# 分部积分法

## 高等数学 I-信息、统计外招

Weiwen Wang(王伟文)

暨南大学

2025 年秋季学期

课程网页

# 分部积法

设函数 $u = u(x)$ 及 $v = v(x)$ 具有连续导数, 则两个函数乘积的导数公式为

$$(uv)' = u'v + uv'$$

移项, 得

$$uv' = (uv)' - u'v$$

对这个不等式求不定积分, 得

$$\int uv' \mathrm{d}x = uv - \int u'v \mathrm{d}x$$

$$\int uv'\mathrm{d}x = uv - \int u'v\mathrm{d}x \qquad (\spadesuit)$$

公式(♠)称为分部积分公式.

$$\int uv'\mathrm{d}x = uv - \int u'v\mathrm{d}x \qquad (\spadesuit)$$

公式(♠)称为分部积分公式.

- 求不定积分$\int uv'\mathrm{d}x$转换为求不定积分$\int u'v\mathrm{d}x$

$$\int uv'\mathrm{d}x = uv - \int u'v\mathrm{d}x \qquad (\spadesuit)$$

公式(♠)称为分部积分公式.

- 求不定积分$\int uv'\mathrm{d}x$转换为求不定积分$\int u'v\mathrm{d}x$
- 因为

$$\mathrm{d}u = u'\mathrm{d}x \quad \mathrm{d}v = v'\mathrm{d}x,$$

分部积分公式可以表述为

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u \qquad (\clubsuit)$$

**例 1**

求 $\int x\cos x\mathrm{d}x$

**例 1**

求 $\int x\cos x\mathrm{d}x$

解

$$\begin{aligned}\int x\cos x\mathrm{d}x &= \int x(\sin x)'\mathrm{d}x \\ &= \int x\mathrm{d}\sin x\end{aligned}$$

**例 1**

求 $\int x\cos x\mathrm{d}x$

解

$$\begin{aligned}\int x\cos x\mathrm{d}x &= \int x(\sin x)'\mathrm{d}x \\ &= \int x\mathrm{d}\sin x \\ &= x\sin x - \int \sin x\mathrm{d}x\end{aligned}$$

**例 1**

求 $\int x\cos x\mathrm{d}x$

解

$$\begin{aligned}\int x\cos x\mathrm{d}x &= \int x(\sin x)'\mathrm{d}x\\ &= \int x\mathrm{d}\sin x\\ &= x\sin x - \int \sin x\mathrm{d}x\\ &= x\sin x + \cos x + C\end{aligned}$$

**例 2**

求 $\int xe^x \mathrm{d}x$

**例 2**

求 $\int xe^x \mathrm{d}x$

解

$$\begin{aligned}\int xe^x \mathrm{d}x &= \int x(e^x)' \mathrm{d}x \\ &= \int x \mathrm{d}e^x\end{aligned}$$

**例 2**

求 $\int xe^x \mathrm{d}x$

解

$$\begin{aligned}\int xe^x \mathrm{d}x &= \int x(e^x)' \mathrm{d}x \\ &= \int x \mathrm{d}e^x \\ &= xe^x - \int e^x \mathrm{d}x\end{aligned}$$

**例 2**

求 $\int xe^x \mathrm{d}x$

解

$$\begin{aligned}\int xe^x \mathrm{d}x &= \int x(e^x)' \mathrm{d}x \\ &= \int x \mathrm{d}e^x \\ &= xe^x - \int e^x \mathrm{d}x \\ &= xe^x - e^x + C\end{aligned}$$

**例 3**

$\int x^2 e^x \mathrm{d}x$

## 例 3

$\int x^2 e^x \mathrm{d}x$

解

$$\int x^2 e^x \mathrm{d}x = \int x^2 \mathrm{d}e^x$$

## 例 3

$\int x^2 e^x \mathrm{d}x$

解

$$\begin{aligned}\int x^2 e^x \mathrm{d}x &= \int x^2 \mathrm{d}e^x \\ &= x^2 e^x - \int e^x \mathrm{d}x^2\end{aligned}$$

## 例 3

$\int x^2 e^x \mathrm{d}x$

解

$$\begin{aligned}\int x^2 e^x \mathrm{d}x &= \int x^2 \mathrm{d}e^x \\ &= x^2 e^x - \int e^x \mathrm{d}x^2 \\ &= x^2 e^x - 2\int x e^x \mathrm{d}x \\ &= x^2 e^x - 2\int x \mathrm{d}e^x\end{aligned}$$

## 例 3

$\int x^2 e^x \mathrm{d}x$

解

$$
\begin{aligned}
\int x^2 e^x \mathrm{d}x &= \int x^2 \mathrm{d}e^x \\
&= x^2 e^x - \int e^x \mathrm{d}x^2 \\
&= x^2 e^x - 2\int x e^x \mathrm{d}x \\
&= x^2 e^x - 2\int x \mathrm{d}e^x \\
&= x^2 e^x - 2\left(x e^x - \int e^x \mathrm{d}x\right) \\
&= x^2 e^x - 2\left(x e^x - e^x\right) + C
\end{aligned}
$$

**例 3** 说明在求不定积分过程中可以多次应用分部积分法.

- 若 $u$ 是幂函数, 则它在分部积分法中发生降幂

$$\int x^2 e^x \mathrm{d}x \Longrightarrow \int x e^x \mathrm{d}x$$

**例 4**

求 $\int x \ln x dx$

**例 4**

求 $\int x \ln x \mathrm{d}x$

解

$$\begin{aligned}\int x \ln x \mathrm{d}x &= \frac{1}{2}\int \ln x \cdot (x^2)' \mathrm{d}x \\ &= \frac{1}{2}\int \ln x \mathrm{d}x^2\end{aligned}$$

**例 4**

求 $\int x\ln x\mathrm{d}x$

解

$$\begin{aligned}\int x\ln x\mathrm{d}x &= \frac{1}{2}\int \ln x\cdot(x^2)'\mathrm{d}x\\ &= \frac{1}{2}\int \ln x\mathrm{d}x^2\\ &= \frac{1}{2}\left(x^2\ln x-\int x^2\mathrm{d}\ln x\right)\end{aligned}$$

**例 4**

求 $\int x \ln x \mathrm{d}x$

解

$$\begin{aligned}
\int x \ln x \mathrm{d}x &= \frac{1}{2} \int \ln x \cdot (x^2)' \mathrm{d}x \\
&= \frac{1}{2} \int \ln x \mathrm{d}x^2 \\
&= \frac{1}{2} \left( x^2 \ln x - \int x^2 \mathrm{d} \ln x \right) \\
&= \frac{1}{2} \left( x^2 \ln x - \int x^2 \frac{1}{x} \mathrm{d}x \right)
\end{aligned}$$

**例 4**

求 $\int x\ln x\mathrm{d}x$

解

$$\begin{aligned}\int x\ln x\mathrm{d}x &= \frac{1}{2}\int \ln x\cdot(x^2)'\mathrm{d}x\\ &= \frac{1}{2}\int \ln x\mathrm{d}x^2\\ &= \frac{1}{2}\left(x^2\ln x-\int x^2\mathrm{d}\ln x\right)\\ &= \frac{1}{2}\left(x^2\ln x-\int x^2\frac{1}{x}\mathrm{d}x\right)\\ &= \frac{1}{2}\left(x^2\ln x-\int x\mathrm{d}x\right)=\frac{1}{2}\left(x^2\ln x-\frac{1}{2}x^2\right)+C\end{aligned}$$

**例 5**

求 $\int \arccos x dx$

**例 5**

求 $\int \arccos x \mathrm{d}x$

解

$$\begin{aligned}\int \arccos x \mathrm{d}x = \int \arccos x \cdot (x)' \mathrm{d}x &= x \arccos x - \int x \mathrm{d} \arccos x \\ &= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x\end{aligned}$$

**例 5**

求 $\int \arccos x \mathrm{d}x$

解

$$\int \arccos x \mathrm{d}x = \int \arccos x \cdot (x)' \mathrm{d}x = x \arccos x - \int x \mathrm{d} \arccos x$$

$$= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x \quad \text{(分部积分法)}$$

**例 5**

求 $\int \arccos x \mathrm{d}x$

解

$$\begin{aligned}\int \arccos x \mathrm{d}x = \int \arccos x \cdot (x)' \mathrm{d}x &= x \arccos x - \int x \mathrm{d} \arccos x \\ &= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x \quad \text{(分部积分法)} \\ &= x \arccos x - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} \mathrm{d}(1-x^2)\end{aligned}$$

**例 5**

求 $\int \arccos x \mathrm{d}x$

解

$$\begin{aligned}\int \arccos x \mathrm{d}x = \int \arccos x \cdot (x)' \mathrm{d}x &= x \arccos x - \int x \mathrm{d} \arccos x \\ &= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x \quad \text{(分部积分法)} \\ &= x \arccos x - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} \mathrm{d}(1-x^2) \quad \text{(换元积分法)}\end{aligned}$$

**例 5**

求 $\int \arccos x \mathrm{d}x$

解

$$\begin{aligned}
\int \arccos x \mathrm{d}x = \int \arccos x \cdot (x)' \mathrm{d}x &= x \arccos x - \int x \mathrm{d} \arccos x \\
&= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x \quad \text{(分部积分法)} \\
&= x \arccos x - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} \mathrm{d}(1-x^2) \quad \text{(换元积分法)} \\
&= x \arccos x - \sqrt{1-x^2} + C
\end{aligned}$$

**例 6**

求 $\int x \arctan x \mathrm{d}x$

**例 6**

求 $\int x \arctan x \mathrm{d}x$

解

$$\int x \arctan x \mathrm{d}x = \frac{1}{2} \int \arctan x \mathrm{d}x^2$$

**例 6**

求 $\int x \arctan x \mathrm{d}x$

解

$$\begin{aligned}\int x \arctan x \mathrm{d}x &= \frac{1}{2}\int \arctan x \mathrm{d}x^2 \\ &= \frac{1}{2}\left(x^2 \arctan x - \int x \mathrm{d}\arctan x\right)\end{aligned}$$

**例 6**

求 $\int x\arctan x\mathrm{d}x$

解

$$\begin{aligned}\int x\arctan x\mathrm{d}x &= \frac{1}{2}\int \arctan x\mathrm{d}x^2\\ &= \frac{1}{2}\left(x^2\arctan x-\int x\mathrm{d}\arctan x\right)\\ &= \frac{1}{2}\left(x^2\arctan x-\int \frac{x}{1+x^2}\mathrm{d}x\right)\end{aligned}$$

**例 6**

求 $\int x\arctan x\mathrm{d}x$

解

$$\begin{aligned}\int x\arctan x\mathrm{d}x &= \frac{1}{2}\int \arctan x\mathrm{d}x^2\\ &= \frac{1}{2}\left(x^2\arctan x-\int x\mathrm{d}\arctan x\right)\\ &= \frac{1}{2}\left(x^2\arctan x-\int \frac{x}{1+x^2}\mathrm{d}x\right)\\ &= \frac{1}{2}\left[x^2\arctan x-\frac{1}{2}\int \frac{1}{1+x^2}\mathrm{d}(1+x^2)\right]\end{aligned}$$

**例 6**

求 $\int x\arctan x\mathrm{d}x$

解

$$\begin{aligned}\int x\arctan x\mathrm{d}x &= \frac{1}{2}\int \arctan x\mathrm{d}x^2\\ &= \frac{1}{2}\left(x^2\arctan x-\int x\mathrm{d}\arctan x\right)\\ &= \frac{1}{2}\left(x^2\arctan x-\int \frac{x}{1+x^2}\mathrm{d}x\right)\\ &= \frac{1}{2}\left[x^2\arctan x-\frac{1}{2}\int \frac{1}{1+x^2}\mathrm{d}(1+x^2)\right]\\ &= \frac{x^2\arctan x}{2}-\frac{\ln(1+x^2)}{4}+C\end{aligned}$$

**例 7**

求 $\int e^x \sin x dx$

## 例 7

求 $\int e^x \sin x \mathrm{d}x$

解

$$\int e^x \sin x \mathrm{d}x = \int \sin x \mathrm{d}e^x = e^x \sin x - \int e^x \mathrm{d}\sin x = e^x \sin x - \int e^x \cos x \mathrm{d}x$$

## 例 7

求 $\int e^x \sin x \mathrm{d}x$

解

$$\int e^x \sin x \mathrm{d}x = \int \sin x \mathrm{d}e^x = e^x \sin x - \int e^x \mathrm{d}\sin x = e^x \sin x - \int e^x \cos x \mathrm{d}x$$
$$= e^x \sin x - \int \cos x \mathrm{d}e^x$$

## 例 7

求 $\int e^x \sin x \mathrm{d}x$

解

$$\begin{aligned}\int e^x \sin x \mathrm{d}x = \int \sin x \mathrm{d}e^x = e^x \sin x - \int e^x \mathrm{d}\sin x = e^x \sin x - \int e^x \cos x \mathrm{d}x \\ = e^x \sin x - \int \cos x \mathrm{d}e^x \\ = e^x \sin x - \left(e^x \cos x - \int e^x \mathrm{d}\cos x\right)\end{aligned}$$

## 例 7

求 $\int e^x \sin x \mathrm{d}x$

解

$$\begin{aligned}\int e^x \sin x \mathrm{d}x = \int \sin x \mathrm{d}e^x = e^x \sin x - \int e^x \mathrm{d}\sin x &= e^x \sin x - \int e^x \cos x \mathrm{d}x \\ &= e^x \sin x - \int \cos x \mathrm{d}e^x \\ &= e^x \sin x - \left(e^x \cos x - \int e^x \mathrm{d}\cos x\right) \\ &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \mathrm{d}x\right)\end{aligned}$$

## 例 7

求 $\int e^x \sin x \mathrm{d}x$

解

$$\begin{aligned}\int e^x \sin x \mathrm{d}x = \int \sin x \mathrm{d}e^x = e^x \sin x - \int e^x \mathrm{d}\sin x = e^x \sin x - \int e^x \cos x \mathrm{d}x \\ = e^x \sin x - \int \cos x \mathrm{d}e^x \\ = e^x \sin x - \left(e^x \cos x - \int e^x \mathrm{d}\cos x\right) \\ = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \mathrm{d}x\right)\end{aligned}$$

解得

$$\int e^x \sin x \mathrm{d}x = \frac{1}{2} e^x (\sin x - \cos x) + C$$

**例 9**

求 $\int e^{\sqrt{x}}\mathrm{d}x$

**例 9**

求 $\int e^{\sqrt{x}}\mathrm{d}x$

解　令 $t=\sqrt{x}$, 则 $x=t^2$, 代入所求不定积分

$$\int e^{\sqrt{x}}\mathrm{d}x=\int e^t\mathrm{d}t^2=2\int te^t\mathrm{d}t$$

**例 9**

求 $\int e^{\sqrt{x}}\mathrm{d}x$

解　令 $t=\sqrt{x}$, 则 $x=t^2$, 代入所求不定积分

$$\begin{aligned}\int e^{\sqrt{x}}\mathrm{d}x &= \int e^t \mathrm{d}t^2 = 2\int te^t\mathrm{d}t \quad \text{(换元积分法)}\\ &= 2\int t\mathrm{d}e^t\end{aligned}$$

**例 9**

求 $\int e^{\sqrt{x}}\mathrm{d}x$

解　令 $t=\sqrt{x}$, 则 $x=t^2$, 代入所求不定积分

$$\begin{aligned}\int e^{\sqrt{x}}\mathrm{d}x &= \int e^t \mathrm{d}t^2 = 2\int te^t\mathrm{d}t \quad \text{(换元积分法)}\\ &= 2\int t\mathrm{d}e^t\\ &= 2\left(te^t - \int e^t\mathrm{d}t\right) \quad \text{(分部积分法)}\end{aligned}$$

**例 9**

求 $\int e^{\sqrt{x}}\mathrm{d}x$

解　令 $t=\sqrt{x}$, 则 $x=t^2$, 代入所求不定积分

$$\begin{aligned}\int e^{\sqrt{x}}\mathrm{d}x=\int e^t\mathrm{d}t^2&=2\int te^t\mathrm{d}t \quad \text{(换元积分法)}\\&=2\int t\mathrm{d}e^t\\&=2\left(te^t-\int e^t\mathrm{d}t\right) \quad \text{(分部积分法)}\\&=2te^t-2e^t+C \quad \text{(回代}t=\sqrt{x}\text{)}\\&=2\sqrt{x}e^x-2e^{\sqrt{x}}+C\end{aligned}$$

# 随堂练习

**求下列不定积分**

- $\int x\cos 2x\mathrm{d}x$
- $\int e^{\sqrt{3x+9}}\mathrm{d}x$
- $\int \ln(1+x^2)\mathrm{d}x$
- $\int x\ln^2 x\mathrm{d}x$
- $\int \sin\ln x\mathrm{d}x$

## 求 $\int x\cos 2x\mathrm{d}x$

解:

$$\begin{aligned}\int x\cos 2x\mathrm{d}x &= \frac{1}{2}\int x(\sin 2x)'\mathrm{d}x = \frac{1}{2}\int x\mathrm{d}(\sin 2x)\\ &= \frac{1}{2}\left(x\sin 2x - \int \sin 2x\mathrm{d}x\right)\\ &= \frac{1}{2}\left(x\sin 2x - \frac{1}{2}\int \sin 2x\mathrm{d}(2x)\right)\\ &= \frac{x\sin 2x}{2} + \frac{\cos 2x}{4} + C\end{aligned}$$

## 求 $\int e^{\sqrt{3x+9}}\mathrm{d}x$

解: 令 $t=\sqrt{3x+9}$, $x=\frac{t^2-9}{3}$, $\mathrm{d}x=\frac{2}{3}t\mathrm{d}t$, 代入所求不定积分

$$\begin{aligned}\int e^{\sqrt{3x+9}}\mathrm{d}x &= \int e^t\cdot\frac{2}{3}t\mathrm{d}t=\frac{2}{3}\int e^t t\mathrm{d}t\\ &=\frac{2}{3}\int t\mathrm{d}e^t\\ &=\frac{2}{3}\left(te^t-\int e^t\mathrm{d}t\right)\\ &=\frac{2}{3}\left(te^t-e^t\right)+C\\ &=\frac{2e^{\sqrt{3x+9}}}{3}\left(\sqrt{3x+9}-1\right)+C\end{aligned}$$

## 求 $\int \ln(1+x^2)\mathrm{d}x$

解:

$$\begin{aligned}
\int \ln(1+x^2)\mathrm{d}x &= x\ln(1+x^2) - \int x\mathrm{d}\ln(1+x^2) \\
&= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\mathrm{d}x \\
&= x\ln(1+x^2) - \int \frac{2x^2+2-2}{1+x^2}\mathrm{d}x \\
&= x\ln(1+x^2) - \int 2\mathrm{d}x + 2\int \frac{1}{1+x^2}\mathrm{d}x \\
&= x\ln(1+x^2) - 2x + 2\arctan x + C.
\end{aligned}$$

## 求 $\int x\ln^2 x\mathrm{d}x$

解:

$$\begin{aligned}\int x\ln^2 x\mathrm{d}x &= \frac{1}{2}\int \ln^2 x\mathrm{d}x^2\\ &= \frac{1}{2}\left[x^2\ln^2 x - \int x^2\mathrm{d}\ln^2 x\right]\\ &= \frac{1}{2}\left[x^2\ln^2 x - \int 2x\ln x\mathrm{d}x\right]\\ &= \frac{1}{2}\left[x^2\ln^2 x - \int \ln x\mathrm{d}x^2\right]\\ &= \frac{1}{2}\left[x^2\ln^2 x - x^2\ln x + \int x^2\mathrm{d}\ln x\right]\\ &= \frac{1}{2}\left[x^2\ln^2 x - x^2\ln x + \int x\mathrm{d}x\right] = \frac{x^2\ln x}{2}\left(\ln x - 1\right) + \frac{x^2}{4} + C.\end{aligned}$$

## 求 $\int \sin\ln x\mathrm{d}x$

解:

$$\begin{aligned}
\int \sin\ln x\mathrm{d}x &= x\cdot\sin\ln x - \int x\mathrm{d}\sin\ln x \\
&= x\cdot\sin\ln x - \int x\cos\ln x\cdot\frac{1}{x}\mathrm{d}x \\
&= x\cdot\sin\ln x - \int \cos\ln x\mathrm{d}x \\
&= x\cdot\sin\ln x - \left(x\cos\ln x - \int x\mathrm{d}\cos\ln x\right) \\
&= x\cdot\sin\ln x - \left(x\cos\ln x + \int \sin\ln x\mathrm{d}x\right)
\end{aligned}$$

解得

$$\int \sin\ln x = \frac{x\cdot\sin\ln x - x\cos\ln x}{2}$$

# 作业

- 教材习题 4-3: 1-9; 11; 12; 16; 19; 22; 23.