☆ Why Poisson with large λ can be approximated by Normal ?

Prop. If $X_1 \sim Pois(\lambda_1)$, $X_2 \sim Pois(\lambda_2)$, X_1 is independent of X_2, then $X_1 + X_2 \sim Pois(\lambda_1+\lambda_2)$

Pf. For any $k \in \mathbb{Z}_+$,
$$\begin{aligned}
\mathbb{P}(X_1+X_2=k) &= \sum_{i=0}^{k} \mathbb{P}(X_1=i, X_2=k-i) \\
&= \sum_{i=0}^{k} \mathbb{P}(X_1=i)\cdot\mathbb{P}(X_2=k-i) \\
&= \sum_{i=0}^{k} \frac{e^{-\lambda_1}\cdot\lambda_1^i}{i!}\cdot\frac{e^{-\lambda_2}\cdot\lambda_2^{k-i}}{(k-i)!} \\
&= \sum_{i=0}^{k} \frac{e^{-(\lambda_1+\lambda_2)}\cdot\lambda_1^i\lambda_2^{k-i}}{i!(k-i)!} \\
&= \sum_{i=0}^{k} \frac{e^{-(\lambda_1+\lambda_2)}\cdot\lambda_1^i\lambda_2^{k-i}}{k!}\cdot\frac{k!}{i!(k-i)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}\cdot\sum_{i=0}^{k}\binom{k}{i}\lambda_1^i\lambda_2^{k-i} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}(\lambda_1+\lambda_2)^k.
\end{aligned}$$

Thm. (Central Limit Theorem).

X_i i.i.d. $\mathbb{E}X_i=\mu$. $var(X_i)=\sigma^2<\infty$. Then
$$\frac{\frac{1}{n}\sum_{i=1}^{n}X_i-\mu}{\sigma/\sqrt{n}} \xrightarrow{d.} N(0,1)$$

" $\xrightarrow{d.}$ " means "converge in distribution".

It would not be so helpful to deal with it too rigorously as we do not have enough tools.

For intuition, it is said that the distribution (say, the pdf/pmf and cdf) of $\frac{\frac{1}{n}\sum_{i=1}^{n}X_i-\mu}{\sigma/\sqrt{n}}$ will be more and more close to $N(0,1)$ as $n\to\infty$.

From the proposition: for a large λ, suppose $\lambda = n \in \mathbb{Z}_+$

we can see $Y \sim Pois(n)$ as $Y = X_1 + \cdots + X_n$ where $X_i \overset{i.i.d.}{\sim} Pois(1)$

for any $i = 1, \cdots, n$, $\mathbb{E}X_i = 1$ $Var(X_i) = 1$

by CLT. $$\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}X_i}{\sqrt{var(\frac{1}{n}\sum_{i=1}^{n} X_i)}} \xrightarrow{d.} N(0,1)$$

i.e. $$\frac{\frac{1}{n}\cdot Y - 1}{\frac{1}{\sqrt{n}}} \xrightarrow{d.} N(0,1)$$

recall $\lambda = n$, then this is: $$\frac{Y - \lambda}{\sqrt{\lambda}} \xrightarrow{d.} N(0,1)$$

$$Y - \lambda \xrightarrow{d.} N(0, \lambda)$$

$$Y \xrightarrow{d.} N(\lambda, \lambda)$$

A typical proof of CLT is via MGF. WLOG. set X_i i.i.d. mean 0. $var(X_i) = \sigma^2$.

$$\mathbb{E}[e^{t\frac{S_n}{\sqrt{n}\cdot\sigma}}] = \mathbb{E}[e^{t(\frac{X_1}{\sqrt{n}\cdot\sigma} + \cdots + \frac{X_n}{\sqrt{n}\cdot\sigma})}] = \mathbb{E}[\prod_{i=1}^{n} e^{t(\frac{X_i}{\sqrt{n}\cdot\sigma})}] = \prod_{i=1}^{n} \mathbb{E}[e^{t(\frac{X_1}{\sqrt{n}\cdot\sigma})}] = \left(\mathbb{E}[e^{t(\frac{X_1}{\sqrt{n}\cdot\sigma})}]\right)^n$$

$$\mathbb{E}[e^{t\cdot\frac{X_1}{\sqrt{n}\cdot\sigma}}] = \mathbb{E}\left[1 + t\cdot\frac{X_1}{\sqrt{n}\cdot\sigma} + \frac{(t\cdot\frac{X_1}{\sqrt{n}\cdot\sigma})^2}{2!} + \frac{(t\cdot\frac{X_1}{\sqrt{n}\cdot\sigma})^3}{3!} + \cdots\right]$$

$$= 1 + 0 + \frac{t^2}{2}\cdot\frac{var(X_1)}{n\sigma^2} + \frac{1}{n^{3/2}} f(t,n)$$

$$= 1 + \frac{t^2}{2n} + \frac{1}{n^{3/2}} f(t,n)$$ where $\limsup_{n\to\infty} f(t,n) \leq C$. for some $C > 0$.

Thus, $$\left(\mathbb{E}[e^{t\cdot\frac{X_1}{\sqrt{n}\cdot\sigma}}]\right)^n = \left(1 + \frac{t^2}{2n} + \frac{1}{n^{3/2}}\cdot f(t,n)\right)^n$$

$$= \left(1 + \frac{(\frac{t^2}{2})}{n}\right)^n + \sum_{k=1}^{n} \binom{n}{k} \left(\frac{1}{n^{3/2}} f(t,n)\right)^k \left(1 + \frac{(\frac{t^2}{2})}{n}\right)^{n-k}$$

$$\lim_{n\to\infty}\left(1 + \frac{(\frac{t^2}{2})}{n}\right)^n = e^{\frac{t^2}{2}}$$

$$\lim_{n\to\infty} | \boxed{} | = 0$$

$\underline{\mathbb{E}|X_1|^3} = \infty$ might happen!

for $X \sim N(0,1)$

$$\mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+2tx+t^2-t^2)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+t)^2} dx$$

$s = x+t$
$ds = dx$

$$\stackrel{\checkmark}{=} e^{-\frac{t^2}{2}} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} ds}_{=1}$$

$$= e^{-\frac{t^2}{2}}$$

Thm. If two r.v. has the same MGF, then they are identically distributed.

if we want to change the order to $\begin{cases}\text{take expectation and}\\ \text{do summation}\end{cases}$

In the Taylor expansion step, it is required that all $\mathbb{E}[|X|^k] < \infty$ for all $k \in \mathbb{Z}_+$, which is not the assumption in Central Limit Theorem. (only $\mathbb{E}X^2 < \infty$ is provided).

It could be solved by truncation and convergence theorem with careful treatment.

for X_i, we could define a sequence $\{Y_N\}_{N=1}^{\infty}$ by

$Y_N = X_i \wedge N = \min\{X_i, N\}$ (truncate it)

then $\{Y_N\}$ is a sequence of bounded random variables, thus are sub-Gaussian, with all moments being finite.

e.g. Fatou's Lemma.
Monotone Convergence Theorem.

1\. (a) $X \sim Pois(\lambda)$.

$\mathbb{E}X = \lambda$. $Var(X) = \lambda$. So we should use $\boxed{N(\lambda, \lambda)}$ to approximate $Pois(\lambda)$.

(b) $X \sim Pois(14)$

$Y \sim N(14, 14)$ could used for approximating X.

$\mathbb{P}(X = k) \approx \mathbb{P}(k - \frac{1}{2} < Y \leq k + \frac{1}{2})$

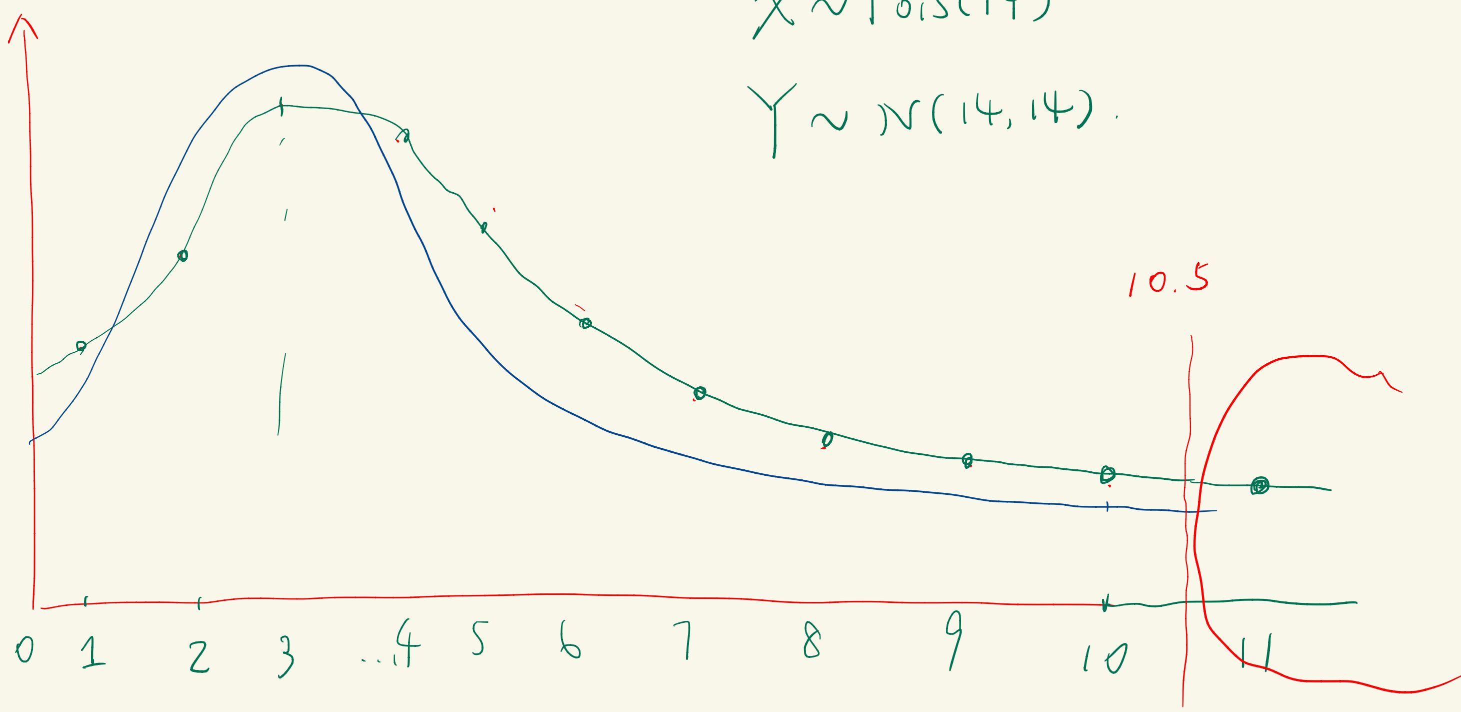
Thus. $\mathbb{P}(X > k) \approx \mathbb{P}(Y > k + \frac{1}{2})$

$\mathbb{P}(X \geq k) \approx \mathbb{P}(Y > k - \frac{1}{2})$

$$\mathbb{P}(X > 10) \approx \mathbb{P}(Y > 10.5)$$
$$= \mathbb{P}\left(\frac{Y-14}{\sqrt{14}} > \frac{10.5-14}{\sqrt{14}}\right)$$
$$\approx \mathbb{P}(Z > -0.94)$$
$$\approx \boxed{0.8264}$$

$X \sim Pois(14)$

$Y \sim N(14, 14)$.



Continuous correction.

2. (a) 8 are supporters of ST102 FC.

Define X to be the number of supporters in this four people.

then $X \in \{0,1,2,3,4\}$. $(\frac{12}{20})^4$

$$\mathbb{P}(X=0) = \frac{\#\{\text{pick } 4 \text{ non-Supporters}\}}{\#\{\text{pick any } 4\}} = \frac{\binom{8}{0}\binom{12}{4}}{\binom{20}{4}} = \frac{12\times 11\times 10\times 9}{20\times 19\times 18\times 17}$$

$$\mathbb{P}(X=1) = \frac{\#\{\text{pick } 4, \text{ where } 3 \text{ are non-Supporters}, 1 \text{ is Supporter}\}}{\#\{\text{pick any } 4\}} = \frac{\binom{8}{1}\binom{12}{3}}{\binom{20}{4}}$$

$$\mathbb{P}(X=2) = \frac{\binom{8}{2}\binom{12}{2}}{\binom{20}{4}}$$

$$\mathbb{P}(X=3) = \frac{\binom{8}{3}\binom{12}{1}}{\binom{20}{4}} = \frac{\frac{8\times 7\times 6}{\cancel{3\times 2\times 1}}\cdot\frac{12}{1}}{5\,\frac{\cancel{20}\times 19\times 18\times 17}{\cancel{4\times 3\times 2\times 1}}} = \frac{\overset{4}{\cancel{12}}\times 8\times 7\times\cancel{6}}{5\times 19\times\underset{3}{\cancel{18}}\times 17} = \frac{4\times 8\times 7}{5\times 19\times 17} = \frac{224}{1615} \approx \boxed{0.1387}$$

X follows Hypergeometric distribution.

(b) Define X to be the number of supporters.

$$\mathbb{P}(X=20) = \frac{\binom{0.4n}{20}\binom{0.6n}{20}}{\binom{n}{40}}$$

$$= \frac{\frac{0.4n\cdot(0.4n-1)\cdot\cdots\cdot(0.4n-19)}{20!}\cdot\frac{0.6n(0.6n-1)\cdot\cdots\cdot(0.6n-19)}{20!}}{\frac{n(n-1)\cdot\cdots\cdot(n-39)}{40!}}$$

$$= \frac{40!}{20!\cdot 20!}\cdot\left(\frac{0.4n}{n}\cdot\frac{0.4n-1}{n-1}\cdot\cdots\cdot\frac{0.4n-19}{n-19}\right)\cdot\left(\frac{0.6n}{n-20}\cdot\frac{0.6n-1}{n-21}\cdot\cdots\cdot\frac{0.6n-19}{n-39}\right)$$

$$\approx \binom{40}{20}\cdot 0.4^{20}\cdot 0.6^{20}$$

0.0553

$$f(x) = \frac{0.4n-x}{n-x} = \frac{0.4n-0.4x-0.6x}{n-x}$$

$$= 0.4 - \frac{0.6x}{n-x} \approx 0.4 \text{ if } n \text{ is large.}$$

By treating "sampling without replacement" as "sampling with replacement",

X approximately follows Binomial $(40, 0.4)$

$n\pi = 40 \times 0.4 = 16$ $\quad Y \sim \text{Pois}(16)$ could be used to approximate X.

$$\mathbb{P}(Y=20) = \frac{e^{-16} \cdot 16^{20}}{20!} \approx \boxed{0.056}$$ (avoid calculating $\binom{40}{20}$ in calculator.)

0.0557

(c)

Denote X as the number of supporters.

approximately, $X \sim \text{Binomial}(100, 0.4)$

$\mathbb{E}X = n\pi = 40 \quad \text{Var}(X) = n\pi(1-\pi) = 100 \times 0.4 \times 0.6 = 24$

Thus, $Y \sim N(40, 24)$ could approximate X well.

$$\mathbb{P}(X \geq 30) \approx \mathbb{P}(Y > 29.5)$$
$$= \mathbb{P}\left(\frac{Y-40}{\sqrt{24}} > \frac{29.5-40}{\sqrt{24}}\right)$$
$$\approx \mathbb{P}(Z > -2.14)$$
$$\approx \boxed{0.98382}$$

if $X \sim \text{Binomial}(n, p)$ with large n.

$$\mathbb{P}(X=k) \approx \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$
$$\parallel$$
$$\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$\mathbb{P}(X>k) = \int_k^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$
$$\parallel$$
$$\sum_{i=k}^{+\infty} \frac{e^{-\lambda} \cdot \lambda^{i+1}}{(i+1)!}$$

(d) If you have used a suitable approximation in any of the previous parts, explain why it is appropriate in each case.

(e) Comment on the differences, if any, in the assumptions and methods you have used when calculating the probabilities obtained above.

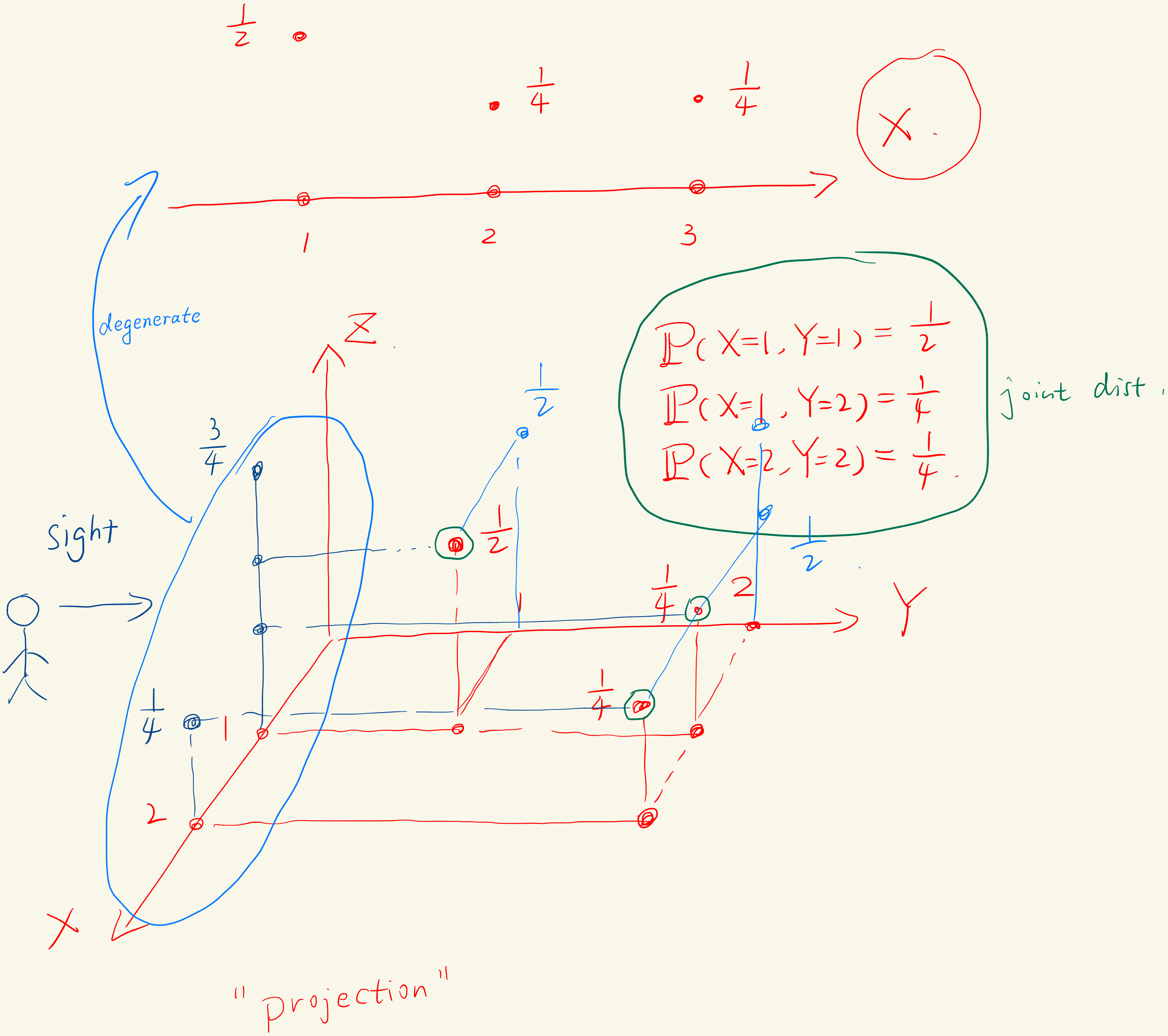
From the solution:

(d) The only approximation used here is the normal approximation to the binomial distribution, used in (b) and (c), and it is justified because:

- n is 'large' (although whether $n = 40$ in (b) can be considered large is debatable)
- the population is large enough to justify using the binomial in the first place
- $n\pi > 5$ and $n(1-\pi) > 5$ in each case.

(e) Three possible comments are the following.

- For (a), if we had (wrongly) used Bin(4, 0.4) then we would have obtained $P(X = 3) = 0.1536$, which is quite a long way from the true value (roughly 11%, proportionally) – we might have reached the wrong conclusion.
- For (b) and (c) it was important, to justify using the binomial distribution, that the population was 'large'.
- The true value (to 4 decimal places) for (c) is 0.9852, so the approximation obtained is pretty good – it is very unlikely that we might have reached the wrong conclusion.



X.
degenerate
Z
sight
$\mathbb{P}(X=1, Y=1) = \frac{1}{2}$
$\mathbb{P}(X=1, Y=2) = \frac{1}{4}$
$\mathbb{P}(X=2, Y=2) = \frac{1}{4}$
joint dist.
Y
X
"projection"

3. (a)

X	-1	0	1
p	0.45	0.3	0.25

Y	0	1
p	0.65	0.35

(b) $\mathbb{E}X = \sum_{x \in S} x \cdot \mathbb{P}(X=x)$

$= (-1) \times 0.45 + 0 + 1 \times 0.25 = \boxed{-0.2}$

$\mathbb{E}X^2 = \sum_{x\in S} x^2 \cdot \mathbb{P}(X=x)$

$= (-1)^2 \times 0.45 + 0 + 1^2 \times 0.25 = 0.7$

$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 0.7 - 0.04 = \boxed{0.66}$

$\mathbb{E}Y = 1 \times 0.35 + 0 = \boxed{0.35}$

$\mathbb{E}Y^2 = 1^2 \times 0.35 = 0.35$

$Var(Y) = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = 0.35 - 0.35^2 = \boxed{0.2275}$

(c) when $X=-1$. Y can take 0 and 1.

$\mathbb{P}(Y=0 \mid X=-1) = \frac{\mathbb{P}(Y=0, X=-1)}{\mathbb{P}(X=-1)} = \frac{0.3}{0.45} = \frac{2}{3}$

$\mathbb{P}(Y=1 \mid X=-1) = \frac{1}{3}$.

when $Y=0$, X can take $-1, 0, 1$.

$X \mid Y=0$	-1	0	1
p	$\frac{6}{13}$	$\frac{5}{13}$	$\frac{2}{13}$

(d) $\mathbb{E}_{Y|X}(Y \mid X=-1) = 0 \cdot \mathbb{P}(Y=0 \mid X=-1)$

$+ 1 \cdot \mathbb{P}(Y=1 \mid X=-1) = \boxed{\frac{1}{3}}$.

$\mathbb{E}_{X|Y}(X \mid Y=0) = (-1) \cdot \mathbb{P}(X=-1 \mid Y=0)$

$+ 0 \cdot \mathbb{P}(X=0 \mid Y=0)$

$+ 1 \cdot \mathbb{P}(X=1 \mid Y=0) = -\frac{4}{13}$

$\approx \boxed{-0.3077}$

$\mathbb{E}[g(X)] = \sum_{x\in S} g(x) \cdot \mathbb{P}(X=x)$.

$X \in \mathbb{R}^2 \quad X = \binom{x_1}{x_2}. \quad g: \mathbb{R}^2 \to \mathbb{R} \quad g(x_1, x_2) = x_1 x_2$.

(e) $\mathbb{E}[XY] = \sum_{(x,y)\in S} xy \cdot \mathbb{P}(X=x, Y=y)$

$= (-1) \times 1 \times \mathbb{P}(X=-1, Y=1) + 1 \times 1 \times \mathbb{P}(X=1, Y=1)$

$= (-1) \times 0.15 + 1 \times 0.15$

$= \boxed{0}$

$Cov(X,Y) = \mathbb{E}[XY] - (\mathbb{E}X) \cdot (\mathbb{E}Y)$

where $\mathbb{E}X = (-1) \times 0.45 + 1 \times 0.25 = -0.2$

$\mathbb{E}Y = 1 \times 0.35 = 0.35$

$Cov(X,Y) = 0 - (-0.2) \times (0.35) = \boxed{0.07}$

$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}} = \frac{0.07}{\sqrt{0.66} \times \sqrt{0.2275}} = \boxed{0.1807}$

(f) $\mathbb{P}(X>Y) = \mathbb{P}(X>Y, Y=0) + \mathbb{P}(X>Y, Y=1)$

$= \mathbb{P}(X>Y, Y=0) + 0$

$= \mathbb{P}(X=1, Y=0)$

$= \boxed{0.1}$

$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2$

$\mathrm{cov}(X,Y) = \mathbb{E}[XY] - (\mathbb{E}[X])(\mathbb{E}[Y])$.

set $Y=X$. $\mathrm{cov}(X,X) = \mathbb{E}[X\cdot X] - (\mathbb{E}X)(\mathbb{E}X)$

$= \mathrm{var}(X)$.

$\mathrm{cov}(\cdot,\cdot)$ is a bilinear function.

$\mathrm{cov}(z, ax+by) = \mathrm{cov}(z, ax) + \mathrm{cov}(z, by)$

$= a\cdot \mathrm{cov}(z,x) + b\cdot \mathrm{cov}(z,y)$.

$$\mathrm{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{var}(X_i) + 2\sum_{1\le i<j\le n} \mathrm{cov}(X_i, X_j)$$

$$\mathbb{P}(X^2>Y^2) = \mathbb{P}(X^2>Y^2,\ Y=0) + \mathbb{P}(X^2>Y^2,\ Y=1)$$
$$= \mathbb{P}(X^2>Y^2,\ Y=0) + 0$$
$$= \mathbb{P}(X^2>0,\ Y=0)$$
$$= \mathbb{P}(X=1,\ Y=0) + \mathbb{P}(X=-1,\ Y=0)$$
$$= \boxed{0.4}$$

(g)

Def. X and Y are independent if

$$\mathbb{P}(X=x, Y=y) = \mathbb{P}(X=x)\cdot\mathbb{P}(Y=y)$$

holds for all $x, y \in \mathbb{R}$.

take negation:

X and Y are independent: $\forall x, y \in \mathbb{R}$, $\mathbb{P}(X=x, Y=y) = \mathbb{P}(X=x)\mathbb{P}(Y=y)$

X and Y are not independent: $\exists x_0, y_0 \in \mathbb{R}$, $\mathbb{P}(X=x_0, Y=y_0) \neq \mathbb{P}(X=x_0)\mathbb{P}(Y=y_0)$

Remark. for those (x,y) such that $\mathbb{P}(X=x)=0$ or $\mathbb{P}(Y=y)=0$.

LHS $= 0 =$ RHS obviously.

So what really matters are those (x,y) s.t. $\mathbb{P}(X=x)\neq 0$ and $\mathbb{P}(Y=y)\neq 0$.

$$\mathbb{P}(X=1, Y=1) = 0.15$$
$$\mathbb{P}(X=1)\,\mathbb{P}(Y=1) = 0.25\times 0.35 = 0.0875 \neq \mathbb{P}(X=1, Y=1).$$

Thus, X, Y are not independent.

if $\mathbb{E}[XY] = \mathbb{E}[X]\cdot\mathbb{E}[Y]$.

is X and Y independent ?

not necessary!

e.g.
$$\begin{cases} X \sim \text{Unif}[0, 2\pi], \\ Y = \sin(X). \end{cases}$$

$$\mathbb{E}[X\cdot Y]=0, \qquad \mathbb{E}[Y]=0. \implies \mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$$

4.

(a) A natural restriction is that:

$$\begin{cases} X \le 2 \\ Y \le 2 \\ X+Y \le 2 \end{cases}$$

		X		
		0	1	2
Y	0	$\frac{1}{15}$	$\frac{4}{15}$	$\frac{2}{15}$
	1	$\frac{1}{5}$	$\frac{4}{15}$	×
	2	$\frac{1}{15}$	×	×

check: $\frac{1+4+2+3+4+1}{15} = 1$ ✓

$\mathbb{P}(X=0, Y=0) = \mathbb{P}(\text{two balls are blue})$

$$= \frac{\overset{\text{red}}{\binom{4}{0}}\overset{\text{green}}{\binom{3}{0}}\overset{\text{blue}}{\binom{3}{2}}}{\binom{10}{2}} = \frac{3\times 2}{10\times 9} = \frac{1}{15}$$

$$\mathbb{P}(X=0, Y=1) = \frac{\binom{4}{0}\binom{3}{1}\binom{3}{1}}{\binom{10}{2}} = \frac{3\times 3}{\frac{10\times 9}{2\times 1}} = \frac{1}{5}$$

$$\mathbb{P}(X=0, Y=2) = \frac{\binom{4}{0}\binom{3}{2}\binom{3}{0}}{\binom{10}{2}} = \frac{3\times 2}{10\times 9} = \frac{1}{15}$$

$$\mathbb{P}(X=1, Y=0) = \frac{\binom{4}{1}\binom{3}{0}\binom{3}{1}}{\binom{10}{2}} = \frac{4\times 3}{\frac{10\times 9}{2}} = \frac{4}{15}$$

$$\mathbb{P}(X=1, Y=1) = \frac{\binom{4}{1}\binom{3}{1}\binom{3}{0}}{\binom{10}{2}} = \frac{4}{15}$$

$$\mathbb{P}(X=2, Y=0) = \frac{\binom{4}{2}\binom{3}{0}\binom{3}{0}}{\binom{10}{2}} = \frac{4\times 3}{10\times 9} = \frac{2}{15}$$

(b) the number of blue balls in the sample.

(c) $\operatorname{cov}(X,Y) = \mathbb{E}[XY] - (\mathbb{E}X)(\mathbb{E}Y) = \frac{4}{15} - \frac{4}{5}\times\frac{3}{5} = \frac{20-12\times 3}{75} = \boxed{\frac{-16}{75} \approx -0.2133}$

$$\mathbb{E}[XY] = \sum_{(x,y)\in S} xy\cdot\mathbb{P}(X=x, Y=y) = 1\times 1\times\frac{4}{15} = \frac{4}{15}.$$

marginal distribution of X:

X	0	1	2
p	$\frac{1}{3}$	$\frac{8}{15}$	$\frac{2}{15}$

$$\mathbb{E}X = 1\times\frac{8}{15} + 2\times\frac{2}{15} = \frac{4}{5}$$

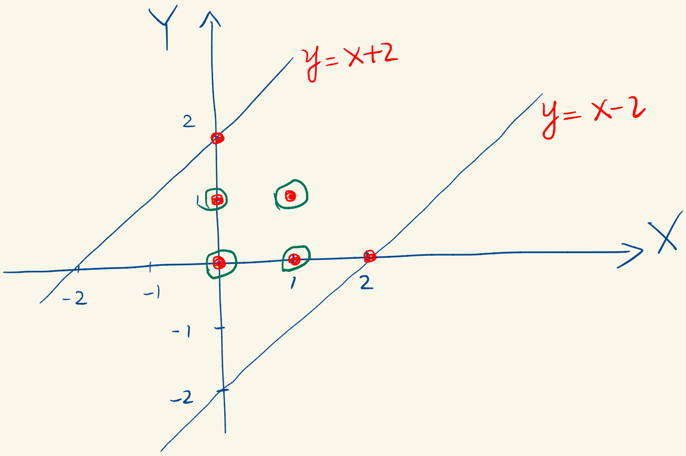
marginal distribution of Y:

Y	0	1	2
p	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

$$\mathbb{E}Y = 1\times\frac{7}{15} + 2\times\frac{1}{15} = \frac{3}{5}$$

(d) $\mathbb{P}(X=1 \mid -2 < X-Y < 2) = \dfrac{\mathbb{P}(\{X=1\} \cap \{-2 < X-Y < 2\})}{\mathbb{P}(-2 < X-Y < 2)} = \dfrac{\frac{8}{15}}{\frac{4}{5}} = \boxed{\frac{2}{3}}$

$-2 < X-Y < 2 \iff \begin{cases} Y < X+2 \\ Y > X-2 \end{cases}$



Numerator:

$\mathbb{P}(\{X=1\} \cap \{-2 < X-Y < 2\})$

$= \mathbb{P}(\{(1,0), (1,1)\})$

$= \frac{4}{15} + \frac{4}{15}$

$= \frac{8}{15}$

Denominator:

$\mathbb{P}(-2 < X-Y < 2)$

$= \mathbb{P}(\{(0,0), (0,1), (1,0), (1,1)\})$

$= \frac{1}{15} + \frac{1}{5} + \frac{4}{15} + \frac{4}{15}$

$= \frac{4}{5}$