1. Define $X := \#$ of the components have failed.

then $X \sim Bin(15, 0.4)$, $X \in \{0, 1, 2 \cdots 15\}$

$$\mathbb{P}(\underbrace{X \geq 3}_{A} \mid \underbrace{X \geq 2}_{B}) = \frac{\mathbb{P}(\{X\geq 3\} \cap \{X \geq 2\})}{\mathbb{P}(X \geq 2)} = \frac{\mathbb{P}(X = 3, 4, \cdots 15)}{\mathbb{P}(X = 2, 3, 4 \cdots 15)} = \frac{1 - \mathbb{P}(X = 0, 1, 2)}{1 - \mathbb{P}(X = 0, 1)} \approx 0.9780$$

$$\mathbb{P}(X = 0) = \binom{15}{0} 0.4^0 \cdot 0.6^{15} = (\frac{3}{5})^{15}$$

$$\mathbb{P}(X = 1) = \binom{15}{1} 0.4^1 \cdot 0.6^{14} = 15 \cdot \frac{2}{5} \cdot (\frac{3}{5})^{14}$$

$$\mathbb{P}(X = 2) = \binom{15}{2} \cdot 0.4^2 \cdot 0.6^{13} = \frac{15 \times 14}{2} \cdot (\frac{2}{5})^2 \cdot (\frac{3}{5})^{13}$$

$X \sim Bernoulli(p)$

X has p prob. to take 1, and $1-p$ prob. to take 0.

$$X = \begin{cases} 1 & w.p. \ p. \\ 0 & w.p. \ 1-p. \end{cases} \qquad x \in \{0, 1\}.$$

$Y \sim Binomial(n, \pi)$

Do independent Bernoulli experiment n times.

Y is the # of success. $\quad Y \in \{0, 1, 2 \cdots n\}$.

if $n=1$. $\quad Binomial(1, \pi) \overset{d.}{=} Bernoulli(\pi)$

prop. If $X_1 — X_n \overset{i.i.d.}{\sim} Bernoulli(p)$

then $\sum_{i=1}^{n} X_i \sim Binomial(n, p)$

There are $X_1, X_2 — X_n$, $Y_1, Y_2 — Y_m$.
if X_i is independent of $Y_1, Y_2 — Y_m$
for $i = 1, 2, 3 \cdots n$
Then, $f(X_1, \cdots, X_n)$ is independent of $Y_1, Y_2 — Y_m$.

2. $$\mathbb{P}(X_1 = 1 \mid \sum_{i=1}^{n} X_i = k) \overset{\text{definition}}{=} \frac{\mathbb{P}(\sum_{i=1}^{n} X_i = k, X_1 = 1)}{\mathbb{P}(\sum_{i=1}^{n} X_i = k)}$$

now we compute the numerator and the denominator separately.

$$\text{numator} = \mathbb{P}(\sum_{i=2}^{n} X_i = k-1, X_1 = 1) = \mathbb{P}(\sum_{i=2}^{n} X_i = k-1)\,\mathbb{P}(X_1 = 1) = \binom{n-1}{k-1} \pi^{k-1} (1-\pi)^{n-k} \cdot \pi \quad \text{——— (1)}$$

independence between $\sum_{i=2}^{n} X_i$ and X_1

$\sum_{i=2}^{n} X_i \sim Bin(n-1, \pi)$

$\begin{cases} \sum_{i=1}^{n} x_i = k \\ x_1 = 1 \end{cases}$ and $\begin{cases} \sum_{i=2}^{n} x_i = k-1 \\ x_1 = 1 \end{cases}$ has the same solution as linear equation systems.

$$\mathbb{P}(\sum_{i=1}^{n} X_i = k, X_1 = 1) = \mathbb{P}(\{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} X_i = k, X_1 = 1\})$$

probability is actually the measure of set,
use another way to describe the set, does not change the probability.

$$\text{denominator} = \mathbb{P}(\sum_{i=1}^{n} X_i = k) = \binom{n}{k} \pi^k (1-\pi)^{n-k} \quad \text{——— (2)}$$

$\sum_{i=1}^{n} X_i \sim Bin(n, \pi)$

Thus,

$$\mathbb{P}(X_1 = 1 \mid \sum_{i=1}^{n} X_i = k) \overset{\text{definition}}{=} \frac{\mathbb{P}(\sum_{i=1}^{n} X_i = k, X_1 = 1)}{\mathbb{P}(\sum_{i=1}^{n} X_i = k)} \overset{\text{(1) and (2) below}}{=} \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{(n-1)!}{(k-1)!\,(n-k)!}}{\frac{n!}{k!\,(n-k)!}} = \frac{k}{n}$$

factorial: $n! = n \times (n-1) \times \cdots \times 1$

permutation $P_n^m = \frac{n!}{m!} = n \times (n-1) \times \cdots (m+1)$

Conbination: $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$

Toss a fair coin. $X = \begin{cases} 1 & \text{if result is head} \\ 0 & \text{—— tail.} \end{cases}$ $S = \{ \text{head}, \text{tail} \}$

$\mathbb{P}$ is defined on S.

$$\mathbb{P}(X=1) = \mathbb{P}(\text{result is head}) = \mathbb{P}(\{\text{head}\}) = \frac{1}{2}$$

$$\begin{cases} \mathbb{P}(\{\text{head}\}) = \frac{1}{2}. \\ \mathbb{P}(\{\text{tail}\}) = \frac{1}{2} \end{cases}$$

Review distributions related to this seminar:

① Binomial distribution.

$X \sim \text{Bin}(n, \pi)$

$$\mathbb{P}(X=k) = \binom{n}{k} \pi^k (1-\pi)^{n-k}, \quad \text{for } k \in \{0, 1, 2 \cdots n\}.$$

② Bernoulli distribution

$Y \sim \text{Bern}(\pi) \quad (\pi \in [0,1])$

$\mathbb{P}(Y=1) = \pi$, $\mathbb{P}(Y=0) = 1-\pi$.

The next proposition reveals the relationship between Bernoulli and Binomial.

Prop. $Y_1, Y_2 \cdots Y_n$ is n independent copy of Y. $Y_i \overset{\text{i.i.d.}}{\sim} \text{Bern}(\pi)$.
$Z = Y_1 + Y_2 + \cdots + Y_n$. then $Z \sim \text{Bin}(n, \pi)$

$$\mathbb{P}(Z=k) = \mathbb{P}(Y_1 + \cdots + Y_n = k) = \mathbb{P}(k \text{ of them get result '1', the other get '0'})$$

$$= \binom{n}{k} (\mathbb{P}(\text{'1'}))^k (\mathbb{P}(\text{get '0'}))^{n-k}$$

It could also be proved by moment generating function. (see Q4.)

③ Poisson. $X \sim \text{Pois}(\lambda)$

$X \in \{0, 1 \cdots\}$.

$$\mathbb{P}(X=k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$1 = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$

3. Denote X_i as the number of typos at page i. $i=1,2\cdots n$.

then $X_i \sim Pois(\lambda)$

It is fair to assume X_i and X_j are independent for $i\neq j$.

$$\mathbb{P}(X_i > k) = \sum_{j=k+1}^{\infty} \mathbb{P}(X_i = j) = \sum_{j=k+1}^{\infty} \frac{e^{-\lambda} \cdot \lambda^j}{j!}$$

$\mathbb{P}(X_i = k+1) + \mathbb{P}(X_i = k+2) + \cdots$

denote $\pi = \mathbb{P}(X_i > k)$ which does not depend on i.

then each page has probability π to have typos more than k.

Define $Y = \#\{X_i : X_i > k, i=1,2\cdots n\}$

Then $Y \sim Bin(n, \pi)$

$$\mathbb{P}(Y \geq m) = \sum_{j=m}^{n} \binom{n}{m} \pi^m (1-\pi)^{n-m}$$

Binomial Theorem.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

e.g. $(a+b)^2 = a^2 + 2ab + b^2$
$= \binom{2}{0} a^2 b^0 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^0 b^2$

X takes values in S.
$\mathbb{E}[g(X)] = \sum_{x \in S} g(x) \cdot \mathbb{P}(X=x)$

4. (a) $X \sim Bin(n, \pi)$ has the pmf:

$$\mathbb{P}(X=k) = \binom{n}{k} \pi^k (1-\pi)^{n-k} \quad \text{for } k=0,1,\cdots n$$

$\Rightarrow S = \{0,1,2 \cdots n\}$

Then, $M_X(t) = \mathbb{E}e^{tX} = \sum_{k=0}^{n} e^{tk} \cdot \mathbb{P}(X=k)$

$$= \sum_{k=0}^{n} e^{tk} \cdot \binom{n}{k} \pi^k (1-\pi)^{n-k}$$

$$= \sum_{k=0}^{\infty} \binom{n}{k} (\underbrace{\pi \cdot e^t}_{a})^k \underbrace{(1-\pi)}_{b.}^{n-k}$$

$$= (\underbrace{\pi e^t}_{a} + \underbrace{1-\pi}_{b})^n \qquad (a+b)^n.$$

(b) $\dfrac{\partial M_X(t)}{\partial t} = n \cdot (e^t \pi + 1 - \pi)^{n-1} \cdot \pi e^t$

Set $t=0$. $\left.\dfrac{\partial M_X(t)}{\partial t}\right|_{t=0} = n\pi \cdot (\pi + 1 - \pi)^{n-1} \cdot e^0 = n\pi.$

Thus, $\mathbb{E}X = n\pi$

$$\mathbb{E}X^2 = \sum_{k=0}^{n} k^2 \mathbb{P}(X=k)$$
$g(x) = x^2$

$$\frac{\partial^2 M_X(t)}{\partial t^2} = n \cdot (e^t \pi + 1 - \pi)^{n-1} \cdot \pi e^t + n(n-1) \cdot (e^t \pi + 1 - \pi)^{n-1} \cdot (\pi e^t)^2$$

Set $t=0$, $\left.\dfrac{\partial^2 M_X(t)}{\partial t^2}\right|_{t=0} = n\pi + n(n-1) \cdot \pi^2$

Thus, $\mathbb{E}X^2 = n\pi + n(n-1)\pi^2$

$$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = n\pi + n(n-1)\pi^2 - n^2\pi^2$$
$$= n\pi - n\pi^2$$
$$= n\pi(1-\pi)$$

(a)* another way to do (a)

$$X = Y_1 + Y_2 + \cdots + Y_n.$$

$$Y_i \overset{i.i.d.}{\sim} Bern(\pi). \qquad X \sim Bin(n, \pi)$$

$$M_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}[e^{t(Y_1 + \cdots + Y_n)}]$$

$$= \mathbb{E}[e^{tY_1} \cdot e^{tY_2} \cdot \cdots \cdot e^{tY_n}]$$

independance

$$= \mathbb{E}[e^{tY_1}] \cdot \mathbb{E}[e^{tY_2}] \cdot \cdots \cdot \mathbb{E}[e^{tY_n}]$$

identically distributed.

$$= (\mathbb{E}[e^{tY_1}])^n$$

$$Y_1 = \begin{cases} 1 & w.p.\ \pi \\ 0 & w.p.\ (1-\pi). \end{cases}$$

$$\mathbb{E}[e^{tY_1}] = e^{t \cdot 1} \cdot \mathbb{P}(Y_1 = 1) + e^{t \cdot 0} \cdot \mathbb{P}(Y_1 = 0)$$

$$= e^t \cdot \pi + 1 - \pi.$$

Thus $M_X(t) = (\pi e^t + 1 - \pi)^n$

5. (a) Poisson distribution.

① " on average "

② independent events.

(b) Define $X_1 = \#$ of icebergs calved in 5 weeks.

then $X_1 \sim \text{Poisson}(2)$.

Prop. If $X_1 = \#$ of $\cdots$ in period with length t, $X_1 \sim \text{Pois}(\lambda)$

then, for any positive value a.

Define $X_2 = \#$ of $\cdots$ in period with length at, $X_2 \sim \text{Pois}(a\lambda)$

Define $X_2 = \#$ of icebergs fall into the sea in next 3 weeks.

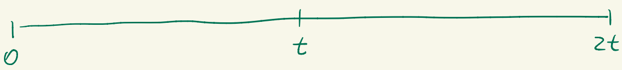
then $X_2 \sim \text{Pois}\left(\frac{6}{5}\right)$

$\mathbb{P}$(no ice is calved in next 3 weeks)

$= \mathbb{P}(X_2 = 0)$

$= \dfrac{e^{-\lambda} \cdot \lambda^0}{0!} = e^{-\lambda} = e^{-\frac{6}{5}} \approx 0.3012$

(c) still $e^{-\frac{6}{5}} \approx 0.3012$



events are independent, what happen in $[0, t]$ does not affect $[t, 2t]$.

memoriless property:

In two disjoint periods, the counts of events are independent.

(d) Define $X_3 = \#$ of icebergs calved in the next four weeks.

then $X_3 \sim \text{Pois}(\frac{8}{5})$

$$\mathbb{P}(\underline{\qquad\qquad}) = \mathbb{P}(X_3 = 5)$$
$$= \frac{e^{-\lambda} \cdot \lambda^5}{5!} = \frac{e^{-\frac{8}{5}} \cdot (\frac{8}{5})^5}{5 \times 4 \times 3 \times 2} \approx 0.0176$$

(e) Define $X_4 = \#$ of ______________ in the four-week period after the next four weeks.

$X_4 \sim \text{Pois}(\frac{8}{5})$

Independent.

$$\mathbb{P}(\underline{\qquad\qquad}) = \sum_{k=6}^{\infty} \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$
$$= 1 - \sum_{k=0}^{5} \frac{e^{-\frac{8}{5}} \cdot (\frac{8}{5})^k}{k!}$$

Define $X_5 = \#$ of icebergs calved in the next 8 weeks,

$X_4 = \#$ ______________________ 4 weeks.

$$\mathbb{P}(X_5 = 10 \mid X_4 = 5)$$
$$= \frac{\mathbb{P}(X_5 = 10, X_4 = 5)}{\mathbb{P}(X_4 = 5)} = \mathbb{P}(X_5 - X_4 = 5).$$

$X_5 - X_4 \sim \text{Poisson}(\frac{8}{5}).$

numerator: $\mathbb{P}(X_5 = 10, X_4 = 5)$

$$= \mathbb{P}(X_4 = 5, X_5 - X_4 = 5)$$
$$= \mathbb{P}(X_4 = 5) \cdot \mathbb{P}(X_5 - X_4 = 5)$$