Summary Sheet.

For exam purposes, one has to at least know how to use these formulas properly.

① $\mathbb{E}[g(X)] = \begin{cases} \sum_{x\in S} g(x)\cdot \mathbb{P}(X=x) \\ \int_{-\infty}^{\infty} g(x)\cdot f_X(x)\,dx \end{cases}$, where $f_X(x)$ is the pdf of X.

② $Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$Var(g(X)) = \mathbb{E}[(g(X))^2] - (\mathbb{E}[g(X)])^2$

③ For continuous-type X:

$\mathbb{P}(X \in (a,b)) = \int_a^b f_X(x)\,dx$ (a, b could be $-\infty, \infty$ here as long as the integral is well-defined.)

or $[a,b]$
$[a,b)$
$(a,b]$

because:

④ If X is continuous-type, $\forall x \in \mathbb{R}$, $\mathbb{P}(X=x)=0$.

⑤ cdf of X is defined as: $F_X(x) := \mathbb{P}(X \leq x)$

$$= \begin{cases} \sum_{y\in(-\infty,x]\cap S} \mathbb{P}(X=y) \\ \int_{-\infty}^{x} f_X(y)\,dy \end{cases}$$

$$\mathbb{E}[g(X)] = \begin{cases} \sum_{x \in S} g(x) \cdot \mathbb{P}(X=x) \\ \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx \end{cases}$$

$f_X(\cdot)$ is the pdf of X.

1. (a) $M_X(t) := \mathbb{E}[e^{tX}]$

$$= e^{t \cdot 1} \cdot P(X=1) + e^{t \cdot 0} \cdot P(X=0)$$

$$= e^{t} \cdot \pi + (1-\pi)$$

(b) $\mathbb{E}X = \left.\frac{\partial M_X(t)}{\partial t}\right|_{t=0} = (\pi \cdot e^{t})\Big|_{t=0} = \pi$

$\frac{\partial M_X(t)}{\partial t} = \mathbb{E}[\frac{\partial}{\partial t} e^{tX}] = \mathbb{E}[Xe^{tX}]$, Set $t=0$, get $\mathbb{E}[X]$

$$\frac{\partial^2 M_X(t)}{\partial t^2} = \pi \cdot e^{t}$$

$$\mathbb{E}[X^2] = \left.\frac{\partial^2 M_X(t)}{\partial t^2}\right|_{t=0} = \pi$$

$\frac{\partial^2 M_X(t)}{\partial t^2} = \mathbb{E}[\frac{\partial}{\partial t}(Xe^{tX})] = \mathbb{E}[X^2 \cdot e^{tX}]$, Set $t=0$, get $\mathbb{E}[X^2]$.

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \pi - \pi^2$$

$$= \pi(1-\pi)$$

do not suggest writing

$\frac{\partial}{\partial t} M_Y(0)$

$\left(\frac{\partial}{\partial t} M_Y(t)\right)\Big|_{t=0}$.

2. $\mathbb{E}[Y] = \left(\frac{\partial}{\partial t} M_Y(t)\right)\Big|_{t=0}$

$$\frac{\partial}{\partial t} M_Y(t) = \underbrace{e^{c(M_X(t)-1)}}_{f(t)} \cdot \underbrace{c \cdot \frac{\partial}{\partial t} M_X(t)}_{g(t)}$$

Set $t=0$ and notice $M_X(t)\big|_{t=0} = \mathbb{E}[e^{0 \cdot X}] = \mathbb{E}[1] = 1$.

$$\mathbb{E}[Y] = \left(\frac{\partial}{\partial t} M_Y(t)\right)\Big|_{t=0} = e^{c(1-1)} \cdot c \cdot \left(\frac{\partial}{\partial t} M_X(t)\right)\Big|_{t=0}$$

$$= 1 \cdot c \cdot \mathbb{E}X. = c\mu$$

$(u(x)v(x))' = u(x)v'(x) + u'(x)v(x)$

$$\frac{\partial^2}{\partial t^2} M_Y(t) = \frac{\partial}{\partial t}(f(t)\, g(t))$$

$$= f(t) \cdot \frac{\partial}{\partial t} g(t) + g(t) \cdot \frac{\partial}{\partial t} f(t)$$

$$= \left(e^{c(M_X(t)-1)} \cdot c \cdot \frac{\partial}{\partial t} M_X(t)\right) \cdot c \cdot \frac{\partial}{\partial t} M_X(t)$$

$$+ e^{c(M_X(t)-1)} \cdot \left(c \cdot \frac{\partial^2}{\partial t^2} M_X(t)\right)$$

$$\mathbb{E}[Y^2] = \left(\frac{\partial^2}{\partial t^2} M_Y(t)\right)\Big|_{t=0} = \left(e^{c(1-1)} \cdot c \cdot \mathbb{E}X\right) \cdot c \cdot \mathbb{E}X$$

$$+ e^{c(1-1)} \cdot c \cdot \mathbb{E}X^2$$

$$= c^2 \cdot \mu^2 + c \cdot (\mu^2 + \sigma^2)$$

$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2$

$\Downarrow$

$\mathbb{E}[X^2] = Var(X) + (\mathbb{E}X)^2$

$= \sigma^2 + \mu^2$.

$$Var(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$= c^2\mu^2 + c(\mu^2+\sigma^2) - (c\mu)^2$$

$$= c(\mu^2+\sigma^2)$$

3. Guess: $X = \begin{cases} 1 & \text{w.p. } \frac{2}{11} \\ 2 & \text{w.p. } \frac{4}{11} \\ 4 & \text{w.p. } \frac{4}{11} \\ 8 & \text{w.p. } \frac{1}{11} \end{cases}$

formally:

$P(X=1) = \frac{2}{11}$

$P(X=2) = \frac{4}{11}$

$P(X=4) = \frac{4}{11}$

$P(X=8) = \frac{1}{11}$

for $x \notin \{1,2,4,8\}$, $P(X=x) = 0$

The uniqueness of MGF?

If $M_X(t) = M_Y(t)$, can we conclude $X = Y$? ✗

or $\mathbb{P}(X \le x) = \mathbb{P}(Y \le x)$ ✓ identically distributed.

σ-algebra

measure theory.

e.g. toss a fair coin.

$X = \#\{\text{heads}\}$.

$Y = \#\{\text{tail}\}$.

$\mathbb{E}[e^{tX}]$.

i.i.d. = independent and identically distributed.

MGF: $M_X(t) = \mathbb{E}[e^{t \cdot X}]$

characteristic function $\varphi_X(t) := \mathbb{E}[e^{it \cdot X}]$ $i = \sqrt{-1}$.

Fourier transform

$Var(\underset{\sim}{X}) = \mathbb{E}[X^2] - (\mathbb{E}X)^2$

$Var(g(X)) = \mathbb{E}[(g(X))^2] - (\mathbb{E}[g(X)])^2$

4. (a) $\begin{cases} \int_{-\infty}^{\infty} f(x)\,dx = 1 \\ \mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{25}{36} \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 (\alpha x + \beta x^2)\,dx = \left(\alpha \cdot \frac{x^2}{2} + \beta \cdot \frac{x^3}{3}\right)\Big|_0^1 = \frac{\alpha}{2} + \frac{\beta}{3} = 1$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 (\alpha x^2 + \beta x^3)\,dx = \left(\alpha \cdot \frac{x^3}{3} + \beta \cdot \frac{x^4}{4}\right)\Big|_0^1 = \frac{\alpha}{3} + \frac{\beta}{4} = \frac{25}{36}$$

$$\Rightarrow \begin{cases} \frac{\alpha}{2} + \frac{\beta}{3} = 1 & \text{— ①} \\ \frac{\alpha}{3} + \frac{\beta}{4} = \frac{25}{36} & \text{— ②} \end{cases}$$

① $\Rightarrow$ $\alpha = 2 - \frac{2}{3}\beta$, substitute into ②:

$$\frac{2 - \frac{2}{3}\beta}{3} + \frac{\beta}{4} = \frac{25}{36}$$

$$\Leftrightarrow \frac{2}{3} - \frac{2}{9}\beta + \frac{1}{4}\beta = \frac{25}{36}$$

$$\Leftrightarrow 24 - 8\beta + 9\beta = 25$$

$$\Leftrightarrow \beta = 1$$

thus, $\alpha = 2 - \frac{2}{3}\beta = \frac{4}{3}$.

$$f(x) = \begin{cases} \frac{4}{3}x + x^2 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(b). $Var(\frac{1}{X}) = \mathbb{E}[\frac{1}{X^2}] - (\mathbb{E}[\frac{1}{X}])^2$

$$\mathbb{E}[\tfrac{1}{X}] = \int_{-\infty}^{\infty} \frac{1}{x} f(x)\,dx = \int_0^1 \left(\frac{4}{3} + x\right) dx = \left(\frac{4}{3}x + \frac{x^2}{2}\right)\Big|_0^1$$

$$= \frac{4}{3} + \frac{1}{2} = \frac{11}{6}$$

$$\mathbb{E}[\tfrac{1}{X^2}] = \int_{-\infty}^{\infty} \frac{1}{x^2} \cdot f(x)\,dx = \int_0^1 \left(\frac{4}{3} \cdot \frac{1}{x} + 1\right) dx$$

$$= \frac{4}{3} \int_0^1 \frac{1}{x}\,dx + 1$$

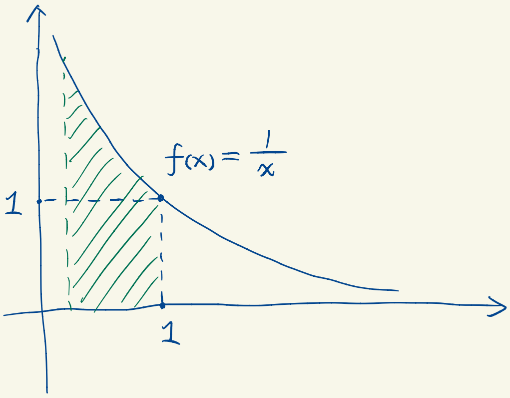
$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x}\,dx$

defined by

$\int \frac{1}{x}\,dx = \ln|x| + C.$

$\int_0^1 \frac{1}{x}\,dx = \ln|1| - \ln|0| = +\infty$.

why $\int_0^1 \frac{1}{x}\,dx$ does not exist ?



why $\int \frac{1}{x}\,dx = \ln x + C$?

$$(\ln x)' = \lim_{t\to 0^+} \frac{\ln(x+t)-\ln x}{t} = \lim_{t\to 0^+} \frac{\ln(1+\frac{t}{x})}{t} = \lim_{y\to 1^+} \frac{y}{(e^y-1)x}$$

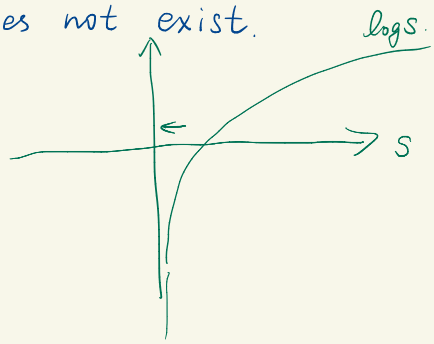
$\ln x$ is defined by e^x.

set $y=\ln(1+\frac{t}{x})$ $e^y = 1+\frac{t}{x}$

while $t\to 0^+$, $y\to 1^+$ $t=(e^y-1)x$

① $$\int_0^1 \frac{1}{x}\,dx = \lim_{s\to 0^+}\int_s^1 \frac{1}{x}\,dx = \lim_{s\to 0^+}(\log 1 - \log s)$$

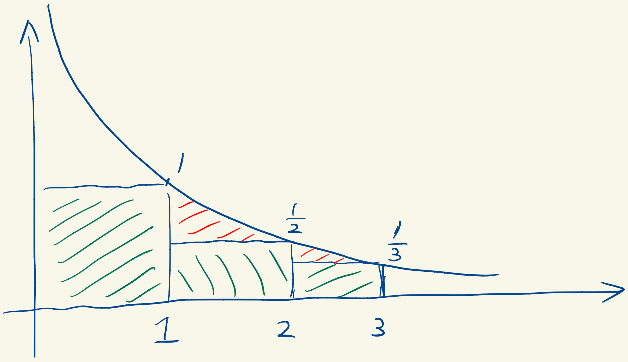
$$= \lim_{s\to 0^+} \log s \quad \text{does not exist.}$$



② $f(x)=\frac{1}{x}$ is symmetric w.r.t. $f(x)=x$.

thus, $\int_0^1 \frac{1}{x}\,dx = \int_1^\infty \frac{1}{x}\,dx + 1$.

it is equivalent to show $\int_1^\infty \frac{1}{x}\,dx$ does not exist. $\int_a^d = \int_a^b + \int_b^c + \int_c^d$



$$\int_1^{2^m} \frac{1}{x}\,dx = \sum_{k=0}^{m-1} \int_{2^k}^{2^{k+1}} \frac{1}{x}\,dx$$

$\frac{1}{x}$ decreasing

$$> \sum_{k=0}^{m-1} \int_{2^k}^{2^{k+1}} \frac{1}{2^{k+1}}\,dx$$

$\int_a^b c\,dx = c\cdot(b-a)$

$$= \sum_{k=0}^{m-1} \frac{1}{2^{k+1}}\cdot(2^{k+1}-2^k)$$

$$= \frac{m}{2} \qquad \text{———— } (*)$$

$\forall\, C>0$, take $n_0 = 2^{2\lceil C\rceil}$, $\forall\, n>n_0$, $\int_1^n \frac{1}{x}\,dx > \int_1^{n_0} \frac{1}{x}\,dx$

$$= \int_1^{2^{2\lceil C\rceil}} \frac{1}{x}\,dx$$

use $(*)$

$$> \frac{2\lceil C\rceil}{2}$$

$$= \lceil C\rceil$$

$$\geq C$$

Thus, $\forall\, C>0$, $\exists\, n_0\in\mathbb{N}$, $\forall\, n>n_0$, $\int_1^n \frac{1}{x}\,dx > C$.

$\int_1^{+\infty} \frac{1}{x}\,dx$ does not exist.

(C) $\mathbb{P}\left((2X-1)^2<\frac{1}{4} \,\middle|\, X<\frac{1}{2}\right) = \dfrac{\mathbb{P}\left((2X-1)^2<\frac{1}{4} \text{ and } X<\frac{1}{2}\right)}{\mathbb{P}\left(X<\frac{1}{2}\right)} = \dfrac{\frac{31}{192}}{\frac{5}{24}} = \dfrac{31}{40} = 0.7750$

numerator: $\begin{cases} X<\frac{1}{2} \Leftrightarrow 2X-1<0 \\ (2X-1)^2<\frac{1}{4} \Leftrightarrow -\frac{1}{2}<2X-1<\frac{1}{2} \end{cases}$

$x^2 \leq C \Rightarrow |x| \leq \sqrt{C}$

$\Downarrow$

$-\sqrt{C} \leq x \leq \sqrt{C}$

take intersection: $-\frac{1}{2} < 2X-1 < 0$

$\Downarrow \qquad \Downarrow$

$X>\frac{1}{4} \qquad X<\frac{1}{2}$

i.e. $X \in \left(\frac{1}{4}, \frac{1}{2}\right)$

$$\mathbb{P}\left(X\in\left(\tfrac{1}{4},\tfrac{1}{2}\right)\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} f(x)\,dx = \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\tfrac{4}{3}x + x^2\right)dx = \left(\tfrac{2}{3}x^2 + \tfrac{x^3}{3}\right)\Big|_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \tfrac{2}{3}\left(\tfrac{1}{4}-\tfrac{1}{16}\right) + \tfrac{1}{3}\left(\tfrac{1}{8}-\tfrac{1}{64}\right)$$

$$= \tfrac{2}{3}\cdot\tfrac{3}{16} + \tfrac{1}{3}\cdot\tfrac{7}{64}$$

$$= \tfrac{24+7}{192} = \tfrac{31}{192}$$

$$\mathbb{P}\left(X\in\left[0,\tfrac{1}{2}\right)\right) = \left(\tfrac{2}{3}x^2+\tfrac{x^3}{3}\right)\Big|_0^{\frac{1}{2}} = \tfrac{2}{3}\cdot\tfrac{1}{4} + \tfrac{1}{3}\cdot\tfrac{1}{8} = \tfrac{1}{3}\cdot\tfrac{5}{8} = \tfrac{5}{24}$$

For continuous X.

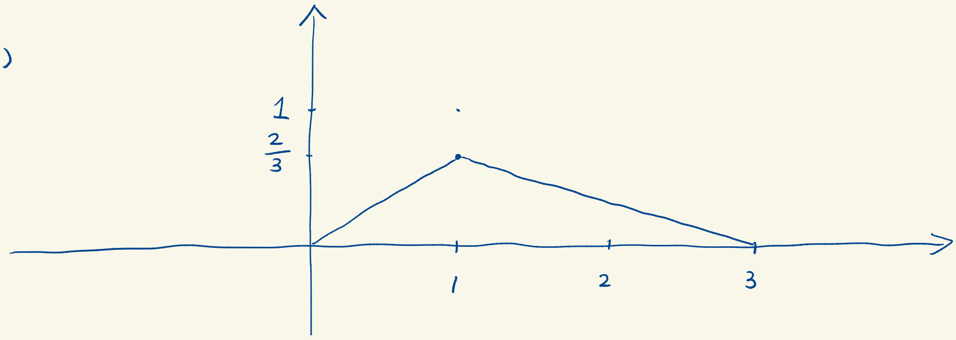
$$\mathbb{P}\left(X\in[a,b]\right) = \int_a^b f_X(x)\,dx$$

$\downarrow$

$\left.\begin{matrix}(a,b] \\ [a,b) \\ (a,b)\end{matrix}\right\}$ the same

If X continuous, $\forall x \in \mathbb{R}$, $\mathbb{P}(X=x)=0$

5. (a)



(b) for $x\in[0,1)$, $F(x)=\int_{-\infty}^{x} f(t)\,dt = \int_0^x \frac{2}{3}t\,dt = \frac{t^2}{3}\Big|_0^x = \frac{x^2}{3}$

$x\in[1,3)$, $F(x)=\int_{-\infty}^{x} f(t)\,dt = \int_0^1 \frac{2}{3}t\,dt + \int_1^x \frac{3-t}{3}\,dt$

$= (\frac{t^2}{3})\Big|_0^1 + (t-\frac{t^2}{6})\Big|_1^x$

$= \frac{1}{3} + (x-\frac{x^2}{6}) - (1-\frac{1}{6})$

$= x - \frac{x^2}{6} - \frac{1}{2}$

check: while $x=3$ $F(x) = 3 - \frac{9}{6} - \frac{1}{2} = 1$.

Thus, $F(x) = \begin{cases} 0 & x<0 \\ \frac{x^2}{3} & 0\le x<1 \\ x-\frac{x^2}{6}-\frac{1}{2} & 1\le x<3 \\ 1 & x\ge 3 \end{cases}$

(c) $\mathbb{E}X = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\cdot\frac{2}{3}x\,dx + \int_1^3 x\cdot\frac{3-x}{3}\,dx$

$= (\frac{2}{9}x^3)\Big|_0^1 + (\frac{x^2}{2}-\frac{x^3}{9})\Big|_1^3$

$= \frac{2}{9} + \left[(\frac{9}{2}-3)-(\frac{1}{2}-\frac{1}{9})\right]$

$= \frac{1}{3}+4-3 = \frac{4}{3}$

$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\cdot\frac{2}{3}x\,dx + \int_1^3 x^2\cdot\frac{3-x}{3}\,dx$

$= (\frac{2}{3}\cdot\frac{x^4}{4})\Big|_0^1 + (\frac{x^3}{3}-\frac{1}{3}\cdot\frac{x^4}{4})\Big|_1^3$

$= \frac{1}{6} + (9-\frac{27}{4}) - (\frac{1}{3}-\frac{1}{12})$

$= \frac{1}{6}+9-7 = \frac{13}{6}$

$\mathrm{Var}(X) = \mathbb{E}[X^2]-(\mathbb{E}X)^2 = \frac{13}{6}-\frac{16}{9} = \frac{39-32}{18} = \frac{7}{18}$

$\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)} \approx 0.6236$