1. (a) $X \in \{0,1,2,3\}$.

set W = the number of heads on the last toss.

then $W \in \{0,1\}$ and $Y = X + W$.

$$\mathbb{P}(X=k, Y=\ell) = \mathbb{P}(X=k, X+W=\ell)$$
$$= \mathbb{P}(X=k, W=\ell-k)$$
$$= \mathbb{P}(X=k)\cdot\mathbb{P}(W=\ell-k)$$
$$= \binom{3}{k}\left(\tfrac{1}{2}\right)^3 \cdot \binom{1}{\ell-k}\left(\tfrac{1}{2}\right)$$
$$= \frac{3!}{k!(3-k)!}\cdot\frac{1}{16} =: g(k)$$

for $k \in \{0,1,2,3\}$, $\ell - k \in \{0,1\}$.

$Y \backslash X$	0	1	2	3
0	$\frac{1}{16}$			
1	$\frac{1}{16}$	$\frac{3}{16}$		
2		$\frac{3}{16}$	$\frac{3}{16}$	
3			$\frac{3}{16}$	$\frac{1}{16}$
4				$\frac{1}{16}$
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

numerically, $\mathbb{P}(X=0, Y=0) = \mathbb{P}(X=0, Y=1) = g(0) = \frac{1}{16}$

$\mathbb{P}(X=1, Y=1) = \mathbb{P}(X=1, Y=2) = g(1) = \frac{3}{16}$

$\mathbb{P}(X=2, Y=2) = \mathbb{P}(X=2, Y=3) = g(2) = \frac{3}{16}$

$\mathbb{P}(X=3, Y=3) = \mathbb{P}(X=3, Y=4) = g(3) = \frac{1}{16}$

(b) $X \sim Bin(3, \frac{1}{2})$

$\mathbb{E}X = n\pi = 3 \times \frac{1}{2} = \frac{3}{2}$

$Var(X) = n\pi(1-\pi) = 3 \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$.

$X \sim Bin(n, \pi)$

$\mathbb{E}X = n\pi$

$Var(X) = n\pi(1-\pi)$.

$Y \sim Bin(n, 1-\pi)$

$var(Y) = var(X)$.

(c) Given $X=2$, Y can only take value 2 or 3.

$\mathbb{P}(X=2) = \mathbb{P}(X=2, Y=2) + \mathbb{P}(X=2, Y=3) = \frac{3}{16} + \frac{3}{16} = \frac{3}{8}$.

$$\mathbb{P}(Y=2 \mid X=2) = \frac{\mathbb{P}(Y=2, X=2)}{\mathbb{P}(X=2)} = \frac{\frac{3}{16}}{\frac{3}{8}} = \frac{1}{2}.$$

$$\mathbb{P}(Y=3 \mid X=2) = \frac{\mathbb{P}(Y=3, X=2)}{\mathbb{P}(X=2)} = \frac{\frac{3}{16}}{\frac{3}{8}} = \frac{1}{2}.$$

(d) $Y_1 := Y \mid X=2$ follows a probability distribution, takes 2 and 3 with probability $\frac{1}{2}$.

$\mathbb{E}[Y \mid X=2] = \mathbb{E}[Y_1] = 2 \times \frac{1}{2} + 3 \times \frac{1}{2} = \frac{5}{2}$

$Var(Y \mid X=2) = Var(Y_1) = \mathbb{E}[Y_1^2] - (\mathbb{E}[Y_1])^2 = (2^2 \times \frac{1}{2} + 3^2 \times \frac{1}{2}) - \frac{5}{2} = 4$

$$\mathbb{E}[Y \mid X=2] = \sum_{y \in S} y \cdot \mathbb{P}(Y=y \mid X=2).$$

2. (a) Denote X_1 as the number of cars pass in a 2-minute period.

X_2 as the number of other motor vehicle pass in a 2-minute period.

$X_1 \sim Pois(\frac{150}{30})$, $X_2 \sim Pois(\frac{75}{30})$

$$\mathbb{P}(X_1 = 1, X_2 = 1) = \mathbb{P}(X_1 = 1) \cdot \mathbb{P}(X_2 = 1)$$
$$= \frac{e^{-\lambda_1} \cdot \lambda_1}{1!} \cdot \frac{e^{-\lambda_2} \cdot \lambda_2}{1!}$$
$$= e^{-5} \times 5 \times e^{-2.5} \times 2.5$$
$$\approx 0.0069$$

(b) $\mathbb{P}(X_1 + X_2 = 2) = \mathbb{P}(X_1 = 0, X_2 = 2) + \mathbb{P}(X_1 = 1, X_1 = 1) + \mathbb{P}(X_1 = 2, X_2 = 0)$

$X_1 + X_2 \sim Pois(7.5)$

$$= \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 2) + \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1) + \mathbb{P}(X_1 = 2)\mathbb{P}(X_2 = 0)$$
$$= e^{-\lambda_1} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^2}{2!} + e^{-\lambda_1} \cdot \lambda_1 \cdot e^{-\lambda_2} \cdot \lambda_2 + e^{-\lambda_1} \cdot \frac{\lambda_1^2}{2} \cdot e^{-\lambda_2}$$
$$= e^{-\lambda_1 - \lambda_2} \cdot (\lambda_1 + \lambda_2)^2 \cdot \frac{1}{2}$$
$$= e^{-7.5} \times 7.5^2 \times \frac{1}{2}$$
$$\approx 0.0156$$

In general,

$X_1 \sim Pois(\lambda_1) \cdots X_n \sim Pois(\lambda_n)$ X_1 — X_n are independent.

$X_1 + \cdots + X_n \sim Pois(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$

Prop. $X_1 \sim Pois(\lambda_1)$, $X_2 \sim Pois(\lambda_2)$, X_1 and X_2 are independent.

then $X_1 + X_2 \sim Pois(\lambda_1 + \lambda_2)$

proof. for any $k \in \mathbb{Z}_+ \cup \{0\}$,

$$\mathbb{P}(X_1 + X_2 = k) = \sum_{i=0}^{k} \mathbb{P}(X_1 = i, X_1 + X_2 = k)$$
$$= \sum_{i=0}^{k} \mathbb{P}(X_1 = i, X_2 = k - i)$$
$$= \sum_{i=0}^{k} \mathbb{P}(X_1 = i) \cdot \mathbb{P}(X_2 = k - i)$$

check! $= \cdots$

$$= e^{-(\lambda_1 + \lambda_2)} \cdot \frac{(\lambda_1 + \lambda_2)^k}{k!} \qquad \square.$$

That is the reason why $Pois(\lambda) \xrightarrow{d} Normal$.

3. (a) $X+Y \in \{2,3,4,5,6,7\}$

$Y \backslash X$	1	2	3
1	2	3	4
2	3	4	5
3	4	5	6
4	5	6	7

$\mathbb{P}(X+Y=2) = \mathbb{P}(\{(1,1)\}) = \frac{1}{12}$.

$\mathbb{P}(X+Y=3) = \mathbb{P}(\{(1,2),(2,1)\}) = \frac{2}{12}$

$\mathbb{P}(X+Y=4) = \mathbb{P}(\{(1,3),(2,2),(3,1)\}) = \frac{3}{12}$

$\mathbb{P}(X+Y=5) = \mathbb{P}(\{(1,4),(2,3),(3,2)\}) = \frac{3}{12}$

$\mathbb{P}(X+Y=6) = \mathbb{P}(\{(2,4),(3,3)\}) = \frac{2}{12}$

$\mathbb{P}(X+Y=7) = \mathbb{P}(\{(3,4)\}) = \frac{1}{12}$

(b) $\mathbb{E}[\underbrace{X+Y}_{W}] = 2\times\frac{1}{12} + 3\times\frac{2}{12} + 4\times\frac{3}{12} + 5\times\frac{3}{12} + 6\times\frac{2}{12} + 7\times\frac{1}{12}$

$= \frac{1}{12}(2+6+12+15+12+7)$

$= 2 + \frac{30}{12} = 4.5$

$\mathbb{E}[X+2X] = \mathbb{E}[X] + \mathbb{E}[2X] = 3\cdot\mathbb{E}[X]$.

or $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 2 + 2.5 = 4.5$

$W = X+Y$

$\mathbb{E}[W^2]$ $\quad Var(W) = \mathbb{E}[W^2] - (\mathbb{E}[W])^2$

To compute $Var(X+Y)$, one can first compute $\mathbb{E}[(X+Y)^2] = 2^2\times\frac{1}{12} + 3^2\times\frac{2}{12} + \cdots + 7^2\times\frac{1}{12}$

then apply $Var(X+Y) = \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2$

Or:

$Var(X+Y) = Cov(X+Y, X+Y) \overset{\text{linearity in 2nd place}}{=} Cov(X+Y, X) + Cov(X+Y, Y) \overset{\text{linearity in 1st place}}{=} (Cov(X,X) + Cov(Y,X)) + (Cov(X,Y) + Cov(Y,Y))$

$= Cov(X,X) + Cov(Y,Y) + 2Cov(X,Y)$

$= var(X) + var(Y) + 0$

$var(X) = (1-2)^2\times\frac{1}{3} + (2-2)^2\times\frac{1}{3} + (3-2)^2\times\frac{1}{3} = \frac{2}{3}$

$var(Y) = \mathbb{E}[(Y-\mathbb{E}Y)^2] = \frac{1}{4}\cdot(1.5^2+0.5^2+0.5^2+1.5^2) = \frac{5}{4}$.

Thus, $var(X+Y) = \frac{2}{3} + \frac{5}{4} = \frac{23}{12}$.

4. (a) Use X to denote the piston diameter, $X \sim N(6.84, 0.03^2)$

Use Y to denote the cylinder diameter, $Y \sim N(6.94, 0.04^2)$

$X - Y \sim N(6.84-6.94, 0.03^2+0.04^2)$ i.e. $X-Y \sim N(-0.1, 0.05^2)$

$$\mathbb{P}(X>Y) = \mathbb{P}(X-Y>0)$$
$$= \mathbb{P}\left(\frac{(X-Y)-(-0.1)}{0.05} > \frac{0-(-0.1)}{0.05}\right)$$
$$= \mathbb{P}(Z>2)$$
$$\approx 0.02275$$

(b) $(X_1, Y_1) \cdots (X_n, Y_n)$ where $n = 75$

$$\mathbb{P}(\text{ every pair fit })$$
$$= \mathbb{P}(X_1 \le Y_1, X_2 \le Y_2, \cdots, X_n \le Y_n)$$
$$= \mathbb{P}(X_1 \le Y_1) \cdot \mathbb{P}(X_2 \le Y_2) \cdot \cdots \cdot \mathbb{P}(X_n \le Y_n)$$
$$= \left(\mathbb{P}(X_1 \le Y_1)\right)^{75}$$
$$= (1 - 0.02275)^{75}$$
$$\approx 0.178$$

Or: Define W as the number of pairs (X_i, Y_i) such that $X_i \le Y_i$

$W \sim Bin(75, \pi)$

where $\pi = \mathbb{P}(X \le Y) = 0.97725$

$$\mathbb{P}(\text{all fits}) = \mathbb{P}(W=75) = \binom{75}{75} \pi^{75} (1-\pi)^0 = 0.97725^{75}$$

Prop. If X_1, X_2 are two normal random variables, then $X_1 + X_2$ is still normal!

In addition, if X_1 and X_2 are independent,

Prop. $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$

X_1 and X_2 are independent, then:

$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

$X_3 = -X_2$ $\quad X_3 \sim N(-\mu_2, \sigma_2^2)$ $\quad X_1 + X_3 \sim N(\mu_1 + (-\mu_2), \sigma_1^2 + \sigma_2^2)$

pf. $\mathbb{E}[X_1 + X_2] = \mathbb{E}X_1 + \mathbb{E}X_2 = \mu_1 + \mu_2$.

$Var(X_1 + X_2) = Var(X_1) + Var(X_2) = \sigma_1^2 + \sigma_2^2$. $\square$

In particular, $X_2 \sim N(\mu_2, \sigma_2^2) \Rightarrow \boxed{-X_2} \sim N(-\mu_2, \sigma_2^2)$

Apply the proposition for X_1 and $-X_2$:

$X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

$\mathbb{E}[-X_2] = -\mathbb{E}[X_2] = -\mu_2$.

$Var(-X_2) = \mathbb{E}[(-X_2)^2] - (\mathbb{E}[-X_2])^2$

$= \mathbb{E}X_2^2 - (\mathbb{E}X_2)^2$

$= var(X_2)$

Prop: $X \sim N(\mu, \sigma^2)$

$$\frac{X-\mu}{\sigma} \sim N(0,1)$$

5. (a) We show this by showing two more essential results:

Prop. bilinearity

For any random variables X, X_1, X_2, Y, any constant $a, b \in \mathbb{R}$:

(1) (Symmetry) $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$.

(2) (linearity) $\begin{cases} \mathrm{Cov}(X_1+X_2, Y) = \mathrm{Cov}(X_1, Y) + \mathrm{Cov}(X_2, Y) \\ \mathrm{Cov}(aX, Y) = a \cdot \mathrm{Cov}(X, Y) \end{cases}$

$\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$

pf. (1). $\mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] = \mathbb{E}[(Y-\mu_Y)(X-\mu_X)] = \mathrm{Cov}(Y,X)$

in other words, the symmetry is implied by the commutativity of product of two real numbers.

(2). $\mathrm{Cov}(X_1+X_2, Y) = \mathbb{E}[(X_1+X_2-\mu_{X_1}-\mu_{X_2})(Y-\mu_Y)]$

$= \mathbb{E}[(X_1-\mu_{X_1})(Y-\mu_Y) + (X_2-\mu_{X_2})(Y-\mu_Y)]$

$= \mathbb{E}[(X_1-\mu_{X_1})(Y-\mu_Y)] + \mathbb{E}[(X_2-\mu_{X_2})(Y-\mu_Y)]$

$= \mathrm{Cov}(X_1, Y) + \mathrm{Cov}(X_2, Y)$

$\mathrm{Cov}(aX, Y) = \mathbb{E}[(aX-\mathbb{E}[aX])(Y-\mathbb{E}[Y])]$

$= \mathbb{E}[a(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$

$= a \cdot \mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)]$

$= a \cdot \mathrm{Cov}(X,Y)$. $\square$.

Prop.

a is a constant, X is any r.v.

then $\mathrm{Cov}(a, X) = 0$

pf. $\mathrm{Cov}(a, X) = \mathbb{E}[(a-\mathbb{E}a)(X-\mathbb{E}X)]$

$= \mathbb{E}[0] = 0$. $\square$.

$\mathrm{Cov}(aX+b, cY+d) = \mathrm{Cov}(aX, cY+d) + \mathrm{Cov}(b, cY+d)$

$= \mathrm{Cov}(aX, cY) + \mathrm{Cov}(aX, d) + 0$

$= \mathrm{Cov}(aX, cY) + 0 + 0$

$= ac\,\mathrm{Cov}(X,Y)$

Use these two properties:

$\mathrm{Cov}(aX+b, cY+d) = \mathrm{Cov}(aX, cY) + \mathrm{Cov}(aX, d) + \mathrm{Cov}(b, cY) + \mathrm{Cov}(b, d)$

$= ac \cdot \mathrm{Cov}(X,Y) + 0 + 0 + 0$.

$\forall$ r.v. X, Y.

$\mathrm{Corr}(X,Y) = \dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}}$

$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0. \end{cases}$

(b) $\mathrm{Corr}(aX+b, cY+d) = \dfrac{\mathrm{Cov}(aX+b, cY+d)}{\sqrt{\mathrm{Var}(aX+b) \cdot \mathrm{Var}(cY+d)}} = \dfrac{ac \cdot \mathrm{Cov}(X,Y)}{\sqrt{a^2c^2 \cdot \mathrm{Var}(X) \cdot \mathrm{Var}(Y)}} = \dfrac{a}{|a|} \cdot \dfrac{c}{|c|} \cdot \mathrm{Corr}(X,Y)$

$\mathrm{Var}(X) = \mathrm{Cov}(X,X)$

$\sqrt{a^2} = \begin{cases} a & \text{if } a \geq 0. \\ -a & \text{if } a<0 \end{cases} = |a|$

here $\mathrm{Var}(aX+b) = \mathrm{Cov}(aX+b, aX+b) = a^2 \cdot \mathrm{Cov}(X,X) = a^2 \cdot \mathrm{Var}(X)$.

$\mathrm{Var}(cY+d) = \cdots = c^2 \cdot \mathrm{Var}(Y)$

(c) now $a=1$, $b=0$, then $\dfrac{a}{|a|} \cdot \dfrac{c}{|c|} = 1 \cdot \dfrac{c}{|c|} = \begin{cases} 1 & \text{for } c>0 \\ -1 & \text{for } c<0 \end{cases}$