UROPS Project Presentation 6

Wang Zexin

Derivation of Black-Scholes PDE

Quantitative Finance
National University of Singapore

February 26, 2017

Today's Agenda

1. Basics
   - Black-Scholes World
   - Itô's Lemma

2. Derivation
   - Delta-hedging Argument
   - Replication of portfolio

Black-Scholes World

We assume that the following two SDE hold:

$$dM_t = rM_t dt$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Itô's Lemma

We assume that the following equation hold:

$$\text{As } dS_t = \mu S_t dt + \sigma S_t dW_t,$$

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t$$

Delta-hedging Argument

Our first aim is to find ϕ_t

for $\Pi_t = V_t - \phi_t S_t$

such that

$$d\Pi_t = dV_t - \phi_t dS_t (\text{Self-financing})$$
$$d\Pi_t = r\Pi_t dt (\text{risk free})$$

From the equations, we can obtain that

$$r\Pi_t dt = dV_t - \phi_t dS_t$$
$$r(V_t - \phi_t S_t)dt = dV_t - \phi_t dS_t$$
$$dV_t = r(V_t - \phi_t S_t)dt + \phi_t dS_t$$

Compare with Itô's Lemma

$$dV_t = r(V_t - \phi_t S_t)dt + \phi_t dS_t$$

By Itô's Lemma,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t$$

Hence we obtain two equations:

$$\phi_t = \frac{\partial V}{\partial S}$$

$$r(V_t - \phi_t S_t) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S}$$

Last step of Delta-hedging

$$r(V_t - \frac{\partial V}{\partial S} S_t) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S_t - rV_t = 0$$

At time t, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S - rV = 0$$

First step of replication

Our aim is to find a_t and b_t such that

$\Pi_t = a_t S_t + b_t M_t$ can entirely replicate V_t

And also, the self-financing condition holds:

$$d\Pi_t = a_t dS_t + b_t dM_t$$

As $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $dM_t = rM_t dt$,

$$\begin{aligned} d\Pi_t &= a_t(\mu S_t dt + \sigma S_t dW_t) + b_t(rM_t dt) \\ &= (a_t \mu S_t + rb_t M_t)dt + (\sigma a_t S_t)dW_t \end{aligned}$$

Bring in Itô′s Lemma

By Itô′s Lemma,

$$
\begin{aligned}
dV_t &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t \\
&= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S}(\mu S_t dt + \sigma S_t dW_t) \\
&= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t)dt + (\sigma S_t \frac{\partial V}{\partial S})dW_t
\end{aligned}
$$

As Π_t fully replicates V_t,

$$d\Pi_t = (a_t \mu S_t + r b_t M_t)dt + (\sigma a_t S_t)dW_t = dV_t$$

Also by Itô's Lemma,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t)dt + (\sigma S_t \frac{\partial V}{\partial S})dW_t$$

Hence we obtain that,

$$a_t = \frac{\partial V}{\partial S}$$

$$a_t \mu S_t + r b_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t$$

Last step in replication

$$\frac{\partial V}{\partial S}\mu S_t + rb_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t$$

$$rb_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S}$$

$$ra_t S_t + rb_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + ra_t S_t$$

$$rV_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S_t$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S_t - rV_t = 0$$

Hence,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S - rV = 0$$

Thank You

E0007424@u.nus.edu