

换元积分法

高等数学 I-信息、统计外招

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课程网页

第一类换元法

设 $f(u)$ 具有原函数 $F(u)$, 即

$$F'(u) = f(u) \quad \int f(u) du = F(u) + C$$

考虑形如

$$\int f[\varphi(x)]\varphi'(x)dx \quad (\spadesuit)$$

的积分.

第一类换元法

因为 $d\varphi(x) = \varphi'(x)dx$, 代入不定积分(\spadesuit), 得到

$$\int f[\varphi(x)]d\varphi(x) \quad (\clubsuit)$$

第一类换元法

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$$\int f[\varphi(x)]d\varphi(x) \quad (\clubsuit)$$

记 $\varphi(x) = u$, 不定积分(\clubsuit)可改写为

$$\int f(u)du$$

第一类换元法

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记 $\varphi(x) = u$, 不定积分(\clubsuit)可改写为

$$\int f(u)du = F(u) + C = F[\varphi(x)] + C.$$

第一类换元法

定理 1

设 $f(u)$ 具有原函数, $u = \varphi(x)$ 可导, 则有换元公式

$$\int f[\varphi(x)]\varphi'(x)dx = \left[\int f(u)du \right]_{u=\varphi(x)}$$

第一类换元法

定理 1

设 $f(u)$ 具有原函数, $u = \varphi(x)$ 可导, 则有换元公式

$$\int f[\varphi(x)]\varphi'(x)dx = \left[\int f(u)du \right]_{u=\varphi(x)}$$

- 对一般的不定积分 $\int g(x)dx$, 有一些可以化为 $\int f[\varphi(x)]\varphi'(x)dx$ 的形式.

例 1

求 $\int 2\cos 2x dx$

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解

$$\int 2 \cos 2x dx = \int \cos 2x \cdot \underbrace{(2x)'}_{d(2x)} dx$$

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记 $u = 2x$, 代入上式可以得到

$$\int \cos 2x d(2x) = \int \cos u du = \sin u + C$$

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$$\begin{aligned}\int \frac{1}{3+2x} dx &= \int \frac{1}{2} \cdot \frac{1}{3+2x} \underbrace{(3+2x)' dx}_{d(3+2x)} \\ &= \frac{1}{2} \int \frac{1}{3+2x} d(3+2x)\end{aligned}$$

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记 $u = 3 + 2x$, 代入上面的不定积分

$$\frac{1}{2} \int \frac{1}{3+2x} d(3+2x) = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C$$

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例 3

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解

$$\int \frac{x^2}{(x+2)^3} dx = \int \frac{(x+2)^2 - 4(x+2) + 4}{(x+2)^3} \cdot \underbrace{(x+2)' dx}_{d(x+2)}$$

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求 $\int \frac{x^2}{(x+2)^3} dx$

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记 $u = x + 2$, 代入上式

$$\int \frac{(x+2)^2 - 4(x+2) + 4}{(x+2)^3} d(x+2) = \int \frac{u^2 - 4u + 4}{u^3} du = \int \frac{1}{u} du - 4 \int \frac{1}{u^2} du + 4 \int \frac{1}{u^3} du$$

例 3

求 $\int \frac{x^2}{(x+2)^3} dx$

解

$$\begin{aligned}\int \frac{x^2}{(x+2)^3} dx &= \int \frac{(x+2)^2 - 4(x+2) + 4}{(x+2)^3} \underbrace{\cdot (x+2)' dx}_{d(x+2)} \\ &= \int \frac{(x+2)^2 - 4(x+2) + 4}{(x+2)^3} d(x+2)\end{aligned}$$

记 $u = x + 2$, 代入上式

$$\begin{aligned}\int \frac{(x+2)^2 - 4(x+2) + 4}{(x+2)^3} d(x+2) &= \int \frac{u^2 - 4u + 4}{u^3} du = \int \frac{1}{u} du - 4 \int \frac{1}{u^2} du + 4 \int \frac{1}{u^3} du \\ &= \ln |u| + \frac{4}{u} - \frac{2}{u^2} + C \\ &= \ln |x+2| + \frac{4}{x+2} - \frac{2}{(x+2)^2} + C\end{aligned}$$

例 4

求 $\int 2xe^{x^2} dx$

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$$\int 2xe^{x^2} dx = \int e^{x^2} (x^2)' dx = \int e^{x^2} d(x^2)$$

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$$\int e^{x^2} d(x^2) = \int e^u du = e^u + C$$

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求 $\int x\sqrt{1-x^2}dx$

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$$\int x\sqrt{1-x^2}dx = -\frac{1}{2} \int \sqrt{1-x^2}(1-x^2)'dx = -\frac{1}{2} \int \sqrt{1-x^2}d(1-x^2)$$

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$$\int x\sqrt{1-x^2}dx = -\frac{1}{2} \int \sqrt{1-x^2}(1-x^2)'dx = -\frac{1}{2} \int \sqrt{1-x^2}d(1-x^2)$$

令 $u = 1-x^2$, 代入上式得到

$$-\frac{1}{2} \int \sqrt{1-x^2}d(1-x^2) = -\frac{1}{2} \int \sqrt{u}du$$

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$$\int x\sqrt{1-x^2}dx = -\frac{1}{2} \int \sqrt{1-x^2}(1-x^2)'dx = -\frac{1}{2} \int \sqrt{1-x^2}d(1-x^2)$$

令 $u = 1-x^2$, 代入上式得到

$$-\frac{1}{2} \int \sqrt{1-x^2}d(1-x^2) = -\frac{1}{2} \int \sqrt{u}du = -\frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + C$$

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求 $\int x\sqrt{1-x^2}dx$

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$$\int x\sqrt{1-x^2}dx = -\frac{1}{2} \int \sqrt{1-x^2}(1-x^2)'dx = -\frac{1}{2} \int \sqrt{1-x^2}d(1-x^2)$$

令 $u = 1-x^2$, 代入上式得到

$$-\frac{1}{2} \int \sqrt{1-x^2}d(1-x^2) = -\frac{1}{2} \int \sqrt{u}du = -\frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + C = -\frac{1}{3} (1-x^2)^{\frac{3}{2}} + C$$

例 6

求 $\int \frac{1}{a^2+x^2} dx (a \neq 0)$

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解

$$\int \frac{1}{a^2+x^2} dx = \int \frac{1}{a^2} \frac{1}{1+\left(\frac{x}{a}\right)^2} dx = \frac{1}{a} \int \frac{1}{1+\left(\frac{x}{a}\right)^2} d\left(\frac{x}{a}\right)$$

例 6

求 $\int \frac{1}{a^2+x^2} dx (a \neq 0)$

解

$$\begin{aligned}\int \frac{1}{a^2+x^2} dx &= \int \frac{1}{a^2} \frac{1}{1+\left(\frac{x}{a}\right)^2} dx = \frac{1}{a} \int \frac{1}{1+\left(\frac{x}{a}\right)^2} d\left(\frac{x}{a}\right) \\ &= \frac{1}{a} \arctan \frac{x}{a} + C.\end{aligned}$$

例 7

求 $\int \frac{dx}{\sqrt{a^2-x^2}} (a > 0)$

例 7

求 $\int \frac{dx}{\sqrt{a^2 - x^2}} (a > 0)$

解

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{1}{a} \frac{dx}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} = \int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} d\left(\frac{x}{a}\right)$$

例 7

求 $\int \frac{dx}{\sqrt{a^2-x^2}} (a > 0)$

解

$$\begin{aligned}\int \frac{dx}{\sqrt{a^2-x^2}} &= \int \frac{1}{a} \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} d\left(\frac{x}{a}\right) \\ &= \arcsin \frac{x}{a} + C\end{aligned}$$

例 9

求 $\int \frac{dx}{x(1+2\ln x)}$

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解

$$\int \frac{dx}{x(1+2\ln x)} = \frac{1}{2} \int \frac{d(1+2\ln x)}{1+2\ln x}$$

例 9

求 $\int \frac{dx}{x(1+2\ln x)}$

解

$$\int \frac{dx}{x(1+2\ln x)} = \frac{1}{2} \int \frac{d(1+2\ln x)}{1+2\ln x} = \frac{1}{2} \ln |1+2\ln x| + C.$$

例 10

求 $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}} dx$

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解

$$\int \frac{e^{3\sqrt{x}}}{\sqrt{x}} dx = \frac{2}{3} \int e^{3\sqrt{x}} d(3\sqrt{x})$$

例 10

求 $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}} dx$

解

$$\int \frac{e^{3\sqrt{x}}}{\sqrt{x}} dx = \frac{2}{3} \int e^{3\sqrt{x}} d(3\sqrt{x}) = \frac{2}{3} e^{3\sqrt{x}} + C.$$

随堂练习

使用第一类换元积分法求解下列不定积分

(1) $\int \sin(ax)dx \ (a \neq 0)$

(2) $\int \frac{1}{2+2x}dx$

(3) $\int \cos(\cos x) \sin x dx$

(4) $\int e^{\sin x} \cos x dx$

(5) $\int \frac{\sin x}{1+\cos x} dx$

(6) $\int \frac{1}{\sqrt[3]{x^2}[1+(\sqrt[3]{x})^2]} dx$

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使用第一类换元积分法求解下列不定积分

$$(1) \int \sin(ax) dx \ (a \neq 0) = -\frac{1}{a} \cos(ax) + C$$

$$(2) \int \frac{1}{2+2x} dx = \frac{1}{2} \ln |2+2x| + C$$

$$(3) \int \cos(\cos x) \sin x dx = -\sin(\cos x) + C$$

$$(4) \int e^{\sin x} \cos x dx = e^{\sin x} + C$$

$$(5) \int \frac{\sin x}{1+\cos x} dx = -\ln |1+\cos x| + C$$

$$(6) \int \frac{1}{\sqrt[3]{x^2}[1+(\sqrt[3]{x})^2]} dx = 3 \arctan \sqrt[3]{x} + C$$

第二类换元积分法

$$\int f(x)dx \xrightarrow{\text{令 } x=\varphi(t)}$$

第二类换元积分法

$$\int f(x)dx \xrightarrow{\text{令 } x=\varphi(t)} \int f[\varphi(t)]d\varphi(t) = \int f[\varphi(t)]\varphi'(t)dt$$

定理 2

设 $x = \varphi(t)$ 是单调的可导函数, 并且 $\varphi'(t) \neq 0$. 又设 $f[\varphi(t)]\varphi'(t)$ 具有原函数, 则有换元公式

$$\int f(x)dx = \left[\int f[\varphi(t)]\varphi'(t)dt \right]_{t=\varphi^{-1}(x)}$$

其中 $\varphi^{-1}(x)$ 是 $x = \varphi(t)$ 的反函数.

- 将 $t = \varphi^{-1}(x)$ 代入等式右边, 两端同时关于 x 求导, 利用反函数求导法则即可证明等式成立.
- $\varphi(t)$ 的单调性确保其反函数 $\varphi^{-1}(x)$ 存在.

随堂练习

使用第二类换元积分法求解以下不定积分

$$(1) \int \frac{1}{1+\sqrt{2x}} dx$$

$$(2) \int \frac{1}{\sqrt{1+e^x}} dx$$

$$(3) \int \frac{x}{(1-x)^3} dx$$

$$(4) \int \frac{dx}{x^2 \sqrt{x^2-1}}$$

$$(5) \int \frac{\sqrt{1-x^2}}{x^2} dx$$

(1) 求 $\int \frac{1}{1+\sqrt{2x}} dx$

解: 令 $\sqrt{2x} = t$, $x = \frac{t^2}{2}$, $dx = t dt$, 代入所求积分

$$\begin{aligned}\int \frac{1}{1+\sqrt{2x}} dx &= \int \frac{t}{1+t} dt \\&= \int \frac{t+1-1}{1+t} dt \\&= \int 1 dt - \int \frac{1}{1+t} dt \\&= \int 1 dt - \int \frac{1}{1+t} d(1+t) \\&= t - \ln(1+t) + C \\&= \sqrt{2x} - \ln(1+\sqrt{2x}) + C\end{aligned}$$

(2) 求 $\int \frac{1}{\sqrt{1+e^x}} dx$

解: 令 $\sqrt{1+e^x} = u$, $x = \ln(u^2 - 1)$ ($u > 1$), $dx = \frac{2u}{u^2-1} du$. 代入所求积分

$$\begin{aligned}\int \frac{1}{\sqrt{1+e^x}} dx &= \int \frac{1}{u} \cdot \frac{2u}{u^2-1} du \\&= \int \frac{2}{u^2-1} du \\&= \int \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du \\&= \int \frac{1}{u-1} d(u-1) - \int \frac{1}{u+1} d(u+1) \\&= \ln \frac{u-1}{u+1} + C = \ln \frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1} + C\end{aligned}$$

(3) 求 $\int \frac{x}{(1-x)^3} dx$

解: 令 $1-x=t$, $x=1-t$, $dx=-dt$, 代入所求积分

$$\begin{aligned}\int \frac{x}{(1-x)^3} dx &= \int \frac{t-1}{t^3} dt \\&= \int \frac{1}{t^2} dt - \int \frac{1}{t^3} dt \\&= -\frac{1}{t} + \frac{1}{2} \frac{1}{t^2} + C \\&= \frac{1}{x-1} + \frac{1}{2(1-x)^2} + C\end{aligned}$$

(4) 求 $\int \frac{dx}{x^2\sqrt{x^2-1}}$

解: 令 $x = \frac{1}{t}$, $dx = -\frac{1}{t^2}dt$, 代入所求不定积分

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-1}} &= \int \frac{t^2|t|}{\sqrt{1-t^2}} \cdot \left(-\frac{1}{t^2}\right) dt \\ &= \int -\frac{|t|}{\sqrt{1-t^2}} dt\end{aligned}$$

当 $t > 0$ 时,

$$\begin{aligned}\int -\frac{|t|}{\sqrt{1-t^2}} dt &= \int \frac{-t}{\sqrt{1-t^2}} dt = \frac{1}{2} \int \frac{1}{\sqrt{1-t^2}} d(1-t^2) = (1-t^2)^{\frac{1}{2}} + C = \sqrt{1-\frac{1}{x^2}} + C \\ &= \frac{\sqrt{x^2-1}}{x} + C\end{aligned}$$

(4) 求 $\int \frac{dx}{x^2\sqrt{x^2-1}}$

当 $t < 0$ 时

$$\begin{aligned}\int -\frac{|t|}{\sqrt{1-t^2}} dt &= \int \frac{t}{\sqrt{1-t^2}} dt = -\frac{1}{2} \int \frac{1}{\sqrt{1-t^2}} d(1-t^2) = -(1-t^2)^{\frac{1}{2}} + C \\ &= -\sqrt{1-\frac{1}{x^2}} + C \\ &= -\frac{\sqrt{x^2-1}}{-x} + C = \frac{\sqrt{x^2-1}}{x} + C\end{aligned}$$

综上

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x} + C$$

$$(5) \int \frac{\sqrt{1-x^2}}{x^2} dx$$

解: 令 $x = \sin t (-\frac{\pi}{2} \leq t \leq \frac{\pi}{2})$, $dx = \cos t dt$, $\sqrt{1-x^2} = \cos t$, 代入所求不定积分

$$\begin{aligned} \int \frac{\sqrt{1-x^2}}{x^2} dx &= \int \frac{\cos t}{\sin^2 t} \cos t dt \\ &= \int \frac{\cos^2 t}{\sin^2 t} dt \\ &= \int \frac{\cos^2 t + \sin^2 t - \sin^2 t}{\sin^2 t} dt \\ &= \int \frac{1}{\sin^2 t} dt - \int 1 dt \\ &= -\frac{\cos t}{\sin t} - t + C = -\frac{\sqrt{1-x^2}}{x} - \arcsin x + C. \end{aligned}$$

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求 $\int \sqrt{a^2 - x^2} dx$ ($a > 0$)

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令 $x = a \sin t$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, 则

$$\begin{aligned}\int \sqrt{a^2 - x^2} dx &= \int \sqrt{a^2 - a^2 \sin^2 t} \, d(a \sin t) = a^2 \int \cos^2 t dt \\ &= a^2 \int \frac{1 + \cos 2t}{2} dt\end{aligned}$$

例 21

求 $\int \sqrt{a^2 - x^2} dx$ ($a > 0$)

解 要使上述积分有意义, 则 $-a \leq x \leq a$.

令 $x = a \sin t$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, 则

$$\begin{aligned}\int \sqrt{a^2 - x^2} dx &= \int \sqrt{a^2 - a^2 \sin^2 t} \, d(a \sin t) = a^2 \int \cos^2 t \, dt \\&= a^2 \int \frac{1 + \cos 2t}{2} dt \\&= a^2 \int \frac{1}{2} dt + \frac{a^2}{4} \int \cos(2t) \, d(2t) \\&= \frac{a^2 t}{2} + \frac{a^2 \sin 2t}{4} + C\end{aligned}$$

又因为 $x = a \sin t$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, 故

$$\begin{aligned} t &= \arcsin \frac{x}{a} & -a \leq x \leq a \\ a \cos t &= \sqrt{a^2 - a^2 \sin^2 t} = \sqrt{a^2 - x^2} & -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \end{aligned}$$

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因此

$$\begin{aligned} \int \sqrt{a^2 - x^2} dx &= \frac{a^2 t}{2} + \frac{a^2 \sin 2t}{4} + C \\ &= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{a^2}{4} 2 \cos t \sin t + C \end{aligned}$$

又因为 $x = a \sin t$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, 故

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因此

$$\begin{aligned} \int \sqrt{a^2 - x^2} dx &= \frac{a^2 t}{2} + \frac{a^2 \sin 2t}{4} + C \\ &= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{a^2}{4} 2 \cos t \sin t + C \\ &= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{1}{2} x \sqrt{a^2 - x^2} + C. \end{aligned}$$

例 24

求 $\int \frac{\sqrt{a^2-x^2}}{x^4} dx$ ($a \neq 0$)

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解 令 $x = \frac{1}{t}$, 代入所求不定积分.

$$\int \frac{\sqrt{a^2-x^2}}{x^4} dx = \int \sqrt{a^2 - \frac{1}{t^2}} \cdot t^4 \cdot d\left(\frac{1}{t}\right)$$

例 24

求 $\int \frac{\sqrt{a^2-x^2}}{x^4} dx$ ($a \neq 0$)

解 令 $x = \frac{1}{t}$, 代入所求不定积分.

$$\begin{aligned}\int \frac{\sqrt{a^2-x^2}}{x^4} dx &= \int \sqrt{a^2 - \frac{1}{t^2}} \cdot t^4 \cdot d\left(\frac{1}{t}\right) \\ &= \int \sqrt{a^2 t^2 - 1} \cdot \frac{1}{|t|} \cdot t^4 \left(-\frac{1}{t^2}\right) dt \\ &= -\int (a^2 t^2 - 1)^{\frac{1}{2}} |t| dt\end{aligned}$$

$$\int \frac{\sqrt{a^2 - x^2}}{x^4} dx = - \int (a^2 t^2 - 1)^{\frac{1}{2}} |t| dt$$

当 $x > 0$, 即 $t > 0$,

$$\begin{aligned} \int \frac{\sqrt{a^2 - x^2}}{x^4} dx &= \int (a^2 t^2 - 1)^{\frac{1}{2}} t dt \\ &= - \int (a^2 t^2 - 1)^{\frac{1}{2}} \cdot \frac{1}{2a^2} \cdot (a^2 t^2 - 1)' dt \\ &= - \frac{1}{2a^2} \int (a^2 t^2 - 1)^{\frac{1}{2}} d(a^2 t^2 - 1) \end{aligned}$$

$$\int \frac{\sqrt{a^2 - x^2}}{x^4} dx = - \int (a^2 t^2 - 1)^{\frac{1}{2}} |t| dt$$

当 $x > 0$, 即 $t > 0$,

$$\begin{aligned} \int \frac{\sqrt{a^2 - x^2}}{x^4} dx &= \int (a^2 t^2 - 1)^{\frac{1}{2}} t dt \\ &= - \int (a^2 t^2 - 1)^{\frac{1}{2}} \cdot \frac{1}{2a^2} \cdot (a^2 t^2 - 1)' dt \\ &= - \frac{1}{2a^2} \int (a^2 t^2 - 1)^{\frac{1}{2}} d(a^2 t^2 - 1) \\ &= - \frac{1}{2a^2} \cdot \frac{2}{3} (a^2 t^2 - 1)^{\frac{3}{2}} + C \end{aligned}$$

$$\int \frac{\sqrt{a^2 - x^2}}{x^4} dx = - \int (a^2 t^2 - 1)^{\frac{1}{2}} |t| dt$$

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$$\begin{aligned} \int \frac{\sqrt{a^2 - x^2}}{x^4} dx &= \int (a^2 t^2 - 1)^{\frac{1}{2}} t dt \\ &= - \int (a^2 t^2 - 1)^{\frac{1}{2}} \cdot \frac{1}{2a^2} \cdot (a^2 t^2 - 1)' dt \\ &= - \frac{1}{2a^2} \int (a^2 t^2 - 1)^{\frac{1}{2}} d(a^2 t^2 - 1) \\ &= - \frac{1}{2a^2} \cdot \frac{2}{3} (a^2 t^2 - 1)^{\frac{3}{2}} + C \\ &= - \frac{1}{3a^2 x^3} (a^2 - x^2)^{\frac{3}{2}} + C \end{aligned}$$

当 $x < 0$ 时, 有相同的结果. 综上,

$$\int \frac{\sqrt{a^2 - x^2}}{x^4} dx = -\frac{1}{3a^2 x^3} (a^2 - x^2)^{\frac{3}{2}} + C$$

新增常用不定积分公式 ($a>0$)

$$(1) \int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$(2) \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

$$(3) \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C$$

$$(4) \int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2}) + C$$

$$(5) \int \frac{dx}{\sqrt{x^2-a^2}} = \ln |x + \sqrt{x^2-a^2}| + C$$

例 26

求 $\int \frac{dx}{\sqrt{4x^2+9}}$

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$$\int \frac{dx}{\sqrt{4x^2+9}} = \int \frac{dx}{\sqrt{(2x)^2+3^2}}$$

例 26

求 $\int \frac{dx}{\sqrt{4x^2+9}}$

$$\begin{aligned}\int \frac{dx}{\sqrt{4x^2+9}} &= \int \frac{dx}{\sqrt{(2x)^2+3^2}} \\ &= \int \frac{1}{2} \frac{d(2x)}{\sqrt{(2x)^2+3^2}}\end{aligned}$$

例 26

求 $\int \frac{dx}{\sqrt{4x^2+9}}$

$$\begin{aligned}\int \frac{dx}{\sqrt{4x^2+9}} &= \int \frac{dx}{\sqrt{(2x)^2+3^2}} \\ &= \int \frac{1}{2} \frac{d(2x)}{\sqrt{(2x)^2+3^2}} \\ &= \frac{1}{2} \ln(2x + \sqrt{4x^2+9}) + C\end{aligned}$$

作业

- 教材习题 4-2: $1(1)(3)(5)(7)(14);$
 $2(1)(3)(4)(7)(8)(9)(10)(21)(30)(31)(34)(36)(37)$