1. $Y = \#\{\text{blue balls}\}$.

$$\begin{cases} X, Y \in \mathbb{Z}_+ . \\ X + Y = 8 \\ X \leq 13 \\ Y \leq 5 \Rightarrow 8 - X \leq 5 \Rightarrow X \geq 3 . \end{cases} \Longrightarrow X \in \{3,4,5,6,7,8\} =: S$$

$S := \{3,4,5,6,7,8\}$.

$$\forall x \in S, \quad \mathbb{P}(X = x) = \mathbb{P}(X = x, Y = 8 - x)$$
$$= \frac{\binom{13}{x}\binom{5}{8-x}}{\binom{18}{8}}$$

2. Define X_i to be the mutiple which multiplied his fortune after i-th play.

$$X_i = \begin{cases} 2 & \text{if he wins} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X_i] = 2 \cdot \mathbb{P}(\text{he wins}) + \frac{1}{2} \cdot \mathbb{P}(\text{he loses})$$
$$= 2 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$$
$$= \frac{5}{4} = 1.25$$

$$\mathbb{E}[X_1 \cdots X_n] = \mathbb{E}[X_1] \cdots \mathbb{E}[X_n] = \left(\frac{5}{4}\right)^n$$

☆ $\mathbb{E}[g(X)] = \sum_{x \in S} g(x) \cdot \mathbb{P}(X = x)$

e.g. take $g(\cdot)$ to be $g(t) = t$: $\mathbb{E}[X] = \sum_{x \in S} x \cdot \mathbb{P}(X = x)$

$g(u) = u^2$: $\mathbb{E}[X^2] = \sum_{x \in S} x^2 \cdot \mathbb{P}(X = x)$.

☆ $\operatorname{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

3. X = the number of tosses needed to obtain the first head.

$X \in \{1, 2, \cdots\} =: S$

$$\forall x \in S,\ \mathbb{P}(X = x) = \mathbb{P}(\text{get tail in 1-st, 2nd, } \cdots \text{ (x-1)-th try, and get head in x-th try})$$

independence

$$= \prod_{i=1}^{x-1} \mathbb{P}(\text{get tail in i-th try}) \cdot \mathbb{P}(\text{get head in x-th try})$$

$$= \left(\frac{1}{2}\right)^{x-1} \cdot \left(\frac{1}{2}\right)$$

$$= \left(\frac{1}{2}\right)^{x}$$

We need to verify $\sum_{x \in S} \mathbb{P}(X = x) = 1$ which is $\sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^{x} = 1$.

By definition, $\sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^{x} = \lim_{n \to \infty} \sum_{x=1}^{n} \left(\frac{1}{2}\right)^{x}$

$$= \lim_{n \to \infty} \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)}{1 - \frac{1}{2}}$$

$$= \lim_{n \to \infty} 1 - \left(\frac{1}{2}\right)^{n+1}$$

$$= 1.$$

prop. $1 + a + a^2 + \cdots + a^n = \dfrac{1 - a^{n+1}}{1 - a}$ $(a \neq 1)$

if $|a| < 1$, $\lim_{n \to \infty} \dfrac{1 - a^{n+1}}{1 - a} = \dfrac{1}{1 - a}$

proof. Denote $f(n) = 1 + a + a^2 + \cdots + a^n$ ——— ①

$a \cdot f(n) = a + a^2 + \cdots + a^n + a^{n+1}$ ——— ②

① − ② :

$$f(n) - a \cdot f(n) = 1 - a^{n+1}$$

$$\Rightarrow \quad f(n) = \frac{1 - a^{n+1}}{1 - a}$$

3^{*}. A fair coin. solve for expected number of tosses to see the first head.

X !!

$$p(x) = \begin{cases} \frac{1}{2^x} & x = 1, 2, \cdots \\ 0 & \text{otherwise}. \end{cases}$$

$$\mathbb{E}[X] = \sum_{x=1}^{\infty} x \cdot \frac{1}{2^x} =: f$$

$$f = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{1}{2^3} + 4 \cdot \frac{1}{2^4} + \cdots \quad \text{——— } ①$$

$$\frac{1}{2} f = \qquad 1 \cdot \frac{1}{2^2} + 2 \cdot \frac{1}{2^3} + 3 \cdot \frac{1}{2^4} + \cdots \quad \text{——— } ②$$

① − ②

$$\frac{1}{2} f = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

$$= 1$$

$$f = 2$$

3^{**}. a fair 6-face die, what is the expected number of rolls to see all 6 faces?

3^{***}. a fair coin. what is the expected number of tosses to see both sides?

$\mathbb{E}[\#$ of tosses to see a new side $|$ have seen 0 different sides$] = 1$

$\mathbb{E}[\#$ of tosses to see a new side $|$ have seen 1 different side$] =$

X !!

$X \in \{1, 2, 3 \cdots\} = S, \forall x \in S$

$\mathbb{P}(X=1) = \mathbb{P}($ got a different in the 2nd $) = \frac{1}{2}$

$\mathbb{P}(X=2) = \mathbb{P}($ got an existing side in 2nd, a different side in the 3rd $) = \frac{1}{2} \cdot \frac{1}{2}$.

$\vdots$

$\mathbb{E}[\#$ toss to see all $] = \mathbb{E}[\#$ toss $0 \to 1] + \mathbb{E}[\#$ toss $1 \to 2]$

$= 1 + 2$

$= 3$.

$\mathbb{E}[\#$ rolls $] = \mathbb{E}[0 \to 1] + \mathbb{E}[1 \to 2] + \cdots + \mathbb{E}[5 \to 6]$.

$= 1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1}$

$= 6\left(1 + \frac{1}{2} + \cdots + \frac{1}{6}\right)$

4. $\mathbb{E}[2^X] = \sum_{x=1}^{\infty} 2^x \cdot \mathbb{P}(X=x)$

$g(t) = 2^t$

$\mathbb{E}[g(X)] = \sum_{x\in S} g(x)\cdot \mathbb{P}(X=x)$

$$= \sum_{x=1}^{\infty} 2^x \cdot \frac{1}{2^x}$$

$$= \sum_{x=1}^{\infty} 1$$

$$= \lim_{n\to\infty} \sum_{x=1}^{n} 1$$

$$= \lim_{n\to\infty} n \quad \text{does not exist.}$$

4*. How to prove the limit of a sequence does not exist mathematically?

"$a_n \longrightarrow a$" is defined as:

$\forall \varepsilon > 0, \ \exists n_0 \in \mathbb{N}, \ \forall n > n_0, \ |a_n - a| < \varepsilon$

take the negation:

"a_n does not converge to a" is defined as:

$\exists \varepsilon_0 > 0, \ \forall n \in \mathbb{N}, \ \exists n_1 > n, \ |a_n - a| \geqslant \varepsilon$

...

4** St. Petersburg paradox.

	tail.					head.
1.	2.	3.	4	$\cdots$	n	$n+1$
you me 1 1.	2^1 2^1	2^2 2^2	2^3 2^3		$2^{n-1}, 2^{n-1}$	$2^n, 2^n$
lose	lose	lose				
-1	-2	-2^2	-2^3	$\cdots$	-2^{n-1}	$+2^n$

$$\sum_{i=0}^{n-1} 2^i = \frac{1-2^n}{1-2} = 2^n - 1$$

St. Petersburg Paradox.

$$(-1) + (-2) + (-4) + \cdots + (-2^{n-1}) + 2^n$$

$$= -\left(\sum_{i=0}^{n-1} 2^i\right) + 2^n$$

$$= -\frac{2^n - 1}{2-1} + 2^n = 1$$

Stopping time. T is r.v. such that at each round, you can be sure about whether T is achieved.

e.g. $T = \{$ got the 3rd head $\}$.

$T = \{$ no head in the subsequent 3 times tosses $\}$.

$$\mathbb{E}[C_T] = \mathbb{E}[C_0].$$

5. $\sum_{x=0}^{\infty} p(x) = 1$ holds since it is a probability distribution.

$$\iff \sum_{x=0}^{\infty} \frac{e^{-\lambda}\cdot\lambda^x}{x!} = 1$$

$\boxed{0! = 1}$

$$\iff e^{-\lambda}\cdot\left(\sum_{x=0}^{\infty}\frac{\lambda^x}{x!}\right) = 1$$

$$\iff e^{-\lambda}\cdot\left(1+\sum_{x=1}^{\infty}\frac{\lambda^x}{x!}\right) = 1$$

Taylor expansion of e^λ

$$\iff \boxed{\sum_{x=1}^{\infty}\frac{\lambda^x}{x!}} = e^\lambda - 1$$

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x\cdot p(x) = 0 + p(1) + \sum_{x=2}^{\infty} x\cdot p(x)$$

$$= \frac{e^{-\lambda}\cdot\lambda}{1} + \sum_{x=2}^{\infty} x\cdot\frac{e^{-\lambda}\cdot\lambda^x}{x!} = e^{-\lambda}\cdot\lambda + (\lambda - e^{-\lambda}\cdot\lambda) = \lambda.$$

$$\sum_{x=2}^{\infty} x\cdot\frac{e^{-\lambda}\cdot\lambda^x}{x!} = e^{-\lambda}\cdot\sum_{x=2}^{\infty}\frac{\lambda^{x-1}\cdot\lambda}{(x-1)!}$$

$y = x-1$

$x = 2, 3, \cdots \downarrow y = 1, 2, \cdots$

$$= e^{-\lambda}\cdot\lambda\cdot\sum_{y=1}^{\infty}\frac{\lambda^y}{y!}$$

$$= e^{-\lambda}\cdot\lambda\cdot(e^\lambda - 1)$$

$$= \lambda - e^{-\lambda}\cdot\lambda$$

$$\sum_{i=0}^{10} f(i)$$

$\parallel$ $j = i$

$$\sum_{j=0}^{10} f(j)$$

$\parallel$ $\underline{k = j-2}$

$\downarrow$

$j = 0,\ k = -2$

$j = 10,\ k = 8$

$$\sum_{k=-2}^{8} f(k+2)$$

5*. Taylor expansion.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots \qquad \forall\, x \in \mathbb{R}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad x \in (-1, 1)$$

for smooth $f(x)$.

$$f'(x) = f^{(1)}(x) = \frac{d}{dx}(f(x))$$

$$f''(x) = f^{(2)}(x) = \frac{d}{dx}\left(\frac{d}{dx} f(x)\right)$$

$$\vdots$$

under regularity conditions, $\sum_{i=0}^{n} \frac{f^{(i)}(x)}{i!} \longrightarrow f(x)$. as $n \to \infty$.