复变量微分

Wirtinger导数

1.一元复变函数的微分
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3.多元向量值函数的微分与导数
4.矩阵函数的微分与导数

1 一元函数的Wirtinger导数

定义.对应于一元复变函数 $f(z)=U(z)+i\cdot V(z)$ (其中 $z=x+i\cdot y$)关于实值 x 和 y 的二元函数 $F(x,y)=u(x,y)+i\cdot v(x,y)$ 的微分写为

$$dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial v}dy$$

定义 (Wirtinger 导数)

$$\frac{\partial}{\partial z}=\frac{1}{2}\left[\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right]$$

$$\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left[\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right]$$

$$df=\frac{\partial f}{\partial z}dz+\frac{\partial f}{\partial \bar{z}}d\bar{z}$$

利用

$dz=dx+i\cdot dy$，$d\bar{z}=dx-i\cdot dy$ 可得

$$dx=\frac{1}{2}(dz+d\bar{z})\ ,\ dy=\frac{1}{2\cdot i}(dz-d\bar{z})$$

$$dF=\frac{1}{2}\left[\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right]F(x,y)\cdot dz+\frac{1}{2}\left[\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right]F(x,y)\cdot d\bar{z}$$

Wirtinger导数的基本性质（1）

设f, g为一元复变函数，关于二元实变量x,y可微，那么

1. $$\frac{\partial}{\partial z}(f+g)=\frac{\partial f}{\partial z}+\frac{\partial g}{\partial z};\frac{\partial}{\partial \overline{z}}(f+g)=\frac{\partial f}{\partial \overline{z}}+\frac{\partial g}{\partial \overline{z}}$$

2. $$\frac{\partial}{\partial z}(f\cdot g)=\frac{\partial f}{\partial z}g+f\cdot\frac{\partial g}{\partial z};\frac{\partial}{\partial \overline{z}}(f\cdot g)=\frac{\partial f}{\partial \overline{z}}g+f\cdot\frac{\partial g}{\partial \overline{z}}$$

3. $$\frac{\partial}{\partial z}(f(g))=\left.\frac{\partial f}{\partial w}\right|_{w=g(z)}\frac{\partial g}{\partial z}+\left.\frac{\partial f}{\partial \overline{w}}\right|_{w=g(z)}\frac{\partial \overline{g}}{\partial z}$$

$$\frac{\partial}{\partial \overline{z}}(f(g))=\left.\frac{\partial f}{\partial w}\right|_{w=g(z)}\frac{\partial g}{\partial \overline{z}}+\left.\frac{\partial f}{\partial \overline{w}}\right|_{w=g(z)}\frac{\partial \overline{g}}{\partial \overline{z}}$$

Wirtinger导数的基本性质(2)

1. $\dfrac{\partial}{\partial z}\overline{z}=0$, $\dfrac{\partial}{\partial \overline{z}}z=0$;

2 推论: 共轭函数 $\overline{f}(z)$ 的导数满足关系

$$\frac{\partial \overline{f}}{\partial z}=\overline{\left(\frac{\partial f}{\partial \overline{z}}\right)},\quad \frac{\partial \overline{f}}{\partial \overline{z}}=\overline{\left(\frac{\partial f}{\partial z}\right)}$$

3 定理:对于一元复变量的实值函数 $f(z)\in R$ (其中 $z=x+i\cdot y$)的微分

1). $df=2\operatorname{Re}\left\{\dfrac{\partial f}{\partial z}dz\right\}=2\operatorname{Re}\left\{\dfrac{\partial f}{\partial \overline{z}}d\overline{z}\right\}$ 2). $df=0\Leftrightarrow \dfrac{\partial f}{\partial z}=0$

复变元的实值函数最速下降方向：

对于实值函数 $f(z)$ 我们知道其最速下降方向为梯度方向,即

$$\Delta x = -\frac{\partial F(x,y)}{\partial x}, \quad \Delta y = -\frac{\partial F(x,y)}{\partial y},$$

从而

$$\Delta z = \frac{1}{2}(\Delta x + i \cdot \Delta y) = -\frac{1}{2}\left(\frac{\partial f}{\partial x} + i \cdot \frac{\partial f}{\partial y}\right)$$

$$= -\frac{1}{2}\left(\frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y}\right) f = -\frac{\partial f}{\partial \bar{z}}$$

由此可见，$f(z)$的最速下降方向为 $-\frac{\partial f}{\partial \bar{z}}$

2.多元函数的微分与Wirtinger导数

对于多复变量函数 $f(z_1,\ldots,z_n)$ ，同样可以定义

$$df = \sum_{k=1}^{n}\frac{\partial f(\boldsymbol{z})}{\partial z_k}dz_k + \sum_{k=1}^{n}\frac{\partial f(\boldsymbol{z})}{\partial \bar{z}_k}d\bar{z}_k \quad = \frac{\partial f}{\partial \boldsymbol{z}^T}d\boldsymbol{z} + \frac{\partial f}{\partial \boldsymbol{z}^H}d\bar{\boldsymbol{z}}$$

其中 $\boldsymbol{z} = (z_1,\ldots,z_n)^T$,

定理： $\boxed{df = \boldsymbol{a}^T dz + \boldsymbol{b}^T d\bar{z}} \Leftrightarrow \boxed{\frac{\partial f}{\partial \boldsymbol{z}} = \boldsymbol{a}, \quad \frac{\partial f}{\partial \bar{\boldsymbol{z}}} = \boldsymbol{b}}$

证明:由上式可得.

定理：对于多元复变量的实值函数 $f(z) \in R$ (其中 $z \in C^n$)的微分为 0 的充要条件是其 Wirtinger 导数为 0,即

$$df = 0 \Leftrightarrow \frac{\partial f}{\partial z} = 0$$

实值函数的梯度下降方向:

对于实值函数 $f(z)$ 我们知道其最速下降方向为梯度方向,即

$$\Delta x_k = -\frac{\partial F(\boldsymbol{x}, \boldsymbol{y})}{\partial x_k}, \quad \Delta y_k = -\frac{\partial F(\boldsymbol{x}, \boldsymbol{y})}{\partial y_k},$$

$$\Delta z_k = \frac{1}{2}(\Delta x_k + i \cdot \Delta y_k) = -\frac{1}{2}\left(\frac{\partial f}{\partial x_k} + i \cdot \frac{\partial f}{\partial y_k}\right)$$

$$= -\frac{1}{2}\left(\frac{\partial}{\partial x_k} + i \cdot \frac{\partial}{\partial y_k}\right) f = -\frac{\partial f}{\partial z_k}$$

因此 $\Delta z = (\Delta z_1, \Delta z_2, \ldots, \Delta z_n)^T = -(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \ldots, \frac{\partial f}{\partial z_n})^T = -\frac{\partial f}{\partial \boldsymbol{z}}$

由此可见最速的 $\boldsymbol{z}$ 的最速下降方向为 $\dfrac{\partial f}{\partial \boldsymbol{z}}$。

例: $f(\mathbf{z}) = \mathbf{z}^H A\mathbf{z}$, $\mathrm{d}f = d\mathbf{z}^H A\mathbf{z} + \mathbf{z}^H A d\mathbf{z}$,

$$\mathrm{d}f = d\overline{\mathbf{z}}^T A\mathbf{z} + (A^T\overline{\mathbf{z}})^T d\mathbf{z},$$

因此 $\dfrac{\partial f}{\partial \mathbf{z}} = A^T\overline{\mathbf{z}}, \dfrac{\partial f}{\partial \overline{\mathbf{z}}} = A\mathbf{z}$

3　多元向量值函数

对于多元复向量函数 $\boldsymbol{f}=(f_1(\boldsymbol{z}),f_2(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z}))^T$ 定义其微分为

$$d\boldsymbol{f}=\begin{pmatrix}\sum_{k=1}^{n}\frac{\partial f_1(\boldsymbol{z})}{\partial z_k}dz_k\\ \vdots\\ \sum_{k=1}^{n}\frac{\partial f_m(\boldsymbol{z})}{\partial z_k}dz_k\end{pmatrix}+\begin{pmatrix}\sum_{k=1}^{n}\frac{\partial f_1(\boldsymbol{z})}{\partial \overline{z}_k}d\overline{z}_k\\ \vdots\\ \sum_{k=1}^{n}\frac{\partial f_m(\boldsymbol{z})}{\partial \overline{z}_k}d\overline{z}_k\end{pmatrix}=\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{z}^T}d\boldsymbol{z}+\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{z}^H}d\overline{\boldsymbol{z}}$$

定理：$\boxed{d\boldsymbol{f}=\boldsymbol{A}d\boldsymbol{z}+\boldsymbol{B}d\overline{\boldsymbol{z}}}\Leftrightarrow\boxed{\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{z}^T}=\boldsymbol{A},\qquad\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{z}^H}=\boldsymbol{B}}$

定理：对于多元复变量的实值函数 $f(z)\in R^m$（其中 $z\in C^n$）的微分为 0 的充要条件是其 Wirtinger 导数为 0,即

$$d\boldsymbol{f}=0 \Leftrightarrow \frac{\partial \boldsymbol{f}}{\partial \mathbf{z}^T}=0$$

4 对于复矩阵函数 $f(A)$，定义

$$df=\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{\partial f}{\partial a_{k,l}}da_{k,l}+\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{\partial f}{\partial \bar{a}_{k,l}}d\bar{a}_{k,l}$$

$$=tr\left(\frac{\partial f}{\partial A}\cdot dA^{T}+\frac{\partial f}{\partial \bar{A}}\cdot dA^{\mathrm{H}}\right)$$

定理: $df=tr\left(X\cdot dA^{T}+Y\cdot dA^{\mathrm{H}}\right)=tr\left(X^{T}\cdot dA+Y^{T}\cdot d\bar{A}\right)\Leftrightarrow\frac{\partial f}{\partial A}=X,\frac{\partial f}{\partial \bar{A}}=Y.$

例：$f(A)=\mathrm{tr}((\boldsymbol{I}+AA^H)^{-1})$

解：$df=tr(-(\boldsymbol{I}+AA^H)^{-1}(dA\cdot A^H+A\cdot dA^H)(\boldsymbol{I}+AA^H)^{-1})$

$$= tr(-A^H(\boldsymbol{I}+AA^H)^{-2}\cdot dA-(\boldsymbol{I}+AA^H)^{-2}A\cdot dA^H)$$

因此

$$\frac{\partial f}{\partial A}=\frac{\partial f}{\partial \bar{A}}=-(\boldsymbol{I}+AA^H)^{-2}A$$