Value at Risk

Dai Bo

Department of Mathematics, National University of Singapore
Science Drive 2, Singapore 117543

Abstract

Value at Risk (VaR) is one of the most popular tools used to estimate exposure to market risks, and it measures the worst expected loss at a given confidence level. In this report, we explain the concept of VaR, and then describe in detail some methods of the computation of VaR. We then discuss some VaR tools that are particularly useful for risk management, including marginal VaR, incremental VaR and component VaR. The next consideration is the effect of time varying risk, which can be estimated by a moving average method or a GARCH model. Finally, we introduce some back testing methods to validate the use of VaR model.

Definition of VaR

According to Philippe Jorion, “VaR measures the worst expected loss over a given horizon under normal market conditions at a given level of confidence.

For example, a bank might say that the daily VaR of its trading portfolio is $1 million at the 99 percent confidence level. In other words, under normal market conditions, only one percent of the time, the daily loss will exceed $1 million.” (Jorion 2001, p. xxii).

Computation of VaR

<u>Distribution of portfolio return:</u> If a portfolio return follows a normal distribution with mean μ and standard deviation σ, then the VaR number at the 99 percent confidence level can be obtained as follows:

1. Find the α value corresponding to the 99% confidence level in the normal table;
2. Compute the VaR number $VaR = \alpha\sigma - \mu$.

Note that since VaR is the portfolio return in the worst case, it should correspond to the lower tail level of the distribution. However, we always use a positive number to represent loss, therefore, both α value and the VaR number are positive.

<u>Rate of portfolio return</u>: Suppose the initial value of a portfolio is P_0 and the rate of return R is normally distributed with mean μ and standard deviation σ, then the VaR number at the

confidence level c is $VaR = P_0(\alpha\sigma - \mu)$, where α is the number corresponding to c in the normal table.

If the target time horizon is short, we can assume the mean value of R is zero, thus the formula can be further simplified as $VaR = P_0\alpha\sigma$. (Jorion 2001, pp. 110-113)

Delta-normal valuation: If a financial instrument depends on only one risk factor—the spot price S, and dS/S is a normal variable, delta-normal valuation can be used. Delta-normal method defines a pricing function $P(S)$, and finds the first partial derivative Δ_0 at the initial position as $\Delta_0 = \left.\frac{\partial P}{\partial S}\right|_{S=S_0}$. Then the VaR for this portfolio is $VaR = |\Delta_0| \times (\alpha\sigma S_0)$. (Jorion 2001, pp. 206-209)

Delta-gamma method: It is possible to increase the accuracy of delta-normal valuation. If we use Taylor expansion up to the second term, it is called delta-gamma.

We can approximate dP as $dP = \Delta dS + 1/2\Gamma dS^2$, by similar procedure as Delta-normal, we have the formula $VaR = |\Delta|\alpha\sigma S - 1/2\Gamma(\alpha\sigma S)^2$. (Jorion 2001, pp. 211-214)

Portfolio VaR: If a portfolio consists of N assets, the portfolio return is a linear combination of returns on underlying assets.

Define the weight $w_i = P_i / P_0$ with P_0 the portfolio value and P_i the value of asset i. Let $\mathbf{w}$ be the column vector of weights, and $\mathbf{R}$ be the column vector of the rate of return, then the rate of return for the whole portfolio is $R_p = \mathbf{w}^T\mathbf{R}$, where $\mathbf{w}^T$ is the transpose of the weight matrix.

The variance for R_p is $\sigma_p^2 = \mathbf{w}^T \sum \mathbf{w}$, where Σ is the covariance matrix and the VaR value is therefore $VaR_p = \alpha\sigma_p P_0 = \alpha\sqrt{\mathbf{x}^T \sum \mathbf{x}}$. (Jorion 2001, pp. 148-153)

VaR tools

Marginal VaR: Marginal VaR is the partial derivative with respect to the component weight, and it measures the change in portfolio VaR resulting from adding additional dollar to a component. For a given portfolio, the marginal VaR of its i^{th} component is $\Delta VaR_i = \alpha\sigma_{ip} / \sigma_p$, which shows that how much the portfolio VaR will change if adding one dollar to the i^{th} component. (Jorion 2001, pp. 154-155)

Incremental VaR: Incremental VaR measures the change in the VaR due to a new position on the portfolio. Let $\mathbf{a}$ be the new position added, the incremental VaR is the difference between the two VaRs, i.e. Incremental $VaR = VaR_{p+a} - VaR_p$. However, to calculate VaR_{p+a}, we need to compute a new covariance matrix, which might be time consuming. Instead, we could use the approximation Incremental $VaR = (\Delta\mathbf{VaR})^{\mathbf{T}} \times \mathbf{a}$. (Jorion 2001, pp. 155-159)

Component VaR: Component VaR is a partition of the portfolio VaR that indicates the change of VaR if a given component was deleted. The Component VaR is defined in term of marginal VaR, and for the i^{th} component, Component $VaR = \Delta VaR_i \times w_i P_0$. Note that the sum of all component VaRs is the portfolio VaR. (Jorion 2001, pp. 159-161)

Time varying risk

The risk of basic financial variables, such as interest rate, exchange rate, appears to change over time. To forecast the risk factor, moving average, GARCH estimation and RiskMetrics approach can be used.

Moving average: Moving average (MA) is one of the most popular and easy to use tools available to measure time varying risk. By using an average of prices, moving average provides a smooth trend line, which can be used to predict future changes in the risk factor.

Suppose we have the data of returns r_t over n days and we choose an M-day average. Then the average variance is $\sigma_t^2 = \frac{1}{M}\sum_{i=0}^{M-1} r_{t-i}^2$ for $t = M, M + 1, \ldots, n$. (Jorion 2001, pp. 186-187)

GARCH estimation: GARCH model can be used to forecast the new variance by using previous data. In a GARCH(1,1) model, the standard deviation at time t depends on the return and the standard deviation at time t – 1. With parameters $\alpha_0, \alpha_1,$ and β, the forecast of the standard deviation is $\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta\sigma_{t-1}^2$.

Given a set of returns, the parameters can be determined by the maximization of logarithm of the likelihood function: $\max F(\alpha_0, \alpha_1, \beta \mid r) = \sum_{t=1}^{T} \ln f(r_t) = \sum_{t=1}^{T} (\ln \frac{1}{\sqrt{2\pi}\sigma_t} - \frac{r_t^2}{2\sigma_t^2})$. Once the parameters are determined, the average variance for that period is $\sigma^2 = \alpha_0 / (1 - \alpha_1 - \beta)$, and the forecast of volatility at time t + 1 is $\sigma_{t+1}^2 = \alpha_0 \sum_{k=0}^{t} \beta^k + \alpha_1 \sum_{k=0}^{t} \beta^k r_{t-k}^2 + \beta^{k+1}\sigma_0^2$. (Jorion 2001, pp.187-193)

RiskMetrics approach: In the RiskMetrics approach, the forecast of the standard deviation at time t is a weighted average of the previous forecast, using weight λ, and of the latest squared innovation, using weight $(1 - \lambda)$, i.e. $\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)r_{t-1}^2$.

RiskMetrics is actually a special case of GARCH(1, 1) model with parameters $\alpha_0 = 0, \alpha_1 + \beta = 1$, and decay factor $\lambda = \beta$. (Jorion 2001, pp.193-196)

Back testing VaR models

Back testing with exceptions: Suppose a VaR number is reported at the confidence interval c, then an exception occurs if its losses exceed the VaR number. The expected number of exceptions in a total of T observations is $T(1-c)$ and we will accept the VaR model if the number of exception is within the range $-\alpha\sqrt{Tc(1-c)} + T(1-c) < x < \alpha\sqrt{Tc(1-c)} + T(1-c)$, and reject it otherwise. (Jorion 2001, pp.130-136)

Back testing with conditional coverage: There are two parts in the conditional coverage model: the test of unconditional coverage (uc) and the test of independence (ind). (Chatfield 2001)

For T observations, let T_0 be the number of exceptions, T_1 be the number of nonexceptions, and c be the confidence level. Then we can perform the likelihood ratio test as $\mathrm{LR}_{\mathrm{uc}} = -2T_0 \ln(1-c) - 2T_1 \ln c + 2T_0 \ln(T_0 / T) + 2T_1 \ln(T_1 / T)$. If $\mathrm{LR}_{\mathrm{uc}} > 3.84$, we can reject the model at the 95 percent confidence level.

Let T_{00} be the occurrence of two consecutive exceptions, T_{01} be the occurrence of an exception followed by a nonexception, T_{11} and T_{10} be defined similarly. The independence test is $\mathrm{LR}_{\mathrm{ind}} = -2T_0 \ln(1-c) - 2T_1 \ln c + 2T_{00} \ln(T_{00} / T_0) + 2T_{11} \ln(T_{01} / T_0) + 2T_{10} \ln(T_{10} / T_1) + 2T_{11} \ln(T_{11} / T_1)$.

References

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