A STACKY p-ADIC RIEMANN–HILBERT CORRESPONDENCE ON HITCHIN-SMALL LOCUS

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ABSTRACT. Let C be a algebraically closed perfectoid field over $\mathbb{Q}_p$ with the ring of integer $\mathcal{O}_C$ and the infinitesimal thickening $\mathbf{A}_{\mathrm{inf}}$. Let $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{O}_C$ with a fixed smooth lifting $\widetilde{\mathfrak{X}}$ over $\mathbf{A}_{\mathrm{inf}}$. Let X be the generic fiber of $\mathfrak{X}$ and $\widetilde{X}$ be its lifting over $\mathbf{B}_{\mathrm{dR}}^+$ induced by $\widetilde{\mathfrak{X}}$. Let $\mathrm{MIC}_r(\widetilde{X})^{\text{H-small}}$ and $\mathrm{LS}_r(X,\mathbb{B}_{\mathrm{dR}}^+)^{\text{H-small}}$ be the v-stacks of rank-r Hitchin-small integrable connections on $X_{\mathrm{\acute{e}t}}$ and $\mathbb{B}_{\mathrm{dR}}^+$-local systems on X_v, respectively. In this paper, we establish an equivalence between this two stacks by introducing a new period sheaf with connection $(\mathcal{O}\mathbb{B}_{\mathrm{dR,pd}}^+,\mathrm{d})$ on X_v.

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1. INTRODUCTION

1.1. **Overview.** For a projective smooth variety over the the field $\mathbb{C}$ of complex numbers, the Riemann–Hilbert correspondence describes an equivalence between the category $\mathbb{C}$-local systems and the category of integrable connections on X. This equivalence upgrades to the moduli level; that is, there is an homeomorphism between the moduli space $\mathbf{M}_B$ of $\mathbb{C}$-local systems and the moduli space $\mathbf{M}_{\mathrm{dR}}$ of integrable connections.

The p-adic Riemann–Hilbert correspondence aims to give an analogue of the above correspondence for rigid spaces; that is, it suggests an equivalence between the category of certain local systems and the category of certain integrable connections on a rigid spaces over a complete p-adic field. The first step towards this direction is due to Scholze [Sch13]. In *loc.cit.*, for a smooth rigid variety X over a complete discrete valuation field K of mixed characteristic $(0,p)$ with the perfect residue field, he introduced so-call $\mathbb{B}_{\mathrm{dR}}^+$-*local systems* on the pro-étale site $X_{\mathrm{pro\acute{e}t}}$, constructed a period sheaf with connection $(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+,\mathrm{d})$ on $X_{\mathrm{pro\acute{e}t}}$ whose de Rham complex gives a resolution of $\mathbb{B}_{\mathrm{dR}}^+$, and established an equivalence between the category of *de Rham* $\mathbb{B}_{\mathrm{dR}}^+$-local systems and the category of *filtered integrable connections* on $X_{\mathrm{\acute{e}t}}$. Based on his work, Liu and Zhu constructed two functors $\mathcal{RH}$ and D_{dR} from the category of $\mathbb{Q}_p$-local systems on $X_{\mathrm{\acute{e}t}}$ to the category of G_K-equivariant filtered integrable connections on "$(X\widehat{\otimes}_K\mathbf{B}_{\mathrm{dR}}^+)_{\mathrm{\acute{e}t}}$" and the category of filtered integrable connections on $X_{\mathrm{\acute{e}t}}$, respectively. Using this, they proved the rigidity for "being de Rham" of $\mathbb{Q}_p$-local systems [LZ17]. Their approach also works in the theory of log-geometry [DLLZ23]. By taking grades, the functor $\mathcal{RH}$ induces a functor $\mathcal{H}$ from the category of $\mathbb{Q}_p$-local systems on $X_{\mathrm{\acute{e}t}}$ to the category of G_K-equivariant Higgs bundles on $X_{\widehat{\overline{K}},\mathrm{\acute{e}t}}$. More generally, it was shown that $\mathcal{H}$ upgrades to an equivalence from the category of generalised representations on $X_{\mathrm{pro\acute{e}t}}$ to the category of G_K-equivariant Higgs bundles on $X_{\widehat{\overline{K}},\mathrm{\acute{e}t}}$ (cf. [MW22]). The similar phenomenon also occurs when studying Riemann–Hilbert correspondence, in [GMW], one can establish an equivalence between the category of $\mathbb{B}_{\mathrm{dR}}^+$-local systems on $X_{\mathrm{pro\acute{e}t}}$ and the category of G_K-equivariant integrable connections on "$X\widehat{\otimes}_K\mathbf{B}_{\mathrm{dR}}^+$", which can be used to classify $\mathbb{B}_{\mathrm{dR}}^+$-local systems with prismatic source (see [MW22], [AHLB23b] for relevant results in "mod t"

2020 *Mathematics Subject Classification.* Primary .

Key words and phrases. p-adic Riemman-Hilbert correspondence, period sheaf, $\mathbb{B}_{\mathrm{dR}}^+$-local systems, flat connections, Hitchin-small locus.

case). On the other hand, in the lecture note of his ICM talk, Bhatt described his joint work with Lurie [Bha21, Th. 5.4], a Riemann–Hilbert functor from the category of certain $\mathbb{Q}_p$-sheaves to the category of certain D-modules. Parts of their results are also obtained by Li [Li] independently by using a certain variant of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ and applying the approach of Liu–Zhu [LZ17]. It is worth emphasizing that all results above require X is defined over a complete discrete valuation field K containing $\mathbb{Q}_p$ with the perfect residue field. The advantage in this case is that all $\mathbb{B}_{\mathrm{dR}}^+$-local systems and integrable connections are contained in the fibers of the corresponding muduli stacks at the origin of the Hitchin base (cf. Remark 1.8 below).

Now, let C be a algebraically closed perfectoid field containing $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_C$ and the infinitesimal thickening $\mathbf{A}_{\mathrm{inf}}$. Very recently, starting with a smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_C$ admitting a smooth lifting $\widetilde{\mathfrak{X}}$ over $\mathbf{A}_{\mathrm{inf}}$, Chen, Liu, Zhu and the last author [CLWZ24] can constructed an overconvergent de Rham sheaf with connection $(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}, \mathrm{d})$ whose de Rham complex is again a resolution of $\mathbb{B}_{\mathrm{dR}}^+$. Using this, they are able to establish an equivalence between the category of *(Faltings-)small* $\mathbb{B}_{\mathrm{dR}}^+$-local systems on $X_{\mathrm{pro\acute{e}t}}$ and the category of *(Faltings-)small* integrable connections on $\widetilde{X}$, where X is the generic fiber of $\mathfrak{X}$ and $\widetilde{X}$ is the lifting of X over $\mathbf{B}_{\mathrm{dR}}^+$ induced by $\widetilde{\mathfrak{X}}$. Modulo t, this equivalence coincides with Faltings' p-adic Simpson correspondence (cf. [Fal05], [AGT16], [Wan23], and *etc.*). More generally, in *loc.cit.*, the authors even constructed a period sheaf with connection $(\mathcal{O}\widetilde{\mathbb{C}}^I, \mathrm{d})$ which induces a resolution of relative Robba ring $\widetilde{\mathbb{C}}^I$ and established a kind of Riemann–Hilbert correspondence for local systems and integrable connections over Robba rings.

In this paper, let $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{O}_C$ with a fixed smooth lifting $\widetilde{\mathfrak{X}}$ over $\mathbf{A}_{\mathrm{inf}}$. We shall establish an equivalence between the category of *Hitchin-small* $\mathbb{B}_{\mathrm{dR}}^+$-local systems on X_v, the v-site of X introduced in [Sch17], and the category of *Hitchin-small* integrable connections on $\widetilde{X}$. Indeed, we shall prove this on the moduli level; that is, we shall give an equivalence between the moduli stacks of Hitchin-small $\mathbb{B}_{\mathrm{dR}}^+$-local systems and Hitchin-small integrable connections by constructing a new period sheaf with connection $(\mathcal{O}\mathbb{B}_{\mathrm{dR,pd}}^+, \mathrm{d})$, whose reduction behaves like the period sheaves $\mathcal{B}_{\widetilde{\mathfrak{X}}}$ and $\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}}$ considered in [AHLB23b] and [MW23]. In particular, our work generalises a stacky p-adic Simpson correspondence in [AHLB23b]. As Faltings-smallness always implies Hithcin-smallness, our work also gives a slightly generalisation of the p-adic Riemann–Hilbert correspondence in [CLWZ24].

1.2. **Main results.** Now let us state our main theorem. From now on, we always let C be a algebraically closed perfectoid field containing $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_C$, maximal ideal $\mathfrak{m}_C$ of $\mathcal{O}_C$ and the infinitesimal thickening $\mathbf{A}_{\mathrm{inf}}$. Fix a compatible sequence $\{\zeta_{p^n}\}_{n\geq 0}$ of primitive p^n-th roots of unity in C. We also let $\mathfrak{X}$ be a liftable smooth formal scheme over $\mathcal{O}_C$ with a fixed lifting $\widetilde{\mathfrak{X}}$ over $\mathbf{A}_{\mathrm{inf}}$. Let X be the generic fiber of $\mathfrak{X}$ and $\widetilde{X}$ be its lifting over $\mathbf{B}_{\mathrm{dR}}^+$ induced by $\widetilde{\mathfrak{X}}$. For any $n\geq 1$, let $\widetilde{X}_n$ be the reduction of $\widetilde{X}$ modulo t^n. Let Perfd be the v-site of all affinoid perfectoid spaces over $\mathrm{Spa}(C, \mathcal{O}_C)$.

Theorem 1.1. *For any $n\geq 1$, there is an equivalence*

$$\rho_{\widetilde{\mathfrak{X}}}: \mathrm{LS}_r(X, \mathbb{B}_{\mathrm{dR},n}^+)^{\text{H-small}} \simeq \mathrm{MIC}_r(\widetilde{X}_n)^{\text{H-small}}$$

between the moduli stack $\mathrm{LS}_r(X, \mathbb{B}_{\mathrm{dR},n}^+)^{\text{H-small}}$ of Hitchin-small $\mathbb{B}_{\mathrm{dR},n}^+$-local systems of rank r on X_v and the moduli stack $\mathrm{MIC}_r(\widetilde{X}_n)^{\text{H-small}}$ of Hitchin-small integrable connections of rank r on $\widetilde{X}_n$.

Remark 1.2. The $n=1$ case for above theorem was already obtained by Anschütz, Heuer and Le Bras [AHLB23b], where the functor $\rho_{\widetilde{\mathfrak{X}}}$ (in the $n=1$ case) is denoted by $S_{\widetilde{\mathfrak{X}}}$. We remark the injectivity of $S_{\widetilde{\mathfrak{X}}}$ (that is, the full faithfulness of its evaluation at every $\mathrm{Spa}(A, A^+)\in \mathrm{Perfd}$) was proved in [AHLB23a], and the surjectivity was established in [AHLB23b] by checking this locally and then gluing together. Compared with theirs, our construction provides an *explicit* and *global* description of $\rho_{\widetilde{\mathfrak{X}}}^{-1}$.

Now, we explain some notions appearing in above Theorem 1.1. Let $S=\mathrm{Spa}(A, A^+)\in \mathrm{Perfd}$ be an affinoid perfectoid space. Let $\mathfrak{Z}$ be a smooth formal scheme over $\mathrm{Spf}(A^+)$ with a smooth lifting

$\widetilde{\mathfrak{Z}}$ over $\mathbb{A}_{\mathrm{inf}}(A,A^+)$. Let Z be the generic fiber of $\mathfrak{Z}$ with the lifting $\widetilde{Z}$ over $\mathbf{B}^+_{\mathrm{dR}}$ induced by $\widetilde{\mathfrak{Z}}$. For any $n\geq 1$, let $\widetilde{Z}_n$ be the reduction of $\widetilde{Z}$ modulo t^n and $\mathbb{B}^+_{\mathrm{dR},n}:=\mathbb{B}^+_{\mathrm{dR}}/t^n$.

By a $\mathbb{B}^+_{\mathrm{dR},n}$*-local system* of rank r on Z_v, the v-site of Z in the sense of [Sch17], we mean a locally finite free $\mathbb{B}^+_{\mathrm{dR},n}$-module of rank r. When $n=1$, the $\mathbb{B}^+_{\mathrm{dR},1}$ is the structure sheaf $\widehat{\mathcal{O}}_Z$, and the $\mathbb{B}^+_{\mathrm{dR},1}$-local systems reduce to the v-vector bundles on Z_v. One can similarly define $\mathbb{B}^+_{\mathrm{dR}}$-local systems of rank r on Z_v.

Let $\widetilde{\mathrm{d}}:\widetilde{\mathfrak{Z}}\to\Omega^1_{\widetilde{\mathfrak{Z}}}$ be the usual (p,ξ)-complete derivation on $\widetilde{\mathfrak{Z}}$ and let d be the following composite

$$\mathrm{d}:\mathcal{O}_{\widetilde{\mathfrak{Z}}}\xrightarrow{\widetilde{\mathrm{d}}}\Omega^1_{\widetilde{\mathfrak{Z}}}\hookrightarrow\Omega^1_{\widetilde{\mathfrak{Z}}}\{-1\}.$$

Here $\Omega^1_{\widetilde{\mathfrak{Z}}}\{-1\}$ denotes the Breuil–Kisin–Fargues twist of $\Omega^1_{\widetilde{\mathfrak{Z}}}$ (cf. §1.4). By an *integrable connection* of rank r on $\widetilde{Z}_n$, we mean a pair $(\mathcal{M},\nabla)$ consisting of a locally finite free $\mathcal{O}_{\widetilde{Z}_n}$-module $\mathcal{M}$ together with a $\mathbb{B}^+_{\mathrm{dR},n}(A,A^+)$-linear map

$$\nabla:\mathcal{M}\to\mathcal{M}\otimes_{\mathcal{O}_{\widetilde{\mathfrak{Z}}}}\Omega^1_{\widetilde{\mathfrak{Z}}}\{-1\}$$

satisfying the Leibniz rule with respect to d such that $\nabla\wedge\nabla=0$. One can similarly define integrable connections of rank r on $\widetilde{Z}$. We remark that if we trivialize the Tate twist $\Omega^1_{\widetilde{Z}}(-1)$ by $\Omega^1_{\widetilde{Z}}\cdot t^{-1}$, then via the identification

$$\mathcal{M}\otimes_{\mathcal{O}_{\widetilde{\mathfrak{Z}}}}\Omega^1_{\widetilde{\mathfrak{Z}}}\{-1\}=\mathcal{M}\otimes_{\mathcal{O}_{\widetilde{\mathfrak{Z}}}}\Omega^1_{\widetilde{\mathfrak{Z}}}(-1),$$

locally on $\widetilde{Z}_n$, ∇ behaves like a t*-connection* in the sense of [Yu24]. In particular, when $n=1$, we see that $\mathrm{d}=0$ and the integrable connections on $\widetilde{Z}_1=Z$ reduce to the Higgs bundles on Z.

For a smooth $\mathfrak{X}$ over $\mathcal{O}_C$ with the fixed lifting $\widetilde{\mathfrak{X}}$ over $\mathbf{A}_{\mathrm{inf}}$, for any $S=\mathrm{Spa}(A,A^+)\in\mathrm{Perfd}$, let $\mathfrak{X}_A$, $\widetilde{\mathfrak{X}}_A$, X_A and $\widetilde{X}_A$ be the base-changes of $\mathfrak{X}$, $\widetilde{\mathfrak{X}}$, X and $\widetilde{X}$ to A^+, $\mathbb{A}_{\mathrm{inf}}(A,A^+)$, A and $\mathbb{B}^+_{\mathrm{dR}}(A,A^+)$, respectively. Consider the following two functors:

$$\mathrm{LS}_r(X,\mathbb{B}^+_{\mathrm{dR},n}):\mathrm{Perfd}\to\mathrm{Groupoid},\quad S\mapsto\{\mathbb{B}^+_{\mathrm{dR},n}\text{-local systems on }(X_A)_v\text{ of rank }r\}$$

and

$$\mathrm{MIC}_r(\widetilde{X}_n):\mathrm{Perfd}\to\mathrm{Groupoid},\quad S\mapsto\{\text{integrable connections on }\widetilde{X}_{A,n}\text{ of rank }r\}.$$

Remark 1.3. For $n=1$, the above functors were first considered by Heuer [Heu22] without assuming X has a good reduction. In *loc.cit.*, he introduced the functors $v\mathrm{Bun}_r(X)$ (resp. $\mathrm{HIG}_r(X)$) of v-vector bundles (resp. Higgs bundles) of rank r on $X_{\text{proét}}$ (resp. $X_{\text{ét}}$), which is exactly the $\mathrm{LS}_r(X,\mathbb{B}^+_{\mathrm{dR},1})$ (resp. $\mathrm{MIC}_r(\widetilde{X}_1)$) above. He proved these two stacks are small v-stacks on Perfd and isomorphic after taking étale sheafifications. For general n, the above functors are introduced by Yu [Yu24]. In *loc.cit.*, Yu also proved all stacks above are small v-stacks, and $\mathrm{LS}_r(X,\mathbb{B}^+_{\mathrm{dR},n})$ is isomorphic to $\mathrm{MIC}_r(\widetilde{X}_n)$ after taking étale sheafifications.

Clearly, we have the following diagram:
(1.1)

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] \\
\mathrm{LS}(X, \mathbb{B}^+_{\mathrm{dR},n+1}) \arrow[d] & & \mathrm{MIC}_r(\widetilde{X}_{n+1}) \arrow[d] \\
\mathrm{LS}(X, \mathbb{B}^+_{\mathrm{dR},n}) \arrow[d] & & \mathrm{MIC}_r(\widetilde{X}_n) \arrow[d] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
v\mathrm{Bun}_r(X) = \mathrm{LS}(X, \mathbb{B}^+_{\mathrm{dR},1}) \arrow[dr, "\widetilde{h}"'] & & \mathrm{MIC}_r(\widetilde{X}_1) = \mathrm{HIG}_r(X) \arrow[dl, "h"] \\
& \mathcal{A}_r &
\end{tikzcd}$$

where all vertical maps are induced by taking the obvious reduction, h and $\widetilde{h}$ denote the Hitchin fibrations introduced in [Heu22], and $\mathcal{A}_r$ is the Hitchin base, which is defined as a functor

$$\mathcal{A}_r : \mathrm{Perfd} \to \mathrm{Sets}, \quad \mathrm{Spa}(A, A^+) \mapsto \oplus_{i=1}^r \mathrm{H}^0(X_A, \mathrm{Sym}^i(\Omega^1_{X_A/A}\{-1\})).$$

One can define the *Hitchin-small locus* $\mathcal{A}_r^{\text{H-small}} \subset \mathcal{A}_r$ by

$$\mathcal{A}_r^{\text{H-small}} : \mathrm{Perfd} \to \mathrm{Sets}, \quad S = \mathrm{Spa}(A, A^+) \mapsto \oplus_{i=1}^r p^{<\frac{i}{p-1}} \mathrm{H}^0(\mathfrak{X}_A, \mathrm{Sym}^i(\Omega^1_{\mathfrak{X}_A/A^+}\{-1\}))$$

as in [AHLB23b], where $p^{<\frac{i}{p-1}} := (\zeta_p - 1)^i \mathfrak{m}_C \subset \mathcal{O}_C$. The $\mathcal{A}_r$ and $\mathcal{A}_r^{\text{H-small}}$ both make sense as v-sheaves. For any stack Z lying over $\mathcal{A}_r$, define its Hitchin-small locus $\mathrm{Z}^{\text{H-small}}$ as the sub-stack

$$\mathrm{Z}^{\text{H-small}} := \mathrm{Z} \times_{\mathcal{A}_r} \mathcal{A}_r^{\text{H-small}}.$$

Now, we have explained all notions involved in Theorem 1.1. It is still worth pointing out that one can check the Hitchin-smallness by taking reduction modulo t:

Remark 1.4. A $\mathbb{B}^+_{\mathrm{dR},n}$-local system (resp. integrable connection on $\widetilde{X}_n$) is Hitchin-small *if and only if* its reduction modulo t is a Hitchin-small v-bundle on $X_{\text{proét}}$ (resp. Higgs bundle on $X_{\text{ét}}$).

We list some immediate corollaries. Again, let $\mathfrak{X}$ be a liftable smooth formal scheme over $\mathcal{O}_C$ with a fixed lifting $\widetilde{\mathfrak{X}}$ over $\mathbf{A}_{\text{inf}}$. First, by letting n go to ∞, we obtain the following equivalence of stacks:

Corollary 1.5. *There exists an equivalence*

$$\rho_{\widetilde{\mathfrak{X}}} : \mathrm{LS}_r(X, \mathbb{B}^+_{\mathrm{dR}})^{\text{H-small}} \simeq \mathrm{MIC}_r(\widetilde{X})^{\text{H-small}}$$

between the moduli stack $\mathrm{LS}_r(X, \mathbb{B}^+_{\mathrm{dR}})^{\text{H-small}}$ *of Hitchin-small* $\mathbb{B}^+_{\mathrm{dR}}$*-local systems of rank* r *and the moduli stack* $\mathrm{MIC}_r(\widetilde{X})^{\text{H-small}}$ *of Hitchin-small integrable connections of rank* r*.*

To simplify the notations, set $\mathbb{B}^+_{\mathrm{dR}} := \mathbb{B}^+_{\mathrm{dR},\infty}$ and $\widetilde{X}_\infty := \widetilde{X}$. The next corollary follows from by taking C-points in Theorem 1.1 and Corollary 1.5 immediately.

Corollary 1.6. *For any* $1 \leq n \leq \infty$*, there exists an equivalence*

$$\mathrm{LS}(X, \mathbb{B}^+_{\mathrm{dR},n})^{\text{H-small}}(C) \simeq \mathrm{MIC}(\widetilde{X}_n)^{\text{H-small}}(C)$$

between the category $\mathrm{LS}(X, \mathbb{B}^+_{\mathrm{dR},n})^{\text{H-small}}(C)$ *of Hitchin-small* $\mathbb{B}^+_{\mathrm{dR},n}$*-local systems on* X_v *and the category* $\mathrm{MIC}(\widetilde{X}_n)^{\text{H-small}}(C)$ *of Hitchin-small integrable connections on* $\widetilde{X}_n$*.*

This generalizes the equivalence between the categories of (Faltings-)small objects in [CLWZ24, Th.7.11]. See also Theorem 1.10 for a more explicit statement.

Corollary 1.7. *Let $\mathbb{P}^d$ be the projective space over C of dimension d. For any $1 \leq n \leq \infty$, there exists an equivalence of the whole stacks*

$$\mathrm{LS}_r(\mathbb{P}^d, \mathbb{B}^+_{\mathrm{dR},n}) \simeq \mathrm{MIC}_r(\widetilde{\mathbb{P}}^d_n).$$

Note that $\mathbb{P}^d$ always admits a liftable smooth formal model $\mathfrak{P}^d$ over $\mathcal{O}_C$, and $\widetilde{\mathbb{P}}^d_n$ is the lifting of $\mathbb{P}^d$ coming from a lifting of $\mathfrak{P}^d$.

Proof. This holds true as any $\mathbb{B}^+_{\mathrm{dR},n}$-local systems (resp. intergable connections on $\widetilde{\mathbb{P}}^d_n$) are automatically Hitchin-small (and even belong to the fiber of corresponding stacks at the origin $0 \in \mathcal{A}_r$). See [AHLB23b, Cor. 3.26] for details. $\square$

Remark 1.8. In Corollary 1.7, it is not necessary to work with the smooth integral model of $\mathbb{P}^n$. More generally, for any smooth rigid variety X over C with a smooth lifting $\widetilde{X}$ over $\mathbf{B}^+_{\mathrm{dR}}$, using the period sheaf with connection $(\mathcal{O}\mathbb{B}_{\mathrm{dR}}, \mathrm{d})$ in [Yu24, Def. 2.36] and the same argument in this paper, we can give an equivalence of stacks

$$\mathrm{LS}_r(X, \mathbb{B}^+_{\mathrm{dR},n})^0 \simeq \mathrm{MIC}_r(\widetilde{X}_n)^0$$

where Z^0 denotes the fiber at the origin $0 \in \mathcal{A}_r$ for any stack Z over $\mathcal{A}_r$. See [LMNQ24] for more details. When X is the base-change along $K \to C$ for some smooth X_0 defined over a complete discrete valuation sub-field $K \subset C$, the base-change $\widetilde{X}$ of X_0 along $K \to \mathbf{B}^+_{\mathrm{dR}}$ is a lifting of X. In this case, all $\mathbb{B}^+_{\mathrm{dR}}$-local systems on $X_{0,\text{ét}}$ and all G_K-equivariant integrable connections on $\widetilde{X}_{\text{ét}}$ belong to the fibers of the corresponding stacks at the origin $0 \in \mathcal{A}_r$ (See [GMW] or [MW22, Rem. 3.2]).

Remark 1.9. For general X, it is still a question if there exists an equivalence of the whole stacks

$$\mathrm{LS}_r(X, \mathbb{B}^+_{\mathrm{dR},n}) \simeq \mathrm{MIC}_r(\widetilde{X}_n). \tag{1.2}$$

Note that for $n = 1$; that is, in the case for v-vector bundles and Higgs bundles, it was prove by Heuer and Xu [HX24] that if X is a smooth curve, such an equivalence (1.2) exists up to a certain normalisation. Beyond curves, we only get an equivalence between the category of v-vector bundles on X_v and the category Higgs bundles on $X_{\text{ét}}$ for *proper* smooth X [Heu23]. It seems the problem (at least in the $n = 1$ case) might be solved by using the Simpson gerbe claimed by Bhargav Bhatt and Mingjia Zhang. However, for general $n > 1$, we know *nothing* on (1.2) besides results in this paper.

1.3. **Strategy for the proof and organization.** Here, we sketch the idea for the proof of Theorem 1.1 and explain how the paper is organized.

Fix a smooth $\mathfrak{X}$ with the fixed lifting $\widetilde{\mathfrak{X}}$. For any $S = \mathrm{Spa}(A, A^+) \in \mathrm{Perfd}$, let $\mathfrak{X}_A$, $\widetilde{\mathfrak{X}}_A$, X_A and $\widetilde{X}_A$ be the base-changes of $\mathfrak{X}$, $\widetilde{\mathfrak{X}}$, X and $\widetilde{X}$ to A^+, $\mathbb{A}_{\mathrm{inf}}(A, A^+)$, A and $\mathbb{B}^+_{\mathrm{dR}}(A, A^+)$, respectively. Recall that in order to get the stacky Simpson correspondence [AHLB23b, Th. 1.1], one have to construct an equivalence of categories

$$S_{\widetilde{\mathfrak{X}}_A} : \mathrm{HIG}^{\text{H-small}}_r(X)(A) \xrightarrow{\simeq} v\mathrm{Bun}_r(X, \widehat{\mathcal{O}})^{\text{H-small}}(A)$$

which is functorial in A. To define $S_{\widetilde{\mathfrak{X}}_A}$, one have to use the period sheaf with Higgs field $(\mathcal{B}_{\widetilde{X}_A}, \Theta)$ constructed in [AHLB23a] (depending on the lifting of $\mathfrak{X}$ over $\mathbf{A}_{\mathrm{inf}}/\xi^2$ and coming from prismatic theory of Bhatt–Lurie [BL22a] and [BL22b]), which locally behaves like a pd-polynomial ring over $\widehat{\mathcal{O}}_X$ with the usual derivation on the coordinates.

The key observation is that if one could construct a period sheaf with connection $(\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},A}, \mathrm{d})$ for each $\mathfrak{X}_A$ functorial in A, of which the reduction modulo t behaves (at least locally) like $(\mathcal{B}_{\widetilde{\mathfrak{X}}_A}, \Theta)$, then it seems possible to generalize the stacky Simpson correspondence to a stacky Riemann–Hilbert correspondence. The §2 is devoted to doing this. We also remark that our construction of $(\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},A}, \mathrm{d})$ is *self-contained*; that is, unlike the construction in [AHLB23a], we do not need to use any prismatic theory. It is indeed inspired of the constructions in [MW23, §2] and [CLWZ24, §3].

Using $(\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},A}, \mathrm{d})$, one can prove the following Riemann–Hilbert correspondence:

Theorem 1.10 (Theorem 4.1)**.** *Fix a* $\mathrm{Spa}(A, A^+) \in \mathrm{Perfd}$. *Let $\nu : X_{A,v} \to X_{A,\text{ét}}$ be the natural morphism of sites. Let $1 \leq n \leq \infty$.*

(1) *For any* $\mathbb{L} \in \mathrm{LS}^{\text{H-small}}(X, \mathbb{B}^+_{\mathrm{dR},n})(A)$ *of rank* r*, we have*

$$\mathrm{R}^n\nu_*(\mathbb{L} \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},A}) = \begin{cases} \mathcal{D}(\mathbb{L}), & n = 0 \\ 0, & n \geq 1 \end{cases}$$

where $\mathcal{D}(\mathbb{L})$ *is a locally finite free* $\mathcal{O}_{\widetilde{X}_{A,n}}$*-module of rank* r *on* $X_{A,\text{ét}}$ *such that the*

$$\mathrm{id}_{\mathbb{L}} \otimes \mathrm{d} : \mathbb{L} \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},A} \to \mathbb{L} \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},A} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}_A}} \Omega^1_{\widetilde{\mathfrak{X}}_A}\{-1\}$$

induces a flat connection $\nabla_{\mathbb{L}}$ *on* $\mathcal{D}(\mathbb{L})$ *making* $(\mathcal{D}(\mathbb{L}), \nabla_{\mathbb{L}})$ *an object in* $\mathrm{MIC}^{\text{H-small}}(\widetilde{X}_n)(A)$*.*

(2) *For any* $(\mathcal{D}, \nabla) \in \mathrm{MIC}^{\text{H-small}}(\widetilde{X}_n)(A)$ *of rank* r*, define*

$$\nabla_{\mathcal{D}} := \nabla \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{d} : \mathcal{D} \otimes_{\mathcal{O}_{\widetilde{X}_A}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},A} \to \mathcal{D} \otimes_{\mathcal{O}_{\widetilde{X}_A}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},A} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}_A}} \Omega^1_{\widetilde{\mathfrak{X}}_A}\{-1\}.$$

Then

$$\mathbb{L}(\mathcal{D}, \nabla) := (\mathcal{D} \otimes_{\mathcal{O}_{\widetilde{X}_A}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},A})^{\nabla_{\mathcal{D}}=0}$$

is an object of rank r *in* $\mathrm{LS}^{\text{H-small}}(X, \mathbb{B}^+_{\mathrm{dR},n})(A)$*.*

(3) *The functors* $\mathbb{L} \mapsto (\mathcal{D}(\mathbb{L}), \nabla_{\mathbb{L}})$ *and* $(\mathcal{D}, \nabla) \mapsto \mathbb{L}(\mathcal{D}, \nabla)$ *in Items (1) and (2) respectively define an equivalence of categories*

$$\rho_{\widetilde{\mathfrak{X}}_A} : \mathrm{LS}^{\text{H-small}}(X, \mathbb{B}^+_{\mathrm{dR},n})(A) \xrightarrow{\simeq} \mathrm{MIC}^{\text{H-small}}(\widetilde{X}_n)(A)$$

which preserves ranks, tensor products and dualities. Moreover, for any $\mathbb{L} \in \mathrm{LS}^{\text{H-small}}(X, \mathbb{B}^+_{\mathrm{dR},n})(A)$ *with corresponding* $(\mathcal{D}, \nabla) \in \mathrm{MIC}^{\text{H-small}}(\widetilde{X}_n)(A)$*, there exists a quasi-isomorphism*

$$\mathrm{R}\nu_*\mathbb{L} \simeq \mathrm{DR}(\mathcal{D}, \nabla).$$

In particular, we have a quasi-isomorphism

$$\mathrm{R}\Gamma(X_{A,v}, \mathbb{L}) \simeq \mathrm{R}\Gamma(X_{A,\text{ét}}, \mathrm{DR}(\mathcal{D}, \nabla)).$$

It is clearly the construction of $\rho_{\widetilde{\mathfrak{X}}_A}$ is functorial in A and thus Theorem 1.1 follows immediately. The §4 is devoted to proving 1.10. The approach is standard: We reduce the proof to the case where $\mathfrak{X}$ admits a chart and then check everything locally. We shall exhibit the details in §3.

Remark 1.11. The construction of $(\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}, \mathrm{d})$ is no doubt under the influence of the constructions in [CLWZ24, §3]. All results in this paper should be viewed as a continuation of the Riemann–Hilbert correspondence (for *Faltings-small* objects) [CLWZ24, Th. 7.11] in *loc.cit.*. It is worth pointing out Theorem 1.10 is also compatible with [CLWZ24, Th. 7.11] as we can compare period sheaves used here and in *loc.cit.* (cf. Remark 2.12).

1.4. **Notations.** Throughout this paper, let C be a algebraically closed perfectoid field containing $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}_C$. Let $\mathbf{A}_{\mathrm{inf}}$ and $\mathbf{B}^+_{\mathrm{dR}}$ be the corresponding infinitesimal and de Rham period ring. Fix an embedding $p^{\mathbb{Q}} \subset C^\times$, which induces an embedding $\varpi^{\mathbb{Q}} \subset C^{\flat\times}$, where $\varpi = (p, p^{1/p}, p^{1/p^2}, \dots) \in C^\flat$. Fix a coherent system $\{\zeta_{p^n}\}_{n\geq 0}$ of primitive p^n-th roots of unity in C, and let $\epsilon := (1, \zeta_p, \zeta_p^2, \dots) \in C^\flat$. Let $u = [\epsilon^{\frac{1}{p}}] - 1 \in \mathbf{A}_{\mathrm{inf}}$, and then the canonical surjection $\theta : \mathbf{A}_{\mathrm{inf}} \to \mathcal{O}_C$ is principally generated by $\xi := \frac{\phi(u)}{u}$. Let $t = \log([\epsilon]) \in \mathbf{B}^+_{\mathrm{dR}}$ be the Fontaine's p-adic analogue of "$2\pi i$". For any (sheaf of) $\mathbf{A}_{\mathrm{inf}}$-module M and any $n \in \mathbb{Z}$, denote by $M\{n\}$ its Breuil–Kisin–Fargues twist

$$M\{n\} := M \otimes_{\mathbf{A}_{\mathrm{inf}}} \mathrm{Ker}(\theta)^{\otimes n},$$

which can be trivialized by ξ^n; that is, we have the identification $M\{n\} = M \cdot \xi^n$. Using this, we may regard M as a sub-$\mathbf{A}_{\mathrm{inf}}$-module of $M\{-1\}$ via the identification $M = \xi M\{-1\}$.

Fix a ring R. If an element $x \in R$ admits arbitrary pd-powers, we denote by $x^{[n]}$ its n-th pd-power (i.e. analogue of $\frac{x^n}{n!}$) in R. Put $E_i = (0, \dots, 1, \dots, 0) \in \mathbb{N}^d$ with 1 appearing at exactly the i-th component. For any $J = (j_1, \dots, j_d) \in \mathbb{N}^d$ and any $x_1, \dots, x_d \in R$, we put

$$\underline{x}^J := x_1^{j_1} \cdots x_d^{j_d}$$

and if moreover x_i admits arbitrary pd-powers in A for all i, we put

$$\underline{x}^{[J]} := x_1^{[j_1]} \cdots x_d^{[j_d]}.$$

For any $\alpha \in \mathbb{N}[1/p] \cap [0,1)$, we put

$$\zeta^{\alpha} = \zeta_{p^n}^m$$

if $\alpha = \frac{m}{p^n}$ such that p, m are coprime in $\mathbb{N}$. If $x \in A$ admits compatible p^n-th roots $x^{\frac{1}{p^n}}$, we put

$$x^{\alpha} = x^{\frac{m}{p^n}}$$

for $\alpha = \frac{m}{p^n}$ as above. In general, for any $\underline{\alpha} := (\alpha_1, \dots, \alpha_d) \in (\mathbb{N}[1/p] \cap [0,1))^d$ and any $x_1, \dots, x_d$ admitting compatible p^n-th roots in A, we put

$$\underline{x}^{\underline{\alpha}} := x_1^{\alpha_1} \cdots x_d^{\alpha_d}.$$

Acknowledgement. The work started as a research project of 2024 summer school on algebra and number theory at Peking university, held by School of Mathematical Science, Peking University and Academy of Mathematics and System Science, Chinese Academy of Science. The authors would like to thank the organizers and institutes for providing with a great environment for our collaborating. We would like to thank Jiahong Yu for explaining his paper [Yu24] and thank Yue Chen and Wenrong Zou for their interests in this project. Special thank goes to Ruochuan Liu for his valuable comments and suggestions on this paper. The idea for constructing $(\mathcal{O}\mathbb{B}_{\mathrm{dR,pd}}^+, \mathrm{d})$ occurs during the period of the last author's collaboration [CLWZ24] with Zekun Chen, Ruochuan Liu and Xinwen Zhu, and we would like to thank them for allowing us to use some results in *loc.cit.* and exhibit Theorem 1.1 here. Y.W. is partially supported by CAS Project for Young Scientists in Basic Research, Grant No. YSBR-032.

2. The period sheaf with connection $(\mathcal{O}\mathbb{B}_{\mathrm{dR,pd}}^+, \mathrm{d})$

Fix a $\mathrm{Spa}(A, A^+) \in \mathrm{Perfd}$ and put $\mathbf{A}_{\mathrm{inf}} := \mathbb{A}_{\mathrm{inf}}(A, A^+)$ and $\mathbf{B}_{\mathrm{dR}}^+ := \mathbb{B}_{\mathrm{dR}}^+(A, A^+)$ for short in what follows. Let $\mathfrak{X}$ be a smooth formal scheme over A^+ with the generic fiber X. We say $\mathfrak{X}$ is *liftable* if it admits a smooth lifting $\widetilde{\mathfrak{X}}$ over $\mathbf{A}_{\mathrm{inf}}$ along the canonical map $\theta : \mathbf{A}_{\mathrm{inf}} \to A^+$. From now on, we assume $\mathfrak{X}$ is liftable and fix such a lifting $\widetilde{\mathfrak{X}}$ and a fortiori a lifting $\widetilde{X}$ of X over $\mathbf{B}_{\mathrm{dR}}^+$.

For any affinoid perfectoid $U = \mathrm{Spa}(S, S^+) \in X_v$, let Σ_U be the set of all étale morphisms

$$i_{\mathfrak{Y}} : \mathfrak{Y} = \mathrm{Spf}(\mathcal{R}) \to \mathfrak{X}$$

over A^+ such that the natural map $U \to X$ factors through the generic fiber of $i_{\mathfrak{Y}}$. By the étaleness of $i_{\mathfrak{Y}}$, the $\mathfrak{Y}$ admits a unique lifting $\widetilde{\mathfrak{Y}} = \mathrm{Spf}(\widetilde{\mathcal{R}})$ over $\widetilde{\mathfrak{X}}$ such that $i_{\mathfrak{Y}}$ uniquely lifts to a morphism

$$i_{\widetilde{\mathfrak{Y}}} : \widetilde{\mathfrak{Y}} = \mathrm{Spf}(\widetilde{\mathcal{R}}) \to \widetilde{\mathfrak{X}}$$

over $\mathbf{A}_{\mathrm{inf}}$. Then we have a natural morphism

$$\theta_{\mathfrak{Y}} : \widetilde{\mathcal{R}} \widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}} \mathbb{A}_{\mathrm{inf}}(U) \to S^+$$

lifting the natural map $\mathcal{R} \widehat{\otimes}_{A^+} S^+ \to S^+$ along θ.

Lemma 2.1. *The kernel* $\mathrm{Ker}(\theta_{\mathfrak{Y}})$ *of* $\theta_{\mathfrak{Y}}$ *is a finite generated ideal of* $\widetilde{\mathcal{R}} \widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}} \mathbb{A}_{\mathrm{inf}}(U)$.

Proof. The kernel of the multiplicative map $\mathcal{R} \widehat{\otimes}_{A^+} \mathcal{R} \to \mathcal{R}$ is finitely generated by the smoothness of $\mathcal{R}$. Let $x_1, \dots, x_d$ be its generators and then theie images in $\mathcal{R} \widehat{\otimes}_{A^+} \mathbb{A}_{\mathrm{inf}}(U)$ generate the ideal $\mathrm{Ker}(\mathcal{R} \widehat{\otimes}_{A^+} S^+) \to S^+$. Then $\mathrm{Ker}(\theta_{\mathfrak{Y}})$ is generated by any liftings of x_i's together with ξ. $\square$

Let $A_{\mathfrak{Y}}$ be the $(p, \mathrm{Ker}(\theta_{\mathfrak{Y}}))$-adic completion of $\widetilde{\mathcal{R}} \widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}} \mathbb{A}_{\mathrm{inf}}(U)$. Denote by $B_{\mathfrak{Y}} = A_{\mathfrak{Y}}[u \cdot \mathrm{Ker}(\theta_{\mathfrak{Y}})\{-1\}] \subset A_{\mathfrak{Y}}[\frac{1}{\xi}]$ the blow-up algebra

$$B_{\mathfrak{Y}} := A_{\mathfrak{Y}}[\frac{u \cdot \mathrm{Ker}(\theta_{\mathfrak{Y}})}{\xi}]$$

and define

$$\Gamma_{\mathfrak{Y}} := \left(B_{\mathfrak{Y}}[u \cdot \mathrm{Ker}(\theta_{\mathfrak{Y}})\{-1\}]_{\mathrm{pd}}\right)^{\wedge} = \left(B_{\mathfrak{Y}}[\frac{u \cdot \mathrm{Ker}(\theta_{\mathfrak{Y}})}{\xi}]_{\mathrm{pd}}\right)^{\wedge}$$

the $(p, \mathrm{Ker}(\theta_{\mathfrak{Y}}))$-adic completion of the pd-envelope of $B_{\mathfrak{Y}}$ with respect to the ideal generated by $\frac{u \cdot \mathrm{Ker}(\theta_{\mathfrak{Y}})}{\xi}$. Clearly, it is an algebra over $\mathbb{A}_{\mathrm{inf}}(U)[u]_{\mathrm{pd}}^{\wedge}$, the (p, ξ)-adic completion of the pd-envelope of $\mathbb{A}_{\mathrm{inf}}(U)$ with respect to the ideal (u).

Before we move on, let us give a quick review of the relative Robba rings $\widetilde{\mathbb{C}}^1(U)$. Let

$$\mathrm{Spa}(\widetilde{\mathbb{C}}^1(U),\widetilde{\mathbb{C}}^{1,+}(U))\to \mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}(U)[\frac{1}{p}],\mathbb{A}_{\mathrm{inf}}(U))$$

be the rational localisation with respect to the condition $|[\varpi]|=p^{-1}=|p|$. Let $|\cdot|_1$ be the norm on $\mathbb{A}_{\mathrm{inf}}(U)[\frac{1}{p[\varpi]}]$ such that for any $x=\sum_{n\gg 0}[x_n]p^n$ with $x_n\in S^{\flat+}$,

$$|x|_1:=\sup_{n\gg 0}\|x_n\|p^{-n},$$

where $\|\cdot\|$ denotes the spectral norm on $S^\flat$. Then we have the following result:

Lemma 2.2. (1) *The $\widetilde{\mathbb{C}}^{1,+}(U)=\mathbb{A}_{\mathrm{inf}}(U)[(\frac{[\varpi]}{p})^{\pm 1}]^\wedge$ is the p-adic completion of $\mathbb{A}_{\mathrm{inf}}(U)[(\frac{[\varpi]}{p})^{\pm 1}]$ and $\widetilde{\mathbb{C}}^1(U)=\widetilde{\mathbb{C}}^{1,+}(U)[\frac{1}{p}]$.*

(2) *The $\widetilde{\mathbb{C}}^1(U)$ is the completion of $\mathbb{A}_{\mathrm{inf}}(U)[\frac{1}{p[\varpi]}]$ with respect to the norm $|\cdot|_1$, which is thus a Banach $\mathbb{Q}_p$-algebra with the closed unit ball*

$$\widetilde{\mathbb{C}}^{1,+}(U)=\{x\in\widetilde{\mathbb{C}}^1(U)\mid |x|_1\leq 1\}.$$

(3) *The $t=\log[\epsilon]=\sum_{n\geq 1}\frac{(1-[\epsilon])^n}{n}$ converges in $\widetilde{\mathbb{C}}^{1,+}(U)$ and is a unit-multiple of $u\xi$.*

(4) *The u is a unit-multiple of $[\epsilon^{1/p}-1]$ in $\widetilde{\mathbb{C}}^{1,+}(U)$, and thus admits arbitrary pd-powers in $\widetilde{\mathbb{C}}^{1,+}(U)$.*

(5) *The natural map $\theta:\mathbb{A}_{\mathrm{inf}}(U)\to S^+$ extends to a surjection $\widetilde{\mathbb{C}}^{1,+}(U)\to S^+$ with kernel principally generated by $\frac{\xi}{p}$, which induces a canonical bounded morphism*

$$\widetilde{\mathbb{C}}^1(U)\to\mathbb{B}_{\mathrm{dR}}^+(U)$$

identifying the target with the ξ-adic completion of the source.

Proof. Item (1) follows from the definition. Item (2) is [CLWZ24, Lem. 2.7(1)]. Item (3) and the first part of Item (4) is [CLWZ24, Prop. 3.12]. For the second part of Item (4), just note that for any $n\geq 1$,

$$|\frac{u^n}{n!}|_1=|[\frac{\epsilon^{1/p}-1]^n}{n!}|_1=p^{\nu_p(n!)-\frac{n}{p-1}}\leq p^{-\frac{1}{p-1}}.$$

Here, we use the well-known fact that $\nu_p(n!)\geq\frac{n-1}{p-1}$ for any $n\geq 1$. All statements in Item (5) follows from [CLWZ24, Lem. 2.12], except that

$$\mathrm{Ker}(\widetilde{\mathbb{C}}^{1,+}(U)\to S^+)=(\frac{\xi}{p}).$$

To see this, note that ξ is a unit-multiple of $\beta:=p-[\varpi]$ in $\mathbb{A}_{\mathrm{inf}}(U)$. It suffices to show that

$$\mathrm{Ker}(\widetilde{\mathbb{C}}^{1,+}(U)\to S^+)=(1-\frac{[\varpi]}{p}).$$

This follows from Item (1) easily. □

By Lemma2.2(4), we see the natural inclusion $\mathbb{A}_{\mathrm{inf}}(U)\to\widetilde{\mathbb{C}}^{1,+}(U)$ uniquely extends to the inclusion

$$\mathbb{A}_{\mathrm{inf}}(U)[u]_{\mathrm{pd}}^\wedge\to\widetilde{\mathbb{C}}^{1,+}(U).$$

Now, for any $?\in\{+,\emptyset\}$, we put

$$C_{\mathfrak{Y}}^?:=\Gamma_{\mathfrak{Y}}\widehat{\otimes}_{\mathbb{A}_{\mathrm{inf}}(U)[u]_{\mathrm{pd}}^\wedge}\widetilde{\mathbb{C}}^{1,?}(U),$$

and then we have $C_{\mathfrak{Y}}=C_{\mathfrak{Y}}^+[\frac{1}{p}]$. Define

$$B_{\mathrm{dR,pd},\mathfrak{Y}}^+:=\xi\text{-adic completion of } C_{\mathfrak{Y}}.$$

Then $B_{\mathrm{dR,pd},\mathfrak{Y}}^+$ is a $\mathbb{B}_{\mathrm{dR}}^+(U)$-algebra by Lemma 2.2(5).

Let $\mathrm{d}:\widetilde{\mathcal{R}}\to\Omega^1_{\widetilde{\mathcal{R}}}$ be the usual derivation on $\widetilde{\mathcal{R}}$, where $\Omega^1_{\widetilde{\mathcal{R}}}$ denotes the module of (p,ξ)-adically continuous differentials over $\widetilde{\mathcal{R}}$ over $\mathbf{A}_{\mathrm{inf}}$. It extends uniquely to an $\mathbb{A}_{\mathrm{inf}}(U)$-linear derivation

$$\mathrm{d}:\widetilde{\mathcal{R}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}}\mathbb{A}_{\mathrm{inf}}(U)\to\Omega^1_{\widetilde{\mathcal{R}}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}}\mathbb{A}_{\mathrm{inf}}(U)$$

such that for any $n \geq 1$,
$$\mathrm{d}(\mathrm{Ker}(\theta_{\mathfrak{Y}})^n) \subset \mathrm{Ker}(\theta_{\mathfrak{Y}})^{n-1}.$$
Thus it extends uniquely to an $\mathbb{A}_{\mathrm{inf}}(U)$-linear derivation
$$\mathrm{d} : A_{\mathfrak{Y}} \to A_{\mathfrak{Y}} \otimes_{\widetilde{\mathcal{R}}} \Omega^1_{\widetilde{\mathcal{R}}},$$
and then an $\mathbb{A}_{\mathrm{inf}}(U)$-linear derivation
$$\mathrm{d} : B_{\mathfrak{Y}} \to B_{\mathfrak{Y}} \otimes_{\widetilde{\mathcal{R}}} \Omega^1_{\widetilde{\mathcal{R}}}\{-1\}$$
where $\Omega^1_{\widetilde{\mathcal{R}}}\{-1\} = \Omega^1_{\widetilde{\mathcal{R}}} \cdot \xi^{-1}$. The d also extends uniquely to an $\mathbb{A}_{\mathrm{inf}}(U)[u]^{\wedge}_{\mathrm{pd}}$-linear derivation
$$\mathrm{d} : \Gamma_{\mathfrak{Y}} \to u\Gamma_{\mathfrak{Y}} \otimes_{\widetilde{\mathcal{R}}} \Omega^1_{\widetilde{\mathcal{R}}}\{-1\}.$$
By base-change to $\widetilde{\mathbb{C}}^{1,+}(U)$ (and taking ξ-adic completion), it uniquely extends to a $\widetilde{\mathbb{C}}^{1,+}(U)$-linear (resp. $\widetilde{\mathbb{C}}^{1}(U)$-linear, $\mathbb{B}^+_{\mathrm{dR}}(U)$-linear) derivation
$$\mathrm{d} : C^+_{\mathfrak{Y}} \to uC^+_{\mathfrak{Y}} \otimes_{\widetilde{\mathcal{R}}} \Omega^1_{\widetilde{\mathcal{R}}}\{-1\} \text{ (resp. } \mathrm{d} : C_{\mathfrak{Y}} \to C_{\mathfrak{Y}} \otimes_{\widetilde{\mathcal{R}}} \Omega^1_{\widetilde{\mathcal{R}}}\{-1\},\ \mathrm{d} : B^+_{\mathrm{dR,pd},\mathfrak{Y}} \to B^+_{\mathrm{dR,pd},\mathfrak{Y}} \otimes_{\widetilde{\mathcal{R}}} \Omega^1_{\widetilde{\mathcal{R}}}\{-1\}).$$
Recall that u is invertible in $\widetilde{\mathbb{C}}^{1}(U)$ and $\mathbb{B}^+_{\mathrm{dR}}(U)$.

Definition 2.3. Let $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}$ be the sheaves corresponding to the sheafification of the presheaf sending each affinoid perfectoid $U \in X_{\mathrm{pro\acute{e}t}}$ to
$$\mathrm{colim}_{\mathfrak{Y}\in\Sigma_U} B^+_{\mathrm{dR,pd},\mathfrak{Y}},$$
and let
$$\mathrm{d} : \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} u\Omega^1_{\widetilde{\mathfrak{X}}}\{-1\}$$
be the derivation induced from the usual (continuous) derivation on $\widetilde{\mathfrak{X}}$ as constructed above. Denote by $\mathrm{DR}(\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}, \mathrm{d})$ the corresponding de Rham complex
$$\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}} \xrightarrow{\mathrm{d}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} u\Omega^1_{\widetilde{\mathfrak{X}}}\{-1\} \xrightarrow{\mathrm{d}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} u^2\Omega^2_{\widetilde{\mathfrak{X}}}\{-2\} \to \cdots \xrightarrow{\mathrm{d}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} u^d\Omega^d_{\widetilde{\mathfrak{X}}}\{-d\}.$$
We remark that u is invertible for $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}$.

Then we have the following Poincaré's Lemma:

Theorem 2.4 (Poincaré's Lemma)**.** *The natural morphism $\mathbb{B}^+_{\mathrm{dR}} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}$ induces an exact sequence*
$$0 \to \mathbb{B}^+_{\mathrm{dR}} \to \mathrm{DR}(\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}, \mathrm{d}).$$

Remark 2.5. One can similarly let $\mathcal{O}\widetilde{\mathbb{C}}^{1,+}_{\mathrm{pd}}$ and $\mathcal{O}\widetilde{\mathbb{C}}^{1}_{\mathrm{pd}}$ be the sheafifications of the presheaves sending each affinoid perfectoid $U \in X_{\mathrm{pro\acute{e}t}}$ to
$$\mathrm{colim}_{\mathfrak{Y}\in\Sigma_U} C^+_{\mathfrak{Y}} \text{ and } \mathrm{colim}_{\mathfrak{Y}\in\Sigma_U} C_{\mathfrak{Y}}$$
respectively. Then the usual (continuous) derivation on $\widetilde{\mathfrak{X}}$ induces a derivation
$$\mathrm{d} : \mathcal{O}\mathbb{P}_{\mathrm{pd}} \to \mathcal{O}\mathbb{P}_{\mathrm{pd}} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} u\Omega^1_{\widetilde{\mathfrak{X}}}\{-1\}$$
for $\mathbb{P} \in \{\widetilde{\mathbb{C}}^{1,+}, \widetilde{\mathbb{C}}^{1}\}$. Then the arguments below also proves that there exists an exact sequence
$$0 \to \mathbb{P} \to \mathrm{DR}(\mathcal{O}\mathbb{P}_{\mathrm{pd}}, \mathrm{d})$$
when $p \geq 3$. Here, we have to work with $p \geq 3$ as in this case, we have $\frac{\xi}{u}$ is topologically nilpotent in $\widetilde{\mathbb{C}}^{1,+}(U)$. This will be used to construct the morphism j in the proof of Proposition 2.7. We leave details to interested readers.

We shall prove the above theorem later by giving the local description of $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}$.

Suppose that $\mathfrak{X} = \mathrm{Spf}(\mathcal{R})$ is *small affine*; that is, there is a (p-completely) étale morphism of formal schemes over A^+
$$\psi : \mathfrak{X} = \mathrm{Spf}(\mathcal{R}) \to \mathrm{Spf}(A^+\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle) = \mathrm{Spf}(A^+\langle \underline{T}^{\pm 1}\rangle).$$
Then ψ lifts uniquely to a ((p,ξ)-completely) étale map
$$\widetilde{\psi} : \widetilde{\mathfrak{X}} = \mathrm{Spf}(\widetilde{\mathcal{R}}) \to \mathbf{A}_{\mathrm{inf}}\langle \underline{T}^{\pm 1}\rangle.$$

Let $X_\infty := \mathrm{Spa}(\widehat{R}_\infty, \widehat{R}_\infty^+)$ be the base-change of $X = \mathrm{Spa}(R = \mathcal{R}[\frac{1}{p}], R^+ = \mathcal{R}) \to \mathrm{Spa}(A\langle \underline{T}^{\pm 1}\rangle, A^+\langle \underline{T}^{\pm 1}\rangle)$ along the pro-étale Γ-torsor

$$\mathrm{Spa}(A\langle \underline{T}^{\pm 1/p^\infty}\rangle, A^+\langle \underline{T}^{\pm 1/p^\infty}\rangle) \to \mathrm{Spa}(A\langle \underline{T}^{\pm 1}\rangle, A^+\langle \underline{T}^{\pm 1}\rangle),$$

where $\Gamma = \oplus_{i=1}^d \mathbb{Z}_p \gamma_i$ acts on $A\langle \underline{T}^{\pm 1/p^\infty}\rangle$ via

$$\gamma_i(T_j^{\frac{1}{p^n}}) = \zeta_{p^n}^{\delta_{ij}} T_j^{\frac{1}{p^n}},\ \forall\ 1 \le i, j \le d\ \&\ n \ge 0.$$

Put $T_i^\flat := (T_i, T_i^{1/p}, \dots) \in \widehat{R}_\infty^{\flat +}$. Then for any $1 \le i, j \le d$ and any $n \ge 0$, we have

$$\gamma_i((T_j^\flat)^{\frac{1}{p^n}}) = \epsilon^{\delta_{ij}\frac{1}{p^n}} (T_j^\flat)^{\frac{1}{p^n}}.$$

Lemma 2.6. *For any affinoid perfectoid $U = \mathrm{Spa}(S, S^+) \in X_{\infty, v}$ and any $\mathfrak{Y} \in \Sigma_U$, the map*

$$\iota : \mathbb{A}_{\mathrm{inf}}(U)[[V_1, \dots, V_d]] \to A_{\mathfrak{Y}}$$

sending each V_i to $\frac{T_i - [T_i^\flat]}{T_i}$ is well-defined and an isomorphism of $\mathbb{A}_{\mathrm{inf}}(U)$-algebras. Here, $\mathbb{A}_{\mathrm{inf}}(U)[[V_1, \dots, V_d]]$ is the ring of formal power series over $\mathbb{A}_{\mathrm{inf}}(U)$ freely generated by $V_1, \dots, V_d$. Moreover, the isomorphism carries the ideal $(\xi, V_1, \dots, V_d)$ onto the ideal $\mathrm{Ker}(\theta_{\mathfrak{Y}})$.

Proof. Without loss of generality, we may assume $\mathfrak{Y} = \mathfrak{X} = \mathrm{Spf}(\mathcal{R})$. To see ι is well-defined, consider the map

$$i : \mathbb{A}_{\mathrm{inf}}(U)[V_1, \dots, V_d] \to A_{\mathfrak{Y}}$$

sending V_i to $\frac{T_i - [T_i^\flat]}{T_i}$. As $i(V_j) \in \mathrm{Ker}(\theta_{\mathfrak{Y}})$ and $A_{\mathfrak{Y}}$ is $\mathrm{Ker}(\theta_{\mathfrak{Y}})$-complete, the i uniquely extends to a map of $\mathbb{A}_{\mathrm{inf}}(U)$-algebras

$$i : \mathbb{A}_{\mathrm{inf}}(U)[[V_1, \dots, V_d]] \to A_{\mathfrak{Y}},$$

which is nothing but ι. To conclude, it remains to construct a map

$$j : A_{\mathfrak{Y}} \to \mathbb{A}_{\mathrm{inf}}(U)[[V_1, \dots, V_d]]$$

which is the inverse of ι.

Consider the morphism of $\mathbf{A}_{\mathrm{inf}}$-algebras

$$j : \mathbf{A}_{\mathrm{inf}}\langle \underline{T}^{\pm 1}\rangle \to \mathbb{A}_{\mathrm{inf}}(U)[[V_1, \dots, V_d]]$$

sending each T_i to $[T_i^\flat](1 - V_i)^{-1}$. Then the composite

$$\mathbf{A}_{\mathrm{inf}}\langle \underline{T}^{\pm 1}\rangle \xrightarrow{j} \mathbb{A}_{\mathrm{inf}}(U)[[V_1, \dots, V_d]] \xrightarrow{\bmod (\xi, V_1, \dots, V_d)} S^+$$

coincides with the natural morphism $A^+\langle \underline{T}^{\pm 1}\rangle \to \mathcal{R} \to S^+$. By the étaleness of $\widetilde{\psi}$, the j uniquely lifts to a morphism of $\mathbf{A}_{\mathrm{inf}}$-algebras

$$\widetilde{\mathcal{R}} \to \mathbb{A}_{\mathrm{inf}}(U)[[V_1, \dots, V_d]]$$

and thus by scalar extension, uniquely extends to a morphism of $\mathbb{A}_{\mathrm{inf}}(U)$-algebras (still denoted by)

$$j : \widetilde{\mathcal{R}} \widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}} \mathbb{A}_{\mathrm{inf}}(U) \to \mathbb{A}_{\mathrm{inf}}(U)[[V_1, \dots, V_d]]$$

such that the composite

$$\widetilde{\mathcal{R}} \widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}} \mathbb{A}_{\mathrm{inf}}(U) \xrightarrow{j} \mathbb{A}_{\mathrm{inf}}(U)[[V_1, \dots, V_d]] \xrightarrow{\bmod (\xi, V_1, \dots, V_d)} S^+$$

coincides with the composite

$$\widetilde{\mathcal{R}} \widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}} \mathbb{A}_{\mathrm{inf}}(U) \xrightarrow{\bmod \xi} \mathcal{R} \widehat{\otimes}_{A^+} S^+ \to S^+,$$

which is nothing but $\theta_{\mathfrak{Y}}$. In particular, j carries $\mathrm{Ker}(\theta_{\mathfrak{Y}})$ into the ideal $(\xi, V_1, \dots, V_d)$. Thus, as $\mathbb{A}_{\mathrm{inf}}(U)[[V_1, \dots, V_d]]$ is $(p, \xi, V_1, \dots, V_d)$-adically complete, j uniquely extends to a morphism of $\mathbb{A}_{\mathrm{inf}}(U)$-algebras (still denoted by)

$$j : A_{\mathfrak{Y}} \to \mathbb{A}_{\mathrm{inf}}(U)[[V_1, \dots, V_d]].$$

To complete the proof, we only need to check j is the inverse of ι. Granting this, it follows from the above construction that

$$\iota((\xi, V_1, \dots, V_d)) \subset \mathrm{Ker}(\theta_{\mathfrak{Y}}) \text{ and } j(\mathrm{Ker}(\theta_{\mathfrak{Y}})) \subset (\xi, V_1, \dots, V_d),$$

yielding the "moreover" part of the result.

It is clear from the construction that $j\circ\iota=\mathrm{id}$. We have to show $\iota\circ j=\mathrm{id}$. Note that by construction, the composite

$$\mathbf{A}_{\mathrm{inf}}\langle\underline{T}^{\pm1}\rangle\xrightarrow{i\circ j}A_{\mathfrak{Y}}\xrightarrow{\mathrm{mod}\ \mathrm{Ker}(\theta_{\mathfrak{Y}})}S^+$$

coincides with natural map $A^+\langle\underline{T}^{\pm1}\rangle\to\mathcal{R}\to S^+$, and thus by the étaleness of $\widetilde{\psi}$, the $i\circ j$ unique extends to a morphism of $\mathbf{A}_{\mathrm{inf}}$-algebras

$$\widetilde{\mathcal{R}}\to A_{\mathfrak{Y}}.$$

However, as $i\circ j$ coincides with natural map

$$\mathbf{A}_{\mathrm{inf}}\langle\underline{T}^{\pm}\rangle\to\widetilde{\mathcal{R}}\to\widetilde{\mathcal{R}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}}\mathbb{A}_{\mathrm{inf}}(U)\to A_{\mathfrak{Y}},$$

by the uniqueness, the above map $\widetilde{\mathcal{R}}\to A_{\mathfrak{Y}}$ is also the natural map

$$\widetilde{\mathcal{R}}\to\widetilde{\mathcal{R}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}}\mathbb{A}_{\mathrm{inf}}(U)\to A_{\mathfrak{Y}}.$$

As argued in the second paragraph, this map uniquely extends to a morphism $\mathbb{A}_{\mathrm{inf}}(U)$-algebras $A_{\mathfrak{Y}}\to A_{\mathfrak{Y}}$, which is nothing but $\iota\circ j$ by construction. Then the uniqueness criterion forces $\iota\circ j=\mathrm{id}$ because the identity id is another extension of the natural map $\widetilde{\mathcal{R}}\to A_{\mathfrak{Y}}$ above. □

Proposition 2.7. *The map of $\widetilde{\mathbb{C}}^{1,+}(U)$-algebras*

$$\iota_n:\widetilde{\mathbb{C}}^{1,+}(U)[U_1,\dots,U_d]^{\wedge}_{\mathrm{pd}}/(\xi p^{-1})^n\to C^+_{\mathfrak{Y}}/(\xi p^{-1})^n$$

sending each U_i to $\frac{u(T_i-[T_i^\flat])}{\xi}$ is a well-defined isomorphism for any $n\geq1$ such that the derivation d *on $C^+_{\mathfrak{Y}}$ reads*

$$\mathrm{d}=\sum_{i=1}^d u\frac{\partial}{\partial U_i}\otimes\frac{\mathrm{d}T_i}{\xi}:\widetilde{\mathbb{C}}^{1,+}[U_1,\dots,U_d]^{\wedge}_{\mathrm{pd}}/(\xi p^{-1})^n\to\oplus_{i=1}^d\widetilde{\mathbb{C}}^{1,+}[U_1,\dots,U_d]^{\wedge}_{\mathrm{pd}}/(\xi p^{-1})^n\cdot\frac{\mathrm{d}T_i}{\xi}$$

via the above isomorphism and the isomorphism $\Omega^1_{\widetilde{\mathcal{R}}}\{-1\}\cong\oplus_{i=1}^d\widetilde{\mathcal{R}}\cdot\frac{\mathrm{d}T_i}{\xi}$.

Proof. Fix an $n\geq1$. It suffices to show that ι_n is a well-defined isomorphism, and the rest follows directly.

As T_i is invertible in $\widetilde{\mathcal{R}}$, by Lemma 2.6, we have the isomorphism

$$\mathbb{A}_{\mathrm{inf}}(U)[[V_1,\dots,V_d]]\xrightarrow{V_i\mapsto(T_i-[T_i^\flat])}A_{\mathfrak{Y}}.$$

Thus, by construction of $C^+_{\mathfrak{Y}}$, we have an isomorphism

$$\mathbb{A}_{\mathrm{inf}}(U)[[V_1,\dots,V_d]][u,\frac{uV_1}{\xi},\dots,\frac{uV_d}{\xi}]^{\wedge}_{\mathrm{pd}}\widehat{\otimes}_{\mathbb{A}_{\mathrm{inf}}(U)}\widetilde{\mathbb{C}}^{1,+}(U)\xrightarrow{V_i\mapsto(T_i-[T_i^\flat])}C^+_{\mathfrak{Y}},$$

where $\mathbb{A}_{\mathrm{inf}}(U)[[V_1,\dots,V_d]][u,\frac{uV_1}{\xi},\dots,\frac{uV_d}{\xi}]^{\wedge}_{\mathrm{pd}}$ is the $(p,\xi,V_1,\dots,V_d)$-adic completion of the pd-algebra $\mathbb{A}_{\mathrm{inf}}(U)[[V_1,\dots,V_d]][u,\frac{uV_1}{\xi},\dots,\frac{uV_d}{\xi}]_{\mathrm{pd}}$. Thus, to show ι is an isomorphism, it suffices to show that the map

$$i:\widetilde{\mathbb{C}}^{1,+}(U)[U_1,\dots,U_d]^{\wedge}_{\mathrm{pd}}/(p^{-1}\xi)^n\to(\mathbb{A}_{\mathrm{inf}}(U)[[V_1,\dots,V_d]][u,\frac{uV_1}{\xi},\dots,\frac{uV_d}{\xi}]^{\wedge}_{\mathrm{pd}}\widehat{\otimes}_{\mathbb{A}_{\mathrm{inf}}(U)}\widetilde{\mathbb{C}}^{1,+}(U))/(p^{-1}\xi)^n$$

sending U_i to $\frac{uV_i}{\xi}$ is a well-defined isomorphism.

The well-definedness of i is trivial as $\mathbb{A}_{\mathrm{inf}}(U)[[V_1,\dots,V_d]][u,\frac{uV_1}{\xi},\dots,\frac{uV_d}{\xi}]^{\wedge}_{\mathrm{pd}}\widehat{\otimes}_{\mathbb{A}_{\mathrm{inf}}(U)}\widetilde{\mathbb{C}}^{1,+}(U)$ is a p-adic complete $\widetilde{\mathbb{C}}^{1,+}(U)$-algebra and $\frac{uV_i}{\xi}$ admits arbitrary pd-powers. Now, we construct the inverse of i as follows:

As $\frac{\xi}{u}=\frac{\xi}{p}\frac{p}{u}$ and $\frac{p}{u}\in\widetilde{\mathbb{C}}^{1,+}(U)$, we see that $\frac{\xi}{u}\in\widetilde{\mathbb{C}}^{1,+}(U)/(p^{-1}\xi)^n$ is nilpotent. Thus, the map of $\mathbb{A}_{\mathrm{inf}}(U)$-algebras

$$j:\mathbb{A}_{\mathrm{inf}}(U)[V_1,\dots,V_d]\to\widetilde{\mathbb{C}}^{1,+}(U)[U_1,\dots,U_d]^{\wedge}_{\mathrm{pd}}/(p^{-1}\xi)^n$$

sending V_i to $\frac{\xi U_i}{u}$ uniquely extends to a map (still denoted by)

$$j:\mathbb{A}_{\mathrm{inf}}(U)[[V_1,\dots,V_d]]\to\widetilde{\mathbb{C}}^{1,+}(U)[U_1,\dots,U_d]^{\wedge}_{\mathrm{pd}}/(p^{-1}\xi)^n.$$

Since $u, U_1, \dots, U_d$ admit arbitrary pd-powers in $\widetilde{\mathbb{C}}^{1,+}(U)[U_1, \dots, U_d]^{\wedge}_{\mathrm{pd}}$, as argued in the proof of Lemma 2.6, it is easy to see that j uniquely extends to a morphism of $\widetilde{\mathbb{C}}^{1,+}(U)$-algebras

$$j : (\mathbb{A}_{\mathrm{inf}}(U)[[V_1, \dots, V_d]][u, \frac{uV_1}{\xi}, \dots, \frac{uV_d}{\xi}]^{\wedge}_{\mathrm{pd}} \widehat{\otimes}_{\mathbb{A}_{\mathrm{inf}}(U)} \widetilde{\mathbb{C}}^{1,+}(U))/(p^{-1}\xi)^n \to \widetilde{\mathbb{C}}^{1,+}(U)[U_1, \dots, U_d]^{\wedge}_{\mathrm{pd}}/(p^{-1}\xi)^n$$

sending V_i to $\frac{\xi U_i}{u}$. It is clear that j is exactly the inverse of i.

This completes the proof. □

Corollary 2.8. *The morphism of sheaves of $\mathbb{B}^+_{\mathrm{dR}|_{X_\infty}}$-algebras*

$$\iota : \mathbb{B}^+_{\mathrm{dR}|_{X_\infty}}\langle U_1, \dots, U_d\rangle_{\mathrm{pd}} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}|_{X_\infty}}$$

sending U_i to $\frac{u(T_i - [T_i^\flat])}{\xi}$ is an isomorphism, where $\mathbb{B}^+_{\mathrm{dR}|_{X_\infty}}\langle U_1, \dots, U_d\rangle_{\mathrm{pd}}$ denotes the ξ-adic completion of

$$\widetilde{\mathbb{C}}^1_{|_{X_\infty}}[U_1, \dots, U_d]^{\wedge}_{\mathrm{pd}} = \widetilde{\mathbb{C}}^{1,+}_{|_{X_\infty}}[U_1, \dots, U_d]^{\wedge}_{\mathrm{pd}}[\frac{1}{p}].$$

Proof. As we have

$$\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}(U) = \varprojlim_n C^+_{\mathfrak{Y}}[\frac{1}{p}]/(\xi p^{-1})^n/(\xi p^{-1})^n,$$

the result follows from Corollary 2.7 directly. □

Now, we are able to prove Poincaré's Lemma 2.4

Proof of Theorem 2.4. Since the problem is local on both $\mathfrak{X}_{\text{ét}}$ and $X_{\text{proét}}$, we may assume $\mathfrak{X} = \mathrm{Spf}(\mathcal{R})$ with X_∞ as above. Then the result follows from Corollary 2.8 directly. □

Now we still assume $\mathfrak{X} = \mathrm{Spf}(\mathcal{R})$ is small and keep the notations as above.

Proposition 2.9. *The series $\frac{u^2}{t}\log\frac{[T_i^\flat]}{T_i} = \frac{u^2}{t}\sum_{n\geq 1}\frac{(1-\frac{[T_i^\flat]}{T_i})^n}{n}$ converges in $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}|_{X_\infty}}$. The morphism of the sheaves of $\mathbb{B}^+_{\mathrm{dR}|_{X_\infty}}$-algebras*

$$\iota : \mathbb{B}^+_{\mathrm{dR}|_{X_\infty}}\langle W_1, \dots, W_d\rangle_{\mathrm{pd}} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}|_{X_\infty}}$$

sending W_i to $\frac{u^2}{t}\log\frac{[T_i^\flat]}{T_i}$ is an isomorphism such that the derivation d on $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}|_{X_\infty}}$ reads

$$\mathrm{d} = \sum_{i=1}^d -u\frac{\partial}{\partial W_i} \otimes \frac{u}{t}\mathrm{dlog}T_i,$$

where we trivialize $\Omega^1_{\widetilde{\mathcal{R}}}\{-1\}$ by $\oplus_{i=1}^d \widetilde{\mathcal{R}} \cdot \frac{u}{t}\mathrm{dlog}T_i$, and the Galois group Γ acts on $\mathcal{O}\widetilde{\mathbb{C}}^+_{\mathrm{pd}|_{X_\infty}}$ such that

$$\gamma_i(W_j) = W_j + \delta_{ij}u^2, \ \forall\ 1 \leq i, j \leq d.$$

Proof. Recall t is a unit-multiple of ξu in $\widetilde{\mathbb{C}}^{1,+}(U)$ (cf. Lemma 2.2) and T_i is invertible in $\widetilde{\mathcal{R}}$. It follows from Corollary 2.8 that the morphism

$$\mathbb{B}^+_{\mathrm{dR,pd}|_{X_\infty}}\langle U_1, \dots, U_d\rangle_{\mathrm{pd}} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}|_{X_\infty}}$$

sending U_i to $\frac{u^2}{t}(1 - \frac{[T_i^\flat]}{T_i})$ is an isomorphism. Thus, we need to show the series

$$\frac{u^2}{t}\sum_{n\geq 1}\frac{(\frac{t}{u^2}U_i)^n}{n}$$

converges in $\mathbb{B}^+_{\mathrm{dR,pd}|_{X_\infty}}\langle U_1, \dots, U_d\rangle_{\mathrm{pd}}$ and the morphism

$$i : \mathbb{B}^+_{\mathrm{dR,pd}|_{X_\infty}}\langle W_1, \dots, W_d\rangle_{\mathrm{pd}} \to \mathbb{B}^+_{\mathrm{dR,pd}|_{X_\infty}}\langle U_1, \dots, U_d\rangle_{\mathrm{pd}}$$

sending W_i to $\frac{u^2}{t}\sum_{n\geq 1}\frac{(\frac{t}{u^2}U_i)^n}{n}$ is an isomorphism. However, the

$$\frac{u^2}{t}\sum_{n\geq 1}\frac{(\frac{t}{u^2}U_i)^n}{n} = \sum_{n\geq 1}(\frac{t}{u^2})^{n-1}(n-1)!U_i^{[n]}$$

reduces to a finite sum modulo t^n for any $n \geq 1$, yielding the desired convergence. Similarly, the series

$$\frac{u^2}{t}(1-\exp(\frac{t}{u^2}W_i)) = -\sum_{n\geq 1}(\frac{t}{u^2})^{n-1}W_i^{[n]}$$

converges in $\mathbb{B}^+_{\mathrm{dR,pd}|X_\infty}\langle W_1,\dots,W_d\rangle_{\mathrm{pd}}$. So the map

$$j : \mathbb{B}^+_{\mathrm{dR,pd}|X_\infty}\langle U_1,\dots,U_d\rangle_{\mathrm{pd}} \to \mathbb{B}^+_{\mathrm{dR,pd}|X_\infty}\langle W_1,\dots,W_d\rangle_{\mathrm{pd}}$$

sending U_i to $\frac{u^2}{t}(1-\exp(\frac{t}{u^2}W_i)) = -\sum_{n\geq 1}(\frac{t}{u^2})^{n-1}W_i^{[n]}$ is well-defined as well. Now one can conclude by checking j is the inverse of i directly. $\square$

The above construction is compatible with base-change of $(A, A^+) \in \mathrm{Perfd}$.

Proposition 2.10. *Let $(A, A^+) \to (B, B^+)$ be a morphism in* Perfd. *Let $\mathfrak{X}_A$ be a smooth formal scheme over A^+ with a fixed lifting $\widetilde{\mathfrak{X}}_A$ over $\mathbb{A}_{\mathrm{inf}}(A, A^+)$. Let $\mathfrak{X}_B := \mathfrak{X}_A \times_{\mathrm{Spf}(A^+)} \mathrm{Spf}(B^+)$ be the induced smooth formal scheme over B^+ with the lifting $\widetilde{\mathfrak{X}}_B = \widetilde{\mathfrak{X}}_A \times_{\mathrm{Spf}(\mathbb{A}_{\mathrm{inf}}(A,A^+))} \mathrm{Spf}(\mathbb{A}_{\mathrm{inf}}(B, B^+))$. Let $(\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},A}, \mathrm{d})$ and $(\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},B}, \mathrm{d})$ be the period sheaves with connections corresponding to $\widetilde{\mathfrak{X}}_A$ and $\widetilde{\mathfrak{X}}_B$, respectively. Then there exists a natural isomorphism*

$$(\mathbb{B}^+_{\mathrm{dR}}(B, B^+)\widehat{\otimes}_{\mathbb{B}^+_{\mathrm{dR}}(A,A^+)}\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},A}, \mathrm{id}\otimes \mathrm{d}) \to (\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},B}, \mathrm{d})$$

compatible with connections.

Proof. The existence of the morphism

$$\mathbb{B}^+_{\mathrm{dR}}(B, B^+)\widehat{\otimes}_{\mathbb{B}^+_{\mathrm{dR}}(A,A^+)}\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},A} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd},B}$$

follows directly from the construction of $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}$ above. To see it is an isomorphism, we are reduced to the case $\mathfrak{X}_A = \mathrm{Spf}(\mathcal{R})$ is small. Then $\mathfrak{X}_B = \mathrm{Spf}(\mathcal{R}\widehat{\otimes}_{A^+}B^+)$. Note that $X_{\infty,B} = X_{\infty,A} \times_{\mathrm{Spa}(A,A^+)} \mathrm{Spa}(B, B^+)$. One can conclude by using Proposition 2.9. $\square$

Remark 2.11. If we put $\mathbb{A}_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{A}_{\mathrm{dR}}$ the (ξp^{-1})-adic completion of $\widetilde{\mathbb{C}}^{1,+}$ and $\mathcal{O}\widetilde{\mathbb{C}}^{1,+}_{\mathrm{pd}}$ (cf. Remark 2.5), respectively, then the derivation d on $\mathcal{O}_{\widetilde{\mathfrak{X}}}$ induces a derivation

$$u^{-1}\mathrm{d} : \mathcal{O}\mathbb{A}_{\mathrm{dR}} \to \mathcal{O}\mathbb{A}_{\mathrm{dR}} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \cdot\Omega^1_{\widetilde{\mathfrak{X}}}\{-1\}.$$

The last paragraph in the proof of Proposition 2.9 together with Proposition 2.7 tells us when $\mathfrak{X} = \mathrm{Spf}(\mathcal{R})$ is small, then the morphism

$$\iota : \mathbb{A}_{\mathrm{dR}|X_\infty}[W_1,\dots,W_d]^\wedge_{\mathrm{pd}} \to \mathcal{O}\mathbb{A}_{\mathrm{dR}|X_\infty}$$

sending W_i to $\frac{u^2}{t}\log\frac{[T_i^\flat]}{T_i}$ is a well-defined isomorphism and gives the identification $u^{-1}\mathrm{d} = -\sum_{i=1}^d \frac{\partial}{\partial W_i}\otimes \frac{u}{t}\mathrm{dlog}T_i$. One can check directly the sequence

$$0 \to \mathbb{A}_{\mathrm{dR}} \to \mathcal{O}\mathbb{A}_{\mathrm{dR}} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \cdot\Omega^1_{\widetilde{\mathfrak{X}}}\{-1\} \to \cdots \to \mathcal{O}\mathbb{A}_{\mathrm{dR}} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \cdot\Omega^d_{\widetilde{\mathfrak{X}}}\{-d\}$$

is exact; that is, the Poincaré's Lemma holds true in this case. This leads to an integral version of p-adic Riemann–Hilbert correspondence lifting the integral p-adic Simpson correspondence in [MW23].

Remark 2.12. In [CLWZ24, §3], given a lifting $\widetilde{\mathfrak{X}}$ of $\mathfrak{X}$, one can construct an overconvergent de Rham period sheaf with connection $(\mathcal{O}\mathbb{B}^{\dagger,+}_{\mathrm{dR}}, \mathrm{d} : \mathcal{O}\mathbb{B}^{\dagger,+}_{\mathrm{dR}} \to \mathcal{O}\mathbb{B}^{\dagger,+}_{\mathrm{dR}} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \Omega^1_{\widetilde{\mathfrak{X}}}\{-1\})$. One can check $(\mathcal{O}\mathbb{B}^{\dagger,+}_{\mathrm{dR}}, \mathrm{d})$ can be viewed as a sub-$\mathbb{B}^+_{\mathrm{dR}}$-algebra of $(\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}, \mathrm{d})$ compatible with connections. Indeed, let $(\mathcal{O}\widehat{\mathbb{B}}^+_{\mathrm{dR}}, \mathrm{d})$ be the ξ-adic completion of the period sheaf with connection $(\mathcal{O}\widetilde{\mathbb{C}}^{1,1}, \mathrm{d})$ in [CLWZ24, Def. 2.29] with the d-preserving lattice $\mathcal{O}\widetilde{\mathbb{C}}^{1,1,+}$, and then it clearly contains $(\mathcal{O}\mathbb{B}^{\dagger,+}_{\mathrm{dR}}, \mathrm{d})$ as a sub-$\mathbb{B}^+_{\mathrm{dR}}$-algebra compatible with connections. On the other hand, as u admits pd-powers in $\mathcal{O}\widetilde{\mathbb{C}}^{1,1,+}$, one can construct a natural map $\mathcal{O}\widetilde{\mathbb{C}}^{1,+}_{\mathrm{pd}} \to \mathcal{O}\widetilde{\mathbb{C}}^{1,1,+}$ compatible with connections, yielding a well-defined injection $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}} \to \mathcal{O}\widehat{\mathbb{B}}^+_{\mathrm{dR}}$ compatible with connections by inverting p and taking ξ-adic completion. One can conclude by showing that the natural inclusion $\mathcal{O}\mathbb{B}^{\dagger,+}_{\mathrm{dR}} \hookrightarrow \mathcal{O}\widehat{\mathbb{B}}^+_{\mathrm{dR}}$ factors through $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}$ (by working locally; that is, working with small affine $\mathfrak{X}$).

Remark 2.13. The $(\mathcal{O}\widetilde{\mathbb{C}}^{1,+}_{\mathrm{pd}}, \mathrm{d})$ and $(\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}, \mathrm{d})$ reduce to the $(\mathcal{O}\widehat{\mathbb{C}}^+_{\mathrm{pd}}, \Theta)$ and $(\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}}, \Theta)$ in [MW23] modulo (ξp^{-1}). This follows by combining arguments in Remark 2.12 and [CLWZ24, Rem. 3.24], together with the identification in [MW23, Prop. 2.10]. We now check this on $X_{\infty,\text{proét}}$ for $\mathfrak{X}$ small affine: Put $Y_i = \frac{u}{t}\log\frac{[T_i^\flat]}{T_i}$, and then the reduction of $(\mathcal{O}\widetilde{\mathbb{C}}^{1,+}_{\mathrm{pd}}, \mathrm{d})$ is given by

$$(\widehat{\mathcal{O}}^+_X[(\zeta_p-1)\underline{Y}]^\wedge_{\mathrm{pd}}, -\sum_{i=1}^d \frac{\partial}{\partial Y_i}\otimes(\zeta_p-1)\cdot\frac{1}{t}\mathrm{dlog}T_i)$$

with the Γ-action such that

$$\gamma_i(Y_j) = Y_j + \delta_{ij}(\zeta_p-1),\ \forall\ 1\leq i,j\leq d$$

by noting that the reduction of u is $\zeta_p - 1$. Now, one can conclude by applying [MW23, Cor. 2.6]. One can also check the reduction of $\mathcal{O}\widetilde{\mathbb{C}}^{1,+}_{\mathrm{pd}}$ modulo ξp^{-1} behaves as $\mathcal{B}_{\widetilde{\mathfrak{X}}}$ appearing in [AHLB23b, Lem. 3.8] for small affine $\mathfrak{X}$ (see also [AHLB23a, Prop. 4.8]).

3. Local small Riemann–Hilbert correspondence

In this section, we fix a $\mathrm{Spa}(A, A^+) \in \mathrm{Perfd}$ and let $\mathfrak{X}$ be a liftable smooth formal scheme over A^+ with a fixed lifting $\widetilde{\mathfrak{X}}$ over $\mathbf{A}_{\mathrm{inf}} = \mathbb{A}_{\mathrm{inf}}(A, A^+)$. Let $(\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}, \mathrm{d})$ be the period sheaf with connection defined in the previous section and denote by $(\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}}, \Theta)$ its reduction modulo t.

3.1. A criterion for being Hitchin-small.

Denote by $v\mathrm{Bun}_r(X)(A)^{\text{H-small}}$ and by $\mathrm{HIG}_r(X)(A)^{\text{H-small}}$ the category of Hitchin-small v-vector bundles on X_v of rank r and the category of Hitchin-small Higgs bundles on $X_{\text{ét}}$ of rank r. Then

$$v\mathrm{Bun}(X)(A)^{\text{H-small}} := \bigcup_{r\geq 1} v\mathrm{Bun}_r(X)(A)^{\text{H-small}} \text{ and } \mathrm{HIG}(X)(A)^{\text{H-small}} := \bigcup_{r\geq 1}\mathrm{HIG}_r(X)(A)^{\text{H-small}}$$

are the whole categories of v-vector bundles and Higgs bundles. Let $\mathcal{A}_r(A)$ and $\mathcal{A}_r^{\text{H-small}}(A)$ be the Hitchin base and its Hitchin-small locus associated to X with respect to the fixed rank r. Recall a v-vector bundle $\mathcal{L}$ (resp. a Higgs bundle $(\mathcal{H},\theta)$) of rank r is Hitchin-small if and only if its image $\widetilde{h}(\mathcal{L})$ (resp. $h(\mathcal{H},\theta)$) in $\mathcal{A}_r(A)$ belongs to $\mathcal{A}_r^{\text{H-small}}(A)$.

We first give a more convenient description of Hitchin-small Higgs bundles in [AHLB23a].

Proposition 3.1. *A Higgs bundle $(\mathcal{H},\theta)$ on $X_{\text{ét}}$ is Hitchin-small if and only if étale locally on $\mathfrak{X}$, there is a Higgs bundle on $\mathfrak{X}$, i.e. a locally finite free $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{H}^+$ with a Higgs field $\theta^+ : \mathcal{H}^+ \to \mathcal{H}^+\otimes_{\mathcal{O}_{\mathfrak{X}}}\Omega^1_{\mathfrak{X}}\{-1\}$, such that*

1. *$(\mathcal{H},\theta_{\mathcal{H}}) = (\mathcal{H}^+,\theta^+)[\frac{1}{p}]$, and that*
2. *the (ζ_p-1)-twist of θ^+ is topologically nilpotent; that is, θ^+ takes values in $(\zeta_p-1)\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\Omega^1_{\mathfrak{X}}\{-1\}$ such that*
$$\frac{1}{\zeta_p-1}\theta^+ : \mathcal{H}\to\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\Omega^1_{\mathfrak{X}}\{-1\}$$
is (p-adically) topologically nilpotent.

Indeed, one can always find such an $(\mathcal{H}^+,\theta^+)$ on any $\mathfrak{Y} = \mathrm{Spf}(\mathcal{R}) \in \mathfrak{X}_{\text{ét}}$ which is small affine.

Proof. Recall one can check Hitchin-smallness of Higgs bundle by passing to geometric points in the sense of [AHLB23b, Def. 3.2]. Compared with [AHLB23a, Def. 6.24(2)], the result follows from [AHLB23a, Th. 7.13(3)]. Indeed, one can find the desired lattice $(\mathcal{H}^+,\theta^+)$ on any affine $\mathfrak{Y} = \mathrm{Spf}(\mathcal{R}) \in \mathfrak{X}_{\text{ét}}$ splitting the Hodge–Tate gerbe $\mathfrak{Y}^{\mathrm{HT}}$ (cf. [AHLB23a, Th. 7.11(3)]). Note that the Hodge–Tate gerbe $\mathfrak{Y}$ always splits when $\mathfrak{Y}$ is affine small by [BL22b, Prop. 5.12]. $\square$

We also want to give an explicit description of Hitchin-small v-vector bundles. Let $(\mathcal{B},\Theta_{\mathcal{B}})$ be the sheaf with Higgs field in [AHLB23a, Def. 7.17]. Recall the following constructions in *loc.cit.*: For any $(\mathcal{H},\theta)\in\mathrm{HIG}(X)(A)^{\text{H-small}}$, define

$$\Theta'_{\mathcal{H}} := \theta\otimes\mathrm{id} + \mathrm{id}\otimes\Theta_{\mathcal{B}} : \mathcal{H}\otimes_{\mathcal{O}_X}\mathcal{B}\to\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\Omega^1_{\mathfrak{X}}\{-1\} \tag{3.1}$$

and define

$$\mathcal{L}(\mathcal{H},\theta_{\mathcal{H}}) := (\mathcal{H}\otimes_{\mathcal{O}_X}\mathcal{B})^{\Theta'_{\mathcal{H}}=0}.$$

Then we have the following result:

Theorem 3.2 ([AHLB23b, Th. 3.20], [AHLB23a, Th. 7.13(1)]). *The functor $(\mathcal{H},\theta_{\mathcal{H}})\mapsto\mathcal{L}(\mathcal{H},\theta_{\mathcal{H}})$ above induces an equivalence between the categories*

$$v\mathrm{Bun}(X)(A)^{\text{H-}small}\simeq\mathrm{HIG}(X)(A)^{\text{H-}small}$$

which is compatible with cohomology in the sense that we have a quasi-isomorphism

$$\mathrm{R}\nu_*(\mathcal{L}(\mathcal{H},\theta_{\mathcal{H}}))\simeq\mathrm{HIG}(\mathcal{H},\theta_{\mathcal{H}}).$$

Corollary 3.3. *Assume $\mathfrak{X}$ is small affine. For any $(\mathcal{H},\theta)\in\mathrm{HIG}(X)^{\text{H-}small}(A)$, define*

$$\Theta_{\mathcal{H}}:=\theta\otimes\mathrm{id}+\mathrm{id}\otimes\Theta:\mathcal{H}\otimes_{\mathcal{O}_X}\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}}\to\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\Omega^1_{\mathfrak{X}}\{-1\}.$$

Then $\mathcal{L}(\mathcal{H},\theta_{\mathcal{H}})=(\mathcal{H}\otimes_{\mathcal{O}_X}\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}})^{\Theta_{\mathcal{H}}=0}$. In particular, the functor

$$\mathrm{HIG}(X)(A)^{\text{H-}small}\to v\mathrm{Bun}(X)(A)^{\text{H-}small},\quad(\mathcal{H},\theta)\mapsto(\mathcal{H}\otimes_{\mathcal{O}_X}\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}})^{\Theta_{\mathcal{H}}=0}$$

is an equivalence of categories which is compatible with cohomology.

Proof. Recall we have an identification $(\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}},\Theta)_{|X_\infty}=(\mathcal{B},\Theta_{\mathcal{B}})_{|X_\infty}$ by Remark 2.13. The result follows from Theorem 3.2 directly. □

Now, let $\mathfrak{X}=\mathrm{Spf}(\mathcal{R})$ be small affine with the generic fiber $\mathrm{Spa}(R,R^+)$ and write $\Omega^1_{\mathfrak{X}}\{-1\}=\oplus_{i=1}^d\mathcal{R}\cdot(\zeta_p-1)\frac{\mathrm{dlog}T_i}{t}$. Define

$$(B^+_\infty,\Theta):=(\mathcal{O}\widehat{\mathbb{C}}^+_{\mathrm{pd}},\Theta)(X_\infty)$$

and then we have

$$(B^+_\infty,\Theta)=(\widehat{R}^+_\infty[\underline{W}]^\wedge_{\mathrm{pd}},-\sum_{i=1}^d(\zeta_p-1)\frac{\partial}{\partial W_i}\otimes(\zeta_p-1)\cdot\frac{1}{t}\mathrm{dlog}T_i)\tag{3.2}$$

by Proposition 2.9 such that $\Gamma=\oplus_{i=1}^d\mathbb{Z}_p\gamma_i$ acts on B^+_∞ via

$$\gamma_i(W_j)=W_j+\delta_{ij}(\zeta_p-1)^2,\ \forall\ 1\le i,j\le d.$$

Set $B^+=R^+[\underline{W}]^\wedge_{\mathrm{pd}}$, and then it is Θ-preserving and stable under the action of Γ.

Now let (H,θ) be the global section of a Hitchin-small Higgs bundle on $X=\mathrm{Spa}(R,R^+)$, and

$$(H^+,\theta^+:H^+\to H^+\cdot(\zeta_p-1)\frac{\mathrm{dlog}T_i}{t})$$

be the lattice of (H,θ) satisfying the condition in Proposition 3.1. Then one can write

$$\theta^+=\sum_{i=1}^d(\zeta_p-1)\theta_i\otimes(\zeta_p-1)\frac{\mathrm{dlog}T_i}{t}$$

with $\theta_i\in\mathrm{End}_{R^+}(H^+)$ topologically nilpotent. Put

$$\Theta^+_H:=\theta^+\otimes\mathrm{id}+\mathrm{id}\otimes\Theta:H^+\otimes_{\mathcal{R}}B^+_\infty\to H^+\otimes_{\mathcal{R}}B^+_\infty\otimes_{\mathcal{R}}\Omega^1_{\mathcal{R}}\{-1\}$$

and then $H^+\otimes_{\mathcal{R}}B^+_\infty$ is Θ^+_H-preserving. Put

$$L^+(H^+,\theta^+):=(H^+\otimes_{\mathcal{R}}B^+)^{\Theta^+_H=0}\text{ and }L^+_\infty(H^+,\theta^+):=(H^+\otimes_{\mathcal{R}}B^+_\infty)^{\Theta^+_H=0}.$$

Then $L^+(H^+,\theta^+)$ and $L^+_\infty(H^+,\theta^+)$ inherit the Γ-actions from B^+_∞ satisfying

$$L^+_\infty(H^+,\theta^+)=L^+(H^+,\theta^+)\otimes_{\mathcal{R}}\widehat{R}^+_\infty.$$

Lemma 3.4. *The $L^+:=L^+(H^+,\theta^+)$ is a finite projective R^+-module which is isomorphic to H^+ such that for any $x\in L^+$ and any $1\le i\le d$, we have*

$$L^+=\exp(\sum_{i=1}^d\theta_iW_i)(H^+)=\{x\in H^+\otimes_{\mathcal{R}}B^+\mid x=\sum_{J\in\mathbb{N}^d}\underline{\theta}^J(y)\underline{W}^{[J]},y\in H^+\}\tag{3.3}$$

$$\gamma_i(x)=\exp(-\theta_i(\zeta_p-1)^2)(x).$$

Proof. Let $x=\sum_{J\in\mathbb{N}^d}x_J\underline{W}^{[J]}$ be an element in $H^+\otimes_{\mathcal{R}}B^+\cong H^+[\underline{W}]^\wedge_{\mathrm{pd}}$. Then we have

$$\begin{aligned}\Theta_H^+(x)=&\sum_{i=1}^d\sum_{J\in\mathbb{N}^d}\left((\zeta_p-1)\theta_i(x_J)\underline{W}^{[J]}-(\zeta_p-1)x_J\underline{W}^{[J-E_i]}\right)\otimes(\zeta_p-1)\frac{\mathrm{dlog}T_i}{t}\\=&(\zeta_p-1)\sum_{i=1}^d\sum_{J\in\mathbb{N}^d}\left(\theta_i(x_J)-x_{J+E_i}\right)\underline{W}^{[J]}\otimes(\zeta_p-1)\frac{\mathrm{dlog}T_i}{t}.\end{aligned}$$

Thus, $x\in L^+$ if and only if for any $J\in\mathbb{N}^d$ and any $1\leq i\leq d$, $x_{J+E_i}=\theta_i(x_J)$. Using this, by iteration, we see that $x\in L^+$ if and only if

$$x=\sum_{J\in\mathbb{N}^d}\underline{\theta}^J(x_0)\underline{W}^{[J]}=\exp(\sum_{i=1}^d\theta_iW_i)(x).$$

So we have

$$L^+=\exp(\sum_{i=1}^d\theta_iW_i)(H^+)\cong H^+$$

and the description of Γ on L^+ follows as $\gamma_i(W_j)=W_j+\delta_{ij}(\zeta_p-1)^2$ for any $1\leq i,j\leq d$. $\square$

Note that $L_\infty(H,\theta):=L_\infty^+(H^+,\theta^+)$ is exactly $\mathcal{L}(\mathcal{H},\theta)(X_\infty)$. Thus, if $\mathcal{L}\in v\mathrm{Bun}(X)^{\text{H-small}}(A)$, then there must be a finite projective $\mathcal{R}$-module H^+ endowed with an action of Γ such that

$$\mathcal{L}(X_\infty)\cong H^+\otimes_{\mathcal{R}}\widehat{R}_\infty$$

and that for any $1\leq i\leq d$, there exists a topologically nilpotent $\theta_i\in\mathrm{End}_{\mathcal{R}}(H^+)$ such that for any $x\in H^+$, we have $\gamma_i(x)=\exp(-\theta_i(\zeta_p-1)^2)(x)$. We will show this condition is also sufficient. The arguments in what follows are closely related with [AHLB23b, §3] and [MW23, §3 and Th. 4.1].

Definition 3.5. Let $C\in\{\mathcal{R}=R^+,R,\widehat{R}_\infty^+,\widehat{R}_\infty\}$

(1) By an *Hitchin-small representation* of Γ over $\mathcal{R}$ of rank r, we mean a finite projective $\mathcal{R}$-module L^+ endowed with an action of Γ such that for any $1\leq i\leq d$, there exists a topologically nilpotent $\theta_i\in\mathrm{End}_{\mathcal{R}}(L^+)$ such that for any $x\in L^+$, we have $\gamma_i(x)=\exp(-\theta_i(\zeta_p-1)^2)(x)$. Denote by $\mathrm{Rep}_\Gamma^{\text{H-small}}(\mathcal{R})$ the category of Hitchin-small representations of Γ over $\mathcal{R}$.

(2) In general, by an *Hitchin-small representation* of Γ over C of rank r, we mean a finite projective C-module L endowed with an action of Γ such that there exists some $L^+\in\mathrm{Rep}_\Gamma^{\text{H-small}}(\mathcal{R})$ such that
$$L\cong L^+\otimes_{\mathcal{R}}C.$$
Denote by $\mathrm{Rep}_\Gamma^{\text{H-small}}(C)$ the category of Hitchin-small representations of Γ over C.

(3) By a *Hitchin-small Higgs module* over $\mathcal{R}$ of rank r, we mean a pair
$$(H^+,\theta^+:H^+\to H^+\otimes_{\mathcal{R}}\Omega^1_{\mathcal{R}}\{-1\}=\oplus_{i=1}^dH^+\cdot(\zeta_p-1)\frac{\mathrm{dlog}T_i}{t})$$
satisfying $\theta^+\wedge\theta^+=0$ such that H^+ is a finite projective $\mathcal{R}$-module of rank r and $\theta=\sum_{i=1}^d(\zeta_p-1)\theta_i\otimes(\zeta_p-1)\frac{\mathrm{dlog}T_i}{t})$ with $\theta_i\in\mathrm{End}_{\mathcal{R}}(H^+)$ topologically nilpotent. Denote by $\mathrm{HIG}^{\text{H-small}}(\mathcal{R})$ the category of Hitchin-small Higgs module over $\mathcal{R}$. A Higgs module (H,θ) over R of rank r is called *Hitchin-small* if it is of the form $(H,\theta)=(H^+,\theta^+)[\frac{1}{p}]$ for some $(H^+,\theta^+)\in\mathrm{HIG}^{\text{H-small}}(\mathcal{R})$ of rank r. Denote by $\mathrm{HIG}^{\text{H-small}}(R)$ the category of Hitchin-small Higgs module over R.

The following lemma is obvious.

Lemma 3.6. *The base-change $L^+\mapsto L^+_\infty:=L^+\otimes_{\mathcal{R}}\widehat{R}^+_\infty$ induces an equivalence of categories*

$$\mathrm{Rep}_\Gamma^{\text{H-small}}(\mathcal{R})\simeq\mathrm{Rep}_\Gamma^{\text{H-small}}(\widehat{R}^+_\infty)$$

such that the natural map

$$\mathrm{R}\Gamma(\Gamma,L^+)\to\mathrm{R}\Gamma(\Gamma,L^+_\infty)$$

identifies the former with a direct summand of the latter whose complement is concentrated in degree ≥ 1 and killed by ζ_p-1.

Proof. To get the desired equivalence, we need to show the base-change functor is fully faithful. Note that L_∞ admits a Γ-equivariant decomposition:

$$L_\infty = \widehat{\bigoplus}_{\underline{\alpha}=(\alpha_1,\dots,\alpha_d)\in(\mathbb{N}\cap[0,1))^d]} L^+ \cdot \underline{T}^\alpha, \tag{3.4}$$

where $T^\alpha = T_1^{\alpha_1}\cdots T_d^{\alpha_d}$. Then the desired full faithfulness and the whole lemma will follows from that for any $\underline{\alpha}\neq 0$,

$$\mathrm{R}\Gamma(\Gamma, L^+\cdot \underline{T}^\alpha)$$

is concentrated in degree ≥ 1 and killed by $\zeta_p - 1$. To do so, without loss of generality, we may assume $\alpha_1 \neq 0$. Then $\gamma_1 - 1$ acts on $L^+\cdot\underline{T}^\alpha$ via

$$\zeta^\alpha \exp(-\theta_1(\zeta_p-1)^2) - 1 = (\zeta^\alpha - 1)\big(1+\zeta^\alpha\sum_{n\geq 1}\frac{(1-\zeta_p)^n}{n!}\frac{(\zeta_p-1)^n}{\zeta^\alpha-1}\theta_1^n\big)$$

with θ_1 topologically nilpotent. As $1+\zeta^\alpha\sum_{n\geq1}\frac{(1-\zeta_p)^n}{n!}\frac{(\zeta_p-1)^n}{\zeta^\alpha-1}\theta_1^n$ is invertible, we see that $\mathrm{H}^i(\Gamma, L^+\cdot\underline{T}^\alpha)$ is killed by ζ_p-1 for any i and vanishes for $i=0$. Now one can conclude by using Hochschild–Serre spectral sequence. □

We also need the following lemma.

Lemma 3.7. *For any $L^+\in\mathrm{Rep}_\Gamma^{\text{H-small}}(\mathcal{R})$ with $L_\infty^+ = L^+\otimes_{\mathcal{R}}\widehat{R}_\infty^+$, the natural inclusion*

$$\mathrm{R}\Gamma(\Gamma, L^+\otimes_{\mathcal{R}} B^+)\to \mathrm{R}\Gamma(\Gamma, L_\infty^+\otimes_{\widehat{R}_\infty^+} B_\infty^+)$$

identifies the former with a direct summand of the latter whose complement is concentrated in degree ≥ 1 and killed by ζ_p-1. Moreover, let $\theta_i\in\mathrm{End}_{\mathcal{R}}(L^+)$ be as in Definition 3.5(1), we have

$$\begin{aligned}\mathrm{H}^+(L^+) &:= \mathrm{H}^0(\Gamma, L\otimes_{\mathcal{R}} B^+) = \exp(\sum_{i=1}^d \theta_i W_i)(L^+)\\ &:= \{x\in L^+\otimes_{\mathcal{R}} B^+ \mid x = \sum_{J\in\mathbb{N}^d}\underline{\theta}^J(y)\underline{W}^{[J]}, y\in L^+.\}\end{aligned}\tag{3.5}$$

with a natural isomorphism

$$\mathrm{H}^+(L^+)\otimes_{\mathcal{R}} B^+ \xrightarrow{\cong} L^+\otimes_{\mathcal{R}} B^+,\tag{3.6}$$

and that $\mathrm{H}^n(\Gamma, L^+\otimes_{\mathcal{R}} B^+)$ is killed by $(\zeta_p-1)^2$ for any $n\geq 1$.

Proof. The decomposition (3.4) induces a Γ-equivariant decomposition

$$L_\infty^+\otimes_{\widehat{R}_\infty^+} B_\infty^+ = \widehat{\bigoplus}_{\underline{\alpha}=(\alpha_1,\dots,\alpha_d)\in(\mathbb{N}\cap[0,1))^d]}(L^+\otimes_{\mathcal{R}} B^+)\cdot\underline{T}^\alpha.$$

For the first claim, it suffices to show that for any $\underline{\alpha}\neq 0$,

$$\mathrm{R}\Gamma(\Gamma, (L^+\otimes_{\mathcal{R}} B^+)\cdot\underline{T}^\alpha)$$

is concentrated in degree ≥ 1 and killed by $(\zeta_p-1)^2$. But this follows from [MW23, Prop. 3.17(1)] by using Hochschild–Serre spectral sequence as in the proof of Lemma 3.6. For the "moreover" part follows from [MW23, Prop. 3.17(2)] directly. □

Now, we are going to prove the following Simpson correspondence, which can be viewed as an analogue of Theorem 3.2 on the integral level.

Proposition 3.8. (1) *For any $L_\infty^+\in\mathrm{Rep}_\Gamma^{\text{H-small}}(\widehat{R}_\infty^+)$ of rank r, we have*

$$\mathrm{H}^i(\Gamma, L_\infty^+\otimes_{\widehat{R}_\infty^+} B_\infty^+) = \begin{cases} H^+(L_\infty^+), & i=0\\ (\zeta_p-1)^2\text{-torsion}, & i\geq 1,\end{cases}$$

where $\mathrm{H}^+(L_\infty^+)$ is a finite projective $\mathcal{R}$-module of rank r. The restriction of

$$\Theta_{L_\infty}^+ : L_\infty^+\otimes_{\widehat{R}_\infty^+} B_\infty^+ \to L_\infty^+\otimes_{\widehat{R}_\infty^+} B_\infty^+\otimes_{\mathcal{R}}\Omega_{\mathcal{R}}^1\{-1\}$$

to $\mathrm{H}^+(L_\infty^+)$ defines a Higgs field θ^+ on $\mathrm{H}^+(L_\infty^+)$ making $(H^+(L_\infty^+),\theta^+)\in\mathrm{HIG}^{\text{H-small}}(\mathcal{R})$.

(2) *For any $(H^+,\theta^+)\in\mathrm{HIG}^{\text{H-small}}(\mathcal{R})$ of rank r, put*

$$\Theta_H^+ := \theta^+\otimes\mathrm{id} + \mathrm{id}\otimes\Theta : H^+\otimes_{\mathcal{R}} B_\infty^+ \to H^+\otimes_{\mathcal{R}} B_\infty^+\otimes_{\mathcal{R}}\Omega_{\mathcal{R}}^1\{-1\}.$$

Then $L^+_\infty(H^+,\theta^+) := (H^+\otimes_{\mathcal{R}} B^+_\infty)^{\Theta^+_H=0}$ *equipped with the induced* Γ*-action is a well-defined object in* $\mathrm{Rep}_\Gamma^{\text{H-small}}(\widehat{R}^+_\infty)$.

(3) The functors $L^+_\infty \mapsto (H^+(L^+_\infty),\theta^+)$ *and* $(H^+,\theta^+)\mapsto L^+_\infty(H^+,\theta^+)$ *induces an equivalence of categories*

$$\mathrm{Rep}_\Gamma^{\text{H-small}}(\widehat{R}^+_\infty)\simeq \mathrm{HIG}^{\text{H-small}}(\mathcal{R})$$

which preserves ranks, tensor products and dualities. Moreover, for any $L^+_\infty\in\mathrm{Rep}_\Gamma^{\text{H-small}}(\widehat{R}^+_\infty)$ *with corresponding* $(H^+,\theta^+)\in\mathrm{HIG}^{\text{H-small}}(\mathcal{R})$*, we have a natural isomorphism*

$$(L^+_\infty\otimes_{\widehat{R}^+_\infty}B^+_\infty,\Theta^+_{L_\infty})\cong(H^+\otimes_{\mathcal{R}}B^+_\infty,\Theta^+_H) \tag{3.7}$$

compatible with Higgs fields, and a quasi-isomorphism

$$\mathrm{R}\Gamma(\Gamma,L_\infty)\simeq\mathrm{HIG}(H,\theta)$$

where $L_\infty=L^+_\infty[\frac{1}{p}]$ *and* $(H,\theta)=(H^+,\theta^+)[\frac{1}{p}]$.

Proof. For Item (1): Let $L^+\in\mathrm{Rep}_\Gamma^{\text{H-small}}(\mathcal{R})$ be the representation corresponding to L^+_∞ in the sense of Lemma 3.6. Let $\theta_i\in\mathrm{End}_{\mathcal{R}}(L^+)$ be topologically nilpotent as in Definition 3.5(1). By Lemma 3.7, we have

$$H^+(L^+_\infty)=H^+(L^+)=\exp(\sum_{i=1}^d\theta_iW_i)(L^+)$$

is finite projective of rank r and θ^+ is the restriction of Θ^+_L to $H^+(L^+)$. It then follows from (3.2) that

$$\theta^+=\sum_{i=1}^d-(\zeta_p-1)\theta_i\otimes(\zeta_p-1)\frac{\mathrm{dlog}T_i}{t}.$$

So $(\mathrm{H}^+(L^+),\theta^+)$ is a well-defined object in $\mathrm{HIG}^{\text{H-small}}(\mathcal{R})$.

Item (2) follows from Lemma 3.4.

For Item (3): Granting we have already obtained the desired equivalence, then (3.7) follows from (3.6). Note that Item (1) implies $\mathrm{H}^i(\Gamma,L_\infty\otimes_{\widehat{R}_\infty}B^\infty)=0$ for any $i\geq1$. As the Higgs complex $\mathrm{HIG}(B_\infty,\Theta)$ is a resolution of $\widehat{R}_\infty$ by Theorem 2.4, we then have quasi-isomorphisms

$$\mathrm{R}\Gamma(\Gamma,L_\infty)\simeq\mathrm{R}\Gamma(\Gamma,\mathrm{HIG}(L_\infty\otimes_{\widehat{R}_\infty}B_\infty,\Theta_{L_\infty}))\simeq\mathrm{HIG}(H,\theta_H)$$

as desired. It remains to establish the desired equivalence.

For any $L^+_\infty\in\mathrm{Rep}_\Gamma^{\text{H-small}}(\widehat{R}^+_\infty)$, put $(H^+,\theta^+)=(H^+(L_\infty),\theta^+)$. Then we have a canonical morphism

$$\iota_{L^+_\infty}:L_\infty(H^+,\theta^+)\to L_\infty$$

compatible with Γ-actions defined by the composites

$$\begin{aligned}L_\infty(H^+,\theta^+)=&(H^+\otimes_{\mathcal{R}}B^+_\infty)^{\Theta^+_H:=\theta^+\otimes\mathrm{id}+\mathrm{id}\otimes\Theta=0}\\ \xrightarrow{=}&((L^+_\infty\otimes_{\widehat{R}^+_\infty}B^+_\infty)^\Gamma\otimes_{\mathcal{R}}B^+_\infty)^{\Theta^+_L\otimes\mathrm{id}+(\mathrm{id}\otimes\mathrm{id})\otimes\Theta=0}\\ \hookrightarrow&(L^+_\infty\otimes_{\widehat{R}^+_\infty}B^+_\infty\otimes_{\widehat{R}^+_\infty}B^+_\infty)^{\mathrm{id}\otimes\Theta\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{id}\otimes\Theta=0}\\ \to&(L^+_\infty\otimes_{\widehat{R}^+_\infty}B^+_\infty)^{\mathrm{id}\otimes\Theta=0}\\ =&L_\infty.\end{aligned}$$

Here, the last but second arrow is induced by the multiplication on B_∞.

For any $(H^+,\theta^+)\in\mathrm{HIG}^{\text{H-small}}(\mathcal{R})$, put $L^+_\infty=L^+_\infty(H^+,\theta^+)$. Then we have a canonical morphism

$$\iota_{(H^+,\theta^+)}:H^+(L^+_\infty)\to H^+$$

compatible with Higgs fields defined by the composites

$$\begin{aligned}H^+(L^+_\infty) =&(L^+_\infty \otimes_{\widehat{R}^+_\infty} B^+_\infty)^\Gamma\\ \xrightarrow{=}&((H^+ \otimes_{\mathcal{R}} B^+_\infty)^{\Theta^+_H=0} \otimes_{\widehat{R}^+_\infty} B^+_\infty)^\Gamma\\ \hookrightarrow&(H^+ \otimes_{\mathcal{R}} B^+_\infty \otimes_{\widehat{R}^+_\infty} B^+_\infty)^\Gamma\\ \to&(H \otimes_{\mathcal{R}} B^+_\infty)^\Gamma\\ =&H^+.\end{aligned}$$

Here the last but second arrow is again induced by the multiplication on B_∞ while the last follows as $(B^+_\infty)^\Gamma = (B^+)^\Gamma = \mathcal{R}$ by Lemma 3.7. To conclude, it is enough to show both $\iota_{L^+_\infty}$ and $\iota_{(H^+,\theta^+)}$ are isomorphism. But this follows from the explicit description (3.3) and (3.5) immediately. $\square$

Now, we are able to give a criterion for being Hitchin-small v-vector bundles.

Proposition 3.9. *Let $\mathfrak{X}$ be a liftable smooth formal scheme (not necessarily small affine) over A^+ with the fixed lifting $\widetilde{\mathfrak{X}}$ as before. Then an v-vector bundle $\mathcal{L}$ on X_v is Hitchin-small if and only if there exists an étale covering $\{\mathfrak{X}_i \to \mathfrak{X}\}_{i\in I}$ by affine small $\mathfrak{X}_i = \mathrm{Spf}(\mathcal{R}_i)$ with the corresponding $X_{i,\infty} = \mathrm{Spa}(\widehat{R}^+_\infty, \widehat{R}^+_{i,\infty})$ such that $\mathcal{L}(X_{i,\infty}) \in \mathrm{Rep}^{\text{H-small}}_\Gamma(\widehat{R}_{i,\infty})$ for all i.*

Proof. Recall for each affine small $\mathfrak{X}_i = \mathrm{Spf}(\mathcal{R}_i)$ with the generic fiber $X_i = \mathrm{Spa}(R_i, R_i^+ = \mathcal{R}_i)$, taking global sections induces an equivalence of categories

$$\mathrm{HIG}^{\text{H-small}}(X_i)(A) \simeq \mathrm{HIG}^{\text{H-small}}(R_i).$$

The result then follows from Corollary 3.3 and Proposition 3.8 immediately. $\square$

Remark 3.10. Thanks to existence of geometric Sen operator of Rodriguez-Carmago [RC22] (and of Heuer and Xu [HX24, Th. 3.2.1] in our setting), one can give another criterion for being a Hitchin-small v-vector bundle $\mathcal{L}$ in terms of its geometric Sen operator $\phi_{\mathcal{L}}$, just like Proposition 3.1. However, this definition is not good enough for in our setting as up to now, there is no "geometric Sen operator" for general $\mathbb{B}^+_{\mathrm{dR},n}$-local systems.

3.2. **Local small Riemann–Hilbert correspondence.** In this part, we always assume $\mathfrak{X} = \mathrm{Spf}(\mathcal{R})$ is small affine with the chart ψ over A^+ with the given lifting $\mathrm{Spf}(\widetilde{\mathcal{R}})$ over $\mathbb{A}_{\mathrm{inf}}(A, A^+)$. Let $X = \mathrm{Spa}(R, R^+)$ be the generic fiber of $\mathfrak{X}$ and $\widetilde{R}$ be the induced lifting of R over $\mathbb{B}^+_{\mathrm{dR}}(A, A^+)$. Then the lifting (cf. the paragraph above Lemma 2.6)

$$\widetilde{\psi} : \mathbb{A}_{\mathrm{inf}}(A, A^+)\langle \underline{T}^{\pm 1}\rangle \to \widetilde{\mathcal{R}}$$

induces a unique morphism

$$\widetilde{\psi}_{\mathrm{dR}} : \widetilde{R} \to \mathbb{B}^+_{\mathrm{dR}}(X_\infty).$$

Put $\widetilde{\mathbf{B}}_{\psi,\infty} = \mathbb{B}^+_{\mathrm{dR}}(X_\infty)$ and then it is equipped with a Γ-action such that for any $1 \le i,j \le d$,

$$\gamma_i([T_j^\flat]) = [\epsilon]^{\delta_{ij}}[T_j^\flat].$$

The $\widetilde{\psi}_{\mathrm{dR}}$ identifies $\widetilde{R}$ with a Γ-stable sub-$\mathbb{B}^+_{\mathrm{dR}}(A, A^+)$-algebra, denoted by $\widetilde{\mathbf{B}}_\psi$, such that $\widetilde{\mathbf{B}}_{\psi,\infty}$ admits a Γ-equivariant decomposition

$$\widetilde{\mathbf{B}}_{\psi,\infty} = \widehat{\oplus}_{\underline{\alpha}=(\alpha_1,\dots,\alpha_d)\in(\mathbb{N}[1/p]\cap[0,1))^d}\widetilde{\mathbf{B}}_\psi \cdot [T_1^{\flat\alpha_1}]\cdots[T_d^{\flat\alpha_d}].$$

Clearly, $\widetilde{\mathbf{B}}_{\psi,\infty}/t = \widehat{R}_\infty$ and $\widetilde{\mathbf{B}}_\psi/t = R$. For simplicity, in what follows, we put

$$\widetilde{\mathbf{B}}_{\psi,\infty}/t^\infty := \widetilde{\mathbf{B}}_{\psi,\infty}, \widetilde{\mathbf{B}}_\psi/t^\infty = \widetilde{\mathbf{B}}_\psi \text{ and } \widetilde{R}/t^\infty = \widetilde{R}.$$

Definition 3.11. Fix an $1 \le n \le \infty$.

(1) Let $C \in \{\widetilde{\mathbf{B}}_{\psi,\infty}/t^n, \widetilde{\mathbf{B}}_\psi/t^n\}$ By an *Hitchin-small representation* of Γ over C of rank r, we mean a finite projective C-module L endowed with an action of Γ such that its reduction $L/t \in \mathrm{Rep}^{\text{H-small}}_\Gamma(C/t)$. Denote by $\mathrm{Rep}^{\text{H-small}}_\Gamma(C)$ the category of Hitchin-small representation of Γ over C.

(2) By a *integrable connection* over $\widetilde{R}/t^n$ of rank r, we mean a pair

$$(D,\nabla: D\to D\otimes_{\widetilde{\mathcal{R}}}\Omega^1_{\widetilde{\mathcal{R}}}\{-1\})$$

satisfying $\nabla\wedge\nabla=0$ such that D is a finite projective $\widetilde{R}/t^n$-module of rank r and ∇ satisfies the Leibniz rule with respect to the derivation $\mathrm{d}:\widetilde{\mathcal{R}}\to\Omega^1_{\widetilde{\mathcal{R}}}\{-1\}$. An integrable connection (D,∇) is called *Hitchin-small* if it reduction $(D,\nabla)/t\in\mathrm{HIG}^{\text{H-small}}(R)$ modulo t. Denote by $\mathrm{MIC}^{\text{H-small}}(\widetilde{R}/t^n)$ the category of Hitchin-small integrable connection over $\widetilde{R}/t^n$. For an integrable connection (D,∇), denoted by

$$\mathrm{DR}(D,\nabla):=[D\xrightarrow{\nabla}D\otimes_{\widetilde{\mathcal{R}}}u\Omega^1_{\widetilde{\mathcal{R}}}\{-1\}\to\cdots\to D\otimes_{\widetilde{\mathcal{R}}}u^d\Omega^d_{\widetilde{\mathcal{R}}}\{-d\}]$$

the induced de Rham complex and put

$$\mathrm{H}^n_{\mathrm{dR}}(D,\nabla):=\mathrm{H}^n(\mathrm{DR}(D,\nabla)).$$

Recall we have deduce the following ismorphism from Proposition 2.9:

$$(\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}},\mathrm{d})(X_\infty)=(\widetilde{\mathbf{B}}_{\psi,\infty}\langle\underline{W}\rangle_{\mathrm{pd}}=\widetilde{\mathbf{B}}_{\psi,\infty}\langle W_1,\dots W_d\rangle_{\mathrm{pd}},\sum_{i=1}^d-u\frac{\partial}{\partial W_i}\otimes\frac{u}{t}\mathrm{dlog}T_i)$$

such that

$$\gamma_i(W_j)=W_j+\delta_{ij}u^2,\ \forall\ 1\le i,j\le d.$$

Now, we can give the local Riemann–Hilbert correspondence for Hitchin-small representations of Γ over $\widetilde{\mathbf{B}}_{\psi,\infty}/t^n$ and Hitchin-small integrable connections over $\widetilde{R}/t^n$.

Proposition 3.12. *Fix $1\le n\le\infty$.*

(1) *For any $L_\infty\in\mathrm{Rep}^{\text{H-small}}_\Gamma(\widetilde{\mathbf{B}}_{\psi,\infty}/t^n)$ of rank r, we have*

$$\mathrm{H}^n(\Gamma,L_\infty\otimes_{\widetilde{\mathbf{B}}_{\psi,\infty}}\widetilde{\mathbf{B}}_{\psi,\infty}\langle\underline{W}\rangle_{\mathrm{pd}})=\begin{cases}D(M_\infty), & n=0\\ 0, & n\ge1,\end{cases}$$

where $D(L_\infty)$ is a finite free $\widetilde{R}/t^n$-module of rank r. Moreover, the restriction of d to $D(L_\infty)$ induces a flat connection ∇ on $D(L_\infty)$ making it an object in $\mathrm{MIC}^{\text{H-small}}(\widetilde{R}/t^n)$.

(2) *For any $(D,\nabla)\in\mathrm{MIC}^{\text{H-small}}(\widetilde{R}/t^n)$ of rank r, define*

$$\nabla_D=\nabla\otimes\mathrm{id}+\mathrm{id}\otimes\nabla: D\otimes_{\widetilde{R}}\widetilde{\mathbf{B}}_{\psi,\infty}\langle\underline{W}\rangle_{\mathrm{pd}}\to D\otimes_{\widetilde{R}}\widetilde{\mathbf{B}}_{\psi,\infty}\langle\underline{W}\rangle_{\mathrm{pd}}\otimes_{\widetilde{R}}\Omega^1_{\widetilde{R}}\{-1\}.$$

Then it satisfies $\nabla_D\wedge\nabla_D=0$ and we have

$$\mathrm{H}^n_{\mathrm{dR}}(D\otimes_{\widetilde{R}}\widetilde{\mathbf{B}}_{\psi,\infty}\langle\underline{W}\rangle_{\mathrm{pd}},\nabla_D)=\begin{cases}L_\infty(D,\nabla), & n=0\\ 0, & n\ge1,\end{cases}$$

where $L_\infty(D,\nabla)$ is a finite free $\widetilde{\mathbf{B}}_{\psi,\infty}/t^n$-module of rank r. Moreover, the Γ-action on $\widetilde{\mathbf{B}}_{\psi,\infty}\langle\underline{W}\rangle_{\mathrm{pd}}$ induces a Γ-action on $L_\infty(D,\nabla)$ making it an object in $\mathrm{Rep}^{\text{H-small}}_\Gamma(\widetilde{\mathbf{B}}_{\psi,\infty}/t^n)$.

(3) *The functors $L_\infty\mapsto(D(L_\infty),\nabla)$ and $(D,\nabla)\mapsto L_\infty(D,\nabla)$ induce an equivalence of categories*

$$\mathrm{Rep}^{\text{H-small}}_\Gamma(\widetilde{\mathbf{B}}_{\psi,\infty}/t^n)\simeq\mathrm{MIC}^{\text{H-small}}(\widetilde{R}/t^n)$$

preserving ranks, tensor products and dualities. Moreover for any $L_\infty\in\mathrm{Rep}^{\text{H-small}}_\Gamma(\widetilde{\mathbf{B}}_{\psi,\infty}/t^n)$ corresponding to (D,∇), we have a quasi-isomorphism

$$\mathrm{R}\Gamma(\Gamma,L_\infty)\simeq\mathrm{DR}(D,\nabla).$$

Before proving Proposition 3.12, we first recall some well-known lemmas.

Lemma 3.13. *Let B be a ring with a non-zero divisor t and $B_m:=B/t^m$ for any $m\ge1$.*

(1) *Let M be a B_m-module. If M/tM is a finite free B_1-module of rank r and for any $0\le n\le m-1$, the multiplication by t^n induces an isomorphism of B_1-modules*

$$M/tM\xrightarrow{\cong}t^nM/t^{n+1}M,$$

then M is a finite free B_m-module of rank r.

(2) *Assum moreover B is t-adically complete. Let M be a B-module such that M/tM is a finite free B_1-module of rank r and for any $n \geq 0$, the multiplication by t^n induces an isomorphism of B_1-modules*
$$M/tM \xrightarrow{\cong} t^nM/t^{n+1}M,$$
then M is a finite free B-module of rank r.

Proof. The Item (2) is a consequence of Item (1) together with [MT20, Lem. 1.9]. So it is enough to prove Item (1).

Let $e_1, \dots, e_r$ be elements in M whose reductions modulo t induces an B_1-basis of M/tM. We claim that $e_1, \dots, e_r$ form a B_m-basis of M. We now prove this by induction on m. We may assume $m > 1$ and then by inductive hypothesis, the reduction of $e_1, \dots, e_r$ modulo t^{m-1} form a B_{m-1}-basis of $M/t^{m-1}M$. Thus, for any $x \in M$, there exist $b_1, \dots, b_r \in B_m$ such that
$$x \equiv b_1e_1 + \cdots + b_re_r \mod t^{m-1}M.$$
As $t^{m-1}M \cong M/tM$, one can find $c_1, \dots, c_r \in B_m$ such that
$$x - (b_1e_1 + \cdots + b_re_r) = t^{m-1}c_1e_1 + \cdots + t^{m-1}c_re_r.$$
Put $a_i = b_i + t^{m-1}c_i$ for any $1 \leq i \leq r$ and then we have
$$x = a_1e_1 + \cdots + a_re_r.$$
So $e_1, \dots, e_r$ generate M over B_m. To conclude, it remains to show for any $d_1, \dots, d_r \in B_m$ such that
$$d_1e_1 + \cdots + d_re_r = 0,$$
we must have
$$d_1 = d_2 = \cdots = d_r = 0.$$
By inductive hypothesis, we have
$$d_1 \equiv d_2 \equiv \cdots \equiv d_r \mod t^{m-1}.$$
That is, there are $f_1, \dots, f_r \in B_m$ such that $d_i = t^{m-1}f_i$ for any $1 \leq i \leq r$. Thus, we have
$$f_1t^{m-1}e_1 + \cdots + f_rt^{m-1}e_r = 0.$$
Using $t^{m-1}M \cong M/tM$ again, we conclude that
$$f_1 \equiv f_2 \equiv \cdots \equiv f_r \mod t,$$
which forces that for any $1 \leq i \leq r$,
$$d_i = t^{m-1}f_i = 0$$
as desired. This completes the proof. □

Lemma 3.14. (1) *Fix $B \in \{\widetilde{\mathbf{B}}_{\psi,\infty}, \widetilde{\mathbf{B}}_\psi\}$. Then the functor $M \mapsto \{M/t^n\}_{n\geq 1}$ induces an equivalence of categories*
$$\mathrm{Rep}_\Gamma^{\text{H-}small}(B) \simeq \varprojlim_n \mathrm{Rep}_\Gamma^{\text{H-}small}(B/t^n)$$
which is compatible with cohomologies; that is, we have a canonical quasi-isomorphism
$$\mathrm{R}\Gamma(\Gamma, M) \xrightarrow{\simeq} \varprojlim_n \mathrm{R}\Gamma(\Gamma, M/t^n).$$
(2) *Then the functor $(D, \nabla) \mapsto (D/t^n, \nabla)$ induces an equivalence of categories*
$$\mathrm{MIC}^{\text{H-}small}(\widetilde{R}) \simeq \varprojlim_n \mathrm{MIC}^{\text{H-}small}(\widetilde{R}/t^n),$$
which is compatible with cohomologies; that is, we have a canonical quasi-isomorphism
$$\mathrm{DR}(D, \nabla) = \varprojlim_n \mathrm{DR}(D/t^n\nabla)$$

Proof. Note that $\widetilde{\mathbf{B}}_\psi$, $\widetilde{\mathbf{B}}_{\psi,\infty}$ and $\widetilde{R}$ are all t-adically complete and t-torsion free. The desired equivalences follows immediately from Lemma 3.13(2) while the cohomological comparison follows from the (derived) Nakayama's Lemma. □

Proof of Proposition 3.12: By Lemma 3.14, it suffices to deal with the case where $n < \infty$. Note that the $n = 1$ case has been established in Proposition 3.8. We will finish the proof by induction on n. From now on, assume there exists some $n \geq 1$ such that the result holds true for any $1 \leq m \leq n$.

For Item (1): Fix an $M_\infty \in \mathrm{Rep}_\Gamma^{\text{H-small}}(\widetilde{\mathbf{B}}_{\psi,\infty}/t^{n+1})$ of rank r. Then we have

$$tM_\infty \in \mathrm{Rep}_\Gamma^{\text{H-small}}(\widetilde{\mathbf{B}}_{\psi,\infty}/t^n) \text{ and } M_\infty/t \in \mathrm{Rep}_\Gamma^{\text{H-small}}(\widetilde{\mathbf{B}}_{\psi,\infty}/t).$$

The short exact sequence

$$0 \to tM_\infty \to M_\infty \to M_\infty/t \to 0$$

gives rise to an exact triangle

$$\mathrm{R}\Gamma(\Gamma, tM_\infty \otimes_{\widetilde{\mathbf{B}}_{\psi,\infty}} \widetilde{\mathbf{B}}_{\psi,\infty}\langle \underline{W}\rangle_{\mathrm{pd}}) \to \mathrm{R}\Gamma(\Gamma, M_\infty \otimes_{\widetilde{\mathbf{B}}_{\psi,\infty}} \widetilde{\mathbf{B}}_{\psi,\infty}\langle \underline{W}\rangle_{\mathrm{pd}}) \to \mathrm{R}\Gamma(\Gamma, M_\infty/t \otimes_{\widetilde{\mathbf{B}}_{\psi,\infty}} \widetilde{\mathbf{B}}_{\psi,\infty}\langle \underline{W}\rangle_{\mathrm{pd}}).$$

Considering the induced long exact sequence, by inductive hypothesis, we have

$$\mathrm{H}^n(\Gamma, M_\infty \otimes_{\widetilde{\mathbf{B}}_{\psi,\infty}} \widetilde{\mathbf{B}}_{\psi,\infty}\langle \underline{W}\rangle_{\mathrm{pd}}) = 0$$

for any $n \geq 1$ and there is a short exact sequence

$$0 \to D(tM_\infty) \to D(M_\infty) \to D(M_\infty/t) \to 0.$$

We claim that

$$D(tM_\infty) = tD(M_\infty) \text{ and } D(M_\infty/t) = D(M_\infty)/t.$$

Indeed, consider the exact sequence

$$\cdots \to M_\infty \xrightarrow{\times t} M_\infty \xrightarrow{\times t^n} M_\infty \xrightarrow{\times t} M_\infty \to M_\infty/t \to 0.$$

Applying $\mathrm{R}\Gamma(\Gamma, - \otimes_{\widetilde{\mathbf{B}}_{\psi,\infty}} \widetilde{\mathbf{B}}_{\psi,\infty}\langle \underline{W}\rangle_{\mathrm{pd}})$ to the above sequence, we obtain an exact sequence

$$\cdots \to D(M_\infty) \xrightarrow{\times t} D(M_\infty) \xrightarrow{\times t^n} D(M_\infty) \xrightarrow{\times t} D(M_\infty) \to D(M_\infty/t) \to 0.$$

So we have the short exact sequence

$$0 \to tD(M_\infty) \to D(M_\infty) \to D(M_\infty/t) \to 0.$$

Thus the claim follows.

A similar argument implies that for any $0 \leq m \leq n$, we have $D(t^m M_\infty) \cong t^m D(M_\infty)$, yielding that

$$t^m D(M_\infty)/t^{m+1}D(M_\infty) \cong D(t^m M_\infty/t^{m+1}M_\infty) \cong D(M_\infty/t).$$

By Lemma 3.13, we see that $D(M_\infty)$ is finite free of rank r over $\widetilde{R}/t^{n+1}$ as desired. Finally, as

$$(D(M_\infty/t), \nabla) = (D(M_\infty)/t, \nabla) \in \mathrm{MIC}^{\text{H-small}}(\widetilde{R}/t)$$

we have $(D(M_\infty/t), \nabla) \in \mathrm{MIC}^{\text{H-small}}(\widetilde{R}/t^{n+1})$ as desired. This completes the proof of Item (1).

Item (2) can be deduced from the similar argument above.

For Item (3): Similar to the proof of Proposition 3.8 (3), for any $M_\infty \in \mathrm{Rep}_\Gamma^{\text{H-small}}(\widetilde{\mathbf{B}}_{\psi,\infty}/t^n)$ and any $(D, \nabla) \in \mathrm{MIC}^{\text{H-small}}(\widetilde{R}/t^{n+1})$, one can construct natural morphisms

$$\iota_{M_\infty} : M_\infty(D(M_\infty), \nabla) \to M_\infty$$

and

$$\iota_{(D,\nabla)} : (D(M_\infty(D, \nabla)), \nabla) \to (D, \nabla).$$

To see they are both isomorphism, by Nakayama's Lemma, one can check this modulo t, and thus reduces to Theorem Proposition 3.8 (3). This establishes the desired equivalence of categories. The cohomological comparison follows easily. $\square$

To complete the local theory, we state the following result, which will not be used in this paper.

Corollary 3.15. *For any $1 \leq n \leq \infty$, the base-change $L \mapsto L_\infty := L \otimes_{\widetilde{\mathbf{B}}_\psi} \widetilde{\mathbf{B}}_{\psi,\infty}$ induces an equivalence of categories*

$$\mathrm{Rep}_\Gamma^{\textit{H-small}}(\widetilde{\mathbf{B}}_\psi/t^n) \xrightarrow{\simeq} \mathrm{Rep}_\Gamma^{\textit{H-small}}(\widetilde{\mathbf{B}}_{\psi,\infty}/t^n)$$

which is compatible with cohomologies; that is, we have a quasi-isomorphism

$$\mathrm{R}\Gamma(\Gamma, L) \cong \mathrm{R}\Gamma(\Gamma, L_\infty).$$

Proof. The full faithfulness together with the cohomological comparison follows from the $n=1$ case (cf. Lemma 3.6) together with the derived Nakayama's Lemma. It remains to show the essential surjectivity. Note that

$$\widetilde{\mathbf{B}}_{\psi}\langle \underline{W}\rangle_{\mathrm{pd}} \subset \widetilde{\mathbf{B}}_{\psi,\infty}\langle \underline{W}\rangle_{\mathrm{pd}}$$

is a d-preserving sub-ring stable under the action of Γ. For any $L_\infty \in \mathrm{Rep}_\Gamma^{\text{H-small}}(\widetilde{\mathbf{B}}_{\psi,\infty}/t^n)$ with the induced $(D,\nabla)\in \mathrm{MIC}^{\text{H-small}}(\widetilde{R}/t^n)$, one can achieve an $L\in \mathrm{Rep}_\Gamma^{\text{H-small}}(\widetilde{\mathbf{B}}_{\psi}/t^n)$ by using $\widetilde{\mathbf{B}}_{\psi}\langle \underline{W}\rangle_{\mathrm{pd}}$ instead of $\widetilde{\mathbf{B}}_{\psi,\infty}\langle \underline{W}\rangle_{\mathrm{pd}}$ in Proposition 3.12. Then one can conclude by checking $L\otimes_{\widetilde{\mathbf{B}}_\psi}\widetilde{\mathbf{B}}_{\psi,\infty}\cong L_\infty$ directly. □

4. The stacky Riemann–Hilbert correspondence: Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. To do so, we need the following result:

Theorem 4.1. *Fix a* $\mathrm{Spa}(A,A^+)\in \mathrm{Perfd}$. *Let* $\mathfrak{X}$ *be a liftable smooth formal scheme over* A^+ *with a fixed lifting* $\widetilde{\mathfrak{X}}$ *over* $\mathbb{A}_{\mathrm{inf}}(A,A^+)$. *Let* X *be the generic fiber of* $\mathfrak{X}$ *and* $\widetilde{X}$ *be the lifting of* X *over* $\mathbb{B}_{\mathrm{dR}}^+(A,A^+)$ *induced by* $\widetilde{\mathfrak{X}}$. *Let* $(\mathcal{O}\mathbb{B}_{\mathrm{dR,pd}}^+,\mathrm{d})$ *be the period sheaf with connection constructed in §2. Let* $\nu: X_v\to X_{\mathrm{\acute{e}t}}$ *be the natural morphism of sites. Let* $1\le n\le\infty$.

(1) *For any* $\mathbb{L}\in \mathrm{LS}^{\text{H-small}}(X,\mathbb{B}_{\mathrm{dR},n}^+)(A)$ *of rank* r, *we have*

$$\mathrm{R}^n\nu_*(\mathbb{L}\otimes_{\mathbb{B}_{\mathrm{dR}}^+}\mathcal{O}\mathbb{B}_{\mathrm{dR,pd}}^+)=\begin{cases}\mathcal{D}(\mathbb{L}), & n=0\\ 0, & n\ge 1\end{cases}$$

where $\mathcal{D}(\mathbb{L})$ *is a locally finite free* $(\mathcal{O}_{\widetilde{X}_n})$*-module of rank* r *on* $X_{\mathrm{\acute{e}t}}$ *such that the*

$$\mathrm{id}_{\mathbb{L}}\otimes\mathrm{d}:\mathbb{L}\otimes_{\mathbb{B}_{\mathrm{dR}}^+}\mathcal{O}\mathbb{B}_{\mathrm{dR,pd}}^+\to \mathbb{L}\otimes_{\mathbb{B}_{\mathrm{dR}}^+}\mathcal{O}\mathbb{B}_{\mathrm{dR,pd}}^+\otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}}\Omega^1_{\widetilde{\mathfrak{X}}}\{-1\}$$

induces a flat connection $\nabla_{\mathbb{L}}$ *on* $\mathcal{D}(\mathbb{L})$ *making* $(\mathcal{D}(\mathbb{L}),\nabla_{\mathbb{L}})$ *an object in* $\mathrm{MIC}^{\text{H-small}}(\widetilde{X}_n)(A)$.

(2) *For any* $(\mathcal{D},\nabla)\in \mathrm{MIC}^{\text{H-small}}(\widetilde{X}_n)$ *of rank* r, *define*

$$\nabla_{\mathcal{D}}:=\nabla\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{d}:\mathcal{D}\otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}^I}}\mathcal{O}\mathbb{B}_{\mathrm{dR,pd}}^+\to\mathcal{D}\otimes_{\mathcal{O}_{\widetilde{X}}}\mathcal{O}\mathbb{B}_{\mathrm{dR,pd}}^+\otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}}\Omega^1_{\widetilde{\mathfrak{X}}}\{-1\}.$$

Then

$$\mathbb{L}(\mathcal{D},\nabla):=(\mathcal{D}\otimes_{\mathcal{O}_{\widetilde{X}}}\mathcal{O}\mathbb{B}_{\mathrm{dR,pd}}^+)^{\nabla_{\mathcal{D}}=0}$$

is an object of rank r *in* $\mathrm{LS}^{\text{H-small}}(X,\mathbb{B}_{\mathrm{dR},n}^+)(A)$.

(3) *The functors* $\mathbb{L}\mapsto(\mathcal{D}(\mathbb{L}),\nabla_{\mathbb{L}})$ *and* $(\mathcal{D},\nabla)\mapsto\mathbb{L}(\mathcal{D},\nabla)$ *in Items (1) and (2) respectively define an equivalence of categories*

$$\rho_{\widetilde{\mathfrak{X}}}:\mathrm{LS}^{\text{H-small}}(\mathfrak{X},\mathbb{B}_{\mathrm{dR},n}^+)(A)\xrightarrow{\simeq}\mathrm{MIC}^{\text{H-small}}(\widetilde{X}_n)(A)$$

which preserves ranks, tensor products and dualities. Moreover, for any $\mathbb{L}\in \mathrm{LS}^{\text{H-small}}(X,\mathbb{B}_{\mathrm{dR},n}^+)$ *with corresponding* $(\mathcal{D},\nabla)\in\mathrm{MIC}^{\text{H-small}}(\widetilde{X}_n)(A)$, *there exists a quasi-isomorphism*

$$\mathrm{R}\nu_*\mathbb{L}\simeq \mathrm{DR}(\mathcal{D},\nabla).$$

In particular, we have a quasi-isomorphism

$$\mathrm{R}\Gamma(X_v,\mathbb{L})\simeq\mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}},\mathrm{DR}(\mathcal{D},\nabla)).$$

Before proving this theorem, we explain how to use it to obtain Theorem 1.1.

Proof of Theorem 1.1: Let $\mathfrak{X}$ be a liftable smooth formal scheme over $\mathcal{O}_C$ with the fixed lifting $\widetilde{\mathfrak{X}}$ over $\mathbf{A}_{\mathrm{inf}}$, the generic fiber X and the lifting $\widetilde{X}$ of X over $\mathbf{B}_{\mathrm{dR}}^+$. For any $\mathrm{Spa}(A,A^+)\in\mathrm{Perfd}$, we denote by $\mathfrak{X}_A$, $\widetilde{\mathfrak{X}}_A$, X_A and $\widetilde{X}_A$ their corresponding base-changes to A^+, $\mathbb{A}_{\mathrm{inf}}(A,A^+)$, A and $\mathbb{B}_{\mathrm{dR}}^+(A,A^+)$. Let $(\mathcal{O}\mathbb{B}_{\mathrm{dR,pd},A}^+,\mathrm{d})$ be the period sheaf with connection on $X_{A,v}$ corresponding to the lifting $\widetilde{\mathfrak{X}}_A$. Then it follows from Proposition 2.10 that for any $f:\mathrm{Spa}(B,B^+)\to\mathrm{Spa}(A,A^+)$, we have an isomorphism

$$\mathcal{O}\mathbb{B}_{\mathrm{dR,pd},B}^+=\mathcal{O}\mathbb{B}_{\mathrm{dR,pd},A}^+\widehat{\otimes}_{\mathbb{B}_{\mathrm{dR}}^+(A,A^+)}\mathbb{B}_{\mathrm{dR}}^+(B,B^+)$$

compatible with connection. Note that f induces the obvious base-change functors

$$f_{\mathrm{LS}}:\mathrm{LS}(X,\mathbb{B}_{\mathrm{dR},n}^+)^{\text{H-small}}(A)\to\mathrm{LS}(X),\mathbb{B}_{\mathrm{dR},n}^+)^{\text{H-small}},\quad \mathbb{L}\mapsto\mathbb{L}\widehat{\otimes}_{\mathbb{B}_{\mathrm{dR}}^+(A,A^+)}\mathbb{B}_{\mathrm{dR}}^+(B,B^+)$$

and

$$f_{\mathrm{MIC}}:\mathrm{MIC}(\widetilde{X}_n)^{\text{H-small}}(A)\to\mathrm{MIC}^{\text{H-small}}(\widetilde{X}_n)(B),\quad(\mathcal{D},\nabla)\mapsto(\mathcal{D}\widehat{\otimes}_{\mathbb{B}^+_{\mathrm{dR}}(A,A^+)}\mathbb{B}^+_{\mathrm{dR}}(B,B^+),\nabla\otimes\mathrm{id}).$$

As the construction in Theorem 4.1 is clearly functorial in $\mathrm{Spa}(A,A^+)$, we have

$$f_{\mathrm{MIC}}\circ\rho_{\widetilde{\mathfrak{X}}_A}=\rho_{\widetilde{\mathfrak{X}}_B}\circ f_{\mathrm{LS}}.$$

As $\rho_{\widetilde{\mathfrak{X}}_A}$ preserves ranks, the equivalence criterion of $\rho_{\widetilde{\mathfrak{X}}_A}$ yields the desired equivalence of stacks

$$\rho_{\widetilde{\mathfrak{X}}}:\mathrm{LS}_r(X,\mathbb{B}^+_{\mathrm{dR},n})^{\text{H-small}}\xrightarrow{\simeq}\mathrm{MIC}_r(\widetilde{X}_n)^{\text{H-small}}.$$

□

Now, we focus on the proof of Theorem 4.1 by proceeding as in the proof of [CLWZ24, Th. 7.11].

Lemma 4.2. *Suppose that $\mathfrak{X}$ is small affine and let $\mathbb{L}\in\mathrm{LS}(X,\mathbb{B}^+_{\mathrm{dR},n})(A)$. Then for any affinoid perfectoid $U=\mathrm{Spa}(S,S^+)\in X_{\infty,v}$ and for any $i\geq 1$, we have*

$$\mathrm{H}^i(U,\mathbb{L}\otimes_{\mathbb{B}^+_{\mathrm{dR}}}\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}})=0.$$

Proof. We need to show that $\mathrm{R}\Gamma(U,\mathbb{L}\otimes_{\mathbb{B}^+_{\mathrm{dR}}}\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}})$ is concentrated in degree 0. By derived Nakayama's Lemma, we are reduced to showing the case for $n=1$. That is, we have to show for any v-vector bundle $\mathbb{L}$,

$$\mathrm{R}\Gamma(U,\mathbb{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}})$$

is concentrated in degree 0. It follows from [KL16, Th. 3.5.8] that there is a finite projective S-module L such that $\mathbb{L}_{|U}\cong L\otimes_{S^+}\widehat{\mathcal{O}}_U$. So without loss of generality, we may assume $\mathbb{L}_{|U}=\widehat{\mathcal{O}}_U$, and are reduced to showing that

$$\mathrm{R}\Gamma(U,\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}})=\mathrm{R}\Gamma(U,\widehat{\mathcal{O}}_U[\underline{W}]^\wedge_{\mathrm{pd}})$$

is concentrated in degree 0, by using Proposition 2.9. It suffices to show that

$$\mathrm{R}\Gamma(U,\widehat{\mathcal{O}}^+_U[\underline{W}]^\wedge_{\mathrm{pd}})$$

has cohomologies killed by $\mathfrak{m}_C$ in degree ≥ 1. However, as $\widehat{\mathcal{O}}^+_U[\underline{W}]^\wedge_{\mathrm{pd}}$ is the p-adic completion of a directly limit of finite free $\widehat{\mathcal{O}}^+_U$-modules, by the quasi-compactness of U, we are reduced to show that

$$\mathrm{R}\Gamma(U,\widehat{\mathcal{O}}^+_U)$$

has cohomologies killed by $\mathfrak{m}_C$ in degree ≥ 1. This is well-known (cf. [Sch17, Prop. 8.8]). □

Lemma 4.3. *Suppose that $\mathfrak{X}$ is small affine and let $\mathbb{L}\in\mathrm{LS}(X,\mathbb{B}^+_{\mathrm{dR},n})(A)$. Then there exists a natural quasi-isomorphism*

$$\mathrm{R}\Gamma(\Gamma,\mathbb{L}\otimes_{\mathbb{B}^+_{\mathrm{dR}}}\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}(X_\infty))\to\mathrm{R}\Gamma(X_v,\mathbb{L}\otimes_{\mathbb{B}^+_{\mathrm{dR}}}\mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}).$$

Proof. This follows from the same argument for the proof of [Wan23, Lem. 5.11] by using Lemma 4.2 instead of [Wan23, Lem. 5.7]. □

Now, we are prepared to show Theorem 4.1.

Proof of Theorem 4.1: For Item (1): Fix an $\mathbb{L}\in\mathrm{LS}^{\text{H-small}}(X,\mathbb{B}^+_{\mathrm{dR},n})(A)$ of rank r. Since the problem is local on $\mathfrak{X}_{\text{ét}}$, we may assume $\mathfrak{X}=\mathrm{Spf}(\mathcal{R})$ is small affine. According to Remark 1.4 and Proposition 3.9, comparing Definition 3.5(2) with Definition 3.11(1), we have $\mathbb{L}(X_\infty)\in\mathrm{Rep}^{\text{H-small}}_\Gamma(\widetilde{\mathbf{B}}_{\psi,\infty}/t^n)$. Then one can deduce Item (1) from Lemma 4.3 together with Proposition 3.12(1).

For Item (2): Fix an $(\mathcal{D},\nabla)\in\mathrm{MIC}^{\text{H-small}}(\widetilde{X}_n)(A)$. Since the problem is again local on $\mathfrak{X}_{\text{ét}}$, we may assume $\mathfrak{X}=\mathrm{Spf}(\mathcal{R})$ is small affine. According to Remark 1.4 and Proposition 3.1, comparing Definition 3.5(3) with Definition 3.11(2), we see that the global section

$$(D,\nabla):=(\mathcal{D},\nabla)(X)\in\mathrm{MIC}^{\text{H-small}}(\widetilde{R}/t^n).$$

Then one can deduce Item (2) from Proposition 3.12(2).

For Item (3): For any $\mathbb{L}\in\mathrm{LS}^{\text{H-small}}(X,\mathbb{B}^+_{\mathrm{dR},n})(A)$ and any $(\mathcal{D},\nabla)\in\mathrm{MIC}^{\text{H-small}}(\widetilde{X}_n)(A)$, similar to the proof of Proposition 3.12(3), one can construct two canonical morphisms

$$\iota_{\mathbb{L}}:\mathbb{L}(\mathcal{D}(\mathbb{L}),\nabla_{\mathbb{L}})\to\mathbb{L}$$

and

$$\iota_{(\mathcal{D},\nabla)} : (\mathcal{D}(\mathbb{L}(\mathcal{D},\nabla)), \nabla_{\mathbb{L}(\mathcal{D},\nabla)}) \to (\mathcal{D},\nabla).$$

To get desired equivalence of categories, we have to show both $\iota_{\mathbb{L}}$ and $\iota_{(\mathcal{D},\nabla)}$ are isomorphisms. But this is still a local problem and thus reduces to the proof of Proposition 3.12(3). Finally, we have to show that for any $\mathbb{L} \in \mathrm{LS}^{\text{H-small}}(X, \mathbb{B}^+_{\mathrm{dR},n})(A)$ with corresponding $(\mathcal{D},\nabla) \in \mathrm{MIC}^{\text{H-small}}(\widetilde{X}_n)(A)$, we have

$$\mathrm{R}\nu_*\mathbb{L} \simeq \mathrm{DR}(\mathcal{D},\nabla).$$

By Poincaré's Lemma (cf. Theorem 2.4), we have

$$\mathrm{R}\nu_*\mathbb{L} \simeq \mathrm{R}\nu_*(\mathrm{DR}(\mathbb{L}\otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}, \mathrm{id}_{\mathbb{L}}\otimes \mathrm{d})).$$

It follows from Item (1) that

$$\mathrm{R}\nu_*(\mathrm{DR}(\mathbb{L}\otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}, \mathrm{id}_{\mathbb{L}}\otimes \mathrm{d})) \simeq \nu_*(\mathrm{DR}(\mathbb{L}\otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,pd}}, \mathrm{id}_{\mathbb{L}}\otimes \mathrm{d})) \simeq \mathrm{DR}(\mathcal{D},\nabla_{\mathcal{D}}).$$

This completes the proof. □

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