1. Use F to denote the profit.

Without loss of generality, assume $n<200$.

indicator function.

$$\mathbb{1}_{\{x\in A\}} = \begin{cases} 1 & \text{if } x\in A \\ 0 & \text{if } x\notin A \end{cases}$$

If $X \le n$, $F = X\cdot 1 - 0.5(n-X)$

If $X>n$, $F = n\cdot 1 - 5(X-n)$

Thus, $F = \left(X - \frac{1}{2}(n-X)\right)\cdot \mathbb{1}_{\{X\le n\}} + \left(n-5(X-n)\right)\cdot \mathbb{1}_{\{X>n\}}$

linearity $\begin{cases} \mathbb{E}[a+b] = \mathbb{E}[a]+\mathbb{E}[b]. \\ \mathbb{E}[c\cdot X] = c\cdot \mathbb{E}[X]. \quad c \text{ is constant.} \end{cases}$

$$\mathbb{E}[F] = \mathbb{E}\left[\left(X-\tfrac{1}{2}(n-X)\right)\cdot \mathbb{1}\{X\le n\}\right] + \mathbb{E}\left[\left(n-5(X-n)\right)\cdot \mathbb{1}\{X>n\}\right] \quad (*)$$

$$\mathbb{E}[g(X)] = \begin{cases} \int g(x)\cdot f_X(x)\,dx & \text{if } X \text{ is continuous with pdf } f_X(x). \\ \sum g(x)\cdot \mathbb{P}(X=x) & \text{if } X \text{ is discrete.} \end{cases}$$

denote $g(x) = \left(x-\frac{1}{2}(n-x)\right)\cdot \mathbb{1}\{x\le n\}$

$\int_{-\infty}^{0} 0\,dt = 0$

$$\mathbb{E}\left[\left(X-\tfrac{1}{2}(n-X)\right)\cdot \mathbb{1}\{X\le n\}\right] = \mathbb{E}[g(X)] = \int_0^n \left(t-\tfrac{1}{2}(n-t)\right)\cdot \tfrac{1}{200}\,dt = \tfrac{1}{200}\left(\tfrac{3}{4}t^2 - \tfrac{1}{2}nt\right)\Big|_0^n = \tfrac{1}{800}n^2$$

$\int_{n}^{\infty} 0\,dt = 0$.

$$\mathbb{E}\left[\left(n-5(X-n)\right)\cdot \mathbb{1}\{X>n\}\right] = \int_n^{200} (6n-5t)\,\tfrac{1}{200}\,dt = \tfrac{1}{200}\left(6nt - \tfrac{5}{2}t^2\right)\Big|_n^{200} = \tfrac{1}{200}\left[\left(1200n - \tfrac{5}{2}\cdot 200^2\right) - \left(6n^2 - \tfrac{5}{2}n^2\right)\right]$$

$$= 6n - 500 - \tfrac{7}{400}n^2$$

Substitute (1)(2) in $(*)$:

$$\mathbb{E}[F] = \tfrac{1}{800}n^2 - \tfrac{7}{400}n^2 + 6n - 500$$

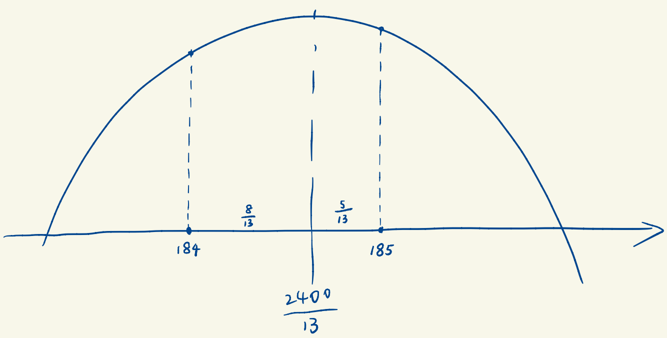
$$= -\tfrac{13}{800}n^2 + 6n - 500$$

$$= -\tfrac{13}{800}\left(n^2 - \tfrac{800}{13}\times 6n + \left(\tfrac{400}{13}\times 6\right)^2\right) - 500 + \left(\tfrac{400\times 6}{13}\right)^2 \times \tfrac{13}{800}$$

$$= -\tfrac{13}{800}\left(n - \tfrac{2400}{13}\right)^2 + \tfrac{18\times 400}{13} - 500$$

$$\begin{array}{r} 184 \\ 13\,\overline{)\,2400} \\ 13 \\ \hline 110 \\ 104 \\ \hline 60 \\ 52 \\ \hline 8 \end{array}$$

$\frac{2400}{13} = 184 + \frac{8}{13}$



$n \in \operatorname{argmax} \mathbb{E}[F] \Rightarrow n^* = 185$

$\mathbb{E}X = 100$

If $X \le n$, $F(X) = X \cdot 1 - 0.5(n-X)$

If $X > n$, $F(X) = n \cdot 1 - 5(X-n)$

$\underbrace{\mathbb{E}[F(X)]} \neq F[\mathbb{E}(X)]$

2. (a) for any $t>0$, $\mathbb{P}(T>t) = \mathbb{P}$(no calls in period $[0, t$ minutes$]$)

define X = the number of calls in period $[0, t$ minutes$]$.

then $X \sim \text{Pois}(\frac{3}{5}t)$, using X we can further translate the event.

$\mathbb{P}$(no calls in period $[0,t]$) $= \mathbb{P}(X=0)$

$$= \frac{e^{-\lambda} \cdot \lambda^0}{0!} = e^{-\frac{3}{5}t}$$

Thus, we get $\mathbb{P}(T>t) = e^{-\frac{3}{5}t}$ for any $t>0$.

$\Rightarrow \mathbb{P}(T \leq t) = 1 - e^{-\frac{3}{5}t}$ cdf.

pdf. $f_T(t) = \frac{d\,\mathbb{P}(T \leq t)}{dt} = \frac{3}{5} \cdot e^{-\frac{3}{5}t}$ for $t>0$.

exponential distribution.

(b) $\mathbb{P}(T>4) = e^{-\frac{3}{5}\cdot 4} = e^{-\frac{12}{5}} \approx 0.0907$

$= \int_4^{+\infty} f_T(t)\,dt$

Prop. If we define X = # of arrivals in period $[0, t]$.

$X \sim \text{Pois}(\lambda)$.

then, if we define X_1 = # of arrivals in period $[0, at]$ for some $a>0$.

then $X_1 \sim \text{Pois}(a\lambda)$

$\forall s.$ $[s, s+at]$.

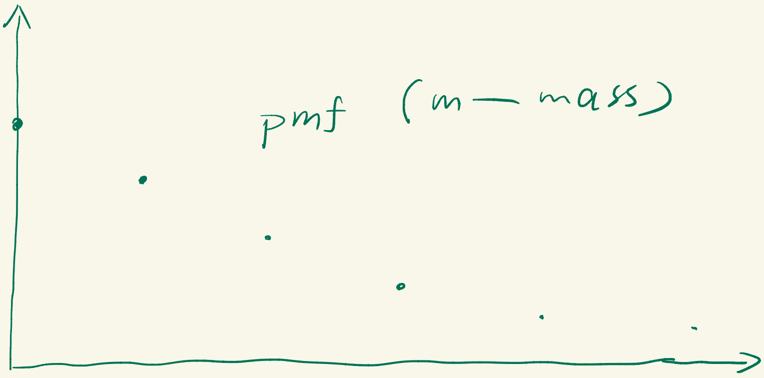
discrete distribution

Poisson distribution

$X \sim \mathrm{Pois}(\lambda)$.

$X \in \{0, 1, 2, \cdots\}$

$$\mathbb{P}(X = k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$



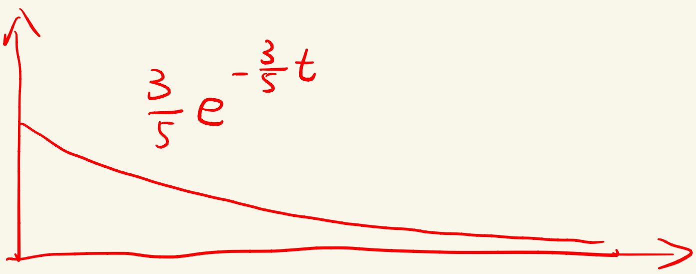
Exponential distribution — continuous distribution.

$T \sim \mathrm{Exp}(\lambda)$

$T \in \mathbb{R}_+ \cup \{0\}$

$$\mathbb{P}(T \leq t) = 1 - e^{-\lambda t}, \quad t \geq 0. \qquad \text{cdf.}$$

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0. \end{cases} \qquad \text{pdf.}$$



3. $96/\text{hour} = \frac{96}{60}/\text{minute} = 1.6/\text{minute}$

(a) (i) Define X to be the number of arrivals in a period of 2 minutes.

then $X \sim \text{Pois}(3.2)$

$$\mathbb{P}(X=5) = \frac{e^{-\lambda}\cdot\lambda^5}{5!} = \frac{e^{-3.2}\cdot(3.2)^5}{5!} \approx 0.1140$$

(ii) Define X_2 to be the number of arrivals in 45 seconds.

then $X_2 \sim \text{Pois}(1.2)$

$$\mathbb{P}(X_2>2) = 1-\mathbb{P}(X_2\le 2)$$

$$= 1-\mathbb{P}(X_2=0)-\mathbb{P}(X_2=1)-\mathbb{P}(X_2=2) \approx 0.1205$$

(iii) Define X_3 to be the number of arrivals in 1 min.

$X_3 \sim \text{Pois}(1.6)$

$\mathbb{P}$(the time to arrival of next customer is less than 1 min)

$$= \mathbb{P}(X_3 \ge 1)$$

$$= 1-\mathbb{P}(X_3=0) = 1-e^{-1.6} \approx 0.7981$$

(b)
$$\mathbb{P}(T>2.4 \mid T>1) = \frac{\mathbb{P}(\{T>\underbrace{2.4}_{t+s}\}\cap\{T>\underbrace{1}_{t}\})}{\mathbb{P}(T>\underbrace{1}_{t})} = \frac{\mathbb{P}(T>\underbrace{2.4}_{t+s})}{\mathbb{P}(T>\underbrace{1}_{t})} = \frac{e^{-2.4\times1.6}}{e^{-1.6}}$$

$$= e^{-1.4\times1.6} \approx 0.1065$$

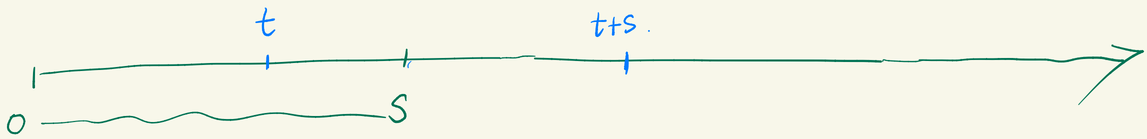
$e^{-\lambda(t+s)}$

$e^{-\lambda t}$

$= e^{-\lambda s}$.

In fact. exponential distribution has a "no-memory" property.

$$\mathbb{P}(T>t+s \mid T>t) = \mathbb{P}(T>s)$$



4. (a). $\begin{cases} \text{independent } \checkmark \\ \text{rate remains constant ?} \end{cases}$

(b) Define X to be the number of crashes will occur next year.

$X \sim Pois(2.75)$

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X < 2)$$
$$= 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) \approx 0.7603$$

(c) Define X_2 to be the number of crashes will occur within three months of the one mentioned in question.

then $X_2 \sim Pois(\frac{11}{16})$

$$\mathbb{P}(X_2 \geq 1) = 1 - \mathbb{P}(X_2 = 0) \approx 0.4972$$

$$\mathbb{P}(X=k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$

5.

X = the life, in hours, of a light bulb.

$X \sim N(185, \sigma^2)$

Now we want: $\mathbb{P}(X \geq 160) \geq 0.95$

Prop. If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$.

$$LHS = \mathbb{P}\left(\frac{X-185}{\sigma} \geq \frac{160-185}{\sigma} \right)$$

$$= \mathbb{P}\left(Z \geq -\frac{25}{\sigma} \right)$$

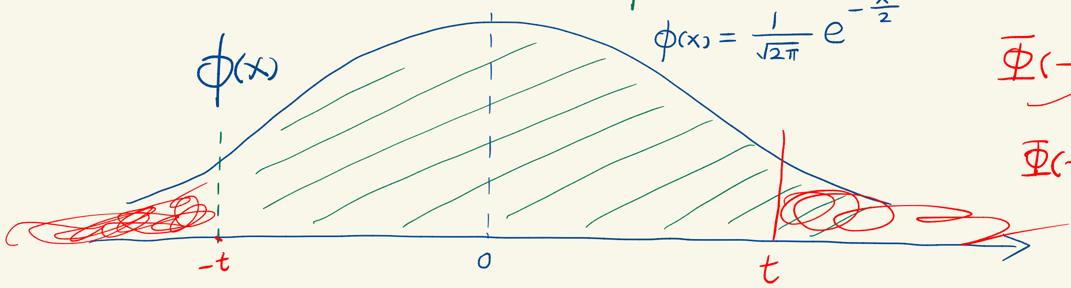
$$= \int_{-\frac{25}{\sigma}}^{+\infty} \phi(x)\, dx$$

$\phi(\cdot)$: pdf of standard normal

$\Phi(\cdot)$: cdf of standard normal.

Symmetricity of pdf $\phi(-x) = \phi(x)$.

p.d.f. $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$



$\Phi(-t) = 1 - \Phi(t)$

$\Phi(-t) + \Phi(t) = 1$.

the larger σ, the smaller $\frac{25}{\sigma}$, the large $-\frac{25}{\sigma}$. the smaller the probability.

check the table to find z_α s.t. $\mathbb{P}(Z \geq z_\alpha) = 0.95$

$$z_\alpha = -1.645$$

$$\frac{25}{\sigma} = 1.645 \implies \sigma = 15.20$$

$X \sim N(185, \sigma^2)$.

$$\Downarrow$$

$X - 185 \sim N(0, \sigma^2)$

$$\Downarrow$$

$\frac{X-185}{\sigma} \sim N(0, 1)$ —— standard normal.