Undergraduate Research Opportunity Programme in Science

Financial Mathematics With Python

WANG ZEXIN

ZHOU CHAO

Department of Mathematics

National University of Singapore

AY2016/17 Semester 2

Contents

Abstract

In this paper, we implemented the library packages for Python to solve problems in Financial Mathematics such as derivatives valuation and simulation as suggested by the book "*Python for Finance*". Those results correspond to the different approaches in pricing financial derivatives. The key effort is on the development of valuation scheme for the options, in particular, the strongly path-dependent options. More importantly, our model is open to extend for various assumptions about the market. We attempt to apply these valuation schemes onto the path-independent options with available closed-form formulain order to verify the valuation results from Monte Carlo simulations as well as the Finite Difference Model.

1 Introduction

The study upon using Python to carry out derivatives valuation has made much progress over the years as there have already been available libraries built for this purpose. The derivatives analytics library suggested in the book "*Python for Finance*" has its advantages on the coverage upon the various aspects of possible analysis for financial derivatives. Still, we can make justifiable modifications and extensions to improve on the accuracy and speed of computations for estimations.

2 Derivation of Black Scholes PDE

2.1 Basics

In the Black-Scholes World, we assume that the following two SDE hold:

$$dM_t = rM_t dt$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

We assume that the It$\hat{o}'$s Lemma hold:

As $dS_t = \mu S_t dt + \sigma S_t dW_t$,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t$$

2.2 Delta-hedging Argument

Our first aim is to find ϕ_t

for $\Pi_t = V_t - \phi_t S_t$

such that

$d\Pi_t = dV_t - \phi_t dS_t$(Self-financing)

$d\Pi_t = r\Pi_t dt$(risk free)

From the equations, we can obtain that

$$r\Pi_t dt = dV_t - \phi_t dS_t$$

$$r(V_t - \phi_t S_t)dt = dV_t - \phi_t dS_t$$

$$dV_t = r(V_t - \phi_t S_t)dt + \phi_t dS_t$$

By It$\hat{o}'$s Lemma,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t$$

Hence we obtain two equations:

$$\phi_t = \frac{\partial V}{\partial S}$$

$$r(V_t - \phi_t S_t) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S}$$

$$r(V_t - \frac{\partial V}{\partial S} S_t) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S_t - rV_t = 0$$

At time t, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S - rV = 0$$

2.3 Replicating portfolio

Our aim is to find a_t and b_t such that $\Pi_t = a_tS_t + b_tM_t$ can entirely replicate V_t while the self-financing condition still holds: $d\Pi_t = a_tdS_t + b_tdM_t$

As $dS_t = \mu S_tdt + \sigma S_tdW_t$ and $dM_t = rM_tdt$,

$$\begin{aligned} d\Pi_t &= a_t(\mu S_tdt + \sigma S_tdW_t) + b_t(rM_tdt) \\ &= (a_t\mu S_t + rb_tM_t)dt + (\sigma a_tS_t)dW_t \end{aligned}$$

By It$\hat{o}'$s Lemma,

$$\begin{aligned} dV_t &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S}dS_t \\ &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S}(\mu S_tdt + \sigma S_tdW_t) \\ &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t)dt + (\sigma S_t\frac{\partial V}{\partial S})dW_t \end{aligned}$$

As Π_t fully replicates V_t,

$$d\Pi_t = (a_t\mu S_t + rb_tM_t)dt + (\sigma a_tS_t)dW_t = dV_t$$

Also by It$\hat{o}'$s Lemma,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t)dt + (\sigma S_t\frac{\partial V}{\partial S})dW_t$$

Hence we obtain that,

$$a_t = \frac{\partial V}{\partial S}$$

$$a_t\mu S_t + rb_tM_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t$$

$$\begin{aligned} \frac{\partial V}{\partial S}\mu S_t + rb_tM_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t \\ rb_tM_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} \\ ra_tS_t + rb_tM_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} + ra_tS_t \\ rV_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S}S_t \\ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} &+ r\frac{\partial V}{\partial S}S_t - rV_t = 0 \end{aligned}$$

Hence,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S}S - rV = 0$$

2.4 Lognormal property

We shall derive the lognormal property of the underlying asset prices. As given by the Geometric Brownian Motion, $dS_t = \mu S_t dt + \sigma S_t dW_t$.
We can define the derivative price function as $V(S_t, t) := \ln S_t$, with

$$\frac{\partial V}{\partial S} = \frac{1}{S}, \frac{\partial^2 V}{\partial S^2} = -\frac{1}{S^2}, \frac{\partial V}{\partial t} = 0$$

By $\text{It}\hat{o}'$s Lemma,

$$\begin{aligned} dV_t &= ((\mu S_t)\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2})dt + b\frac{\partial V}{\partial S} dW_t \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t \end{aligned} \tag{1}$$

As μ and σ are constants, V_t is simply a drifted Brownian Motion.

$$V_t - V_0 = \ln S_t - \ln S_0 \sim \mathcal{N}((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$$

$$\frac{\ln S_t - \ln S_0 - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \sim \mathcal{N}(0, 1)$$

We can also generalize to the case for time t and T. (Let $\tau = T - t$, $\phi \sim \mathcal{N}(0, 1)$)

$$\frac{\ln S_T - \ln S_t - (\mu - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = \phi$$

$$S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi}$$

3 Finite Difference Model for Numerical PDE

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial V}{\partial S}+r\frac{\partial V}{\partial S}S-rV=0$$

If we are to plot out the option prices in a diagram, there should be a 3D diagram with y-axis being the underlying prices, and x-axis being the time, omitting the scenario when the option price is dependent on other stochastic terms such as volatility. In the Black-Scholes formula, we are able to approximate $\Theta=\frac{\partial V}{\partial t}$, $\Delta=\frac{\partial V}{\partial S}$ and $\Gamma=\frac{\partial^2 V}{\partial S^2}$ when the discretization is significantly small and thus it will be easy to simply use the ratio of change in derivatives value with respect to change in time or change in underlying asset value.

$$\begin{aligned}
\frac{\partial V}{\partial t} &\approx \frac{P_t-P_{t-\vartriangle t}}{\vartriangle t}\\
\frac{\partial V}{\partial S} &\approx \frac{P_S-P_{S-\vartriangle S}}{\vartriangle S}\approx\frac{P_{S+\vartriangle S}-P_S}{\vartriangle S}\\
\frac{\partial^2 V}{\partial S^2} &\approx \frac{\partial}{\partial V}\left(\frac{P_{S+\vartriangle S}-P_S}{\vartriangle S}\right)\\
&\approx \frac{\frac{P_{S+\vartriangle S}-P_S}{\vartriangle S}-\frac{P_S-P_{S-\vartriangle S}}{\vartriangle S}}{\vartriangle S}\\
&= \frac{P_{S+\vartriangle S}-2P_S+P_{S-\vartriangle S}}{(\vartriangle S)^2}
\end{aligned} \tag{2}$$

3.1 One factor model using Implicit Scheme

3.2 Explicit Scheme

The difference between explicit and implicit scheme lies in the need to solve an implicit equation at each discretization or there is a closed-form formula for each step.

3.3 Feynman–Kac formula

3.4 Crank–Nicolson method

4 Variance Reduction Techniques for Monte Carlo Simulation

4.1 Control Variate

4.2 Stratified Sampling

4.3 Importance Sampling

5 European call options

5.1 Closed-form formula

We can derive the closed-form formula for the European call options using risk-neutral pricing.
Payoff function of the European call option is : $c_T = (S_T - K)^+$
By the risk-neutral pricing formula,

$$\begin{aligned} e^{-\mu t} c_t &= \mathbb{E}^Q[e^{-\mu T} c_T | \mathcal{F}_t] \\ &= \mathbb{E}_t^Q[e^{-\mu T}(S_T - K)^+] \end{aligned} \tag{3}$$

By the lognormal property, $S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi}$

$$\begin{aligned} c_t &= e^{\mu t}\mathbb{E}_t^Q[e^{-rT}(S_T - K)^+] \\ &= e^{\mu t}\mathbb{E}_t^Q[e^{-rT}(S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} - K)^+] \\ &= e^{-\mu\tau}\mathbb{E}^Q[(S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} - K)^+] \\ &= e^{-\mu\tau}\int_{-\infty}^{\infty}(S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} - K)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= e^{-\mu\tau}\int_{-d_2}^{\infty}(S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} - K) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= e^{-\mu\tau}\int_{-d_2}^{\infty} S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi - e^{-r\tau}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} K\, d\phi \\ &= e^{(\mu - \frac{1}{2}\sigma^2)\tau - \mu\tau} S_t \int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi^2 - 2\sigma\sqrt{\tau}\phi)}\, d\phi - K e^{-r\tau}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= S_t\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi^2 - 2\sigma\sqrt{\tau}\phi + \sigma^2\tau)}\, d\phi - K e^{-r\tau}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= S_t\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi - \sigma\sqrt{\tau}\phi)^2}\, d\phi - K e^{-r\tau}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= S_t\int_{-d_2 - \sigma\sqrt{\tau}}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy - K e^{-r\tau}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= S_t\int_{-\infty}^{d_2 + \sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy - K e^{-r\tau}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= S_t N(d_1) - K e^{-r\tau} N(d_2) \end{aligned} \tag{4}$$

where,

$$d_2 = \frac{\ln\frac{S_t}{K} + (\mu - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_1 = d_2 + \sigma\sqrt{\tau}$$

Also, N is the cumulative density function of standard normal function.

5.2 Numerical PDE

5.3 Monte Carlo Simulation

With the following settings: $\sigma = 0.25$, $\mu = 0.05$, $T = 1$, $S_0 = 100$, when we compare the results from Monte Carlo simulation with the option prices computed using the closed form formula, it is observable that a simulation with 500000 data points still has variation of error within range from -0.02 to 0.02. However, this comparison has ensured that our valuation scheme using Monte Carlo simulation and the variance reduction techniques is on the right way, and may be applied onto other options.
A tabulation of the resultant option prices are shown below:

Closed-form formula	Ordinary Monte Carlo	Control Variate
10.0022021172	10.005252032814727	10.005252032814751
8.02638469385	8.0166707887876427	8.016670788787664
6.37924904693	6.3659464432937911	6.3659464432937733
5.02541348179	5.0148870431746415	5.0148870431746184
3.92690420603	3.9241698581683577	3.9241698581683728
3.04592058431	3.044679765401427	3.0446797654014213
2.34679877596	2.3310307686205207	2.3310307686205225
1.79723400902	1.8055531375480696	1.8055531375480751
1.36889248498	1.3630119824610754	1.3630119824610734
1.03756650489	1.0254470920297398	1.0254470920297361
0.783018613011	0.77819086371547286	0.77819086371547308
0.588637155719	0.5924848566970754	0.59248485669707973
0.440997057228	0.44342591182216823	0.4434259118221664
0.329392108384	0.32449718396190719	0.32449718396190735
0.245381782063	0.2462392632801686	0.24623926328016907
0.182377553986	0.17995020687354496	0.1799502068735449
0.13528073067	0.13478417666883458	0.13478417666883413
0.100175092579	0.099449778570450814	0.099449778570450523
0.0740722950356	0.074452813719303179	0.074452813719303304
0.0547050187389	0.051631534936688268	0.051631534936688074

6 European put options

6.1 Closed-form formula

We can derive the closed-form formula for the European put options using risk-neutral pricing.
Payoff function of the European put option is : $p_T = (K - S_T)^+$
By the risk-neutral pricing formula,

$$\begin{aligned} e^{-\mu t} p_t &= \mathbb{E}^Q[e^{-\mu T} p_T | \mathcal{F}_t] \\ &= \mathbb{E}_t^Q[e^{-\mu T}(K - S_T)^+] \end{aligned} \tag{5}$$

By the lognormal property, $\quad S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi}$

$$\begin{aligned} p_t &= e^{\mu t}\mathbb{E}_t^Q[e^{-rT}(K - S_T)^+] \\ &= e^{\mu t}\mathbb{E}_t^Q[e^{-rT}(K - S_t e^{(\mu-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}\phi})^+] \\ &= e^{-\mu\tau}\mathbb{E}^Q[(K - S_t e^{(\mu-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}\phi})^+] \\ &= e^{-\mu\tau}\int_{-\infty}^{\infty}(K - S_t e^{(\mu-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}\phi})^+ \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\phi^2}\,d\phi \\ &= e^{-\mu\tau}\int_{-\infty}^{-d_2}(K - S_t e^{(\mu-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}\phi}) \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\phi^2}\,d\phi \\ &= e^{-r\tau}\int_{-\infty}^{-d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\phi^2}K\,d\phi - e^{-\mu\tau}\int_{-\infty}^{-d_2}S_t e^{(\mu-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}\phi}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\phi^2}\,d\phi \\ &= Ke^{-r\tau}\int_{-\infty}^{-d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\phi^2}\,d\phi - e^{(\mu-\frac{1}{2}\sigma^2)\tau-\mu\tau}S_t\int_{-\infty}^{-d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\phi^2-2\sigma\sqrt{\tau}\phi)}\,d\phi \\ &= Ke^{-r\tau}\int_{-\infty}^{-d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\phi^2}\,d\phi - S_t\int_{-\infty}^{-d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\phi^2-2\sigma\sqrt{\tau}\phi+\sigma^2\tau)}\,d\phi \\ &= Ke^{-r\tau}\int_{-\infty}^{-d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\phi^2}\,d\phi - S_t\int_{-\infty}^{-d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\phi-\sigma\sqrt{\tau}\phi)^2}\,d\phi \\ &= Ke^{-r\tau}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\phi^2}\,d\phi - S_t\int_{-\infty}^{-d_2-\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\,dy \\ &= Ke^{-r\tau}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\phi^2}\,d\phi - S_t\int_{-\infty}^{-d_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\,dy \\ &= Ke^{-r\tau}N(-d_2) - S_tN(-d_1) \end{aligned} \tag{6}$$

where,

$$d_2 = \frac{\ln\frac{S_t}{K} + (\mu - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_1 = d_2 + \sigma\sqrt{\tau}$$

Also, N is the cumulative density function of standard normal function.

6.2 Numerical PDE

6.3 Monte Carlo Simulation

With the following settings: $\sigma = 0.25$, $\mu = 0.05$, $T = 1$, $S_0 = 100$, when we compare the results from Monte Carlo simulation with the option prices computed using the closed form formula, it is observable that a simulation with 500000 data points still has variation of error within range from -0.02 to 0.02. However, this comparison has ensured that our valuation scheme using Monte Carlo simulation and the variance reduction techniques is on the right way, and may be applied onto other options.

A tabulation of the resultant option prices are shown below:

Closed-form formula	Ordinary Monte Carlo	Control Variate
9.88129168973	9.89121100448	9.89121100448
12.6616213889	12.6718073072	12.6718073072
15.7706328645	15.7786013449	15.7786013449
19.1729444219	19.1488969419	19.1488969419
22.8305822686	22.8396769379	22.8396769379
26.7057457694	26.7349375768	26.7349375768
30.7627710836	30.7487677853	30.7487677853
34.9693534391	34.9407041949	34.9407041949
39.2971590376	39.3339188168	39.3339188168
43.72198018	43.6672929866	43.6672929866
48.2235794106	48.2561306956	48.2561306956
52.7853450758	52.7364667849	52.7364667849
57.3938520998	57.3635053884	57.3635053884
62.0383942735	62.0538619165	62.0538619165
66.7105310697	66.7099933687	66.7099933687
71.4036739641	71.3460094069	71.3460094069
76.1127242633	76.1619698093	76.1619698093
80.8337657477	80.8390739688	80.8390739688
85.5638100727	85.5359768969	85.5359768969
90.3005899189	90.2678714265	90.2678714265

7 Barrier option

7.1 Closed-form formula

7.2 Numerical PDE

7.3 Monte Carlo Simulation