

分部积分法

高等数学 I-信息、统计外招

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课程网页

分部积法

设函数 $u = u(x)$ 及 $v = v(x)$ 具有连续导数, 则两个函数乘积的导数公式为

$$(uv)' = u'v + uv'$$

移项, 得

$$uv' = (uv)' - u'v$$

对这个不等式求不定积分, 得

$$\int uv' dx = uv - \int u'v dx$$

$$\int uv' dx = uv - \int u'v dx \quad (\spadesuit)$$

公式(\spadesuit)称为分部积分公式.

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- 求不定积分 $\int uv' dx$ 转换为求不定积分 $\int u'v dx$

$$\int uv' dx = uv - \int u'v dx \quad (\spadesuit)$$

公式(\spadesuit)称为分部积分公式.

- 求不定积分 $\int uv' dx$ 转换为求不定积分 $\int u'v dx$
- 因为

$$du = u' dx \quad dv = v' dx,$$

分部积分公式可以表述为

$$\int u dv = uv - \int v du \quad (\clubsuit)$$

例 1

求 $\int x \cos x dx$

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解

$$\begin{aligned}\int x \cos x dx &= \int x(\sin x)' dx \\ &= \int x d \sin x\end{aligned}$$

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$$\begin{aligned}\int x \cos x dx &= \int x(\sin x)' dx \\ &= \int x d \sin x \\ &= x \sin x - \int \sin x dx\end{aligned}$$

例 1

求 $\int x \cos x dx$

解

$$\begin{aligned}\int x \cos x dx &= \int x(\sin x)' dx \\ &= \int x d \sin x \\ &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + C\end{aligned}$$

例 2

求 $\int x e^x dx$

例 2

求 $\int xe^x dx$

解

$$\begin{aligned}\int xe^x dx &= \int x(e^x)' dx \\ &= \int x de^x\end{aligned}$$

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解

$$\begin{aligned}\int xe^x dx &= \int x(e^x)' dx \\ &= \int x de^x \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C\end{aligned}$$

例 3

$$\int x^2 e^x dx$$

例 3

$$\int x^2 e^x dx$$

解

$$\int x^2 e^x dx = \int x^2 de^x$$

例 3

$$\int x^2 e^x dx$$

解

$$\begin{aligned}\int x^2 e^x dx &= \int x^2 de^x \\ &= x^2 e^x - \int e^x dx^2\end{aligned}$$

例 3

$$\int x^2 e^x dx$$

解

$$\begin{aligned}\int x^2 e^x dx &= \int x^2 de^x \\ &= x^2 e^x - \int e^x dx^2 \\ &= x^2 e^x - 2 \int x e^x dx \\ &= x^2 e^x - 2 \int x de^x\end{aligned}$$

例 3

$$\int x^2 e^x dx$$

解

$$\begin{aligned}\int x^2 e^x dx &= \int x^2 de^x \\&= x^2 e^x - \int e^x dx^2 \\&= x^2 e^x - 2 \int x e^x dx \\&= x^2 e^x - 2 \int x de^x \\&= x^2 e^x - 2 \left(x e^x - \int e^x dx \right) \\&= x^2 e^x - 2(x e^x - e^x) + C\end{aligned}$$

例 3 说明在求不定积分过程中可以多次应用分部积分法.

- 若 u 是幂函数, 则它在分部积分法中发生降幂

$$\int x^2 e^x dx \Rightarrow \int x e^x dx$$

例 4

求 $\int x \ln x dx$

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解

$$\begin{aligned}\int x \ln x dx &= \frac{1}{2} \int \ln x \cdot (x^2)' dx \\ &= \frac{1}{2} \int \ln x dx^2\end{aligned}$$

例 4

求 $\int x \ln x dx$

解

$$\begin{aligned}\int x \ln x dx &= \frac{1}{2} \int \ln x \cdot (x^2)' dx \\&= \frac{1}{2} \int \ln x dx^2 \\&= \frac{1}{2} \left(x^2 \ln x - \int x^2 d \ln x \right)\end{aligned}$$

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求 $\int x \ln x dx$

解

$$\begin{aligned}\int x \ln x dx &= \frac{1}{2} \int \ln x \cdot (x^2)' dx \\&= \frac{1}{2} \int \ln x dx^2 \\&= \frac{1}{2} \left(x^2 \ln x - \int x^2 d \ln x \right) \\&= \frac{1}{2} \left(x^2 \ln x - \int x^2 \frac{1}{x} dx \right)\end{aligned}$$

例 4

求 $\int x \ln x dx$

解

$$\begin{aligned}\int x \ln x dx &= \frac{1}{2} \int \ln x \cdot (x^2)' dx \\&= \frac{1}{2} \int \ln x dx^2 \\&= \frac{1}{2} \left(x^2 \ln x - \int x^2 d \ln x \right) \\&= \frac{1}{2} \left(x^2 \ln x - \int x^2 \frac{1}{x} dx \right) \\&= \frac{1}{2} \left(x^2 \ln x - \int x dx \right) = \frac{1}{2} \left(x^2 \ln x - \frac{1}{2} x^2 \right) + C\end{aligned}$$

例 5

求 $\int \arccos x dx$

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解

$$\begin{aligned}\int \arccos x dx &= \int \arccos x \cdot (x)' dx = x \arccos x - \int x d \arccos x \\ &= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx\end{aligned}$$

例 5

求 $\int \arccos x dx$

解

$$\begin{aligned}\int \arccos x dx &= \int \arccos x \cdot (x)' dx = x \arccos x - \int x d \arccos x \\ &= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx \quad (\text{分部积分法})\end{aligned}$$

例 5

求 $\int \arccos x dx$

解

$$\begin{aligned}\int \arccos x dx &= \int \arccos x \cdot (x)' dx = x \arccos x - \int x d \arccos x \\&= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx \quad (\text{分部积分法}) \\&= x \arccos x - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} d(1-x^2)\end{aligned}$$

例 5

求 $\int \arccos x dx$

解

$$\begin{aligned}\int \arccos x dx &= \int \arccos x \cdot (x)' dx = x \arccos x - \int x d \arccos x \\&= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx \quad (\text{分部积分法}) \\&= x \arccos x - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} d(1-x^2) \quad (\text{换元积分法})\end{aligned}$$

例 5

求 $\int \arccos x dx$

解

$$\begin{aligned}\int \arccos x dx &= \int \arccos x \cdot (x)' dx = x \arccos x - \int x d \arccos x \\&= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx \quad (\text{分部积分法}) \\&= x \arccos x - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} d(1-x^2) \quad (\text{换元积分法}) \\&= x \arccos x - \sqrt{1-x^2} + C\end{aligned}$$

例 6

求 $\int x \arctan x dx$

例 6

求 $\int x \arctan x dx$

解

$$\int x \arctan x dx = \frac{1}{2} \int \arctan x dx^2$$

例 6

求 $\int x \arctan x dx$

解

$$\begin{aligned}\int x \arctan x dx &= \frac{1}{2} \int \arctan x dx^2 \\ &= \frac{1}{2} \left(x^2 \arctan x - \int x^2 d \arctan x \right)\end{aligned}$$

例 6

求 $\int x \arctan x dx$

解

$$\begin{aligned}\int x \arctan x dx &= \frac{1}{2} \int \arctan x dx^2 \\&= \frac{1}{2} \left(x^2 \arctan x - \int x^2 d \arctan x \right) \\&= \frac{1}{2} \left(x^2 \arctan x - \int \frac{x^2}{1+x^2} dx \right)\end{aligned}$$

例 6

求 $\int x \arctan x dx$

解

$$\begin{aligned}\int x \arctan x dx &= \frac{1}{2} \int \arctan x dx^2 \\&= \frac{1}{2} \left(x^2 \arctan x - \int x^2 d \arctan x \right) \\&= \frac{1}{2} \left(x^2 \arctan x - \int \frac{x^2}{1+x^2} dx \right) \\&= \frac{1}{2} \left(x^2 \arctan x - \int \frac{1+x^2-1}{1+x^2} dx \right)\end{aligned}$$

例 6

求 $\int x \arctan x dx$

解

$$\begin{aligned}\int x \arctan x dx &= \frac{1}{2} \int \arctan x dx^2 \\&= \frac{1}{2} \left(x^2 \arctan x - \int x^2 d \arctan x \right) \\&= \frac{1}{2} \left(x^2 \arctan x - \int \frac{x^2}{1+x^2} dx \right) \\&= \frac{1}{2} \left(x^2 \arctan x - \int \frac{1+x^2-1}{1+x^2} dx \right) \\&= \frac{1}{2} \left(x^2 \arctan x - \int dx + \frac{1}{1+x^2} dx \right) = \frac{(x^2+1)\arctan x}{2} - \frac{1}{2}x + C\end{aligned}$$

例 7

求 $\int e^x \sin x dx$

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求 $\int e^x \sin x dx$

解

$$\int e^x \sin x dx = \int \sin x de^x = e^x \sin x - \int e^x d \sin x = e^x \sin x - \int e^x \cos x dx$$

例 7

求 $\int e^x \sin x dx$

解

$$\begin{aligned}\int e^x \sin x dx &= \int \sin x de^x = e^x \sin x - \int e^x d \sin x = e^x \sin x - \int e^x \cos x dx \\ &= e^x \sin x - \int \cos x de^x\end{aligned}$$

例 7

求 $\int e^x \sin x dx$

解

$$\begin{aligned}\int e^x \sin x dx &= \int \sin x de^x = e^x \sin x - \int e^x d \sin x = e^x \sin x - \int e^x \cos x dx \\ &= e^x \sin x - \int \cos x de^x \\ &= e^x \sin x - \left(e^x \cos x - \int e^x d \cos x \right)\end{aligned}$$

例 7

求 $\int e^x \sin x dx$

解

$$\begin{aligned}\int e^x \sin x dx &= \int \sin x de^x = e^x \sin x - \int e^x d \sin x = e^x \sin x - \int e^x \cos x dx \\&= e^x \sin x - \int \cos x de^x \\&= e^x \sin x - \left(e^x \cos x - \int e^x d \cos x \right) \\&= e^x \sin x - \left(e^x \cos x + \int e^x \sin x dx \right)\end{aligned}$$

例 7

求 $\int e^x \sin x dx$

解

$$\begin{aligned}\int e^x \sin x dx &= \int \sin x de^x = e^x \sin x - \int e^x d \sin x = e^x \sin x - \int e^x \cos x dx \\&= e^x \sin x - \int \cos x de^x \\&= e^x \sin x - \left(e^x \cos x - \int e^x d \cos x \right) \\&= e^x \sin x - \left(e^x \cos x + \int e^x \sin x dx \right)\end{aligned}$$

解得

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

例 9

求 $\int e^{\sqrt{x}} dx$

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求 $\int e^{\sqrt{x}} dx$

解 令 $t = \sqrt{x}$, 则 $x = t^2$, 代入所求不定积分

$$\int e^{\sqrt{x}} dx = \int e^t dt^2 = 2 \int te^t dt$$

例 9

求 $\int e^{\sqrt{x}} dx$

解 令 $t = \sqrt{x}$, 则 $x = t^2$, 代入所求不定积分

$$\begin{aligned}\int e^{\sqrt{x}} dx &= \int e^t dt^2 = 2 \int t e^t dt \quad (\text{换元积分法}) \\ &= 2 \int t de^t\end{aligned}$$

例 9

求 $\int e^{\sqrt{x}} dx$

解 令 $t = \sqrt{x}$, 则 $x = t^2$, 代入所求不定积分

$$\begin{aligned}\int e^{\sqrt{x}} dx &= \int e^t dt^2 = 2 \int t e^t dt \quad (\text{换元积分法}) \\ &= 2 \int t de^t \\ &= 2 \left(t e^t - \int e^t dt \right) \quad (\text{分部积分法})\end{aligned}$$

例 9

求 $\int e^{\sqrt{x}} dx$

解 令 $t = \sqrt{x}$, 则 $x = t^2$, 代入所求不定积分

$$\begin{aligned}\int e^{\sqrt{x}} dx &= \int e^t dt^2 = 2 \int t e^t dt \quad (\text{换元积分法}) \\ &= 2 \int t de^t \\ &= 2 \left(t e^t - \int e^t dt \right) \quad (\text{分部积分法}) \\ &= 2 t e^t - 2 e^t + C \quad (\text{回代 } t = \sqrt{x}) \\ &= 2 \sqrt{x} e^x - 2 e^{\sqrt{x}} + C\end{aligned}$$

随堂练习

求下列不定积分

- $\int x \cos 2x dx$
- $\int e^{\sqrt{3x+9}} dx$
- $\int \ln(1+x^2) dx$
- $\int x \ln^2 x dx$
- $\int \sin \ln x dx$

求 $\int x \cos 2x dx$

解:

$$\begin{aligned}\int x \cos 2x dx &= \frac{1}{2} \int x (\sin 2x)' dx = \frac{1}{2} \int x d(\sin 2x) \\&= \frac{1}{2} \left(x \sin 2x - \int \sin 2x dx \right) \\&= \frac{1}{2} \left(x \sin 2x - \frac{1}{2} \int \sin 2x d(2x) \right) \\&= \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + C\end{aligned}$$

求 $\int e^{\sqrt{3x+9}} dx$

解: 令 $t = \sqrt{3x+9}$, $x = \frac{t^2-9}{3}$, $dx = \frac{2}{3}t dt$, 代入所求不定积分

$$\begin{aligned}\int e^{\sqrt{3x+9}} dx &= \int e^t \cdot \frac{2}{3}t dt = \frac{2}{3} \int e^t t dt \\&= \frac{2}{3} \int t de^t \\&= \frac{2}{3} \left(te^t - \int e^t dt \right) \\&= \frac{2}{3} (te^t - e^t) + C \\&= \frac{2e^{\sqrt{3x+9}}}{3} (\sqrt{3x+9} - 1) + C\end{aligned}$$

求 $\int \ln(1+x^2)dx$

解:

$$\begin{aligned}\int \ln(1+x^2)dx &= x \ln(1+x^2) - \int x d \ln(1+x^2) \\&= x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx \\&= x \ln(1+x^2) - \int \frac{2x^2+2-2}{1+x^2} dx \\&= x \ln(1+x^2) - \int 2dx + 2 \int \frac{1}{1+x^2} dx \\&= x \ln(1+x^2) - 2x + 2 \arctan x + C.\end{aligned}$$

求 $\int x \ln^2 x dx$

解:

$$\begin{aligned}\int x \ln^2 x dx &= \frac{1}{2} \int \ln^2 x dx^2 \\&= \frac{1}{2} \left[x^2 \ln^2 x - \int x^2 d \ln^2 x \right] \\&= \frac{1}{2} \left[x^2 \ln^2 x - \int 2x \ln x dx \right] \\&= \frac{1}{2} \left[x^2 \ln^2 x - \int \ln x dx^2 \right] \\&= \frac{1}{2} \left[x^2 \ln^2 x - x^2 \ln x + \int x^2 d \ln x \right] \\&= \frac{1}{2} \left[x^2 \ln^2 x - x^2 \ln x + \int x dx \right] = \frac{x^2 \ln x}{2} (\ln x - 1) + \frac{x^2}{4} + C.\end{aligned}$$

求 $\int \sin \ln x dx$

解:

$$\begin{aligned}\int \sin \ln x dx &= x \cdot \sin \ln x - \int x d \sin \ln x \\&= x \cdot \sin \ln x - \int x \cos \ln x \cdot \frac{1}{x} dx \\&= x \cdot \sin \ln x - \int \cos \ln x dx \\&= x \cdot \sin \ln x - \left(x \cos \ln x - \int x d \cos \ln x \right) \\&= x \cdot \sin \ln x - \left(x \cos \ln x + \int \sin \ln x dx \right)\end{aligned}$$

解得

$$\int \sin \ln x = \frac{x \cdot \sin \ln x - x \cos \ln x}{2}$$

作业

- 教材习题 4-3: 1-9; 11; 12; 16; 19; 22; 23.