UROPS Project 16124 Full report
Financial Mathematics with Python

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1 Derivation of Black Scholes PDE

1.1 Basics

We assume that the following two SDE hold:

$$dM_t = rM_t dt$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

We assume that the following Itô's Lemma hold:

$$\text{As } dS_t = \mu S_t dt + \sigma S_t dW_t,$$

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t$$

1.2 Delta-hedging Argument

Our first aim is to find ϕ_t

for $\Pi_t = V_t - \phi_t S_t$

such that

$$d\Pi_t = dV_t - \phi_t dS_t (\text{Self-financing})$$

$$d\Pi_t = r\Pi_t dt (\text{risk free})$$

From the equations, we can obtain that

$$r\Pi_t dt = dV_t - \phi_t dS_t$$

$$r(V_t - \phi_t S_t)dt = dV_t - \phi_t dS_t$$

$$dV_t = r(V_t - \phi_t S_t)dt + \phi_t dS_t$$

By Itô's Lemma,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t$$

Hence we obtain two equations:

$$\phi_t = \frac{\partial V}{\partial S}$$

$$r(V_t - \phi_t S_t) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S}$$

$$r(V_t - \frac{\partial V}{\partial S} S_t) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S_t - rV_t = 0$$

At time t, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S - rV = 0$$

1.3 Replicating portfolio

Our aim is to find a_t and b_t such that

$\Pi_t = a_t S_t + b_t M_t$ can entirely replicate V_t

And also, the self-financing condition holds:

$$d\Pi_t = a_t dS_t + b_t dM_t$$

As $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $dM_t = rM_t dt$,

$$\begin{aligned} d\Pi_t &= a_t(\mu S_t dt + \sigma S_t dW_t) + b_t(rM_t dt) \\ &= (a_t \mu S_t + r b_t M_t)dt + (\sigma a_t S_t)dW_t \end{aligned}$$

By Itô's Lemma,

$$\begin{aligned} dV_t &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t \\ &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S}(\mu S_t dt + \sigma S_t dW_t) \\ &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t)dt + (\sigma S_t \frac{\partial V}{\partial S})dW_t \end{aligned}$$

As Π_t fully replicates V_t,

$$d\Pi_t = (a_t \mu S_t + r b_t M_t)dt + (\sigma a_t S_t)dW_t = dV_t$$

Also by Itô's Lemma,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t)dt + (\sigma S_t \frac{\partial V}{\partial S})dW_t$$

Hence we obtain that,

$$a_t = \frac{\partial V}{\partial S}$$

$$a_t \mu S_t + r b_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t$$

$$\frac{\partial V}{\partial S}\mu S_t + r b_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t$$

$$r b_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S}$$

$$r a_t S_t + r b_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + r a_t S_t$$

$$r V_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S_t$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S_t - r V_t = 0$$

Hence,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S - rV = 0$$

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5.1 Closed-form formula

5.2 Numerical PDE

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6.1 Closed-form formula

6.2 Numerical PDE

6.3 Monte Carlo Simulation

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8.1 Closed-form formula

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