2.d : $R^k = R^{k+1}$

assume $(a, c) \in R^{k+1}$

and we also assume $(a, c) \in R^k$

$\because R^{k+1} = R^k \cup (R ; R^k)$

and $(a, c) \notin R^k$

$\therefore (a, c) \in (R ; R^k)$

$\therefore \exists (a, m_k) \in R \quad (m_k, c) \in R^k$

$\because (m_k, c) \notin R^i_{i \leq k-1}$ [if $(m_k, c) \in R^{k-1}$ and $(a, m_k) \in R$ we can get $(a, c) \in R^k$]

so $(m_k, c) \in R ; R^{k-2}$

$\therefore \exists (m_k, m_{k-1}) \in R \quad (m_{k-1}, c) \in R^{k-1}$

we still have $(m_{k-1}, c) \notin R^i \; (i \leq k-2)$

[because, if $(m_{k-1}, c) \in R^{k-2}$, then $(m_k, c) \in R^{k-1}$]

and so on ...

arrange the result

(a, m_k) $\quad\quad\quad$ $(m_k, c) \notin R^i \quad i \leq k-1$

(m_k, m_{k-1}) $\quad\quad\quad$ $(m_{k-1}, c) \notin R^i \quad i \leq k-2$

(m_{k-1}, m_{k-2}) $\quad\quad\quad$ $(m_{k-2}, c) \notin R^i \quad i \leq k-3$

$\vdots$

(m_3, m_2) $\quad\quad\quad$ $(m_2, c) \notin R^i \quad i \leq 1$

(m_2, m_1) $\quad\quad\quad$ $(m_1, c) \notin R^i \quad i \leq 1$

(m_1, m_0) $\quad\quad\quad$ $(m_0, c) \notin R^0$

and $m_0 \neq m_1 \neq m_2 \cdots \neq m_k$

$m_i \in S$

$|\{m_0, m_1, m_2 \cdots m_k\}| = K = |S|$

$\therefore a \in \{m_0, m_1 \cdots m_k\}$

$\therefore (a, c) \in R^k$

与假设矛盾 (假设 $(a, c) \notin R^k$)

$\therefore (a, c) \in R^k$

此题得证

2、e. 证明 R^k is T

assume (a,b), $(b,c) \in R^k$

we assume $(a,c) \notin R^k$

① (a,b), $(b,c) \in R$

矛盾，$(a,c) \in R^k$

② $(a,b) \in R$ or $(b,c) \in R$

两者有一个成立

当 $(a,b) \in R$，显然矛盾

当 $(b,c) \in R$，$(a,b) \Rightarrow [(a,m_k), (m_k,b) \in R^k$

$m_k \neq b$ 且 $(m_k,b) \notin R$ [if $(m_k,b) \in R$, $(a,c) \in R^k$]

∴ $(m_k,b) \Rightarrow (m_k,m_{k-1}) \in R$, $(m_{k-1},b) \in R^{k-2}$ [$(m_{k-1},b) \in R$] ∴

以此类推，最终可得 $|\{m_k, m_{k-1}, m_{k-2}, m_{k-3} \dots m_1\}| = |S| = k$

其中有 $m_i = a$

∴ $(a,b) \in R$

与假设矛盾

∴ 此情况也矛盾

③ $(a,b) \notin R$ 和 $(b,c) \in R$

(a,m_k) $\quad$ $(m_k,b) \in R^i$

(b,n_k) $\quad$ $(n_k,c) \in R^i$

∵ $(a,b) \in R$ ∴ $m_k \neq b$, $n_k \neq c$

即 $(b,b) \notin R^i$ $\quad$ $(c,c) \in R^i$

矛盾，因此可证 $(a,c) \in R^k$，T

2. f: $Q = (R \cup R^{\leq})$

(1) R: $Q_0 = \{(x,x) \mid x \in S\}$

$Q_0 \subseteq Q^R$

∴ R 成立

S.

① $(a,b) \in Q$

$\exists (b,a) \in Q$

due to $R \cup R^{\leq}$

② $(a,b) \notin Q$

∴ $\exists (a, m_k) \in Q$

$(m_k, b) \in Q^R$

∵ $Q = R \cup R^{\leq}$

∴ $(m_k, a) \in Q$.

③ if $(m_k, b) \in Q$ 得证

if $(m_k, b) \notin Q$, ∃

$\exists (m_k, m_{k-1}) \in Q$, $(m_{k-1}, b) \in Q^R$

重复步骤③

最终我们找到一个 $\in R$

的 (m_{i_Q}, b) (

这样有 $(a, m_k) \in R$

$(m_k, m_{k-1}) \in R$

$(m_{k-1}, m_{k-2}) \in R$

$\vdots$

$(m_i, b) \in R$.

$\because R \cup R^{\leftarrow}$

$\therefore (m_k, a), (m_{k-1}, m_k), (m_{k-2}, m_{k-1}) \ldots (b_i, m_i) \in Q$

$\therefore (b, a) \in Q$

因此，S 成立

T: 由上一问可证

T成立

since 是 equavilance 关系