1.

① You select $i \in \{1, \cdots, n\}$ (for simplicity, we call it 1.)

② Monty open $n-2$ empty boxes. $\begin{cases} \text{if prize is in 1, Monty randomly select } n-2 \text{ out of } n-1. \\ \text{if prize is not in 1, Monty does not have choices.} \end{cases}$

$$\mathbb{P}(\text{prize in } 1 \mid \underbrace{\text{Monty opens } 3 \sim n}_{\text{does not open } 2}) = \frac{\mathbb{P}(\text{prize in } 1 \cap \text{does not open } 2)}{\mathbb{P}(\text{does not open } 2)}$$

$2, 3, \cdots n-1$	n
$2, 3, \cdots n-2, n$	$n-1$
$\vdots$	
$3, \cdots n$	2

$$\text{numerator} = \mathbb{P}(\text{does not open } 2 \mid \text{prize in } 1) \cdot \mathbb{P}(\text{prize in } 1)$$

$$= \mathbb{P}(\text{randomly choose } n-2 \text{ elements from } \{2, 3, \cdots n\}, \text{ not contain } 2) \cdot \mathbb{P}(\text{prize in } 1)$$

$$= \frac{1}{n-1} \cdot \frac{1}{n}$$

$$\frac{\#\{\text{does not choose } 2\}}{\#\{\text{choose } n-2 \text{ out of } n-1\}} = \frac{1}{\binom{n-1}{n-2}} = \frac{1}{n-1}$$

$$\text{denominator} = \mathbb{P}(\text{does not open } 2)$$

$$\mathbb{P}(A) = \frac{\#\{A\}}{\#\{\text{all posibilities}\}}$$

$$= \sum_{i=1}^{n} \mathbb{P}(\text{does not open } 2 \mid \text{prize in } i) \cdot \mathbb{P}(\text{prize in } i)$$

$$= \Big(\underbrace{\frac{1}{n-1}}_{i=1} + \underbrace{1}_{i=2} + \underbrace{0}_{i=3} + \cdots + 0 \Big) \cdot \frac{1}{n}$$

$$= \frac{n}{n-1} \cdot \frac{1}{n}$$

$$\binom{n-1}{n-2} = \binom{n-1}{1}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

thus. $\mathbb{P}(\text{prize in } 1 \mid \text{does not open } 2) = \frac{1}{n}$

the same procedure gives $\mathbb{P}(\text{prize in } 2 \mid \text{does not open } 2) = \frac{n-1}{n}$

1^*. Let everything being the same, except "Monty know where the prize is".

Now assume Monty does not know where the prize is, and he choose $\{3, 4, \cdots n\}$ to open.

It turns out these are empty.

Now solve for $\mathbb{P}(\text{prize in } 1 \mid 3 - n \text{ are empty})$

$$\mathbb{P}(\text{prize in } 1 \mid 3 \sim n \text{ are empty}) = \frac{\mathbb{P}(3 \sim n \text{ are empty} \mid \text{prize in } 1) \cdot \mathbb{P}(\text{prize in } 1)}{\mathbb{P}(3 \sim n \text{ are empty})} = \frac{1 \cdot \frac{1}{n}}{\frac{2}{n}} = \frac{1}{2}$$

$$\mathbb{P}(3 \sim n \text{ are empty}) = \mathbb{P}(\text{prize in } 1 \text{ or } 2) = \frac{2}{n}$$

$$\mathbb{P}(3 \sim n \text{ are empty}) = \sum_{i=1}^{n} \mathbb{P}(3 \sim n \text{ are empty} \mid \text{prize in } i) \cdot \mathbb{P}(\text{prize in } i)$$

$$= (1 + 1 + 0 + \cdots + 0) \cdot \frac{1}{n} = \frac{2}{n}$$

1^{**} what if we don't know whether Monty knows where the prize is.

Bayesians' view: ① probability is Subjective.

② with more information, one could update his/her/their probability

2.

$$\mathbb{P}(\text{the second is empty} \mid \text{the first is empty})$$

$$= \frac{\mathbb{P}(2^{nd} \text{ is empty and } 1^{st} \text{ is empty})}{\mathbb{P}(1^{st} \text{ is empty})} \qquad = \left(\frac{n-2}{n-1}\right)^k$$

$$\text{numerator} = \mathbb{P}(\text{all objects are not in } 1^{st} \text{ or } 2^{nd})$$

$$= \prod_{i=1}^{k} \mathbb{P}(\text{object } i \text{ is not in } 1^{st} \text{ or } 2^{nd})$$

$$= \prod_{i=1}^{k} \frac{n-2}{n} \qquad P(A) = \frac{\#\{A\}}{\#\{all\}}.$$

$$= \left(\frac{n-2}{n}\right)^k$$

$$\text{denominator} = \mathbb{P}(\text{all objects are not in } 1^{st})$$

$$= \cdots$$

$$= \left(\frac{n-1}{n}\right)^k$$

3.

key point to use Bayes' Theorem: find a proper partition of Sample space.

(a) Set $B = \#\{\text{boys}\}$. then $B \in \{0, 1, 2, 3\}$

$$\mathbb{P}(\text{all three are boys} \mid \text{at least 2 are boys})$$

$$= \frac{\mathbb{P}(\text{at least 2 are boys} \mid \text{all three are boys}) \cdot \mathbb{P}(\text{all three are boys})}{\sum_{i=0}^{3} \mathbb{P}(\text{at least 2 are boys} \mid B = i) \cdot \mathbb{P}(B = i)}$$

$$= \frac{1 \cdot \frac{1}{8}}{0 + 0 + 1 \cdot \mathbb{P}(B=2) + 1 \cdot \mathbb{P}(B=3)}$$

$$= \frac{\frac{1}{8}}{\frac{3}{8} + \frac{1}{8}} = \frac{1}{4}$$

BBB 2^3
BBG ✓
BGB ✓
BGG $\frac{3}{8}$
GBB ✓
GBG
GGB
GGG

(b) $\mathbb{P}(\text{all three are boys} \mid \text{two oldest are boys})$

$$= \frac{\mathbb{P}(\text{———} \mid B=3) \cdot \mathbb{P}(B=3)}{\sum_{i=0}^{3} \mathbb{P}(\text{two oldest are boys} \mid B=i) \cdot \mathbb{P}(B=i)}$$

$$= \frac{1 \cdot \frac{1}{8}}{0 + 0 + \frac{1}{3} \cdot \frac{3}{8} + 1 \cdot \frac{1}{8}}$$

$$= \frac{1}{2}$$

now if $B=2$.
list them s.t. age increasing
B B G
B G B
G B B ✓ $\frac{1}{3}$.

4. a key question: do judges vote independently?

answer: no!

(a) $\mathbb{P}(3 \text{ vote } g \mid 1 \text{ and } 2 \text{ vote } g) = \dfrac{\mathbb{P}(\text{all vote } g)}{\mathbb{P}(1 \text{ and } 2 \text{ vote } g)} = 0.8286$

$$nu = \mathbb{P}(\text{all vote } g) = \mathbb{P}(\text{all vote } g \mid \text{true } g)\cdot\mathbb{P}(\text{true } g) + \mathbb{P}(\text{all vote } g \mid \text{false } g)\cdot\mathbb{P}(\text{false } g)$$

$$= \prod_{i=1}^{3}\mathbb{P}(i \text{ vote } g \mid \text{true } g) \times 0.7 \quad + \prod_{i=1}^{3}\mathbb{P}(i \text{ vote } g \mid \text{false } g) \times 0.3$$

$$= 0.85^3 \times 0.7 + 0.25^3 \times 0.3 = 0.434575$$

$$de = \mathbb{P}(1 \text{ and } 2 \text{ vote } g) = \cdots = 0.85^2 \times 0.7 + 0.25^2 \times 0.3 = 0.5245$$

If A is independent with B, then we should have:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A)$$

meaning that: if judges vote independently, we should have:

$$\mathbb{P}(3 \text{ vote guilty} \mid 1 \text{ and } 2 \text{ vote guilty}) = \mathbb{P}(3 \text{ vote guilty})$$

However, $\mathbb{P}(3 \text{ vote guilty} \mid 1 \text{ and } 2 \text{ vote guilty}) = 0.8286$,

$$\mathbb{P}(3 \text{ vote } g) = 0.85 \times 0.7 + 0.25 \times 0.3 = 0.67 \neq 0.8286$$

☆ $\boxed{\text{Conditional independence} \neq \text{independence}}$

(b) $\mathbb{P}(3g \mid \underline{1g\ 2ng} \text{ or } \underline{1ng\ 2g}) = \dfrac{\mathbb{P}(3g \cap (1g2ng \cup 1ng2g))}{\mathbb{P}(1g2ng \text{ or } 1ng2g)} = 0.6180$

$$\text{numerator} = \mathbb{P}(3g \cap (1g2ng \cup 1ng2g))$$

$$= \mathbb{P}((3g \cap 1g2ng) \cup (3g \cap 1ng2g))$$

$$= \mathbb{P}(3g1g2ng) + \mathbb{P}(3g1ng2g)$$

$$= 2\mathbb{P}(3g1g2ng)$$

$$= 2 \times [0.7 \times 0.85^2 \times 0.15 + 0.3 \times 0.25^2 \times 0.75]$$

$$\text{denominator} = \mathbb{P}(1g2ng \text{ or } 1ng2g)$$

$$= \mathbb{P}(1g2ng) + \mathbb{P}(1ng2g)$$

$$= 2\mathbb{P}(1g2ng)$$

$$= 2 \times [0.7 \times 0.85 \times 0.15 + 0.3 \times 0.25 \times 0.75]$$

(c) $\mathbb{P}(3g \mid 1ng\, 2ng) = \dfrac{\mathbb{P}(3g\, 1ng\, 2ng)}{\mathbb{P}(1ng\, 2ng)} = 0.3012$

$\mathbb{P}(1ng\, 2ng\, 3g) = 0.7 \times 0.15^2 \times 0.85 + 0.3 \times 0.75^2 \times 0.25$

$\mathbb{P}(1ng\, 2ng) = 0.7 \times 0.15^2 + 0.3 \times 0.75^2$