换元积分法

高等数学 I-信息、统计外招

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课程网页

第一类换元法

设 $f(u)$ 具有原函数 $F(u)$, 即

$$F'(u) = f(u) \quad \int f(u)\mathrm{d}u = F(u) + C$$

考虑形如

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x \tag{$\spadesuit$}$$

的积分.

第一类换元法

因为 $d\varphi(x) = \varphi'(x)\mathrm{d}x$, 代入不定积分(♠), 得到

$$\int f[\varphi(x)]\mathrm{d}\varphi(x) \tag{♣}$$

第一类换元法

因为 $d\varphi(x)=\varphi'(x)\mathrm{d}x$, 代入不定积分(♠), 得到

$$\int f[\varphi(x)]\mathrm{d}\varphi(x) \tag{♣}$$

记 $\varphi(x)=u$, 不定积分(♣)可改写为

$$\int f(u)\mathrm{d}u$$

第一类换元法

因为 $d\varphi(x) = \varphi'(x)\mathrm{d}x$，代入不定积分(♠)，得到

$$\int f[\varphi(x)]\mathrm{d}\varphi(x) \qquad (♣)$$

记 $\varphi(x) = u$，不定积分(♣)可改写为

$$\int f(u)\mathrm{d}u = F(u) + C = F[\varphi(x)] + C.$$

第一类换元法

定理 1

设 $f(u)$ 具有原函数, $u=\varphi(x)$ 可导, 则有换元公式

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \left[\int f(u)\mathrm{d}u\right]_{u=\varphi(x)}$$

第一类换元法

定理 1

设 $f(u)$ 具有原函数，$u=\varphi(x)$ 可导，则有换元公式

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=\left[\int f(u)\mathrm{d}u\right]_{u=\varphi(x)}$$

- 对一般的不定积分 $\int g(x)\mathrm{d}x$，有一些可以化为 $\int f[\varphi(x)]\varphi'(x)\mathrm{d}x$ 的形式.

例 1

求 $\int 2\cos 2x\mathrm{d}x$

例 1

求 $\int 2\cos 2x \mathrm{d}x$

解

$$\int 2\cos 2x \mathrm{d}x = \int \cos 2x \underbrace{\cdot (2x)' \mathrm{d}x}_{\mathrm{d}(2x)}$$

例 1

求 $\int 2\cos 2x\mathrm{d}x$

解

$$\int 2\cos 2x\mathrm{d}x = \int \cos 2x \cdot \underbrace{(2x)'\mathrm{d}x}_{\mathrm{d}(2x)}$$

$$= \int \cos 2x\mathrm{d}(2x)$$

记 $u = 2x$, 代入上式可以得到

$$\int \cos 2x\mathrm{d}(2x) = \int \cos u\mathrm{d}u = \sin u + C$$

例 1

求 $\int 2\cos 2x \mathrm{d}x$

解

$$\int 2\cos 2x \mathrm{d}x = \int \cos 2x \cdot \underbrace{(2x)' \mathrm{d}x}_{\mathrm{d}(2x)}$$
$$= \int \cos 2x \mathrm{d}(2x)$$

记 $u = 2x$, 代入上式可以得到

$$\int \cos 2x \mathrm{d}(2x) = \int \cos u \mathrm{d}u = \sin u + C = \sin 2x + C.$$

例 2

求 $\int \frac{1}{3+2x}\mathrm{d}x$

例 2

求 $\int \frac{1}{3+2x}\mathrm{d}x$

解

$$\int \frac{1}{3+2x}\mathrm{d}x = \int \frac{1}{2} \cdot \frac{1}{3+2x} \underbrace{(3+2x)'\mathrm{d}x}_{\mathrm{d}(3+2x)}$$

$$= \frac{1}{2}\int \frac{1}{3+2x}\mathrm{d}(3+2x)$$

例 2

求 $\int \frac{1}{3+2x} \mathrm{d}x$

解

$$\int \frac{1}{3+2x} \mathrm{d}x = \int \frac{1}{2} \cdot \frac{1}{3+2x} \underbrace{(3+2x)' \mathrm{d}x}_{\mathrm{d}(3+2x)}$$

$$= \frac{1}{2} \int \frac{1}{3+2x} \mathrm{d}(3+2x)$$

记 $u = 3+2x$, 代入上面的不定积分

$$\frac{1}{2} \int \frac{1}{3+2x} \mathrm{d}(3+2x) = \frac{1}{2} \int \frac{1}{u} \mathrm{d}u = \frac{1}{2} \ln|u| + C$$

例 2

求 $\int \frac{1}{3+2x}\mathrm{d}x$

解

$$\int \frac{1}{3+2x}\mathrm{d}x = \int \frac{1}{2} \cdot \frac{1}{3+2x} \underbrace{(3+2x)'\mathrm{d}x}_{\mathrm{d}(3+2x)}$$

$$= \frac{1}{2}\int \frac{1}{3+2x}\mathrm{d}(3+2x)$$

记 $u = 3+2x$, 代入上面的不定积分

$$\frac{1}{2}\int \frac{1}{3+2x}\mathrm{d}(3+2x) = \frac{1}{2}\int \frac{1}{u}\mathrm{d}u = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|3+2x| + C.$$

例 3

求 $\int \frac{x^2}{(x+2)^3} \mathrm{d}x$

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求 $\int \frac{x^2}{(x+2)^3} \mathrm{d}x$

解

$$\int \frac{x^2}{(x+2)^3} \mathrm{d}x = \int \frac{(x+2)^2 - 4(x+2) + 4}{(x+2)^3} \underbrace{\cdot (x+2)' \mathrm{d}x}_{\mathrm{d}(x+2)}$$

例 3

求 $\int \frac{x^2}{(x+2)^3}\mathrm{d}x$

解

$$\int \frac{x^2}{(x+2)^3}\mathrm{d}x = \int \frac{(x+2)^2-4(x+2)+4}{(x+2)^3} \underbrace{\cdot(x+2)'\mathrm{d}x}_{\mathrm{d}(x+2)}$$

$$= \int \frac{(x+2)^2-4(x+2)+4}{(x+2)^3}\mathrm{d}(x+2)$$

记 $u = x+2$, 代入上式

$$\int \frac{(x+2)^2-4(x+2)+4}{(x+2)^3}\mathrm{d}(x+2) = \int \frac{u^2-4u+4}{u^3}\mathrm{d}u = \int \frac{1}{u}\mathrm{d}u - 4\int \frac{1}{u^2}\mathrm{d}u + 4\int \frac{1}{u^3}\mathrm{d}u$$

例 3

求 $\int \frac{x^2}{(x+2)^3} \mathrm{d}x$

解

$$\int \frac{x^2}{(x+2)^3} \mathrm{d}x = \int \frac{(x+2)^2 - 4(x+2) + 4}{(x+2)^3} \underbrace{\cdot (x+2)' \mathrm{d}x}_{\mathrm{d}(x+2)}$$

$$= \int \frac{(x+2)^2 - 4(x+2) + 4}{(x+2)^3} \mathrm{d}(x+2)$$

记 $u = x+2$, 代入上式

$$\int \frac{(x+2)^2 - 4(x+2) + 4}{(x+2)^3} \mathrm{d}(x+2) = \int \frac{u^2 - 4u + 4}{u^3} \mathrm{d}u = \int \frac{1}{u} \mathrm{d}u - 4 \int \frac{1}{u^2} \mathrm{d}u + 4 \int \frac{1}{u^3} \mathrm{d}u$$

$$= \ln|u| + \frac{4}{u} - \frac{2}{u^2} + C$$

$$= \ln|x+2| + \frac{4}{x+2} - \frac{2}{(x+2)^2} + C$$

例 4

求 $\int 2xe^{x^2}\,\mathrm{d}x$

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求 $\int 2xe^{x^2}\mathrm{d}x$

解

$$\int 2xe^{x^2}\mathrm{d}x = \int e^{x^2}(x^2)'\mathrm{d}x = \int e^{x^2}\mathrm{d}(x^2)$$

例 4

求 $\int 2xe^{x^2}\mathrm{d}x$

解

$$\int 2xe^{x^2}\mathrm{d}x = \int e^{x^2}(x^2)'\mathrm{d}x = \int e^{x^2}\mathrm{d}(x^2)$$

记 $u = x^2$, 代入上式得到

$$\int e^{x^2}\mathrm{d}(x^2) = \int e^u \mathrm{d}u = e^u + C$$

例 4

求 $\int 2xe^{x^2}\,\mathrm{d}x$

解

$$\int 2xe^{x^2}\,\mathrm{d}x = \int e^{x^2}(x^2)'\,\mathrm{d}x = \int e^{x^2}\,\mathrm{d}(x^2)$$

记 $u = x^2$, 代入上式得到

$$\int e^{x^2}\,\mathrm{d}(x^2) = \int e^u\,\mathrm{d}u = e^u + C = e^{x^2} + C$$

例 5

求 $\int x\sqrt{1-x^2}\mathrm{d}x$

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求 $\int x\sqrt{1-x^2}\mathrm{d}x$

解

$$\int x\sqrt{1-x^2}\mathrm{d}x=-\frac{1}{2}\int\sqrt{1-x^2}(1-x^2)'\mathrm{d}x=-\frac{1}{2}\int\sqrt{1-x^2}\mathrm{d}(1-x^2)$$

例 5

求 $\int x\sqrt{1-x^2}\mathrm{d}x$

解

$$\int x\sqrt{1-x^2}\mathrm{d}x = -\frac{1}{2}\int \sqrt{1-x^2}(1-x^2)'\mathrm{d}x = -\frac{1}{2}\int \sqrt{1-x^2}\mathrm{d}(1-x^2)$$

令 $u = 1-x^2$, 代入上式得到

$$-\frac{1}{2}\int \sqrt{1-x^2}\mathrm{d}(1-x^2) = -\frac{1}{2}\int \sqrt{u}\mathrm{d}u$$

例 5

求 $\int x\sqrt{1-x^2}\mathrm{d}x$

解

$$\int x\sqrt{1-x^2}\mathrm{d}x = -\frac{1}{2}\int \sqrt{1-x^2}(1-x^2)'\mathrm{d}x = -\frac{1}{2}\int \sqrt{1-x^2}\mathrm{d}(1-x^2)$$

令 $u = 1-x^2$, 代入上式得到

$$-\frac{1}{2}\int \sqrt{1-x^2}\mathrm{d}(1-x^2) = -\frac{1}{2}\int \sqrt{u}\mathrm{d}u = -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C$$

例 5

求 $\int x\sqrt{1-x^2}\mathrm{d}x$

解

$$\int x\sqrt{1-x^2}\mathrm{d}x = -\frac{1}{2}\int \sqrt{1-x^2}(1-x^2)'\mathrm{d}x = -\frac{1}{2}\int \sqrt{1-x^2}\mathrm{d}(1-x^2)$$

令 $u=1-x^2$, 代入上式得到

$$-\frac{1}{2}\int \sqrt{1-x^2}\mathrm{d}(1-x^2) = -\frac{1}{2}\int \sqrt{u}\mathrm{d}u = -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}+C = -\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C$$

例 6

求 $\int \frac{1}{a^2+x^2}\mathrm{d}x(a \neq 0)$

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求 $\int \frac{1}{a^2+x^2}\mathrm{d}x(a \neq 0)$

解

$$\int \frac{1}{a^2+x^2}\mathrm{d}x = \int \frac{1}{a^2}\frac{1}{1+\left(\frac{x}{a}\right)^2}\mathrm{d}x = \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\mathrm{d}\left(\frac{x}{a}\right)$$

例 6

求 $\int \frac{1}{a^2+x^2}\mathrm{d}x(a \neq 0)$

解

$$\int \frac{1}{a^2+x^2}\mathrm{d}x = \int \frac{1}{a^2}\frac{1}{1+\left(\frac{x}{a}\right)^2}\mathrm{d}x = \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\mathrm{d}\left(\frac{x}{a}\right)$$

$$= \frac{1}{a}\arctan\frac{x}{a} + C.$$

例 7

求 $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}(a>0)$

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求 $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}(a>0)$

解

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}\mathrm{d}x = \int \frac{1}{a}\frac{\mathrm{d}x}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\mathrm{d}\left(\frac{x}{a}\right)$$

例 7

求 $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}(a>0)$

解

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}\mathrm{d}x = \int \frac{1}{a}\frac{\mathrm{d}x}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\mathrm{d}\left(\frac{x}{a}\right)$$

$$= \arcsin\frac{x}{a} + C$$

例 9

求 $\int \frac{\mathrm{d}x}{x(1+2\ln x)}$

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求 $\int \frac{\mathrm{d}x}{x(1+2\ln x)}$

解

$$\int \frac{\mathrm{d}x}{x(1+2\ln x)} = \frac{1}{2}\int \frac{\mathrm{d}(1+2\ln x)}{1+2\ln x}$$

例 9

求 $\int \frac{\mathrm{d}x}{x(1+2\ln x)}$

解

$$\int \frac{\mathrm{d}x}{x(1+2\ln x)} = \frac{1}{2}\int \frac{\mathrm{d}(1+2\ln x)}{1+2\ln x} = \frac{1}{2}\ln|1+2\ln x| + C.$$

例 10

求 $\int \frac{e^{3\cdot\sqrt{x}}}{\sqrt{x}}\mathrm{d}x$

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求 $\int \frac{e^{3\cdot\sqrt{x}}}{\sqrt{x}}\mathrm{d}x$

解

$$\int \frac{e^{3\cdot\sqrt{x}}}{\sqrt{x}}\mathrm{d}x = \frac{2}{3}\int e^{3\cdot\sqrt{x}}\mathrm{d}(3\sqrt{x})$$

例 10

求 $\int \frac{e^{3\cdot\sqrt{x}}}{\sqrt{x}}\mathrm{d}x$

解

$$\int \frac{e^{3\cdot\sqrt{x}}}{\sqrt{x}}\mathrm{d}x = \frac{2}{3}\int e^{3\cdot\sqrt{x}}\mathrm{d}(3\sqrt{x}) = \frac{2}{3}e^{3\cdot\sqrt{x}} + C.$$

随堂练习

使用第一类换元积分法求解下列不定积分

(1) $\int \sin(ax)\mathrm{d}x \ (a \neq 0)$

(2) $\int \frac{1}{2+2x}\mathrm{d}x$

(3) $\int \cos(\cos x)\sin x \mathrm{d}x$

(4) $\int e^{\sin x}\cos x \mathrm{d}x$

(5) $\int \frac{\sin x}{1+\cos x}\mathrm{d}x$

(6) $\int \frac{1}{\sqrt[3]{x^2}[1+(\sqrt[3]{x})^2]}\mathrm{d}x$

随堂练习

使用第一类换元积分法求解下列不定积分

(1) $\int \sin(ax)\mathrm{d}x \ (a \neq 0)$ $= -\frac{1}{a}\cos(ax) + C$

(2) $\int \frac{1}{2+2x}\mathrm{d}x$ $= \frac{1}{2}\ln|2+2x| + C$

(3) $\int \cos(\cos x)\sin x\mathrm{d}x$ $= -\sin(\cos x) + C$

(4) $\int e^{\sin x}\cos x\mathrm{d}x$ $= e^{\sin x} + C$

(5) $\int \frac{\sin x}{1+\cos x}\mathrm{d}x$ $= -\ln|1+\cos x| + C$

(6) $\int \frac{1}{\sqrt[3]{x^2}[1+(\sqrt[3]{x})^2]}\mathrm{d}x$ $= 3\arctan\sqrt[3]{x} + C$

第二类换元积分法

$$\int f(x)\mathrm{d}x \xRightarrow{\text{令}x=\varphi(t)}$$

第二类换元积分法

$$\int f(x)\mathrm{d}x \xlongequal{令x=\varphi(t)} \int f[\varphi(t)]\mathrm{d}\varphi(t) = \int f[\varphi(t)]\varphi'(t)\mathrm{d}t$$

定理 2

设 $x=\varphi(t)$ 是单调的可导函数，并且 $\varphi'(t)\neq 0$. 又设 $f[\varphi(t)]\varphi'(t)$ 具有原函数，则有换元公式

$$\int f(x)\mathrm{d}x=\left[\int f[\varphi(t)]\varphi'(t)\mathrm{d}t\right]_{t=\varphi^{-1}(x)}$$

其中 $\varphi^{-1}(x)$ 是 $x=\varphi(t)$ 的反函数.

- 将 $t=\varphi(t)$ 代入等式右边，两端同时关于 x 求导，利用反函数求导法则即可证明等式成立.
- $\varphi(t)$ 的单调性确保其反函数 $\varphi^{-1}(x)$ 存在.

随堂练习

使用第二类换元积分法求解以下不定积分

(1) $\int \frac{1}{1+\sqrt{2x}} \mathrm{d}x$

(2) $\int \frac{1}{\sqrt{1+e^x}} \mathrm{d}x$

(3) $\int \frac{x}{(1-x)^3} \mathrm{d}x$

(4) $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}}$

(5) $\int \frac{\sqrt{1-x^2}}{x^2} \mathrm{d}x$

(1) 求 $\int \frac{1}{1+\sqrt{2x}}\mathrm{d}x$

解: 令 $\sqrt{2x}=t$, $x=\frac{t^2}{2}$, $\mathrm{d}x=t\mathrm{d}t$, 代入所求积分

$$
\begin{aligned}
\int \frac{1}{1+\sqrt{2x}}\mathrm{d}x &= \int \frac{t}{1+t}\mathrm{d}t \\
&= \int \frac{t+1-1}{1+t}\mathrm{d}t \\
&= \int 1\mathrm{d}t - \int \frac{1}{1+t}\mathrm{d}t \\
&= \int 1\mathrm{d}t - \int \frac{1}{1+t}\mathrm{d}(1+t) \\
&= t-\ln(1+t)+C \\
&= \sqrt{2x}-\ln(1+\sqrt{2x})+C
\end{aligned}
$$

(2) 求 $\int \frac{1}{\sqrt{1+e^x}}\mathrm{d}x$

解: 令 $\sqrt{1+e^x}=u$, $x=\ln(u^2-1)$ $(u>1)$, $\mathrm{d}x=\frac{2u}{u^2-1}\mathrm{d}u$. 代入所求积分

$$\begin{aligned}
\int \frac{1}{\sqrt{1+e^x}}\mathrm{d}x &= \int \frac{1}{u}\cdot\frac{2u}{u^2-1}\mathrm{d}u \\
&= \int \frac{2}{u^2-1}\mathrm{d}u \\
&= \int \left(\frac{1}{u-1}-\frac{1}{u+1}\right)\mathrm{d}u \\
&= \int \frac{1}{u-1}\mathrm{d}(u-1)-\int \frac{1}{u+1}\mathrm{d}(u+1) \\
&= \ln\frac{u-1}{u+1}+C=\ln\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}+C
\end{aligned}$$

(3) 求 $\int \frac{x}{(1-x)^3}\mathrm{d}x$

解: 令 $1-x=t$, $x=1-t$, $\mathrm{d}x=-\mathrm{d}t$, 代入所求积分

$$\begin{aligned}
\int \frac{x}{(1-x)^3}\mathrm{d}x &= \int \frac{t-1}{t^3}\mathrm{d}t \\
&= \int \frac{1}{t^2}\mathrm{d}t - \int \frac{1}{t^3}\mathrm{d}t \\
&= -\frac{1}{t} + \frac{1}{2}\frac{1}{t^2} + C \\
&= \frac{1}{x-1} + \frac{1}{2(1-x)^2} + C
\end{aligned}$$

(4) 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}}$

解: 令 $x=\frac{1}{t}$, $\mathrm{d}x=-\frac{1}{t^2}\mathrm{d}t$, 代入所求不定积分

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} = \int \frac{t^2|t|}{\sqrt{1-t^2}}\cdot\left(-\frac{1}{t^2}\right)\mathrm{d}t$$
$$= \int -\frac{|t|}{\sqrt{1-t^2}}\mathrm{d}t$$

当 $t>0$ 时,

$$\int -\frac{|t|}{\sqrt{1-t^2}}\mathrm{d}t = \int \frac{-t}{\sqrt{1-t^2}}\mathrm{d}t = \frac{1}{2}\int \frac{1}{\sqrt{1-t^2}}\mathrm{d}(1-t^2) = (1-t^2)^{\frac{1}{2}}+C = \sqrt{1-\frac{1}{x^2}}+C$$
$$= \frac{\sqrt{x^2-1}}{x}+C$$

(4) 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}}$

当 $t<0$ 时

$$\int -\frac{|t|}{\sqrt{1-t^2}}\mathrm{d}t = \int \frac{t}{\sqrt{1-t^2}}\mathrm{d}t = -\frac{1}{2}\int \frac{1}{\sqrt{1-t^2}}\mathrm{d}(1-t^2) = -(1-t^2)^{\frac{1}{2}} + C$$

$$= -\sqrt{1-\frac{1}{x^2}} + C$$

$$= -\frac{\sqrt{x^2-1}}{-x} + C = \frac{\sqrt{x^2-1}}{x} + C$$

综上

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x} + C$$

(4) 求 $\int \frac{dx}{x^2\sqrt{x^2-1}}$(解法二)

因为 $x^2-1>0$, 故 $x>1$ 或 $x<-1$.

(1) 当 $x>1$ 时, 令 $x=\sec t \ (0<t<\frac{\pi}{2})$,

$$dx = d\sec t = d\frac{1}{\cos t} = \frac{\sin t}{\cos^2 t}dt = \frac{\sin t}{\cos t}\frac{1}{\cos t} = \tan t \sec t dt,$$

因为 $\sec^2 t - \tan^2 t = 1$, 故

$$\sqrt{x^2-1} = \sqrt{\tan^2 t} = |\tan t| = \tan t \ (0<t<\frac{\pi}{2}).$$

代入所求不定积分得到

$$\int \frac{1}{x^2\sqrt{x^2-1}}dx = \int \frac{\tan t \sec t}{\sec^2 t \tan t}dt = \int \frac{1}{\sec t}dt = \int \cos t dt = \sin t + C.$$

(4) 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}}$(解法二)

接下来用 x 表示 $\sin t$ $(0 < t < \frac{\pi}{2})$,
因为 $\sec t = x$, $\tan t = \sqrt{x^2-1}$, 故

$$\sin t = \frac{\sin t}{\cos t}\cos t = \frac{\tan t}{\sec t} = \frac{\sqrt{x^2-1}}{x},$$

即

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} = \sin t + C = \frac{\sqrt{x^2-1}}{x} + C.$$

(2) 当 $x < -1$ 时, 令 $u = -x$, 则 $u > 1$, 且 $\mathrm{d}x = -\mathrm{d}u$. 代入所求不定积分,
并由 **(1)** 的结论得到

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} = -\int \frac{\mathrm{d}u}{u^2\sqrt{u^2-1}} = -\frac{\sqrt{u^2-1}}{u} + C = \frac{\sqrt{x^2-1}}{x} + C.$$

(4) 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}}$(解法二)

综合 **(1)** 和 **(2)**, 无论 $x>1$ 或 $x<-1$, 均有

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x} + C.$$

(5) $\int \frac{\sqrt{1-x^2}}{x^2}\mathrm{d}x$

解: 令 $x=\sin t(-\frac{\pi}{2}\le t\le \frac{\pi}{2})$, $\mathrm{d}x=\cos t\mathrm{d}t$, $\sqrt{1-x^2}=\cos t$, 代入所求不定积分

$$\begin{aligned}
\int \frac{\sqrt{1-x^2}}{x^2}\mathrm{d}x &= \int \frac{\cos t}{\sin^2 t}\cos t\mathrm{d}t \\
&= \int \frac{\cos^2 t}{\sin^2 t}\mathrm{d}t \\
&= \int \frac{\cos^2 t+\sin^2 t-\sin^2 t}{\sin^2 t}\mathrm{d}t \\
&= \int \frac{1}{\sin^2 t}\mathrm{d}t-\int 1\mathrm{d}t \\
&= -\frac{\cos t}{\sin t}-t+C=-\frac{\sqrt{1-x^2}}{x}-\arcsin x+C.
\end{aligned}$$

例 21

求 $\int\sqrt{a^2-x^2}\mathrm{d}x\ (a>0)$

例 21

求 $\int \sqrt{a^2 - x^2}\mathrm{d}x \ (a > 0)$

解　要使上述积分有意义, 则 $-a \le x \le a$.

例 21

求 $\int \sqrt{a^2 - x^2} \mathrm{d}x \ (a > 0)$

解　要使上述积分有意义, 则 $-a \le x \le a$.

令 $x = a\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, 则

$$\int \sqrt{a^2 - x^2} \mathrm{d}x = \int \sqrt{a^2 - a^2 \sin^2 t}\ \mathrm{d}(a\sin t) = a^2 \int \cos^2 t \mathrm{d}t$$

$$= a^2 \int \frac{1 + \cos 2t}{2} \mathrm{d}t$$

例 21

求 $\int \sqrt{a^2 - x^2} \mathrm{d}x \ (a > 0)$

解　要使上述积分有意义, 则 $-a \le x \le a$.

令 $x = a \sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, 则

$$\begin{aligned}
\int \sqrt{a^2 - x^2} \mathrm{d}x = \int \sqrt{a^2 - a^2 \sin^2 t} \ \mathrm{d}(a \sin t) &= a^2 \int \cos^2 t \mathrm{d}t \\
&= a^2 \int \frac{1 + \cos 2t}{2} \mathrm{d}t \\
&= a^2 \int \frac{1}{2} \mathrm{d}t + \frac{a^2}{4} \int \cos(2t) \ \mathrm{d}(2t) \\
&= \frac{a^2 t}{2} + \frac{a^2 \sin 2t}{4} + C
\end{aligned}$$

又因为 $x=a\sin t$, $-\frac{\pi}{2}\le t\le\frac{\pi}{2}$, 故

$$t=\arcsin\frac{x}{a} \qquad -a\le x\le a$$

$$a\cos t=\sqrt{a^2-a^2\sin^2 t}=\sqrt{a^2-x^2} \qquad -\frac{\pi}{2}\le t\le\frac{\pi}{2}$$

又因为 $x = a\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, 故

$$t = \arcsin\frac{x}{a} \qquad -a \le x \le a$$

$$a\cos t = \sqrt{a^2 - a^2\sin^2 t} = \sqrt{a^2 - x^2} \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

因此

$$\begin{aligned}\int\sqrt{a^2-x^2}\mathrm{d}x &= \frac{a^2 t}{2} + \frac{a^2\sin 2t}{4} + C \\ &= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{4}2\cos t\sin t + C\end{aligned}$$

又因为 $x = a\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, 故

$$t = \arcsin\frac{x}{a} \qquad -a \le x \le a$$

$$a\cos t = \sqrt{a^2 - a^2\sin^2 t} = \sqrt{a^2 - x^2} \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

因此

$$\begin{aligned}\int \sqrt{a^2 - x^2}\mathrm{d}x &= \frac{a^2 t}{2} + \frac{a^2 \sin 2t}{4} + C \\ &= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{4} 2\cos t\sin t + C \\ &= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{1}{2}x\sqrt{a^2 - x^2} + C.\end{aligned}$$

例 24

求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x\ (a \neq 0)$

例 24

求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x \ (a \neq 0)$

解 令 $x = \frac{1}{t}$, 代入所求不定积分.

$$\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x = \int \sqrt{a^2-\frac{1}{t^2}} \cdot t^4 \cdot \mathrm{d}\left(\frac{1}{t}\right)$$

例 24

求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x \ (a \neq 0)$

解　令 $x = \frac{1}{t}$, 代入所求不定积分.

$$\begin{aligned}
\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x &= \int \sqrt{a^2-\frac{1}{t^2}} \cdot t^4 \cdot \mathrm{d}\left(\frac{1}{t}\right) \\
&= \int \sqrt{a^2t^2-1} \cdot \frac{1}{|t|} \cdot t^4 \left(-\frac{1}{t^2}\right)\mathrm{d}t \\
&= -\int (a^2t^2-1)^{\frac{1}{2}}|t|\mathrm{d}t
\end{aligned}$$

$$\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x = -\int (a^2t^2-1)^{\frac{1}{2}}|t|\mathrm{d}t$$

当 $x>0$，即 $t>0$，

$$\begin{aligned}\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x- &= \int (a^2t^2-1)^{\frac{1}{2}}t\mathrm{d}t\\ &= -\int (a^2t^2-1)^{\frac{1}{2}}\cdot\frac{1}{2a^2}\cdot(a^2t^2-1)'\mathrm{d}t\\ &= -\frac{1}{2a^2}\int (a^2t^2-1)^{\frac{1}{2}}\mathrm{d}(a^2t^2-1)\end{aligned}$$

$$\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x = -\int (a^2t^2-1)^{\frac{1}{2}}|t|\mathrm{d}t$$

当 $x>0$, 即 $t>0$,

$$\begin{aligned}
\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x- &= \int (a^2t^2-1)^{\frac{1}{2}}t\mathrm{d}t \\
&= -\int (a^2t^2-1)^{\frac{1}{2}}\cdot \frac{1}{2a^2}\cdot (a^2t^2-1)'\mathrm{d}t \\
&= -\frac{1}{2a^2}\int (a^2t^2-1)^{\frac{1}{2}}\mathrm{d}(a^2t^2-1) \\
&= -\frac{1}{2a^2}\cdot\frac{2}{3}\left(a^2t^2-1\right)^{\frac{3}{2}}+C
\end{aligned}$$

$$\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x = -\int (a^2t^2-1)^{\frac{1}{2}}|t|\mathrm{d}t$$

当 $x>0$, 即 $t>0$,

$$\begin{aligned}
\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x- &= \int (a^2t^2-1)^{\frac{1}{2}}t\mathrm{d}t \\
&= -\int (a^2t^2-1)^{\frac{1}{2}}\cdot \frac{1}{2a^2}\cdot (a^2t^2-1)'\mathrm{d}t \\
&= -\frac{1}{2a^2}\int (a^2t^2-1)^{\frac{1}{2}}\mathrm{d}(a^2t^2-1) \\
&= -\frac{1}{2a^2}\cdot\frac{2}{3}\left(a^2t^2-1\right)^{\frac{3}{2}}+C \\
&= -\frac{1}{3a^2x^3}\left(a^2-x^2\right)^{\frac{3}{2}}+C
\end{aligned}$$

当 $x<0$ 时, 有相同的结果. 综上,

$$\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x = -\frac{1}{3a^2x^3}\left(a^2-x^2\right)^{\frac{3}{2}}+C$$

新增常用不定积分公式 (a>0)

(1) $\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C$

(2) $\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

(3) $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$

(4) $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) + C$

(5) $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}| + C$

例 26

求 $\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$

例 26

求 $\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$

$$\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}} = \int \frac{\mathrm{d}x}{\sqrt{(2x)^2+3^2}}$$

例 26

求 $\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}} &= \int \frac{\mathrm{d}x}{\sqrt{(2x)^2+3^2}} \\ &= \int \frac{1}{2} \frac{\mathrm{d}(2x)}{\sqrt{(2x)^2+3^2}}\end{aligned}$$

例 26

求 $\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}} &= \int \frac{\mathrm{d}x}{\sqrt{(2x)^2+3^2}} \\
&= \int \frac{1}{2} \frac{\mathrm{d}(2x)}{\sqrt{(2x)^2+3^2}} \\
&= \frac{1}{2} \ln(2x+\sqrt{4x^2+9})+C
\end{aligned}
$$

作业

- 教材习题 4-2: 1(1)(3)(5)(7)(14);
  2(1)(3)(4)(7)(8)(9)(10)(21)(30)(31)(34)(36)(37)