

**Undergraduate Research Opportunity
Programme in Science**

Financial Mathematics With Python

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Contents

1	Introduction	2
2	Derivation of Black Scholes PDE	3
2.1	Basics	3
2.2	Delta-hedging Argument	4
2.3	Replicating portfolio	5
2.4	Lognormal property	6
3	Finite Difference Model for Numerical PDE	7
3.1	One factor model using Implicit Scheme	7
3.2	Explicit Scheme	8
3.3	Feynman–Kac formula	9
3.4	Crank–Nicolson method	10
4	Variance Reduction Techniques for Monte Carlo Simulation	11
4.1	Control Variate	11
4.2	Stratified Sampling	12
4.3	Importance Sampling	13
5	European call options	14
5.1	Closed-form formula	14
5.2	Numerical PDE	15
5.3	Monte Carlo Simulation	16
6	European put options	17
6.1	Closed-form formula	17
6.2	Numerical PDE	18
6.3	Monte Carlo Simulation	19
7	Barrier option	20
7.1	Closed-form formula	20
7.2	Numerical PDE	21
7.3	Monte Carlo Simulation	22

Abstract

In this paper, we implemented the library packages for Python to solve problems in Financial Mathematics such as derivatives valuation and simulation as suggested by the book “*Python for Finance*”. Those results correspond to the different approaches in pricing financial derivatives. The key effort is on the development of valuation scheme for the options, in particular, the strongly path-dependent options. More importantly, our model is open to extend for various assumptions about the market. We attempt to apply these valuation schemes onto the path-independent options with available closed-form formula in order to verify the valuation results from Monte Carlo simulations as well as the Finite Difference Model.

1 Introduction

The study upon using Python to carry out derivatives valuation has made much progress over the years as there have already been available libraries built for this purpose. The derivatives analytics library suggested in the book “*Python for Finance*” has its advantages on the coverage upon the various aspects of possible analysis for financial derivatives. Still, we can make justifiable modifications and extensions to improve on the accuracy and speed of computations for estimations.

2 Derivation of Black Scholes PDE

2.1 Basics

In the Black-Scholes World, we assume that the following two SDE hold:

$$dM_t = r M_t dt$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

We assume that the Itô's Lemma hold:

$$\text{As } dS_t = \mu S_t dt + \sigma S_t dW_t,$$

$$dV_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS_t$$

2.2 Delta-hedging Argument

Our first aim is to find ϕ_t

for $\Pi_t = V_t - \phi_t S_t$

such that

$$d\Pi_t = dV_t - \phi_t dS_t \text{ (Self-financing)}$$

$$d\Pi_t = r\Pi_t dt \text{ (risk free)}$$

From the equations, we can obtain that

$$r\Pi_t dt = dV_t - \phi_t dS_t$$

$$r(V_t - \phi_t S_t) dt = dV_t - \phi_t dS_t$$

$$dV_t = r(V_t - \phi_t S_t) dt + \phi_t dS_t$$

By Itô's Lemma,

$$dV_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS_t$$

Hence we obtain two equations:

$$\phi_t = \frac{\partial V}{\partial S}$$

$$r(V_t - \phi_t S_t) = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}$$

$$r(V_t - \frac{\partial V}{\partial S} S_t) = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + r \frac{\partial V}{\partial S} S_t - r V_t = 0$$

At time t, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r \frac{\partial V}{\partial S} S - r V = 0$$

2.3 Replicating portfolio

Our aim is to find a_t and b_t such that $\Pi_t = a_t S_t + b_t M_t$ can entirely replicate V_t while the self-financing condition still holds: $d\Pi_t = a_t dS_t + b_t dM_t$

As $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $dM_t = r M_t dt$,

$$\begin{aligned} d\Pi_t &= a_t(\mu S_t dt + \sigma S_t dW_t) + b_t(r M_t dt) \\ &= (a_t \mu S_t + r b_t M_t) dt + (\sigma a_t S_t) dW_t \end{aligned}$$

By Itô's Lemma,

$$\begin{aligned} dV_t &= \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS_t \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} (\mu S_t dt + \sigma S_t dW_t) \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} \mu S_t \right) dt + (\sigma S_t \frac{\partial V}{\partial S}) dW_t \end{aligned}$$

As Π_t fully replicates V_t ,

$$d\Pi_t = (a_t \mu S_t + r b_t M_t) dt + (\sigma a_t S_t) dW_t = dV_t$$

Also by Itô's Lemma,

$$dV_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} \mu S_t \right) dt + (\sigma S_t \frac{\partial V}{\partial S}) dW_t$$

Hence we obtain that,

$$a_t = \frac{\partial V}{\partial S}$$

$$a_t \mu S_t + r b_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} \mu S_t$$

$$\frac{\partial V}{\partial S} \mu S_t + r b_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} \mu S_t$$

$$r b_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}$$

$$r a_t S_t + r b_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + r a_t S_t$$

$$r V_t = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + r \frac{\partial V}{\partial S} S_t$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + r \frac{\partial V}{\partial S} S_t - r V_t = 0$$

Hence,

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r \frac{\partial V}{\partial S} S - r V = 0$$

2.4 Lognormal property

We shall derive the lognormal property of the underlying asset prices. As given by the Geometric Brownian Motion, $dS_t = \mu S_t dt + \sigma S_t dW_t$.

We can define the derivative price function as $V(S_t, t) := \ln S_t$, with

$$\frac{\partial V}{\partial S} = \frac{1}{S}, \frac{\partial^2 V}{\partial S^2} = -\frac{1}{S^2}, \frac{\partial V}{\partial t} = 0$$

By Itô's Lemma,

$$\begin{aligned} dV_t &= ((\mu S_t) \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}) dt + b \frac{\partial V}{\partial S} dW_t \\ &= (\mu - \frac{1}{2} \sigma^2) dt + \sigma dW_t \end{aligned} \tag{1}$$

As μ and σ are constants, V_t is simply a drifted Brownian Motion.

$$\begin{aligned} V_t - V_0 &= \ln S_t - \ln S_0 \sim \mathcal{N}((\mu - \frac{1}{2} \sigma^2)t, \sigma^2 t) \\ \frac{\ln S_t - \ln S_0 - (\mu - \frac{1}{2} \sigma^2)t}{\sigma \sqrt{t}} &\sim \mathcal{N}(0, 1) \end{aligned}$$

We can also generalize to the case for time t and T . (Let $\tau = T - t$, $\phi \sim \mathcal{N}(0, 1)$)

$$\begin{aligned} \frac{\ln S_T - \ln S_t - (\mu - \frac{1}{2} \sigma^2)\tau}{\sigma \sqrt{\tau}} &= \phi \\ S_T &= S_t e^{(\mu - \frac{1}{2} \sigma^2)\tau + \sigma \sqrt{\tau} \phi} \end{aligned}$$

3 Finite Difference Model for Numerical PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r \frac{\partial V}{\partial S} S - rV = 0$$

If we are to plot out the option prices in a diagram, there should be a 3D diagram with y-axis being the underlying prices, and x-axis being the time, omitting the scenario when the option price is dependent on other stochastic terms such as volatility. In the Black-Scholes formula, we are able to approximate $\Theta = \frac{\partial V}{\partial t}$, $\Delta = \frac{\partial V}{\partial S}$ and $\Gamma = \frac{\partial^2 V}{\partial S^2}$ when the discretization is significantly small and thus it will be easy to simply use the ratio of change in derivatives value with respect to change in time or change in underlying asset value.

$$\begin{aligned} \frac{\partial V}{\partial t} &\approx \frac{P_t - P_{t-\Delta t}}{\Delta t} \\ \frac{\partial V}{\partial S} &\approx \frac{P_S - P_{S-\Delta S}}{\Delta S} \approx \frac{P_{S+\Delta S} - P_S}{\Delta S} \\ \frac{\partial^2 V}{\partial S^2} &\approx \frac{\partial}{\partial V} \left(\frac{P_{S+\Delta S} - P_S}{\Delta S} \right) \\ &\approx \frac{\frac{P_{S+\Delta S} - P_S}{\Delta S} - \frac{P_S - P_{S-\Delta S}}{\Delta S}}{\Delta S} \\ &= \frac{P_{S+\Delta S} - 2P_S + P_{S-\Delta S}}{(\Delta S)^2} \end{aligned} \tag{2}$$

3.1 One factor model using Implicit Scheme

3.2 Explicit Scheme

The difference between explicit and implicit scheme lies in the need to solve an implicit equation at each discretization or there is a closed-form formula for each step.

3.3 Feynman–Kac formula

3.4 Crank–Nicolson method

4 Variance Reduction Techniques for Monte Carlo Simulation

4.1 Control Variate

4.2 Stratified Sampling

4.3 Importance Sampling

5 European call options

5.1 Closed-form formula

We can derive the closed-form formula for the European call options using risk-neutral pricing.

Payoff function of the European call option is : $c_T = (S_T - K)^+$

By the risk-neutral pricing formula,

$$\begin{aligned} e^{-\mu t} c_t &= \mathbb{E}^Q[e^{-\mu T} c_T | \mathcal{F}_t] \\ &= \mathbb{E}_t^Q[e^{-\mu T} (S_T - K)^+] \end{aligned} \quad (3)$$

By the lognormal property, $S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi}$

$$\begin{aligned} c_t &= e^{\mu t} \mathbb{E}_t^Q[e^{-rT} (S_T - K)^+] \\ &= e^{\mu t} \mathbb{E}_t^Q[e^{-rT} (S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} - K)^+] \\ &= e^{-\mu\tau} \mathbb{E}^Q[(S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} - K)^+] \\ &= e^{-\mu\tau} \int_{-\infty}^{\infty} (S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} - K)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi \\ &= e^{-\mu\tau} \int_{-d_2}^{\infty} (S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} - K) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi \\ &= e^{-\mu\tau} \int_{-d_2}^{\infty} S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi - e^{-r\tau} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} K d\phi \\ &= e^{(\mu - \frac{1}{2}\sigma^2)\tau - \mu\tau} S_t \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi^2 - 2\sigma\sqrt{\tau}\phi)} d\phi - K e^{-r\tau} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi \quad (4) \\ &= S_t \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi^2 - 2\sigma\sqrt{\tau}\phi + \sigma^2\tau)} d\phi - K e^{-r\tau} \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi \\ &= S_t \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi - \sigma\sqrt{\tau})^2} d\phi - K e^{-r\tau} \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi \\ &= S_t \int_{-d_2 - \sigma\sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy - K e^{-r\tau} \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi \\ &= S_t \int_{-\infty}^{d_2 + \sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy - K e^{-r\tau} \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi \\ &= S_t N(d_1) - K e^{-r\tau} N(d_2) \end{aligned}$$

where,

$$\begin{aligned} d_2 &= \frac{\ln \frac{S_t}{K} + (\mu - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\ d_1 &= d_2 + \sigma\sqrt{\tau} \end{aligned}$$

Also, N is the cumulative density function of standard normal function.

5.2 Numerical PDE

5.3 Monte Carlo Simulation

6 European put options

6.1 Closed-form formula

We can derive the closed-form formula for the European put options using risk-neutral pricing.

Payoff function of the European put option is : $p_T = (K - S_T)^+$

By the risk-neutral pricing formula,

$$\begin{aligned} e^{-\mu t} p_t &= \mathbb{E}^Q[e^{-\mu T} p_T | \mathcal{F}_t] \\ &= \mathbb{E}_t^Q[e^{-\mu T} (K - S_T)^+] \end{aligned} \quad (5)$$

By the lognormal property, $S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi}$

$$\begin{aligned} p_t &= e^{\mu t} \mathbb{E}_t^Q[e^{-rT} (K - S_T)^+] \\ &= e^{\mu t} \mathbb{E}_t^Q[e^{-rT} (K - S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi})^+] \\ &= e^{-\mu\tau} \mathbb{E}^Q[(K - S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi})^+] \\ &= e^{-\mu\tau} \int_{-\infty}^{\infty} (K - S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi})^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi \\ &= e^{-\mu\tau} \int_{-\infty}^{-d_2} (K - S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi}) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi \\ &= e^{-r\tau} \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} K d\phi - e^{-\mu\tau} \int_{-\infty}^{-d_2} S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi \\ &= K e^{-r\tau} \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi - e^{(\mu - \frac{1}{2}\sigma^2)\tau - \mu\tau} S_t \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi^2 - 2\sigma\sqrt{\tau}\phi)} d\phi \\ &= K e^{-r\tau} \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi - S_t \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi^2 - 2\sigma\sqrt{\tau}\phi + \sigma^2\tau)} d\phi \\ &= K e^{-r\tau} \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi - S_t \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi - \sigma\sqrt{\tau})^2} d\phi \\ &= K e^{-r\tau} \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi - S_t \int_{-\infty}^{-d_2 - \sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\ &= K e^{-r\tau} \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi - S_t \int_{-\infty}^{-d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\ &= K e^{-r\tau} N(-d_2) - S_t N(-d_1) \end{aligned} \quad (6)$$

where,

$$d_2 = \frac{\ln \frac{S_t}{K} + (\mu - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_1 = d_2 + \sigma\sqrt{\tau}$$

Also, N is the cumulative density function of standard normal function.

6.2 Numerical PDE

6.3 Monte Carlo Simulation

7 Barrier option

7.1 Closed-form formula

7.2 Numerical PDE

7.3 Monte Carlo Simulation