Summary

Q1, Q2, Q3 are about classic probability. (all outcomes happen with the same probability)

$$P\{A\} = \frac{\#\{A\}}{\#\{\text{all outcomes}\}}$$

typically $\#\{\text{all outcomes}\}$ is easy, then the problem becomes a counting problem.

methods for counting: ① list all outcomes, count one by one (according to whether they satisfy 'A')
② detect 'must-have' structure, then look at the freedom variable.
⋮

Q4. Basic proof in probability.

The more rigorous your answers are, the less likely will you lose marks.

Try your best to mention why you can proceed each step (which theorem/proposition/corollary)
It is also extremely helpful to think why you proceed towards this direction.

Q5. Get familiar with conditional probability and the law of total probability.

about notation "#":

cardinality of a set.

$\#\{1,2,3\} = 3$.

$\#\{a,b\} = 2$

1. $P = \dfrac{\#\{\text{satisfying condition } A\}}{\#\{\text{all possible outcomes}\}} = \dfrac{\binom{2}{2}\cdot\binom{13}{4}}{\binom{15}{6}} = \dfrac{6\times5\times4\times\cancel{3\times2\times1}}{15\times14\times13\times\cancel{12\times11}\times10}\cdot\dfrac{13\times\cancel{12\times11}\times10}{4\times\cancel{3\times2}\times1}$

$= \dfrac{6\times5}{15\times14} = \dfrac{1}{7}$ $\qquad \dfrac{2}{14}$ X

Simplify fractions

1^*. 30 bulbs. 11 are blue

9 are red.

10 are yellow.

choose 8, solve for $\mathbb{P}$(exactly 2 are blue, 3 are red)

$= \mathbb{P}$(2 are blue, 3 are red, $\underset{\substack{\uparrow \\ 8-2-3}}{3}$ are yellow)

$$= \frac{\binom{11}{2}\cdot\binom{9}{3}\cdot\binom{10}{3}}{\binom{30}{8}}$$

$= \cdots$

☆ From the formula, think of:

why $\binom{n}{m} = \dfrac{n!}{m!\,(n-m)!}$ is always an integer?

2. If we directly calculate the probability:

$$\mathbb{P}(|X_1-X_2|>2) = \mathbb{P}(|X_1-X_2|=3)+\mathbb{P}(|X_1-X_2|=4)+\cdots+\mathbb{P}(|X_1-X_2|=19)$$

too many terms to calculate!

use $\mathbb{P}(A)=1-\mathbb{P}(A^c)$.

$$\mathbb{P}(|X_1-X_2|>2) = 1-\mathbb{P}(|X_1-X_2|\leq 2)$$

$$\mathbb{P}(|X_1-X_2|\leq 2) = \mathbb{P}(|X_1-X_2|=0)+\mathbb{P}(|X_1-X_2|=1)+\mathbb{P}(|X_1-X_2|=2)$$

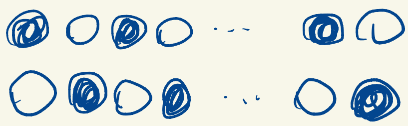
$$= 0 + \frac{\#\{(X_1,X_2) \mid |X_1-X_2|=1\}}{\binom{20}{2}} + \frac{\#\{(X_1,X_2) \mid |X_1-X_2|=2\}}{\binom{20}{2}}$$

$$\#\{(X_1,X_2) \mid |X_1-X_2|=1\} = \#\{(1,2) \sim (19,20)\} = 19.$$

$$\#\{(X_1,X_2) \mid |X_1-X_2|=2\} = \#\{(1,3),(2,4),\cdots(18,20)\} = 18$$

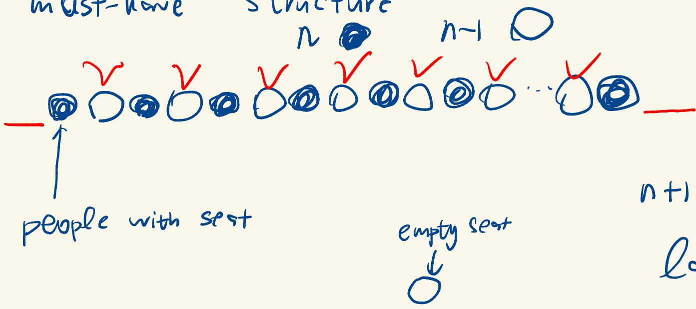
3. $\mathbb{P}$(no two people get adjacent seats) $= \dfrac{\#\{\text{ satisfying }\}}{\#\{\text{ all }\}}$.

$$= \frac{?}{\binom{2n}{n}} = \frac{n+1}{\binom{2n}{n}}$$



To count the number, consider

"must-have" structure



$n+1$ ways to assign the last free empty seat.

4. $\boxed{\begin{array}{ll} \text{A and B are independent} \Rightarrow & \text{A and } B^c \text{ are independent.} \\ \mathbb{P}(A\cap B) = \mathbb{P}(A)\mathbb{P}(B) & \mathbb{P}(A\cap B^c) = \mathbb{P}(A)\mathbb{P}(B^c) \end{array}}$

Pf. $A = (A\cap B)\cup(A\cap B^c)$ is a partition.

$$\mathbb{P}(A) = \mathbb{P}(A\cap B) + \mathbb{P}(A\cap B^c)$$

$$= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A\cap B^c)$$

re-arrange:

$$\mathbb{P}(A\cap B^c) = \mathbb{P}(A)\cdot\left(1 - \mathbb{P}(B)\right)$$

$$= \mathbb{P}(A)\cdot\mathbb{P}(B^c) \qquad \square.$$

Prop: For any events. C, D.

if C and D are ind. then C and D^c are ind.

if we take $C = B^c$.

$D = A$.

Symmetry.

then 4(b) is obvious.

5. recap of tools:

conditional probability. For two events A, B, s.t. $P(B) \neq 0$,

then $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$

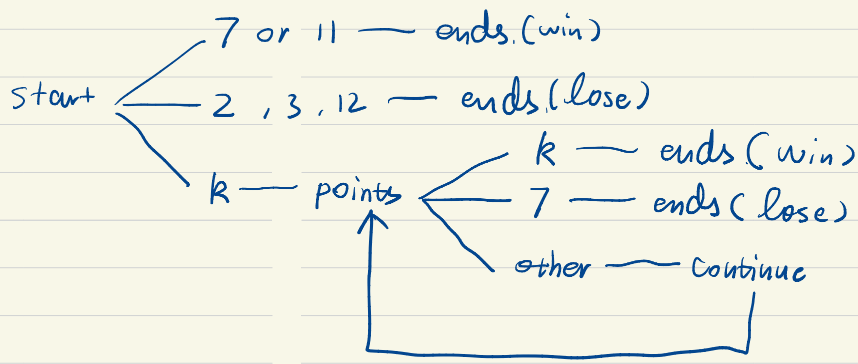
law of total probability. If $A_1 \cdots A_n \cdots$ s.t. $\begin{cases} \bigcup_{i=1}^{n} A_i = S \\ A_i \cap A_j = \emptyset \quad \forall\, i \neq j \end{cases}$.

then $P(B) = \sum_{i=1}^{\infty} P(B\cap A_i)$

$= \sum_{i=1}^{\infty} P(B|A_i)\cdot P(A_i)$

Bayes' formula. $P(B|A) = \dfrac{P(A|B)P(B)}{P(A)} = \dfrac{P(A|B)P(B)}{\sum_{i=1}^{\infty} P(A|B_i)P(B_i)}$

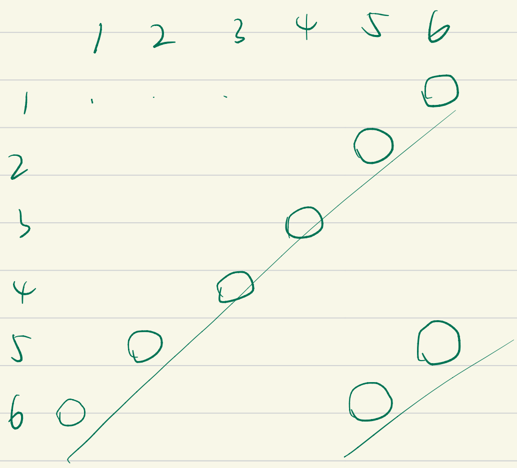
structure of the game:



$\mathbb{P}(\text{win}) = \mathbb{P}(\text{win 1st round}) + \sum_{i=2}^{\infty} \mathbb{P}(\text{win } i\text{-th round})$ (events are disjoint)

$\mathbb{P}(\text{win 1st round}) = \mathbb{P}(\text{get 7 or 11})$

$= \mathbb{P}(\text{get } 7) + \mathbb{P}(\text{get } 11) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}$



$\mathbb{P}(\text{get } 7) = \dfrac{6}{6^2} = \dfrac{1}{6}$

$\mathbb{P}(\text{get } 11) = \dfrac{2}{6^2} = \dfrac{1}{18}$

For $i \geq 2$, It seems calculating $\mathbb{P}(\text{win } i\text{-th round})$ would be easier if we know the point. So let us consider the conditional probability first. Take 'point = 4' for example

restate the condition 'win in i-th round'

independence of rolling dices

$$\begin{aligned}
&\mathbb{P}(\text{win } i\text{-th round} \mid \text{point is } 4) \\
&= \mathbb{P}(\boxed{\text{not win, not lose}} \text{ in } 2\text{—}(i-1)\text{th round } \& \text{ get } 4 \text{ in } i\text{-th round} \mid \text{point is } 4) \\
&= \mathbb{P}(\boxed{\text{not get } 4 \text{ or } 7} \text{ in } 2\text{—}(i-1)\text{-th round}, \ \& \text{ get } 4 \text{ in } i\text{-th round} \mid \text{point is } 4) \\
&= \mathbb{P}(\text{not get } 4 \text{ or } 7 \text{ in } 2\text{-nd round}) \\
&\quad \cdot \mathbb{P}(\text{________ } 3\text{-rd round}) \\
&\quad \vdots \\
&\quad \cdot \mathbb{P}(\text{________ } (i-1)\text{-th round}) \\
&\quad \cdot \mathbb{P}(\text{get } 4 \text{ in } i\text{-th round})
\end{aligned}$$

be careful.

$$\begin{aligned}
&= \left[1 - \mathbb{P}(\text{get } 4) - \mathbb{P}(\text{get } 7)\right]^{\boxed{i-2}} \cdot \mathbb{P}(\text{get } 4) \\
&= \left(1 - \pi - \tfrac{1}{6}\right)^{i-2} \cdot \pi
\end{aligned}$$

Apply law of total probability. Denote $K = \{4, 5, 6, 8, 9, 10\}$, the set of all possible 'point'.

$$\begin{aligned}
&\mathbb{P}(\text{win in } i\text{-th round}) \\
&= \sum_{j \in K} \mathbb{P}(\text{win in } i\text{-th round} \mid \text{point is } j) \cdot \mathbb{P}(\text{point is } j) \\
&= \sum_{j \in K} \left(1 - \pi_j - \tfrac{1}{6}\right)^{i-2} \cdot \pi_j \cdot \pi_j
\end{aligned}$$

Then, $\mathbb{P}(\text{win after the 1st round})$

$$\begin{aligned}
&= \sum_{i=2}^{\infty} \mathbb{P}(\text{win in } i\text{-th round}) \\
&= \sum_{i=2}^{\infty} \sum_{j \in K} \left(1 - \pi_j - \tfrac{1}{6}\right)^{i-2} \cdot \pi_j^2 \\
&= \sum_{j \in K} \pi_j^2 \cdot \sum_{i=2}^{\infty} \left(1 - \pi_j - \tfrac{1}{6}\right)^{i-2}
\end{aligned}$$

Set $k = i-2$

for $|a| < 1$, $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$

$$\begin{aligned}
&= \sum_{j \in K} \pi_j^2 \cdot \sum_{k=0}^{\infty} \left(1 - \pi_j - \tfrac{1}{6}\right)^{k} \\
&= \sum_{j \in K} \pi_j^2 \cdot \frac{1}{1 - (1 - \pi_j - \frac{1}{6})} \\
&= \sum_{j \in K} \frac{\pi_j^2}{\pi_j + \frac{1}{6}}
\end{aligned}$$

$$K = \{4, 5, 6, 8, 9, 10\}$$

Corresponding $\pi = \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}$

$1+a+a^2+a^3+\cdots+a^n+\cdots$

Prop.

$$\boxed{1+a+a^2+\cdots+a^n=\frac{1-a^{n+1}}{1-a}}$$

pf. $f(a)=1+\cancel{a}+\cancel{\cdots}+\cancel{a^n}$. —— ①

$\Rightarrow af(a)=\cancel{a}+\cancel{a^2}+\cancel{\cdots}+a^{n+1}$ —— ②

② − ①: $af(a)-f(a)=a^{n+1}-1$

$\Rightarrow f(a)=\frac{a^{n+1}-1}{a-1}=\frac{1-a^{n+1}}{1-a}\quad(a\neq 1)$ Q.E.D.

$$\sum_{i=1}^{\infty}a^i := \lim_{n\to\infty}\sum_{i=1}^{n}a^i=\lim_{n\to\infty}\frac{1-a^{n+1}}{1-a}=\begin{cases}\frac{1}{1-a} & \text{if } |a|<1.\\ \text{do not exist} & |a|\geq 1.\end{cases}$$