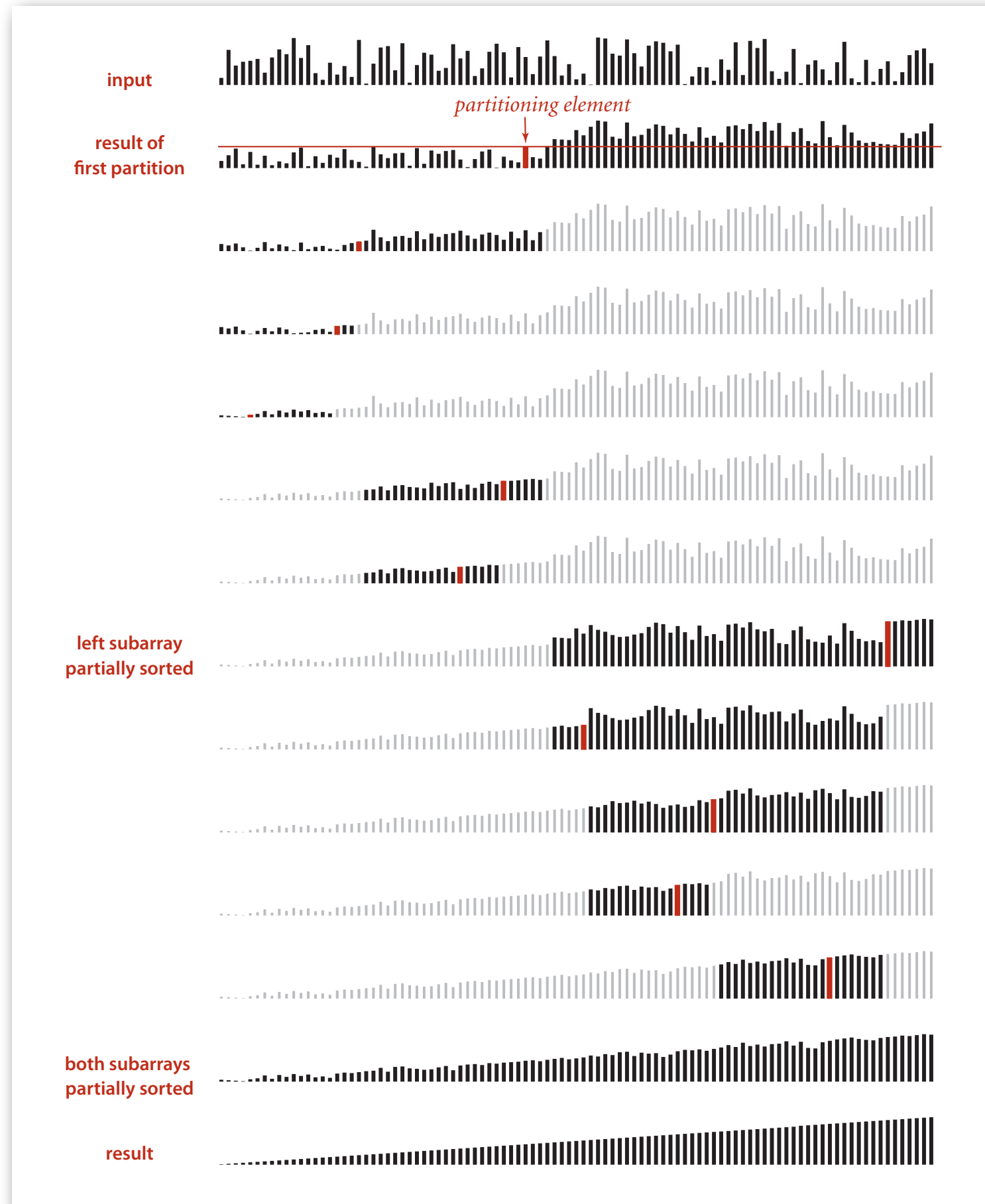
Optimize parameters.

- Median-of-3 (random) elements.
- Cutoff to insertion sort for ≈ 10 elements.

$\sim 12/7 \ N \ln N$ compares (slightly fewer)
 $\sim 12/35 \ N \ln N$ exchanges (slightly more)



Quicksort with median-of-3 and cutoff to insertion sort: visualization



Quicksort

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Selection

Goal. Find the k^{th} largest element.

Ex. Min ($k = 0$), max ($k = N - 1$), median ($k = N / 2$).

Applications.

- Order statistics.
- Find the “top k .”

Use theory as a guide.

- Easy $O(N \log N)$ upper bound. How?
- Easy $O(N)$ upper bound for $k = 1, 2, 3$. How?
- Easy $\Omega(N)$ lower bound. Why?

Which is true?

- $\Omega(N \log N)$ lower bound?  is selection as hard as sorting?
- $O(N)$ upper bound?  is there a linear-time algorithm for all k ?

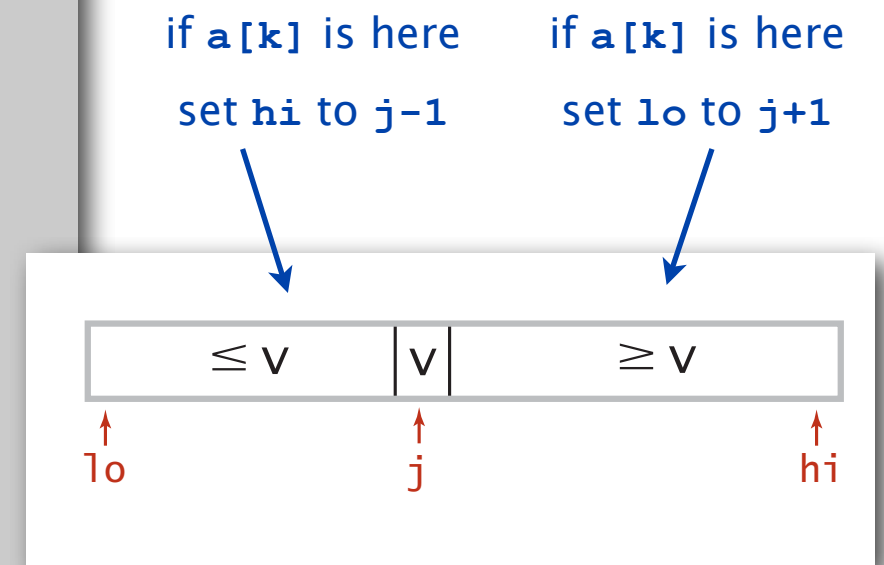
Quick-select

Partition array so that:

- Element **a[j]** is in place.
- No larger element to the left of **j**.
- No smaller element to the right of **j**.

Repeat in **one** subarray, depending on **j**; finished when **j** equals **k**.

```
public static Comparable select(Comparable[] a, int k)
{
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo)
    {
        int j = partition(a, lo, hi);
        if (j < k) lo = j + 1;
        else if (j > k) hi = j - 1;
        else
            return a[k];
    }
    return a[k];
}
```



Quick-select: mathematical analysis

Proposition. Quick-select takes **linear** time on average.

Pf sketch.

- Intuitively, each partitioning step splits array approximately in half:
 $N + N/2 + N/4 + \dots + 1 \sim 2N$ compares.
- Formal analysis similar to quicksort analysis yields:

$$C_N = 2N + k \ln(N/k) + (N-k) \ln(N/(N-k))$$

Ex. $(2 + 2 \ln 2) N$ compares to find the median.

Remark. Quick-select uses $\sim \frac{1}{2} N^2$ compares in the worst case, but (as with quicksort) the random shuffle provides a probabilistic guarantee.

Theoretical context for selection

Proposition. [Blum, Floyd, Pratt, Rivest, Tarjan, 1973] There exists a compare-based selection algorithm whose worst-case running time is linear.

Time Bounds for Selection

by .

Manuel Blum, Robert W. Floyd, Vaughan Pratt,
Ronald L. Rivest, and Robert E. Tarjan

Abstract

The number of comparisons required to select the i -th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than $5.4305 n$ comparisons are ever required. This bound is improved for

Remark. But, constants are too high \Rightarrow not used in practice.

Use theory as a guide.

- Still worthwhile to seek **practical** linear-time (worst-case) algorithm.
- Until one is discovered, use quick-select if you don't need a full sort.

Generic methods

In our `select()` implementation, client needs a cast.

```
Double[] a = new Double[N];  
for (int i = 0; i < N; i++)  
    a[i] = StdRandom.uniform();  
Double median = (Double) Quick.select(a, N/2);
```

← unsafe cast
required in client

The compiler complains.

```
% javac Quick.java  
Note: Quick.java uses unchecked or unsafe operations.  
Note: Recompile with -Xlint:unchecked for details.
```

Q. How to fix?

Generic methods

Pedantic (safe) version. Compiles cleanly, no cast needed in client.

```
public class QuickPedantic
{
    public static <Key extends Comparable<Key>> Key select(Key[] a, int k)
    { /* as before */ }

    public static <Key extends Comparable<Key>> void sort(Key[] a)
    { /* as before */ }

    private static <Key extends Comparable<Key>> int partition(Key[] a, int lo, int hi)
    { /* as before */ }

    private static <Key extends Comparable<Key>> boolean less(Key v, Key w)
    { /* as before */ }

    private static <Key extends Comparable<Key>> void exch(Key[] a, int i, int j)
    { Key swap = a[i]; a[i] = a[j]; a[j] = swap; }
}
```

generic type variable
(value inferred from argument a[])

return type matches array type

can declare variables of generic type

<http://www.cs.princeton.edu/algs4/23quicksort/QuickPedantic.java.html>

Remark. Obnoxious code needed in system sort; not in this course (for brevity).

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Duplicate keys

Often, purpose of sort is to bring records with duplicate keys together.

- Sort population by age.
- Find collinear points.
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.

```
Chicago 09:25:52
Chicago 09:03:13
Chicago 09:21:05
Chicago 09:19:46
Chicago 09:19:32
Chicago 09:00:00
Chicago 09:35:21
Chicago 09:00:59
Houston 09:01:10
Houston 09:00:13
Phoenix 09:37:44
Phoenix 09:00:03
Phoenix 09:14:25
Seattle 09:10:25
Seattle 09:36:14
Seattle 09:22:43
Seattle 09:10:11
Seattle 09:22:54
```

↑
key

Duplicate keys

Mergesort with duplicate keys. Always between $\frac{1}{2} N \lg N$ and $N \lg N$ compares.

Quicksort with duplicate keys.

- Algorithm goes **quadratic** unless partitioning stops on equal keys!
- 1990s C user found this defect in **qsort()**.

several textbook and system
implementation also have this defect

S T O P O N E Q U A L K E Y S

↑ swap

↑ if we don't stop on equal keys

↑ if we stop on equal keys

Duplicate keys: the problem

Mistake. Put all keys equal to the partitioning element on one side.

Consequence. $\sim \frac{1}{2} N^2$ compares when all keys equal.

B A A B A B B B C C C

A A A A A A A A A A A A

Recommended. Stop scans on keys equal to the partitioning element.

Consequence. $\sim N \lg N$ compares when all keys equal.

B A A B A B C C B C B

A A A A A A A A A A A A

Desirable. Put all keys equal to the partitioning element in place.

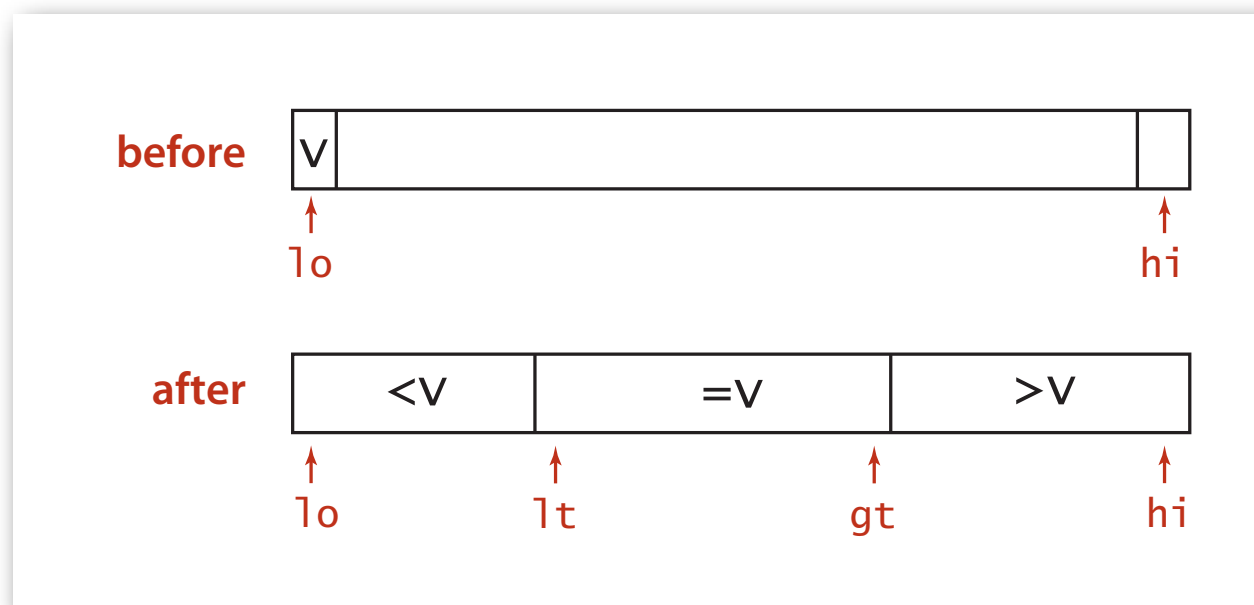
A A A B B B B B C C C

A A A A A A A A A A A A

3-way partitioning

Goal. Partition array into 3 parts so that:

- Elements between **lt** and **gt** equal to partition element **v**.
- No larger elements to left of **lt**.
- No smaller elements to right of **gt**.



Dutch national flag problem. [Edsger Dijkstra]

- Conventional wisdom until mid 1990s: not worth doing.
- New approach discovered when fixing mistake in C library **qsort()**.
- Now incorporated into **qsort()** and Java system sort.

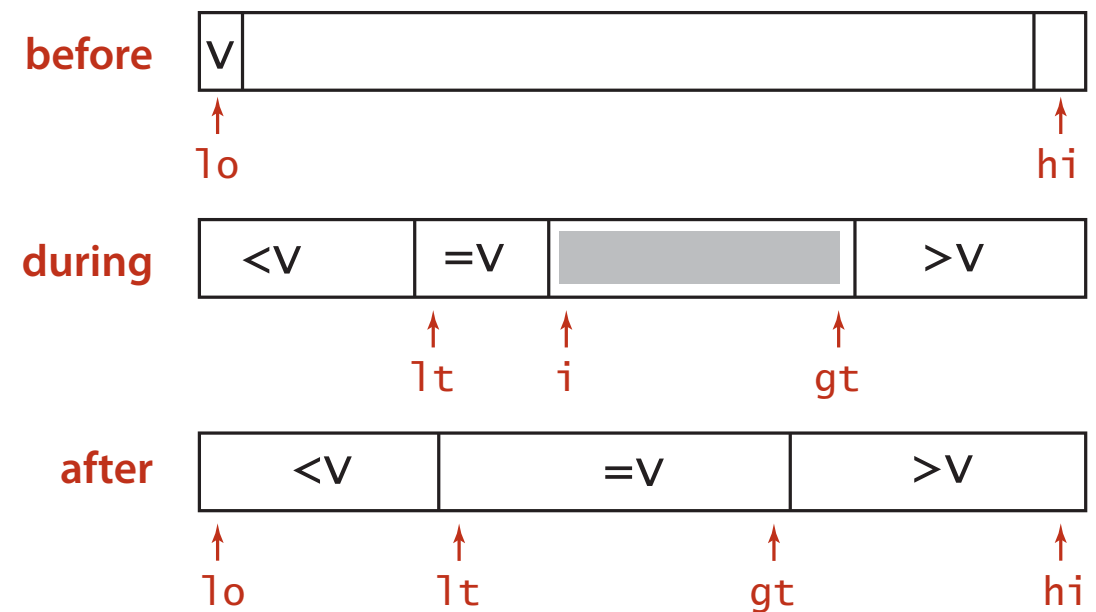
Dijkstra 3-way partitioning algorithm

3-way partitioning.

- Let v be partitioning element $a[l_o]$.
- Scan i from left to right.
 - $a[i]$ less than v : exchange $a[l_t]$ with $a[i]$ and increment both l_t and i
 - $a[i]$ greater than v : exchange $a[gt]$ with $a[i]$ and decrement gt
 - $a[i]$ equal to v : increment i

All the right properties.

- In-place.
- Not much code.
- Small overhead if no equal keys.



3-way partitioning: trace

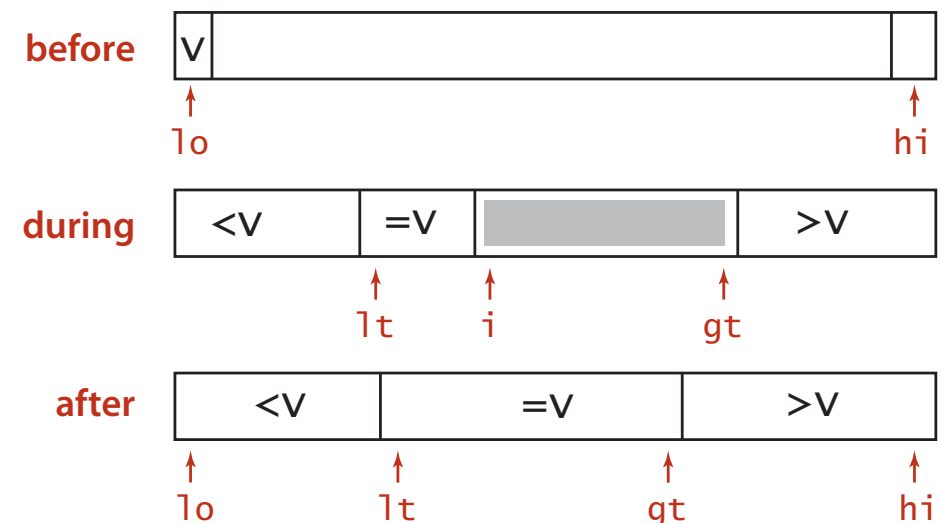
			a[]												
l	t	i	gt	0	1	2	3	4	5	6	7	8	9	10	11
0		0	11	R	B	W	W	R	W	B	R	R	W	B	R
0		1	11	R	B	W	W	R	W	B	R	R	W	B	R
1		2	11	B	R	W	W	R	W	B	R	R	W	B	R
1		2	10	B	R	R	W	R	W	B	R	R	W	B	W
1		3	10	B	R	R	W	R	W	B	R	R	W	B	W
1		3	9	B	R	R	B	R	W	B	R	R	W	W	W
2		4	9	B	B	R	R	R	W	B	R	R	W	W	W
2		5	9	B	B	R	R	R	W	B	R	R	W	W	W
2		5	8	B	B	R	R	R	W	B	R	R	W	W	W
2		5	7	B	B	R	R	R	R	B	R	W	W	W	W
2		6	7	B	B	R	R	R	R	B	R	W	W	W	W
3		7	7	B	B	B	R	R	R	R	R	W	W	W	W
3		8	7	B	B	B	R	R	R	R	R	W	W	W	W
3		8	7	B	B	B	R	R	R	R	R	W	W	W	W

3-way partitioning trace (array contents after each loop iteration)

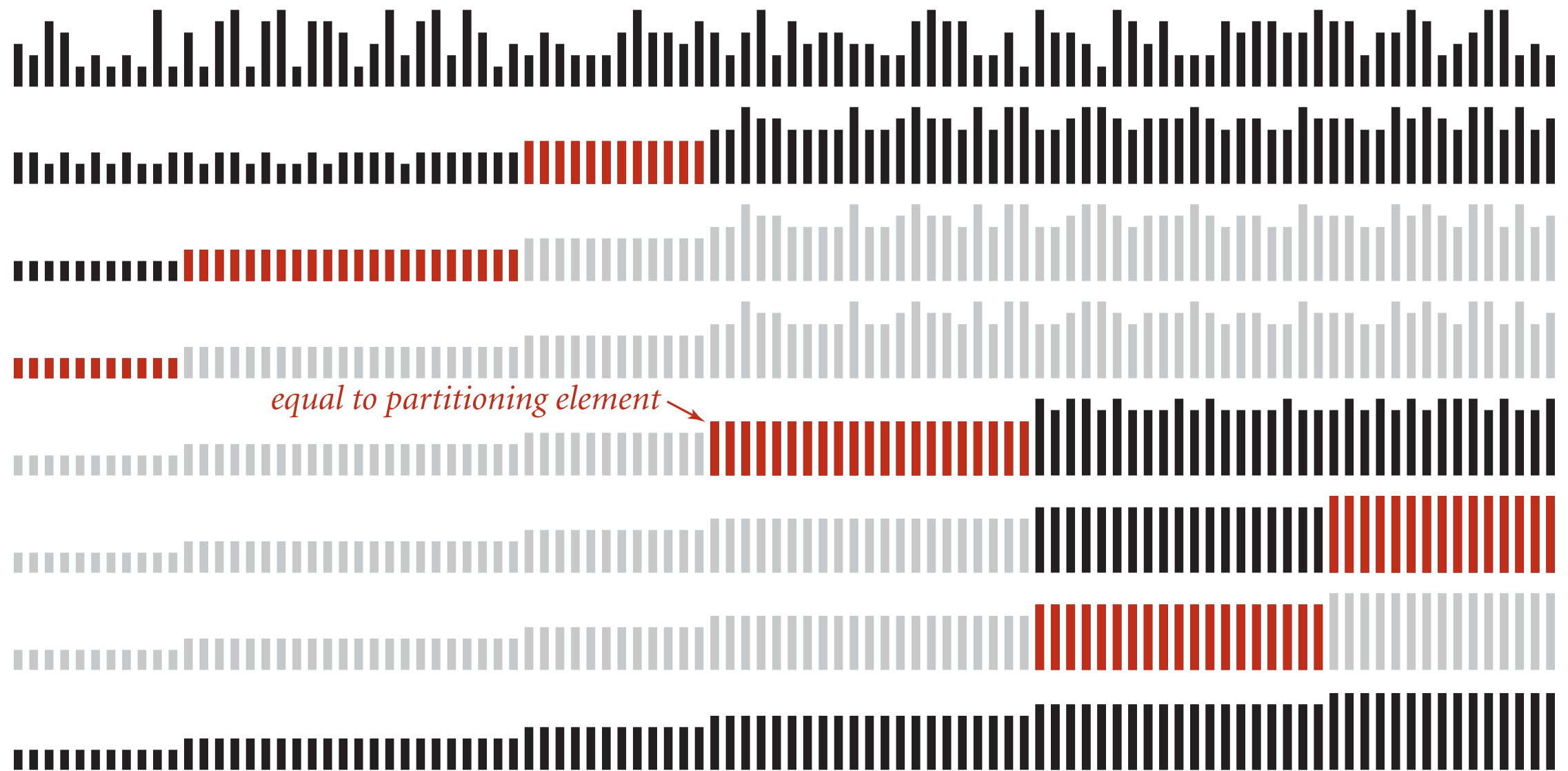
3-way quicksort: Java implementation

```
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo) return;
    int lt = lo, gt = hi;
    Comparable v = a[lo];
    int i = lo;
    while (i <= gt)
    {
        int cmp = a[i].compareTo(v);
        if (cmp < 0) exch(a, lt++, i++);
        else if (cmp > 0) exch(a, i, gt--);
        else i++;
    }

    sort(a, lo, lt - 1);
    sort(a, gt + 1, hi);
}
```



3-way quicksort: visual trace



Duplicate keys: lower bound

Sorting lower bound. If there are n distinct keys and the i^{th} one occurs x_i times, any compare-based sorting algorithm must use at least

$\lg \left(\frac{N!}{x_1! x_2! \cdots x_n!} \right)$ compares in the worst case. $\sim - \sum_{i=1}^n x_i \lg \frac{x_i}{N}$

$N \lg N$ when all distinct;
linear when only a constant number of distinct keys

Proposition. [Sedgewick-Bentley, 1997]

Quicksort with 3-way partitioning is **entropy-optimal**.

Pf. [beyond scope of course]

proportional to lower bound

Bottom line. Randomized quicksort with 3-way partitioning reduces running time from linearithmic to linear in broad class of applications.

Quicksort

- quicksort
- selection
- duplicate keys
- system sorts

Sorting applications

Sorting algorithms are essential in a broad variety of applications:

- Sort a list of names.
- Organize an MP3 library.
- Display Google PageRank results.
- List RSS feed in reverse chronological order.

obvious applications

- Find the median.
- Find the closest pair.
- Binary search in a database.
- Identify statistical outliers.
- Find duplicates in a mailing list.

problems become easy once
elements

are in sorted order

- Data compression.
- Computer graphics.
- Computational biology.
- Supply chain management.
- Load balancing on a parallel computer.

...

Every system needs (and has) a system sort!

non-obvious applications

Java system sorts

Java uses both mergesort and quicksort.

- **Arrays.sort()** sorts an array of **Comparable** or of any primitive type.
- Uses tuned quicksort for primitive types; tuned mergesort for objects.

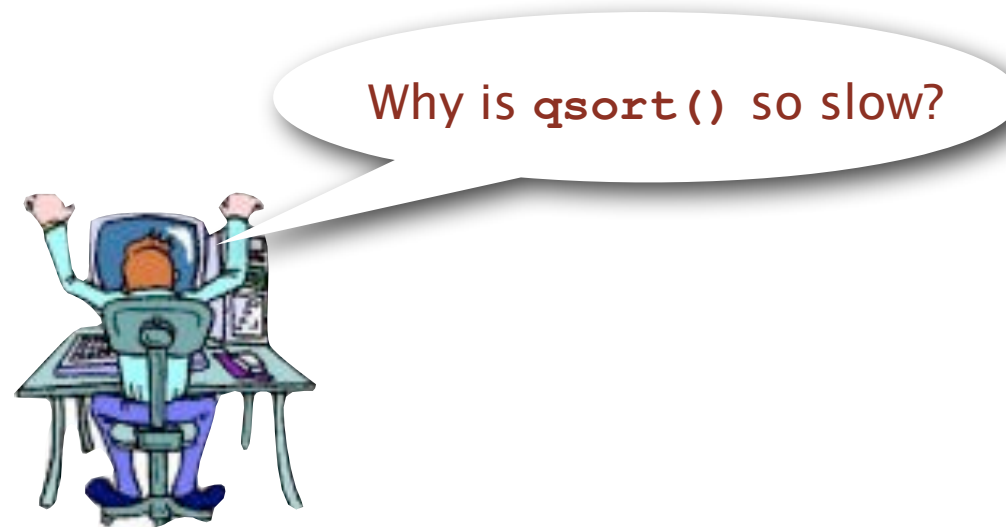
```
import java.util.Arrays;

public class StringSort
{
    public static void main(String[] args)
    {
        String[] a = StdIn.readAll().split("\\s+");
        Arrays.sort(a);
        for (int i = 0; i < N; i++)
            StdOut.println(a[i]);
    }
}
```

Q. Why use different algorithms, depending on type?

War story (C qsort function)

[AT&T Bell Labs \(1991\)](#). Allan Wilks and Rick Becker discovered that a **qsort()** call that should have taken a few minutes was consuming hours of CPU time.



At the time, almost all **qsort()** implementations based on those in:

- Version 7 Unix (1979): quadratic time to sort organ-pipe arrays.
- BSD Unix (1983): quadratic time to sort random arrays of 0s and 1s.



Engineering a system sort

Basic algorithm = quicksort.

- Cutoff to insertion sort for small subarrays.
- Partitioning scheme: Bentley-McIlroy 3-way partitioning. [ahead]
- Partitioning element.
 - small arrays: middle element
 - medium arrays: median of 3
 - large arrays: Tukey's ninther [next slide]

Engineering a Sort Function

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SUMMARY

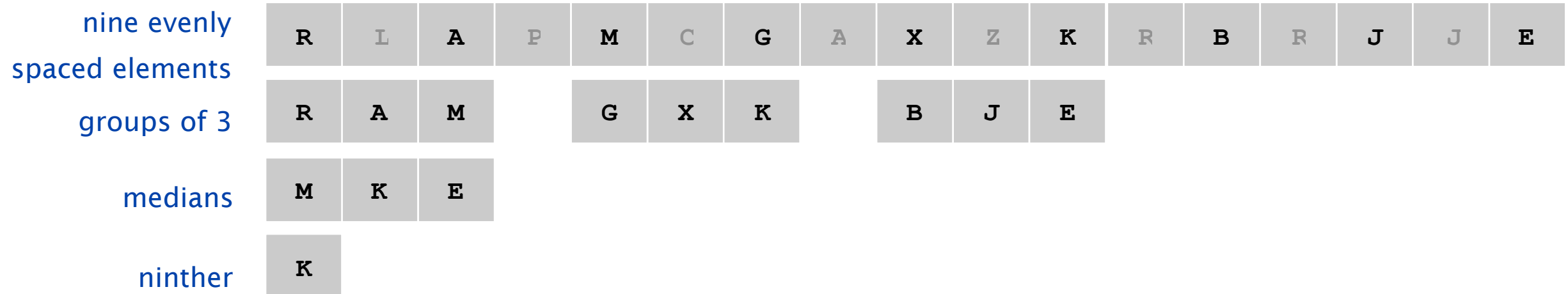
We recount the history of a new `qsort` function for a C library. Our function is clearer, faster and more robust than existing sorts. It chooses partitioning elements by a new sampling scheme; it partitions by a novel solution to Dijkstra's Dutch National Flag problem; and it swaps efficiently. Its behavior was assessed with timing and debugging testbeds, and with a program to certify performance. The design techniques apply in domains beyond sorting.

Now widely used. C, C++, Java,

Tukey's ninther

Tukey's ninther. Median of the median of 3 samples, each of 3 elements.

- Approximates the median of 9.
- Uses at most 12 compares.



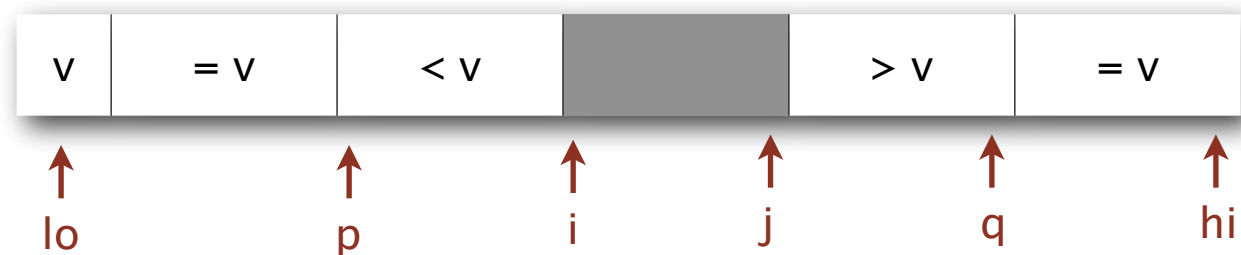
Q. Why use Tukey's ninther?

A. Better partitioning than random shuffle and less costly.

Bentley-McIlroy 3-way partitioning

Partition elements into **four** parts:

- No larger elements to left of **i**.
- No smaller elements to right of **j**.
- Equal elements to left of **p**.
- Equal elements to right of **q**.



Afterwards, swap equal keys into center.

All the right properties.

- In-place.
- Not much code.
- Linear time if keys are all equal.
- Small overhead if no equal keys.

Achilles heel in Bentley-McIlroy implementation (Java system sort)

Q. Based on all this research, Java's system sort is solid, **right?**

more disastrous consequences in C

A. No: a killer input.

- Overflows function call stack in Java and crashes program.
- Would take quadratic time if it didn't crash first.

```
% more 250000.txt
0
218750
222662
11
166672
247070
83339
...
```

250,000 integers
between 0 and 250,000

```
% java IntegerSort 250000 < 250000.txt
Exception in thread "main"
java.lang.StackOverflowError
    at java.util.Arrays.sort1 (Arrays.java:562)
    at java.util.Arrays.sort1 (Arrays.java:606)
    at java.util.Arrays.sort1 (Arrays.java:608)
    at java.util.Arrays.sort1 (Arrays.java:608)
    at java.util.Arrays.sort1 (Arrays.java:608)
    ...
```

Java's sorting library crashes, even if
you give it as much stack space as Windows allows

Achilles heel in Bentley-McIlroy implementation (Java system sort)

McIlroy's devious idea. [A Killer Adversary for Quicksort]

- Construct malicious input **on the fly** while running system quicksort, in response to the sequence of keys compared.
- Make partitioning element compare low against all keys not seen during selection of partitioning element (but don't commit to their relative order).
- Not hard to identify partitioning element.



Consequences.

- Confirms theoretical possibility.
- Algorithmic complexity attack: you enter linear amount of data; server performs quadratic amount of work.

Good news. Attack is not effective if **sort()** shuffles input array.

Q. Why do you think **Arrays.sort()** is deterministic?

System sort: Which algorithm to use?

Many sorting algorithms to choose from:

Internal sorts.

- Insertion sort, selection sort, bubblesort, shaker sort.
- Quicksort, mergesort, heapsort, samplesort, shellsort.
- Solitaire sort, red-black sort, splaysort, Dobosiewicz sort, psort, ...

External sorts. Poly-phase mergesort, cascade-merge, oscillating sort.

String/radix sorts. Distribution, MSD, LSD, 3-way string quicksort.

Parallel sorts.

- Bitonic sort, Batcher even-odd sort.
- Smooth sort, cube sort, column sort.
- GPU sort.

System sort: Which algorithm to use?

Applications have diverse attributes.

- Stable?
- Parallel?
- Deterministic?
- Keys all distinct?
- Multiple key types?
- Linked list or arrays?
- Large or small records?
- Is your array randomly ordered?
- Need guaranteed performance?

		attributes									
		1	2	3	4	M	
algorithm	A	•			•						
	B			•		•				•	
	C		•		•						
	D						•				
	E			•							
	F		•			•		•			
	G	•								•	
	.			•		•		•			
	.		•	•				•			
	.						•			•	
	K	•				•					

many more combinations of
attributes than algorithms

Elementary sort may be method of choice for some combination.
Cannot cover **all** combinations of attributes.

Q. Is the system sort good enough?

A. Usually.

Sorting summary

	inplace?	stable?	worst	average	best	remarks
selection	x		$N^2 / 2$	$N^2 / 2$	$N^2 / 2$	N exchanges
insertion	x	x	$N^2 / 2$	$N^2 / 4$	N	use for small N or partially ordered
shell	x		?	?	N	tight code, subquadratic
merge		x	$N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee, stable
quick	x		$N^2 / 2$	$2 N \ln N$	$N \lg N$	$N \log N$ probabilistic guarantee fastest in practice
3-way quick	x		$N^2 / 2$	$2 N \ln N$	N	improves quicksort in presence of duplicate keys
???	x	x	$N \lg N$	$N \lg N$	$N \lg N$	holy sorting grail

Which sorting algorithm?

lifo	find	data	data	data	data	hash	data
fifo	fifo	fifo	fifo	exch	fifo	fifo	exch
data	data	find	find	fifo	lifo	data	fifo
type	exch	hash	hash	find	type	link	find
hash	hash	heap	heap	hash	hash	leaf	hash
heap	heap	lifo	lifo	heap	heap	heap	heap
sort	less	link	link	leaf	link	exch	leaf
link	left	list	list	left	sort	node	left
list	leaf	push	push	less	find	lifo	less
push	lifo	root	root	lifo	list	left	lifo
find	push	sort	sort	link	push	find	link
root	root	type	type	list	root	path	list
leaf	list	leaf	leaf	sort	leaf	list	next
tree	tree	left	tree	tree	null	next	node
null	null	node	null	null	path	less	null
path	path	null	path	path	tree	root	path
node	node	path	node	node	exch	sink	push
left	link	tree	left	type	left	swim	root
less	sort	exch	less	root	less	null	sink
exch	type	less	exch	push	node	sort	sort
sink	sink	next	sink	sink	next	type	swap
swim	swim	sink	swim	swim	sink	tree	swim
next	next	swap	next	next	swap	push	tree
swap	swap	swim	swap	swap	swim	swap	type
original	?	?	?	?	?	?	sorted

Which sorting algorithm?

lifo fifo data type hash heap sort link list push find root leaf tree null path node left less exch sink swim next swap	find fifo data exch hash heap less left leaf lifo push root list tree null path node link sort type sink swim next swap	data fifo find hash heap lifo link list push root sort type leaf left node null path tree exch less next sink swap swim	data fifo find hash heap lifo link list push root sort type leaf tree null path node left less exch sink swim next swap	data exch fifo find hash heap leaf left less lifo link list sort tree null path node type root push sink swim next swap	data fifo lifo type hash heap link sort find list push root leaf null path tree exch left less node next sink swap swim	hash fifo data link leaf heap exch node lifo left find path list next less root sink swim null sort type tree push swap	data exch fifo find hash heap leaf left less lifo link list next node null path push root sink sort swap swim tree type
original	quicksort	mergesort	insertion	selection	merge BU	shellsort	sorted