Problem 1:

(a)

In order to prove (R1;R2);R3 = R1;(R2;R3), we should first prove (R1;R2);R3 $\subseteq$ R1;(R2;R3), and then prove R1;(R2;R3) $\subseteq$ (R1;R2);R3. From the condition given(R1;R2 = {(a,c) : there is b with (a,b) $\in$ R1 and (b,c) $\in$ R2}), we can suppose that $R_1 \subseteq A\times B$, $R_2 \subseteq B\times C$, $R_3 \subseteq C\times D$:

Proof for (R1;R2);R3 $\subseteq$ R1;(R2;R3):

(R1;R2);R3, if we set (a,d) $\subseteq$ A $\times$ D, from the definition given above,

$\Rightarrow$ $\exists$ c ((a,c) $\in$ (R1;R2) ; (c,d) $\in$ R3)

$\Rightarrow$ $\exists$ c [$\exists$ b (a,b) $\in$ R1 (b,c) $\in$ R2] ; (c,d) $\in$ R3

$\Rightarrow$ $\exists$ b, $\exists$ c [(a,b) $\in$ R1 ; (b,c) $\in$ R2 ; (c,d) $\in$ R3]

$\Rightarrow$ $\exists$ b [(a,b) $\in$ R1; $\exists$ c[(b,c) $\in$ R2 ; (c,d) $\in$ R3]

$\Rightarrow$ $\exists$ b [(a,b) $\in$ R1; [(b,d) $\in$ R2; R3]

$\Rightarrow$ (a,d) $\in$ R1;(R2; R3)

$\Rightarrow$ (R1;R2);R3 $\subseteq$ R1;(R2;R3)

Proof for R1;(R2;R3) $\subseteq$ (R1;R2);R3

R1;(R2;R3), if we set (a,d) $\subseteq$ A $\times$ D, from the definition given above,

$\Rightarrow$ (a,d) $\in$ R1;(R2; R3)

$\Rightarrow$ $\exists$ b [(a,b) $\in$ R1; [(b,d) $\in$ R2; R3]

$\Rightarrow$ $\exists$ b [(a,b) $\in$ R1; $\exists$ c[(b,c) $\in$ R2 ; (c,d) $\in$ R3]

$\Rightarrow$ $\exists$ b, $\exists$ c [(a,b) $\in$ R1 (b,c) $\in$ R2 ; (c,d) $\in$ R3]

$\Rightarrow$ $\exists$ c [$\exists$ b (a,b) $\in$ R1 (b,c) $\in$ R2] ; (c,d) $\in$ R3

$\Rightarrow$ $\exists$ c ((a,c) $\in$ (R1;R2) ; (c,d) $\in$ R3)

$\Rightarrow$ (a,d) $\subseteq$ (R1;R2);R3

$\Rightarrow$ R1;(R2;R3) $\subseteq$ (R1;R2);R3

Therefore (R1;R2);R3 = R1;(R2;R3)

(b)

To prove question (b), from the condition given(I = {(x,x) : x $\in$ S}) we can take R1, as a matrix, and I = (x,x), which means that I is the unitary matrix, no matter what kinds of R , we will still get R (R1;I= R1)

Proof for I;R1 = R1

So I;R1 = {(a,c) : $\exists$b[(a,b)$\in$I , (b,c)$\in$R1}

Because (a,b) $\in$I ,therefore b = a, and because (b,c) $\in$R1

$\Rightarrow$ (a,c) $\in$R1 therefore I;R1 = (a,c) = R1

Proof for R1;I = R1

So R1;I = {(a,c) : $\exists$b[(a,b)$\in$R1 , (b,c)$\in$I}

Because (b,c) $\in$I ,therefore b = c, and because (a,b) $\in$R1

$\Rightarrow$ (a,c) $\in$R1 therefore R1; I = (a,c) = R1

(c)

The assumption $(R1;R2)^{\leftarrow} = R1^{\leftarrow};R2^{\leftarrow}$ is false ,the counterexample as follow:

From the conclusion in question (b), we take R1, R2 as arbitrary matrix as well,

We suppose R1= $\begin{matrix} & 1 & 0 & 1 & \\ [& 0 & 1 & 1 &] \\ & 1 & 1 & 0 & \end{matrix}$, R2 = $\begin{matrix} & 0 & 0 & 1 & \\ [& 1 & 1 & 0 &] \\ & 1 & 0 & 1 & \end{matrix}$,

and we get :R1;R2 = $\begin{matrix} & 1 & 0 & 1 & \\ [& 1 & 1 & 1 &] \\ & 1 & 1 & 1 & \end{matrix}$, $(R1;R2)^{\leftarrow}$ = $\begin{matrix} & 1 & 1 & 1 & \\ [& 0 & 1 & 1 &] \\ & 1 & 1 & 1 & \end{matrix}$,

for $R1^{\leftarrow}$ = $\begin{matrix} & 1 & 0 & 1 & \\ [& 0 & 1 & 1 &] \\ & 1 & 1 & 0 & \end{matrix}$, $R2^{\leftarrow}$ = $\begin{matrix} & 0 & 1 & 1 & \\ [& 0 & 1 & 0 &] \\ & 1 & 0 & 1 & \end{matrix}$, $R1^{\leftarrow};R2^{\leftarrow}$ = $\begin{matrix} & 1 & 1 & 1 & \\ [& 1 & 1 & 1 &] \\ & 0 & 1 & 1 & \end{matrix}$

we can find that: $\begin{matrix} & 1 & 1 & 1 & \\ [& 0 & 1 & 1 &] \\ & 1 & 1 & 1 & \end{matrix} \neq \begin{matrix} & 1 & 1 & 1 & \\ [& 1 & 1 & 1 &] \\ & 0 & 1 & 1 & \end{matrix}$ therefore $(R1;R2)^{\leftarrow} \neq R1^{\leftarrow};R2^{\leftarrow}$

(d)

Proof for (R1$\cup$R2);R3 $\subseteq$ (R1;R3)$\cup$(R2;R3)

If we suppose (a,c) $\in$ (R1$\bigcup$R2); R3, From the definition given,

we know that $\exists$b$\in$B such that [(a,b) $\in$R3];[(b,c)$\in$R1$\bigcup$R2]

$\Rightarrow$ $\exists$b [(a,b) $\in$R3; (b,c) $\in$R1]$\bigcup$[(a,b) $\in$R3; (b,c) $\in$R2]

$\Rightarrow$ $\exists$b [(a,b) $\in$R3; (b,c) $\in$R1]$\bigcup$ $\exists$b[(a,b) $\in$R3; (b,c) $\in$R2]

$\Rightarrow$ [(a,c) $\in$R3];[R1$\bigcup$(a,c) $\in$R3;R2]

$\Rightarrow$ (a,c) $\in$[R3;R1$\bigcup$R3;R2] = R3; (R1$\bigcup$R2) = (R1$\bigcup$R2);R3

$\Rightarrow$ R1$\cup$R2);R3 $\subseteq$ (R1;R3)$\cup$(R2;R3)

Proof for (R1;R3)$\cup$(R2;R3) $\subseteq$ (R1$\cup$R2);R3

The proof for (R1;R3)$\cup$(R2;R3) $\subseteq$ (R1$\cup$R2);R3 is similar to prove (R1$\cup$R2);R3 $\subseteq$ (R1;R3)$\cup$(R2;R3) in opposite direction as follow:

$\Rightarrow$ (R1;R3)$\cup$(R2;R3)

$\Rightarrow$ (a,c) $\in$R3;R1$\bigcup$R3;R2 = R3; (R1$\bigcup$R2) = (R1$\bigcup$R2);R3

$\Rightarrow$ (a,c) $\in$R3;R1$\bigcup$(a,c) $\in$R3;R2

$\Rightarrow$ $\exists$b [(a,b) $\in$R3; (b,c) $\in$R1]$\bigcup$ $\exists$b[(a,b) $\in$R3; (b,c) $\in$R2]

$\Rightarrow$ $\exists$b [(a,b) $\in$R3; (b,c) $\in$R1]$\bigcup$[(a,b) $\in$R3; (b,c) $\in$R2]

$\Rightarrow$ (R3;R1) $\bigcup$(R3;R2)

$\Rightarrow$ (R1;R3) $\bigcup$(R2;R3)

$\Rightarrow$ (R1;R3)$\cup$(R2;R3) $\subseteq$ (R1$\cup$R2);R3

Therefore (R1$\cup$R2);R3 = (R1;R3)$\cup$(R2;R3)

(e)

The assumption : R1;(R2$\cap$R3) = (R1;R2)$\cap$(R1;R3) is false , the counterexample as follow:

$$\text{We suppose R1} = \begin{matrix}1&0&0\\ [0&1&1]\\ 1&1&0\end{matrix},\quad \text{R2} = \begin{matrix}0&1&1\\ [1&0&0]\\ 0&0&0\end{matrix},\quad \text{R3} = \begin{matrix}1&1&1\\ [0&0&0]\\ 1&1&0\end{matrix}$$

$$\text{So R2}\cap\text{R3} = \begin{matrix}0&1&1\\ [0&0&0]\\ 0&0&0\end{matrix},\quad \text{R1;(R2}\cap\text{R3)} = \begin{matrix}0&1&1\\ [0&0&0]\\ 0&1&1\end{matrix}$$

$$\text{And R1;R2} = \begin{matrix}0&1&1\\ [1&0&0]\\ 1&1&1\end{matrix},\quad \text{R1;R3} = \begin{matrix}1&1&1\\ [1&1&0]\\ 1&1&1\end{matrix},\quad \text{(R1;R2)}\cap\text{(R1;R3)} = \begin{matrix}1&1&1\\ [1&1&0]\\ 1&1&1\end{matrix}$$

$$\text{We can find that: } \begin{matrix}0&1&1\\ [0&0&0]\\ 0&1&1\end{matrix} \neq \begin{matrix}1&1&1\\ [1&1&0]\\ 1&1&1\end{matrix}$$

Therefore R1;(R2$\cap$R3) $\neq$ (R1;R2)$\cap$(R1;R3) is false

Problem2:

(a)

Proof:

In order to prove $R^j = R^i$ for $j \geqslant i$, we can prove $R^j=R^i$ then prove $R^{j+1}=R^i$ by mathematical induction , the details as follw:

When j = i: for all $i \geqslant 0$, Ri = Ri always stand up, therefore for j = i, $R^i = R^i+1$, then Rj = Ri (Base Case)

When j >i: to take j = k; therefore $R^{k+1} = R^k \cup (R;\ R^k)$

$= R^i \cup (R;\ R^i) = R^i$(Recursion)

Therefore for j = i , $R^j=R^i$ stands, then j+1 , $R^i = R^i+1$, then $R^j = R^i$ for all $j \geqslant i$.

(b)

From the conclusion in question(a), we know that if $R^i = R^i+1$, $R^j = R^i$, for all $j \geqslant i$,so when $k \geqslant i$, $R^k=R^i$, $R^k \subseteq R^i$

when $i \geqslant k \geqslant 0$, because $R^{i+1} := R^i \cup (R;R^i)$ for $i \geqslant 0$,we can find R^i cover less area than R^{i+1} , therefore $R^i \subseteq R^{i+1}$,by mathematical induction, $R^0 \subseteq R^1 \subseteq R^2 \subseteq R^3 \subseteq \ldots R^k \subseteq \ldots R^i$, therefore when $i \geqslant k \geqslant 0$, $R^k \subseteq R^i$

therefore for all $k \geqslant 0$, the assumption is true.

(c)

When n = 0, $P(0) = R^0;R^m=R^m$, because $R^0=I$ ($I = \{(x,x) : x \in S\}$)

$=I;R^m = R^m$

Therefore P(n) holds for all $n \in N$.

When n = k, from the condition given: $R^n;R^m = R^{n+m}$,

$R^k;R^m=R^{k+m}$

Because $R^k;R^m = [R^k \cup (R;R^k)];R^m$

$=(R^k;R^m) \cup [(R;R^k);R^m]$

$=R^{k+m} \cup [R; (R^k;R^m)]$

$=R^{k+m+1}$

Because n = k satisfy ,then n = k+1 satisfy,

Therefore if P(n) holds

(d) the following proof probably is wrong, I try to prove it, but somewhere still not make sense.

Proof:

If we suppose $(a,c) \in R^{k+1}$, and $(a,c) \notin R^k$

Beacause $R^{k+1} = R^k \cup (R;R^k)$, $(a,c) \notin R^k$,therefore $(a,c) \in (R;R^k)$

From the conclusion above, there must be b_1 ,$(a,b1) \in R$, $(b1,c) \in R_k$

The same for the (b_1,c), $(b_1,c) \in (R;R^{k-1})$, by the Incursion, we can find the number of elements in $(a,c) \in R$, is k+2, therefore $(a,c) \in R^k$

This contradicts the hypothesis, therefore $R^{k+1} \subseteq R^k$

And $Rk+1 = Rk \cup (R; Rk)$ therefore $R^k \subseteq R^{k+1}$

Therefore $R^k = R^{k+1}$

(e)

In order to prove R^k is transitive, we need to prove $(a,b) \in R^k, (b,c) \in R^k \Rightarrow (a,c) \in R^{2k}$

According the conclusion in (c), $R^k;R^k = R^{2k}$

According the conclusion in (d), if $|S| = k$, $R^k = R^{k+1}$

Suppose there is i , $R^i = R^{i+1}$, we know when $j \geqslant i$, $R^k = R^i$, (conclusion in (a))

Therefore $j \geqslant k$, $R^j = R^k$

If $j = 2k$,$2k \geqslant k$, therefore $R^{2k} = R^k$

Therefore $(a,b) \in R^k, (b,c) \in R^k \Rightarrow (a,c) \in R^{2k}$

Therefore R^k is transitive, if $|S| = k$,

(f)

In order to prove $(R \cup R^{\leftarrow})^k$ is equivalence relation ,we should show $(R \cup R^{\leftarrow})^k$ is R, S, T

For property T:

from the question (e), we know that if $|S| = k$, R^k is transitive, to take $R \cup R^{\leftarrow}$ as a whole part, therefore $(R \cup R^{\leftarrow})^k$ is also transitive.

For property R

Because $R^0 = I$ ($I = \{(x,x) : x \in S\}$) , from the conclusion in question (b), we know that $R^0 \subseteq R^1 \subseteq R^2 \subseteq R^3 \subseteq \ldots R^k \subseteq (R \cup R^{\leftharpoonup})^k$

cause I is reflexivity so the same for $(R \cup R^{\leftharpoonup})^k$ is reflexivity.

For property S

I do not know how to prove it……sorry……

Problem3:

(a)

We define the binary tree data as follow:

(B) an empty binary tree or

(R) an ordered binary tree with left tree or right tree.

(b)

I did not learn Java so I write in python instead, hope it does not matters, thank you! And I also write it in a mathematic way.

```
count(T):
def count(T):
    if T.isempty():                                                (B)
        return 0
    else:                                                          (R)
        return (1 + nodes(left) + nodes(right))
```

In a mathematic way:

Count(T):

(B): If T is a empty binary tree: count(T) = 0

(R): If T is not a empty binary tree: count(T) = 1+count(left)+count(right)

(count(left) means count the nodes in the left tree, count(right) means count the nodes in the right tree.)

(c)

```
leaves(T):
def leaves(T):
    if T.isempty():                                                (B)
        return 0
    elif T_left.isempty() and T_right.isempty():                   (R)
        return 1
    else:
        return (leaves(left) + leaves(right))
```

In a mathematic way:

leaves(T):

(B): If T is a empty binary tree: leaves(T) = 0
(R): If T is not a empty binary tree, and both left tree and right tree is empty:
leaves(T) = 1
If T is not a empty binary tree, either the left tree is empty or right tree is empty:
leaves(T) =leaves(left) +leaves(right)
(leaves(left) means count the leaves in the left tree, leaves(right) means count the leaves in the right tree.)

Internal(T):

```
def Internal(T):
    if T.isempty():                                          (B)
        return 0
    elif T_left.isempty() or T_right.isempty():              (R)
        return (Internal(left) + Internal(right))
    else:
        return (1 + Internal(left) + Internal(right))
```

In a mathematic way:

Internals:

(B): If T is a empty binary tree: Internals(T) = 0

(R): If T is not a empty binary tree and either the left tree is empty or right tree is empty:

Internals(T) = Internal(left) + Internal(right)

If T is not a empty binary tree, and both left tree and right tree are not empty:

Internals(T) = Internal(left) + Internal(right) + 1

(f)

We set n as sum of node, n0 as empty tree(leaves(T)), n1 as a binary with one successor, n2 as a binary with two successors(Internal(T))

And we set line be the $\rightarrow$ in the binary tree.

We know that: $n = n0+n1+n2$

$b = n-1$

$b = n1+2*n2$

and then $n0+n1+n2-1 = n1+2*n2$

$n0 = n2 +1$

therefore leaves(T) = Internal(T) +1

Problem 4:

(a)

We set HA means Alpha using channel hi, LA means Alpha using channel lo;

HB means Bravo using channel hi, LB means Bravo using channel lo;

HC means Charlie using channel hi, LC means Charlie using channel lo;

HD means Delta using channel hi, LD means Delta using channel lo

(i)

$(HA \lor LA) \land (HB \lor LB) \land (HB \lor LB) \land (HD \lor LD)$

(ii)

$((HA \land \neg LA) \lor (\neg HA \land LA)) \land ((HB \land \neg LB) \lor (\neg HB \land LB))$

$\land ((HC \land \neg LC) \lor (\neg HC \land LC)) \land ((HD \land \neg LD) \lor (\neg HD \land LD))$

(iii)

$(HA \land LB \land HC \land LD) \lor (LA \land HB \land LC \land HD)$

(b) (i)

Part of the True Assignment：

HA	LA	HB	LB	HC	LC	HD	LD	satisfiable output
0	1	1	0	0	1	1	0	1
1	0	0	1	1	0	0	1	1
0	1	1	0	1	1	0	1	0
1	0	1	0	1	0	1	0	0

In order to satisfy $\phi 1 \land \phi 2 \land \phi 3$, the allocation as follows:

$LA \land HB \land LC \land HD$

Which means Alpha using Hi, Bravo using Lo, Charlie using Hi, and Delta using Lo.

(ii) In order to avoid interference, the adjacent networks should not use the same channel, so the solution in previous question also satisfy this problem.

So the answer is also: $LA \land HB \land LC \land HD$

Which means Alpha using Hi, Bravo using Lo, Charlie using Hi, and Delta using Lo.

Or it can also assign like: HA $\wedge$ LB $\wedge$ HC $\wedge$ LD

Which means Alpha using Lo, Bravo using Hi, Charlie using Lo, and Delta using Hi.