

PARTIAL DIFFERENTIAL EQUATIONS FOR OPTION PRICING

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Introduction

Option pricing is one of the many problems of financial mathematics or financial engineering as it is called now. It all started in the seventies with the celebrated model of Merton, Black and Scholes [81, 18] and their Nobel consecration later. Some of these ideas were already in the thesis of Bachelier[13] in 1900, but everyone forgot because the era of fast electronic transactions had not come. Today financial assets (stocks, bonds, commodities, etc...) are used as a base for thousands of more complex financial products known as financial derivatives. The simplest example may be the European put option on a given asset which enables its holder to sell the asset at a future time T , the *maturity*, for a price K , the *strike*. If the asset is worth S_T at time T , the option will be exercised only if $K > S_T$, generating a profit $K - S_T$. In the other case, the option will not be exercised, and the profit will be 0. Therefore, the profit generated by the option at T will be $(K - S_T)^+$. Assuming that the market is liquid and that arbitrage is not possible (one cannot make an instantaneous benefit without taking a risk), the price of the option at T will be $(K - S_T)^+$. Option pricing at time $t < T$ is more difficult because S_T is not known. The Black-Scholes model makes the above assumptions and supposes furthermore that the market is made of two assets, the previously mentioned risky asset and a riskless asset whose price evolves with a known interest rate r . It allows for pricing the above mentioned put option as the expectation of $(K - S_T)^+$ discounted at the interest rate r . Another assumption of the Black-Scholes model is that $S_{t+\delta t}$ evolves from S_t with a mean *tendency* μ and a random fluctuation of intensity σ , the *volatility*:

$$S_{t+\delta t} = S_t(1 + \mu\delta t) + \sigma S_t N(0, \delta t),$$

where $N(0, v)$ is a normal distribution with mean zero and variance v , and that $S_{t+\delta t} - S_t$ is independent from the events before t . A very simple set of ideas indeed! But as the model should not depend on the time increment δt , one should use continuous-time stochastic processes: therefore the European put P_t is priced by

$$P_t = e^{-r(T-t)} \mathbb{E}(K - S_T)^+, \quad \text{with} \quad dS_t = S_t(\mu dt + \sigma dB_t), \quad (1)$$

where B_t is a Brownian motion. Since S_t is a Markov process, there exists a two variable function P , called *the pricing function*, such that $P_t = P(S_t, t)$, and P

solves the partial differential equation (PDE):

$$\frac{\partial P}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0, \quad (2)$$

for $t \in [0, T)$ and $S > 0$.

There are three important classes of numerical methods in financial engineering: Monte-Carlo methods, tree based methods and deterministic methods based on the PDE (2). The goal of the present paper is to focus on the latter.

Of course, it may seem that numerical methods for (2) are now well known, but there are special difficulties in finance, because traders require a quick and accurate response. Furthermore, there are much more complicated contracts than the one described above, and the PDE may become much more complex: for example, American option pricing involves variational inequalities, stochastic volatility models lead to multidimensional PDEs; for *e.g.* basket options the problem can become numerically formidable because set in a space whose dimension is the number of assets in the basket; finally models involving more general Lévy processes lead to partial integro-differential equations, [30].

The diversity of the models for financial derivatives has grown to such a point that it is not possible to discuss all of them in one book chapter. Here only local volatility (the volatility may depend on time and on the price of the underlying asset) and stochastic volatility models will be considered, and models based on jump process will just be briefly described. Similarly the types of contracts on the markets are too numerous and we will mostly deal with European, American options with or without barriers, possibly multidimensional (basket options for example).

Our objective in this paper is numerical: what is a good method for the computer implementations?

One may choose between finite difference methods, [98], finite element methods, [28, 27, 113], finite volume methods, [42], spectral methods, [92, 17], etc... We have decided to work with the Finite Element Method (FEM) because it is very flexible on the one hand and supported by a strong theory on the other hand. In financial engineering, the PDEs often have a parabolic character (the first order terms in the equations are usually not dominant) which makes FEMs well adapted.

In this restricted context, what is the best way to implement the methods, namely what polynomial degree, what mesh, what linear system solver, etc...? As usual the answer goes through a mathematical analysis of the variational formulation of the problem, on which one can prove existence, uniqueness and qualitative properties of the solution. Then error estimates, especially a posteriori estimates are most useful and the FEM is best suited for such analysis. The outcome of such a study is that one can guarantee the precision of the calculations, a property appreciated in banking where, in view of the large sums involved, an error greater than 0. 1% is often unacceptable.

For clarity we present the material by increasing order of mathematical complexity. The first chapter deals with plain vanilla European options. The variational formulation is given and studied and the posteriori estimates derived in [6] are

restated. A C++ implementation on an arbitrary mesh is given at the end. The second chapter deals with higher dimensional models. We consider in particular European and American options on baskets and stochastic volatility models. In this context, we discuss the variational analysis of the boundary value problems. When the dimension of the problem is rather small (≤ 4), FEMs are competitive; we describe several techniques concerning the solution procedure, in particular for American options. We also review on a promising and new class of methods which may be used for parabolic PDEs when the dimension of the problem lies between 4 and 20, the sparse grid and sparse Galerkin methods.

In the third chapter, we recall the method given in [6] for computing the sensitivity of the solution with respect to the parameters of the problem, the *Greeks* (called so because practitioners have used Greek letters). The method is based on automatic differentiation of computer programs, [54, 57], a very powerful technique particularly appropriate to financial engineering. The operator overloading feature of the C++ language makes it easy to implement this approach. It is also useful in the context of *parameters calibration*, where the gradients of the least square functionals with respect to the model parameters are needed. This brings us to another important topic of financial engineering: better models or better parameters? As an example of calibration, we consider the calibration of local volatility . We discuss Dupire's equation [37] and the use of least squares methods for calibration.

Chapter 1

One Dimensional Partial Differential Equations For Option Pricing

1.1 The Partial Differential Equation

A European vanilla call (respectively put) option is a contract giving its owner the right to buy (respectively sell) a share of a specific common stock at a fixed price K at a certain date T . The specific stock is called the *underlying asset*. The fixed price K is termed the *strike* and T is called the *maturity*. The term *vanilla* is used to notify that this kind of option is the simplest among possibly complicated contracts. The price of the underlying asset at time t will be referred to as the *spot price* and will be noted S_t . Assuming that the market rules out arbitrage (the possibility to make an instantaneous risk-free benefit), it is easy to see that the price of a call (resp. put) option at maturity is $C_0(S_T) = (S_T - K)^+$, (resp. $P_0(S_T) = (K - S_T)^+$). The payoff of the option at maturity is a function of S_T , called the *payoff function*. Naturally, other payoff functions than the ones mentioned above are possible and used in practice.

In order to price the option before maturity, some assumptions have to be made on the spot price S_t : the Black-Scholes model assumes the existence of a risk-free asset whose price at time t is $S_t^0 = S_0^0 \exp\left(\int_0^t r(s)ds\right)$, where $r(t)$ is the interest rate; the model assumes that the price of the risky asset satisfies the following stochastic differential equation,

$$dS_t = S_t(\mu dt + \sigma_t dB_t), \quad (1.1)$$

where B_t is a standard Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Here σ_t is a positive number called the *volatility*. With the Black-Scholes assumption,

tions, it is possible to prove that the option's price at time t is given by

$$P_t = \exp\left(\int_t^T r(s)ds\right) \mathbb{E}^*(P_0(S_T)|F_t), \quad (1.2)$$

where the expectation \mathbb{E}^* is taken with respect to the so-called risk-neutral probability \mathbb{P}^* (equivalent to \mathbb{P} and under which $dS_t = S_t(rdt + \sigma_t dW_t)$, W_t being a standard Brownian motion under \mathbb{P}^* and F_t being the natural filtration of W_t).

From (1.2) and since S_t is a Markov process, it can be shown that the option's price P_t is a function of t and of S_t , i.e., that there exists a two variable function P , called *the pricing function*, such that $P_t = P(S_t, t)$.

Assuming that $\sigma_t = \sigma(S_t, t)$, where σ is a smooth enough function, it can be seen that the pricing function P solves the backward in time parabolic partial differential equation:

$$\frac{\partial P}{\partial t} + \frac{\sigma^2(S, t)S^2}{2} \frac{\partial^2 P}{\partial S^2} + r(t)S \frac{\partial P}{\partial S} - r(t)P = 0 \quad (1.3)$$

for $t \in [0, T)$ and $S > 0$, and satisfies the final time condition

$$P(S, t = T) = P_0(S) \quad (1.4)$$

for $S > 0$. Problem (1.3), (1.4) is called a *final value problem*.

The volatility is the difficult parameter of the Black-Scholes model. It is convenient to take it to be constant but then the computed options' prices do not match the prices given by the market. There are essentially three ways for improving on the Black-Scholes model with a constant volatility:

- Use a *local volatility*, i.e., assume that the volatility is a function of time and of the stock price. Then one has to *calibrate the volatility* from the market data, i.e., to find a volatility function which permits to recover the prices of the options available on the market.
- assume that the volatility is itself a stochastic process, see for example [58, 44] and §2.4.
- generalize the Black-Scholes model by assuming that the spot price is for example a Lévy process, see [30] and references therein.

There is much discussion among specialists in finance on comparing the merits of the three kinds of models above. In this paragraph, we will focus on the first one.

1.1.1 Changes of Variables

Several changes of variables and of unknown functions can be used.

Step 1 Consider the function v such that $P(S, t) = v(S, t)e^{-\lambda(t)}$, then (1.3) can be written

$$\frac{\partial P}{\partial t} = -\lambda'(t)e^{-\lambda(t)}v + e^{-\lambda(t)}\frac{\partial v}{\partial t}, \quad \frac{\partial P}{\partial S} = e^{-\lambda(t)}\frac{\partial v}{\partial S}, \quad \frac{\partial^2 P}{\partial S^2} = e^{-\lambda(t)}\frac{\partial^2 v}{\partial S^2}.$$

Choosing $\lambda(t) = -\int_t^T r(s)ds$ leads to

$$\frac{\partial v}{\partial t} + rS\frac{\partial v}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 v}{\partial S^2} = 0.$$

Step 2 Now set $x = \log S$ and check that $\frac{\partial v}{\partial S} = \frac{1}{S}\frac{\partial v}{\partial x}$ and $\frac{\partial^2 v}{\partial S^2} = -\frac{1}{S^2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}$. We also set $\tau = T - t$ and $w(x, \tau) = v(e^x, T - \tau)$. Calling \tilde{r} and $\tilde{\sigma}$ the functions defined by $\tilde{r}(\tau) = r(t)$ and $\tilde{\sigma}(x, \tau) = \sigma(e^x, t)$, we have

$$\frac{\partial w}{\partial \tau} - \frac{\tilde{\sigma}^2(x, \tau)}{2}\frac{\partial^2 w}{\partial x^2} + (\tilde{r}(\tau) - \frac{\tilde{\sigma}^2(x, \tau)}{2})\frac{\partial w}{\partial x} = 0, \quad \text{in } \mathbb{R} \times (0, T). \quad (1.5)$$

Step 3 When σ depends on t only, one may use the change of variable $(x, \tau) \mapsto (y, \tau)$, where $y = x - \int_0^\tau (\tilde{r}(\theta) - \frac{\tilde{\sigma}^2(\theta)}{2})d\theta$ and set $W(y, \tau) = w(x, \tau)$: it is easy to see that

$$\frac{\partial W}{\partial \tau}(y, \tau) - \frac{\tilde{\sigma}^2(\tau)}{2}\frac{\partial^2 W}{\partial y^2}(y, \tau) = 0, \quad \text{in } \mathbb{R} \times (0, T),$$

and that $W(y, 0) = w(y, 0)$. When σ is a positive constant, this equation is the heat equation.

A similar idea can be used if $x \mapsto \tilde{\sigma}^2(x, \tau)$ is Lipschitz continuous uniformly with respect to τ : we call $X(\theta; x, \tau)$ the solution of the ordinary differential equation

$$\frac{d}{d\theta}X(\theta; x, \tau) = (\tilde{r}(\theta) - \frac{\tilde{\sigma}^2(X(\theta; x, \tau), \theta)}{2}), \quad \theta \in (0, T), \quad X(\tau; x, \tau) = x.$$

Assuming that $(x, \theta) \mapsto X(\theta; x, \tau)$ is regular enough and introducing $W(x, \theta) = w(X(\theta; x, \tau), \theta)$, we obtain

$$\frac{\partial W}{\partial \theta}(x, \theta) = \frac{\partial w}{\partial t}(X(\theta; x, \tau), \theta) + (\tilde{r}(\theta) - \frac{\tilde{\sigma}^2(X(\theta; x, \tau), \theta)}{2})\frac{\partial w}{\partial x}(X(\theta; x, \tau), \theta)$$

and

$$\begin{aligned} \frac{\partial W}{\partial x}(x, \theta) &= \frac{\partial w}{\partial x}(X(\theta; x, \tau), \theta)\frac{\partial X(\theta; x, \tau)}{\partial x}, \\ \frac{\partial^2 W}{\partial x^2}(x, \theta) &= \frac{\partial^2 w}{\partial x^2}(X(\theta; x, \tau), \theta)\left(\frac{\partial X(\theta; x, \tau)}{\partial x}\right)^2 + \frac{\partial w}{\partial x}(X(\theta; x, \tau), \theta)\frac{\partial^2 X(\theta; x, \tau)}{\partial x^2}. \end{aligned}$$

Taking $\theta = \tau - \delta t$ for δt small, we have that $\frac{\partial X(\tau; x, \tau - \delta t)}{\partial x} \sim 1$ and $\frac{\partial^2 X(\tau; x, \tau - \delta t)}{\partial x^2} \sim 0$. Then using (1.5), we obtain the following semi-discrete scheme:

$$\frac{1}{\delta t}(W(x, \tau) - W(x, \tau - \delta t)) - \frac{\tilde{\sigma}^2(x, \tau)}{2}\frac{\partial^2 W}{\partial x^2}(x, \tau) \sim 0,$$

i.e.

$$\frac{1}{\delta t} (w(x, \tau) - w(X(\tau - \delta t; x, \tau), \tau - \delta t)) - \frac{\tilde{\sigma}^2(x, \tau)}{2} \frac{\partial^2 w}{\partial x^2}(x, \tau) \sim 0,$$

which is known as the *method of characteristics* and often used in fluid mechanics.

1.1.2 The Black-Scholes Formulas

Calling $P(S, t)$ the price of an option with maturity T and payoff function P_0 and assuming that r and $\sigma > 0$ are constant, the Black-Scholes formula is

$$P(S, t) = e^{-r(T-t)} \mathbb{E}^*(P_0(S e^{r(T-t)} e^{\sigma(W_T - W_t) - \frac{\sigma^2}{2}(T-t)})), \quad (1.6)$$

and since under P^* , $W_T - W_t$ is a centered Gaussian distribution with variance $T - t$,

$$P(S, t) = \frac{1}{\sqrt{2\pi}} e^{-r(T-t)} \int_{\mathbb{R}} P_0(S e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma x \sqrt{T-t}}) e^{-\frac{x^2}{2}} dx. \quad (1.7)$$

When the option is a vanilla European option, noting C the price of the call and P the price of the put), a more explicit formula can be deduced from (1.2). Take for example a call:

$$\begin{aligned} C(S, t) &= \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{+\infty} \left(S e^{-\frac{\sigma^2}{2}(T-t) + \sigma x \sqrt{T-t}} - K e^{-r(T-t)} \right) e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} \left(S e^{-\frac{\sigma^2}{2}(T-t) - \sigma x \sqrt{T-t}} - K e^{-r(T-t)} \right) e^{-\frac{x^2}{2}} dx, \end{aligned} \quad (1.8)$$

where

$$d_1 = \frac{\log(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T-t}. \quad (1.9)$$

Finally, introducing the upper tail of the Gaussian function

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-\frac{x^2}{2}} dx, \quad (1.10)$$

and using (1.8), (1.9), we obtain the *Black-Scholes formula*:

Proposition 1.1 *When σ and r are constant, the price of the call is given by*

$$C(S, t) = S N(d_1) - K e^{-r(T-t)} N(d_2), \quad (1.11)$$

and the price of the put is given by

$$P(S, t) = -S N(-d_1) + K e^{-r(T-t)} N(-d_2), \quad (1.12)$$

where d_1 and d_2 are given by (1.9) and N is given by (1.10).

Remark 1.1 *If r is a function of time (1.9) must be replaced by*

$$d_1 = \frac{\log(\frac{S}{K}) + \int_t^T r(\tau) d\tau + \frac{\sigma^2}{2}(T-t)}{\sigma \sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T-t}. \quad (1.13)$$

1.1.3 Classical Solutions

In the previous paragraph, we have seen that if the coefficients are constant (with a positive volatility), then (1.3) (1.4) has a solution given by (1.7).

In this paragraph, we give a classical existence and uniqueness result for the final value problem (1.3) (1.4) in the general case when $r = r(t)$ and $\sigma = \sigma(S, t)$. It is necessary to restrict the growth of the solutions when $S \rightarrow 0$ or $S \rightarrow +\infty$. Here, we will impose that the solution is bounded but this restriction can be relaxed (for example, depending on P_0 , one can look for solutions with linear growth as $S \rightarrow +\infty$).

Definition 1.1 We fix a positive number ρ_0 . Let α be a real number such that $0 < \alpha < 1$. We call $\mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ the space of continuous real valued functions $v \in \mathcal{C}^0(\mathbb{R}^d)$ such that

$$\|v\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} |v(x)| + \sup_{x,y \in \mathbb{R}^d, |x-y| \leq \rho_0} \frac{|v(x) - v(y)|}{|x-y|^\alpha} < +\infty.$$

The space $\mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ endowed with the norm $\|\cdot\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}$ is a Banach space.

We call $\mathcal{C}^{\alpha,\alpha/2}(\mathbb{R}^d \times [0, T])$ the space of continuous real valued functions $v \in \mathcal{C}^0(\mathbb{R}^d \times [0, T])$ such that

$$\begin{aligned} \|v\|_{\mathcal{C}^{\alpha,\alpha/2}(\mathbb{R}^d \times [0, T])} = & \\ & \sup_{(x,t) \in \mathbb{R}^d \times [0, T]} |v(x, t)| + \sup_{\substack{(x,t), (y,s) \in \mathbb{R}^d \times [0, T], \\ |x-y| + |t-s| \leq \rho_0}} \frac{|v(x, t) - v(y, s)|}{(|x-y|^2 + |t-s|)^{\frac{\alpha}{2}}} < +\infty. \end{aligned}$$

The space $\mathcal{C}^{\alpha,\alpha/2}(\mathbb{R}^d \times [0, T])$ endowed with the norm $\|\cdot\|_{\mathcal{C}^{\alpha,\alpha/2}(\mathbb{R}^d \times [0, T])}$ is a Banach space.

Theorem 1.1 Under the following assumptions on the coefficients and the final value:

1. the real valued function defined on $\mathbb{R} \times [0, T]$, $(x, t) \mapsto \sigma^2(e^x, t)$ belongs to $\mathcal{C}^{\alpha,\alpha/2}(\mathbb{R} \times [0, T])$,
2. the function $t \mapsto r(t)$ belongs to $\mathcal{C}^{\alpha/2}([0, T])$,
3. the function \tilde{P}_0 defined on \mathbb{R} by $\tilde{P}_0(x) = P_0(e^x)$ is such that \tilde{P}_0 , $\frac{\partial \tilde{P}_0}{\partial x}$, and $\frac{\partial^2 \tilde{P}_0}{\partial x^2}$ belong to $\mathcal{C}^\alpha(\mathbb{R})$,
4. there exists a positive constant $\underline{\sigma}$ such that, for all $x \in \mathbb{R}$, $t \in [0, T]$, $\sigma(t, e^x) \geq \underline{\sigma}$,

the final value problem (1.3) (1.4) has a unique solution P such that, calling \tilde{P} the function defined on $[0, T] \times \mathbb{R}$ by $\tilde{P}(x, t) = P(e^x, t)$, the functions \tilde{P} , $\frac{\partial \tilde{P}}{\partial x}$, $\frac{\partial \tilde{P}}{\partial t}$, $\frac{\partial^2 \tilde{P}}{\partial x^2}$ belong to $\mathcal{C}^{\alpha,\alpha/2}(\mathbb{R} \times [0, T])$.

Under the previous assumptions except the one on \tilde{P}_0 and assuming that \tilde{P}_0 is a bounded function, (1.3), (1.4) has a unique solution \tilde{P} such that for all $\tau < T$, the functions \tilde{P} , $\frac{\partial \tilde{P}}{\partial x}$, $\frac{\partial \tilde{P}}{\partial t}$, $\frac{\partial^2 \tilde{P}}{\partial x^2}$ belong to $C^{\alpha, \alpha/2}(\mathbb{R} \times [0, \tau])$.

This theorem is proved in [73], see also [45, 72].

1.1.4 Variational Framework

Weighted Sobolev Norms

The theory of variational formulations of parabolic equations is well known, see the work of Lions and Magenes [76]. It is particularly useful when strong solutions do not exist either because of some singularity in the data or the domain boundary or the coefficients or nonlinearity. Such situations are very frequent in physics and engineering. Even when the boundary value problem has a classical solution, the variational theory is interesting for several reasons:

- it provides global estimates, often called *energy estimates*.
- It has strong connections with the finite element method which will be advocated below.
- it is the most natural way to study obstacle problems, see the section § 2.3 devoted to American options on baskets.

Note that there are other theories of weak solutions, in particular the theory of *viscosity solutions*, which may also be quite useful in the context of quantitative finance. We will not discuss viscosity solutions here, and we refer the reader to [35, 14, 43].

Almost all the proofs of the results below are omitted for brevity; they can be found in [6].

Variational formulations of parabolic PDE rely on suitable Sobolev spaces. We are going to introduce the Sobolev space useful for the initial value problem (1.3) (1.4) posed in the price variable S .

We denote by $L^2(\mathbb{R}_+)$ the Hilbert space of square integrable functions on \mathbb{R}_+ endowed with the norm $\|v\|_{L^2(\mathbb{R}_+)} = (\int_{\mathbb{R}_+} v(S)^2 dS)^{\frac{1}{2}}$ and the inner product $(v, w)_{L^2(\mathbb{R}_+)} = \int_{\mathbb{R}_+} v(S)w(S) dS$. Calling $\mathcal{D}(\mathbb{R}_+)$ the space of the smooth functions with compact support in \mathbb{R}_+ , we know that $\mathcal{D}(\mathbb{R}_+)$ is dense in $L^2(\mathbb{R}_+)$.

Let us introduce the space

$$V = \left\{ v \in L^2(\mathbb{R}_+) : S \frac{dv}{dS} \in L^2(\mathbb{R}_+) \right\}, \quad (1.14)$$

where the derivative must be understood in the sense of the distributions on \mathbb{R}_+ . A natural scalar product for V is $(v, w)_V = (v, w) + (S \frac{dv}{dS}, S \frac{dw}{dS})$; The space V endowed with the norm $\|v\|_V = \sqrt{(v, v)_V}$ is a Hilbert space.

We have the following properties (see [6])

Theorem 1.2 • The space $\mathcal{D}(\mathbb{R}_+)$ is dense in V .

- (Poincaré's inequality) If $v \in V$, then

$$\|v\|_{L^2(\mathbb{R}_+)} \leq 2 \left\| S \frac{dv}{dS} \right\|_{L^2(\mathbb{R}_+)} \quad (1.15)$$

so the semi-norm $|v|_V = \|S \frac{dv}{dS}\|_{L^2(\mathbb{R}_+)}$ is also a norm on V , equivalent to $\|\cdot\|_V$.

- For any $w \in L^2(\mathbb{R}_+)$ the function $S \rightarrow v(S) = \frac{1}{S} \int_0^S w(s)ds$ belongs to V , and $\|v\|_V \leq C \|w\|_{L^2(\mathbb{R}_+)}$ for some positive constant C independent of w .

We denote by V' the topological dual space of V , and for $w \in V'$, $\|w\|_{V'} = \sup_{v \in V \setminus \{0\}} \frac{\langle w, v \rangle}{\|v\|_V}$.

The Weak Formulation of the Black-Scholes Equation

Consider a vanilla put option with maturity T and payoff function u_0 . Let u be the pricing function, i.e., the price of the option at time $T - t$ and when the spot price is S is $u(S, t)$. The function u solves the initial value problem

$$\frac{\partial u}{\partial t} - \frac{\sigma^2 S^2}{2} \frac{\partial^2 u}{\partial S^2} - rS \frac{\partial u}{\partial S} + ru = 0 \text{ in } \mathbb{R}_+ \times (0, T), \quad u(S, 0) = u_0(S) \text{ in } \mathbb{R}_+. \quad (1.16)$$

Let us multiply (1.16) by a smooth real valued function w defined on \mathbb{R}_+ and integrate in the variable S on \mathbb{R}_+ . Assuming that integrations by part are allowed, we obtain

$$\frac{d}{dt} \left(\int_{\mathbb{R}_+} u(S, t) w(S) dS \right) + a_t(v, w) = 0,$$

where the bilinear form a_t is defined by

$$\begin{aligned} a_t(v, w) &= \int_{\mathbb{R}_+} \left(\frac{1}{2} S^2 \sigma^2(S, t) \frac{\partial v}{\partial S} \frac{\partial w}{\partial S} + r(t) v w \right) dS \\ &\quad + \int_{\mathbb{R}_+} \left(-r(t) + \sigma^2(S, t) + S \sigma(S, t) \frac{\partial \sigma}{\partial S}(S, t) \right) S \frac{\partial v}{\partial S} w dS. \end{aligned} \quad (1.17)$$

Assume that the coefficient $r \geq 0$ is bounded and that σ is sufficiently regular so that the following makes sense:

Assumption 1.1 1. There exists two positive constants, $\underline{\sigma}$ and $\bar{\sigma}$ such that for all $t \in [0, T]$ and all $S \in \mathbb{R}_+$,

$$0 < \underline{\sigma} \leq \sigma(S, t) \leq \bar{\sigma}. \quad (1.18)$$

2. There exists a positive constant C_σ such that for all $t \in [0, T]$ and all $S \in \mathbb{R}_+$,

$$|S \frac{\partial \sigma}{\partial S}(S, t)| \leq C_\sigma. \quad (1.19)$$

Lemma 1.1 Under Assumption 1.1, the bilinear form a_t is continuous on V , i.e. there exists a positive constant μ such that for all $v, w \in V$,

$$|a_t(v, w)| \leq \mu |v|_V |w|_V. \quad (1.20)$$

It also satisfies Gårding's inequality : there exists a non negative constant λ such that for all $v \in V$,

$$a_t(v, v) \geq \frac{\sigma^2}{4} |v|_V^2 - \lambda \|v\|_{L^2(\mathbb{R}_+)}^2. \quad (1.21)$$

One associates with the bilinear form a_t the continuous linear operator A_t : $V \rightarrow V'$; for all $v, w \in V$, $(A_t v, w) = a_t(v, w)$. The interpretation of A_t is as follows:

$$A_t v = -\frac{1}{2} \sigma^2(S, t) S^2 \frac{\partial^2 v}{\partial S^2} - r(t) S \frac{\partial v}{\partial S} + r(t) v.$$

We define $C^0([0, T]; L^2(\mathbb{R}_+))$ as the space of continuous functions on $[0, T]$ with values in $L^2(\mathbb{R}_+)$, and $L^2(0, T; V)$ as the space of square integrable functions on $(0, T)$ with values in V . Assuming that $u_0 \in L^2(\mathbb{R}_+)$, and following [76], it is possible to write a weak formulation for (1.16):

Weak formulation of (1.16)

Find $u \in C^0([0, T]; L^2(\mathbb{R}_+)) \cap L^2(0, T; V)$ with $\frac{\partial u}{\partial t} \in L^2(0, T; V')$, and

$$u|_{t=0} = u_0 \quad \text{in } \mathbb{R}_+, \text{ and for a.e. } t \in (0, T), \quad (1.22)$$

$$\forall v \in V, \quad \left(\frac{\partial u}{\partial t}(t), v \right) + a_t(u(t), v) = 0. \quad (1.23)$$

Theorem 1.3 under Assumption 1.1 and if $u_0 \in L^2(\mathbb{R}_+)$, the weak formulation (1.22) (1.23) has a unique solution, and we have the estimate, for all t , $0 < t < T$

$$e^{-2\lambda t} \|u(t)\|_{L^2(\mathbb{R}_+)}^2 + \frac{1}{2} \sigma^2 \int_0^t e^{-2\lambda \tau} |u(\tau)|_V^2 d\tau \leq \|u_0\|_{L^2(\mathbb{R}_+)}^2. \quad (1.24)$$

Note that Theorem 1.3 does not apply to a European call option because the payoff is not a function of $L^2(\mathbb{R}_+)$; one must either use the put-call parity, see § 1.1.4, or deduce the price of the call from that of the put or work with a different Sobolev space with a weight decaying at infinity.

Regularity of the Weak Solutions

If the interest rate, the volatility, and the payoff are smooth enough, then it is possible to prove additional regularity for the solution to (1.22) (1.23). In particular for all $t \in [0, T]$ and for λ given in Lemma 1.1, the domain of $A_t + \lambda$ is

$$D = \{v \in V; S^2 \frac{\partial^2 v}{\partial S^2} \in L^2(\mathbb{R}_+)\}. \quad (1.25)$$

Let us assume the following:

Assumption 1.2 There exists a positive constant C and $0 \leq \alpha \leq 1$ such that for all $t_1, t_2 \in [0, T]$ and $S \in \mathbb{R}_+$,

$$|r(t_1) - r(t_2)| + |\sigma(S, t_1) - \sigma(S, t_2)| + S \left| \frac{\partial \sigma}{\partial S}(S, t_1) - \frac{\partial \sigma}{\partial S}(S, t_2) \right| \leq C|t_1 - t_2|^\alpha. \quad (1.26)$$

Theorem 1.4 Under Assumptions 1.1 and 1.2, for all s , $0 < t \leq T$, the solution u of (1.22) (1.23) satisfies $u \in \mathcal{C}^0([t, T]; D)$ and $\frac{\partial u}{\partial t} \in \mathcal{C}^0([t, T]; L^2(\mathbb{R}_+))$, and there exists a constant C such that for all t , $0 < t \leq T$,

$$\|A_t u(t)\|_{L^2(\mathbb{R}_+)} \leq \frac{C}{t}.$$

If $u_0 \in D$, then the solution u of (1.22) (1.23) belongs to $\mathcal{C}^0([0, T]; D)$ and $\frac{\partial u}{\partial t} \in \mathcal{C}^0([0, T]; L^2(\mathbb{R}_+))$.

Furthermore if $u_0 \in V$ then the solution to (1.22) (1.23) belongs to $\mathcal{C}^0([0, T]; V) \cap L^2(0, T; D)$, $\frac{\partial u}{\partial t} \in L^2(0, T; L^2(\mathbb{R}_+))$, and there exists a non negative constant $\tilde{\lambda}$ such that

$$e^{-2\tilde{\lambda}t} \|S \frac{\partial u}{\partial S}(t)\|_{L^2(\mathbb{R}_+)}^2 + \frac{\sigma^2}{2} \int_0^t e^{-2\tilde{\lambda}\tau} |S \frac{\partial u}{\partial S}(\tau)|_V^2 d\tau \leq \|S \frac{\partial u_0}{\partial S}\|_{L^2(\mathbb{R}_+)}^2. \quad (1.27)$$

The Maximum Principle for weak solutions

We refer to [91] for a monograph on the maximum principle. The solutions of (1.16) may not vanish for $S \rightarrow +\infty$, therefore, we are going to state the maximum principle for a class of functions much larger than V , i.e.,

$$\mathcal{V} = \{v : \forall \epsilon > 0, v(S)e^{-\epsilon \log^2(S+2)} \in V\}. \quad (1.28)$$

Note that the polynomial functions belong to \mathcal{V} .

Theorem 1.5 (Weak Maximum Principle) Let $u(S, t)$ be such that for all positive number ϵ ,

- $ue^{-\epsilon \log^2(S+2)} \in \mathcal{C}^0([0, T]; L^2(\mathbb{R}_+)) \cap L^2(0, T; V)$,
- $u|_{t=0} \geq 0$ a.e.,
- $\frac{\partial u}{\partial t} + A_t u \geq 0$, (in the sense of distributions)

then $u \geq 0$ almost everywhere.

Various Bounds The maximum principle is an extremely powerful tool for proving estimates on the solutions of elliptic and parabolic partial differential equations. Here we give easy examples of its application to option pricing.

Proposition 1.2 Under Assumption 1.1, let u be the weak solution to (1.16), with $u_0 \in L^2(\mathbb{R}_+)$ being a bounded positive function, i.e., $0 \leq \underline{u}_0 \leq u_0(S) \leq \overline{u}_0$. Then, a.e.

$$\underline{u}_0 e^{-\int_0^t r(\tau)d\tau} \leq u(S, t) \leq \overline{u}_0 e^{-\int_0^t r(\tau)d\tau}. \quad (1.29)$$

Proof. We know that $\underline{u}_0 e^{-\int_0^t r(\tau)d\tau}$ and $\overline{u}_0 e^{-\int_0^t r(\tau)d\tau}$ are two solutions of (1.16). Therefore, we can apply the maximum principle to $u - \underline{u}_0 e^{-\int_0^t r(\tau)d\tau}$ and to $\overline{u}_0 e^{-\int_0^t r(\tau)d\tau} - u$. ■

Remark 1.2 In the case of a vanilla put option: $u_0(S) = (K - S)^+$, Proposition 1.2 just says that $0 \leq u(S, t) \leq K e^{-\int_0^t r(\tau)d\tau}$, which is certainly not a surprise.

For the vanilla put option as in Remark 1.2, we have more information:

Proposition 1.3 Under Assumption 1.1, let u be the weak solution to (1.16), with $u_0(S) = (K - S)^+$, then

$$(K e^{-\int_0^t r(\tau)d\tau} - S)^+ \leq u(S, t) \leq K e^{-\int_0^t r(\tau)d\tau}. \quad (1.30)$$

Proof. Observe that $K e^{-\int_0^t r(\tau)d\tau} - S$ is a solution to (1.16) and apply the maximum principle to $u(S, t) - (K e^{-\int_0^t r(\tau)d\tau} - S)$. We have $K e^{-\int_0^t r(\tau)d\tau} - S \leq u(S, t)$. Then (1.30) is obtained by combining this estimate with the one given in Remark 1.2. Note that (1.30) yields $u(0, t) = K e^{-\int_0^t r(\tau)d\tau}$ for all $t \leq T$. ■

The Put-Call Parity Let u be the pricing function of a vanilla put option with strike K , and consider $C(S, t)$ given by:

$$C(S, t) = S - K e^{-\int_0^t r(\tau)d\tau} + u(S, t). \quad (1.31)$$

From the fact that u and $S - K e^{-\int_0^t r(\tau)d\tau}$ satisfy (1.16), it is clear that C is a solution to (1.16) with the Cauchy condition $C(S, 0) = (S - K)^+$. This is precisely the boundary value problem for the European vanilla call option. Furthermore, from the maximum principle, we know that a well behaved solution of this boundary value problem (in the sense of Theorem 1.5) is unique.

Convexity of u in the Variable S

Assumption 1.3 There exists a positive constant C such that

$$|S^2 \frac{\partial^2 \sigma}{\partial S^2}(S, t)| \leq C, \quad a.e. \quad (1.32)$$

Proposition 1.4 Under Assumptions 1.1 and 1.3, let u be the weak solution to (1.16) where $u_0 \in V$ is a convex function such that $\frac{\partial^2 u_0}{\partial S^2}$ has a compact support. Then, for all $t > 0$, $u(S, t)$ is a convex function of S .

As a consequence, we see that under Assumptions 1.1 and 1.3, the price of a vanilla European put option is convex with respect to S , and thanks to the call-put parity, this is also true for the vanilla European call.

More Bounds

We focus on a vanilla put with a local volatility σ . By using Proposition 1.4, it is possible to compare u with the pricing function of vanilla puts with constant volatilities:

Proposition 1.5 *Under Assumptions 1.1, we have for all $t \in [0, T]$ and for all $S > 0$,*

$$\underline{u}(S, t) \leq u(S, t) \leq \bar{u}(S, t), \quad (1.33)$$

where \underline{u} , (respectively \bar{u}) is the solution to (1.16) with $\sigma = \underline{\sigma}$, (respectively $\bar{\sigma}$).

Localization Again, we focus on a vanilla put. For a numerical approximation to u , one has to limit the domain in the variable S , i.e. to consider only $S \in (0, \bar{S})$ for \bar{S} large enough and to impose some artificial boundary condition at $S = \bar{S}$. Imposing that the new function vanishes on the artificial boundary, we obtain the new boundary value problem:

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \tilde{u}}{\partial S^2} - rS \frac{\partial \tilde{u}}{\partial S} + r\tilde{u} &= 0, \quad t \in (0, T], \quad S \in (0, \bar{S}), \\ \tilde{u}(\bar{S}, t) &= 0, \quad t \in (0, T], \end{aligned} \quad (1.34)$$

with the Cauchy data $\tilde{u}(S, 0) = (K - S)^+$ in $(0, \bar{S})$. The theory of Lions-Magenes applies to this new boundary value problem, but one has to work in the new Sobolev space:

$$\tilde{V} = \{v, S \frac{\partial v}{\partial S} \in L^2((0, \bar{S})), v(\bar{S}) = 0\}.$$

The theory of weak solutions can be applied to problem (1.34). The question is to estimate the error between u and \tilde{u} .

Proposition 1.6 *Under Assumptions 1.1, the error $\max_{t \in [0, T], S \in [0, \bar{S}]} |u(S, t) - \tilde{u}(S, t)|$ decays faster than any negative power of \bar{S} as $\bar{S} \rightarrow \infty$, i.e., faster than $\bar{S}^{-\eta}$ for any positive number η .*

Proof. From the maximum principle applied to weak solutions of (1.16) in $(0, \bar{S}) \times (0, T]$, we immediately see that $u \geq \tilde{u}$ in $(0, \bar{S}) \times (0, T)$, because $u(\bar{S}, t) \geq \tilde{u}(\bar{S}, t) = 0$. On the other hand, from Proposition 1.5, $u \leq \bar{u}$, which implies that $u(\bar{S}, t) \leq \bar{u}(\bar{S}, t)$. Call $\pi(\bar{S}) = \max_{t \in [0, T]} \bar{u}(\bar{S}, t)$. The maximum principle applied to the function $E(S, t) = \pi(\bar{S}) - u(S, t) + \tilde{u}(S, t)$ yields that $\pi(\bar{S}) \geq u - \tilde{u}$ in $[0, \bar{S}] \times [0, T]$. At this point, we have proved that

$$0 \leq u - \tilde{u} \leq \pi(\bar{S}), \quad \text{in } [0, \bar{S}] \times [0, T].$$

But $\pi(\bar{S})$ can be computed semi-explicitly by the Black-Scholes formula (1.12), and it is easy to see that for all $\eta > 0$, $\lim_{\bar{S} \rightarrow \infty} \pi(\bar{S}) \bar{S}^\eta = 0$.

Therefore, $\max_{t \in [0, T], S \in [0, \bar{S}]} |u(S, t) - \tilde{u}(S, t)|$ decays faster than any power $\bar{S}^{-\eta}$ as $\bar{S} \rightarrow \infty$. ■

1.2 A Finite Element Method

1.2.1 Description of the Method

Consider the boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2(S, t)S^2 \frac{\partial^2 u}{\partial S^2} - \alpha(t)S \frac{\partial u}{\partial S} + \beta(t)u &= 0, \quad t \in (0, T), S \in (0, \bar{S}), \\ u(S, 0) = u_0(S) \quad S \in (0, \bar{S}), \quad u(\bar{S}, t) &= 0 \quad t \in (0, T]. \end{aligned} \quad (1.35)$$

This generalization of (1.34) is used for pricing European puts with possibly continuously paid dividends: this corresponds to the choice $\alpha = r(t) - q(t)$ and $\beta = r(t)$, where r is the interest rate and q is the dividend yield. A problem of the form (1.35) also arises when one looks for the option's price as a function of the maturity and the strike at a fixed spot price (the partial differential equation is known as Dupire's equation [36, 37, 6], see § 3.2.5): this corresponds to the choice $\alpha = -r(t) + q(t)$ and $\beta = q(t)$.

To apply the Finite Element Method of degree one we start with the variational formulation introduced in § 1.1.4, given in (1.22), (1.23).

We introduce a partition of the interval $[0, \bar{S}]$ into subintervals $\kappa_i = [S_{i-1}, S_i]$, $1 \leq i \leq N+1$, such that $0 = S_0 < S_1 < \dots < S_N < S_{N+1} = \bar{S}$. We call $h_i = S_i - S_{i-1}$ and $h = \max_{i=1, \dots, N+1} h_i$. We define the mesh \mathcal{T}_h of $[0, \bar{S}]$ as the set $\{\kappa_1, \dots, \kappa_{N+1}\}$. In what follows, we will assume that the strike K coincides with some node of \mathcal{T}_h , *i.e.*, there $S_{k_0} = K$ for some admissible k_0 .

We define the discrete space V_h by

$$V_h = \{v_h \in C^0([0, \bar{S}]), v_h(\bar{S}) = 0; \forall \kappa \in \mathcal{T}_h, v_h|_\kappa \text{ is affine}\}. \quad (1.36)$$

The assumption on the mesh ensures that $u_0 \in V_h$ when $u_0 = (K - S)^+$. The discrete problem obtained by applying the Euler implicit scheme in time reads:

$$\text{find } (u_h^m)_{1 \leq m \leq M}, u_h^m \in V_h \text{ with } u_h^0(S_i) = u_0(S_i), i = 0 \dots N+1, \text{ and}$$

$$\text{for } m = 1 \dots M, \forall v_h \in V_h, (u_h^m - u_h^{m-1}, v_h) + \delta t_m a_{t_m}(u_h^m, v_h) = 0, \quad (1.37)$$

where

$$\begin{aligned} a_t(v, w) &= \int_0^{\bar{S}} \frac{1}{2} S^2 \sigma^2(S, t) \frac{\partial v}{\partial S} \frac{\partial w}{\partial S} \\ &+ \int_0^{\bar{S}} \left(-\alpha(t) + \sigma^2(S, t) + S \sigma(S, t) \frac{\partial \sigma}{\partial S}(S, t) \right) S \frac{\partial v}{\partial S} w + \beta(t) \int_0^{\bar{S}} v w. \end{aligned} \quad (1.38)$$

Note that we have a simpler expression for $a_t(v, w)$ for $v, w \in V_h$ when σ is continuous with respect to S :

$$a_t(v, w) = -\sum_{i=1}^N \frac{1}{2} S_i^2 \sigma^2(S_i, t) [\frac{\partial v}{\partial S}] (S_i) w(S_i) - \alpha(t) \int_0^{\bar{S}} S \frac{\partial v}{\partial S} w + \beta(t) \int_0^{\bar{S}} v w, \quad (1.39)$$

where $[\cdot]$ denotes the jump

$$[\frac{\partial v}{\partial S}] (S_i) = \frac{\partial v}{\partial S}(S_i^+) - \frac{\partial v}{\partial S}(S_i^-). \quad (1.40)$$

For $i = 0, \dots, N+1$, let w_i be the piecewise linear function on the mesh which takes the value 1 at S_i and 0 at S_j , $j \neq i$, $j = 0, \dots, N+1$. Then $(w_i)_{i=0, \dots, N}$ is the nodal basis of V_h , and

$$u_h^m(S) = \sum_0^N u_h^m(S_i) w_i(S). \quad (1.41)$$

Let \mathbf{M} and \mathbf{A}^m in $\mathbb{R}^{N \times N}$ be respectively the mass and stiffness matrices defined by $\mathbf{M}_{i,j} = (w_i, w_j)$, $\mathbf{A}_{i,j}^m = a_{t_m}(w_j, w_i)$, $0 \leq i, j \leq N$. Calling $\mathbf{u}^m = (u_h^m(S_0), \dots, u_h^m(S_N))^T$ (1.37) is equivalent to

$$(\mathbf{M} + \delta t_m \mathbf{A}^m) \mathbf{u}^m = \mathbf{M} \mathbf{u}^{m-1}, \quad (1.42)$$

The shape functions w_i corresponding to vertex S_i is supported in $[S_{i-1}, S_{i+1}]$. This implies that the matrices \mathbf{M} and \mathbf{A}^m are tridiagonal because when $|i-j| > 1$, the intersection of the supports of w_i and w_j has measure 0. Furthermore, for $i \leq N$,

$$\begin{aligned} w_i(S) &= \frac{S - S_{i-1}}{h_i}, & \frac{\partial w_i}{\partial S} &= \frac{1}{h_i}, & \forall S \in (S_{i-1}, S_i), \\ w_i(S) &= \frac{S_{i+1} - S}{h_{i+1}}, & \frac{\partial w_i}{\partial S} &= -\frac{1}{h_{i+1}}, & \forall S \in (S_i, S_{i+1}), \end{aligned} \quad (1.43)$$

giving,

$$\begin{aligned} \int_0^{\bar{S}} w_{i-1} w_i &= \frac{h_i}{6}, & \int_0^{\bar{S}} S w_i \frac{\partial w_{i-1}}{\partial S} &= -\frac{S_{i-1}}{6} - \frac{S_i}{3}, \\ \int_0^{\bar{S}} w_i w_i &= \frac{h_i + h_{i+1}}{3}, & \int_0^{\bar{S}} S w_i \frac{\partial w_i}{\partial S} &= -\frac{1}{2} \int_0^{\bar{S}} w_i^2 = -\frac{h_i + h_{i+1}}{6}, & \text{if } i > 0, \\ \int_0^{\bar{S}} w_0 w_0 &= \frac{h_1}{3}, & \int_0^{\bar{S}} S w_0 \frac{\partial w_0}{\partial S} &= -\frac{1}{2} \int_0^{\bar{S}} w_0^2 = -\frac{h_1}{6}, \\ \int_0^{\bar{S}} w_{i+1} w_i &= \frac{h_{i+1}}{6}, & \int_0^{\bar{S}} S w_i \frac{\partial w_{i+1}}{\partial S} &= \frac{S_{i+1}}{6} + \frac{S_i}{3}. \end{aligned}$$

From this, a few calculations show that the entries of \mathbf{A}^m are

$$\begin{aligned}\mathbf{A}_{i,i-1}^m &= -\frac{S_i^2 \sigma^2(S_i, t_m)}{2h_i} + \frac{\alpha(t_m)S_i}{2} + (\beta(t_m) - \alpha(t_m))\frac{h_i}{6}, \quad 1 \leq i \leq N, \\ \mathbf{A}_{i,i}^m &= \frac{S_i^2 \sigma^2(S_i, t_m)}{2} \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) + \frac{\alpha(t_m)}{2}(h_{i+1} + h_i) + (\beta(t_m) - \alpha(t_m))\frac{h_i + h_{i+1}}{3}, \quad 1 \leq i \leq N, \\ \mathbf{A}_{0,0}^m &= \frac{\alpha(t_m)}{2}h_1 + (\beta(t_m) - \alpha(t_m))\frac{h_1}{3}, \\ \mathbf{A}_{i,i+1}^m &= -\frac{S_i^2 \sigma^2(S_i, t_m)}{2h_{i+1}} - \frac{\alpha(t_m)S_i}{2} + (\beta(t_m) - \alpha(t_m))\frac{h_{i+1}}{6}, \quad 0 \leq i \leq N-1.\end{aligned}$$

When the mesh is uniform, this matrix is close (but not proportional) to the stiffness matrix obtained by using the finite difference method with a centered scheme, see [6]. The entries of \mathbf{M} are

$$\begin{aligned}\mathbf{M}_{i,i-1} &= \frac{h_i}{6}, \quad 1 \leq i \leq N, \\ \mathbf{M}_{i,i} &= \frac{h_i + h_{i+1}}{3}, \quad 1 \leq i \leq N, \quad \mathbf{M}_{0,0} = \frac{h_1}{3}, \\ \mathbf{M}_{i,i+1} &= \frac{h_{i+1}}{6} \quad 0 \leq i \leq N-1.\end{aligned}$$

Remark 1.3 The value of u at $S = 0$ is known for all time, because the equation degenerates into $\frac{\partial u}{\partial S} + \beta(t)u = 0$. Therefore $u(0, t) = u_0(0) \exp\left(-\int_0^t \beta(s)ds\right)$. hence, it is possible to impose that

$$u_0^m = u_0(0) \exp\left(-\int_0^{t_m} \beta(s)ds\right),$$

and plug this into (1.42). In this case, since u_0^m is known, (1.42) can be rewritten as: $\forall i = 1, \dots, N$

$$\sum_{j=1}^N (\mathbf{M}_{i,j} + \delta t_m \mathbf{A}_{i,j}^m) u_j^m = \sum_{j=0}^N \mathbf{M}_{i,j} u_j^{m-1} - (\mathbf{M}_{i,0} + \delta t_m \mathbf{A}_{i,0}^m) u_0^m. \quad (1.44)$$

1.2.2 A C++ Implementation

The following is a simple C++ implementation of the above for a put option with dividend $d(t)$ on a general mesh which may vary at each time step. It can also solve the Dupire equation (see section § 3.2.5).

The boundary condition at zero is implemented as in Remark 1.3. There are two classes, one for the mesh and one for the put option problem. The mesh class has a simple constructor for a mesh that can be refined near the strike and at the origin in time. The calling program is

```
int main()
{
    VarMesh m(50,100,0.5,300.,1.05,0.9,100.,1.02);
```

```

Option p(1,&m,100.,0.05,0.3);
p.calc();
ofstream result("u.txt");
for(int i = 0; i < m.nT; i++)
{
    for(int j = 0; j < m.nX[i]; j++)
        ff << m.x[i][j] << "\t" << m.t[i] << "\t" << p.u[i][j] << endl;
    ff << endl;
}
return 0;
}

```

The mesh is called `m`; it has 50 time steps and at most 100 mesh points maximum over $(0, 300) \times (0, 0.5)$. At each time level the time step is increased by a factor 1.05 and the number of mesh points N_m is decreased by a factor 1.02, namely

$$\delta t^m = 1.05\delta t^{m-1}, \quad N_m = \text{int}(N_{m-1}/1.02), \quad m = 1, \dots, 49.$$

Finally the mesh is not uniform in space, it is refined near the strike $x_S = 100$ by a factor 0.9, namely on the left of x_S , $\delta x_i = 0.9\delta x_{i-1}$ and on the right of x_S , $\delta x_{i-1} = 0.9\delta x_i$.

The put option is called `p`; it is solution of Black-Schole's PDE (hence the 1 as first parameter in the constructor, 0 being for Dupire's equation). The strike is 100, the maturity is obtained from the second parameter in the mesh constructor; the interest rate is constant here and equal to 0.05 and so is the volatility which is here 0.3. The function `calc` solves Black-Scholes or Dupire's equation by FEM with LU factorization at each time step. The results are in `m.u` and written in the file `u.txt` in a format readable by `gnuplot`. The following `gnuplot` statement `splot "u.txt" w 1` displays something like figure 1.1. File `optionhb.hpp` contains the definitions and implementation of the classes `VarMesh` and `Option`.

```

Inappropriate ioctl for device
using namespace std;
                                         // # include "ddouble.h"
typedef double ddouble;                  // for automatic differentiation

class VarMesh {
public:
    const int nT;                         // nb time step
    int *nX;                             // nb of vertices at each time step
    double *t, **x;                      // mesh points x at times t
    double T, xmax;
    double **xx, *vxxx; // holds vertices of 2 mesh levels; holds function val
    int kk, *ixxx; // total nb vertices; holds origin of vertex (<0 if from
level k-1)
    VarMesh(const int nt, const int nx, const double T1, const double xmax1,
    const double tscal=1,const double xc=0, const double xS=1,
    const double xsize=1); // defaults=>uniform mesh
    void interpol(ddouble* v, const int k, ddouble *w); // w=interpol of v[]
on x[k-1], to x[k]
    void intersect(const int k); // intersect level k and k-1, result in
xx, ixx, kk
    void integral(ddouble* v, const int k, ddouble* w); // even if v and w
are on != mesh
};

class Option {
public:
    const int s;                         // s=0 for Dupire and =1 for B&S
    VarMesh *msh;                        // mesh
    const double S,T; // spot price and Maturity
    double *r, *d; // interest rate and dividend
    ddouble **u, **sigma; // solution and volatility(function of K and T)
    ddouble *w, *am, *bm, *cm; // working arrays for M.u and Gauss fact.

    ddouble u0(const double x); // initial condition
    void factLU(const int nX1);
    void solveLU(const int nX1, ddouble* z);
    void calc();
    Option(const int s1, VarMesh* msh1, const double S1,
    double r1, double sigma1, double d1=0);
    ~Option();
};

Option::Option(const int s1, VarMesh* msh1, const double S1,
    double r1, double sigma1, double d1): s(s1), msh(msh1), S(S1), T(msh1->T){
VarMesh& m = *msh;
r = new double[m.nT];
d = new double[m.nT];
u = new ddouble*[m.nT];
sigma = new ddouble*[m.nT];
int nXmax =0;
for(int i = 0; i < m.nT; i++)
{ if(nXmax<m.nX[i] ) nXmax=m.nX[i];
u[i] = new ddouble[m.nX[i]];
sigma[i] = new ddouble[m.nX[i]];
}
}

```

```

r[i]=r1; d[i] = d1;
for(int j=0;j<m.nX[i];j++)sigma[i][j]=sigma1;
}
am = new ddouble[nXmax];
bm = new ddouble[nXmax];
cm = new ddouble[nXmax];
w = new ddouble[nXmax];
}

ddouble Option::u0(const double x1) { return x1<S ? (S-x1) : 0; }

void Option::factLU(const int nX1){
cm[1] /= bm[1];
for(int i=2;i<nX1-1;i++)
{
    bm[i] -= am[i]*cm[i-1];
    cm[i] /= bm[i];
}
}

void Option::solveLU(const int nX1, ddouble *z){
z[1] /= bm[1];
for(int i=2;i<nX1-1;i++)
    z[i] = (z[i] - am[i]*z[i-1])/bm[i];
for(int i=nX1-2;i>0;i--)
    z[i] -= cm[i]*z[i+1];
}

void Option::calc(){
VarMesh& m = *msh;
for(int i=0;i<m.nX[0];i++) u[0][i] = u0(m.x[0][i]);
const double nml=1./6.;           // no mass lumping = 1./6., mass lumping =0
for(int j=1;j<m.nT;j++)          // time loop
{
    double dt = m.t[j]-m.t[j-1];
    double aux = (r[j]*(4*s-1)+d[j]*(3-4*s))/3, auy = (r[j]*(1-s)+d[j]*s)*dt/6;
    m.integral(u[j-1],j,w);           // rhs of PDE
    for(int i=1;i<m.nX[j]-1;i++)
    {
        double hi = m.x[j][i]-m.x[j][i-1], h1i = m.x[j][i+1]-m.x[j][i];
        double xss = m.x[j][i]*sigma[j][i]*sigma[j][i];
        bm[i] =(hi+h1i)*(0.5-nml +dt*(m.x[j][i]*xss/hi/h1i+aux)/2);           // FEM
matrix
        am[i] = nml*hi - dt*m.x[j][i]*(xss/hi - (2*s-1)*(r[j]-d[j]))/2 + auy*hi;
        cm[i] = nml*h1i - dt*m.x[j][i]*(xss/hi1 + (2*s-1)*(r[j]-d[j]))/2 + auy*h1i;
    }
    u[j][m.nX[j]-1]=0;                // C.L.
    u[j][0]=u[j-1][0]*exp(((1-s)*d[j]-s*r[j])*dt);                // C.L.
    double hi = m.x[j][1]-m.x[j][0];
    w[1] -= u[j][0]*(nml*hi - dt*m.x[j][1]*(
        m.x[j][1]*sigma[j][1]*sigma[j][1]/hi - (2*s-1)*(r[j]-d[j]))/2 + auy*hi);
    factLU(m.nX[j]);
    solveLU(m.nX[j],w);
    for(int i=1;i<m.nX[j]-1;i++) u[j][i]=w[i];
}
}
}

```

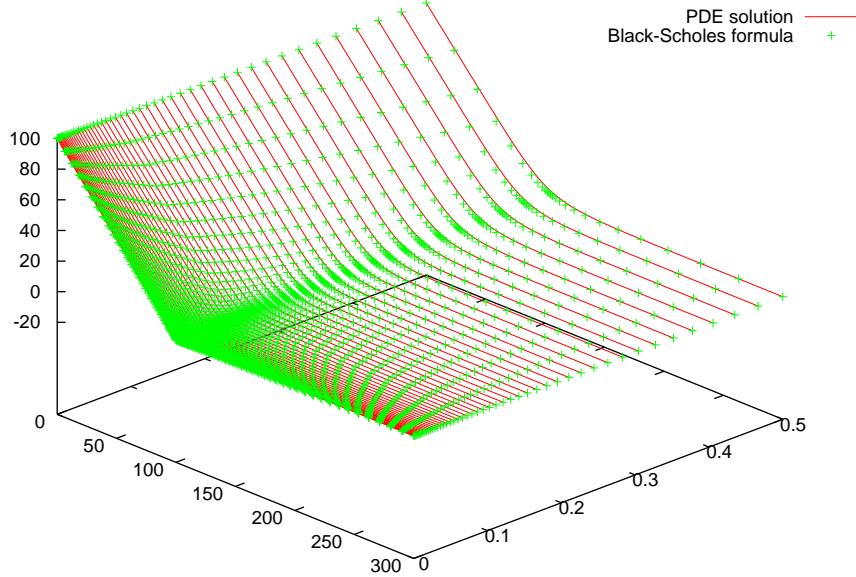


Figure 1.1: Result of the program of Section 1.2.2 and comparison with the Black-Scholes formula at the mesh points (crosses).

1.3 Mesh Adaptivity

Here, we discuss a strategy in order to separately adapt the grids in the variables t and S . Moreover, the mesh in the variable S may vary in time. This method was originally proposed by Bernardi et al [16]. The goal is to find local error indicators which can be explicitly computed from the solution of the discrete problem, and such that their Hilbertian sum is equivalent to the global error. These indicators are said to be optimal if the constants of the norm-equivalence inequalities are independent of the error. We consider two families of error indicators, both of residual type. The first family is global with respect to the spot price variable and local with respect to time: it gives relevant information in order to refine the mesh in time. The second family is local with respect to both spot and time variables, and provides an efficient tool for mesh adaptivity in the price variable at each time step.

All the results below are proved in [6].

1.3.1 Multiple meshes and integration

Consider a parabolic problem in the variables x and t , where t is the time variable. Mesh adaptivity may lead to situations when the mesh used for the variable x varies between two time steps; assume that mesh adaption is performed at time t_m : in this case, u^m is defined on the mesh \mathcal{T}_H used at time t_m and has to be integrated against functions defined on the mesh \mathcal{T}_h used at time t_{m+1} . Let w_i be a nodal basis function associated with the mesh \mathcal{T}_h . To compute the integral of $u^m w_i$ we intersect the meshes \mathcal{T}_H and \mathcal{T}_h ; the result is a new mesh on which both functions u^m and w_i are piecewise linear; therefore, the integral can be computed without error.

1.3.2 Error Indicators for the Semi-Discrete Problem in Time

Our goal is Theorem 1.6 below, but we need first to define appropriate norms. We go back to problem (1.34). From assumption 1.1 and from Gårding's inequality (1.21)), it is possible to prove that (1.34) admits a unique solution that we write u for simplicity. Moreover, introducing the norm

$$[[v]](t) = \left(e^{-2\lambda t} \|v(t)\|^2 + \frac{1}{2} \underline{\sigma}^2 \int_0^t e^{-2\lambda \tau} |v(\tau)|_V^2 d\tau \right)^{\frac{1}{2}}, \quad (1.45)$$

where $\|\cdot\|$ stands for the norm in $L^2(0, \bar{S})$, we have, by multiplying (1.34) by $u(t)e^{-2\lambda t}$ and integrating in $(0, \bar{S}) \times (0, T)$,

$$[[u]](t) \leq \|u_0\|. \quad (1.46)$$

We introduce a partition of the interval $[0, T]$ into subintervals $[t_{n-1}, t_n]$, $1 \leq n \leq N$, such that $0 = t_0 < t_1 < \dots < t_N = T$. We denote by δt_n the length $t_n - t_{n-1}$, and by δt the maximum of the δt_n , $1 \leq n \leq N$. We also define the regularity parameter $\rho_{\delta t}$:

$$\rho_{\delta t} = \max_{2 \leq n \leq N} \frac{\delta t_n}{\delta t_{n-1}}. \quad (1.47)$$

Given $u^0 = u_0 \in L^2(0, \bar{S})$, the semi-discrete problem arising from an implicit Euler scheme is: find $(u^n)_{1 \leq n \leq N} \in V^N$ satisfying

$$\forall n, 1 \leq n \leq N, \quad (u^n - u^{n-1}, v) + \delta t_n a_{t_n}(u^n, v) = 0, \quad \forall v \in V, \quad (1.48)$$

where V is the space

$$V = \left\{ v \in L^2(0, \bar{S}) : S \frac{\partial v}{\partial S} \in L^2(0, \bar{S}), v(\bar{S}) = 0 \right\}. \quad (1.49)$$

For δt smaller than $1/(2\lambda)$, (where λ is the constant appearing in Gårding's inequality (1.21)), the existence and uniqueness of $(u^n)_{1 \leq n \leq N}$ is a consequence of the Lax-Milgram lemma. We call $u_{\delta t}$ the function which is affine on each

interval $[t_{n-1}, t_n]$, and such that $u_{\delta t}(t_n) = u^n$, $0 \leq n \leq N$.

From the standard identity $(a-b, a) = \frac{1}{2}|a|^2 + \frac{1}{2}|a-b|^2 - \frac{1}{2}|b|^2$, a few calculations show that

$$(1 - 2\lambda\delta t_n)\|u^n\|^2 + \frac{1}{2}\delta t_n\sigma^2|u^n|_V^2 \leq \|u^{n-1}\|^2. \quad (1.50)$$

Consider the discrete norm for the sequence $(v^m)_{1 \leq m \leq n}$:

$$[[v^m]]_n = \left(\left(\prod_{i=1}^n (1 - 2\lambda\delta t_i) \right) \|v^n\|^2 + \frac{\sigma^2}{2} \sum_{m=1}^n \delta t_m \left(\prod_{i=1}^{m-1} (1 - 2\lambda\delta t_i) \right) |v^m|_V^2 \right)^{\frac{1}{2}} \quad (1.51)$$

Multiplying equation (1.50) by $\prod_{i=1}^{n-1} (1 - 2\lambda\delta t_i)$ and summing the equations on n , we obtain

$$[[u^m]]_n^2 \leq \|u^0\|^2. \quad (1.52)$$

We will need an equivalence relation between $[(u^m)]_n$ and $[[u_{\delta t}]](t_n)$:

Lemma 1.2 *There exists a positive real number $\alpha \leq \frac{1}{2}$ such that the following equivalence property holds for $\delta t \leq \frac{\alpha}{\lambda}$ and for any family $(v^n)_{0 \leq n \leq N}$ in V_0^{N+1} ,*

$$\frac{1}{8}[[v^m]]_n^2 \leq [[v_{\delta t}]]^2(t_n) \leq \max(2, 1 + \rho_{\delta t})[[v^m]]_n^2 + \frac{1}{2}\sigma^2\delta t_1|v^0|_V^2. \quad (1.53)$$

From (1.52) and (1.53), we deduce that for all n , $1 \leq n \leq N$,

$$[[u_{\delta t}]](t_n) \leq c(u_0) \quad (1.54)$$

where

$$c(u_0) = \left(\max(2, 1 + \rho_{\delta t})\|u_0\|^2 + \frac{1}{2}\sigma^2\delta t_1|u_0|_V^2 \right)^{\frac{1}{2}} \quad (1.55)$$

To evaluate $[[u - u_{\delta t}]](t_n)$ we make a further assumption on the coefficients:

Assumption 1.4 *The function r is Lipschitz continuous in $[0, T]$. The functions σ and $S\frac{\partial\sigma}{\partial S}$ are Lipschitz continuous with respect to t uniformly in the variable S .*

With Assumption 1.4 and the previous set of assumptions there exists constants L_1 , L_2 and L_3 such that, for all t and t' in $[0, T]$,

$$\|\sigma^2(., t) - \sigma^2(., t')\|_{L^\infty(0, \bar{S})} \leq L_1|t' - t|, \quad |r(t) - r(t')| \leq L_3|t' - t| \quad (1.56)$$

and a similar inequality for $\|-r(t) + \frac{1}{2}\sigma^2(., t) + S\sigma(., t)\frac{\partial\sigma}{\partial S}(., t)\|_{L^\infty(0, \bar{S})}$ with L_2 on the right hand side.

Lemma 1.3 *Assume that $u_0 \in V_{0h}$. Then, there exists a constant $\alpha \leq \frac{1}{2}$ such that if $\delta t \leq \frac{\alpha}{\lambda}$, the following a posteriori error estimate holds between the solutions of (1.34) and (1.48):*

$$\begin{aligned} & [[u - u_{\delta t}]](t_n) \leq cb \\ & \text{with } b = \left(\frac{L}{\sigma^2}c(u_0)\delta t + \frac{\mu}{\sigma^2}(1 + \rho_{\delta t})[[u_{\delta t} - u_{h,\delta t}]](t_n) + \frac{\mu}{\sigma^2} \left(\sum_{m=1}^n \eta_m^2 \right)^{\frac{1}{2}} \right), \end{aligned} \quad (1.57)$$

where

$$\eta_m^2 = \delta t_m e^{-2\lambda t_{m-1}} \frac{\sigma^2}{2} |u_h^m - u_h^{m-1}|_V^2, \quad (1.58)$$

and c is a positive constant, $L = 4L_1 + 2L_2 + L_3$ where L_1, L_2, L_3 are given by (1.56), $c(u_0)$ is given by (1.55). Furthermore if the assumptions of Proposition 1.3 are satisfied, the following a posteriori error estimate holds :

$$\|\frac{\partial}{\partial t}(u - u_{\delta t})\|_{L^2(0, t_n, V')} \leq c(\frac{\mu + \sigma^2}{\sigma}) b. \quad (1.59)$$

where μ is the continuity constant of a in (1.20).

1.3.3 Error Indicators for the Fully Discrete Problem

The fully discrete problem has already been defined in (1.37). For each time interval let (\mathcal{T}_{nh}) be the mesh of Ω . Let $h^{(n)}$ denote the maximal size of the intervals in \mathcal{T}_{nh} . For a given element $\omega \in \mathcal{T}_{nh}$, let h_ω be the diameter of ω and let $S_{\min}(\omega), S_{\max}(\omega)$ be the endpoints of ω . We assume that there exists a constant ρ_h such that, for two adjacent elements ω and ω' of (\mathcal{T}_{nh}) ,

$$h_\omega \leq \rho_h h_{\omega'}. \quad (1.60)$$

For each h , we define the discrete spaces by

$$V_{nh} = \{v_h \in V, \forall \omega \in \mathcal{T}_{nh}, v_h|_\omega \in \mathcal{P}_1\}. \quad (1.61)$$

In this study the grids \mathcal{T}_{nh} for different values of n are not independent: indeed, each triangulation \mathcal{T}_{nh} is derived from $\mathcal{T}_{n-1,h}$ by cutting some elements of $\mathcal{T}_{n-1,h}$ into a limited number of smaller intervals or on the contrary by gluing together a limited number of elements of $\mathcal{T}_{n-1,h}$. This permits (w_h^{n-1}, v_h^n) to be evaluated exactly if $w_h^{n-1} \in V_{n-1,h}$ and $v_h^n \in V_{nh}$.

Assuming that $u_0 \in V_{0,h}$, the fully discrete problem reads:

find $(u_h^n)_{1 \leq n \leq N}, u_h^n \in V_{nh}$ satisfying

$$\forall v_h \in V_{nh}, \quad (u_h^n - u_h^{n-1}, v_h) + \delta t_n a_{t_n}(u_h^n, v_h) = 0, \quad (1.62)$$

where $u_h^0 = u_0$. As above, for δt smaller than $1/(2\lambda)$, the existence and uniqueness of $(u_h^n)_{0 \leq n \leq N}$ is a consequence of the Lax-Milgram lemma, and we have the stability estimate

$$[[u_h^m]]_n \leq \|u^0\|. \quad (1.63)$$

We call $u_{h,\delta t}$ the function which is affine on each interval $[t_{n-1}, t_n]$, and such that $u_{h,\delta t}(t_n) = u_h^n$.

We wish to bound the error $[[u - u_{h,\delta t}]](t_n)$, $1 \leq n \leq N$, by indicators computable from $u_{h,\delta t}$. First let us separate the time discretization error from the space discretization by the triangular inequality

$$[[u - u_{h,\delta t}]](t_n) \leq [[u - u_{\delta t}]](t_n) + [[u_{\delta t} - u_{h,\delta t}]](t_n).$$

Lemma 1.4 Assume that $u_0 \in V_{0h}$. Then the following a posteriori error estimate holds between the solution $(u^n)_{1 \leq n \leq N}$ of problem (1.48) and the solution $(u_h^n)_{1 \leq n \leq N}$ of problem (1.62): there exists a constant c such that, for all t_n , $1 \leq n \leq N$,

$$[(u_{\delta t} - u_{h,\delta t})]^2(t_n) \leq \frac{c}{\underline{\sigma}^2} \max(2, 1 + \rho_{\delta t}) \sum_{m=1}^n \delta t_m \prod_{i=1}^{m-1} (1 - 2\lambda\delta t_i) \sum_{\omega \in \mathcal{T}_{mh}} \eta_{m,\omega}^2, \quad (1.64)$$

where

$$\eta_{m,\omega} = \left(\frac{h_\omega}{S_{\max}(\omega)} \left\| \frac{u_h^m - u_h^{m-1}}{\delta t_m} - rS \frac{\partial u_h^m}{\partial S} + ru_h^m \right\|_{L^2(\omega)} \right). \quad (1.65)$$

Combining Lemmas 1.3 and 1.4 leads to the full a posteriori error estimate

Theorem 1.6 Assume that $u_0 \in V_{0h}$ and that $\lambda\delta t \leq \alpha$ is as in Lemma 1.2. Then the following a posteriori error estimate holds between the solution u of problem (1.34) and the solution $u_{h,\delta t}$ of problem (1.62): there exists a constant c such that, for all t_n , $1 \leq n \leq N$,

$$\begin{aligned} & [[u - u_{h,\delta t}]](t_n) \\ & \leq c \left(\frac{L}{\underline{\sigma}^2} c(u_0) \delta t \right. \\ & \quad \left. + \frac{\mu}{\underline{\sigma}^2} \left(\sum_{m=1}^n \eta_m^2 + \frac{\delta t_m}{\underline{\sigma}^2} g(\rho_{\delta t}) \prod_{i=1}^{m-1} (1 - 2\lambda\delta t_i) \sum_{\omega \in \mathcal{T}_{mh}} \eta_{m,\omega}^2 \right)^{\frac{1}{2}} \right), \end{aligned} \quad (1.66)$$

where $L = 4L_1 + 2L_2 + L_3$, L_1 , L_2 , L_3 are given by (1.56), $c(u_0)$ is given by (1.55), η_m is given by (1.58), and $\eta_{m,\omega}$ is given by (1.65), and

$$g(\rho_{\delta t}) = (1 + \rho_{\delta t})^2 \max(2, 1 + \rho_{\delta t}).$$

Conclusion In (1.57), (1.59) and (1.64), we have bounded the norm of the error produced by the finite element method by a Hilbert sum involving the error indicators η_m and $\eta_{m,\omega}$ given in (1.58) and (1.65), which are respectively local in t and local in t and S . Conversely, in Propositions 1.7 and 1.8 below, we will see that the error indicators can be bounded by local norms of the error. This shows that the error indicators are both *reliable* and *efficient*, or in other words that the error produced by the method is well approximated by these indicators. Furthermore, since the indicators are local, they tell us where the mesh should be refined. It is now possible to build a computer program which adapts the mesh so as to reduce the error to a given number ϵ ; from the result of an initial computation $u_{h,\delta t}$, we can adapt separately the meshes in the variables t and S so as to decrease the Hilbert sum in (1.66). The process may be repeated until the desired accuracy is obtained.

1.3.4 Upper Bounds for the Error Indicators

We now investigate the efficiency of the indicators in (1.58) and (1.65). For that, we introduce the notation $[[v^n]]$, for $(v^n)_{1 \leq n \leq N}$, $v^n \in V$:

$$[[v^n]]^2 = \frac{\underline{\sigma}^2}{2} \delta t_n \prod_{i=1}^{n-1} (1 - 2\lambda \delta t_i) |v^n|_V^2. \quad (1.67)$$

Proposition 1.7 Assume that u^0 belong to V , and that $\lambda \delta t \leq \alpha$ as in Lemma 1.2. The following estimate holds for the indicator η_n , $2 \leq n \leq N$,

$$\eta_n \leq c \left(\begin{array}{l} [[u^n - u_h^n]] + \sqrt{\rho_{\delta t}} [[u^{n-1} - u_h^{n-1}]] \\ + \frac{e^{-\lambda t_{n-1}}}{\underline{\sigma}} (\|\frac{\partial}{\partial t}(u - u_{\delta t})\|_{L^2(t_{n-1}, t_n; V')} + \|u - u_{\delta t}\|_{L^2(t_{n-1}, t_n; V)}) \\ + (\frac{L}{\underline{\sigma}^2} (\max(1, \rho_{\delta t}))^{\frac{1}{2}} + \frac{\lambda \mu}{\underline{\sigma}^2}) \delta t_n \|u^0\| \end{array} \right), \quad (1.68)$$

and

$$\eta_1 \leq c \left(\begin{array}{l} [[u^1 - u_h^1]] + \frac{1}{\underline{\sigma}} (\|\frac{\partial}{\partial t}(u - u_{\delta t})\|_{L^2(0, t_1; V')} + \|u - u_{\delta t}\|_{L^2(0, t_1; V)}) \\ + \frac{L+\lambda\mu}{\underline{\sigma}^2} \delta t_1 \|u^0\| + \frac{L}{\underline{\sigma}} (\delta t_1)^{\frac{3}{2}} |u^0|_V \end{array} \right), \quad (1.69)$$

where c is a positive constant.

The most important property of estimate (1.68) is that, up to the last term which depends on the data, all the terms in the right hand side of (1.68) are local in time. More precisely, they involve the solution in the interval $[t_{n-1}, t_n]$.

We need to define a few notations before stating the upper bound result for $\eta_{n,\omega}$.

For $\omega \in \mathcal{T}_{n,h}$, let K_ω be the union of ω and the element that shares a node with ω , and $V_0(K_\omega)$ be the closure of $\mathcal{D}(K_\omega)$ in $V(K_\omega) = \{v \in L^2(K_\omega); S \frac{\partial v}{\partial S} \in L^2(K_\omega)\}$ endowed with the norm $\|v\|_{V(K_\omega)} = (\int_{K_\omega} v^2(S) + S^2(\frac{\partial v}{\partial S}(S))^2)^{\frac{1}{2}}$. We also define $\|v\|_{V_0(K_\omega)} = (\int_{K_\omega} S^2(\frac{\partial v}{\partial S}(S))^2)^{\frac{1}{2}}$, for $v \in V_0(K_\omega)$. We denote by $V'_0(K_\omega)$ the dual space of $V_0(K_\omega)$, endowed with dual norm. We also need the assumption that the meshes do not vary too much between two time steps:

Assumption 1.5 For $n = 1, \dots, N$, there exists a family of grids $(\mathcal{T}_{n,h}^*)_h$ with property (1.60) such that for all h and n each element of $\mathcal{T}_{n,h}$ and of $\mathcal{T}_{n-1,h}$ is the union of at most s elements of $\mathcal{T}_{n,h}^*$, (where s is bounded independently of h and n).

Proposition 1.8 Under Assumption 1.5, the following estimate holds for the indicator $\eta_{n,\omega}$ defined in (1.65), for all $\omega \in \mathcal{T}_{n,h}$, $1 \leq n \leq N$,

$$\eta_{n,\omega} \leq C \left(\left\| \frac{u^{n-1} - u_h^{n-1} - u^n + u_h^n}{\delta t_n} \right\|_{V'_0(K_\omega)} + \mu \left\| S \frac{\partial(u^n - u_h^n)}{\partial S} \right\|_{L^2(K_\omega)} \right). \quad (1.70)$$

1.3.5 Computation of the Bounds

To compute $\rho_{\delta t}$ and $\eta_{n,\omega}$ we may do as follows

```
%{verbatim}
void errorindic(Option& p, double*
rho, double** etamw){
    int nT = p.msh->nT;
    double sigma_m=p.sigma[0][0];
    for(int k=1;k<nT;k++){
        for(int i=0;i<p.msh->nX[k];i++)
            if(sigma_m>p.sigma[k][i]) sigma_m= p.sigma[k][i];
        rho[k]=0;
        double aux = 0.5*(p.msh->t[k]-p.msh->t[k-1])*sqr(sigma_m);
        for(int i=1;i<p.msh->nX[k];i++){
            rho[k] += aux*(
                p.msh->x[k][i]*sqr((p.u[k][i]-p.u[k-1][i]-p.u[k][i-1]+p.u[k-1][i-1])
                /(p.msh->x[k][i]-p.msh->x[k][i-1])) + sqr(p.u[k][i]-p.u[k-1][i]));
            etamw[k][i] = fabs((p.u[k][i]-p.u[k-1][i])/(p.msh->t[k]-p.msh->t[k-1]))
                *(p.msh->x[k][i]-p.msh->x[k][i-1])/p.msh->x[k][i];
        }
    }
}

%} verbatim
```

The graphs below compare these two indicators with the true error.

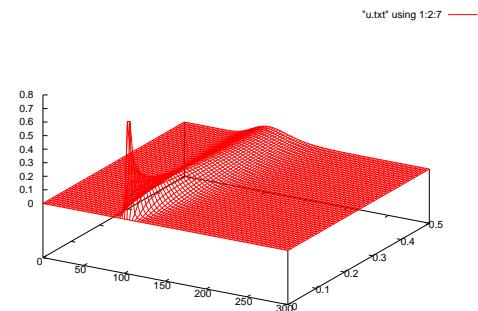
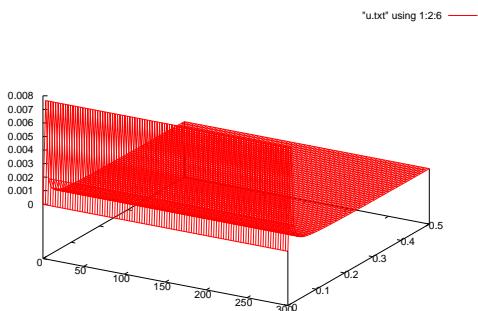
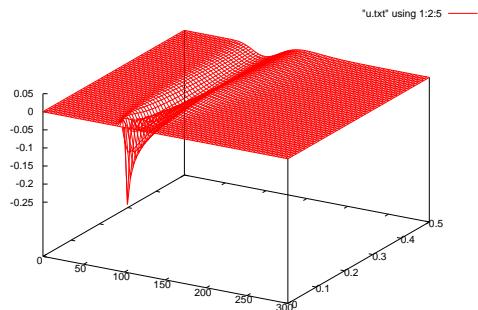


Figure 1.2: The first graph displays the error between the computed solution on a uniform mesh and one computed with Black-Scholes formula. The second graph show $\rho_{\delta t}$; the third graphs shows $\eta_{n,\omega}$ as a function of x and t . The parameters are : $K = 100$, $T = 0.5$, $r = 0$, $\sigma = 0.3$, 50 time steps and 100 mesh points.

Chapter 2

Multidimensional Partial Differential Equations For Option Pricing

2.1 European Basket Options

We consider d risky assets whose prices at time t are called $S_{i,t}$, $i = 1, \dots, d$. We assume that for all i , $1 \leq i \leq d$, $S_{i,t}$ satisfies the stochastic differential equation

$$dS_{i,t} = S_{i,t} (\mu_i dt + \sigma_i dW_{i,t}). \quad (2.1)$$

Here

- $(W_{i,t})$, $1 \leq i \leq d$, are possibly correlated standard Brownian motions on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We call $\rho_{i,j}$ the correlation factor of $(W_{i,t})$ and $(W_{j,t})$. We have $-1 \leq \rho_{i,j} \leq 1$. Of course $\rho_{i,i} = 1$.
- the volatilities σ_i , $1 \leq i \leq d$, are positive constants.

The Black-Scholes model assumes the existence of a risk-free asset whose interest rate r may be supposed constant for simplicity, yet, it is possible to generalize what follows to the case when the volatilities are sufficiently regular functions of t and of the prices $S_{j,t}$, $j = 1, \dots, d$ and when r is a nonnegative function of t .

In what follows, the notation \mathbf{S} is used for the vector (S_1, \dots, S_d) . Consider a European option on a basket containing the above mentioned d risky assets, whose maturity is T and whose payoff function is $\mathbf{S} \mapsto P_o(\mathbf{S})$. As for an option on a single asset, it is possible to find a risk neutral probability \mathbb{P}^* under which the price of the option at time t is

$$P_t = e^{-r(T-t)} \mathbb{E}^*(P_o(S_{1,T}, \dots, S_{d,T}) | F_t). \quad (2.2)$$

The payoff functions actually used in finance may be quite complicated. Among the simplest ones, we can list

- the payoff is a function of a weighted sum of the assets' prices:
 - call option on a weighted sum: $P_o(\mathbf{S}) = \left(\sum_{i=1}^d \alpha_i S_i - K \right)^+$.
 - put option on a weighted sum: $P_o(\mathbf{S}) = \left(K - \sum_{i=1}^d \alpha_i S_i \right)^+$.

Calling P_t the price of the put option and C_t the price of the call option, the put-call parity $C_t - P_t = \sum_{i=1}^d \alpha_i S_{i,t} - K e^{-r(T-t)}$ can be proved either by using (2.2) or by arguments on the partial differential equations used for pricing the options, see § 2.1.1.

- the payoff is a function of $\max_{i=1,\dots,d} S_i$: these options are called *best-of* options.
 - best-of call option: $P_o(\mathbf{S}) = (\max_{i=1,\dots,d} S_i - K)^+$.
 - best-of put option: $P_o(\mathbf{S}) = (K - \max_{i=1,\dots,d} S_i)^+$.

In contrast with the previous case, there is not put-call parity for these two options. Payoff functions depending on $\min_{i=1,\dots,d} S_i$ are used as well.

2.1.1 The Partial Differential Equation

Definition 2.1 Let Ω be an open subset of \mathbb{R}^d . A function $f : \Omega \times (0, T) \rightarrow \mathbb{R}$, continuous and such that its partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial S_i}$ and $\frac{\partial^2 f}{\partial S_i \partial S_j}$, $i, j = 1, \dots, d$, exist and are continuous on $\Omega \times (0, T)$ is said to belong to the class $\mathcal{C}^{2,1}(\Omega \times (0, T))$. If furthermore f and the above mentioned partial derivatives have continuous extensions in $\Omega \times [0, T]$, it is said that $f \in \mathcal{C}^{2,1}(\Omega \times [0, T])$.

It is possible to relate the option's price to the solution of a parabolic partial differential equation with $1 + d$ variables. The partial differential operator appears naturally in the following result:

Proposition 2.1 Call L the partial differential operator with variable coefficients:

$$Lf = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \Xi_{i,j} S_i S_j \frac{\partial^2 f}{\partial S_i \partial S_j} + r \sum_{i=1}^d S_i \frac{\partial f}{\partial S_i}, \quad (2.3)$$

with

$$\Xi_{i,j} = \sigma_i \sigma_j \rho_{i,j}. \quad (2.4)$$

For any function $u : (\mathbf{S}, t) \mapsto u(\mathbf{S}, t)$, $u \in \mathcal{C}^{2,1}(\mathbb{R}_+^d \times [0, T])$ and such that $|S_i \frac{\partial u}{\partial S_i}| \leq C(1 + |S_i|)$, $i = 1, \dots, d$, with C independent of t , the process

$$M_t = e^{-rt} u(S_{1,t}, \dots, S_{d,t}, t) - \int_0^t e^{-r\tau} \left(\frac{\partial u}{\partial t} + Lu - ru \right) (S_{1,\tau}, \dots, S_{d,\tau}, \tau) d\tau$$

is a martingale under \mathbb{P}^* . The partial differential operator L is called the infinitesimal generator of the Markov family $(S_{1,t}, \dots, S_{d,t})$.

Proof. The proof uses the multidimensional Ito's formula. See [69, 74] ■

Theorem 2.1 Consider a continuous function $P \in \mathcal{C}^{2,1}(\mathbb{R}_+^d \times [0, T))$, such that $|S_i \frac{\partial P}{\partial S_i}| \leq C(1 + S_i)$ with C independent of t . Assume that P satisfies

$$\left(\frac{\partial P}{\partial t} + LP - rP \right) (\mathbf{S}, t) = 0, \quad t < T, \quad \mathbf{S} \in \mathbb{R}_+^d \quad (2.5)$$

and

$$P(\mathbf{S}, T) = P_o(\mathbf{S}), \quad \mathbf{S} \in \mathbb{R}_+^d, \quad (2.6)$$

then the price of the European option given by (2.2) satisfies

$$P_t = P(S_{1,t}, \dots, S_{d,t}, t). \quad (2.7)$$

Remark 2.1 All the preceding results can be generalized to the case when r is a bounded and continuous function on $[0, T]$, and when the drifts and volatilities are functions of the variables \mathbf{S} and t such that

1. For all $i = 1, \dots, d$, $(\mathbf{S}, t) \mapsto \mu_i(\mathbf{S}, t)$ and $(\mathbf{S}, t) \mapsto \sigma_i(\mathbf{S}, t)$ are bounded and continuous functions.
2. For all $i = 1, \dots, d$, $\mathbf{S} \mapsto \mu_i(\mathbf{S}, t)S_i$ and $\mathbf{S} \mapsto \sigma_i(\mathbf{S}, t)S_i$ are Lipschitz continuous with a Lipschitz constant C independent of t .

In this case, $\Xi_{i,j}$ and r in (2.3) are functions instead of parameters: $\Xi_{i,j}(\mathbf{S}, t) = \rho_{i,j}\sigma_i(\mathbf{S}, t)\sigma_j(\mathbf{S}, t)$ and $r = r(t)$.

A First Change of Variables It is possible to make the change of variable $x_i = \log(S_i)$, $i = 1, \dots, d$; calling \mathbf{x} the vector (x_1, \dots, x_d) and \tilde{P} the function: $\tilde{P}(\mathbf{x}, t) = P(\mathbf{S}, t)$, the final value problem (2.5) (2.6) becomes

$$\begin{aligned} \left(\frac{\partial \tilde{P}}{\partial t} + \tilde{L}\tilde{P} - r\tilde{P} \right) (\mathbf{x}, t) &= 0, \quad t < T, \quad \mathbf{x} \in \mathbb{R}^d, \\ \tilde{P}(\mathbf{x}, T) &= P_o(e^{x_1}, \dots, e^{x_d}), \quad \mathbf{x} \in \mathbb{R}^d, \end{aligned} \quad (2.8)$$

where

$$\tilde{L}f = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \Xi_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^d \left(r - \frac{\sigma_i^2}{2} \right) \frac{\partial f}{\partial x_i}. \quad (2.9)$$

One sees that when the volatilities and the interest rate are constant, the operator \tilde{L} has constant coefficients. In that case, calling \mathbf{v} the vector of \mathbb{R}^d such that $v_i = \sigma_i^2/2 - r$ and $\bar{P}(\mathbf{y}, t) = e^{rt} \tilde{P}(\mathbf{y} + t\mathbf{v}, T - t)$, we have

$$\frac{\partial \bar{P}}{\partial t} - \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \Xi_{i,j} \frac{\partial^2 \bar{P}}{\partial y_i \partial y_j} = 0, \quad 0 < t \leq T, \quad \mathbf{y} \in \mathbb{R}^d, \quad (2.10)$$

$$\bar{P}(\mathbf{y}, 0) = P_o(e^{y_1}, \dots, e^{y_d}), \quad \mathbf{y} \in \mathbb{R}^d,$$

We make the following assumption: there exists a unitary matrix Θ such that

$$\begin{aligned}\Theta^T \Xi \Theta &= A \text{ is a diagonal positive matrix:} \\ A &= \text{Diag}(a_1, \dots, a_d).\end{aligned}\tag{2.11}$$

Consider the change of variables

$$\mathbf{z} = \sqrt{2}A^{-\frac{1}{2}}\Theta^T \mathbf{y},$$

then the function $\overline{\overline{P}}(z, t) = \overline{P}(y, t)$ satisfies the heat equation

$$\frac{\partial \overline{\overline{P}}}{\partial t} - \Delta \overline{\overline{P}} = 0, \quad 0 < t \leq T, \quad \mathbf{z} \in \mathbb{R}^d.\tag{2.12}$$

From this, calling G the fundamental solution of the heat equation,

$$G(z, t) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|z|^2}{4t}},$$

we have

$$\begin{aligned}e^{r(T-t)} P(\mathbf{S}, t) &= \\ \sqrt{2^d \prod_{i=1}^d a_i} \int_{\mathbb{R}^d} G\left(\sqrt{2}A^{-\frac{1}{2}}\Theta^T (\log(\mathbf{S}) - z - (T-t)v), T-t\right) P_\circ(e^{z_1}, \dots, e^{z_d}) d\mathbf{z},\end{aligned}\tag{2.13}$$

where $\log(\mathbf{S}) = (\log(S_1), \dots, \log(S_d))^T$.

Assume that P_\circ has a compact support contained in the rectangular domain $[0, \overline{S}_\circ]^d$, with $\overline{S}_\circ > 1$, and that there exists a constant K such that $0 \leq P_\circ \leq K$. The identity (2.13) gives information on the decay of $P(\mathbf{S}, t)$ when $t \in [0, T]$ and $\max_{i=1, \dots, d} S_i$ tends to ∞ :

let us introduce

$$\bar{a} = \max(a_1, \dots, a_d),\tag{2.14}$$

and let us assume that $\max_{i=1, \dots, d} S_i \geq \overline{S} \gg 1$, where \overline{S} is a positive number. Let $\mathbf{z} \in \mathbb{R}^d$ be such that $(e^{z_1}, \dots, e^{z_d})$ belongs to the support of P_\circ . Then $-\infty < z_i \leq \log(\overline{S}_\circ)$, and

$$\begin{aligned}P(\mathbf{S}, t) &\leq K \sqrt{2^d \prod_{i=1}^d a_i} \prod_{i=1}^d \int_{-\infty}^{\log(\overline{S}_\circ)} \frac{e^{-\frac{2(\log(S_i) - z_i - (T-t)v_i)^2}{4\bar{a}(T-t)}}}{(4\pi(T-t))^{\frac{1}{2}}} dz_i \\ &\leq K \bar{a}^d \prod_{i=1}^d \int_{\frac{\log(S_i) - \log(\overline{S}_\circ) - (T-t)v_i}{\sqrt{\bar{a}(T-t)}}}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du\end{aligned}$$

Let us assume that

$$\log(\overline{S}) > \log(\overline{S}_\circ) + T|\mathbf{v}|_\infty.\tag{2.15}$$

From the assumptions on \mathbf{S} , there exists at least one index ℓ , $1 \leq \ell \leq d$, such that $S_\ell \geq \bar{S}$; therefore

$$\begin{aligned} P(\mathbf{S}, t) &\leq K\bar{a}^d \int_{\frac{\log(\bar{S}_0) + (t-t)v_\ell - \log(S_\ell)}{\sqrt{\bar{a}(T-t)}}}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\ &\leq K\bar{a}^d \int_{\frac{\log(\bar{S}) - \log(\bar{S}_0) - T|\mathbf{v}|_\infty}{\sqrt{\bar{a}T}}}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\ &\leq \frac{K}{2}\bar{a}^d e^{-\frac{(\log(\bar{S}/\bar{S}_0) - T|\mathbf{v}|_\infty)^2}{2\bar{a}T}}. \end{aligned} \quad (2.16)$$

Solutions of (2.5) (2.6) In the previous paragraph, we have seen that if the coefficients are constant and if assumption (2.11) is satisfied, then (2.5) (2.6) has a solution given by (2.13).

In this paragraph, we give a classical existence and uniqueness result for the final value problem (2.5), (2.6) in the general case when $r = r(t)$ and $\sigma_i = \sigma_i(\mathbf{S}, t)$, $i = 1, \dots, d$. It is necessary to restrict the growth of the solutions when $S_i \rightarrow 0$ or $S_i \rightarrow +\infty$. Here, we will impose that the solution is bounded but this restriction can be relaxed (for example, depending on P_0 , one can look for solutions with linear growth as $S_i \rightarrow +\infty$).

Theorem 2.2 *Under the following regularity assumptions on the coefficients and the final value:*

1. *for $i, j = 0, \dots, d$, the real valued function $\tilde{\Xi}_{i,j}$ defined on $\mathbb{R}^d \times [0, T]$*

$$\tilde{\Xi}_{i,j}(x_1, \dots, x_d, t) = \Xi_{i,j}(e^{x_1}, \dots, e^{x_d}, t)$$

belongs to $\mathcal{C}^{\alpha, \alpha/2}(\mathbb{R}^d \times [0, T])$,

2. *the function $t \mapsto r(t)$ belongs to $\mathcal{C}^{\alpha/2}([0, T])$,*

3. *the function \tilde{P}_0 defined on \mathbb{R}^d by $\tilde{P}_0(x_1, \dots, x_d) = P_0(e^{x_1}, \dots, e^{x_d})$ is such that $\tilde{P}_0, \frac{\partial \tilde{P}_0}{\partial x_i}, \frac{\partial^2 \tilde{P}_0}{\partial x_i \partial x_j}$, $i, j = 1, \dots, d$ belong to $\mathcal{C}^\alpha(\mathbb{R}^d)$,*

and under the following ellipticity assumption:

there exists a positive constant c such that, for all $(x_1, \dots, x_d) \in \mathbb{R}^d$, $t \in [0, T]$, $\xi \in \mathbb{R}^d$,

$$\sum_{i=1}^d \sum_{j=1}^d \tilde{\Xi}_{i,j}(x_1, \dots, x_d, t) \xi_i \xi_j \geq c |\xi|^2,$$

the final value problem (2.8) has a unique solution \tilde{P} such that the functions \tilde{P} , $\frac{\partial \tilde{P}}{\partial x_i}$, $\frac{\partial \tilde{P}}{\partial t}$, $\frac{\partial^2 \tilde{P}}{\partial x_i \partial x_j}$, $i, j = 1, \dots, d$, belong to $\mathcal{C}^{\alpha, \alpha/2}(\mathbb{R}^d \times [0, T])$.

Under the previous assumptions except the one on \tilde{P}_0 and assuming that \tilde{P}_0 is a bounded function, (2.8) has a unique solution $\tilde{P} \in \mathcal{C}^{2,1}(\mathbb{R}^d \times [0, T])$ such that for all $\tau < T$, the functions \tilde{P} , $\frac{\partial \tilde{P}}{\partial x_i}$, $\frac{\partial \tilde{P}}{\partial t}$, $\frac{\partial^2 \tilde{P}}{\partial x_i \partial x_j}$ belong to $\mathcal{C}^{\alpha, \alpha/2}(\mathbb{R}^d \times [0, \tau])$.

This theorem is proved in [73], see also [45, 72].

A Second Change of Variables For basket options with a payoff depending on the weighted sum $\sum_{i=1}^d \alpha_i S_i$, the following change of variables has been proposed by Reisinger[95, 97]:

$$\begin{aligned} y_1 &= \sum_{i=1}^d \alpha_i S_i \in \mathbb{R}_+, \\ y_i &= \frac{\alpha_{i-1} S_{i-1}}{\sum_{k=i-1}^d \alpha_k S_k} \in [0, 1], \quad 2 \leq i \leq d. \end{aligned} \tag{2.17}$$

This mapping is a C^1 diffeomorphism from \mathbb{R}_+^d onto $\mathbb{R}_+ \times (0, 1)^{d-1}$. The inverse change of variables is

$$\begin{aligned} S_1 &= y_1 y_2 / \alpha_1, \\ S_i &= y_1 y_{i+1} / \alpha_i \prod_{k=2}^{i-1} (1 - y_k), \quad 2 \leq i \leq d-1, \\ S_d &= y_1 / \alpha_d \prod_{k=2}^d (1 - y_k). \end{aligned} \tag{2.18}$$

After some calculus, one sees that

$$\begin{aligned} S_i \frac{\partial}{\partial S_i} &= y_1 f_{1,i}(y) \frac{\partial}{\partial y_1} - \sum_{j=2}^d y_j (1 - y_j) f_{j,i}(y) \frac{\partial}{\partial y_j} + y_{i+1} (1 - y_{i+1}) \frac{\partial}{\partial y_{i+1}}, \quad 1 \leq i < d, \\ S_d \frac{\partial}{\partial S_d} &= y_1 f_{1,d}(y) \frac{\partial}{\partial y_1} - \sum_{j=2}^d y_j (1 - y_j) f_{j,d}(y) \frac{\partial}{\partial y_j}, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} f_{j,i}(y) &= y_{i+1} \prod_{k=j+1}^i (1 - y_k), \quad j < i < d, \\ f_{j,d}(y) &= \prod_{k=j+1}^d (1 - y_k), \quad j < d, \\ f_{i,i}(y) &= y_{i+1}, \quad i < d, \\ f_{d,d}(y) &= 1, \\ f_{j,i}(y) &= 0, \quad i < j. \end{aligned} \tag{2.20}$$

The identities

$$\sum_{k=1}^d f_{i,k} = \sum_{k=i}^d f_{i,k} = 1, \quad \forall i = 1, \dots, d, \tag{2.21}$$

are true as well. One can verify that

$$\sum_{i=1}^d S_i \frac{\partial}{\partial S_i} = \frac{\partial}{\partial y_1}. \tag{2.22}$$

Playing with the identities (2.19), (2.20), (2.21) and (2.22), one obtains that in the new variables y_1, \dots, y_d the partial differential equation (2.5) becomes

$$\left(\frac{\partial \check{P}}{\partial t} + \check{L} \check{P} - r \check{P} \right) (y_1, \dots, y_d, t) = 0, \quad t < T, \quad (y_1, \dots, y_d) \in \mathbb{R}_+ \times (0, 1)^{d-1}, \tag{2.23}$$

where

$$\check{L}f = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \check{\Xi}_{i,j}(y) y_i y_j \frac{\partial^2 f}{\partial y_i \partial y_j} + \sum_{i=1}^d \beta_i(y) y_i \frac{\partial f}{\partial y_i}, \quad (2.24)$$

with

$$\begin{aligned} \check{\Xi}_{1,1}(y) &= \sum_{k,l=1}^d \Xi_{k,l} f_{1,k}(y) f_{1,l}(y), \\ \check{\Xi}_{1,j}(y) &= \check{\Xi}_{j,1}(y) = (1 - y_j) \sum_{k,l=1}^d (\Xi_{k,j-1} - \Xi_{k,l}) f_{1,k}(y) f_{j,l}(y), \quad 1 < j \leq d \\ \check{\Xi}_{i,j}(y) &= (1 - y_i)(1 - y_j) \sum_{k,l=1}^d \left(\begin{pmatrix} \Xi_{k,l} + \Xi_{i-1,j-1} \\ -\Xi_{i-1,l} - \Xi_{k,j-1} \end{pmatrix} f_{i,k}(y) f_{j,l}(y) \right), \quad 1 < i, j \leq d, \end{aligned} \quad (2.25)$$

and

$$\begin{aligned} \beta_1(y) &= r, \\ \beta_i(y) &= (1 - y_i) \sum_{k,l=1}^d \left(\begin{pmatrix} 2(1 - y_i)\Xi_{k,l} - 2y_i\Xi_{i-1,i-1} \\ +(2y_i - 1)(\Xi_{i-1,l} + \Xi_{k,i-1}) \end{pmatrix} f_{i,k}(y) f_{i,l}(y) \right), \quad 1 < i \leq d, \end{aligned} \quad (2.26)$$

No boundary conditions are necessary on $y_1 = 0$ and on $y_i = 0$ or $1, 2 \leq i \leq d$, because the partial differential equation is degenerate on these parts of the domain's boundary. The same remark can be made for the boundaries $y_i = 1, i = 2, \dots, d$, because $\check{\Xi}_{i,j}$ and $\check{\Xi}_{i,j}$ vanish, for $j = 1, \dots, d$.

The final condition becomes

$$\check{P}(y_1, \dots, y_d, T) = \check{P}_o(y_1).$$

For numerical computations (with d small enough), the main interest of such a change of variables is the following: if the payoff function has a singularity on the hyperplane $\sum_{i=1}^d \alpha_i S_i = K$, then one needs to refine the mesh in the neighborhood of this hyperplane. This is incompatible with the use of Cartesian grids in the S variables. After the change of variables $S \mapsto y$, one can use a Cartesian grid in the y_1, \dots, y_d variables, with a refinement in the neighborhood of $y_1 = K$. The price to pay is that the operator in the PDE has variable coefficients with rather complicated expressions. Of course, a more general and robust alternative is to use finite elements with adaptive mesh refinement.

Consequences of the Maximum Principle The maximum principle may be applied to (2.5). This has many consequences:

the *super-replication principle* holds for basket options: take two European put options on the above mentioned basket with the same maturity and different payoff functions P_o and Q_o . Call P and Q their respective pricing functions, which both satisfy (2.5). We assume that P_o and Q_o have reasonable growth at infinity, i.e., $\lim_{|S| \rightarrow \infty} \max(P_o(S), Q_o(S)) e^{-\epsilon \log^2(|S|+2)} = 0$ for all $\epsilon > 0$. This

is always satisfied in practice. Applying the maximum principle to (2.5), one sees that if $P_o(\mathbf{S}) \leq Q_o(\mathbf{S})$ for all \mathbf{S} , then for all $t \leq T$ and \mathbf{S} , $P(\mathbf{S}, t) \leq Q(\mathbf{S}, t)$. For example, if P_t is the price of the best-of put option with maturity T and payoff function $(K - \max_{j=1,\dots,d} S_j)^+$, then for all i , $1 \leq i \leq d$, and for all time $t < T$, we have

$$\tilde{Q}_{i,t} \geq P_t \geq Q_t, \quad (2.27)$$

where Q_t is the price of the put option on the weighted sum with the same maturity T and payoff function $(K - \sum_{j=1}^d S_j)^+$, and where $\tilde{Q}_{i,t}$ is the price of the put option on the single asset indexed by i , with maturity and with payoff function $(K - S_i)^+$.

The maximum principle also gives information on the behavior of the pricing function at the boundary of \mathbb{R}_+^d where the operator is degenerate. It can be proved that, for a put option on a weighted sum, (i.e. $P_o(\mathbf{S}) = (K - \sum_{i=1}^d \alpha_i S_i)^+$), the pricing function satisfies

$$P(S_1, \dots, S_{i-1}, 0, S_{i+1}, \dots, S_d, t) = Q(S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_d, t),$$

where Q is the pricing function of the put option on the basket containing all the previous assets but the one indexed by i , with payoff function

$$Q_o(S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_d) = \left(K - \sum_{j=1, j \neq i}^d \alpha_j S_j \right)^+.$$

The same is true for the best-of put option with payoff function $(P_o(\mathbf{S}) = (K - \max_{i=1,\dots,d} S_i)^+)$: we have

$$P(S_1, \dots, S_{i-1}, 0, S_{i+1}, \dots, S_d, t) = Q(S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_d, t),$$

where Q is the pricing function of the best-of put option on the basket containing all the previous assets but the one indexed by i , with payoff function

$$Q_o(S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_d) = \left(K - \max_{j=1, j \neq i}^d S_j \right)^+.$$

2.1.2 Variational Formulation

Here, we assume that the volatilities are functions of the variables \mathbf{S}, t and that r is a function of time.

In order to obtain both global energy estimates and a nice framework for studying more complex situations (i.e. for American options on the same basket), we aim at finding a variational formulation for (2.5) (2.6), assuming that the payoff function is square integrable. For more general payoff functions, variational formulation can be proposed as well with suitable weighting functions as $|\mathbf{S}| \rightarrow \infty$. Variational formulations are also the cornerstone for the finite element method,

see § 2.2.2 below.

The change of variable $T - t \rightarrow t$ yields a forward in time parabolic initial value problem:

$$\left(\frac{\partial P}{\partial t} - LP + rP \right) (\mathbf{S}, t) = 0, \quad 0 < t \leq T, \quad \mathbf{S} \in \mathbb{R}_+^d, \\ P(\mathbf{S}, 0) = P_0(\mathbf{S}), \quad \mathbf{S} \in \mathbb{R}_+^d, \quad (2.28)$$

where L is given by (2.3) with $\Xi_{i,j}(\mathbf{S}, t) = \rho_{i,j}\sigma_i(\mathbf{S}, t)\sigma_j(\mathbf{S}, t)$. Assuming that the coefficients are regular enough (this will be made clear below), we write the operator in divergence form:

$$Lu = \frac{1}{2} \sum_{i=1}^d \frac{\partial}{\partial S_i} \left(\sum_{j=1}^d \Xi_{i,j} S_i S_j \frac{\partial u}{\partial S_j} \right) + \sum_{j=1}^d \left(r(t) S_j - \frac{1}{2} \sum_{i=1}^d \frac{\partial}{\partial S_i} (\Xi_{i,j} S_i S_j) \right) \frac{\partial u}{\partial S_j}. \quad (2.29)$$

Multiplying $-Lu + ru$ by a test function v , integrating on \mathbb{R}_+^d and performing suitable integrations by part, one obtains the bilinear form

$$a_t(u, v) = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_{\mathbb{R}_+^d} \Xi_{i,j} S_i S_j \frac{\partial u}{\partial S_j} \frac{\partial v}{\partial S_i} \\ - \sum_{j=1}^d \int_{\mathbb{R}_+^d} \left(r(t) S_j - \frac{1}{2} \sum_{i=1}^d \frac{\partial}{\partial S_i} (\Xi_{i,j} S_i S_j) \right) \frac{\partial u}{\partial S_j} v + r(t) \int_{\mathbb{R}_+^d} u v. \quad (2.30)$$

We introduce the Hilbert space

$$V = \left\{ v : v \in L^2(\mathbb{R}_+^d), S_i \frac{\partial v}{\partial S_i} \in L^2(\mathbb{R}_+^d), i = 1, \dots, d \right\}, \quad (2.31)$$

with the norm

$$\|v\|_V = \left(\|v\|_{L^2(\mathbb{R}_+^d)}^2 + \sum_{i=1}^d \|S_i \frac{\partial v}{\partial S_i}\|_{L^2(\mathbb{R}_+^d)}^2 \right)^{\frac{1}{2}}, \quad (2.32)$$

one can check the following properties

- The space $\mathcal{D}(\mathbb{R}_+^d)$ of smooth and compactly supported functions in \mathbb{R}_+^d is dense in V
- V is separable.
- The semi-norm $|.|_V$ defined by $|v|_V^2 = \sum_{i=1}^d \|S_i \frac{\partial v}{\partial S_i}\|_{L^2(\mathbb{R}_+^d)}^2$ is in fact a norm on V equivalent to $\|.\|_V$ because $\|v\|_{L^2(\mathbb{R}_+^d)} \leq 2|v|_V$.

We make the following assumptions

- The functions $\Xi_{i,j}$, $1 \leq i, j \leq d$, are bounded by a positive constant $\bar{\sigma}^2$ independent of \mathbf{S} and t :

$$\|\Xi_{i,j}\|_{L^\infty(\mathbb{R}_+^d)} \leq \bar{\sigma}^2. \quad (2.33)$$

- There exists a positive constant $\underline{\sigma}$ such that for all $q \in \mathbb{R}^d$,

$$\sum_{i=1}^d \sum_{j=1}^d \Xi_{i,j} q_i q_j \geq \underline{\sigma}^2 |q|^2. \quad (2.34)$$

- There exists a positive constant M such that, for $i = 1, \dots, d$,

$$\left\| \sum_{i=1}^d \frac{\partial}{\partial S_i} (\Xi_{i,j} S_i) \right\|_{L^\infty(\mathbb{R}_+^d)} \leq M. \quad (2.35)$$

- The function r is nonnegative and bounded by a constant.

Lemma 2.1 *With the assumptions above, the bilinear form a_t is continuous on $V \times V$ and there exists a constant \bar{c} independent of t such that, for any $v, w \in V$,*

$$a_t(v, w) \leq \bar{c} |v|_V |w|_V. \quad (2.36)$$

Moreover, there is a uniform Gårding's inequality for a_t : there exist a positive constant \underline{c} and a nonnegative constant λ such that, for any $v \in V$,

$$a_t(v, v) \geq \underline{c} |v|_V^2 - \lambda \|v\|_{L^2(\mathbb{R}_+^d)}^2. \quad (2.37)$$

From Lemma 2.1, we deduce the existence and uniqueness of a weak solution to the initial value problem (2.28).

Theorem 2.3 *Under the assumptions above, for any $P_\circ \in L^2(\mathbb{R}_+^d)$, there exists a unique P in $L^2(0, T; V) \cap C^0([0, T]; L^2(\mathbb{R}_+^d))$, with $\frac{\partial P}{\partial t} \in L^2(0, T; V')$ such that, for any smooth function $\phi \in \mathcal{D}(0, T)$, for any $v \in V$,*

$$-\int_0^T \phi'(t) \left(\int_{\mathbb{R}_+^d} P(t)v \right) dt + \int_0^T \phi(t) a_t(P, v) dt = 0 \quad (2.38)$$

and

$$P(t = 0) = P_\circ. \quad (2.39)$$

The mapping $P_\circ \mapsto P$ is continuous from $L^2(\mathbb{R}_+^d)$ to $L^2(0, T; V) \cap C^0([0, T]; L^2(\mathbb{R}_+^d))$, and we have the energy estimate

$$e^{-2\lambda t} \|P(t)\|_{L^2(\mathbb{R}_+^d)}^2 + 2\underline{c} \int_0^t e^{-2\lambda\tau} |P(\tau)|_V^2 d\tau \leq \|P_\circ\|_{L^2(\mathbb{R}_+^d)}^2. \quad (2.40)$$

Naturally, there may be barrier options on baskets. For a barrier independent of time, pricing the option then amounts to solving the boundary value problem

$$\begin{aligned} & \left(\frac{\partial P}{\partial t} - LP + rP \right) (\mathbf{S}, t) = 0, \quad 0 < t \leq T, \quad \mathbf{S} \in \Omega, \\ & P(\mathbf{S}, t) = 0, \quad 0 < t \leq T, \quad \mathbf{S} \in \partial\Omega \text{ s.t. } S_i > 0, i = 1, \dots, d, \\ & P(\mathbf{S}, 0) = P_\circ(\mathbf{S}), \quad \mathbf{S} \in \Omega, \end{aligned} \quad (2.41)$$

for a domain Ω of \mathbb{R}_+^d . We restrict ourselves to domains whose boundaries are locally the graph of Lipschitz continuous functions. Then, the Sobolev space to work with is the closure of $\mathcal{D}(\Omega)$ in the space $\{v \in L^2(\Omega); S_i \frac{\partial v}{\partial S_i} \in L^2(\Omega), i = 1, \dots, d\}$ equipped with the norm

$$\left(\|v\|_{L^2(\Omega)}^2 + \sum_{i=1}^d \|S_i \frac{\partial v}{\partial S_i}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

2.2 Numerical Methods for European Basket Options

2.2.1 Localization

Consider the initial value problem (2.28) for pricing the European basket option discussed in § 2.1. For computing a numerical approximation to P , one may

- truncate the domain in the variables \mathbf{S} : the pricing function will be computed for $S_i \in (0, \bar{S})$, with \bar{S} large enough
- impose some boundary condition at the artificial boundaries $S_i = \bar{S}$. Naturally, the choice of these boundary conditions has to depend on the payoff function P_\circ .

We introduce the rectangular domain $\Omega = (0, \bar{S})^d$ and Γ_0 the part of the boundary of Ω given by the following

$$\Gamma_0 = \{\mathbf{S} \in \partial\Omega; \max_{i=1, \dots, d} S_i = \bar{S}\}. \quad (2.42)$$

Let us assume that P_\circ has a compact support contained in the rectangular domain $[0, \bar{S}_\circ]^d$, and that there exists $K \in \mathbb{R}_+$ such that $0 \leq P_\circ(\mathbf{S}) \leq K$ for $\mathbf{S} \in \mathbb{R}_+^d$. This is the case for European put options on weighted sums or for best-of European put options. We choose \bar{S} such that $\bar{S} > \bar{S}_\circ$. From the estimate (2.16), a natural choice of boundary condition is $P = 0$ on Γ_0 . The new boundary value problem becomes

$$\begin{aligned} \left(\frac{\partial P}{\partial t} - LP + rP \right) (\mathbf{S}, t) &= 0, & 0 < t \leq T, \mathbf{S} \in \Omega, \\ P(\mathbf{S}) &= 0, & 0 < t \leq T, \mathbf{S} \in \Gamma_0, \\ P(\mathbf{S}, 0) &= P_\circ(\mathbf{S}), & \mathbf{S} \in \Omega, \end{aligned} \quad (2.43)$$

with L given by (2.3).

One can introduce a variational formulation for (2.43): the only modification to bring to the content of § 2.1.2 is the choice of the space V , which must take into account the change in the domain and the new boundary conditions. We introduce the Hilbert space

$$\tilde{V} = \left\{ v : v \in L^2(\Omega), S_i \frac{\partial v}{\partial S_i} \in L^2(\Omega), i = 1, \dots, d \right\}, \quad (2.44)$$

with the norm $\|v\|_{\tilde{V}} = \left(\|v\|_{L^2(\Omega)}^2 + \sum_{i=1}^d \|S_i \frac{\partial v}{\partial S_i}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$, and V as the completion in \tilde{V} of the space of smooth functions with compact support in Ω . It can be proved that there is a continuous trace operator from \tilde{V} to $L^2(\Gamma_0)$: for a function $v \in \tilde{V}$, we call $v|_{\Gamma_0} \in L^2(\Gamma_0)$ its trace on Γ_0 . The identity

$$V = \{v \in \tilde{V}; v|_{\Gamma_0} = 0\} \quad (2.45)$$

is a consequence of the continuity of the trace operator. Making the same assumptions on the volatilities σ_i and on r as in § 2.1.2, we introduce the bilinear form a_t on $V \times V$:

$$\begin{aligned} a_t(u, v) &= \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_{\Omega} \Xi_{i,j} S_i S_j \frac{\partial u}{\partial S_j} \frac{\partial v}{\partial S_i} \\ &\quad - \sum_{j=1}^d \int_{\Omega} \left(r(t) S_j - \frac{1}{2} \sum_{i=1}^d \frac{\partial}{\partial S_i} (\Xi_{i,j} S_i S_j) \right) \frac{\partial u}{\partial S_j} v + r(t) \int_{\Omega} u v. \end{aligned} \quad (2.46)$$

It can be proved that a_t is continuous on \tilde{V} and that there is a Gårding's inequality in \tilde{V} as in (2.37). The variational formulation for (2.43) is to find P in $L^2(0, T; V) \cap C^0([0, T]; L^2(\Omega))$ with $\frac{\partial P}{\partial t} \in L^2(0, T; V')$ such that, for any smooth function $\phi \in D(0, T)$, for any $v \in V$,

$$-\int_0^T \phi'(t) \left(\int_{\Omega} P(t) v \right) dt + \int_0^T \phi(t) a_t(P, v) dt = 0 \quad (2.47)$$

and

$$P(t = 0) = P_0. \quad (2.48)$$

This problem has a unique solution.

Localization error Of course, the artificial boundary conditions produce an error because the solution P_{exact} of (2.28) does not vanish on Γ_0 .

The maximum principle can be used for proving that

$$P(\mathbf{S}, t) < P_{\text{exact}}(\mathbf{S}, t), \quad \text{for } \mathbf{S} \in \Omega, \quad \text{and } 0 < t \leq T.$$

It can also be proved, at least for the above mentioned two examples of options, that the maximum of $P_{\text{exact}} - P$ in $\bar{\Omega} \times [0, T]$ is reached on $\Gamma_0 \times (0, T]$.

On the other hand, if the volatilities and the interest rate r are constant, we call \mathbf{v} the vector of \mathbb{R}^d whose components are $v_i = \sigma_i^2/2 - r$, and we define \bar{a} by (2.14), (2.11); if \bar{S} satisfies (2.15), then P_{exact} satisfies (2.16) for all $\mathbf{S} \in \Gamma_0$. Thus

$$\|P - P_{\text{exact}}\|_{L^\infty(\Omega \times (0, T))} \leq \|P_{\text{exact}}\|_{L^\infty(\Gamma_0 \times (0, T))} \leq \frac{K}{2} \bar{a}^d e^{-\frac{(\log(\bar{S}/\bar{S}_0) - T|\mathbf{v}|_\infty)^2}{2\bar{a}T}}. \quad (2.49)$$

We see that the error between P and P_{exact} can be made arbitrarily small by letting \bar{S} tend to infinity. In particular, the choice

$$\bar{S} \geq \bar{S}_0 \exp \left(T|\mathbf{v}|_\infty + \sqrt{2\bar{a}T \log(\frac{K\bar{a}^d}{\epsilon})} \right)$$

guarantees an error smaller than ϵ .

For nonconstant coefficients, obtaining such an accurate result is not as easy. Yet, the formula above may be used for a reasonable choice of \bar{S} .

For compactly supported payoff functions, the Neumann boundary conditions

$$\frac{\partial P}{\partial S_i} = 0, \quad \text{on } (\Gamma^0 \cap \{S_i = \bar{S}\})$$

can be used as well. Accurate error bounds may be obtained for put options on a weighted sum and for best-of put options if the coefficients are constant. The variational formulation with the Neumann boundary conditions is (2.47) (2.48) but now, $V = \tilde{V}$.

For a call option on the weighted sum $\sum_{i=1}^d \alpha_i S_i$, the following conditions can be used

- Dirichlet conditions: $P(\mathbf{S}, t) = \sum_{i=1}^d \alpha_i S_i - K e^{-rt}$ on $\Gamma^0 \times (0, T)$. With V given by (2.45), the variational formulation is to find P in $L^2(0, T; \tilde{V}) \cap \mathcal{C}^0([0, T]; L^2(\Omega))$ with $\frac{\partial P}{\partial t} \in L^2(0, T; V')$ such that, for any smooth function $\phi \in \mathcal{D}(0, T)$, for any $v \in V$,

$$-\int_0^T \phi'(t) \left(\int_{\Omega} P(t)v \right) dt + \int_0^T \phi(t) a_t(P, v) dt = 0, \quad (2.50)$$

$$P(t)|_{\Gamma^0} = \sum_{i=1}^d \alpha_i S_i - K e^{-rt} \text{ for a.e. } t, \text{ and (2.48).}$$

- Neumann conditions: $\sum_{j=1}^d \Xi_{i,j} S_j \frac{\partial P}{\partial S_j} = \sum_{j=1}^d \Xi_{i,j} S_j \alpha_j$ on $(\Gamma^0 \cap \{S_i = \bar{S}\}) \times (0, T)$. The variational formulation is to find P in $L^2(0, T; \tilde{V}) \cap \mathcal{C}^0([0, T]; L^2(\Omega))$, with $\frac{\partial P}{\partial t} \in L^2(0, T; V')$ such that, for any smooth function $\phi \in \mathcal{D}(0, T)$, for any $v \in \tilde{V}$,

$$\begin{aligned} & -\int_0^T \phi'(t) \left(\int_{\Omega} P(t)v \right) dt + \int_0^T \phi(t) a_t(P, v) dt \\ &= \frac{\bar{S}}{2} \int_0^T \phi(t) \left(\sum_{i=1}^d \int_{\Gamma^0 \cap \{S_i = \bar{S}\}} \left(\sum_{j=1}^d \Xi_{i,j} S_j \alpha_j \right) v \right) dt. \end{aligned} \quad (2.51)$$

and (2.48).

The error due to artificial boundary conditions can be accurately estimated in the case of constant coefficients, by using the previously obtained estimates for

the corresponding put options and the put-call parity.

For a best-of call option, finding reasonable boundary condition near the regions $S_i = S_j = \overline{S}$, $i \neq j$ is much more difficult. One may have to use an alternative option to artificial boundary conditions, *i.e.*, a change of variables which maps the unbounded domain to a bounded one: one obtains a new boundary value problem in a bounded domain. The partial differential equation becomes degenerate on the part of the boundary which is sent to infinity by the inverse mapping, thus no boundary condition is needed there. An example of such a program is given in § 2.4.2 below in the context of option pricing with stochastic volatility.

2.2.2 Finite Element Methods

Conforming FEM (Finite Element Methods) are numerical approximations closely linked to the theory of variational or weak formulations presented in § 2.1.2. The first finite element method can be attributed to R. Courant, [34].

Conforming finite element methods have the same framework in any dimension of space d : for a weak formulation posed in an infinite dimensional function space V , for example (2.51) (2.48), it consists of choosing a finite dimensional subspace V_h of V , for instance the space of continuous piecewise affine functions on a triangulation of Ω , and of solving the problem with test and trial functions in V_h instead of V . We speak of *conforming* methods because $V_h \subset V$. Non-conforming methods, *i.e.*, $V_h \not\subset V$ are possible too but we will not consider this topic here. In the simpler finite element methods, the construction of the space V_h is done as follows:

- The domain is partitioned into nonoverlapping cells (elements) whose shapes are simple and fixed: for example, intervals in one dimension, triangles or quadrilaterals in two dimensions, tetrahedra, prisms or hexahedra in three dimensions. The set of the elements is in general an unstructured mesh called a *triangulation*.
- The maximal degree k of the polynomial approximation in the elements is chosen.
- V_h is made of continuous functions of V whose restriction to the elements are polynomial of degree less than k .

Programming the method is also somewhat similar in any dimension, but mesh generation is very much dimension dependent. A nice survey on the finite element method, both on the theoretical and practical viewpoints, is proposed in [41].

There is a very well understood theory on error estimates for finite elements. It is possible to distinguish *a priori* and *a posteriori* error estimates: in *a priori* estimates, the error is bounded by some quantity depending on the solution of the continuous problem (which is unknown, but for which estimates are available), whereas, in *a posteriori* estimates, the error is bounded by some quantity

depending on the solution of the discrete problem, which is available.

For a priori error estimates, one can see the books of Raviart and Thomas [93], Strang and Fix [104], Braess [19], Brenner and Scott [23], Ciarlet [28, 27], and Thomée [105] for parabolic problems. By and large, deriving error estimates for finite elements method consists of

1. establishing the stability of the discretization with respect to some norms related to $\|\cdot\|_V$.
2. Once this is done, one sees (at least in simple cases) that the error depends on some distance of the solution of the continuous problem to the space V_h . This quantity cannot be computed exactly since the solution is unknown. However, it can be estimated from a priori knowledge on the regularity of the solution.

When sharp results on the solution of the continuous problem are available, the a priori estimates give very valuable information on how to choose the discretization a priori, see the nice papers by C. Schwab et al [102, 110], in the case of homogeneous parabolic problems with smooth coefficients.

A posteriori error estimates are a precious tool since they give practical information that can be used to refine the mesh when needed. The bibliography on a posteriori error estimates for finite element methods is quite rich: one can see the book of Verfürth [107] and the references therein. For time dependent problems, a posteriori error estimates and mesh adaption for space-time finite element problems have been investigate by Eriksson et al [38, 39, 40]. Another strategy based on decoupled space and time error indicators can be implemented, see [16] and §1.3 for an example with one space variable.

When the space variable is multidimensional, very anisotropic meshes may be useful. A trend in mesh adaptivity consists of building anisotropic meshes by imposing regularity and quasi-uniformity with respect to a new metric constructed from the a posteriori error estimates, see [47]. Below, we show some examples of anisotropic meshes generated with the open source software BAMG of F. Hecht et al.

Example: the Case of a Put Option on a Basket of Two Assets In the sequel, we deal with a simple implementation of the finite element method for approximating the pricing function of an option on a basket containing two assets. Therefore $d = 2$.

The Time Semi-Discrete Problem We introduce a partition of the interval $[0, T]$ into subintervals $[t_{m-1}, t_m]$, $1 \leq m \leq M$, such that $0 = t_0 < t_1 < \dots < t_M = T$. We denote by δt_m the length $t_m - t_{m-1}$, and by δt the maximum of the δt_m , $1 \leq m \leq M$.

For simplicity, we assume that $P_0 \in V$, where V is given by (2.45) with $d = 2$. We discretize (2.47) by means of an implicit Euler scheme, i.e. we look for

$P^m \in V$, $m = 0, \dots, M$, such that $P^0 = P_\circ$, and for $m = 1, \dots, M$, $\forall v \in V$,

$$\frac{1}{\delta t_m} (P^m - P^{m-1}, v)_{L^2(\Omega)} + a_{t_m}(P^m, v) = 0, \quad (2.52)$$

where a_{t_m} is given by (2.46). This scheme is first order.

Remark 2.2 If P_\circ does not belong to V , then we first have to approximate P_\circ by a function in V , at the cost of an additional error.

The Full Discretization: Lagrange Finite Elements Discretization with respect to S_1, S_2 consists of replacing V with a finite dimensional subspace $V_h \subset V$. For example, one may choose V_h as a space of continuous piecewise polynomial functions on a triangulation of Ω : for a positive real number h , consider a partition \mathcal{T}_h of Ω into nonoverlapping closed triangles, (\mathcal{T}_h is the set of all the triangles forming the partition) such that

- $\bar{\Omega} = \cup_{K \in \mathcal{T}_h} K$.
- For all $K \neq K'$ two triangles of \mathcal{T}_h , $K \cap K'$ is either empty, either a vertex of both K and K' , or a whole edge of both K and K' .

Remark 2.3 If Ω is not polygonal but has a smooth boundary, it is possible to find a set \mathcal{T}_h of nonoverlapping triangles of diameters less than h such that the distance between $\bar{\Omega}$ and $\cup_{K \in \mathcal{T}_h} K$ scales like h^2 .

For a positive integer k , we introduce the spaces

$$W_h = \{w_h \in \mathcal{C}^0(\bar{\Omega}) : w_h|_K \in \mathcal{P}^k, \forall K \in \mathcal{T}_h\}, \quad V_h = \{v_h \in W_h, v_h|_{\Gamma_0} = 0\}. \quad (2.53)$$

Hereafter, we focus on the case when $k = 1$, i.e. the functions in W_h are piecewise affine. It is clear that V_h is a finite dimensional subspace of V .

Assuming that $P_\circ \in V_h$, the full discretization of the variational formulation consists of finding $P_h^m \in V_h$, $m = 0, \dots, M$, such that $P_h^0 = P_\circ$, and

$$\forall v_h \in V_h, \quad \frac{1}{\delta t_m} (P_h^m - P_h^{m-1}, v_h)_{L^2(\Omega)} + a_{t_m}(P_h^m, v_h) = 0. \quad (2.54)$$

Here, for simplicity, we assume that $a_{t_m}(u_h, v_h)$ can be computed algebraically for $u_h, v_h \in V_h$, which is the case when the volatilities do not depend on S_1, S_2 , for example. If this is not the case, then quadrature formulas have to be used which induces an additional but controlled source of error.

The Discrete Problem in Matrix Form A basis of V_h is chosen, $(w_i)_{i=1,\dots,N}$. Then, for $1, \dots, M$, u_h^m can be written

$$u_h^m(S_1, S_2) = \sum_1^N u_j^m w_j(S_1, S_2), \quad (2.55)$$

and, using (2.55) in (2.54) with $v_h = w_i$, we obtain a system of linear equations for $\mathbf{U}^m = (u_j^m)_{j=1,\dots,N}^T$:

$$\mathbf{M}(\mathbf{U}^m - \mathbf{U}^{m-1}) + \delta t_m \mathbf{A}^m \mathbf{U}^m = 0, \quad (2.56)$$

where \mathbf{M} and \mathbf{A} are matrices in $\mathbb{R}^{N \times N}$: assuming that the volatilities do not depend on S_1, S_2 ,

$$\begin{aligned} \mathbf{M}_{ij} &= \int_{\Omega} w_i w_j \\ \mathbf{A}_{i,j}^m &= a(w_j, w_i) = \frac{1}{2} \sum_{\ell=1}^2 \sum_{k=1}^2 \int_{\Omega} \Xi_{\ell,k}(t_m) S_{\ell} S_k \frac{\partial w_j}{\partial S_k} \frac{\partial w_i}{\partial S_{\ell}} \\ &\quad - \sum_{k=1}^2 \int_{\Omega} \left(r(t_m) S_k - \frac{1}{2} \sum_{\ell=1}^2 \frac{\partial}{\partial S_{\ell}} (\Xi_{\ell,k}(t_m) S_{\ell} S_k) \right) \frac{\partial w_j}{\partial S_k} w_i + r(t_m) \int_{\Omega} w_j w_i. \end{aligned} \quad (2.57)$$

The matrix \mathbf{M} is called the *mass matrix* and \mathbf{A}^m is called the *stiffness matrix*.

It can be proved thanks to estimates (2.36) (2.37) that if δt is small enough, then $\mathbf{M} + \delta t_m \mathbf{A}^m$ is invertible, and it is possible to solve (2.54).

The Nodal Basis On each triangle $K \in \mathcal{T}_h$, noting by $q^i, i = 1, 2, 3$ the vertices of K , we define for $S \in \mathbb{R}^2$ the barycentric coordinates of S , i.e. the solution of

$$\sum_{i=1,2,3} \lambda_i^K(S) q^i = S, \quad \sum_{i=1,2,3} \lambda_i^K(S) = 1.$$

This 3×3 system of linear equations is never singular because its determinant is twice the area of K . It is obvious that the barycentric coordinates λ_i^K are affine functions of S . Furthermore,

- when $S \in K$, $\lambda_i^K \geq 0$, $i = 1, 2, 3$,
- if $K = [q^{i_1}, q^{i_2}, q^{i_3}]$ and S is aligned with q^{i_1}, q^{i_2} then $\lambda_{i_3}^K = 0$.

Let v_h be a function in V_h : it is easy to check that, on each triangle $K \in \mathcal{T}_h$,

$$v_h(S) = \sum_{j=1,2,3} v_h(q^{i_j}) \lambda_{i_j}^K(S) \quad \forall S \in K.$$

Therefore, a function in V_h is uniquely defined by its values at the nodes of \mathcal{T}_h not located on Γ_0 .

Call $(q^i)_{i=1,\dots,N}$ the nodes of \mathcal{T}_h not located on Γ_0 , and let w^i be the unique function in V_h such that $w^i(q^j) = \delta_{i,j}$, $\forall j = 1, \dots, N$. For a triangle K such that q^i is a vertex of K , it is clear that w^i coincides in K with one of the three barycentric coordinates attached to triangle K . Therefore, we have the identity

$$v_h = \sum_{i=1}^N v_h(q^i) w^i, \quad (2.58)$$

which shows that $(w^i)_{i=1,\dots,N}$ is a basis of V_h . As shown in Figure 2.1, the support of w^i is the union of the triangles of \mathcal{T}_h containing the node q^i , so it is very small when the mesh is fine, and the support of two basis functions w^i and w^j intersect if and only if q^i and q^j are the vertices of a same triangle of \mathcal{T}_h . Therefore, the matrices \mathbf{M} and \mathbf{A}^m constructed with this basis are sparse. This dramatically reduces the complexity when solving properly (2.56). The basis $(w^i)_{i=1,\dots,N}$ is often called the *nodal basis* of V_h . The shape functions w^i are sometimes called *hat functions*. For $v_h \in V_h$, the values $v_i = v_h(q^i)$ are called the degrees of freedom of v_h . If $K = [q^{i_1}, q^{i_2}, q^{i_3}]$, and if b^{i_1} is the point aligned

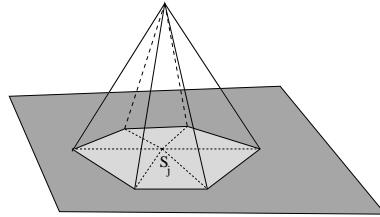


Figure 2.1: The shape function w^j

with q^{i_2} and q^{i_3} and such that $b^{i_1}\vec{q}^{i_1} \perp q^{i_2}\vec{q}^{i_3}$, then

$$\nabla \lambda_{i_1}^K = \frac{1}{|b^{i_1}\vec{q}^{i_1}|^2} b^{i_1}\vec{q}^{i_1}, \quad (2.59)$$

and calling n^{i_1} the unit vector orthogonal to $q^{i_2}\vec{q}^{i_3}$ and pointing to q^{i_1} , i.e. $n^{i_1} = \frac{1}{|b^{i_1}\vec{q}^{i_1}|} b^{i_1}\vec{q}^{i_1}$ and E^{i_1} the length of the edge of K opposite to q^{i_1} , and using the well known identity $E^{i_1}|b^{i_1}\vec{q}^{i_1}| = 2|K|$, we obtain

$$\nabla \lambda_{i_1}^K = \frac{E^{i_1}}{2|K|} n^{i_1}. \quad (2.60)$$

The following integration formula is very important for the numerical implementation of the finite element method:

Proposition 2.2 *Calling λ_i $i = 1, 2, 3$ the barycentric coordinates of the triangle K , and ν_1, ν_2, ν_3 three nonnegative integers and $|K|$ the measure of K ,*

$$\int_K (\lambda_1^K)^{\nu_1} (\lambda_2^K)^{\nu_2} (\lambda_3^K)^{\nu_3} = 2|K| \frac{\nu_1! \nu_2! \nu_3!}{(\nu_1 + \nu_2 + \nu_3 + 2)!}. \quad (2.61)$$

Remark 2.4 *It may be useful to use other bases than the nodal basis, for example bases related to wavelet decompositions, in particular for speeding up the solution of (2.56), see [79, 109].*

Remark 2.5 *The integral of a quadratic function on a triangle K is one third the sum of the values of the function on the mid-edges times $|K|$, therefore (2.61)*

is simpler when $\nu_1 + \nu_2 + \nu_3 = 2$:

$$\int_K \lambda_i^K \lambda_j^K = \frac{|K|}{12} (1 + \delta_{ij}). \quad (2.62)$$

When the system (2.56) becomes large, iterative methods such as gradient methods, GMRES or BICG-stab become attractive. We refer to [12, 49, 100, 50, 82] for good books on this topic. Iterative methods do not need the matrix $\mathbf{M} + \delta t_m \mathbf{A}^m$ but only a function which implements $\mathbf{U} \rightarrow (\mathbf{M} + \delta t_m \mathbf{A}^m)\mathbf{U}$, i.e. which computes

$$\sum_j u_j ((w^j, w^i)_{L^2(\Omega)} + \delta t_m a_{t_m}(w^j, w^i)).$$

Let us show how $\mathbf{A}^m \mathbf{U}$ should be computed, (we take $\mathbf{A}^m \mathbf{U}$ instead of $(\mathbf{M} + \delta t_m \mathbf{A}^m)\mathbf{U}$ only for simplicity). We use the fact that

$$\mathbf{A}^m \mathbf{U} = \sum_K \mathbf{A}^{m,K} \mathbf{U},$$

where $\mathbf{A}^{m,K} \mathbf{U}$ is the vector whose entries are $\sum_j u_j a_{t_m}^K(w^j, w^i)$, $i = 1, \dots, N$ and where

$$\begin{aligned} a_t^K(u, v) &= \frac{1}{2} \sum_{\ell=1}^2 \sum_{k=1}^2 \int_K \Xi_{\ell,k}(t) S_\ell S_k \frac{\partial u}{\partial S_k} \frac{\partial v}{\partial S_\ell} \\ &\quad - \sum_{k=1}^2 \int_K \left(r(t) S_k - \frac{1}{2} \sum_{\ell=1}^2 \frac{\partial}{\partial S_\ell} (\Xi_{\ell,k}(t) S_\ell S_k) \right) \frac{\partial u}{\partial S_k} v + r(t) \int_K u v. \end{aligned} \quad (2.63)$$

Hence

$$(\mathbf{A}^{m,K} \mathbf{U})_i = \sum_j u_j \sum_K \mathbf{A}_{ij}^{m,K}. \quad (2.64)$$

For simplicity only, let us only consider the first term in (2.63), so a_t^K becomes

$$a_t^K(u, v) = \frac{1}{2} \sum_{\ell=1}^2 \sum_{k=1}^2 \int_K \Xi_{\ell,k}(t) S_\ell S_k \frac{\partial u}{\partial S_k} \frac{\partial v}{\partial S_\ell},$$

and

$$\mathbf{A}_{ij}^{m,K} = \frac{1}{2} \sum_{\ell=1}^2 \sum_{k=1}^2 \int_K \Xi_{\ell,k}(t_m) S_\ell S_k \frac{\partial w^j}{\partial S_k} \frac{\partial w^i}{\partial S_\ell}.$$

But $\nabla w^i, \nabla w^j$ are constant on K , and $S_k = \sum_{\nu=1}^3 S_{k,\nu} \lambda_\nu^K$, so from (2.62),

$$\begin{aligned} \mathbf{A}_{ij}^{m,K} &= \frac{1}{2} \sum_{k,\ell=1}^2 \Xi_{\ell,k}(t_m) \frac{\partial w^i}{\partial S_\ell} \frac{\partial w^j}{\partial S_k} \sum_{\nu_1=1}^3 \sum_{\nu_2=1}^3 S_{\ell,\nu_1} S_{k,\nu_2} \int_K \lambda_{\nu_1}^K \lambda_{\nu_2}^K \\ &= \frac{|K|}{24} \sum_{k,\ell=1}^2 \Xi_{\ell,k}(t_m) \frac{\partial w^i}{\partial S_\ell} \frac{\partial w^j}{\partial S_k} \sum_{\nu_1=1}^3 \sum_{\nu_2=1}^3 S_{\ell,\nu_1} S_{k,\nu_2} (1 + \delta_{\nu_1 \nu_2}). \end{aligned} \quad (2.65)$$

The summation (2.64) should *not* be programmed directly like

$$\begin{aligned} & \text{for } i = 1..N \\ & \quad \text{for } j = 1..N \\ & \quad \text{for } K \in \mathcal{T}_h \\ & \quad (\mathbf{A}^m \mathbf{U})_i += \mathbf{A}_{ij}^{m,K} u_j, \end{aligned} \quad (2.66)$$

because the numerical complexity of this loop is of the order of $N^2 N_T$, where N_T is the number of triangles in \mathcal{T}_h . One should rather notice that the sums commute, i.e.

$$\begin{aligned} & \text{for } K \in \mathcal{T}_h \\ & \quad \text{for } j = 1..N \\ & \quad \text{for } i = 1..N \\ & \quad (\mathbf{A}^m \mathbf{U})_i += \mathbf{A}_{ij}^{m,K} u_j, \end{aligned} \quad (2.67)$$

and then see that $\mathbf{A}_{ij}^{m,K}$ is zero when q^i or q^j are not in K . The loop

$$\begin{aligned} & \text{for } K \in \mathcal{T}_h \\ & \quad \text{for } j_{loc} = 1, 2, 3 \\ & \quad \text{for } i_{loc} = 1, 2, 3 \\ & \quad (\mathbf{A}^m \mathbf{U})_{i_{loc} j_{loc}} += \mathbf{A}_{i_{loc} j_{loc}}^{m,K} u_{i_{loc} j_{loc}} \end{aligned} \quad (2.68)$$

has a complexity of the order of $O(N_T)$. This technique is called *assembling*. It has brought up the fact that vertices of triangle K have global indices (their position in the array that store them) and local indices, their position in the triangle K i.e. 1,2 or 3. The notation i_{loc} refers to the map from local to global.

The convergence of iterative methods for solving linear systems depends on the spectral properties of the matrix: for example, if the matrix is symmetric and positive definite, the convergence rate of the conjugate gradient method depends on the condition number of the matrix; for a general matrix, the speed of convergence of the GMRES method depends on the numerical range of the matrix. For the linear systems arising from the discretization of parabolic PDE, it is observed that the convergence deteriorates when the size of the systems increases. Therefore, when solving for example the linear system (2.56), one has better solve instead

$$\mathbf{B}^{-1}(\mathbf{M} + \delta t_m \mathbf{A}^m) \mathbf{U}^m = \mathbf{B}^{-1} \mathbf{M} \mathbf{U}^{m-1}, \quad (2.69)$$

where \mathbf{B} is a matrix such that

- the spectral properties of $\mathbf{B}^{-1}(\mathbf{M} + \delta t_m \mathbf{A}^m)$ are better than those of $\mathbf{M} + \delta t_m \mathbf{A}^m$. This means that \mathbf{B} is in some sense close to $\mathbf{M} + \delta t_m \mathbf{A}^m$.
- the solution of a linear system of the form $\mathbf{B} \mathbf{V} = \mathbf{G}$ can be achieved at a reasonable computational cost.

Such a matrix \mathbf{B} is called a *preconditioner* for (2.56), and the iterative method applied to (2.69) is called a *preconditioned iterative method*. The construction of good preconditioners is an important topic in numerical analysis. We again refer to [12, 49, 100, 50, 82].

Remark 2.6 (Mass Lumping for piecewise linear triangular elements)
Let f be a smooth function and consider the following approximation for the integral of f over $\Omega = \cup_{K \in \mathcal{T}_h} K$, where \mathcal{T}_h is a triangulation of Ω :

$$\int_{\Omega} f = \sum_{K \in \mathcal{T}_h} \int_K f \approx \sum_{K \in \mathcal{T}_h} \frac{|K|}{3} \sum_{i=1}^3 f(q_i^K),$$

where q_1^K, q_2^K, q_3^K are the 3 vertices of K . If f is affine this formula is exact, otherwise it computes the integral with an error $O(h^2)$.

This approximation is called *mass lumping*: for two functions $u_h, v_h \in V_h$, we call \mathbf{U} and \mathbf{V} the vectors of their coordinates in the nodal basis: applying mass lumping, one approximates $\int_{\Omega} u_h v_h$ by $\mathbf{U}^T \tilde{\mathbf{M}} \mathbf{V}$, where $\tilde{\mathbf{M}}$ is a diagonal matrix with positive diagonal entries.

Results The discrete method discussed above has been applied to compute the pricing function of a best-of put option on a two assets basket, $P_o(S_1, S_2) = (100 - \max(S_1, S_2))^+$. The artificial boundary Γ_0 is $\{\max(S_1, S_2) = \bar{S} = 200\}$. Homogeneous Dirichlet conditions have been imposed on Γ_0 . Such a choice of \bar{S} may not be enough for a good accuracy; in fact $\bar{S} = 200$ was chosen to obtain figures with nice proportions.

The parameters of the Black-Scholes model are

$$\sigma_1 = 0.2, \quad \sigma_2 = 0.1, \quad r = 0.05.$$

The correlation factor is either of -0.3 (Figure 2.2) or of -0.9 (Figure 2.3). The first order implicit Euler scheme has been used with a uniform time step of $1/250$ year. Mesh adaption in the (S_1, S_2) variable has been performed every $1/10$ year. For mesh adaption, we have used the software BAMG, see [47]. In Figure 2.2, the adapted mesh and the contours of the pricing function are plotted 0.2 year to maturity (top) and one year to maturity (bottom). The mesh is refined near the lines where the payoff function exhibits singularities. As time to maturity grows, the mesh becomes coarser in these regions. In fact, such a large number of mesh adaptions are not necessary.

It is clearly seen that the pricing function diffuses more in the S_1 variable, which is not surprising, because the volatility of the first asset is higher.

Multigrid Methods

Geometric multigrid method can be applied if there is a hierarchy of nested meshes \mathcal{T}_{h_i} , $i = 1, \dots, q$, in such a way that the corresponding finite element spaces satisfy $V_1 \subset \dots \subset V_i \subset V_{i+1} \subset \dots \subset V_q$. The dimensions of these spaces

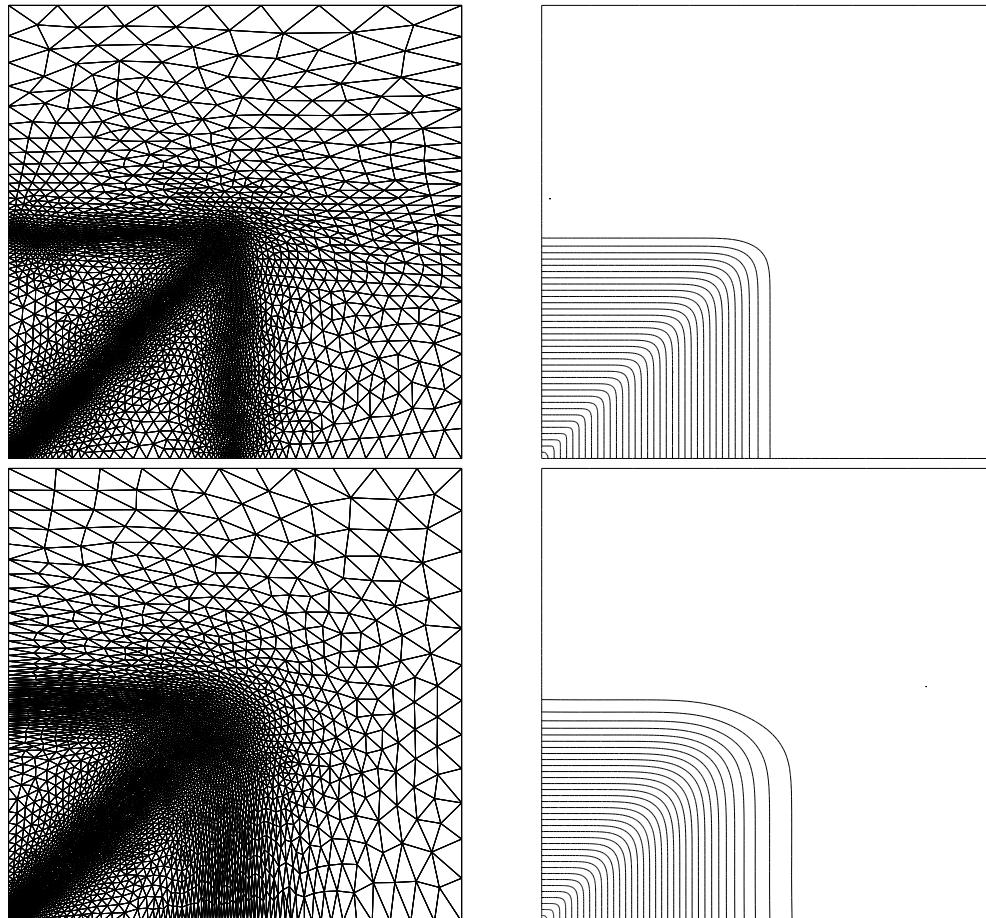


Figure 2.2: The adapted mesh and the contours of P , at the times to maturity 0.2 year (top) and 1 year (bottom). $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $\rho = -0.3$.

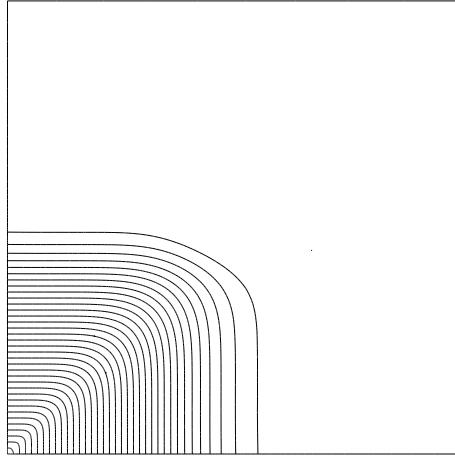


Figure 2.3: The contours of P , 1 year to maturity. $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $\rho = -0.9$.

are $N_1 < \dots < N_i < N_{i+1} < \dots < N_q$. The heuristics supporting multigrid methods for elliptic problem is as follows: the first observation is that with common iterative solvers like Jacobi or SOR, see [82, 24], the components of the error corresponding to high frequencies are usually decreased much faster than those associated with the lower frequencies. This also explains why the convergence rate of such methods deteriorates as the number of unknowns grows. To summarize, the iterative solver makes the error smooth and the smooth part of the error has a slow decay. An iterative solver with this property is called a *smoother*. The second observation is that for a given function $f \in V_{i+1}$, its projection on V_i will be decreased faster by the smoother at level i (operating on V_i) than by the smoother at level $i+1$ (operating on V_{i+1}), because it appears less smooth to the first operator. From these observations, an efficient procedure can be designed by combining the iterations of the smoother with *coarse level corrections*. If this idea is also applied to the coarse level correction, the result is a recursive algorithm.

Assume that a Galerkin method is applied for approximating the solution of a boundary value problem by a function in V_q : the system of linear equation reads $\mathbf{A}^{(q)}\mathbf{u}^{(q)} = \mathbf{f}^{(q)}$. Note that it is also possible to define the similar Galerkin discretizations at the lower levels: using the nodal basis, the corresponding system reads

$$\mathbf{A}^{(i)}\mathbf{u}^{(i)} = \mathbf{f}^{(i)}, \quad 1 \leq i \leq q. \quad (2.70)$$

For simplicity, assume that $\mathbf{A}^{(i)}$ are symmetric and positive definite. Denote by $\mathcal{S}^{(i)}$ the smoother at level i : the vector obtained by performing ν iterations of the smoother at level i for solving (2.70) starting from the initial guess \mathbf{w} is written $\mathcal{S}^{(q)}(\mathbf{f}^{(i)}, \mathbf{w}, \nu)$. An ingredient of the method is the canonical injection

from V^i to V^{i+1} : let \mathbf{I}_i^{i+1} stand for its matrix in the nodal bases. Another ingredient is the restriction operator from V_{i+1} to V_i , whose matrix is \mathbf{I}_{i+1}^i : a possible choice is to take the Galerkin projection, i.e.,

$$(\mathbf{A}^{(i)} \mathbf{I}_{i+1}^i \mathbf{u}, \mathbf{w}) = (\mathbf{A}^{(i+1)} \mathbf{u}, \mathbf{I}_i^{i+1} \mathbf{w}), \quad \forall \mathbf{u} \in \mathbb{R}^{N_{i+1}}, \forall \mathbf{w} \in \mathbb{R}^{N_i}. \quad (2.71)$$

We denote by $MG(\mathbf{f}^{(i)}, \mathbf{w}, i)$ one iteration of the multigrid method at level i for solving (2.70) starting from \mathbf{w} .

One of the most commonly used multigrid algorithm is the *V-cycle*:

One V-cycle : $MG(\mathbf{f}^{(i)}, \mathbf{w}, i) \rightarrow \mathbf{w}$

If $i = 1$, solve the system (2.70) with a direct method, and let \mathbf{w} be the solution.
Else

1. Perform ν_1 iterations of the smoother at level i : $\mathcal{S}^{(i)}(\mathbf{f}^{(i)}, \mathbf{w}, \nu_1) \rightarrow \mathbf{w}$.
2. Compute the residual $\mathbf{r} \in \mathbb{R}^{N_{i-1}}$ on level $i - 1$ by

$$(\mathbf{r}, \mathbf{z}) = (\mathbf{f}^{(i)} - \mathbf{A}^{(i)} \mathbf{w}, \mathbf{I}_{i-1}^i \mathbf{z}), \quad \forall \mathbf{z} \in \mathbb{R}^{N_{i-1}}.$$

Note that \mathbf{r} can be expressed in terms of $\mathbf{A}^{(i-1)} \mathbf{I}_i^{i-1} \mathbf{w}$ and of the projection of \mathbf{f}^i .

3. Apply the multigrid method at level $i - 1$: $MG(\mathbf{r}, 0, i - 1) \rightarrow \tilde{\mathbf{w}}$.
4. Add the coarse level correction to \mathbf{w} : $\mathbf{w} + \mathbf{I}_{i-1}^i \tilde{\mathbf{w}} \rightarrow \mathbf{w}$.
5. Perform another ν_2 iterations of the smoother at level i : $\mathcal{S}^{(i)}(\mathbf{f}^{(i)}, \mathbf{w}, \nu_2) \rightarrow \mathbf{w}$.

The iterative method consists of computing the sequence $\mathbf{w}_{n+1} = MG(\mathbf{f}^{(q)}, \mathbf{w}_n, q)$ until the residual norm becomes smaller than some tolerance ϵ . Under some reasonable assumptions on the elliptic equation, on the mesh and on the smoother, see e.g. [20, 111], it can be proved that the norm $\|\mathbf{u}^q - \mathbf{w}_n\|_{\mathbf{A}}$ decays like ρ^n where $\rho < 1$ does not depend on the mesh parameters.

A very nice introduction to multigrid methods is given in [24]. Multigrid methods can also be used in the construction of preconditioners, see [21, 111]. Finally, the ideas above have been generalized in the so-called *algebraic multigrid methods*, when there is no hierarchy of grids, see [99]. Algebraic multigrid methods are among the most robust and efficient for solving the linear systems arising from the discretization of elliptic and parabolic PDEs. Open source libraries are available, like the library *hypre*, see http://www.llnl.gov/CASC/linear_solvers/.

2.2.3 Sparse Methods

Consider a boundary value problem in the hypercube $\Omega = (0, 1)^d$. One can think of a Poisson problem $\Delta u = -f$ with the Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. For the variational formulation, we need to use the

space $H^1(\Omega)$, equipped with the norm $\|v\|_{H^1(\Omega)} = \sqrt{\|v\|_{L^2(\Omega)}^2 + |v|_{H^1(\Omega)}^2}$ where $|v|_{H^1(\Omega)}^2 = \sum_{i=1}^d \|\frac{\partial v}{\partial x_i}\|_{L^2(\Omega)}^2$, and $H_0^1(\Omega)$, the completion in $H^1(\Omega)$ of the subspace of smooth functions compactly supported in Ω . The previous elliptic problem has a weak or variational formulation in $H_0^1(\Omega)$: find $u \in H_0^1(\Omega)$ such that $\int_{\Omega} \nabla u \cdot \nabla v = \int_{\omega} fv$, for all $v \in H_0^1(\Omega)$.

Assume that the solution of the Poisson problem is approximated by a conforming multilinear finite element method on a Cartesian mesh, more precisely with piecewise linear functions of total degree $\leq d$. This is the lowest order finite element method on this mesh. Assume that the mesh is uniform and that each element is a cube of size n^{-1} . It is easy to see that the dimension of the approximation space is of the order of n^d : the algorithmic complexity grows exponentially with d , which actually forbids the use of this method for $d > 4$. This too rapid growth in complexity is known as the *curse of dimensionality*. Yet, quite recent developments have shown that it may be possible to use deterministic Galerkin methods or grid based methods for elliptic or parabolic problems in dimension d , for $4 \leq d \leq 20$: these methods are based either on sparse grids [112, 53, 51] or on sparse tensor product approximation spaces [52, 109].

In this paragraph, we aim at rapidly describing the principle of sparse approximations. This presentation heavily relies on the review article by H.J. Bungartz and M.Griebel [25]. We concentrate on the previously mentioned Dirichlet boundary value problem in Ω . The solution u will be approximated by a Galerkin method, *i.e.*, a variational problem posed in a finite dimensional approximation space V_n instead of $H_0^1(\Omega)$. The goal is to use approximation spaces V_n whose dimensions do not grow too rapidly with d .

The results below are proved in [25].

Notations and Preliminary Results In this section, bold letters will stand for d -uples: for example, $\boldsymbol{x} = (x_1, \dots, x_d)$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$. We set $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$ and $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$. Take a sufficiently smooth function f defined on $[0, 1]^d$; if $\boldsymbol{\alpha} \in \mathbb{N}^d$, we call $D^{\boldsymbol{\alpha}} f$ the partial derivative

$$D^{\boldsymbol{\alpha}} f = \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}},$$

where $|\boldsymbol{\alpha}| = \sum_{i=1}^d \alpha_i$. For two multi-indices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and a scalar λ , we define

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \sum_{i=1}^d \alpha_i \beta_i, \quad \lambda \boldsymbol{\alpha} = (\lambda \alpha_1, \dots, \lambda \alpha_d), \quad \mathbf{2}^{\boldsymbol{\alpha}} = (2^{\alpha_1}, \dots, 2^{\alpha_d}).$$

We say that $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ if $\alpha_i \leq \beta_i$, $i = 1, \dots, d$, and that $\boldsymbol{\alpha} < \boldsymbol{\beta}$ if $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$.

Let us introduce the function spaces $X^{q,r}(\Omega)$, for $r \in \mathbb{N}$ and $q \in [1, +\infty]$:

$$X^{q,r}(\Omega) = \{u \in L^q(\Omega), \forall \boldsymbol{\alpha} \text{ s.t. } \boldsymbol{\alpha} \leq r\mathbf{1}, D^{\boldsymbol{\alpha}} u \in L^q(\Omega)\}, \quad (2.72)$$

which are endowed with the semi-norms:

$$\begin{aligned}|u|_{q,\alpha} &= \left(\int_{\Omega} |D^{\alpha} u|^q \right)^{\frac{1}{q}}, \quad \alpha \leq r\mathbf{1}, \text{ if } q < \infty, \\ |u|_{\infty,\alpha} &= \|D^{\alpha} u\|_{L^{\infty}(\Omega)}, \quad \alpha \leq r\mathbf{1}, \text{ if } q = \infty.\end{aligned}$$

Note that $X^{q,r}(\Omega)$ is imbedded in the more usual Sobolev space $W^{q,r}(\Omega) = \{u \in L^q(\Omega), \forall \alpha \text{ s.t. } |\alpha| \leq r, D^{\alpha} u \in L^q(\Omega)\}$.

For a multi-index ℓ , consider the Cartesian meshes \mathcal{T}_{ℓ} of $\overline{\Omega}$ with mesh steps $\mathbf{h}_{\ell} = \mathbf{2}^{-\ell} = (2^{-\ell_1}, \dots, 2^{-\ell_d})$. The grid nodes of \mathcal{T}_{ℓ} are the points $\mathbf{x}_i = i \cdot \mathbf{h}_{\ell}$, $0 \leq i \leq \mathbf{2}^{\ell}$.

We note by ϕ the mother hat function:

$$\phi(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

and $\phi_{\ell,i}$ the d -dimensional hat function:

$$\phi_{\ell,i}(\mathbf{x}) = \prod_{k=1}^d \phi(2^{\ell_k} x_k - i_k). \quad (2.73)$$

We call V_{ℓ}

$$V_{\ell} = \text{span}(\phi_{\ell,i}, 1 \leq i \leq \mathbf{2}^{\ell} - \mathbf{1}) \quad (2.74)$$

We also consider the wavelet subspaces:

$$W_{\mathbf{k}} = \text{span}\{\phi_{\mathbf{k},i}, \mathbf{1} \leq i \leq \mathbf{2}^{\mathbf{k}} - \mathbf{1}, i_j \text{ odd }, 1 \leq j \leq d\}. \quad (2.75)$$

We have

$$V_{\ell} = \bigoplus_{1 \leq \mathbf{k} \leq \ell} W_{\mathbf{k}}.$$

The basis of V_{ℓ} obtained by assembling the previously mentioned bases of $W_{\mathbf{k}}$, $1 \leq \mathbf{k} \leq \ell$ is called the hierarchical basis of V_{ℓ} . Calling $\mathbf{I}_{\ell} = \{i \leq \mathbf{2}^{\ell} - \mathbf{1} : i_j \text{ odd }, 1 \leq j \leq d\}$, the hierarchical basis of V_{ℓ} is $\{\phi_{\mathbf{k},i}, i \in \mathbf{I}_{\mathbf{k}}, \mathbf{k} \leq \ell\}$. Note that the completion of $\bigoplus_{1 \leq \mathbf{k}} W_{\mathbf{k}}$ with respect to the $H^1(\Omega)$ norm is exactly $H_0^1(\Omega)$.

Rescaling the $\phi_{\mathbf{k},i}$ as follows

$$\psi_{\mathbf{k},i} = -2^{-(\mathbf{k}+1) \cdot \mathbf{1}} \phi_{\mathbf{k},i}, \quad i \in \mathbf{I}_{\mathbf{k}}, \quad (2.76)$$

we obtain another basis of $W_{\mathbf{k}}$.

If a function u is smooth enough, then the coefficients of its expansion in the hierarchical basis are obtained by a simple integral formula:

Lemma 2.2 *If $u \in H_0^1(\Omega) \cap X^{1,2}(\Omega)$, then*

$$u = \sum_{\mathbf{k} \geq \mathbf{1}} \sum_{i \in \mathbf{I}_{\mathbf{k}}} u_{\mathbf{k},i} \phi_{\mathbf{k},i}, \quad \text{where } u_{\mathbf{k},i} = \int_{\Omega} D^2 u \cdot \psi_{\mathbf{k},i}. \quad (2.77)$$

By using Lemma 2.2, one may evaluate the contribution $u_{\mathbf{k}}$ of a subspace $W_{\mathbf{k}}$ to the hierarchical expansion of u :

Lemma 2.3 *If $u \in H_0^1(\Omega) \cap X^{2,2}(\Omega)$, then the component $u_{\mathbf{k}} \in W_{\mathbf{k}}$ of the expansion of u in the hierarchical representation is such that*

$$\begin{aligned} \|u_{\mathbf{k}}\|_{L^2(\Omega)} &\leq 2^{-2|\mathbf{k}|} 3^{-d} |u|_{2,2}, \\ |u_{\mathbf{k}}|_{H^1(\Omega)} &\leq 2^{-2|\mathbf{k}|} 3^{-d+\frac{1}{2}} \left(\sum_{j=1}^d 2^{2k_j} \right)^{\frac{1}{2}} |u|_{2,2}. \end{aligned} \quad (2.78)$$

Sparse Galerkin Methods It is clear that the dimension of V_{ℓ} is $\prod_{j=1}^d (2^{\ell_j} - 1)$. In particular, $\dim(V_{n1}) = (2^n - 1)^d$. As already mentioned, the full tensor product space V_{n1} is often too large for practical use when $d > 4$.

Let us give an example of a sparse Galerkin method: the discrete space is chosen to be

$$V_n = \bigoplus_{\mathbf{k} \leq \mathbf{n}, |\mathbf{k}| \leq n+d-1} W_{\mathbf{k}} \quad (2.79)$$

instead of the full tensor product space $V_{n1} = \bigoplus_{\mathbf{k} \leq \mathbf{n}} W_{\mathbf{k}}$. One may prove that

$$\dim(V_n) = 2^n \left(\frac{n^{d-1}}{(d-1)!} + O(n^{d-2}) \right). \quad (2.80)$$

Therefore $\dim(V_n)$ is much smaller than $\dim(V_{n1})$. It can be seen that a Galerkin method with V_n is feasible for d of the order of 10. On Figure 2.4, we display the bases of V_{n1} and V_n .

Consider the discretization of the Dirichlet problem in Ω : the discretization error of the Galerkin method with the approximation space V_n (resp. V_{n1}) is of the same order as the best fit error when approximating the solution of the continuous problem by a function of V_n (resp. V_{n1}). Let us assume that u is smooth. We know that $\inf_{v \in V_{n1}} \|v - u\|_{H^1(\Omega)} \leq C 2^{-n} |u|_{W^{2,2}(\Omega)}$, where $|u|_{W^{2,2}(\Omega)}^2 = \sum_{|\alpha|=2} \|D^\alpha u\|_{L^2(\Omega)}^2$. Since V_n is much smaller than V_{n1} , a similar estimate is not true for $\inf_{v \in V_n} \|v - u\|_{H^1(\Omega)}$. Griebel et al have proved the following theorem:

Theorem 2.4 *If $u \in H_0^1(\Omega) \cap X^{2,2}(\Omega)$, and if $u_n \in V_n$ is the component of the expansion of u in the hierarchical representation,*

$$\|u - u_n\|_{L^2(\Omega)} \leq \left(\frac{2^{-2n+1}}{12^d} \sum_{k=0}^{d-1} \binom{n+d-1}{k} \right) |u|_{2,2} = O(2^{-2n} n^{d-1}) |u|_{2,2} \quad (2.81)$$

$$|u - u_n|_{H^1(\Omega)} \leq \left(\frac{2^{-n} d}{\sqrt{3} 6^{d-1}} \right) |u|_{2,2} = O(2^{-n}) |u|_{2,2}. \quad (2.82)$$

Theorem 2.4 says that under the assumption that $u \in H_0^1(\Omega) \cap X^{2,2}(\Omega)$ (which is a rather strong regularity assumption, much stronger than the assumption

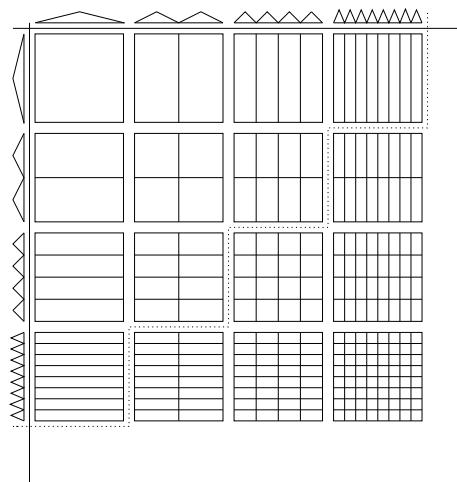


Figure 2.4: The case $d = 2$: each entry of this array corresponds to a pair of integer $\mathbf{k} = (k_1, k_2)$, $1 \leq k_1, k_2 \leq 4$, and contains the grid corresponding to $W_{\mathbf{k}}$. Each space $W_{\mathbf{k}}$ is the tensor product of two spaces whose bases are plotted on the sides of the array. The full tensor space V_{n1} is given by $V_{n1} = \bigoplus_{1 \leq \mathbf{k} \leq n1} W_{\mathbf{k}}$ whereas the sparse tensor space V_n is given by $V_n = \bigoplus_{1 \leq \mathbf{k}, |\mathbf{k}| \leq n+d-1} W_{\mathbf{k}}$, (only the spaces $W_{\mathbf{k}}$ corresponding to the entries above the diagonal are used to construct V_n)

$u \in H_0^1(\Omega) \cap W^{2,2}(\Omega)$ required when the full tensor product space is used), then using the sparse approximation space V_n instead of the full tensor space V_{n1} does not deteriorate the accuracy, at least with respect to the H^1 semi-norm. There is a moderate deterioration for the L^2 norm of the error.

In our presentation, we have focused on sparse methods based on tensorizing one dimensional hierarchical bases made of hat functions. This technique can be generalized to other classes of bases functions, for example higher order piecewise polynomial functions or wavelets as in Figure 2.5.

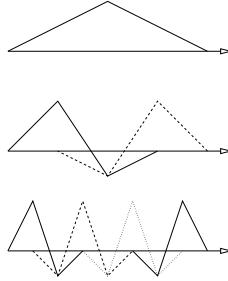


Figure 2.5: An example of wavelets

Sparse Grids Before defining finite difference methods on sparse grids, we need to introduce new notations and concepts.

Consider the one variable shape functions: $\phi_{\ell,i}(x) = \phi(2^\ell x - i)$, $\ell \geq 1$, $1 \leq i \leq 2^\ell - 1$, and call V_ℓ the space spanned by $(\phi_{\ell,i})_{1 \leq i \leq 2^\ell - 1}$. Call W_ℓ the subspace of V_ℓ spanned by $(\phi_{\ell,2i-1})_{1 \leq i \leq 2^\ell - 1}$. We have $V_\ell = W_\ell \oplus V_{\ell-1}$. We have already seen that $V_1 \subset \dots \subset V_\ell \subset V_{\ell+1} \subset \dots$ is a multiresolution analysis of $H_0^1((0, 1))$. For a function $u \in C^0([0, 1])$ s.t. $u(0) = u(1) = 0$, we have

$$u = \sum_{\ell=1}^{\infty} \sum_{i=1}^{2^\ell-1} u_{\ell,i} \phi_{\ell,2i-1},$$

and the projection of u on V_ℓ is

$$\sum_{i=1}^{2^\ell-1} u^{\ell,i} \phi_{\ell,i} = \sum_{k=1}^{\ell} \sum_{i=1}^{2^{k-1}} u_{k,i} \phi_{k,2i-1}.$$

The change of coordinates $(u^{\ell,i})_{i=1,\dots,2^\ell-1} \mapsto (u_{k,i})_{k=1,\dots,\ell, i=1,\dots,2^{k-1}}$ is called T_ℓ . We call U_ℓ and U^ℓ the column vectors: $U^\ell = (u^{\ell,1}, \dots, u^{\ell,2^\ell-1}) \in \mathbb{R}^{2^\ell-1}$ and $U_\ell = (u_{\ell,1}, \dots, u_{\ell,2^{\ell-1}}) \in \mathbb{R}^{2^{\ell-1}}$. We have

$$T_\ell U^\ell = \begin{pmatrix} U_1 \\ \vdots \\ U_\ell \end{pmatrix}.$$

We denote by P^ℓ the restriction operator

$$P^\ell : \mathcal{C}^0([0, 1]) \rightarrow \mathbb{R}^{2^\ell - 1}, \quad P^\ell u = U^\ell. \quad (2.83)$$

Note that T_ℓ^{-1} is the representation of the operator P^ℓ in the wavelet basis, i.e.,

$$P^\ell \left(\sum_{k \leq \ell} \sum_{i=1}^{2^{k-1}} u_{k,i} \phi_{k,2i+1} \right) = T_\ell^{-1} \begin{pmatrix} U_1 \\ \vdots \\ U_\ell \end{pmatrix}.$$

We introduce the interpolation operator I^ℓ :

$$I^\ell : \mathbb{R}^{2^\ell - 1} \rightarrow \mathcal{C}^0([0, 1]), \quad I^\ell U = \sum_{i=1}^{2^\ell - 1} u^i \phi_{\ell,i}. \quad (2.84)$$

We also denote by D^ℓ the finite difference operator for the discretization of $\frac{d^2}{dx^2}$:

$$\begin{aligned} D^\ell : \mathbb{R}^{2^\ell - 1} &\rightarrow \mathbb{R}^{2^\ell - 1}, \\ \forall U, V \in \mathbb{R}^{2^\ell - 1} \quad (D^\ell U, V) &= 2^\ell \int_0^1 (I^\ell U)' (I^\ell V)'. \end{aligned} \quad (2.85)$$

We consider the uniform grids of $(0, 1)$: $\omega^\ell = 2^{-\ell}\{1, \dots, 2^\ell - 1\}$. For $\ell \in \mathbb{N}^d$, $1 \leq \ell$, we introduce the Cartesian grid of Ω : $\Omega^\ell = \prod_{i=1}^d \omega^{\ell_i}$. A grid function on Ω^ℓ is a mapping from Ω^ℓ to \mathbb{R} . The space of the grid functions on Ω^ℓ is exactly $\prod_{i=1}^d \mathbb{R}^{2^{\ell_i}-1}$. The mapping $(u^i)_{1 \leq i \leq 2^\ell - 1} \mapsto u = \sum_{1 \leq i \leq 2^\ell - 1} u^i \phi_{\ell,i}$ is an isomorphism from the space of the grid functions on Ω^ℓ onto V_ℓ defined in (2.74). Moreover, the function u can be written on the wavelet basis $u = \sum_{1 \leq k \leq \ell} \sum_{i \in I_k} u_{k,i} \phi_{k,i}$. Calling U_k the vector $(u_{k,i})_{i \in I_k}$, the grid function will be represented by the family $(U_k)_{1 \leq k \leq \ell}$.

For a positive integer n , we define the sparse grid Ω^n as follows:

$$\Omega^n = \cup_{1 \leq \ell, |\ell| \leq n+d-1} \Omega^\ell \subset \Omega^{n1}. \quad (2.86)$$

An example of a sparse grid in dimension $d = 2$ is presented in Figure 2.6.

A grid function on Ω^n is a mapping from Ω^n to \mathbb{R} . The space of the grid functions on Ω^n is isomorphic to V_n defined in (2.79). As for the full tensor grid, a grid function on Ω^n can be represented on the wavelet basis by $\sum_{1 \leq k, |\kappa| \leq n} \sum_{i \in I_k} u_{k,i} \phi_{k,i}$. Calling U_k the vector $(u_{k,i})_{i \in I_k}$, the sparse grid function will be represented by the family $(U_k)_{1 \leq k, |\kappa| \leq n+d-1}$.

We now define the sparse finite difference discretization of $\frac{\partial^2}{\partial x_1^2}$: given the vectors $\check{k} = (k_2, \dots, k_d) \in \mathbb{N}^{d-1}$, $i \in I_{\check{k}}$ and a sparse grid function represented by $(U_k)_{1 \leq k, |\kappa| \leq n+d-1}$, let \tilde{k} be the positive integer $\tilde{k} = n + d - 1 - |\check{k}|$; we introduce $U_{\check{k}}$ by

$$U_{\check{k},i} = \begin{pmatrix} U_{(1,\check{k},i)} \\ \vdots \\ U_{(\tilde{k},\check{k},i)} \end{pmatrix} \quad \text{where } U_{(j,\check{k},i)} = \left(u_{(j,\check{k}),(m,i)} \right)_{\{m \text{ odd, } 1 \leq m \leq 2^j - 1\}}^T.$$

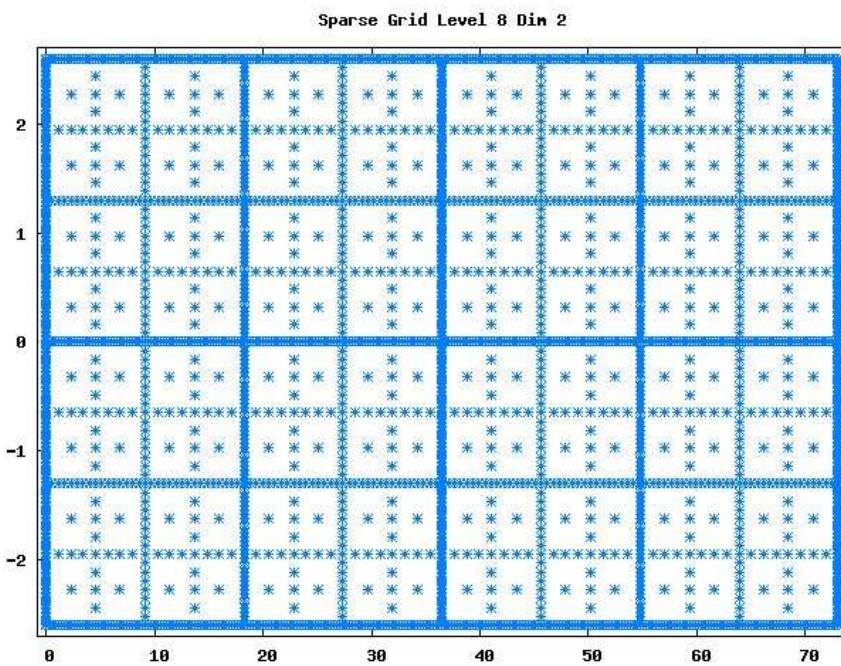


Figure 2.6: An example of a sparse grid for $d = 2$, $n + 1 = 8$

The sparse grid discretization of the operator $\frac{\partial^2}{\partial x_1^2}$ is

$$(U_{\mathbf{k}})_{1 \leq \mathbf{k}, |\mathbf{k}| \leq n+d-1} \mapsto (V_{\mathbf{k}})_{1 \leq \mathbf{k}, |\mathbf{k}| \leq n+d-1} \quad (2.87)$$

such that $V_{\tilde{\mathbf{k}}, \mathbf{i}} = T_{\tilde{\mathbf{k}}} D^{\tilde{\mathbf{k}}} T_{\tilde{\mathbf{k}}}^{-1} U_{\tilde{\mathbf{k}}, \mathbf{i}}, \forall \mathbf{k}, \mathbf{i} \in I_{\mathbf{k}}$.

The sparse grid discretization of the operators $\frac{\partial^2}{\partial x_j^2}, \frac{\partial}{\partial x_j}, j = 1, \dots, d$, can be done in a similar way.

It is natural to define the restriction operator $P^{\ell} : u \mapsto u|_{\Omega^{\ell}}$ and the interpolation operator $I^{\ell} = I^{\ell_1} \otimes \dots \otimes I^{\ell_d} : \prod_{i=1}^d \mathbb{R}^{2^{\ell_i}-1} \rightarrow \mathcal{C}^0(\Omega)$. The finite difference approximation of $\partial_{x_1}^2 u$ on the grid Ω^{ℓ} is $(I^{\ell} \circ (D^{\ell} \otimes Id) \circ P^{\ell})(u)$. It has been proved by Koster, see [71], that the sparse grid approximation of $\partial_{x_1}^2 u$ can be written in terms of these finite difference operators:

Theorem 2.5 *For a function $u \in \mathcal{C}^0(\Omega)$ s.t. $u = 0$ on $\partial\Omega$, we note $D_n(u)$ the function of V_n whose expansion in the wavelet basis is given by $(V_{\mathbf{k}})_{1 \leq \mathbf{k}, |\mathbf{k}| \leq n+d-1}$ in (2.87), where $(U_{\mathbf{k}})_{1 \leq \mathbf{k}, |\mathbf{k}| \leq n+d-1}$ is the expansion on the wavelet basis of the projection of u on V_n . Then*

$$D_n(u) = \left(\sum_{1 \leq \mathbf{k}, |\mathbf{k}| \leq n+d-1} f(\mathbf{k}) I^{\mathbf{k}} \circ (D^{k_1} \otimes Id) \circ P^{\mathbf{k}} \right) (u), \quad (2.88)$$

where $f(\mathbf{k})$ is recursively defined by

$$\begin{aligned} f(\mathbf{k}) &= 0, & \text{if } |\mathbf{k}| > n+d-1 \text{ or } \mathbf{k} < \mathbf{1}, \\ f(\mathbf{k}) &= 1 - \sum_{\ell: \mathbf{k} < \ell} f(\ell), & \text{if } |\mathbf{k}| \leq n+d-1 \text{ and } \mathbf{k} \geq \mathbf{1}. \end{aligned} \quad (2.89)$$

Before stating a consistency estimate, let us introduce some Hölder spaces: let $\boldsymbol{\alpha}$ belong to \mathbb{R}_+^d . Call $[\boldsymbol{\alpha}]$ the vector of \mathbb{N}^d whose i^{th} component is the integer part of α_i . Call $\{\boldsymbol{\alpha}\} = \boldsymbol{\alpha} - [\boldsymbol{\alpha}]$. We note $\mathcal{C}^{\boldsymbol{\alpha}}(\bar{\Omega})$ the space of continuous functions u such that for all $\boldsymbol{\beta} \leq [\boldsymbol{\alpha}]$, $D^{\boldsymbol{\beta}} u$ is continuous and

$$\sup \left\{ \frac{|D^{[\boldsymbol{\alpha}]} u(\mathbf{x} + \mathbf{h}) - D^{[\boldsymbol{\alpha}]}(\mathbf{x})|}{|h_1|^{\{\alpha_1\}} \dots |h_d|^{\{\alpha_d\}}}, \mathbf{x}, \mathbf{x} + \mathbf{h} \in \Omega, |h_i| > 0, i = 1, \dots, d \right\} < +\infty.$$

The last quantity corresponds to a semi-norm on $\mathcal{C}^{\boldsymbol{\alpha}}(\bar{\Omega})$, which we call $|u|_{\mathcal{C}^{\boldsymbol{\alpha}}(\bar{\Omega})}$. Theorem 2.5 is the key to the following consistency estimate, obtained in [71]:

Theorem 2.6 *Assume that $u \in \mathcal{C}^{\boldsymbol{\alpha}}(\bar{\Omega})$, where $\alpha_1 > 2$, $\alpha_i > 0$, $i = 2, \dots, d$, and that $u = 0$ on $\partial\Omega$. Let P^n be the restriction operator on the sparse grid Ω_n : $P^n(u) = u(\Omega_n)$. We have the consistency error estimate*

$$\|P^n\left(\frac{\partial^2 u}{\partial x_1^2}\right) - P^n \circ D_n(u)\|_{\infty} \leq C n^{d-1} 2^{-n \min(\alpha_1-2, \alpha_2, \dots, \alpha_d, 2)} |u|_{\mathcal{C}^{\boldsymbol{\alpha}}(\bar{\Omega})}. \quad (2.90)$$

Similarly, for the sparse discretization of the Laplace operator, the consistency error may be bounded by $Cn^{d-1}2^{-n \min(\alpha_1-2, \alpha_2-2, \dots, \alpha_d-2, 2)}|u|_{\mathcal{C}^\alpha(\bar{\Omega})}$ if $u \in \mathcal{C}^\alpha(\bar{\Omega})$ with $\alpha_i > 2$, $i = 1, \dots, d$.

We see that the sparse grid discretization of Δ is consistent and that the consistency error is almost of the same order (up to the factor n^{d-1}) as the consistency error obtained with a full tensor grid.

We are left with studying the stability of the sparse grid discretization. As far as we know, there is unfortunately no theoretical stability estimates. There is even no proof that the matrix D arising in the discrete problem is invertible. Indeed D does not fall into the well studied classes of matrices: in particular, D is neither a symmetric nor a M matrix. No discrete maximum principle is available. Nevertheless, numerical tests were done in [101], indicating that the stability constant, *i.e.*, $\|D^{-1}\|_\infty$ is bounded by Cn^{d-1} .

If such a stability estimate is true, we see that the sparse grid discretization of the Poisson problem is convergent, with an error of the order of

$$n^{2d-2}2^{-n \min(\alpha_1-2, \alpha_2-2, \dots, \alpha_d-2, 2)},$$

if $u \in \mathcal{C}^\alpha(\bar{\Omega})$ with $\alpha_i > 2$, $i = 1, \dots, d$.

The combination technique For a linear PDE in Ω with say Dirichlet conditions, there is an alternative technique which consists of separately computing the approximations of the solution with standard finite difference schemes on all the Cartesian grids Ω^ℓ , $1 \leq \ell$, $|\ell| \leq n + d - 1$, and suitably combining these solutions, see [112, 53, 97, 94] : the discrete solution is

$$\sum_{\ell \leq \ell, |\ell| \leq n+d-1} f(\ell) u_\ell = \sum_{\ell=n}^{n+d-1} a_{\ell-n} \sum_{\ell, |\ell|=\ell} u_\ell,$$

where u_ℓ is the discrete solution computed with the standard finite difference scheme on Ω^ℓ , $f(\ell)$ is defined in (2.89) and where

$$a_j = (-1)^{d-1-j} \binom{d-1}{j}, \quad 0 \leq j \leq d-1.$$

The choice of the coefficients a_j comes from

- performing a multidimensional Taylor expansion of the error between the solution of the continuous problem and its approximation by a linear finite difference scheme on a Cartesian grid of steps (h_1, \dots, h_d) with respect to h_1, \dots, h_d .
- combining the discrete solutions on the Cartesian grids Ω^ℓ , $1 \leq \ell$, $|\ell| \leq n+d-1$, in order to cancel the larger terms in the above mentioned Taylor expansions.

Doing so, there is an approximation error (and not only a consistency estimate), see [97, 94]: for a second order scheme and a sufficiently smooth u , the error in maximum norm is bounded by

$$C_d(n + 2(d - 1))^{d-1}2^{-2n}.$$

Applications to Option Pricing Sparse methods have been applied for pricing derivatives by several authors, in particular Reisinger [95], Schwab et al [109] with wavelets.

For option pricing, one of the main difficulty is that the payoff function is generally not smooth and furthermore that the locus of its singularity has no relation with the directions of the sparse grid (or sparse tensor product): therefore, the error of the sparse approximation will increase (blow up) near maturity.

For basket options with a payoff depending on the weighted sum $\sum_{i=1}^d \alpha_i S_i$, the change of variable (2.17) proposed by Reisinger [95] may be used; for a Cartesian grid in the new variables $(y_i)_{1 \leq i \leq d}$ or for a sparse grid obtained by removing nodes from the last Cartesian grid, the locus of the singularity is an hyperplane perpendicular to one of the grid's directions. This enables grid refinement in the last direction, which decreases the error while keeping the size of the discrete problem reasonable. The resulting grid is sparse in the directions parallel to the last hyperplane and nonuniformly refined in the remaining direction. The price to pay is a more complicated partial differential equation. Of course, this trick is not possible with other options such as best-of options; more involved refinement strategies have then to be used, see e.g. [51] and the examples below.

To compensate the loss of regularity at maturity, Schwab et al [109] have proposed to use a time stepping with a very nonuniform time-grid suitably refined near maturity.

An even more difficult case is that of American options, see § 2.3, because the pricing function exhibits a singularity at the exercise boundary which is an unknown and cannot be related to the grid's directions.

As an illustration, we plot the pricing function for a put on an average of two assets computed with a sparse grid, see Figure 2.7. Here, the singularity of the payoff is not aligned with the grid. In Figure 2.8, we show an adapted sparse grid for the same baskets on two assets. The sparse grid is computed by progressively enriching an initial coarse grid. The mesh is refined near a node if the corresponding coefficient in the multilevel expansion of the discrete solution is larger than a threshold.

We shall see later that sparse methods prove more useful for option pricing on a single asset but with a multifactor stochastic volatility, see § 2.4.4. In this case, the payoff function depends on the price variable only. Hence, the singularity is located on an hyperplane in the price/volatilities space, and sparse grids can be used in an easy way.

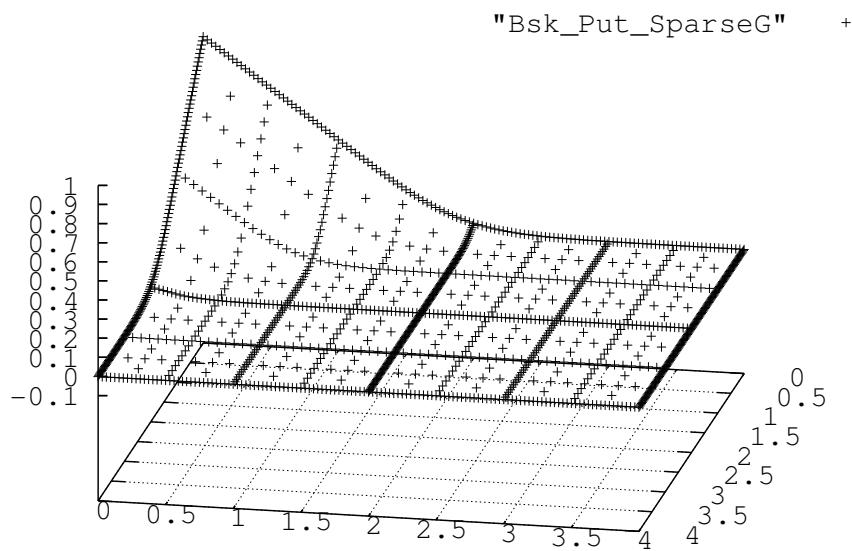


Figure 2.7: The pricing function of a European option on a basket of two assets, computed on a sparse grid (many thanks to David Pommier who wrote the software and gave us this figure).

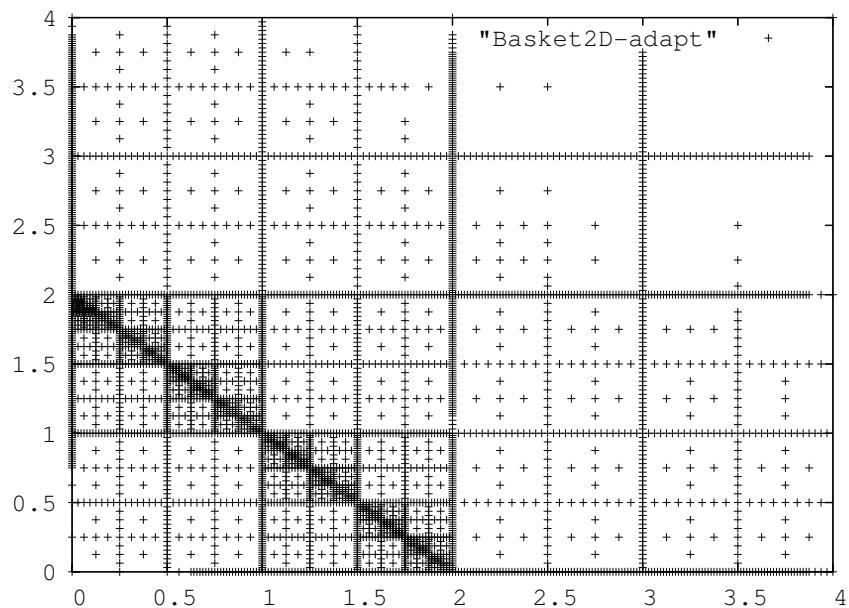


Figure 2.8: Adapted sparse grid for an option on a basket of two assets, (thanks to David Pommier).

2.3 American Basket Options

We consider an American option on the d risky assets whose prices $S_{i,t}$ are the processes described at the beginning of §2.1. The maturity of the option is T and its payoff function is $P_o : \mathbb{R}_+^d \mapsto \mathbb{R}_+$.

The Black-Scholes model leads to the following formula for the price of the American option at time t : under the risk-neutral probability,

$$P_t = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^* \left(e^{-\int_t^\tau r(s)ds} P_o(S_{1,\tau}, \dots, S_{d,\tau}) \middle| F_t \right), \quad (2.91)$$

where $\mathcal{T}_{t,T}$ denotes the set of stopping times in $[t, T]$, (see [68, 67]). It is clear that $P_t \geq P_o(S_{1,t}, \dots, S_{d,t})$.

Under suitable assumptions on the payoff P_o and on the volatilities, it can be proven that P_t is a function of $S_{1,t}, \dots, S_{d,t}$ and t , i.e. $P_t = P(S_{1,t}, \dots, S_{d,t}, t)$, and that the pricing function P is the solution of a variational inequality, see (2.96) below, which is the variational form of the following linear complementarity problem:

$$\begin{aligned} \frac{\partial P}{\partial t} + LP - rP &\leq 0, & \text{in } \mathbb{R}_+^d \times [0, T), \\ P &\geq P_o & \text{in } \mathbb{R}_+^d \times [0, T), \\ \left(\frac{\partial P}{\partial t} + LP - rP \right) (P - P_o) &= 0, & \text{in } \mathbb{R}_+^d \times [0, T), \\ P|_{t=T} &= P_o & \text{in } \mathbb{R}_+^d, \end{aligned} \quad (2.92)$$

where L is given by (2.3). The proof of this result goes beyond the scope of this paragraph: it can be found in Bensoussan et al [15] or in Jaiillet et al [67].

2.3.1 The Variational Inequality

Here, we assume that $P_o \in L^2(\mathbb{R}_+^d)$. It is possible to deal with other payoff functions by using Sobolev spaces with different weights at infinity, see [15] or [67]. Calling t the time to maturity, (2.92) becomes

$$\begin{aligned} \frac{\partial P}{\partial t} - LP + rP &\geq 0, & \text{in } \mathbb{R}_+^d \times (0, T], \\ P &\geq P_o & \text{in } \mathbb{R}_+^d \times (0, T], \\ \left(\frac{\partial P}{\partial t} - LP + rP \right) (P - P_o) &= 0, & \text{in } \mathbb{R}_+^d \times (0, T], \\ P|_{t=0} &= P_o & \text{in } \mathbb{R}_+^d, \end{aligned} \quad (2.93)$$

where L is given by (2.3). To write the variational formulation of (2.93), we use the same Sobolev space V as for the European option, see (2.31). We call \mathbb{K} the closed and convex subset of V :

$$\mathbb{K} = \{v \in V, v \geq P_o \text{ a.e. in } \mathbb{R}_+^d\}. \quad (2.94)$$

and we introduce \mathcal{K} :

$$\mathcal{K} = \{v \in L^2(0, T; V), \text{s.t. } v(t) \in \mathbb{K} \text{ for a.a. } t \in (0, T)\} \quad (2.95)$$

Following Lions, [75], a variational formulation of (2.93) is to

find $P \in \mathcal{K}$ such that $\frac{dP}{dt} \in L^2(0, T; V')$ and $P(t = 0) = P_0$ and satisfying

$$\forall v \in \mathcal{K}, \quad \int_0^T \left\langle \frac{dP}{dt}(t), v(t) - P(t) \right\rangle dt + \int_0^T a_t(P(t), v(t) - P(t)) dt \geq 0, \quad (2.96)$$

where \langle , \rangle is the duality pairing between V' and V and a_t is the bilinear form introduced in (2.30).

By adapting the results in [75], see also [70], one can prove the following theorem:

Theorem 2.7 *Under the assumptions (2.33) (2.34) (2.35) and if r is a bounded function defined on $(0, T)$, the variational inequality (2.96) has a unique solution P . Furthermore, $P \in C^0([0, T]; L^2(\mathbb{R}_+^d))$, $\frac{\partial P}{\partial t} \in L^2(\mathbb{R}_+^d \times (0, T))$, $S_i S_j \frac{\partial^2 P}{\partial S_i \partial S_j} \in L^2(\mathbb{R}_+^d \times (0, T))$, $i, j = 1, \dots, d$, and P satisfies the linear complementarity problem (2.93).*

More properties can be proved under stronger assumptions on the payoff function and on the coefficients: for example, if the volatilities are constant and if the payoff function is piecewise linear and continuous with compact support, then P is continuous on $\mathbb{R}_+^d \times [0, T]$. If the coefficients are constant and under some special assumptions on the payoff, it can be proved that $P(\mathbf{S}, t)$ is nondecreasing with respect to t (t is the time to maturity).

2.3.2 The Exercise Region

The exercise region at time t is the set

$$\{ \mathbf{S} \in \mathbb{R}_+^d, \text{ s.t. } P(\mathbf{S}, t) = P_0(\mathbf{S}) \}.$$

The theoretical results concerning the exercise region for American options on baskets strongly depend on the payoff. S. Villeneuve [108] has proved that if the coefficients are constant with $r > 0$ and if P_0 is bounded and continuous, then the exercise region is nonempty. The shape of the exercise region and its behavior near maturity is studied as well in [108] for a particular class of payoff functions.

Examples of exercise regions for American best-of options computed by finite element methods will be given below, see Figure 2.10. It will be seen that the exercise boundary, *i.e.*, the boundary of the exercise region may exhibit rather strong singularities.

2.3.3 Finite Element Methods

Assuming that P_0 has compact support, we truncate the domain as for the European option, *i.e.*, we consider $\Omega = (0, \bar{S})^d$ (where \bar{S} is large enough so that the support of P_0 is strictly contained in Ω) and Γ_0 given in (2.42). We choose to impose homogeneous Dirichlet artificial boundary conditions on Γ_0 .

The Sobolev space V to work with is given in (2.45) and the new definition of \mathbb{K} is $\mathbb{K} = \{v \in V, v \geq P_\circ \text{ a.e in } \Omega\}$. Changing \mathcal{K} accordingly (see (2.95)), the new variational inequality is (2.96) where a_t is given by (2.46). We are now ready to propose a finite element discretization.

We introduce a partition of the interval $[0, T]$ into subintervals $[t_{m-1}, t_m]$, $1 \leq m \leq M$, with $\delta t_i = t_i - t_{i-1}$, $\delta t = \max_i \delta t_i$. We choose a triangulation \mathcal{T}_h of Ω , and we define V_h by the following:

$$V_h = \{v_h \in V, \forall \omega \in \mathcal{T}_h, v_h|_\omega \in \mathcal{P}_1\}. \quad (2.97)$$

where \mathcal{P}_1 is the space of affine functions. For simplicity, we assume that $P_\circ \in V_h$. We define the closed and convex subset \mathbb{K}_h of V_h by

$$\mathbb{K}_h = \{v \in V_h, v_h \geq P_\circ \text{ in } \bar{\Omega}\}. \quad (2.98)$$

The discrete problem arising from the implicit Euler scheme is:

$$\begin{aligned} & \text{find } (P^m)_{0 \leq m \leq M} \in \mathbb{K}_h \text{ satisfying} \\ & P^0 = P_\circ, \end{aligned} \quad (2.99)$$

and for all m , $1 \leq m \leq M$,

$$\forall v \in \mathbb{K}_h, \quad (P^m - P^{m-1}, v - P^m) + \delta t_m a_{t_m} (P^m, v - P^m) \geq 0. \quad (2.100)$$

Expressing P^m , $0 \leq m \leq M$, and v in the nodal basis of V_h , (2.100) is equivalent to the finite dimensional linear complementary system

$$\begin{array}{rcl} \mathbf{M}(\mathbf{U}^m - \mathbf{U}^{m-1}) + \delta t_m \mathbf{A}^m \mathbf{U}^m & \geq & 0, \\ \mathbf{U}^m & \geq & \mathbf{U}^0, \\ (\mathbf{U}^m - \mathbf{U}^0)^T (\mathbf{M}(\mathbf{U}^m - \mathbf{U}^{m-1}) + \delta t_m \mathbf{A}^m \mathbf{U}^m) & = & 0, \end{array} \quad (2.101)$$

where for two vectors \mathbf{U} and \mathbf{V} , $\mathbf{U} \geq \mathbf{V}$ means that all the coordinates of $\mathbf{U} - \mathbf{V}$ are nonnegative.

Assuming that the coefficients satisfy the assumptions (2.33) (2.34) (2.35), it can be proved that for δt small enough (2.101) has a unique solution for all $m = 1, \dots, M$. Stability and convergence in the natural energy norm can be proved as well.

Mesh adaptivity based on a posteriori error estimates is possible too. The description of the error estimators for parabolic variational inequalities goes beyond the scope of the present article. We refer to [26, 106, 85, 86] for a posteriori error estimates and mesh refinement strategies for elliptic variational inequalities. In the parabolic case, a strategy similar to the one in [16] is studied in [4].

2.3.4 Algorithms

The Projected SOR Algorithm

Let us write (2.101) in the simpler form

$$\begin{array}{rcl} \mathbf{B}\mathbf{u} & \geq & \mathbf{f}, \\ \mathbf{u} & \geq & \mathbf{g}, \\ (\mathbf{u} - \mathbf{g})^T (\mathbf{B}\mathbf{u} - \mathbf{f}) & = & 0, \end{array} \quad (2.102)$$

The projected SOR algorithm is an iterative method for solving (2.102). Let ω be a positive real number. The idea is to approximate \mathbf{u} by using a one step recursion formula $\mathbf{u}^{(k+1)} = \psi(\mathbf{u}^{(k)})$ (starting from an initial guess $\mathbf{u}^{(0)}$), where ψ is the nonlinear mapping in \mathbb{R}^N :

$$\begin{aligned} \psi : \mathbf{v} &\mapsto \mathbf{w} = \psi(\mathbf{v}) : \\ \forall i = 1, \dots, N, \quad w_i &= \max(g_i, y_i), \quad \text{and } y_i \text{ is given by} \\ \frac{1}{\omega} \mathbf{B}_{ii} y_i + \sum_{j < i} \mathbf{B}_{ij} w_j &= f_i + \left(\frac{1}{\omega} - 1\right) \mathbf{B}_{ii} v_i - \sum_{j > i} \mathbf{B}_{ij} v_j. \end{aligned} \quad (2.103)$$

This construction is a modification of the so-called successive over relaxation method (SOR) used for systems of linear equations, see [12], [49], [100] for iterative methods: for solving approximately the system $\mathbf{B}\mathbf{v} = \mathbf{f}$, the SOR algorithm constructs the sequence $(\mathbf{v}^{(k)})_k$ (starting from an initial guess $\mathbf{v}^{(0)}$) by the recursion:

$$\forall i = 1, \dots, N, \quad \frac{1}{\omega} \mathbf{B}_{ii} v_i^{(k+1)} + \sum_{j < i} \mathbf{B}_{ij} v_j^{(k+1)} = f_i + \left(\frac{1}{\omega} - 1\right) \mathbf{B}_{ii} v_i^{(k)} - \sum_{j > i} \mathbf{B}_{ij} v_j^{(k)}.$$

Proposition 2.3 *If \mathbf{B} is a diagonal dominant matrix and if $0 < \omega \leq 1$, then the mapping ψ defined in (2.103) is a contraction in \mathbb{R}^N for the norm $\|\cdot\|_\infty$ ($\|\mathbf{v}\|_\infty = \max_{1 \leq i \leq N} |v_i|$). The fixed point of ψ is \mathbf{u} .*

Under the assumptions of Proposition 2.3, the sequence constructed by the PSOR algorithm converges to \mathbf{u} . The speed convergence of convergence depends on the matrix \mathbf{B} and on the relaxation parameter ω . The convergence is generally slow for ill-conditioned matrices.

Primal-dual Methods

Following [65], we first go back to the semi-discrete problem: find $P^m \in \mathbb{K}$ such that

$$\forall v \in \mathbb{K}, \quad (P^m - P^{m-1}, v - P^m) + \delta t_m a_{t_m}(P^m, v - P^m) \geq 0.$$

For any positive constant c , this is equivalent to finding $P^m \in V$ and a Lagrange multiplier $\mu \in V'$ such that

$$\begin{aligned} \forall v \in V, \quad \frac{1}{\delta t_m} (P^m - P^{m-1}, v) + a_{t_m}(P^m, v) - \langle \mu, v \rangle &= 0, \\ \mu &= \max(0, \mu - c(P^m - P^0)). \end{aligned} \quad (2.104)$$

When using an iterative method for solving (2.104), *i.e.*, when constructing a sequence $(P^{m,j}, \mu^j)$ for approximating (P^m, μ) , the Lagrange multiplier μ^j may not be a function if the gradient of $P^{m,j}$ jumps, whereas μ may be a function. Therefore, a dual method (*i.e.* an iterative method for computing μ) may be

difficult to use. As a remedy, K.Ito and K.Kunisch [65] consider a one parameter family of regularized problems based on smoothing the equation for μ as follows:

$$\mu = \alpha \max(0, \mu - c(P^m - P^0)), \quad (2.105)$$

for $0 < \alpha < 1$, which is equivalent to

$$\mu = \max(0, -\chi(P^m - P^0)), \quad (2.106)$$

for $\chi = c\alpha/(1-\alpha) \in (0, +\infty)$. We may consider a generalized version of (2.106):

$$\mu = \max(0, \bar{\mu} - \chi(P^m - P^0)), \quad (2.107)$$

where $\bar{\mu}$ is a fixed function. This turns out to be useful when the complementarity condition is not strict.

It is now possible to study the fully regularized problem

$$\begin{aligned} \forall v \in V, \quad & \frac{1}{\delta t_m} (P^m - P^{m-1}, v) + a_{t_m}(P^m, v) - \langle \mu, v \rangle = 0, \\ & \mu = \max(0, \bar{\mu} - \chi(P^m - P^0)), \end{aligned} \quad (2.108)$$

and prove that it has a unique solution, with μ a square integrable function. A primal-dual active set algorithm for solving (2.108) is

Primal-Dual Active Set Algorithm

- Choose $P^{m,0}$, set $k = 0$
- Loop

1. Set

$$\begin{aligned} \mathcal{A}^{-,k+1} &= \{S : \bar{\mu}^k(S) - \chi(P^{m,k}(S) - P^0(S)) > 0\}, \\ \mathcal{A}^{+,k+1} &= (0, \bar{S}) \setminus \mathcal{A}^{-,k+1}. \end{aligned}$$

2. Solve for $P^{m,k+1} \in V$: $\forall v \in V$,

$$\begin{aligned} 0 = & \frac{1}{\delta t_m} (P^{m,k+1} - P^{m-1}, v) + a_{t_m}(P^{m,k+1}, v) \\ & - (\bar{\mu} - \chi(P^{m,k+1} - P^0), 1_{\mathcal{A}^{-,k+1}} v). \end{aligned} \quad (2.109)$$

3. Set

$$\mu^{k+1} = \begin{cases} 0 & \text{on } \mathcal{A}^{+,k+1}, \\ \bar{\mu} - \chi(P^{m,k+1} - P^0) & \text{on } \mathcal{A}^{-,k+1} \end{cases} \quad (2.110)$$

4. Set $k = k + 1$.

Calling A_m the operator from V to V' : $\langle A_m v, w \rangle = \frac{1}{\delta t_m} (v, w) + a_{t_m}(v, w)$ and $F : V \times L^2(\mathbb{R}_+^d) \rightarrow V' \times L^2(\mathbb{R}_+^d)$

$$F(v, \mu) = \begin{pmatrix} A_m v + \mu - \frac{P^{m-1}}{\delta t_m} \\ \mu - \max(0, \bar{\mu} - \chi(v - P^0)) \end{pmatrix},$$

it is proved in [65] that $G(v, \mu) : V \times L^2(\mathbb{R}_+^d) \rightarrow V' \times L^2(\mathbb{R}_+^d)$,

$$G(v, \mu)h = \begin{pmatrix} A_m h_1 + h_2 \\ h_2 - \chi 1_{\{\bar{\mu} - \chi(v - P^0) > 0\}} h_1 \end{pmatrix}$$

is a generalized derivative of F in the sense that

$$\lim_{\|h\| \rightarrow 0} \frac{\|F(v + h_1, \mu + h_2) - F(v, \mu) - G(v + h_1, \mu + h_2)h\|}{\|h\|} = 0.$$

Note that

$$G(P^{m,k}, \mu^k)h = \begin{pmatrix} A_m h_1 + h_2 \\ h_2 - \chi 1_{\mathcal{A}^{-,k+1}} h_1 \end{pmatrix}.$$

Thus the primal-dual active set algorithm above can be seen as a semi-smooth Newton method applied to F , *i.e.*,

$$(P^{m,k+1}, \mu^{k+1}) = (P^{m,k}, \mu^k) - G^{-1}(P^{m,k}, \mu^k)F(P^{m,k}, \mu^k). \quad (2.111)$$

Indeed, calling $(\delta P^m, \delta \mu) = (P^{m,k+1} - P^{m,k}, \mu^{k+1} - \mu^k)$, it is straightforward to see that in the primal-dual active set algorithm we have

$$\begin{aligned} A_m \delta P^m + \delta \mu &= -A_m P^{m,k} - \mu^k + \frac{P^{m-1}}{\delta t_m}, \\ \delta \mu &= -\mu^k \text{ on } \mathcal{A}^{+,k+1}, \\ \delta \mu - \chi \delta P^m &= -\mu^k + \bar{\mu} - \chi(P^{m,k} - P^0) \text{ on } \mathcal{A}^{-,k+1}, \end{aligned}$$

which is precisely (2.111).

In [65], by using the results proved in [59], Ito and Kunish establish that the primal-dual active set algorithm converges from any initial guess and that if the initial guess is sufficiently close to the solution of (2.108) then the convergence is superlinear.

To solve (2.104), it is possible to successively compute the solutions $(P^m(\chi_\ell), \mu(\chi_\ell))$ of (2.108) for a sequence of parameters (χ_ℓ) converging to $+\infty$, using $(P^m(\chi_\ell), \mu(\chi_\ell))$ as an initial guess for the primal-dual active set algorithm for $(P^m(\chi_{\ell+1}), \mu(\chi_{\ell+1}))$. Of course, it is possible to use the same algorithm for the fully discrete problem. Convergence results hold in the discrete case if there is a discrete maximum principle. The algorithm amounts to solving a sequence of systems of linear equations and the matrix of the system varies at each iteration.

Examples The discrete method discussed above has been applied to compute the pricing function of an American best-of put option on a two assets basket, $P_o(S_1, S_2) = (K - \max(S_1, S_2))^+$. The artificial boundary Γ_0 is $\{\max(S_1, S_2) = \bar{S} = 200\}$. Homogeneous Dirichlet conditions are imposed on Γ_0 . We have chosen two examples:

1. In the first example, the parameters are

$$\sigma_1 = 0.2, \quad \sigma_2 = 0.1, \quad r = 0.05, \quad \rho = -0.6 \quad K = 100.$$

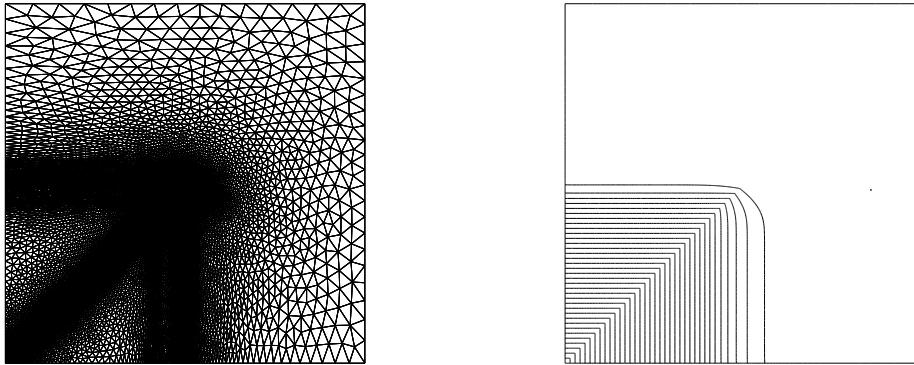


Figure 2.9: The adapted mesh and the contours of P one year to maturity.
 $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $\rho = -0.6$

2. In the second example, the parameters are

$$\sigma_1 = \sigma_2 = 0.2, \quad r = 0.05, \quad \rho = 0 \quad K = 50.$$

The implicit Euler scheme has been used with a uniform time step of 1/250 year. For solving the linear complementarity problems (2.101), we have used the regularized active set method with the regularization parameters $\chi = 10^7$ and $\bar{\mu} = 0$, see (2.107). Mesh adaption in the (S_1, S_2) variable has been performed every 1/10 year. The adaptive strategy is close to the one used in FreeFem, and the mesh is refined in the contact set. This may be unnecessary if the obstacle belongs to the finite element space: we refer to [4] for a better adaptive strategy based on local error indicators where the mesh is not refined in the so-called *strong contact region*, see also [26, 106, 85, 86] for elliptic contact problems.

In Figure 2.9, we have plotted the adapted mesh (left) and the contours of the pricing function (right) one year to maturity for the first example. Note that the contours exhibit right angles in the exercise region. In Figure 2.10, we plot the exercise region one year to maturity for the first example (top) and for the second example (bottom). One sees that the exercise boundary has singularities. It is also visible that the mesh has been adapted near the exercise boundary.

Finally, we consider an American binary option whose payoff is

$$P_{\circ}(S_1, S_2) = K1_{\{\max(S_1, S_2) < K\}},$$

with the parameters of the second example. The exercise region does not evolve. The contours of the pricing function one year to maturity are plotted in Figure 2.11.

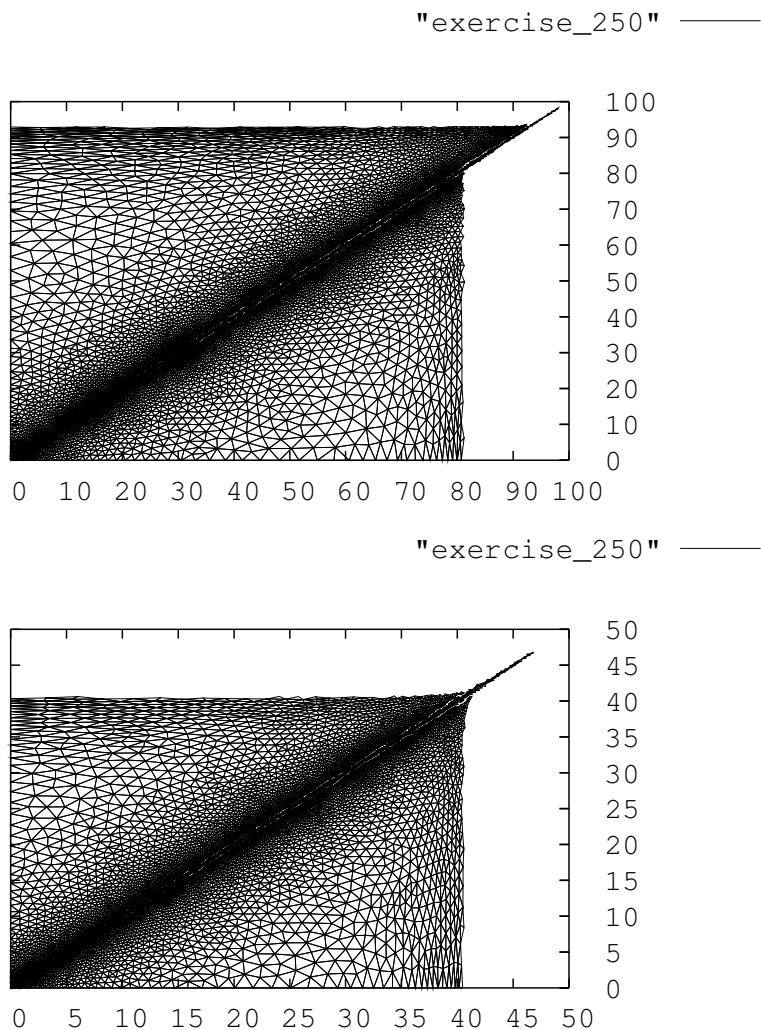


Figure 2.10: The exercise region one year to maturity. Top: $k = 100$, $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $\rho = -0.6$. Bottom: $K = 50$, $\sigma_1 = \sigma_2 = 0.2$, $\rho = 0$

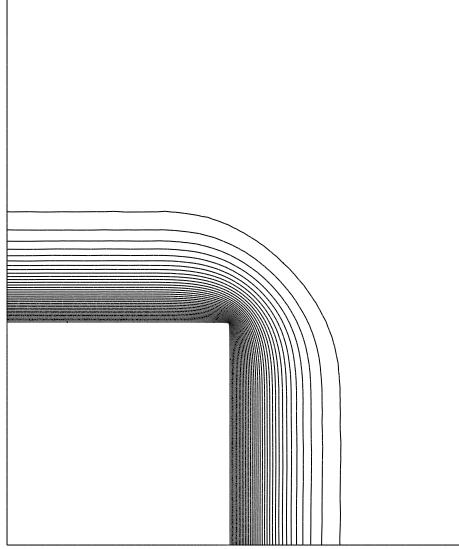


Figure 2.11: The contours of P for an American binary option one year to maturity. $P_o(S_1, S_2) = 50.1_{\{\max(S_1, S_2) < 50\}}$, $\sigma_1 = \sigma_2 = 0.2$, $\rho = 0$

Penalty Methods

Assume that P_o is bounded and that $P_o \in V$ where V is given in (2.31). A penalized version of (2.93) is to look for Q_ϵ such that

$$\begin{aligned} \frac{\partial Q_\epsilon}{\partial t} - LQ_\epsilon + rQ_\epsilon - \mathcal{V}_\epsilon(Q_\epsilon) &= LP_o - rP_o, && \text{in } \mathbb{R}_+^d \times (0, T], \\ Q_\epsilon|_{t=0} &= 0 && \text{in } \mathbb{R}_+^d, \end{aligned} \quad (2.112)$$

where L is given by (2.3), ϵ is a small positive parameter and $\mathcal{V}_\epsilon : \mathbb{R} \rightarrow \mathbb{R}_+$ is a sequence of \mathcal{C}^2 nonincreasing functions such that

- $\mathcal{V}_\epsilon(z) = 0$ if $z > 0$.
- $\mathcal{V}_\epsilon(z) = -\frac{z}{\epsilon}$ if $z < -\epsilon$.
- \mathcal{V}'_ϵ and \mathcal{V}''_ϵ are bounded by $\frac{C}{\epsilon}$.

It is reasonable to think that Q_ϵ is a good approximation of $Q = P - P_o$ as ϵ tends to zero.

The theory of the weak solutions of parabolic partial differential with monotone operators, see [75], can be used because \mathcal{V}_ϵ is nonincreasing. One can prove that (2.112) has a unique weak solution and that Q_ϵ tends to Q in the natural norms associated with the variational problem (2.96). In some cases, error estimates can be found.

The initial value problem (2.112) can in turn be approximated by the finite element method: using e.g. an implicit Euler scheme and piecewise affine functions on a triangulation of Ω , each time step is of the form

$$\mathbf{M}(\mathbf{Q}^m - \mathbf{Q}^{m-1}) + \delta t_m (\mathbf{A}^m \mathbf{Q}^m + V_\epsilon(\mathbf{Q}^m)) = -\delta t_m \mathbf{A}^m \mathbf{U}^0, \quad (2.113)$$

where \mathbf{M} and \mathbf{A} are the matrices in (2.57) and where $V_\epsilon : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a nonlinear operator such that

$$(V_\epsilon(\mathbf{Q}^m))_i \sim \int_{\Omega} w_i \mathcal{V}_\epsilon \left(\sum_{j=1}^N q_j^m w_j \right).$$

Here the sign \sim means that a suitable elementary quadrature rule has been used to approximate $\int_K \mathcal{V}_\epsilon \left(\sum_{j=1}^N q_j^m w_j \right) w_i$, $K \in \mathcal{T}_h$. Newton iterations can be used for solving (2.113), and \mathbf{Q}^{m-1} can be used as an initial guess.

A simpler nonlinear problem is

$$\mathbf{M}(\mathbf{Q}^m - \mathbf{Q}^{m-1}) + \delta t_m (\mathbf{A}^m \mathbf{Q}^m + \mathcal{V}_\epsilon(\mathbf{Q}^m)) = -\delta t_m \mathbf{A}^m \mathbf{U}^0, \quad (2.114)$$

where $\mathcal{V}_\epsilon(\mathbf{Q}^m)$ is the vector defined by $(\mathcal{V}_\epsilon(\mathbf{Q}^m))_i = \mathcal{V}_\epsilon(q_i^m)$, $i = 1, \dots, N$. Convergence may be proved for (2.113) or (2.114).

A similar method has been carefully tested for American options in the slightly different context of a stochastic volatility model, see R. Zvan et al.,[114].

The computing times of the penalty methods and of the primal-dual active set methods are comparable.

Projection Schemes

Let us go back to the sequence of fully discrete linear complementarity problems (2.101) arising from the implicit Euler scheme. It is possible to write them in an equivalent manner using a Lagrange multiplier $\Lambda^m \in \mathbb{R}^N$,

$$\begin{aligned} \mathbf{M}(\mathbf{U}^m - \mathbf{U}^{m-1}) + \delta t_m \mathbf{A}^m \mathbf{U}^m &= \delta t_m \Lambda^m, \\ \mathbf{U}^m &\geq \mathbf{U}^0, \\ \Lambda^m &\geq 0, \\ (\mathbf{U}^m - \mathbf{U}^0)^T \Lambda^m &= 0. \end{aligned} \quad (2.115)$$

Following Lions-Mercier [77], see also [48] and [63], we modify (2.115) in order to obtain a new operator splitting time scheme where each time step is made of two substeps: the first substep is that of the implicit Euler scheme for the corresponding parabolic equation (European option) and the second step is a projection step. Operator splitting schemes are very popular in computational fluid dynamics for incompressible fluids and they are often called *projection schemes*, see e.g [56, 55, 3]. More precisely, the first substep reads:

$$\mathbf{M}(\tilde{\mathbf{U}}^m - \mathbf{U}^{m-1}) + \delta t_m \mathbf{A}^m \tilde{\mathbf{U}}^m = \delta t_m \Lambda^{m-1}, \quad (2.116)$$

where the unknown is $\tilde{\mathbf{U}}^m$ ((2.116) is a system of linear equations) whereas the second substep is

$$\begin{aligned} \mathbf{M}(\mathbf{U}^m - \tilde{\mathbf{U}}^m) &= \delta t_m(\Lambda^m - \Lambda^{m-1}), \\ \mathbf{U}^m &\geq \mathbf{U}^0, \\ \Lambda^m &\geq 0, \\ (\mathbf{U}^m - \mathbf{U}^0)^T \Lambda^m &= 0. \end{aligned} \quad (2.117)$$

The unknowns of the linear complementarity problem (2.117) are \mathbf{U}^m and Λ^m and the matrix of the problem is \mathbf{M} instead of $\mathbf{M} + \delta t_m \mathbf{A}^m$ in (2.115). Since

1. \mathbf{M} is symmetric and positive definite,
2. generally, \mathbf{M} has a much smaller condition number than $\mathbf{M} + \delta t_m \mathbf{A}^m$,

the linear complementarity problem (2.117) is easier to solve than (2.115). For (2.117), either the PSOR algorithm or projected gradient algorithms are efficient. Furthermore, these algorithms can be easily modified in order to use diagonal preconditioning for \mathbf{M} when the finite element mesh is highly nonuniform.

If the operator splitting scheme is used either with mass lumping (see Remark 2.6) or with finite difference schemes instead of finite elements as in [63], then (2.117) becomes easier because \mathbf{M} is replaced with a diagonal matrix and the projection substep consists of solving as many decoupled one dimensional linear complementarity problems as there are nodes in the grid.

Other schemes can be modified to produce operator splitting schemes: for example, the Crank-Nicolson scheme

$$\begin{aligned} \mathbf{M}(\mathbf{U}^m - \mathbf{U}^{m-1}) + \frac{1}{2}\delta t_m(\mathbf{A}^m \mathbf{U}^m + \mathbf{A}^{m-1} \mathbf{U}^{m-1}) &= \delta t_m \Lambda^m, \\ \mathbf{U}^m &\geq \mathbf{U}^0, \\ \Lambda^m &\geq 0, \\ (\mathbf{U}^m - \mathbf{U}^0)^T \Lambda^m &= 0, \end{aligned} \quad (2.118)$$

becomes:

- Linear substep

$$\mathbf{M}(\tilde{\mathbf{U}}^m - \mathbf{U}^{m-1}) + \frac{1}{2}\delta t_m(\mathbf{A}^m \tilde{\mathbf{U}}^m + \mathbf{A}^{m-1} \mathbf{U}^{m-1}) = \delta t_m \Lambda^{m-1}, \quad (2.119)$$

- Projection substep: (2.117).

The second order accurate backward Euler scheme

$$\begin{aligned} \mathbf{M}(\mathbf{U}^m - \frac{4}{3}\mathbf{U}^{m-1} + \frac{1}{3}\mathbf{U}^{m-2}) + \frac{2}{3}\delta t_m \mathbf{A}^m \mathbf{U}^m &= \frac{2}{3}\delta t_m \Lambda^m, \\ \mathbf{U}^m &\geq \mathbf{U}^0, \\ \Lambda^m &\geq 0, \\ (\mathbf{U}^m - \mathbf{U}^0)^T \Lambda^m &= 0, \end{aligned} \quad (2.120)$$

becomes:

- Linear substep

$$\mathbf{M}(\tilde{\mathbf{U}}^m - \frac{4}{3}\mathbf{U}^{m-1} + \frac{1}{3}\mathbf{U}^{m-2}) + \frac{2}{3}\delta t_m \mathbf{A}^m \tilde{\mathbf{U}}^m = \frac{2}{3}\delta t_m \Lambda^{m-1}, \quad (2.121)$$

- Projection substep

$$\begin{aligned} \mathbf{M}(\mathbf{U}^m - \tilde{\mathbf{U}}^m) &= \frac{2}{3}\delta t_m(\Lambda^m - \Lambda^{m-1}), \\ \mathbf{U}^m &\geq \mathbf{U}^0, \\ \Lambda^m &\geq 0, \\ (\mathbf{U}^m - \mathbf{U}^0)^T \Lambda^m &= 0. \end{aligned} \quad (2.122)$$

It is possible to estimate the error produced by using the projection schemes above. This was done in [63] in the context of finite differences: for the Crank-Nicolson scheme with constant time steps and time-invariant coefficients, it was shown that the error between the solution of (2.118) and that of (2.119,2.117) is of the order of δt^2 , if some stability assumptions are satisfied (\mathbf{A} is diagonally dominant with positive diagonal entries). Therefore both the Crank-Nicolson and the projected Crank-Nicolson schemes are second order in time. It is also possible to obtain error estimates in Sobolev norms for the three schemes above by using the techniques in [56, 55].

In terms of computing time, we found that the projection scheme is clearly faster than the primal-dual algorithm above for a comparable accuracy. In our experience, the only difficulty posed by the projection schemes is that they are not fully compatible with a strategy of dynamic mesh adaption (*i.e.*, when the mesh is adapted between two time steps).

Remark 2.7 *Operator splitting can also be applied to the penalized problems (2.113) or (2.114).*

Example We use the projected Euler scheme (2.116,2.117) for computing the pricing in the first example in § 2.3.4. In Figure 2.12, we plot the exercise region one year to maturity. This figure must be compared with Figure 2.10(top). The difference is not visible.

Multigrid Methods

Multigrid methods can also be used for linear complementarity problems: one possibility is to modify the primal-dual algorithm described above: recall that each iteration of such algorithms requires the solution of a linear boundary value problem in a varying subdomain. The idea is to use a multigrid method for these linear substeps. The main difficulty lies in the fact that one cannot obtain a hierarchy of meshes of the subdomains by simply taking subsets of the hierarchical meshes for the whole domain. We refer to [61] for such algorithms for finite difference schemes. In the same context but for finite elements, additive multilevel preconditioners were proposed in [60].

Another possible way is to design a multigrid cycle for the full nonlinear problem. This has been done in [87, 96, 64].

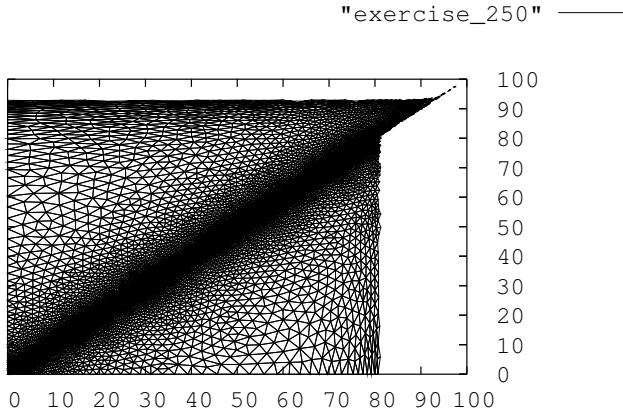


Figure 2.12: The exercise region one year to maturity. $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $\rho = -0.6$. This has to be compared with Figure 2.10

Conclusion

We have presented different algorithms relevant in the context of American options pricing. The PSOR and primal-dual methods are algorithms for solving the linear complementarity problem at each time step. Multigrid methods can be used either as a component of the primal-dual methods or for directly solving the linear complementarity problem. The penalty method replaces the linear complementarity problem by a penalized nonlinear problem. The projection schemes consist of dividing each time step into two substeps, the first one for diffusion and the other one to enforce the constraints.

In our experience, the projection schemes are fast (faster than the primal-dual methods not combined with multigrid) and their implementation is very simple and general. Multigrid methods may be faster but their implementation certainly needs more care.

2.4 Stochastic volatility

2.4.1 Volatility Models with One Stochastic Process

We consider a financial asset whose price is given by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t, \quad (2.123)$$

where $\mu S_t dt$ is a drift term, (W_t) is a Brownian motion, and (σ_t) is the volatility. As we have seen before, the simplest models take a constant deterministic

volatility, but these models are generally too coarse to match real market prices. Here, we assume that (σ_t) is a stochastic process taking nonnegative values, satisfying a stochastic differential equation driven by a second Brownian motion \hat{Z}_t , not perfectly correlated to W_t . More precisely, using the notations of [44], we assume that $\sigma_t = f(Y_t)$, where f is some positive function and (Y_t) is the driving process; the most common choices for (Y_t) are

- lognormal process:

$$dY_t = c_1 Y_t dt + c_2 Y_t d\hat{Z}_t, \quad (2.124)$$

where c_1 and c_2 are positive constants.

- mean reverting Orstein-Uhlenbeck (OU) process:

$$dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t, \quad (2.125)$$

where α and β are positive constants.

- Cox-Ingersoll-Ross (CIR) process:

$$dY_t = \kappa(m - Y_t)dt + \lambda\sqrt{Y_t}d\hat{Z}_t, \quad (2.126)$$

where κ , m and λ are positive constants.

One of the important feature of the second and third processes is their mean-reversion: the drift term in the stochastic differential equation for Y_t pulls the process back to the *long-run mean level* m . For example, if (Y_t) is a mean reverting OU process satisfying (2.125), then the law of Y_t knowing Y_0 is

$$\mathcal{N}\left(m + (Y_0 - m)e^{-\alpha t}, \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})\right).$$

Therefore, m is the limit of the mean value of Y_t as $t \rightarrow +\infty$, and $\frac{1}{\alpha}$ is the characteristic time of mean reversion. The parameter α is called the *rate of mean reversion*. The ratio $\frac{\beta^2}{2\alpha}$ is the limit of the variance of Y_t as $t \rightarrow +\infty$. The *long-run distribution* of the OU process is $\mathcal{N}\left(m, \frac{\beta^2}{2\alpha}\right)$.

The Brownian motion \hat{Z}_t may be correlated to W_t : it can be written as a linear combination of W_t and an independent Brownian motion Z_t :

$$\hat{Z}_t = \rho W_t + \sqrt{1 - \rho^2} Z_t, \quad (2.127)$$

where the *correlation factor* ρ lies in $[-1, 1]$.

The table 2.1 summarizes the most popular choices of stochastic volatility models with one stochastic process

Table 2.1: Frequently used models of stochastic volatilities

Authors	ρ	$f(y)$	Y_t process
Hull-White [62]	$\rho = 0$	$f(y) = \sqrt{y}$	lognormal
Scott	$\rho = 0$	$f(y) = e^y$	mean reverting OU
Stein-Stein [103]	$\rho = 0$	$f(y) = y $	mean reverting OU
Heston [58]	$\rho \neq 0$	$f(y) = \sqrt{y}$	CIR

2.4.2 European Options with Stochastic Volatility

Consider a European option on the previously mentioned asset, with expiration date T and payoff function $P_o(S_T)$. Its price at time $t < T$ will depend on t , on the price of the underlying asset S_t , and on Y_t . We denote by $P(S_t, Y_t, t)$ the price of the option, and by $\tilde{r}(t)$ the interest rate.

The option is priced using a no arbitrage principle and the two dimensional Itô's formula. However, since there are two risk factors, it is not possible to construct a hedged portfolio containing simply one option and shares of the underlying asset. One says that the market is *incomplete*. Instead, one can try to build a hedged portfolio containing options with two different maturities and shares of the underlying assets; only for simplicity, we restrict the discussion to the case when (Y_t) is a OU process satisfying (2.125), with (\hat{Z}_t) given by (2.127), but what follows can be generalized to any Markovian Itô driving process:

$$dY_t = \mu_Y(t, Y_t)dt + \sigma_Y(t, Y_t)d\tilde{Z}_t.$$

Let us try to build a self-financing hedged portfolio containing a_t shares of the underlying asset, one option with expiration date T_1 whose price is

$$P_t^{(1)} = P^{(1)}(S_t, Y_t, t)$$

and b_t options with a larger expiration date $T_2 > T_1$, whose price is

$$P_t^{(2)} = P^{(2)}(S_t, Y_t, t).$$

The value of the portfolio is c_t . The no arbitrage principle yields that for $t < T_1$,

$$dc_t = a_t dS_t + dP_t^{(1)} + b_t dP_t^{(2)} = \tilde{r}_t c_t dt = \tilde{r}_t (a_t S_t + P_t^{(1)} + b_t P_t^{(2)}) dt \quad (2.128)$$

The two-dimensional Itô formula permits $dP_t^{(1)}$ and $dP_t^{(2)}$ to be expressed as combinations of dt , dW_t and dZ_t . The right hand side of (2.128) does not contain dZ_t , thus

$$b_t = -\frac{\frac{\partial P^{(2)}}{\partial y}}{\frac{\partial P^{(1)}}{\partial y}}.$$

From the last equation and since the right hand side of (2.128) does not contain dW_t , we deduce

$$a_t + \frac{\partial P^{(1)}}{\partial S} + b_t \frac{\partial P^{(2)}}{\partial S} = 0.$$

Comparing the dt terms in (2.128) and substituting the values of a_t and b_t , we obtain

$$\frac{1}{\partial P^{(1)} / \partial y} \left(\frac{\partial P^{(1)}}{\partial t} + \frac{1}{2} f(y)^2 S^2 \frac{\partial^2 P^{(1)}}{\partial S^2} + \rho \beta S f(y) \frac{\partial^2 P^{(1)}}{\partial S \partial y} + \frac{1}{2} \beta^2 \frac{\partial^2 P^{(1)}}{\partial y^2} + \tilde{r}(t) \left(S \frac{\partial P^{(1)}}{\partial S} - P^{(1)} \right) \right) = \\ \frac{1}{\partial P^{(2)} / \partial y} \left(\frac{\partial P^{(2)}}{\partial t} + \frac{1}{2} f(y)^2 S^2 \frac{\partial^2 P^{(2)}}{\partial S^2} + \rho \beta S f(y) \frac{\partial^2 P^{(2)}}{\partial S \partial y} + \frac{1}{2} \beta^2 \frac{\partial^2 P^{(2)}}{\partial y^2} + \tilde{r}(t) \left(S \frac{\partial P^{(2)}}{\partial S} - P^{(2)} \right) \right).$$

In the equation above, the left hand side does not depend on T_2 and the right hand side does not depend on T_1 . Therefore, there exists a function $g(S, y, t)$ such that

$$\frac{1}{\partial P / \partial y} \left(\frac{\partial P}{\partial t} + \frac{1}{2} f(y)^2 S^2 \frac{\partial^2 P}{\partial S^2} + \rho \beta S f(y) \frac{\partial^2 P}{\partial S \partial y} + \frac{1}{2} \beta^2 \frac{\partial^2 P}{\partial y^2} + \tilde{r}(t) \left(S \frac{\partial P}{\partial S} - P \right) \right) \\ = g(S, y, t).$$

Writing $g(S, y, t) = \alpha(y - m) + \beta \tilde{\Lambda}(S, y, t)$ makes the infinitesimal generator of the OU process explicit in the last equation. We obtain

$$\frac{\partial P}{\partial t} + \frac{1}{2} f(y)^2 S^2 \frac{\partial^2 P}{\partial S^2} + \rho \beta S f(y) \frac{\partial^2 P}{\partial S \partial y} + \frac{1}{2} \beta^2 \frac{\partial^2 P}{\partial y^2} \\ + \tilde{r}(t) \left(S \frac{\partial P}{\partial S} - P \right) + \left(\alpha(m - y) - \beta \tilde{\Lambda}(S, y, t) \right) \frac{\partial P}{\partial y} = 0, \quad \begin{cases} 0 \leq t < T, \\ S > 0, y \in \mathbb{R}, \end{cases} \quad (2.129)$$

where

$$\tilde{\Lambda}(S, y, t) = \rho \frac{\mu - \tilde{r}(t)}{f(y)} + \sqrt{1 - \rho^2} \tilde{\gamma}(S, y, t), \quad (2.130)$$

with the terminal condition $P(S, y, T) = P_o(S)$.

The function $\tilde{\gamma}(S, y, t)$ (return on the volatility risk) can be chosen arbitrarily. As explained in [44], we can group the differential operator in (2.129) as follows:

$$\underbrace{\frac{\partial P}{\partial t} + \frac{1}{2} f(y)^2 S^2 \frac{\partial^2 P}{\partial S^2} + \tilde{r}(t) \left(S \frac{\partial P}{\partial S} - P \right)}_{BS f(y)} + \underbrace{\rho \beta S f(y) \frac{\partial^2 P}{\partial S \partial y}}_{\text{correlation}} \\ + \underbrace{\frac{1}{2} \beta^2 \frac{\partial^2 P}{\partial y^2}}_{\text{Orstein Uhlenbeck}} + \underbrace{\alpha(m - y) \frac{\partial P}{\partial y} - \beta \tilde{\Lambda}(S, y, t) \frac{\partial P}{\partial y}}_{\text{premium}}. \quad (2.131)$$

The term $\beta \tilde{\Lambda}(S, y, t) \frac{\partial P}{\partial y}$ is a premium on the *volatility risk*: the reason to decompose $\tilde{\Lambda}$ as in (2.130) is that in the perfectly correlated case ($|\rho| = 1$, *complete market*), it is possible to find the equation satisfied by P by a simpler no arbitrage argument with a hedged portfolio containing only the option and shares of the underlying assets. In this case, the equation found for P is:

$$\frac{\partial P}{\partial t} + \frac{1}{2} f(y)^2 S^2 \frac{\partial^2 P}{\partial S^2} + \rho \beta S f(y) \frac{\partial^2 P}{\partial S \partial y} + \frac{1}{2} \beta^2 \frac{\partial^2 P}{\partial y^2} \\ + \tilde{r}(t) \left(S \frac{\partial P}{\partial S} - P \right) + \left(\alpha(m - y) - \beta \rho \frac{\mu - \tilde{r}(t)}{f(y)} \right) \frac{\partial P}{\partial y} = 0, \quad \begin{cases} 0 \leq t < T, \\ S > 0, y \in \mathbb{R}. \end{cases} \quad (2.132)$$

The term $\frac{\mu - \tilde{r}(t)}{f(y)}$ is called the *excess return to risk ratio*.

Finally, with (2.129), the Itô formula and (2.130)

$$dP(S_t, Y_t, t) = (S f(Y_t) \frac{\partial P}{\partial S} + \beta \rho \frac{\partial P}{\partial y}) (\frac{\mu - \tilde{r}}{f(Y_t)} dt + dW_t) + \beta \sqrt{1 - \rho^2} \frac{\partial P}{\partial y} (\tilde{\gamma} dt + dZ_t)$$

from which we see that the function $\tilde{\gamma}$ is the contribution of the second source of randomness dZ_t to the risk premium. The function $\tilde{\gamma}$ is called the *market price of the volatility risk* or the *risk premium factor*.

Similarly, assuming that (Y_t) is a CIR process satisfying (2.126), one obtains

$$\underbrace{\frac{\partial P}{\partial t} + \frac{1}{2} f(y)^2 S^2 \frac{\partial^2 P}{\partial S^2} + \tilde{r}(t) \left(S \frac{\partial P}{\partial S} - P \right)}_{BS_{f(y)}} + \underbrace{\rho \lambda S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y}}_{\text{correlation}} \\ + \underbrace{\frac{1}{2} \lambda^2 y \frac{\partial^2 P}{\partial y^2}}_{\text{CIR}} + \underbrace{\kappa(m - y) \frac{\partial P}{\partial y}}_{\text{premium}} - \lambda \sqrt{y} \tilde{\Lambda}(S, y, t) \frac{\partial P}{\partial y} = 0, \quad (2.133)$$

where $\tilde{\Lambda}(S, y, t)$ is given by (2.130).

Remark 2.8 *It is possible to obtain (2.129) and (2.133) by using a more mathematically sound risk-neutral theory, and the market price of the volatility risk appears from Girsanov's theorem, see [44], §2.5.*

Remark 2.9 *For the Heston model, a closed form solution in terms of integrals is available, see [58].*

The Initial Value Problem for Stein-Stein's Model We discuss the mathematical analysis of the initial value problem with (2.129), in the case when $\rho = 0$ and $f(y) = |y|$ (Stein-Stein's model). The goal is to study variational formulations of (2.129), and obtain global energy estimates. These estimates are useful for studying discrete approximations by e.g. finite element methods. Variational formulations are also particularly useful for the linear complementarity problems obtained when pricing American options. This paragraph summarizes the results contained in [8] and [2].

For simplicity, we assume that the market price of risk $\tilde{\gamma}$ is bounded independently of t , S and y . The variance of the invariant distribution of the OU process, i.e. $\nu^2 = \frac{\beta^2}{2\alpha}$, will play an important role in what follows.

In order to obtain a forward parabolic equation, we work with the time to maturity, i.e. $T - t \rightarrow t$. With the aim of deriving a variational formulation, we make the change of unknown

$$u(S, y, t) = P(S, y, T - t) e^{-(1-\eta) \frac{(y-m)^2}{2\nu^2}}, \quad (2.134)$$

where η is a parameter such that $0 < \eta < 1$; we are going to impose that u tends to 0 as y tends to ∞ . Indeed, if $\tilde{\Lambda} = 0$, then one can find a solution of

(2.129) of the form $g(t)e^{\frac{(y-m)^2}{2\nu^2}}$; imposing that $u(S, y, t)$ tends to 0 as y tends to ∞ prevents such a behavior for large values of y . The parameter η will not be important for practical computations, because in any case, we have to truncate the domain and suppress large values of y .

With the notations $r(t) = \tilde{r}(T-t)$, $\gamma(t) = \tilde{\gamma}(T-t)$ and $\Lambda(t) = \tilde{\Lambda}(T-t)$ the new unknown u satisfies the degenerate parabolic partial differential equation

$$\begin{aligned} & \frac{\partial u}{\partial t} - \frac{1}{2}y^2S^2\frac{\partial^2 u}{\partial S^2} - r(t)\left(S\frac{\partial u}{\partial S} - u\right) - \frac{1}{2}\beta^2\frac{\partial^2 u}{\partial y^2} \\ & + (-\alpha(y-m) + \beta\Lambda(S, y, t))\frac{\partial u}{\partial y} + \left(2\frac{\alpha}{\beta}\Lambda(S, y, t)(y-m) - \alpha\right)u \\ & + \eta\left(2\alpha(y-m)\frac{\partial u}{\partial y} + 2\frac{\alpha^2}{\beta^2}(1-\eta)(y-m)^2u - 2\frac{\alpha}{\beta}\Lambda(y-m)u + \alpha u\right) = 0. \end{aligned} \quad (2.135)$$

The equation is degenerate near the axis $y = 0$ because the coefficient in front of $S^2\frac{\partial^2 u}{\partial S^2}$ vanishes on $y = 0$.

Expanding Λ and denoting by \mathcal{L}_t the linear partial differential operator

$$\begin{aligned} & \mathcal{L}_t v \\ & = -\frac{1}{2}y^2S^2\frac{\partial^2 v}{\partial S^2} - \frac{1}{2}\beta^2\frac{\partial^2 v}{\partial y^2} - r(t)S\frac{\partial v}{\partial S} + ((1-2\eta)\alpha(y-m) + \beta\gamma(S, y, t))\frac{\partial v}{\partial y} \\ & + \left(r(t) + 2\frac{\alpha^2}{\beta^2}\eta(1-\eta)(y-m)^2 + 2(1-\eta)\frac{\alpha}{\beta}(y-m)(\gamma(S, y, t)) - \alpha(1-\eta)\right)v, \end{aligned} \quad (2.136)$$

we obtain

$$\frac{\partial u}{\partial t} + \mathcal{L}_t u = 0. \quad (2.137)$$

We denote by Q the open half plane $Q = \mathbb{R}_+ \times \mathbb{R}$. Let us consider the weighted Sobolev space V :

$$V = \left\{ v : \left(\sqrt{1+y^2}v, \frac{\partial v}{\partial y}, S|y|\frac{\partial v}{\partial S}\right) \in (L^2(Q))^3 \right\}. \quad (2.138)$$

This space with the norm

$$|||v|||_V = \left(\int_Q (1+y^2)v^2 + \left(\frac{\partial v}{\partial y}\right)^2 + S^2y^2\left(\frac{\partial v}{\partial S}\right)^2 \right)^{\frac{1}{2}} \quad (2.139)$$

is a Hilbert space, and it has the following properties:

1. V is separable.
2. Calling $\mathcal{D}(Q)$ the space of smooth functions with compact support in Q , $\mathcal{D}(Q) \subset V$ and $\mathcal{D}(Q)$ is dense in V
3. V is dense in $L^2(Q)$.

The crucial point is point 2 which can be proved by an argument due to Friedrichs (Theorem 4.2 in [46]). We also have

Lemma 2.4 *For any function v in V ,*

$$\int_Q y^2 v^2 \leq 4 \int_Q y^2 S^2 \left(\frac{\partial v}{\partial S} \right)^2. \quad (2.140)$$

The semi-norm

$$\|v\|_V = \left(\int_Q v^2 + \left(\frac{\partial v}{\partial y} \right)^2 + S^2 y^2 \left(\frac{\partial v}{\partial S} \right)^2 \right)^{\frac{1}{2}} \quad (2.141)$$

is in fact a norm in V , equivalent to $\|\cdot\|_V$.

We call V' the dual of V . In order to use the general theory of Lions and Magenes [76] on parabolic equations, we first need to prove the following lemma:

Lemma 2.5 *The operator $v \rightarrow S \frac{\partial v}{\partial S}$ is continuous from V into V' .*

Proof. Call X and Y the differential operators

$$X(v) = Sy \frac{\partial v}{\partial S} + \beta \frac{\partial v}{\partial y}, \quad Y(v) = Sy \frac{\partial v}{\partial S} - \beta \frac{\partial v}{\partial y}, \quad (2.142)$$

The operators X and Y are continuous operators from V into $L^2(Q)$ and their adjoints are

$$\begin{aligned} X^T(v) &= -Sy \frac{\partial v}{\partial S} - \beta \frac{\partial v}{\partial y} - yv = -X(v) - yv, \\ Y^T v &= -Sy \frac{\partial v}{\partial S} + \beta \frac{\partial v}{\partial y} - yv = -Y(v) - yv. \end{aligned} \quad (2.143)$$

Consider the commutator $[X, Y] = XY - YX$: it can be checked that

$$[X, Y](v) = 2\beta S \frac{\partial v}{\partial S}. \quad (2.144)$$

Therefore, for $v \in V$ and $w \in \mathcal{D}(Q)$,

$$(2\beta S \frac{\partial v}{\partial S}, w) = - \int_Q Y(v)(X(w) + yw) + \int_Q X(v)(Y(w) + yw), \quad (2.145)$$

and from (2.140), there exists a constant C such that

$$|(2\beta S \frac{\partial v}{\partial S}, w)| \leq C \|v\|_V \|w\|_V.$$

To conclude, we use the density of $\mathcal{D}(Q)$ into V . ■

Lemma 2.5 implies that the operator \mathcal{L}_t is continuous from V to its dual V' .

Calling a_t the bilinear form defined on $V \times V$ by $a_t(u, v) = \langle \mathcal{L}_t u, v \rangle$, we have

$$\begin{aligned} a_t(u, v) = & \frac{1}{2} \int_Q y^2 S^2 \frac{\partial u}{\partial S} \frac{\partial v}{\partial S} + \int_Q y^2 S \frac{\partial u}{\partial S} v + \frac{\beta^2}{2} \int_Q \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \\ & + \frac{r(t)}{2\beta} \left(\int_Q Y(u)(X(v) + yv) - \int_Q X(u)(Y(v) + yv) \right) \\ & + \int_Q ((2\eta - 1)\alpha(y - m) + \beta\gamma(S, y, t)) \frac{\partial u}{\partial y} v \\ & + \int_Q \left(r(t) + (1 - \eta) \left(2\frac{\alpha^2}{\beta^2}\eta(y - m)^2 + 2\frac{\alpha}{\beta}(y - m)\gamma(S, y, t) - \alpha \right) \right) uv. \end{aligned} \quad (2.146)$$

Proposition 2.4 Assume that r is a bounded function of time and that γ is bounded by a constant. The bilinear form a_t is continuous on $V \times V$, with a continuity constant independent of t .

We also need a Gårding inequality:

Proposition 2.5 Assume that r is a bounded function of time and that γ is bounded by a constant Γ . If $\alpha > \beta$, then there exist two positive constants C and c independent of t and two constants $0 < \eta_1 < \eta_2 < 1$ such that, for $\eta_1 < \eta < \eta_2$ and for any $v \in V$,

$$a_t(v, v) \geq C\|v\|_V^2 - c\|v\|_{L^2(Q)}^2. \quad (2.147)$$

From Propositions 2.4 and 2.5, we can prove the existence and uniqueness of weak solutions to the initial value problem with (2.137).

Theorem 2.8 Assume that $\alpha > \beta$ and that η has been chosen as in Proposition 2.5. Then, for any $u_\circ \in L^2(Q)$, there exists a unique u in $L^2(0, T; V) \cap \mathcal{C}^0([0, T]; L^2(Q))$, with $\frac{\partial u}{\partial t} \in L^2(0, T; V')$ such that, for any smooth function $\phi \in \mathcal{D}(0, T)$, for any $v \in V$,

$$-\int_0^T \phi'(t) \left(\int_Q u(t)v \right) dt + \int_0^T \phi(t) a_t(u, v) dt = 0 \quad (2.148)$$

and

$$u(t = 0) = u_\circ. \quad (2.149)$$

The mapping $u_\circ \mapsto u$ is continuous from $L^2(Q)$ to $L^2(0, T; V) \cap \mathcal{C}^0([0, T]; L^2(Q))$.

Remark 2.10 The ratio $\frac{2\alpha^2}{\beta^2}$ is exactly the ratio between the rate of mean reversion and the asymptotic variance of the volatility. The assumption in Theorem 2.8 says that the rate of mean reversion should not be too small compared with the asymptotic variance of the volatility. This condition is usually satisfied in practice, since α is often much larger than the asymptotic variance $\frac{\beta^2}{2\alpha}$.

It is possible to prove a maximum principle, see[8]: as a consequence, in the case of a vanilla put, we see that the weak solution given by Theorem 2.8 has a financially correct behavior:

Proposition 2.6 *Assume that the coefficients are smooth and bounded, and that $\alpha > \beta$. If $P_o(S, y) = (K - S)^+$, then the function*

$$P(t, S, y) = u(T - t, S, y)e^{(1-\eta)\frac{(y-m)^2}{2\nu^2}},$$

where u is the solution to (2.148), (2.149) with $u_o = e^{(1-\eta)\frac{(y-m)^2}{2\nu^2}} P_o$, satisfies

$$(S - Ke^{-r(T-t)})^- \leq P(t, S, y) \leq Ke^{-r(T-t)}, \quad (2.150)$$

and we have the put-call parity $C(t, S, y) - P(t, S, y) = S - Ke^{-r(T-t)}$, if C is the pricing function of the corresponding call option.

Consider now \mathcal{L}_t as an unbounded operator defined on $L^2(Q)$ and call D_t the domain of \mathcal{L}_t , i.e. $\{v \in V \text{ s.t. } \mathcal{L}_t v \in L^2(Q)\}$. In [2], it is shown that D_t does not depend on t :

Theorem 2.9 *If for all t , $r(t) > 0$, then D_t does not depend on t : $D_t = D$. Moreover, if there exists a constant $r_0 > 0$ such that $r(t) > r_0$ a.e., and if $\frac{\alpha^2}{\beta^2} > 2$, then for well chosen values of η (in particular such that $2\frac{\alpha^2}{\beta^2}\eta(1-\eta) > 1$),*

$$D = \left\{ v \in V; y^2 S^2 \frac{\partial^2 v}{\partial S^2}, \frac{\partial^2 v}{\partial y^2}, yS \frac{\partial^2 v}{\partial S \partial y}, S \frac{\partial v}{\partial S}, y \frac{\partial v}{\partial y}, y^2 v \in L^2(Q) \right\}. \quad (2.151)$$

Then, from general results of Kato, (see [88] Theorem 5.6.8.), regularity results on the solution to (2.148) (2.149) can be obtained:

Theorem 2.10 *Assume that there exists $\zeta, 0 < \zeta \leq 1$ such that γ belongs to $C^\zeta([0, T], L^\infty(Q))$ and r is a Hölder function of time with exponent ζ . Assume also that $r(t) > r_0$ for a positive constant r_0 and that $\frac{\alpha^2}{\beta^2} > 2$. Then for η chosen as in Proposition 2.5 and Theorem 2.9, if u_o belongs to D defined by (2.151), then the solution u of (2.148) (2.149) belongs to $C^1((0, T); L^2(Q)) \cap C^0([0, T]; D)$ and the functional equation in $L^2(Q)$*

$$u'(t) + \mathcal{L}_t u(t) = 0 \quad (2.152)$$

is satisfied pointwise in $[0, T]$.

Furthermore, for $u_o \in L^2(Q)$, the solution of (2.148) (2.149) also belongs to $C^1([\tau, T]; L^2(Q)) \cap C^0([\tau, T]; D)$, for all $\tau > 0$ and satisfies

$$\|u'(t)\|_{L^2(Q)} + \|\mathcal{L}_t u(t)\|_{L^2(Q)} \leq \frac{C}{t}, \quad \text{for } t > 0.$$

Remark 2.11 *The same kind of analysis is possible for an extended Stein-Stein's model with a nonzero correlation factor, but in this case, (still assuming*

that $\tilde{\gamma}$ is bounded) one has to cope with the term $\rho \frac{\mu - \tilde{r}(t)}{|y|} \frac{\partial P}{\partial y}$, which becomes singular on the axis $y = 0$: therefore, one may need to impose a Dirichlet condition on the axis $y = 0$, of the form

$$P(S, 0, t) = g(S, t), \quad 0 \leq t < T, \quad S > 0, \quad (2.153)$$

where

$$\begin{aligned} \frac{\partial g}{\partial t} + \tilde{r}(t) \left(S \frac{\partial g}{\partial S} - g \right) &= 0 \quad 0 \leq t < T, \quad S > 0, \\ g(S, t = T) &= P_0(S, 0). \end{aligned} \quad (2.154)$$

The Initial Value Problem for Heston's Model We aim at making an analysis for Heston's model, in the same spirit as the one proposed above for the Stein-Stein's model. We consider the partial differential equation (2.133), in the simple case when $\rho = 0$. We also assume that $\tilde{\gamma}$ is bounded independently of t , S and y and we define $\gamma(S, y, t) = \tilde{\gamma}(S, y, T - t)$. We need to set the variational formulation in a suitable weighted Sobolev space compatible with the operator degeneracy on the axis $y = 0$. For that, we introduce a smooth positive function ψ defined on \mathbb{R}_+ such that $\psi(y) = \sqrt{y}$ on $(0, m)$ and $\psi(y) = e^{-dy}$ on $(2m, +\infty)$, for some positive parameter d . We will fix d later on.

We introduce the new unknown function

$$u(S, y, t) = \psi(y)P(S, y, T - t). \quad (2.155)$$

Denoting by \mathcal{L}_t the linear partial differential operator

$$\begin{aligned} \mathcal{L}_t v = & -\frac{1}{2}yS^2 \frac{\partial^2 v}{\partial S^2} - r(t) \left(S \frac{\partial v}{\partial S} - v \right) \\ & -\frac{\lambda^2}{2}y \left(\frac{\partial^2 v}{\partial y^2} - 2 \frac{\psi'}{\psi} \frac{\partial v}{\partial y} - \psi \left(\frac{\psi'}{\psi^2} \right)' v \right) - \kappa(m - y) \left(\frac{\partial v}{\partial y} - \frac{\psi'}{\psi} v \right) \\ & + \lambda \sqrt{y} \gamma(S, y, t) \left(\frac{\partial v}{\partial y} - \frac{\psi'}{\psi} v \right), \end{aligned} \quad (2.156)$$

we obtain

$$\frac{\partial u}{\partial t} + \mathcal{L}_t u = 0. \quad (2.157)$$

The equation clearly becomes degenerate on the axis $y = 0$, because the coefficients in front of the two operators $\frac{\partial^2 u}{\partial y^2}$ and $S^2 \frac{\partial^2 u}{\partial S^2}$ vanish. We denote by Q the open sector $Q = \mathbb{R}_+ \times \mathbb{R}_+$. Let us consider the weighted Sobolev space V :

$$V = \left\{ v : \left(v, \frac{v}{\sqrt{y}}, \sqrt{y} \frac{\partial v}{\partial y}, \sqrt{y} S \frac{\partial v}{\partial S} \right) \in (L^2(Q))^4 \right\}. \quad (2.158)$$

This space with the norm

$$\|v\|_V = \left(\int_Q \left(1 + \frac{1}{y} \right) v^2 + y \left(\frac{\partial v}{\partial y} \right)^2 + y S^2 \left(\frac{\partial v}{\partial S} \right)^2 \right)^{\frac{1}{2}} \quad (2.159)$$

is a Hilbert space, and it has the following properties:

1. V is separable.
2. Calling $\mathcal{D}(Q)$ the space of smooth functions with compact support in Q , $\mathcal{D}(Q) \subset V$ and $\mathcal{D}(Q)$ is dense in V
3. V is dense in $L^2(Q)$.
4. For any function v in V ,

$$\int_Q yv^2 \leq 4 \int_Q yS^2 \left(\frac{\partial v}{\partial S}\right)^2. \quad (2.160)$$

Remark 2.12 The reason for imposing that $\frac{v}{\sqrt{y}}$ be square integrable will appear in Lemma 2.6 below. Note that the functions $v(y) = \log^\sigma(y)$, with $0 < \sigma < \frac{1}{2}$, are such that v and $\sqrt{y}v'$ are square integrable near $y = 0$, but $\frac{v}{\sqrt{y}}$ is not square integrable.

Lemma 2.6 The operator $v \rightarrow S \frac{\partial v}{\partial S}$ is continuous from V into V' .

Proof. Call X and Y the differential operators

$$X(v) = \sqrt{y}S \frac{\partial v}{\partial S}, \quad Y(v) = \sqrt{y} \frac{\partial v}{\partial y}, \quad (2.161)$$

The operators X and Y are continuous operators from V into $L^2(Q)$ and their adjoints are

$$X^T(v) = -X(v) - \sqrt{y}v, \quad Y^T v = -Y(v) - \frac{1}{2\sqrt{y}}v. \quad (2.162)$$

It can be checked that

$$[X, Y](v) = -\frac{1}{2}S \frac{\partial v}{\partial S}. \quad (2.163)$$

Therefore, for $v \in V$ and $w \in \mathcal{D}(Q)$,

$$\left(S \frac{\partial v}{\partial S}, w \right) = 2 \int_Q Y(v)(X(w) + \sqrt{y}w) - 2 \int_Q X(v)(Y(w) + \frac{1}{2\sqrt{y}}w), \quad (2.164)$$

and from (2.160), there exists a constant C such that

$$\left| \left(S \frac{\partial v}{\partial S}, w \right) \right| \leq C \|v\|_V \|w\|_V.$$

To conclude, we use the density of $\mathcal{D}(Q)$ into V . ■

Lemma 2.6 and the assumption made on ψ imply that the operator \mathcal{L}_t is continuous from V to its dual V' because, in particular, the functions $\frac{\psi'}{\psi}$, $\frac{1}{y} \frac{d}{dy}(y \frac{\psi'}{\psi})$

and $\psi \frac{d}{dy}(\frac{\psi'}{\psi^2})$ are bounded on $[2m, +\infty)$.

Calling a_t the bilinear form defined on $V \times V$ by $a_t(u, v) = \langle \mathcal{L}_t u, v \rangle$, we have

$$\begin{aligned} a_t(u, v) = & \frac{1}{2} \int_Q y S^2 \frac{\partial u}{\partial S} \frac{\partial v}{\partial S} + \int_Q y S \frac{\partial u}{\partial S} v \\ & + \frac{\lambda^2}{2} \left(\int_Q y \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \int_Q \frac{\partial u}{\partial y} v + 2 \int_Q y \frac{\psi'}{\psi} \frac{\partial u}{\partial y} v + \int_Q y \psi \left(\frac{\psi'}{\psi^2} \right)' uv \right) \\ & - 2r(t) \left(\int_Q Y(u)(X(v) + \sqrt{y}v) - \int_Q X(u)(Y(v) + \frac{1}{2\sqrt{y}}v) \right) + r(t) \int_Q uv \\ & - \kappa \int_Q (m-y) \left(\frac{\partial u}{\partial y} - \frac{\psi'}{\psi} u \right) v + \lambda \int_Q \sqrt{y} \gamma(t, S, y) \left(\frac{\partial u}{\partial y} - \frac{\psi'}{\psi} u \right) v. \end{aligned} \quad (2.165)$$

Assume that r is a bounded function of time and that γ is bounded by a constant. The bilinear form a_t is continuous on $V \times V$, with a continuity constant independent of t . We also need a Gårding type inequality:

Proposition 2.7 *Assume that r is a bounded function of time and that γ is bounded by a constant Γ . If*

$$\kappa \min(m, \kappa) > \frac{3}{4} \lambda^2, \quad (2.166)$$

then one can choose d (see the definition of ψ) such that there exist two positive constants C and c independent of t ,

$$a_t(v, v) \geq C \|v\|_V^2 - c \|v\|_{L^2(Q)}^2, \quad \forall v \in V \quad (2.167)$$

Proof. It is enough to prove (2.167) for $v \in \mathcal{D}(Q)$. Several integrations by part lead to

$$\begin{aligned} a_t(v, v) = & \frac{1}{2} \int_Q y S^2 \left(\frac{\partial v}{\partial S} \right)^2 - \frac{1}{2} \int_Q y v^2 + \frac{3}{2} r(t) \int_Q v^2 + \frac{\lambda^2}{2} \int_Q y \left(\frac{\partial v}{\partial y} \right)^2 \\ & - \int_Q \left(\frac{\lambda^2}{2} \left(y \frac{\psi'}{\psi} \right)' + \kappa y \frac{\psi'}{\psi} \right) v^2 + \lambda \int_Q \sqrt{y} \gamma(t, S, y) \left(\frac{\partial v}{\partial y} - \frac{\psi'}{\psi} v \right) v \\ & + \int_Q \left(\kappa m \frac{\psi'}{\psi} + \frac{\lambda^2}{2} \left(y \psi \left(\frac{\psi'}{\psi^2} \right)' \right) - \frac{\kappa}{2} \right) v^2 \end{aligned}$$

For brevity we skip many details, and we only focus on the main two steps of the proof:

First step Note that, if $y < m$ then $\kappa m \frac{\psi'}{\psi} + \frac{\lambda^2}{2} \left(y \psi \left(\frac{\psi'}{\psi^2} \right)' \right) = \frac{1}{2y} (\kappa m - \frac{3}{4} \lambda^2)$.

From this and (2.166), we see that the quantity $\int_{Q \cap \{y < m\}} \frac{v^2}{y}$ is bounded by a positive factor times $\int_{Q \cap \{y < m\}} \left(\kappa m \frac{\psi'}{\psi} + \frac{\lambda^2}{2} \left(y \psi \left(\frac{\psi'}{\psi^2} \right)' \right) \right) v^2$. On the other

hand, the quantity

$$\begin{aligned} & \int_{Q \cap \{y < 2m\}} \left| -\frac{1}{2}y - \left(\frac{\lambda^2}{2} \left(y \frac{\psi'}{\psi} \right)' + \kappa y \frac{\psi'}{\psi} \right) \right| v^2 \\ & + \int_{Q \cap \{m < y < 2m\}} \left| \kappa m \frac{\psi'}{\psi} + \frac{\lambda^2}{2} \left(y \psi \left(\frac{\psi'}{\psi^2} \right)' \right) \right| v^2 \end{aligned}$$

is bounded by a constant times $\int_{Q \cap \{y < 2m\}} v^2$.

Second step Calling

$$I = \int_{Q \cap \{y > 2m\}} \left(-\frac{1}{2}y - \left(\frac{\lambda^2}{2} \left(y \frac{\psi'}{\psi} \right)' + \kappa y \frac{\psi'}{\psi} \right) + \left(\kappa m \frac{\psi'}{\psi} + \frac{\lambda^2}{2} \left(y \psi \left(\frac{\psi'}{\psi^2} \right)' \right) \right) \right) v^2$$

we choose ψ such that there is an estimate of the form:

$$I \geq -\alpha \int_Q y S^2 \left(\frac{\partial v}{\partial S} \right)^2 - \beta \int_Q v^2,$$

for some $\alpha < \frac{1}{2}$ and $\beta \geq 0$. In view of (2.160), this will be a consequence of the estimate

$$I \geq -\frac{\alpha}{4} \int_Q y v^2 - \beta \int_Q v^2.$$

Therefore, let us look for ψ such that

$$\begin{aligned} & \int_{Q \cap \{y > 2m\}} \left(\left(\frac{\alpha}{4} - \frac{1}{2} \right) y - \left(\frac{\lambda^2}{2} \left(y \frac{\psi'}{\psi} \right)' + \kappa y \frac{\psi'}{\psi} \right) + \left(\kappa m \frac{\psi'}{\psi} + \frac{\lambda^2}{2} \left(y \psi \left(\frac{\psi'}{\psi^2} \right)' \right) \right) \right) v^2 \\ & \geq -\beta \int_Q v^2. \end{aligned}$$

Since $\psi(y) = e^{-dy}$, $y > 2m$, the integrand above is

$$\left(\frac{\alpha}{4} - \frac{1}{2} \right) y - \frac{1}{2} \lambda^2 (d^2 y - d) + \kappa (y - m) d,$$

so the estimate will be true if one can choose d such that $-\frac{1}{2} \lambda^2 d^2 + \kappa d - \frac{3}{8} > 0$. This is the case if $\kappa^2 > \frac{3}{4} \lambda^2$. ■

The continuity of a_t and the Gårding's inequality yield the existence and uniqueness of a weak solution to the initial value problem with equation (2.157).

Theorem 2.11 *Under the assumptions of Proposition 2.7, for any $u_0 \in L^2(Q)$, there exists a unique u in $L^2(0, T; V) \cap C^0([0, T]; L^2(Q))$, with $\frac{\partial u}{\partial t} \in L^2(0, T; V')$ such that, for any smooth function $\phi \in \mathcal{D}(0, T)$, for any $v \in V$,*

$$-\int_0^T \phi'(t) \left(\int_Q u(t)v \right) dt + \int_0^T \phi(t) a_t(u, v) dt = 0 \quad (2.168)$$

and

$$u(t=0) = u_\circ. \quad (2.169)$$

The mapping $u_\circ \mapsto u$ is continuous from $L^2(Q)$ to $L^2(0, T; V) \cap \mathcal{C}^0([0, T]; L^2(Q))$.

Remark 2.13 The assumption (2.166) plays the same role as the assumption $\alpha > \beta$ in the discussion of the Stein-Stein's model.

Remark 2.14 Note that no boundary condition have been imposed on ∂Q .

For example, if $P_\circ(S) = (K - S)^+$ then $u_\circ(S, y) = \psi(y)(K - S)^+$ is a square integrable function, and Theorem 2.11 can be applied to the case of a European put.

A Numerical Method for Pricing Options with Stein-Stein's Model

We consider the partial differential equation (2.131) with $f(y) = |y|$. We assume that the interest rate r is constant, and we take $\rho = 0$, $\tilde{\gamma} = 0$. The partial differential equation is rewritten in divergence form, with the new variable $T - t \rightarrow t$:

$$\begin{aligned} 0 = & \frac{\partial P}{\partial t} - \frac{1}{2} \left(\frac{\partial}{\partial S} \left(y^2 S^2 \frac{\partial P}{\partial S} \right) + \frac{\partial}{\partial y} \left(\beta^2 \frac{\partial P}{\partial y} \right) \right) \\ & + (y^2 S - rS) \frac{\partial P}{\partial S} + \alpha(y - m) \frac{\partial P}{\partial y} + rP \end{aligned} \quad (2.170)$$

A first order implicit Euler scheme is used for time discretization:

$$\begin{aligned} 0 = & \frac{P^m - P^{m-1}}{\delta t} - \frac{1}{2} \left(\frac{\partial}{\partial S} \left(y^2 S^2 \frac{\partial P^m}{\partial S} \right) + \frac{\partial}{\partial y} \left(\beta^2 \frac{\partial P^m}{\partial y} \right) \right) \\ & + (y^2 S - rS) \frac{\partial P^m}{\partial S} + \alpha(y - m) \frac{\partial P^m}{\partial y} + rP^m \end{aligned} \quad (2.171)$$

Aiming at obtaining a discrete version of this equation, we first truncate the domain, *i.e.* we introduce the rectangle $\Omega = (0, \bar{S}) \times (-\bar{y}, \bar{y})$, with \bar{S} and \bar{y} large enough. We are looking for a numerical approximation of P in Ω . For that, we first need to supply reasonable artificial boundary conditions on boundaries of $\partial\Omega$, in agreement with the payoff function. Let us discuss the artificial boundary condition for a vanilla put option, *i.e.* $P_\circ(S) = (K - S)^+$.

- On $\partial\Omega \cap \{S = \bar{S}\}$, we impose $\frac{\partial P}{\partial S}(\bar{S}, y, t) = 0$. Such a condition is reasonable if \bar{S} is large enough compared to K .
- On $\partial\Omega \cap \{y = \pm\bar{y}\}$, finding an accurate artificial boundary condition is not easy. However, if \bar{y} is chosen such that $\alpha(\bar{y} \pm m) \gg \beta^2$, then for $y \sim \pm\bar{y}$, the coefficient of the advection term in the y direction, *i.e.* $\alpha(y - m)$, is much larger in absolute value than the coefficient of the diffusion in the y direction, *i.e.* $\frac{\beta^2}{2}$. Furthermore, near $y = \pm\bar{y}$, the vertical velocity $\alpha(y - m)$ is directed outward Ω . Therefore, the error caused by an artificial condition on $y = \pm\bar{y}$ will be damped away from the boundaries, and

localized in boundary layers whose width is of the order of $\frac{\beta^2}{\alpha\bar{y}}$. Therefore, Dirichlet boundary conditions, e.g. $P(S, \pm\bar{y}, t) = 0$, will not cause a large error for $|y|$ small enough compared to \bar{y} , for example $|y| \leq \frac{\bar{y}}{2}$, even though these conditions are not satisfied at all by the exact solution.

- No boundary condition is needed on $S = 0$, because of the degeneracy of the equation.

Let V_h be the space of continuous piecewise linear functions on a triangulation of Ω , which are equal to zero on $y = \pm\bar{y}$. We consider the following finite element discretization: $\forall v_h \in V_h$,

$$\begin{aligned} \int_{\Omega} \left(\frac{1}{\delta t} + r \right) P^m v + \frac{1}{2} \int_{\Omega} y^2 S^2 \frac{\partial P^m}{\partial S} \frac{\partial v}{\partial S} + \frac{\beta^2}{2} \int_{\Omega} \frac{\partial P^m}{\partial y} \frac{\partial v}{\partial y} \\ + \int_{\Omega} (y^2 S - rS) \frac{\partial P^m}{\partial S} v + \alpha \int_{\Omega} (y - m) \frac{\partial P^m}{\partial y} v = \frac{1}{\delta t} \int_{\Omega} P^{m-1} v \end{aligned} \quad (2.172)$$

To illustrate this, let us take for the parameters

$$r = 0.05, \quad \alpha = 1, \quad \nu = 0.5, \quad m = 0.2, \quad K = 100. \quad (2.173)$$

The goal is to approximate P in the domain $(0, \bar{S}) \times (-1.5, 1.5)$ for t smaller than 1. We choose $\bar{S} = 800$. For computing the solution in $(0, \bar{S}) \times (-1.5, 1.5)$, we choose the larger domain $\Omega = (0, \bar{S}) \times (-3, 3)$, i.e $\bar{y} = 3$.

We compute the pricing function of the put option one year to maturity. The time step has been set to 6 days. The artificial boundary conditions $u(S, \pm\bar{y}, t) = Ke^{-rt}$ have been used. In Figure 2.13, we display the contours of the solution in $\Omega = (0, 800) \times (-3, 3)$. On this figure, it is very well seen that the artificial boundary conditions on $y = \pm 3$ do not affect the solution in the region $|y| < 1.5$.

Remark 2.15 *The choice of $\alpha = 1$ is not quite realistic from a financial viewpoint, if the asset is linked to stocks, because the mean reversion rate is generally larger. When the asset corresponds to interest rates, such values of α are reasonable. When the mean reversion rate is large, it is possible to carry out an asymptotic expansion of the solution as in [44] and we believe that the variational setting introduced above justifies these expansions.*

The analysis of convergence of the finite element method for the elliptic part of operator appearing in (2.170) was performed in [2]. In this paper, it was shown that the finite element error analysis theory as in [28, 27] could be performed, as soon as the family of meshes under consideration satisfy a *regularity assumption* with respect to an intrinsic metric associated with the degenerate operator: more precisely, defining X and Y the smooth vector fields in \mathbb{R}^2 :

$$X = Sy\partial_S, \quad Y = \partial_y, \quad (2.174)$$

we say that an absolutely continuous curve $\gamma : [0, T] \rightarrow \mathbb{R}^2$ is a sub-unit curve with respect to X if for any $\xi \in \mathbb{R}^2$

$$\langle \dot{\gamma}(t), \xi \rangle^2 \leq \langle X(\gamma(t)), \xi \rangle^2 + \langle Y(\gamma(t)), \xi \rangle^2$$

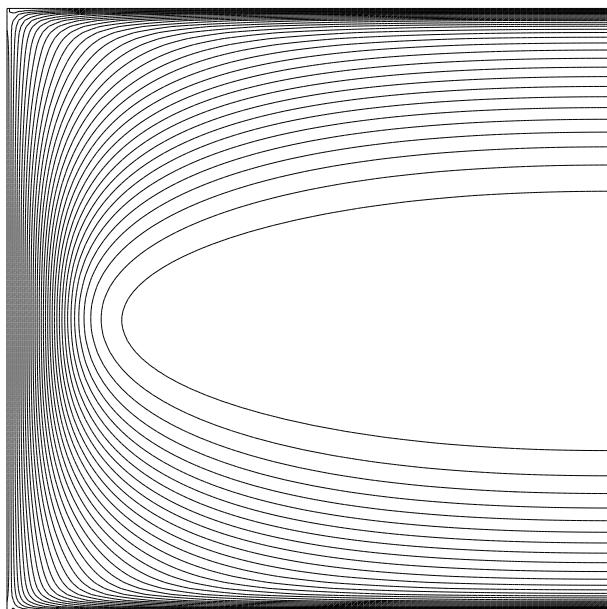


Figure 2.13: The contours of the price computed in $\Omega = (0, 800) \times (-3, 3)$: note the boundary layers due to artificial boundary conditions on $y = \pm 3$.

for a.e. $t \in [0, T]$. Following e.g. [83]), we define the *intrinsic* or *Carnot-Carathéodory* metric d : $\forall P_1, P_2 \in \mathbb{R}^2$,

$$d(P_1, P_2) = \inf \{T > 0 : \text{there exists a sub-unit curve } \gamma, \\ \gamma : [0, T] \longrightarrow \mathbb{R}^2, \quad \gamma(0) = P_1, \quad \gamma(T) = P_2\},$$

If the above set of curves is empty, we take $d(P_1, P_2) = \infty$.

In [2], it was essentially required that the family of meshes be regular with respect to the Carnot-Carathéodory metric: there exists a parameter σ such that, for any triangle T , one can find two Carnot-Carathéodory balls, the first one containing T and the other one contained in T , such that, calling r_1 and r_2 their Carnot-Carathéodory radii, ($r_1 > r_2$), we have $\frac{r_1}{r_2} \leq \sigma$. Under such an assumption, it is possible to construct a local regularization operator similar to that of Clément, [29], and then to obtain optimal error estimates. In [2], examples of meshes satisfying the regularity assumption were given.

A Numerical Method for Pricing Options with Heston's Model With Heston's model, in contrast with the last example, the advection in the y variable does not dominate the diffusion as $y \rightarrow \infty$. Therefore, inexact boundary conditions on an artificial boundary $y = \bar{y}$ may produce large errors. For this reason, another strategy has been chosen: instead of truncating the domain in the variable y , we have used a suitable change of variable in order to map the y -domain, *i.e.*, \mathbb{R}_+ onto the interval $(0, 1)$.

We want to approximate P given by (2.133). The idea is to make the change of variables $z = y/(y + 1)$, which maps \mathbb{R}_+ onto $(0, 1)$. The inverse map is $y = z/(1 - z)$.

In the variables $(T - t, S, z)$, the partial differential equation becomes

$$\frac{\partial \check{P}}{\partial t} + \check{\mathcal{L}}_t \check{P} = 0. \quad (2.175)$$

for $t \in (0, T]$, $S \in \mathbb{R}_+$ and $z \in (0, 1)$, where

$$\begin{aligned} \check{\mathcal{L}}_t v = & -\frac{1}{2} \frac{z}{1-z} S^2 \frac{\partial^2 v}{\partial S^2} - r(t) \left(S \frac{\partial v}{\partial S} - v \right) - \rho \lambda S (1-z) z \frac{\partial^2 v}{\partial S \partial z} \\ & - \frac{\lambda^2}{2} z (1-z)^2 \left((1-z) \frac{\partial^2 v}{\partial z^2} - 2 \frac{\partial v}{\partial z} \right) - \kappa \left(m - \frac{z}{1-z} \right) (1-z)^2 \frac{\partial v}{\partial z}, \end{aligned} \quad (2.176)$$

assuming that $\tilde{\gamma} = 0$.

No boundary condition is needed on the axis $z = 0$ because the partial differential operator becomes degenerate there: indeed, all the coefficients of the second derivatives vanish; moreover, the first order derivatives correspond to an advection with an outgoing velocity, *i.e.*, the coefficient in front of $\frac{\partial v}{\partial z}$, is negative near $z = 0$ (its value is $-\kappa m$). Similarly, no boundary condition is needed on the axis $z = 1$ because $\check{\mathcal{L}}$ becomes degenerate near $z = 1$.

On the other hand, one can truncate the domain in the S variable. The new problem is posed in the rectangle $(0, \bar{S}) \times (0, 1)$, which allows for the use of a

finite element method. We have refined the mesh near the strike.

We have made the following choice of parameters

$$r = 0, \quad \rho = -0.5, \quad \kappa = 2.5, \quad \lambda = 0.5, \quad m = 0.06, \quad K = 1.$$

We have taken $\bar{S} = 4$. An approximation of P solution of (2.133) is obtained from the finite element approximation of \check{P} (or possibly $e^{-\gamma y}\check{P}$) by performing the inverse change of variable $z \mapsto y = z/(1-z)$. The pricing functions of the put and the call half-year to maturity are displayed in Figure 2.14. The computed prices are in good agreement with the closed form obtained by Heston, see [58].

2.4.3 American Options with Stochastic Volatility

In this paragraph, we discuss the pricing of an American put option with the Stein-Stein stochastic volatility model. The parameters of the model are given by (2.173). The domain truncation is also the same as in the example of the European option. Piecewise linear finite element are used for the discretization. We have chosen to use the first order operator splitting or projection scheme described in § 2.3.4. A similar method has been studied in [63] for Heston's model. In order to capture the exercise boundary, we have adapted the mesh in the variables S and y . In Figure 2.15, we have plotted the contours of the pricing function one year to maturity: the exercise zone clearly appears: indeed, it corresponds to the zone where the pricing function matches the function $S \mapsto K - S$, i.e. where the contours are vertical straight lines. Figure 2.15 has to be compared with Figure 2.13 for the European option. In Figure 2.16, we have plotted the exercise region one year to maturity. The mesh is visible too. It is refined near the exercise boundary.

In [64], Ikonen and Toivanen have proposed specific finite difference methods and solution procedures for American options with Heston's model. Special discretization and grid are designed in such a way that the resulting matrix be an M-matrix. The scheme is a seven points scheme and upwinding is used when necessary. A specific alternating direction splitting scheme is proposed. Each sub-step consists of solving a one dimensional linear complementarity problem by the Brennan-Schwartz algorithm, see [22]. Three directions are used: the two axes and the first diagonal. For these algorithms to work, one needs that the tridiagonal matrices used in the substeps are M-matrices. This is not true in general but the scheme and the grid have been designed so that this condition holds. It is also necessary that the exercise boundary intersects the directions once at most. This condition is not proved but no counterexample has been found in the computations. Ikonen and Toivanen have compared this solution procedure with four other methods, in particular the previously described projection scheme and a multigrid algorithm; they have shown that the alternating directions method performs best. On the other hand, the last solution procedure has been tailored for this problem and its robustness has to be assessed.

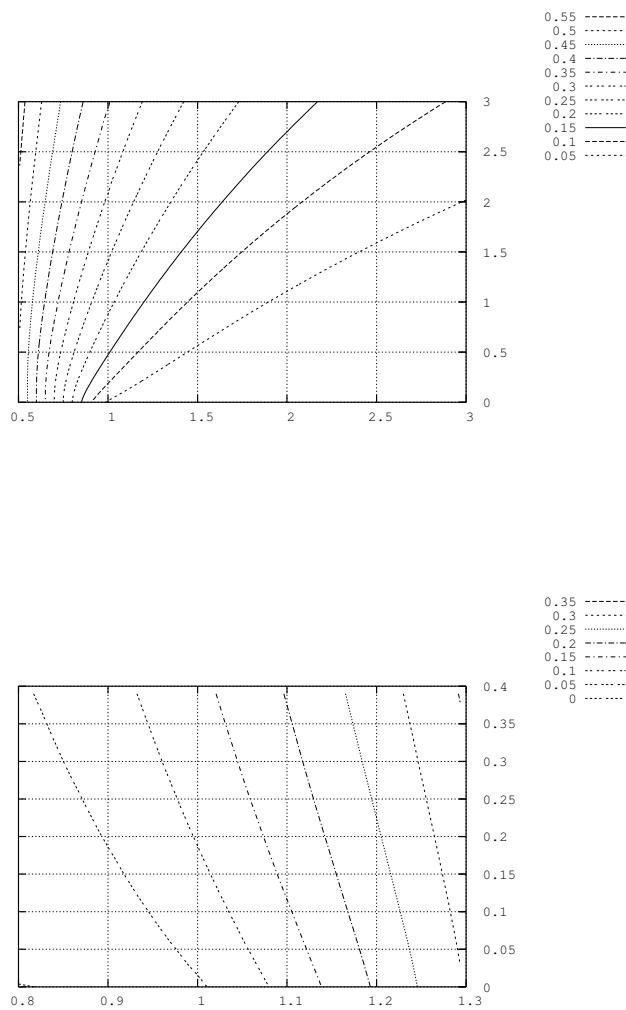


Figure 2.14: The contours of the pricing function of a put(top)/call(bottom) option with Heston model half-year to maturity

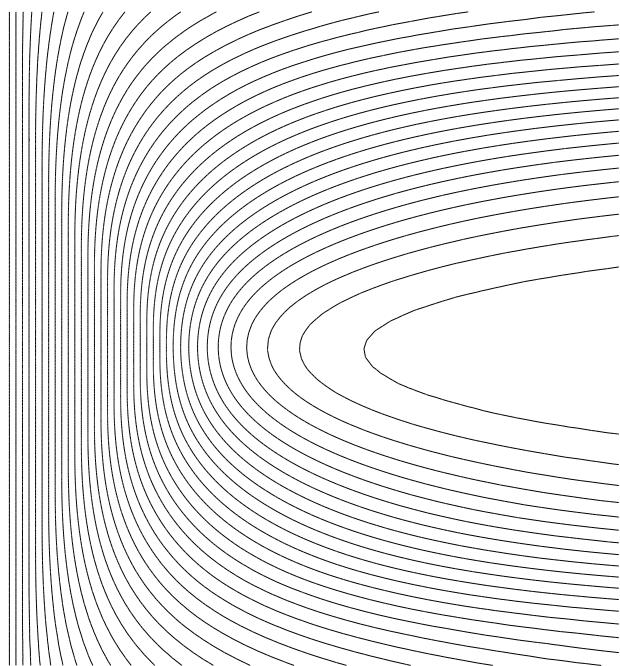


Figure 2.15: The pricing function of the American option one year to maturity.
The exercise zone is clearly visible.

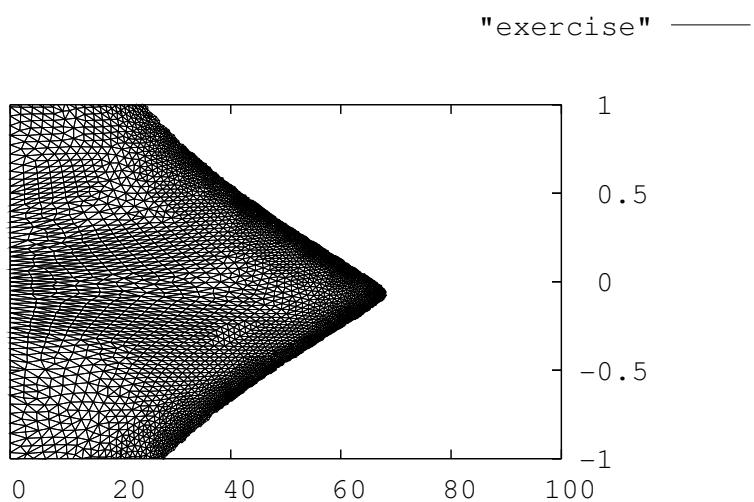


Figure 2.16: The exercise zone one year to maturity. One clearly sees that the mesh has been refined near the exercise boundary.

2.4.4 Volatility Models with Several Stochastic Variables

We give the example of a generalized multifactor Scott model: we consider d fully correlated Ornstein-Uhlenbeck processes

$$dY_t^{(i)} = -\lambda_i Y_t^{(i)} dt + \beta_i dB_t, \quad i = 1, \dots, d, \quad (2.177)$$

where B_t is one dimensional Brownian motion. We consider an asset whose price is a lognormal process:

$$dS_t = r(t)S_t dt + \sigma_t S_t dW_t,$$

where W_t is a Brownian motion independent of B_t and the volatility σ_t is of the form

$$\sigma_t = \sigma \left(\sum_{i=1}^d Y_t^{(i)}, t \right).$$

The price of the option is $P(S_t, Y_t^{(1)}, \dots, Y_t^{(d)}, t)$ where P satisfies the $d+2$ dimensional PDE:

$$\begin{aligned} \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2(z_1, t)S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} \\ + \frac{1}{2} \sum_{i,j} \beta_i \beta_j \frac{\partial^2 P}{\partial y_i \partial y_j} + \rho \beta_i \sigma(z_1, t)S \frac{\partial^2 P}{\partial S \partial y_i} - \sum_{i=1}^d \lambda_i y_i \frac{\partial P}{\partial y_i} - rP = 0, \end{aligned}$$

where $z_1 = \sum_{i=1}^d y_i$. One can make the change of variables $z = Qy$ where

$$Q = \begin{pmatrix} 1 & 1 & \dots & \dots & 1 \\ -\beta_2/\beta_1 & 1 & 0 & \dots & 0 \\ -\beta_3/\beta_1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\beta_d/\beta_1 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and get the PDE

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2(z_1, t)S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} + \frac{1}{2}\beta^2 \frac{\partial^2 P}{\partial z_1^2} + \rho \beta \sigma(z_1, t)S \frac{\partial^2 P}{\partial S \partial z_1} - z^T L^T \nabla_z P - rP = 0$$

where $\beta = \sum_{i=1}^d \beta_i$ and $L = Q \text{Diag}(\lambda_1, \dots, \lambda_d) Q^{-1}$. This linear PDE is parabolic with respect to the variables S and z_1 and hyperbolic with respect to z_i , $1 < i \leq d$.

Sparse grid methods can be used for approximating the five variables function P . We give an example provided: the parameters of the three factors models are

- interest rate: $r = 5\%$,

- spot price-volatility correlation: $\rho = -0.5$,
- mean value of the volatility: $\sigma = 0.2$.
- parameters of the OU processes: $\lambda \approx (29.27, 2.45, 0.108)$, $\beta = (1.26, 0.42, 0.42)$.

With these parameters, one may truncate the domain of computation because the velocity in the advection terms in the PDE are directed outward near the artificial boundaries, and the error produced by the inexact boundary conditions are small sufficiently far from the artificial boundaries.

In Table 2.2, we compute the price of a European call option one year to maturity for several spot values, (the strike is 1), for $y_i = 0$, $i = 1, 2, 3$. The payoff function depends on the variable S only. Hence, the singularity is located on an hyperplane in the spot/volatilities space, and sparse grids are well adapted. A Crank-Nicolson scheme with a time step of 0.01 year has been used.

We compare the results of the sparse grid method with refinement levels of 7, 8 and 9 with a Monte-Carlo simulation. Note that the sparse grid approximation is sharper for spots larger than or equal to the strike, because the sparse grid is relatively coarser for small spots.

Spot	Level = 7	Level = 8	Level = 9	Monte Carlo
0.80	1.51	1.48	1.45	1.41
0.85	2.86	2.81	2.78	2.75
0.90	4.78	4.72	4.71	4.67
0.95	7.29	7.23	7.24	7.22
1.00	10.32	10.31	10.31	10.32
1.05	13.82	13.84	13.85	13.88
1.10	17.72	17.75	17.77	17.81
1.15	21.92	21.94	21.96	22.01
1.20	26.31	26.35	26.38	26.43

Table 2.2: Price of a European call one year to maturity. These results have been obtained by D. Pommier.

The tests have been done on a 2.66GHz Intel Xeon processors with 1.5Gb RAM. The computing time is approximately 4 minutes for $n = 7$, 15 minutes for $n = 8$ and 1 hour for $n = 9$.

Chapter 3

Sensitivity and Calibration

3.1 Sensitivity

It is important to compute the sensitivity of options' prices to parameters such as the spot price or the volatility. The partial derivatives with respect to the relevant parameters are called the *Greeks*: let C be the price of a vanilla European call:

- the δ (*delta*) is its derivative with respect to the stock price S : $\frac{\partial C}{\partial S}$.
- the Θ or *time-decay* is its derivative with respect to time: $\frac{\partial C}{\partial t}$.
- the *vega* κ is its derivative with respect to the volatility σ ,
- the *rho* ρ is its derivative with respect to the interest rate, $\frac{\partial C}{\partial r}$,
- η is its derivative with respect to the strike K
- finally, the *gamma* is the rate of change of its delta : $\frac{\partial^2 C}{\partial S^2}$.

Equations can be derived for these by directly differentiating the partial differential equation and the boundary conditions which define C . Automatic differentiation (AD for brevity) of a computer code for option pricing provides a way to do that efficiently and automatically.

3.1.1 Automatic Differentiation

Automatic differentiation of computer programs is based on the idea that every line of a program can be differentiated analytically. Consider, for instance, the C-program which computes $J = (u^2 - u_d^2)$ for some values of u and u_d

```
int main()
{
    double J, aux, u=2.5, u_d=1;
    aux = u-u_d;
    J=aux*(u+u_d);
    cout<<"J="<<J<<endl;
}
```

It can be made to compute the derivative of J with respect to u as well by adding above each instruction its derivative with respect to u :

```
int main()
{
    double dJdu,J, dauxdu,aux, u=2, u_d=0.1;
    dauxdu=1;
    aux = u-u_d;
    dJdu=dauxdu*(u+u_d) + aux;
    J=aux*(u+u_d);
    cout<<"J="<

However it is more systematic to compute differentials and consider that every double or float variable has an infinitesimal variation, potentially and initially set to zero.


```

Let $J(u) = (u - u_d)(u + u_d)$, then its differential is

$$\delta J = (u - u_d)(\delta u + \delta u_d) + (u + u_d)(\delta u - \delta u_d) \quad (3.1)$$

Obviously the derivative of J with respect to u is obtained by setting $\delta u = 1$ and $\delta u_d = 0$. Now suppose that J is programmed in C/C++ by

```
double J(double u, double u_d){
    double z = u-u_d;
    z = z*(u+u_d);
    return z;
}
int main(){ double u=2, u_d = 0.1;
    cout << J(u,u_d) << endl;
}
```

A program which computes J and its differential can be obtained by writing above each differentiable line its differentiated form:

```
double JandDJ(double u, double u_d, double du, double du_d, double *pdz){
    double dz = du - du_d, z = u-u_d;
    double dJ = dz*(u+u_d) + z*(du + du_d);
    z = z*(u+u_d);
    *pdz = dz;
    return z;
}
int main(){
    double dJ, u=2,u_d = 0.1;
    cout << J(u,u_d,1,0,&dJ) << endl;
}
```

Except for the embarrassing problem of returning both z, dz instead of z , the procedure can be automatized by introducing a structured type holding the value of the variable and of its derivative:

```
struct {double val[2];} dreal;

dreal JandDJ(dreal u, dreal u_d) {
    dreal z;
    z.val[1] = u.val[1]-u_d.val[1];
    z.val[0] = ...;
```

```

z.val[0] = u.val[0]-u_d.val[0];
z.val[1] = z.val[1]*(uval[0]+u_d.val[0])+z.val[0]*(uval[1]+u_d.val[1]);
z.val[0] = z.val[0]*(uval[0]+u_d.val[0]);
return z;
}

int main() {
    dreal u, dJ;
    u.val[0]=2;u_d.val[0]=0.1;u.val[1]=1;u_d.val[1]=0;
    cout <<J(u,u_d).val[0]<<J(u,u_d,1,0).val[1];
}

```

The class ddouble

In C++ the program can be simplified further by redefining the operators `=`, `-`, `+` and `*`. Then a class has to be used instead of a struct!

```

class ddouble {
public: double val[2];
ddouble(double a, double b=0){ v[0] = a; v[1]=b;}
ddouble operator=(const ddouble& a)
{ val[1] = a.val[1]; val[0]=a.val[0];
return *this;
}
friend ddouble operator - (const ddouble& a, const ddouble& b){
ddouble c;
c.v[1] = a.v[1] - b.v[1]; // (a-b)'=a'-b'
c.v[0] = a.v[0] - b.v[0];
return c;
}
friend ddouble operator + (const ddouble& a, const ddouble& b){
ddouble c;
c.v[1] = a.v[1] + b.v[1]; // (a+b)'=a'+b'
c.v[0] = a.v[0] + b.v[0];
return c;
}
friend ddouble operator * (const ddouble& a, const ddouble& b){
ddouble c;
c.v[1] = a.v[1]*b.v[0] + a.v[0]* b.v[1];
c.v[0] = a.v[0] * b.v[0];
return c;
}
};

```

For more complex programs all the operators and all the functions of algebra have to be redefined in the class ddouble. For instance

```

ddouble ddouble::sqrt(const ddouble& x){
ddouble c;
c.v[0] = sqrt(x.v[0]);
c.v[1] = 0.5*x.v[1]/sqrt(x.v[0]);
return c;
}

```

Obviously it creates a problem at $x = 0$ but there the function is non-differentiable anyway.

The library `ddouble.hpp` implements correctly all these; it can be downloaded from <http://www.ann.jussieu.fr/pironeau>.

3.1.2 Sensitivity by Automatic Differentiation

The most obvious application of Automatic Differentiation is for the computation of the sensitivity with respect to the parameters of the models (the *Greeks*). For this there is very little to change in the computer program; for example, consider a local volatility function of S and t :

$$\sigma = \sigma_0(1 + a\sqrt{|S - K|})(1 + bt). \quad (3.2)$$

The following changes are made to computes also the sensitivity with respect to b at $b = 1$ for $a = 0.01$ and $\sigma_0 = 0.3$.

1. In the file `optionhb.hpp` we make now an explicit use of `ddouble.hpp`

```
#include "ddouble.hpp" // for automatic differentiation
// typedef double ddouble; // for compatibility with AD
```

2. and in the main function the definition of b has value 1 in the second argument to indicate that derivatives are with respect to this variable.

```
int main(){
    VarMesh m(50,100,0.5,300.); // uniform nb x, nb t, T, xmax
    const ddouble b(1.,1.); // for derivatives with respect to b at b=1
    const double K=100, a=0.01;
    Option p(1,&m,K,0.05,0.3); // B&S,mesh,strike,r,sigma
    for(int i = 1; i < m.nT; i++)
        for(int j = 1; j < m.nX[i]; j++) // local volatility
            p.sigma[i][j] *= (1 + a*sqr(fabs(m.x[i][j]-K)))*(1+b*m.t[i]);
    p.calc();
}
```

Results are in $p.u[i][j].val[0]$ for the price of the put and in $p.u[i][j].val[1]$ for its derivative with respect to b . Both are shown on Figure 3.1

3.2 Calibration

3.2.1 Limitation of the Black-Scholes Model: the Need for Calibration

Consider a European option on a given stock with a maturity T and a payoff function P_0 , and assume that this option is on the market. Call p its present price. Also, assume the risk-free interest rate is the constant r . One may associate with p the so-called *implied volatility*, *i.e.* the volatility σ_{imp} such that the price given by formula (1.6) at time $t = 0$ with $\sigma = \sigma_{imp}$ coincides with p . If the Black-Scholes model was sharp, then the implied volatility would not depend on the payoff function P_0 . Unfortunately, for *e.g.* vanilla European puts or calls,

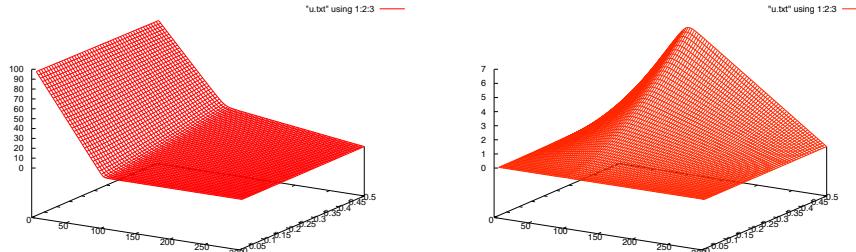


Figure 3.1: The pricing function of the put option (left) and its derivative with respect to the parameter b when σ is given by (3.2)

it is often observed that the implied volatility is far from constant. Rather, it is often a convex function of the strike price. This phenomenon is known as the *volatility smile*. A possible explanation for the volatility smile is that the deeply out-of-the-money options are less liquid, thus relatively more expensive than the options in-the-money.

This shows that the critical parameter in the Black-Scholes model is the volatility σ . Assuming σ constant and using (1.3) often leads to poor predictions of the options' prices. The volatility smile is the price paid for the too great simplicity of Black-Scholes' assumptions. It is thus necessary to use more involved models which must be *calibrated*.

Let us first explain what the term *calibration* means: consider an arbitrage-free market described by a probability measure \mathbb{P} on a scenario space (Ω, \mathcal{A}) . There is a risk-free asset whose price at time τ is $e^{r\tau}$, $r \geq 0$ and a risky asset whose price at time τ is S_τ . Specifying an arbitrage-free option pricing model necessitates the choice of a risk-neutral measure, *i.e.* a probability \mathbb{P}^* equivalent to \mathbb{P} such that the discounted price $(e^{-r\tau} S_\tau)_{\tau \in [0, T]}$ is a martingale under \mathbb{P}^* . Such a probability measure \mathbb{P}^* allows for the pricing of European options; consider a European option with payoff P_o at maturity $t \leq T$: its price at time $\tau \leq t$ is $P_\tau = e^{-r(t-\tau)} \mathbb{E}^{\mathbb{P}^*}(P_o(S_t) | \mathcal{F}_\tau)$, where $(\mathcal{F}_\tau)_{\tau \in [0, T]}$ is the natural filtration.

The pricing model \mathbb{P}^* must be compatible with the prices of the options observed on the market, whose number may be large. *Model calibration* consists of finding \mathbb{P}^* such that the discounted price $(e^{-r\tau} S_\tau)_{\tau \in [0, T]}$ is a martingale, and such that the option prices computed with the model coincide with the observed option prices. This is an *inverse problem*.

Popular extensions to the Black-Scholes model are:

- local volatility models: the volatility is a function of time and of the spot price, *i.e.* $\sigma_t = \sigma(S_t, t)$. With suitable assumptions on the regularity and the behavior at infinity of the function σ , (1.6) holds, and $P_t = p(S_t, t)$, where p satisfies the final value problem (1.3), in which σ varies with t and S . Calibration of local volatility has been much studied, see for example

[37, 9, 66, 5] for volatility calibration with European options and [1, 7] with American options;

- stochastic volatility models: one assumes that $\sigma_t = f(y_t)$, where y_t is a continuous time stochastic process, correlated or not to the process driving S_t , see § 2.4. Stochastic volatility calibration has been performed in [84].
- Lévy driven spot price: one may generalize the Black-Scholes model by assuming that the spot price is driven by a more general stochastic process, *e.g.* a Lévy process [30, 80]. Lévy processes are processes with stationary and independent increments which are continuous in probability. For a Lévy process X_τ on a filtered probability space with probability \mathbb{P} , the Lévy-Khintchine formula says that there exists a function $\chi : \mathbb{R} \rightarrow \mathbb{C}$ such that

$$\mathbb{E}(e^{iuX_\tau}) = e^{\tau\chi(u)},$$

$$\chi(u) = -\frac{\sigma^2 u^2}{2} + i\beta u + \int_{|z|<1} (e^{iuz} - 1 - iuz)\nu(dz) + \int_{|z|>1} (e^{iuz} - 1)\nu(dz),$$

for $\sigma \geq 0$, $\beta \in \mathbb{R}$ and a positive measure ν on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}} \min(1, z^2)\nu(dz) < +\infty$. The measure ν is called the Lévy measure of X . We focus on Lévy measure with a density, $\nu(dz) = k(z)dz$. Assume that the discounted price of the risky asset is a square integrable martingale under \mathbb{P} and that it is represented as the exponential of a Lévy process:

$$e^{-r\tau} S_\tau = S_0 e^{X_\tau}.$$

The martingale property is that $\mathbb{E}(e^{X_\tau}) = 1$, *i.e.*

$$\int_{|z|>1} e^z \nu(dz) < \infty, \quad \text{and} \quad \beta = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^z - 1 - z1_{|z|\leq 1})k(z)(dz).$$

and the square integrability comes from the condition $\int_{|z|>1} e^{2z} k(z)dz < \infty$.

With such models, the pricing function for a European option is obtained by solving a partial integrodifferential equation (PIDE), with a nonlocal term, see [89, 30] for the analysis of this equation and [79, 78, 33, 6] for numerical methods based on the PIDE. Calibration of Lévy models with European options has been discussed in [31, 32].

In this paragraph, we assume that the model is characterized by parameters θ in a suitable class Θ .

The last two classes of models (stochastic volatility and Lévy driven assets) describe incomplete markets, see [31]: the knowledge of the historical price process alone does not allow to compute the option prices in a unique manner. When the option prices do not determine the model completely, additional information may be introduced by specifying a *prior* model. If the historical price process has been estimated statistically from the time series of the underlying asset,

this knowledge has to be injected in the inverse problem; calling \mathbb{P}_0 the prior probability measure obtained as an estimation of \mathbb{P} , the inverse problem may be cast in a least-square formulation of the type: find $\theta \in \Theta$ which minimizes

$$\sum_{i \in I} \omega_i (P^\theta(0, S_0, t_i, x_i) - \bar{p}_i)^2 + \rho J_2(\mathbb{P}^\theta, \mathbb{P}_0), \quad (3.3)$$

where

- ω_i are suitable positive weights,
- S_0 is the price of the underlying asset today,
- $P^\theta(0, S_0, t_i, x_i)$ is the price of the option with maturity t_i strike x_i , computed with the pricing model associated with θ ,
- $\rho J_2(\mathbb{P}^\theta, \mathbb{P}_0)$ is a regularization term which measures the closeness of the model \mathbb{P}^θ to the prior. The number $\rho > 0$ is called the regularization parameter. This functional has two roles: 1) it stabilizes the inverse problem; for that, ρ should be large enough and J_2 should be convex or at least convex in a large enough region; 2) it guarantees that \mathbb{P}^θ remains close to \mathbb{P}_0 in some sense. The choice of J_2 is very important: $J_2(\mathbb{P}^\theta, \mathbb{P}_0)$ is often chosen as the relative entropy of the pricing measure \mathbb{P}^θ with respect to the prior model \mathbb{P}_0 , see [10], because the relative entropy becomes infinite if \mathbb{P}^θ is not equivalent to \mathbb{P}_0 . Some authors have argued that such a choice may be too conservative in some cases, for two reasons: a) the historical data which determine the prior may be missing or partially available -b) in the context of e.g. volatility calibration, once the volatility is specified under \mathbb{P}_0 , then the volatility under \mathbb{P}^θ must be the same for the relative entropy to be finite. A different approach was considered which allowed for volatility calibration, see [11].

Note that local volatility models describe complete markets: however, an additional regularization cost functional is necessary too, as explained in the paragraph below.

3.2.2 The Calibration of Local Volatility

Consider a family of I call options on the same asset with maturities $(T_i)_{1 \leq i \leq I}$ and strike prices $(K_i)_{1 \leq i \leq I}$. Assume that the options are available on the market, so one can observe their prices today. Call $(\bar{C}_i)_{1 \leq i \leq I}$ their prices. Assume that the spot price today is S_0 . Calibrating the *local volatility* amounts to finding a local volatility surface $S, t \rightarrow \sigma(S, t)$ such that if $C_i(S, t, K_i, T_i)$ is computed by solving the boundary value problem

$$\frac{\partial C_i}{\partial t} + \frac{\sigma(S, t)^2}{2} S^2 \frac{\partial^2 C_i}{\partial S^2} + rS \frac{\partial C_i}{\partial S} + rC_i = 0, \quad C_i(S, T_i) = (S - K_i)^+, \quad (3.4)$$

then $C_i(S_0, 0)$ coincides with the observed price \bar{C}_i , for $1 \leq i \leq I$.

A natural idea for this is to use least squares, *i.e.* to minimize a functional

$$J_{LS} : \sigma \mapsto \sum_{i \in I} \omega_i |\bar{C}_i - C_i(S_0, 0)|^2$$

for σ in a suitable function set Σ , where ω_i are positive weights and C_i is computed by solving (3.4). The evaluation of J_{LS} requires the solution of I initial value problems. The set Σ where the volatility is to be found must be chosen in order to ensure that from a minimizing sequence one can extract at least a subsequence that converges in Σ , and that its limit is indeed a solution of the least square problem. For example, Σ may be a compact subset of a Hilbert space W such that the mapping J_{LS} is continuous in W (lower semi-continuous would be enough). In practice, W has a finite dimension and is compactly embedded in the space of bounded and continuous functions σ such that $S\partial_S\sigma$ is bounded. Thus, the existence of a solution to the minimization problem is most often guaranteed. What is more difficult to guarantee is uniqueness and stability: is there a unique solution to the least square problem? If yes, is the solution insensitive to small variations of the data? the answer to these questions is no in general, and we say that the problem is *ill-posed*.

As a possible cure to ill-posedness, one usually modifies the problem by minimizing the functional

$$J : \sigma \mapsto J_{LS}(\sigma) + J_R(\sigma)$$

instead of J_{LS} , where J_R is a sufficiently large strongly convex functional defined on W and containing some financially relevant information. For example, one may choose

$$J_R(\sigma) = \omega \|\sigma - \bar{\sigma}\|^2,$$

where ω is some positive weight, $\|\cdot\|$ is a norm in W and $\bar{\sigma}$ is a prior local volatility, which may come from a historical knowledge. The difficulty is that ω must not be too large not to perturb the inverse problem too much, but not too small to guarantee some stability. The art of the practitioner lies in the choice of J_R .

3.2.3 Gradient Methods

Once the least square problem is chosen, we are left with proposing a strategy for the construction of minimizing sequences. If J_{LS} and J_R are C^1 functional, then gradient methods may be used. The principle of gradient methods is as follows: let $\delta\sigma$ be an admissible variation and assume that J is Frechet differentiable with respect to σ ; then by definition

$$\delta J = J(\sigma + \delta\sigma) - J(\sigma) = \text{grad}J(\sigma) \cdot \delta\sigma + o(|\delta\sigma|). \quad (3.5)$$

So $\delta\sigma = -\rho \text{grad}J(\sigma)$ causes a decrease of J , at least when ρ is small.

Gradient Algorithm with Armijo Rule

1. Choose an initial σ^0 and 2 numbers $0 < \alpha < 1$, $\rho_0 > 0$.
2. Set $H^m = -\text{grad}J(\sigma^m)$

3. Compute by dichotomy a signed integer k such that, with $\rho = \rho_0 2^k$

$$\begin{aligned} J(\sigma^m + \rho H^m) - J(\sigma^m) &\leq \rho \alpha \operatorname{grad} J(\sigma^m) \cdot H^m \\ J(\sigma^m + 2\rho H^m) - J(\sigma^m) &\geq 2\rho \alpha \operatorname{grad} J(\sigma^m) \cdot H^m \end{aligned} \quad (3.6)$$

4. Set $\sigma^{m+1} = \sigma^m + \rho H^m$ and proceed to the next iteration.

Conjugate Gradient Method In the Polak-Ribiere [90] version one changes the definition of H^m , $m > 1$ to

$$H^m = -\operatorname{grad} J(\sigma^m) + \gamma H^{m-1} \text{ with } \gamma = \frac{\operatorname{grad} J(\sigma^m) \cdot (\operatorname{grad} J(\sigma^m) - \operatorname{grad} J(\sigma^{m-1}))}{\|\operatorname{grad} J(\sigma^{m-1})\|^2}$$

Under strong hypothesis such as local convexity and twice differentiability of J , this algorithm converges to a local minimum superlinearly, i.e. faster than any geometric progression.

The drawbacks and advantages of gradient methods are well known: on the one hand, they do not guarantee convergence to the global minimum if the functional is not convex, because the iterates can be trapped near a local minimum. On the other hand, they are fast and accurate when the initial guess is close enough to the minimum. For these reasons, gradient methods are often combined with techniques that permit to localize the global minimum but that are slow, like simulated annealing or evolutionary algorithms.

3.2.4 A Finite Dimensional Example

To reduce the size of the problem we parametrize the volatility surface by a linear-quadratic spline (LQS). The idea is that we want the local volatility to be close to linear in the money and to have a convex shape out of the money; this is desired at each time t but the bounds may depend on t linearly. Hence the surface is given by

$$\sigma(S, t) = \begin{cases} a + (2\frac{\sigma_1 - a}{S_1} - \frac{\sigma_2 - \sigma_1}{S_2 - S_1})S + (\frac{\sigma_2 - \sigma_1}{S_2 - S_1} - \frac{\sigma_1 - a}{S_1})\frac{S^2}{S_1}, & \text{if } S < S_1, \\ \sigma_2 \frac{S - S_1}{S_2 - S_1} + \sigma_1 \frac{S_1 - S}{S_2 - S_1}, & \text{if } S_1 \leq S \leq S_2, \\ \sigma_2 + (S - S_2) \frac{\sigma_2 - \sigma_1}{S_2 - S_1} + \left((S - S_2) \frac{\sigma_2 - \sigma_1}{S_2 - S_1} \right)^2, & \text{if } S > S_2, \end{cases} \quad (3.7)$$

where S_i , σ_i , $i = 1, 2$ are linear with respect to t :

$$S_i = S_{i1}(1 - \frac{t}{T}) + S_{i2}\frac{t}{T}, \quad \sigma_i = \sigma_{i1}(1 - \frac{t}{T}) + \sigma_{i2}\frac{t}{T}. \quad (3.8)$$

The local volatility is thus C^1 regular, linear w.r.t. S for $S_1 \leq S \leq S_2$; it takes the values σ_1 at $S = S_1$, σ_2 at $S = S_2$ and a at $S = 0$. For a, T given, the local

volatility depends on eight parameters: S_{ij} , σ_{ij} . However, the representation (3.7) (3.8) does not make sense unless $0 < S_1(t) < S_2(t)$, so it is better to set

$$\begin{aligned} z_0 &= \sqrt{S_{11}}, \quad z_1 = \sqrt{S_{21}}, \quad z_2 = \sqrt{S_{21} - S_{11}}, \quad z_3 = \sqrt{S_{22} - S_{12}}, \\ z_{2*i+j+1} &= \frac{\sqrt{\sigma}_{ij}}{1 + \sqrt{\sigma}_{ij}}, \quad i, j = 1, 2, \end{aligned} \tag{3.9}$$

so as to work with the set of parameters $\{z_i\}_0^7$; the last formula gives $\sigma_{ij} = z_{2*i+j+1}^2 / (1 + z_{2*i+j+1}^2) \in (0, 1)$ for all possible values of $z_{2*i+j+1}$. This is a simple way to force σ in $(0, 1)$ for all values of the parameters z .

A Test Case Table 3.1 contains the observed prices of several European calls on the same asset. The parameter r is constant: $r = 0.03$. The spot price is 1418.3.

The least square problem is

$$\min_{\sigma \in \Sigma} \sum_{i=1}^I |C_i(S_0, 0) - C_i|^2, \quad \text{subject to (3.4).} \tag{3.10}$$

The conjugate gradient algorithm was used to solve (3.10) with the initial guess for the parameter vector

$$z_0 = \sqrt{1500}, \quad z_1 = \sqrt{1700}, \quad z_2 = z_3 = \sqrt{150}, \quad z_{4+i} = \sqrt{\frac{0.4}{1 + i - 0.4}} \quad i = 0..3.$$

The cost function J has $I = 57$ terms. The other parameters are $a = 4$, $T = 3$ (years). The results are shown on figure 3.2. The strange shape of the volatility surface is due to the parametrization. If in (3.7) we change the definition for $S > S_2$ to be

$$\sigma_2 + (S - S_2) \frac{\sigma_2 - \sigma_1}{S_2 - S_1} + \left((S - S_2) \frac{\sigma_2(0) - \sigma_1(0)}{S_2(0) - S_1(0)} \right)^2, \quad \text{if } S > S_2 \tag{3.11}$$

then the shape is more natural, see figure 3.3, and the same precision is obtained.

3.2.5 Dupire's Equation

Solving (3.10) by a conjugate gradient method is time consuming because the Black-Scholes PDE has to be solved I times at each iterations and for each ρ -trial in the dichotomy (3.6). Dupire[36] noticed that the function $(K, \tau) \mapsto C(S, t, K, \tau)$ (now t and S are fixed) satisfies a forward parabolic PDE. One can obtain this either by reasoning directly on (1.6) or by PDE arguments using adjoint operators. We present the second approach:

Strike	1 Month	2 Months	6 Months	12 Months	24 Months	36 Months
700						733
800						650.6
900						569.8
1000					467.8	
1100					385.3	
1150					345.4	
1175			265.2			
1200			242	266.1	306.6	
1215				253.4		
1225			219	245		
1250			196.6	224.2	269.2	
1275			174.5	203.9	251	
1300			152.9	184.1	233.2	
1325			131.9	164.9	215.8	
1350			111.7	146.3	198.9	
1365			100			
1375	50.6	60	92.5	139	182.6	
1380	46.1	55.8				
1385	41.8	51.8				
1390	37.5	47.9				
1395	33.4	44				
1400	29.4	40.3	74.5	128.4	166.7	215.9
1405	25.6	36.7				
1410	21.9	33.2				
1415	18.7	29.8				
1420	15.4	26.6				
1425	12.7	23.8	58	111.4	151.5	
1430	10	20.7				
1435	8	18.2				
1440	6.3	15.7				
1445	4.4	13.4				
1450	3.1	11.3	43.3	95.2	136.9	187.5
1455	2.05	9.6				
1460	1.45	7.9				
1475			30.6	80.2		
1500			20.3	54	109.6	160.8
1525			12.6	42.7		
1550			7.5	33		
1575				24.7		
1600			1.95	18.2	64.5	113,9
1700					32.7	75,7
1800					15.5	
1900					5.2	

Table 3.1: The prices of a family of calls on the same asset

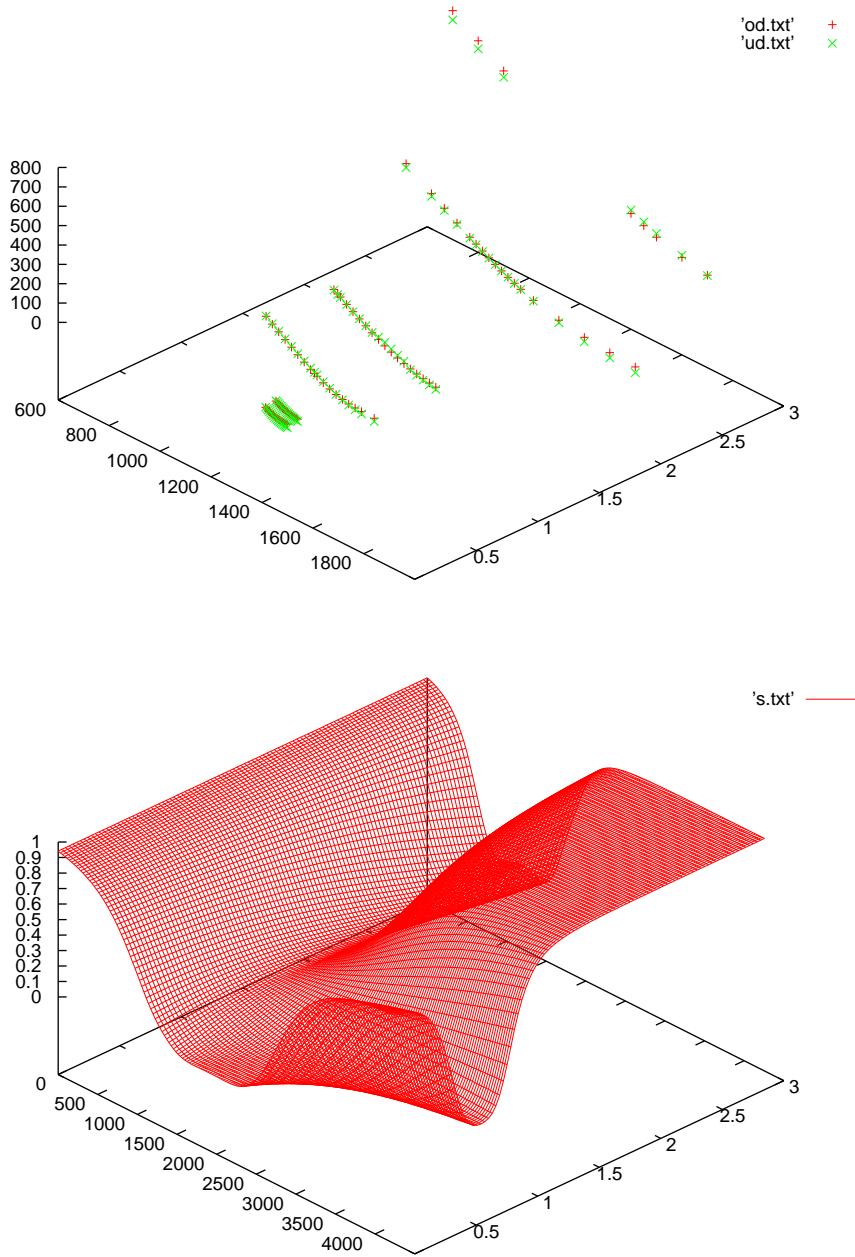


Figure 3.2: Difference between observations and model predictions (top) obtained with the local volatility (bottom) produced by 100 iterations of Conjugate Gradient on (3.10). The cost function is reduced from 500000 to 9000 and the L^2 -norm of the gradient from 10^{11} to 10^7 .

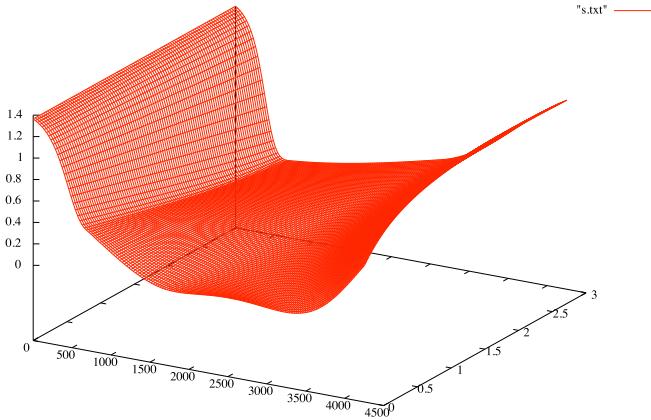


Figure 3.3: Same as Figure 3.2 but with (3.11). Convergence is also not perfect: after 100 iterations J is reduced from 500000 to 12000, giving an average error at each observed point larger than one percent.

Proposition 3.1 (Dupire) Let v be solution in $\mathbb{R}_+ \times (0, T)$ of

$$\frac{\partial v}{\partial t} - \frac{1}{2}\sigma^2(S, t)S^2\frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} = 0, \quad v(S, 0) = (S_0 - S)^+, \quad (3.12)$$

then

$$C(S_0, 0, K, \tau) = v(K, \tau) \quad (3.13)$$

and $p \equiv \frac{\partial^2 v}{\partial S^2}$ satisfies the boundary value problem

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{\sigma^2(S, t)S^2}{2} p \right) + \frac{\partial}{\partial S}(rSp) + rp = 0, \quad p(S, 0) = \delta_{S_0}(S), \quad (3.14)$$

where δ_{S_0} is the Dirac mass at S_0 .

Proof. Call Q the rectangular domain $Q = (S_m, S_M) \times (0, \tau)$, with $0 \leq S_m < S_M$ and consider the boundary value problem in Q

$$\frac{\partial u}{\partial t} + \eta(S, t)\frac{\partial^2 u}{\partial S^2} + \mu\frac{\partial u}{\partial S} - ru = 0, \quad u(S, \tau) = u_\tau(S), \quad (3.15)$$

where $\eta(S, t) = \frac{1}{2}\sigma^2(S, t)S^2$ and $\mu = rS$. Multiplying (3.15) by the solution p of (3.14), integrating on Q and using an integration by part in time and Green's formula in the variable S yield

$$u(S_0, 0) = \int_{S_m}^{S_M} u_\tau(S)p(S, \tau)dS + \int_0^\tau \left[p\eta\frac{\partial u}{\partial S} - u\frac{\partial}{\partial S}(\eta p) + p\mu u \right]_{S_m}^{S_M}. \quad (3.16)$$

We take $u(S, t) = C(S, t, K, \tau)$, $S_m = 0$ and $S_M = +\infty$. The second integral in (3.16) vanishes because $u \sim S$ and p tends to 0 faster than S^{-1} as S tends to infinity. Let v be a double primitive of p , i.e., a function such that $\frac{\partial^2 v}{\partial S^2} = p$ then, integrating (3.14) twice yields

$$\frac{\partial v}{\partial t} - \frac{1}{2}\sigma^2(S, t)S^2\frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} = aS + b, \quad v(S, 0) = c + dS + (S - S_0)^+, \quad (3.17)$$

where a, b, c, d are integration constants. But $u_\tau = (S - K)^+$, so $\frac{\partial^2 u_\tau}{\partial S^2}(S) = \delta_K(S)$, where δ_K is the Dirac mass at K . A double integration by part applied to (3.16) yields

$$u(S_0, 0) = \int_0^\infty u_\tau(S)\frac{\partial^2 v}{\partial S^2}(S)(S, \tau)dS = \int_0^\infty v(S, \tau)\frac{\partial^2 u_\tau}{\partial S^2}(S)dS + \left[u_\tau \frac{\partial v}{\partial S} - v \frac{\partial u_\tau}{\partial S} \right]_0^\infty. \quad (3.18)$$

Therefore, if v vanishes at ∞ , we obtain

$$u(S_0, 0) = v(K, \tau),$$

because $u_\tau(0) = \partial_S u_\tau(0) = 0$.

By choosing $a = b = 0$ and $c = S_0$, $d = -1$, we obtain (3.13) because $(S - S_0)^+ - (S - S_0) = (S - S_0)^- = (S_0 - S)^+$ so v vanishes at infinity and the initial condition in (3.12) is the desired one. ■

Remark 3.1 If the underlying asset yields a distributed dividend, $qS_t dt$, then the pricing function satisfies

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{\sigma^2(S, t)S^2}{2}\frac{\partial^2 C}{\partial S^2} + (r - q)S\frac{\partial C}{\partial S} - rC &= 0 && \text{in } \mathbb{R}_+ \times [0, \tau], \\ C(S, \tau) &= (S - K)^+ && \text{in } \mathbb{R}_+, \end{aligned} \quad (3.19)$$

and the related Dupire's equation is

$$\frac{\partial v}{\partial t} - \frac{1}{2}\sigma^2(S, t)S^2\frac{\partial^2 v}{\partial S^2} + (r - q)S\frac{\partial v}{\partial S} + qv = 0, \quad (3.20)$$

for $\tau \geq t > 0$ and $S \in \mathbb{R}_+$.

Numerical Results A finite difference method implicit in time of order one (Euler's scheme) is used for (3.4) and (3.12); the parameters are $K = 100$, $r = 0.06$, $\sigma = 0.4$, and 200 time steps and 250 mesh points for S . We have compared the numerical results for (3.4) with (3.13) by solving (3.12) for all the values of S_0 used to display $S_0 \rightarrow C(S_0, 0)$. Very good accuracy is found, see figure 3.4.

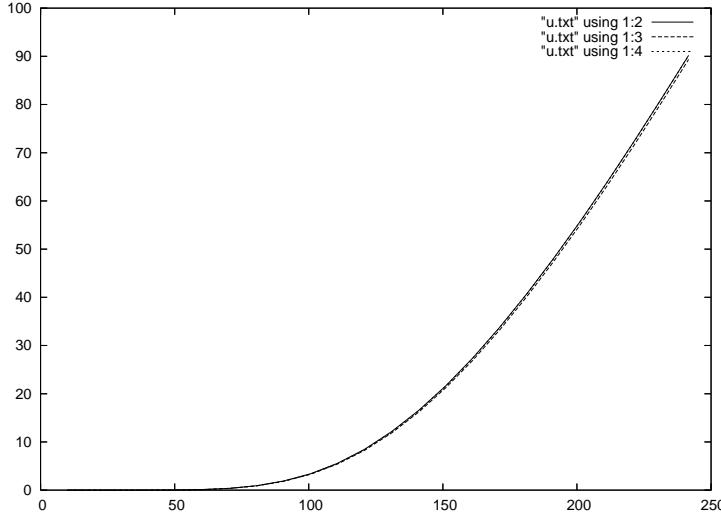


Figure 3.4: The pricing function of European call option one year to maturity computed by 3 different methods: a) by solving (3.4), b) by the Dupire formula (3.12, 3.13), c) by the Black-Scholes analytic formula.

Extension to more complex option is possible as long as the PDE is linear. The previous argument does not work with American options (modeled by a variational inequality). It is not always possible to find a double primitive of the adjoint equation but the discrete equivalent of (3.16) can be found when a variational numerical method such as the Finite Element Method is used. For simplicity assume that the coefficients in (3.15) do not depend of time and consider a numerical discretization of (3.15) with an Euler implicit scheme in time with time step δt and piecewise linear finite elements in the variable S . The scheme can be written in matrix form

$$(\mathbf{B} + \mathbf{A})\mathbf{u}^n - \mathbf{B}\mathbf{u}^{n+1} = 0, \quad (3.21)$$

where \mathbf{u}^n is the vector of the nodal values of the piecewise linear function u_h^n which approximates $u(\cdot, n\delta t)$. Given a vector \mathbf{p}^0 , introduce the sequence of vectors \mathbf{p}^n obtained by iterating

$$(\mathbf{A} + \mathbf{B})^T \mathbf{p}^{n+1} - \mathbf{B}^T \mathbf{p}^n = 0. \quad (3.22)$$

Now notice that (3.21) multiplied by $(\mathbf{p}^{n+1})^T$ gives

$$0 = (\mathbf{p}^{n+1})^T (\mathbf{A} + \mathbf{B})\mathbf{u}^n - (\mathbf{p}^{n+1})^T \mathbf{B}\mathbf{u}^{n+1} = (\mathbf{p}^n)^T \mathbf{B}\mathbf{u}^n - (\mathbf{p}^{n+1})^T \mathbf{B}\mathbf{u}^{n+1}, \quad (3.23)$$

where the last equality has used (3.22). Summing up over all n gives

$$(\mathbf{p}^0)^T \mathbf{B}\mathbf{u}^0 = (\mathbf{p}^N)^T \mathbf{B}\mathbf{u}^N. \quad (3.24)$$

Choosing $\mathbf{p}_j^0 = \delta_{ij}$, $j = 0, \dots, M$ gives the discrete equivalent of (3.16)

$$\left(\sum_{j=0}^M \mathbf{B}_{ij} \right) \mathbf{u}_i^0 \approx (\mathbf{B}\mathbf{u}^0)_i = (\mathbf{p}^N)^T \mathbf{B}\mathbf{u}^N. \quad (3.25)$$

3.2.6 A New Least Square Problem

A natural idea is to somehow interpolate the observed prices by a sufficiently smooth function $\tilde{v} : \mathbb{R}_+ \times [0, \max_{i \in I} T_i] \rightarrow \mathbb{R}_+$, then use (3.20) with $v = \tilde{v}$ in order to obtain an approximation σ . For examples, bicubic splines may be used. This approach has several serious drawbacks:

- it is difficult to design an interpolation process such that $\frac{\partial^2 \tilde{v}}{\partial S^2}$ does not take the value 0, and such that the obtained approximation of the squared volatility is non negative.
- There is an infinity of possible interpolations of \bar{C}_i at (K_i, T_i) , $1 \leq i \leq I$, and for two possible choices, the volatility obtained by (3.20) may differ considerably.

We see that financially relevant additional information have to be added to the interpolation process.

Another natural idea is to consider a new least square problem where the state function satisfies Dupire's equation:

$$\min_{\sigma \in \Sigma} J(\sigma) = \sum_1^I \omega_i |v(K_i, T_i) - \bar{C}_i|^2 + J_R(\sigma) \quad \text{subject to (3.20)} \quad (3.26)$$

The evaluation of J requires solving only one boundary value problem instead of I in the first least square problem.

Numerical Results

The same test problem as in § 3.2.4 was solved: the local volatility is represented by a spline and a gradient method with automatic differentiation is used. Naturally the computing time is more than 50 times faster: on a 2Gig Hz dual core Intel, 50 iterations on a 100×50 grid takes 0.5 seconds. The results are shown on figures 3.5, 3.6 and 3.7.

Adjoint equation

Gradient methods require the differentiation of J with respect to σ . The gradient of J can be computed with automatic differentiation of computer programs if the chosen representation of σ is done with a small number of parameters. Alternatively, when σ has a large number of degrees of freedom, automatic differentiation becomes too expensive, and an analytic procedure is needed. Since J_R explicitly depends on σ , its gradient is easily computed. The gradient

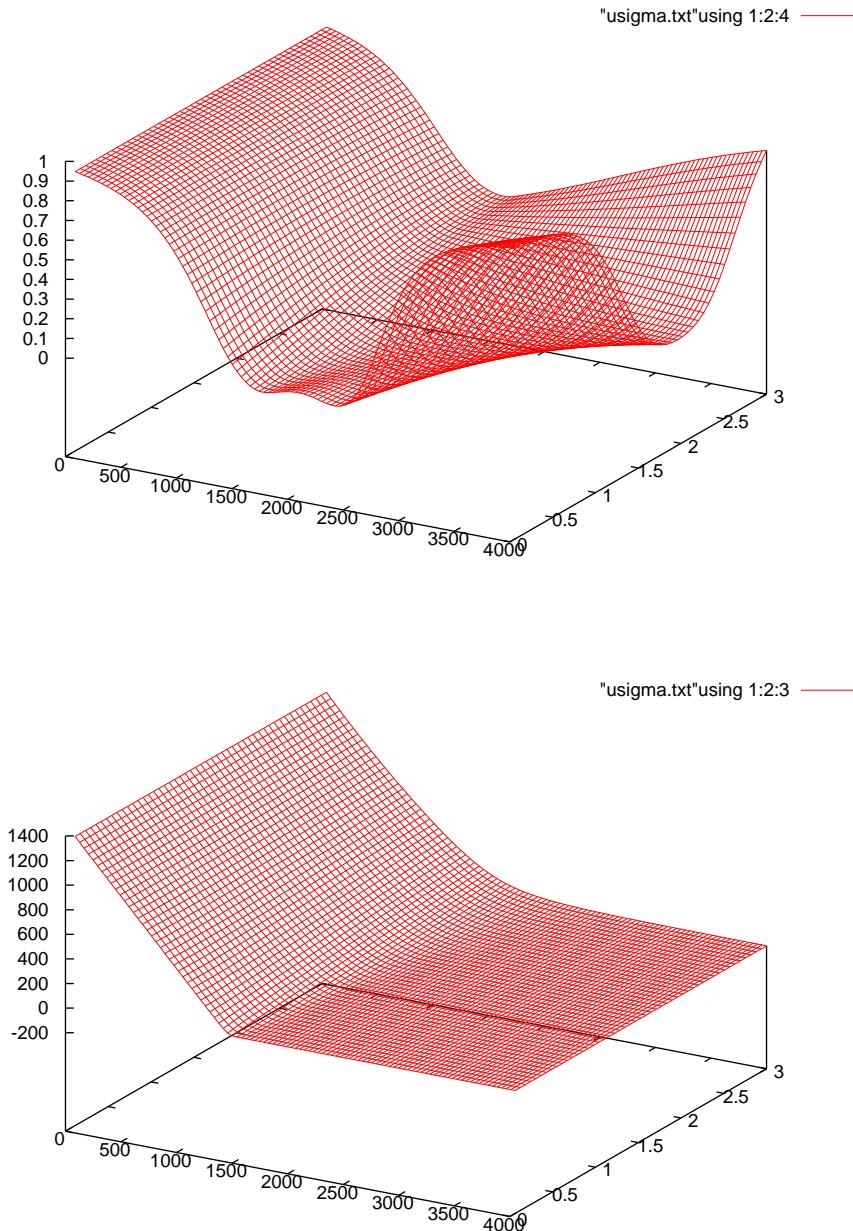


Figure 3.5: Local volatility surface generated by the optimization algorithm from the parametrization (3.7) and using Dupire's equation. Below is the price surface as a function of strike and maturity.

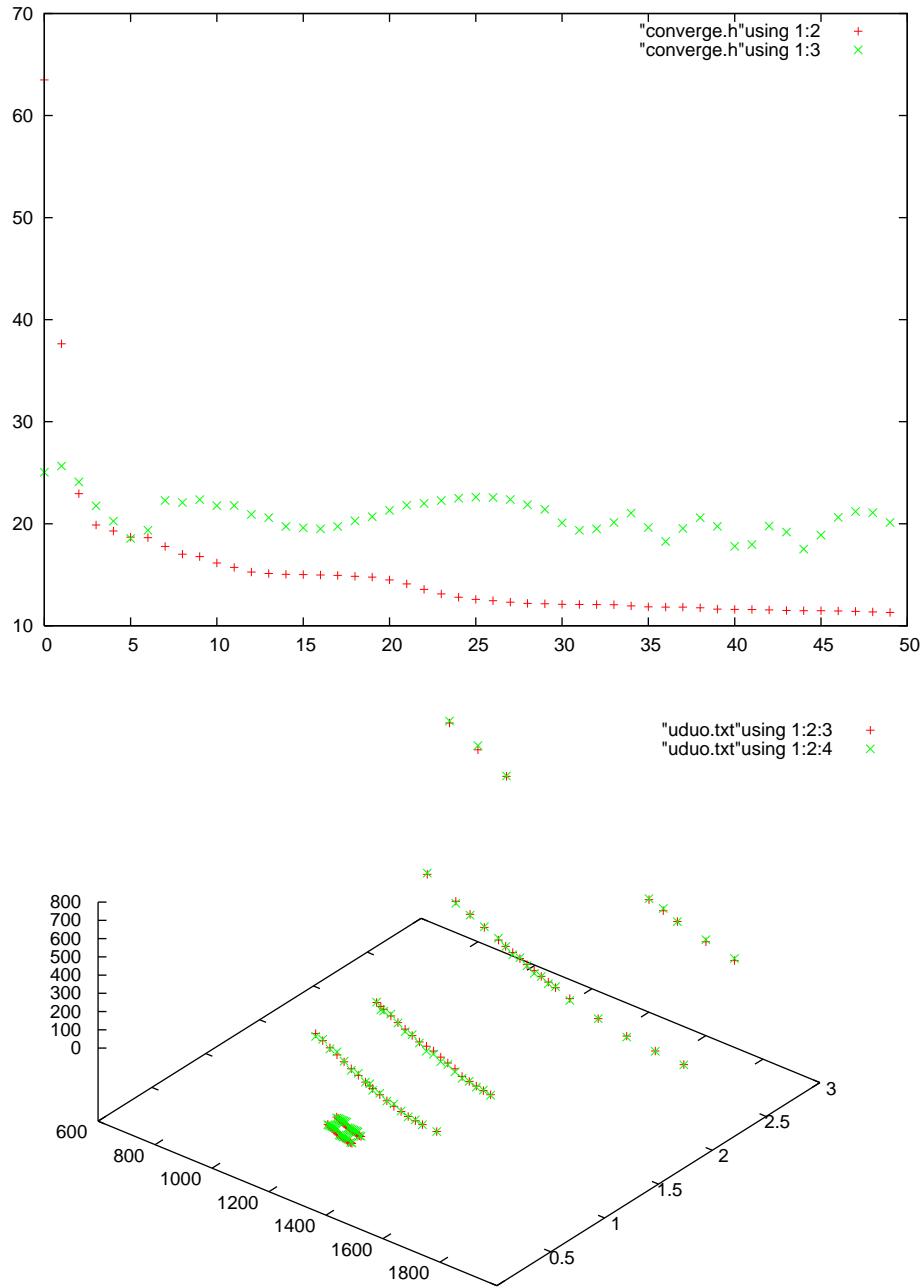


Figure 3.6: Convergence of $J/1000$ to its local minimum and of the logarithm of the squared norm of the gradient to $-\infty$ as a function of iteration number. Below is a comparison between all observed prices (target) and the prices generated by the model.

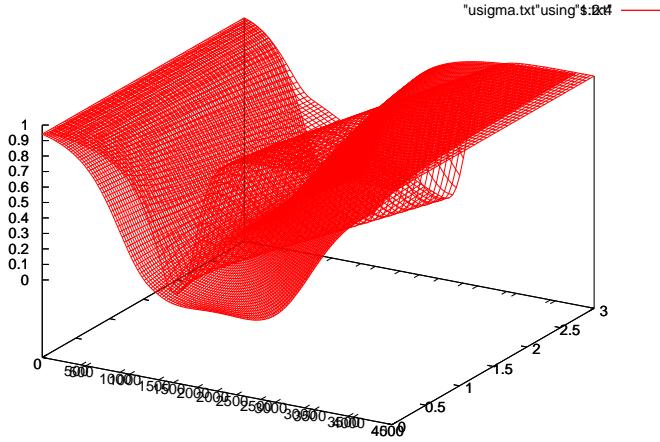


Figure 3.7: Same as Figure 3.5 but with (3.11). 100 iterations of conjugate gradient decreases J from 500000 to 1200, giving an average error smaller than a percent

of $J_{LS} = \sum_1^I |v(K_i, T_i) - \bar{C}_i|^2$ is more difficult to evaluate, because the prices $v(K_i, T_i)$ depend of σ in an indirect way: one needs to evaluate the variations of $v(K_i, T_i)$ caused by a small variation of σ ; calling $\delta\sigma$ the variation of σ and δv the induced variation of v , one sees by differentiating (3.20) that $\delta v(S, t = 0) = 0$ and

$$\partial_t \delta v - \frac{\sigma^2(S, t) S^2}{2} \partial_{SS}^2 \delta v + (r - q) S \partial_S \delta v + q \delta v = \sigma \delta \sigma S^2 \partial_{SS}^2 v. \quad (3.27)$$

To express δJ in terms of $\delta\sigma$, an adjoint state function P is introduced, as the solution to the adjoint problem: find the function P such that $P(\bar{T}, \cdot) = 0$ and for $t < \bar{T}$,

$$\partial_t P + \partial_{SS}^2 \left(\frac{\sigma^2 S^2}{2} P \right) - \partial_S (P(r - q) S) - q P = 2 \sum_{i \in I} \omega_i (v(K_i, T_i) - \bar{C}_i) \delta_{K_i, T_i}, \quad (3.28)$$

where \bar{T} is an arbitrary time greater than $\max_{i \in I} T_i$ and in the right hand side, δ_{K_i, T_i} denote Dirac functions at (K_i, T_i) . The meaning of (3.28) is the following:

$$-\int_Q \left(\partial_t w - \frac{\sigma^2 S^2}{2} \partial_{SS}^2 w + (r - q) S \partial_S w + q w \right) P = 2 \sum_{i \in I} \omega_i (v(K_i, T_i) - \bar{C}_i) w(K_i, T_i). \quad (3.29)$$

where $Q = \mathbb{R}_+ \times (0, \bar{T})$, and w is any function such that $w \in L^2((0, \bar{T}), V)$ with $\partial_t w \in L^2(Q)$ and $S^2 \partial_{SS}^2 w \in L^2(Q)$. Taking $v = \delta v$ in (3.29) and using (3.27),

one finds

$$\begin{aligned} 2 \sum_{i \in I} \omega_i (v(K_i, T_i) - \bar{C}_i) \delta v(K_i, T_i) &= 2 \sum_{i \in I} \omega_i (v(K_i, T_i) - \bar{C}_i) \langle \delta_{K_i, T_i}, \delta v \rangle \\ &= - \int_Q \left(\partial_t \delta v - \frac{\sigma S^2}{2} \partial_{SS}^2 \delta v + (r - q) S \partial_S \delta v + q \delta v \right) P \\ &= - \int_Q \sigma \delta \sigma S^2 P \partial_{SS}^2 v. \end{aligned}$$

We have worked in a formal way, but all the integrations above can be justified. This leads to the estimate

$$\left| \delta J_{LS} + \int_Q \sigma \delta \sigma S^2 P \partial_{SS}^2 v \right| \leq c \|\delta \sigma\|_{L^\infty(Q)}^2,$$

which implies that J_{LS} is differentiable, and that its differential at point σ is given by

$$DJ_{LS}(\sigma) : \eta \mapsto - \int_Q \sigma \eta S^2 P(\sigma) \partial_{SS}^2 v(\sigma),$$

where $P(\sigma)$ satisfies (3.28), and $v(\sigma)$ satisfies (3.20). We see that the gradient of J_{LS} can be evaluated. When (3.20) is discretized with *e.g.* finite elements, all what has been done can be repeated (with a discrete adjoint problem), and the gradient of the functional can be evaluated in the same way. Let us stress that the gradient $DJ_{LS}(\sigma)$ is computed exactly, which would not be the case with *e.g.* a finite difference method.

Local volatility can also be calibrated with American options, but it is not possible to find the analogue of Dupire's equation. Thus, in the context of a least square approach, the evaluation of the cost function requires the solution of I variational inequalities, which is computationally expensive, see [1, 7]. Nevertheless, it is also possible to find necessary optimality conditions involving an adjoint state.

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