Modern approach to packing more carrier frequencies within a given frequency band

→ orthogonal FDM

Conceptual similarity to linear algebra

3-D space: Given two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, they are orthogonal—i.e., perpendicular to each other—if, and only if,

$$x \circ y = x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$$

→ called dot product (or inner product)

→ 3-D: $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ are orthogonal

→ also basis of 3-D

→ called orthonormal if dot product with itself is 1

Lots of other orthogonal basis vectors

For example: $(5, 2, 0)$, $(2, -5, 0)$, $(0, 0, 1)$ are mutually orthogonal

$\rightarrow$ but not orthonormal

$\rightarrow$ how to make them orthonormal?

Relevance to networking:

In CDMA (code division multiple access)—for example, used by Sprint and Verizon for wireless cellular in the U.S.—$(5, 2, 0)$, $(2, -5, 0)$, $(0, 0, 1)$ are called codes

$\rightarrow$ one code per user

$\rightarrow$ 3-D codes: 3 users (say Bob, Mira, Steve)

Suppose each user wants to send a single bit

$\rightarrow$ Bob: 1, Mira: 0, Steve: 0

Bob's cell phone: send $1 \times (5, 2, 0)$ to base station (cell tower)

Mira's cell phone: send $-1 \times (2, -5, 0)$ to base station

Steve's cell phone: send $-1 \times (0, 0, 1)$

$\rightarrow$ common convention: 1 for bit 1, $-1$ for bit 0

Base station receives: $(3, 7, -1)$

$\rightarrow \big(1 \times (5, 2, 0)\big) + \big(-1 \times (2, -5, 0)\big) + \big(-1 \times (0, 0, 1)\big)$

How can base station find what bit Bob has sent?

Base station: compute the dot product of what it has received, $(3, 7, -1)$, and the code of Bob, $(5, 2, 0)$

$\rightarrow (5, 2, 0) \circ (3, 7, -1) = 15 + 14 + 0 = 29$

$\rightarrow$ positive: hence bit 1

$\rightarrow$ what's special about 29?

To find out what Mira has sent:

$\rightarrow (2, -5, 0) \circ (3, 7, -1) = 6 - 35 + 0 = -29$

$\rightarrow$ negative: hence bit 0

To find out what Steve has sent:

$\rightarrow (0, 0, 1) \circ (3, 7, -1) = 0 + 0 + 1 = -1$

$\rightarrow$ negative: hence bit 0

$\rightarrow$ why does this work?

Base station decoding Bob's bit: $(5,2,0) \circ (3,7,-1)$

Since $(3,7,-1) = \big(1 \times (5,2,0)\big) + \big(-1 \times (2,-5,0)\big) + \big(-1 \times (0,0,1)\big)$

$(5,2,0) \circ (3,7,-1)$ equals

$$\begin{aligned}\big(1 \times (5,2,0) \circ (5,2,0)\big) \;&+\; \big(-1 \times (5,2,0) \circ (2,-5,0)\big)\\ &+\; \big(-1 \times (5,2,0) \circ (0,0,1)\big)\end{aligned}$$

which equals $\big(1 \times (5,2,0) \circ (5,2,0)\big)$

$\rightarrow$ the two interference terms are nullified

$\rightarrow$ orthogonality!

Same holds when computing Mira's bit and Steve's bit

$\rightarrow$ CDMA has additional twists (discussed in wireless)

$\rightarrow$ but the above is essential idea

Back to orthogonal FDM (OFDM)

$\rightarrow$ key idea: use carrier waves that are orthogonal

Dot product of two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$

$$x \circ y = \sum_{i=1}^{n} x_i y_i$$

"Dot product" of two sinusoids $x(t) = \sin f_x t$ and $y(t) = \sin f_y t$

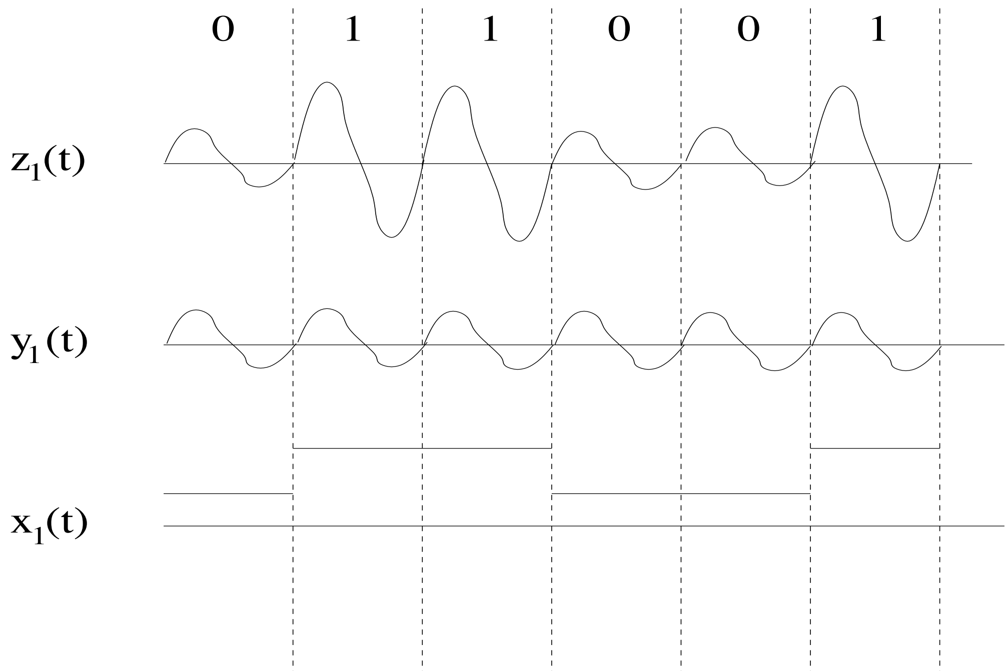
$$x(t) \circ y(t) = \int_{-\infty}^{\infty} (\sin f_x t)\,(\sin f_y t)\,dt$$

$\rightarrow$ again: just a sum of products

More generally: $x(t) \circ y(t) = \int_{-\infty}^{\infty} e^{if_x t} e^{-if_y t} dt$

$\rightarrow$ since Fourier transform involves complex sinusoids

User 1 uses carrier wave $y_1(t)$ to transmit bit stream (high and low) given by $x_1(t)$




Same for users 2 and 3

Suppose carrier waves $y_1(t)$, $y_2(t)$, $y_3(t)$ are orthogonal

Then receiver sees $z_1(t) + z_2(t) + z_3(t)$ which is

$$\sum_{k=1}^{3} x_k(t) y_k(t)$$

To decode what user 1 has sent, receiver computes dot product with $y_1(t)$

$$\begin{aligned} y_1(t) \circ \Big( \sum_{k=1}^{3} x_k(t) y_k(t) \Big) &= \sum_{k=1}^{3} x_k(t) \big( y_1(t) \circ y_k(t) \big) \\ &= x_1(t) \big( y_1(t) \circ y_1(t) \big) \\ &= x_1(t) \end{aligned}$$

$\rightarrow$ last steps holds if also orthonormal

But look at Fourier transform formula (lecture notes, part 2):

$\rightarrow$ just taking dot product!

Root of FDM solution: Joseph Fourier

$\rightarrow$ 18th century idea ("old technology")

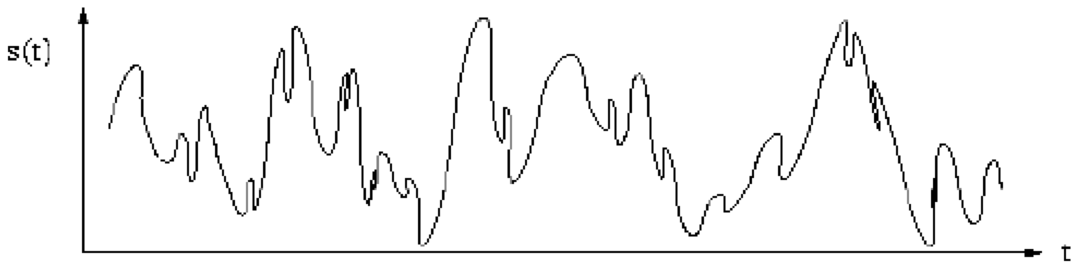
$\rightarrow$ Fourier analysis

$\rightarrow$ engineering bread and butter

$\rightarrow$ worth knowing for its own sake (great idea)

Fourier's key insight:

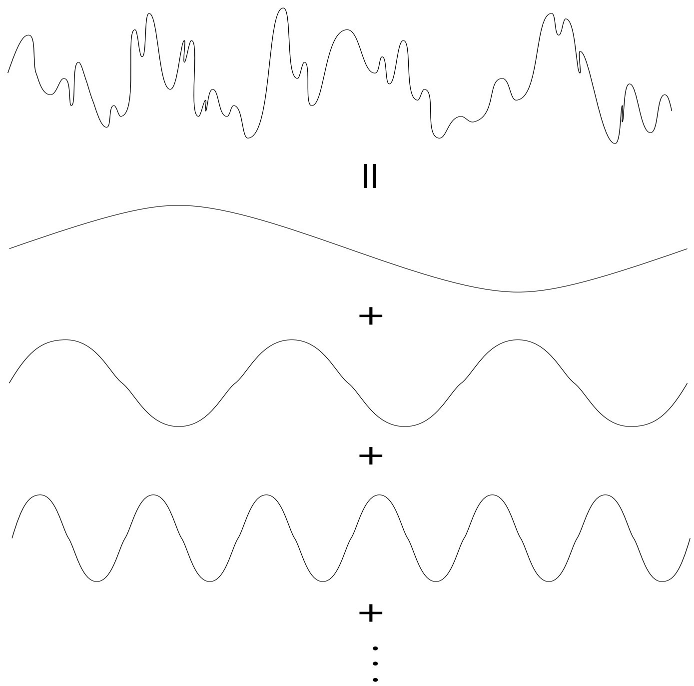
A complicated looking signal $s(t)$ whose shape (i.e., strength) varies over time




is just the sum of very simple building blocks

$\rightarrow$ sinusoids

Thus:



→ may require adding many sinusoids of different frequencies

→ key caveat: before adding sinusoid with frequency $f$, multiply its magnitude by a weight $\alpha$

→ therefore complicated looking $s(t)$ is just the weighted sum of sinusoids

Some conceptual similarity to periodic table and matter

$\rightarrow$ elements of periodic table: building blocks (sinusoids)

$\rightarrow$ matter: complicated looking signal

$\rightarrow$ $H_2O$: 2 parts H and 1 part O

$\rightarrow$ $C_8H_{10}N_4O_2$

$\rightarrow$ of course, matter has additional structure: not simple weighted sum

Obvious consequences:

- The value of the weight $\alpha_f$ of sinusoid $f$ indicates how important sinusoid $f$ is
- For example, if $\alpha_f = 0$ then sinusoid of frequency $f$ is not needed at all for creating $s(t)$
- Since there are an infinite number of frequencies (from 0 to $\infty$) the weighted sum may entail an infinite number of sinusoids

  $\rightarrow$ not relevant for FDM: why?

Fourier's key insight is accompanied by a key technical contribution:

If given some complicated looking signal $s(t)$, then for any sinusoid $f$ Fourier provides a simple formula for finding its weight $\alpha_f$

$\rightarrow$ a way to decompose into building blocks

Let's make use of Fourier's insight for enabling broadband communication: point-to-point link from $A$ to $B$

$A$ wishes to send 3 bits to $B$ in parallel using three carrier frequencies $f_1$, $f_2$, and $f_3$

$\rightarrow$ say 3 bits: 1, 0, 1

$\rightarrow$ carrier frequencies: 1 Hz, 2 Hz, 3 Hz

$\rightarrow$ how to do it?

Fourier's conceptual and technical contribution in more precise language (aka math):

1. Complicated looking signal $s(t)$ is weighted sum of sinusoids

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha_f e^{ift} df$$

$\rightarrow$ called Fourier expansion

$\rightarrow$ integral "$\int$" is just continuous sum

$\rightarrow$ recall: $e^{ift} = \cos ft + i \sin ft$

$\rightarrow$ Euler's formula

$\rightarrow$ why complex sinusoids involving $i = \sqrt{-1}$ ?

2. Given $s(t)$ and frequency $f$, how to find weight $\alpha_f$:

$$\alpha_f = \int_{-\infty}^{\infty} s(t) e^{-ift} dt$$

$\rightarrow$ another weighted sum

$\rightarrow$ called Fourier transform

$\rightarrow$ algorithm to compute Fourier transform quickly: fast Fourier transform (FFT)

$\rightarrow$ see algorithms book