, $\Rightarrow$ and $\Rightarrow \cap$ intersect   or $\Rightarrow \cup$ union.

$A\cap(B\cup C)^c$ is Correct.

1 (a) $\{\text{only } A \text{ occurs}\} = \{A \text{ occurs}, B \text{ not}, C \text{ not}\} = A\cap B^c \cap C^c$.

(b) $\{\text{None occurs}\} = A^c \cap B^c \cap C^c$

(c) $\{\text{exactly one occurs}\} = \{\text{only } A \text{ occurs}\} \cup \{\text{only } B \text{ occurs}\} \cup \{\text{only } C \text{ occurs}\}$

$$= (A\cap B^c\cap C^c) \cup (B\cap A^c \cap C^c) \cup (C\cap A^c \cap B^c)$$

(d) $\{\text{at least two occurs}\} = \{\text{exatly 2 occur}\} \cup \{\text{exactly 3 occur}\}$

$= \{(A,B) \text{ or } (B,C) \text{ or } (C,A) \text{ occur, another not}\} \cup (A\cap B\cap C)$

$$= \left((A\cap B\cap C^c)\cup(B\cap C\cap A^c)\cup(A\cap C\cap B^c)\right) \cup (A\cap B\cap C)$$

$\cap(C\cup C^c)$

$(A\cap B)\cup(A\cap C)\cup(B\cap C)$

$$= \left((A\cap B)\cap(C\cup C^c)\right) \cup \left(A\cap C\cap(B\cup B^c)\right) \cup \left(B\cap C\cap(A\cup A^c)\right)$$

$$= (A\cap B\cap C)\cup(A\cap B\cap C^c)$$

(e) $\{\text{at least two occur}\} = \left((A\cap B\cap C^c)\cup(B\cap C\cap A^c)\cup(A\cap C\cap B^c)\right)$

A typical mistake:

$$\{\text{only } A \text{ occurs}\} \overset{?}{=} \{\underbrace{A \cap B^c \cap C^c}_{Y}\}$$

e.g. $S = \{1, 2, 3, 4\}$.

$Y = \{1, 2\}$.

$\{Y\} = \{\{1, 2\}\}$.

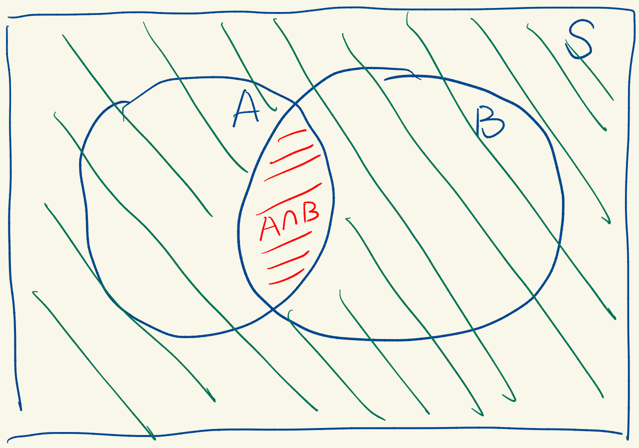
$$\{\varnothing\} \neq \varnothing$$

power set. of $\{1, 2, 3\}$ =

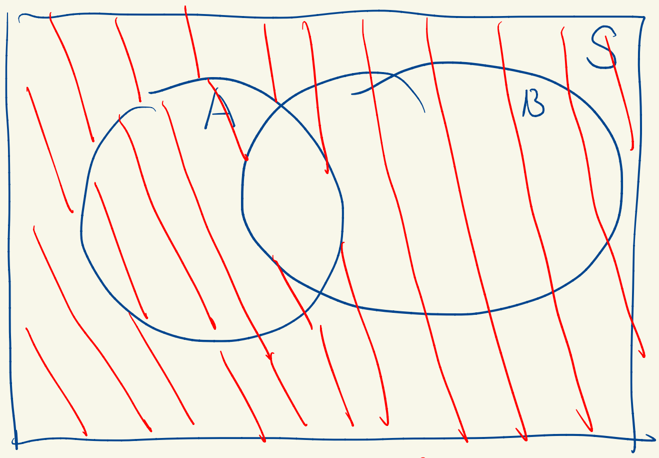
$$\{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

2. $(A \cap B)^c = A^c \cup B^c$.

$(A \cup B)^c = A^c \cap B^c$



green $= (A \cap B)^c$



red $= A^c \cup B^c$.

finite set: $\{1, 2\}$.

Countable set: $\mathbb{N} = \{0, 1, 2 \dots\}$

uncountable set: $\mathbb{R}$.

Prop. de Morgan's Law:

For any index set I, it holds that

$$\left(\bigcup_{\alpha \in I} A_\alpha\right)^c = \bigcap_{\alpha \in I} A_\alpha^c$$

$$\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} A_\alpha^c$$

pf. ① $\left(\bigcap_{i\in I} A_i\right)^c \subseteq \bigcup_{i\in I}(A_i^c)$

Suppose there is a x such that $x\in\left(\bigcap_{i\in I} A_i\right)^c$, but $x\notin \bigcup_{i\in I}(A_i^c)$ (towards contradiction)

$x\notin \bigcup_{i\in I}(A_i^c) \iff \forall i\in I,\ x\notin A_i^c \iff \forall i\in I,\ x\in A_i \implies x\in \bigcap_{i\in I} A_i$

which contradict to $x\in\left(\bigcap_{i\in I} A_i\right)^c$.

② $\bigcup_{i\in I}(A_i^c) \subseteq \left(\bigcap_{i\in I} A_i\right)^c$

Suppose there exist a $x\in \bigcup_{i\in I}(A_i^c)$ but $x\notin\left(\bigcap_{i\in I} A_i\right)^c$ (towards contradiction)

On one hand,

$x\in \bigcup_{i\in I}(A_i^c) \iff$ at least one $i\in I$ satisfying $x\in A_i^c$ — (*)

On the other hand,

$x\notin\left(\bigcap_{i\in I} A_i\right)^c \iff x\in \bigcap_{i\in I} A_i \iff \forall i\in I,\ x\in A_i$ — (**)

(*) contradicts to (**).

3. De Morgan's Law: $(A \cup B)^c = A^c \cap B^c = \{b, c\}$.

A? however, it is not possible to talk about A, since we do not know S

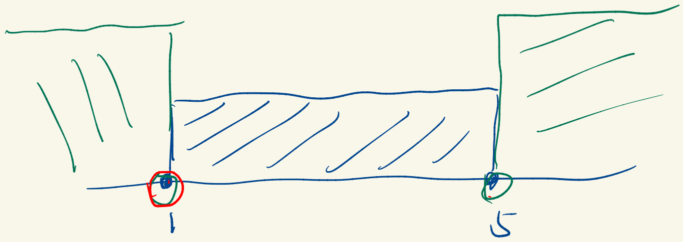
e.g. $S = \{a, b, c, d\}$. $S = \{a, b, c, d, e\}$

$A = \{d\}$. $A = \{d, e\}$

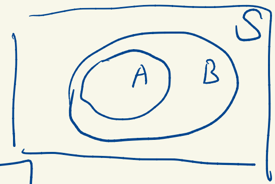
$B = \{a\}$. $B = \{a, e\}$

any solution assuming $S = \dots$ is wrong!

4.



(a) $A^c = \{x \mid x<1 \text{ or } x>5\} = (-\infty, 1) \cup (5, +\infty)$



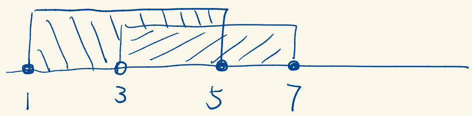
prop.

if $A \subseteq B$

then $A \cap B = A$

$A \cup B = B$

(b)



$A \cup B = [1,7] = \{x \mid 1 \le x \le 7\}$

(c) $C^c = \{x \mid x>0\}$

$B \subset C^c$, thus $B \cap C^c = B = (3,7] = \{x \mid 3 < x \le 7\}$

(d) $A^c \cap B^c \cap C^c \overset{\text{de-Morgan's}}{=} (A \cup B)^c \cap C^c \overset{\text{use (b)}}{=} [1,7]^c \cap C^c$

direct calculation $= \left((-\infty, 1) \cup (7, +\infty)\right) \cap (0, +\infty)$

$(A \cup B) \cap C$
$= (A \cap C) \cup (B \cap C)$

distributive law $= (-\infty, 1) \cap (0, +\infty) \cup (7, +\infty) \cap (0, +\infty)$

$= (0,1) \cup (7, +\infty) = \{x \mid 0 < x < 1 \text{ or } x > 7\}$.

(e) $(A \cup B) \cap C \overset{\text{use(b)}}{=} [1,7] \cap (-\infty, 0] = \emptyset$

$\{\emptyset\}$. wrong!

distinguish $\emptyset$ with $\{\emptyset\}$

$\{A^c \cap B^c \cap C^c\}$

Prop. Suppose X is a set. then $X \neq \{X\}$

$X = \{1,2\}$ $\emptyset$

$\{X\} = \{\{1,2\}\}$ $\{\emptyset\}$

5. (a) $(A\cap B)\cup(A\cap B^c)$

$= A\cap(B\cup B^c)$ (use distributive laws)

$= A\cap S$ ($B\cup B^c = S$ for any set B)

$= A$ ($A\cap S = A$ for any A)

$(A\cap B)\cap(A\cap B^c)$

$= A\cap(B\cap A)\cap B^c$

$= A\cap(A\cap B)\cap B^c$

$= (A\cap A)\cap(B\cap B^c)$

$= A\cap\emptyset$

$= \emptyset$

Ex. write down the property used in each step.

(b).

$$B = (B\cap A^c)\cup(B\cap A) \quad (4)$$

$$A\cup B = (A\cap B^c)\cup(A\cap B)\cup(A^c\cap B). \quad (5)$$

$P(B) = P(B\cap A^c) + P(B\cap A)$ —— (*)

$P(A\cup B) = P(A\cap B^c) + P(A\cap B) + P(A^c\cap B)$ —— (**)

Thus, (**) − (*) gives:

$P(A\cup B) - P(B) = P(A\cap B^c)$

$= P(A) - P(A\cap B)$

rearrange it. Q.E.D.

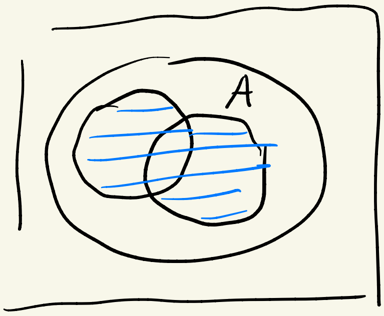
Prop. How to show two sets are equal. $C=D$.

$C \subseteq D$ and $D \subseteq C$

just like: How to define $a=b$, $a,b \in \mathbb{R}$

$a=b$ iff $a \leq b$ and $b \leq a$.

Use the prop. to prove $A = (A \cap B) \cup (A \cap B^c)$.



① $(A \cap B) \cup (A \cap B^c) \subseteq A$.

$A \cap B \subseteq A$

$A \cap B^c \subseteq A$.

then $(A \cap B) \cup (A \cap B^c) \subseteq A$

② $A \subseteq (A \cap B) \cup (A \cap B^c)$

for any $a \in A$, either $a \in B$ or $a \in B^c$ ($B \cup B^c = S$)

if $a \in B$, then $a \in A \cap B$ $\Rightarrow$ $a \in (A \cap B) \cup (A \cap B^c)$

else if $a \in B^c$, then $a \in A \cap B^c$