1. $X \in S := \{1, 2, \cdots k\}$.

$$\mathbb{E}[e^{tX}] = \sum_{i=1}^{k} e^{ti} \cdot \mathbb{P}(X=i)$$

$$= \sum_{i=1}^{k} (e^t)^i \cdot \frac{1}{k}$$

$$= \frac{1}{k} \cdot \sum_{i=1}^{k} (e^t)^i$$

$$= \frac{1}{k} \cdot \frac{e^t(1-(e^t)^k)}{1-e^t}$$

set $a = e^t$, use :

$$1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$$

$(a \neq 1)$

3.(a) no car pass.

rate : 150 per hour

$\Longleftrightarrow$ $\frac{150}{60}$ per minute

$\Longleftrightarrow$ $\frac{300}{60}$ per 2 minute.

define $X = \#\{\text{cars pass in 2 minute}\}$

then $X \sim \text{Poisson}(5)$

$$\mathbb{P}(X=0) = \frac{e^{-\lambda} \cdot \lambda^0}{0!} = e^{-5} \approx 0.0067$$

(b) $Y = \#\{\text{cars pass in 1 minute}\}$

$Y \sim \text{Poisson}(2.5)$

For any $\lambda > 0$, if $Y \sim \text{Poisson}(\lambda)$

then $\mathbb{E}[Y] = \text{var}(Y) = \lambda$.

(c) $$\mathbb{P}(Y > 2.5) = \sum_{k=3}^{\infty} \mathbb{P}(Y=k) = 1 - \mathbb{P}(Y=0) - \mathbb{P}(Y=1) - \mathbb{P}(Y=2)$$

2. for $x \in \{0, 1, 2, \cdots\}$

$$\mathbb{P}(X=x) = \binom{n}{x} \pi^x (1-\pi)^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!} \cdot \left(\frac{\lambda}{n}\right)^x \cdot (1-\pi)^{n-x}$$

$$\mathbb{P}(Y=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$= n \cdot (n-1) \cdot \cdots \cdot (n-x+1)$. x number of factors.

$\lambda = n\pi$

thus, $$\frac{\mathbb{P}(X=x)}{\mathbb{P}(Y=x)} = \frac{n!}{(n-x)!} \cdot \frac{\cancel{\lambda^x}}{n^x} (1-\pi)^{n-x} \cdot e^{\lambda} \cdot \frac{1}{\cancel{\lambda^x}}$$

$$= \frac{n(n-1)\cdot \cdots \cdot (n-x+1)}{n \cdot n \cdot \cdots \cdot n} \cdot \left(1-\frac{\lambda}{n}\right)^{n-x} \cdot e^{\lambda}$$

$$= f(n) \cdot g(n) \cdot e^{\lambda}.$$

e^{λ} does not depend on n, we analyse $f(n)$ and $g(n)$ separately.

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} 1 \cdot \left(1-\frac{1}{n}\right) \cdot \left(1-\frac{2}{n}\right) \cdot \cdots \cdot \left(1-\frac{x-1}{n}\right)$$

$$= \lim_{n\to\infty}\left(1-\frac{1}{n}\right) \cdot \lim_{n\to\infty}\left(1-\frac{2}{n}\right) \cdot \cdots \cdot \lim_{n\to\infty}\left(1-\frac{x-1}{n}\right)$$

$$= 1$$

$$\lim_{n\to\infty} g(n) = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n} \cdot \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= e^{-\lambda} \cdot 1$$

thus, $$\lim_{n\to\infty} \frac{\mathbb{P}(X=x)}{\mathbb{P}(Y=x)} = 1.$$

since $\mathbb{P}(Y=x)$ does not depends on n.

$$\lim_{n\to\infty} \mathbb{P}(X=x) = \lim_{n\to\infty} \frac{\mathbb{P}(X=x)}{\mathbb{P}(Y=x)} \cdot \mathbb{P}(Y=x)$$

$$= \lim_{n\to\infty} \frac{\mathbb{P}(X=x)}{\mathbb{P}(Y=x)} \cdot \lim_{n\to\infty} \mathbb{P}(Y=x)$$

$$= 1 \cdot \mathbb{P}(Y=x).$$

Rules for taking limit.

$$\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

with this, use induction, can generalise to finite case.

$$\lim_{n\to\infty} \prod_{k=1}^{m} a_n^{(k)} = \prod_{k=1}^{m} \lim_{n\to\infty} a_n^{(k)} ,$$

but not infinte. Counter example :

$$\underbrace{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n}_{=\,e} \neq \underbrace{\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)^n}_{=\,1}$$

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$\forall x>0.\ \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\frac{n}{x}\cdot x}$$
$$=\left[\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right]^x$$
$$= e^x$$

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{n}\right)^{-n}$$
$$= \lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n\cdot\frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = 1\cdot\frac{1}{e} = e^{-1}$$

Prop. $(1-x)^k \geq 1-kx$ for any $x<1$. $k\in\mathbb{Z}_+$.

$k=1$ ✓

if for some k ✓. then $(1-x)^{k+1} = (1-x)^k(1-x)$
$$\geq (1-kx)(1-x)$$
$$= 1-(k+1)x+x^2$$
$$\geq 1-(k+1)x \quad \square.$$

use the prop: $\left(1-\frac{1}{n^2}\right)^n \geq 1-\frac{n}{n^2} = 1-\frac{1}{n}$

while $\left(1-\frac{1}{n^2}\right)^n \leq 1$ obviously.

thus. $1-\frac{1}{n} \leq \left(1-\frac{1}{n^2}\right)^n \leq 1$

let $n\to\infty$. $1 \leq \lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n \leq 1$

$$\Longrightarrow \lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n = 1.$$

the same argument could be used in proving:

$\forall x>0,\ \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x}.$

Combine these. we have:

$$\boxed{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \quad \text{for } x\in\mathbb{R}}$$