Undergraduate Research Opportunity Programme in Science

Financial Mathematics With Python

WANG ZEXIN

ZHOU CHAO

Department of Mathematics

National University of Singapore

AY2016/17 Semester 2

Contents

Abstract

In this paper, we implemented the library packages for Python to solve problems in Financial Mathematics such as derivatives valuation and simulation as suggested by the book "*Python for Finance*". Those results correspond to the different approaches in pricing financial derivatives. The key effort is on the development of valuation scheme for the options, in particular, the strongly path-dependent options. More importantly, our model is open to extend for various assumptions about the market. We attempt to apply these valuation schemes onto the path-independent options with available closed-form formulain order to verify the valuation results from Monte Carlo simulations as well as the Finite Difference Model.

1 Introduction

The study upon using Python to carry out derivatives valuation has made much progress over the years as there have already been available libraries built for this purpose. The derivatives analytics library suggested in the book "*Python for Finance*" has its advantages on the coverage upon the various aspects of possible analysis for financial derivatives. Still, we can make justifiable modifications and extensions to improve on the accuracy and speed of computations for estimations.

2 Derivation of Black Scholes PDE

2.1 Basics

In the Black-Scholes World, we assume that the following two SDE hold:

$$dM_t = rM_t dt$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

We assume that the It$\hat{o}'$s Lemma hold:

As $dS_t = \mu S_t dt + \sigma S_t dW_t$,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t$$

2.2 Delta-hedging Argument

Our first aim is to find ϕ_t

for $\Pi_t = V_t - \phi_t S_t$

such that

$d\Pi_t = dV_t - \phi_t dS_t$(Self-financing)

$d\Pi_t = r\Pi_t dt$(risk free)

From the equations, we can obtain that

$r\Pi_t dt = dV_t - \phi_t dS_t$

$r(V_t - \phi_t S_t)dt = dV_t - \phi_t dS_t$

$dV_t = r(V_t - \phi_t S_t)dt + \phi_t dS_t$

By It$\hat{o}'$s Lemma,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t$$

Hence we obtain two equations:

$$\phi_t = \frac{\partial V}{\partial S}$$

$$r(V_t - \phi_t S_t) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S}$$

$$r(V_t - \frac{\partial V}{\partial S} S_t) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S_t - rV_t = 0$$

At time t, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S - rV = 0$$

2.3 Replicating portfolio

Our aim is to find a_t and b_t such that $\Pi_t = a_tS_t + b_tM_t$ can entirely replicate V_t while the self-financing condition still holds: $d\Pi_t = a_tdS_t + b_tdM_t$

As $dS_t = \mu S_tdt + \sigma S_tdW_t$ and $dM_t = rM_tdt$,

$$\begin{aligned} d\Pi_t &= a_t(\mu S_tdt + \sigma S_tdW_t) + b_t(rM_tdt) \\ &= (a_t\mu S_t + rb_tM_t)dt + (\sigma a_tS_t)dW_t \end{aligned}$$

By Itô's Lemma,

$$\begin{aligned} dV_t &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S}dS_t \\ &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S}(\mu S_tdt + \sigma S_tdW_t) \\ &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t)dt + (\sigma S_t\frac{\partial V}{\partial S})dW_t \end{aligned}$$

As Π_t fully replicates V_t,

$$d\Pi_t = (a_t\mu S_t + rb_tM_t)dt + (\sigma a_tS_t)dW_t = dV_t$$

Also by Itô's Lemma,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t)dt + (\sigma S_t\frac{\partial V}{\partial S})dW_t$$

Hence we obtain that,

$$a_t = \frac{\partial V}{\partial S}$$

$$a_t\mu S_t + rb_tM_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t$$

$$\begin{aligned} \frac{\partial V}{\partial S}\mu S_t + rb_tM_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t \\ rb_tM_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} \\ ra_tS_t + rb_tM_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} + ra_tS_t \\ rV_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S}S_t \\ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S_t^2\frac{\partial V}{\partial S} &+ r\frac{\partial V}{\partial S}S_t - rV_t = 0 \end{aligned}$$

Hence,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S}S - rV = 0$$

2.4 Lognormal property

We shall derive the lognormal property of the underlying asset prices. As given by the Geometric Brownian Motion, $dS_t = \mu S_t dt + \sigma S_t dW_t$.
We can define the derivative price function as $V(S_t, t) := \ln S_t$, with

$$\frac{\partial V}{\partial S} = \frac{1}{S}, \frac{\partial^2 V}{\partial S^2} = -\frac{1}{S^2}, \frac{\partial V}{\partial t} = 0$$

By It$\hat{o}'$s Lemma,

$$\begin{aligned} dV_t &= ((\mu S_t)\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2})dt + b\frac{\partial V}{\partial S} dW_t \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t \end{aligned} \tag{1}$$

As μ and σ are constants, V_t is simply a drifted Brownian Motion.

$$V_t - V_0 = \ln S_t - \ln S_0 \sim \mathcal{N}((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$$

$$\frac{\ln S_t - \ln S_0 - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \sim \mathcal{N}(0, 1)$$

We can also generalize to the case for time t and T. (Let $\tau = T - t$, $\phi \sim \mathcal{N}(0, 1)$)

$$\frac{\ln S_T - \ln S_t - (\mu - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = \phi$$

$$S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi}$$

3 Finite Difference Model for Numerical PDE

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial V}{\partial S}+r\frac{\partial V}{\partial S}S-rV=0$$

If we are to plot out the option prices in a diagram, there should be a 3D diagram with y-axis being the underlying prices, and x-axis being the time, omitting the scenario when the option price is dependent on other stochastic terms such as volatility. In the Black-Scholes formula, we are able to approximate $\Theta=\frac{\partial V}{\partial t}$, $\Delta=\frac{\partial V}{\partial S}$ and $\Gamma=\frac{\partial^2 V}{\partial S^2}$ when the discretization is significantly small and thus it will be easy to simply use the ratio of change in derivatives value with respect to change in time or change in underlying asset value.

$$\begin{aligned}
\frac{\partial V}{\partial t} &\approx \frac{P_t-P_{t-\vartriangle t}}{\vartriangle t}\\
\frac{\partial V}{\partial S} &\approx \frac{P_S-P_{S-\vartriangle S}}{\vartriangle S}\approx\frac{P_{S+\vartriangle S}-P_S}{\vartriangle S}\\
\frac{\partial^2 V}{\partial S^2} &\approx \frac{\partial}{\partial V}\left(\frac{P_{S+\vartriangle S}-P_S}{\vartriangle S}\right)\\
&\approx \frac{\frac{P_{S+\vartriangle S}-P_S}{\vartriangle S}-\frac{P_S-P_{S-\vartriangle S}}{\vartriangle S}}{\vartriangle S}\\
&= \frac{P_{S+\vartriangle S}-2P_S+P_{S-\vartriangle S}}{(\vartriangle S)^2}
\end{aligned}\tag{2}$$

3.1 One factor model using Implicit Scheme

3.2 Explicit Scheme

The difference between explicit and implicit scheme lies in the need to solve an implicit equation at each discretization or there is a closed-form formula for each step.

3.3 Feynman–Kac formula

3.4 Crank–Nicolson method

4 Variance Reduction Techniques for Monte Carlo Simulation

Monte Carlo simulations are usually used when there is no readily available closed-form formula and the numerical PDE valuation scheme is not developed yet. However, people have already explored its strenght upon the valuation using high-dimensional scheme. When doing option pricing with Monte Carlo simulations, we simulate a series of possible scenarios and use the payoff function to calculate one option payoff. By repeating this process N times and take the average of all the discounted option payoffs, a more accurate option price can be obtained. When there is time variation in parameters of the model, or the option payoff is path dependent, or there are a number of stochastic variables that the option price depends on, a finite difference scheme will be both space and time consuming. With the advancements of GPU and parallel computing, just as suggestd in the book "*Python for Finance*", the power of Monte Carlo is able to emerge.

As the simulated option payoffs may scatter all over the real number axis, for example when European call option has a particularly low strike price, the variance of the simulated option payoffs can be so large that the accuracy of the valuation is compromised. In order to prevent this from happening, we adopt many different methods to reduce the variance while keeping the estimate unbiased.

4.1 Control Variate

Control Variate is one of the most common methods adopted for the purpose of variance reduction in simulations. Assuming all the Y_i are realizations of the same variable Y, i.e. $\forall i, Y_i$ follows identical and independent distributions. Our aim is to estimate $\mathrm{E}[Y_i]$

Under simulation, we use $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ as an estimate of $\mathrm{E}[Y_i]$. Now we introduce a new variable X, with realizations $X_i, \forall i$ such that $1 \leq i \leq n$

Define the new simulated results $Y_i(\lambda) = Y_i - \lambda(X_i - \mathrm{E}(X))$

$$\bar{Y}(\lambda) = \bar{Y} - \lambda(\bar{X} - \mathrm{E}(X)) = \frac{1}{n}\sum_{i=1}^{n}[Y_i - \lambda(X_i - \mathrm{E}(X))]$$

The new estimate $\bar{Y}(\lambda)$ is unbiased and consistent as proven below.

$$\mathrm{E}(\bar{Y}(\lambda)) = \mathrm{E}[\bar{Y} - \lambda(\bar{X} - \mathrm{E}(X))] = \mathrm{E}(\bar{Y}) - \lambda(\mathrm{E}(\bar{X}) - \mathrm{E}(X)) = \mathrm{E}(Y)$$

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} Y_i(\lambda) &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}[Y_i - \lambda(X_i - \mathrm{E}(X))] \\ &= \mathrm{E}[Y - \lambda(X - \mathrm{E}(X))] \\ &= \mathrm{E}(Y)\end{aligned}$$

We can first express the variance of each new simulated result as:

$$\begin{aligned}\mathrm{Var}[Y_i(\lambda)] &= \mathrm{Var}[Y_i - \lambda(X_i - \mathrm{E}(X))] \\ &= \mathrm{Var}[Y_i - \lambda X_i] \\ &= \mathrm{Var}(Y_i) + \lambda^2\mathrm{Var}(X_i) - 2\lambda\mathrm{Cov}(X_i, Y_i) \\ &= \sigma_Y^2 + \lambda^2\sigma_X^2 - 2\lambda\sigma_X\sigma_Y\rho_{XY}\end{aligned}$$

In order to find the minimum variance by varying λ

Set $\frac{\partial \mathrm{Var}[Y_i(\lambda)]}{\partial \lambda} = 2\lambda\sigma_X^2 - 2\sigma_X\sigma_Y\rho_{XY}$ to 0, $\lambda^* = \frac{2\sigma_X\sigma_Y\rho_{XY}}{2\sigma_X^2} = \frac{\sigma_X\sigma_Y\rho_{XY}}{\sigma_X^2} = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}$

Compare the new variance with the old:

$$\begin{aligned}\frac{\mathrm{Var}[Y_i - \lambda^*(X_i - \mathrm{E}(X))]}{\mathrm{Var}(Y)} &= \frac{\sigma_Y^2 + \lambda^{*2}\sigma_X^2 - 2\lambda^*\sigma_X\sigma_Y\rho_{XY}}{\sigma_Y^2} \\ &= 1 + \frac{\frac{\mathrm{Var}(X)(\mathrm{Cov}(X,Y))^2}{(\mathrm{Var}(X))^2} - \frac{2(\mathrm{Cov}(X,Y))^2}{\mathrm{Var}(X)}}{\sigma_Y^2} \\ &= 1 + \frac{\frac{\sigma_X^4\sigma_Y^2\rho_{XY}^2}{\sigma_X^4} - \frac{2\sigma_X^2\sigma_Y^2\rho_{XY}^2}{\sigma_X^2}}{\sigma_Y^2} \\ &= 1 + \frac{\sigma_Y^2\rho_{XY}^2 - 2\sigma_Y^2\rho_{XY}^2}{\sigma_Y^2} \\ &= 1 - \rho_{XY}^2\end{aligned}$$

A $100(1-\alpha)\%$ Confidence Interval is $[\bar{Y}(\lambda) - \mathcal{Z}_{1-\alpha/2}\frac{\hat{\sigma}_{n,Y_\lambda}}{\sqrt{n}}, \bar{Y}(\lambda) + \mathcal{Z}_{1-\alpha/2}\frac{\hat{\sigma}_{n,Y_\lambda}}{\sqrt{n}}]$

As a conclusion from the theoretical results, we shall see that the stronger the correlation, the better the reduction in variance. We can use random variables with stronger correlation with the option payoffs as the control variates, such as underlying asset prices, tractable option prices, bond prices.
In the case that it is not feasible to calculate using the probability distribution of X and Y, we should work λ^* out as an estimate using a pilot simulation.

Data: Scenarios for simulations of X and Y
Result: Estimation for $\mathrm{E}(Y)$
initialization;
assign N to be the number of simulations to do;
pilot simulation to obtain correlation;
for *i = 1 ... N* **do**
    $generate(X_i, Y_i)$;
end
Assign $\lambda^* = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)} = \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sum_{i=1}^{n}(X_i-\bar{X})^2}$;
for *i = 1 ... N* **do**
    $generate(X_i, Y_i)$;
    set $Y_i(\lambda) = Y_i + \lambda^*(X_i - \mathrm{E}[X])$;
end
$\bar{Y}(\lambda) = \frac{1}{n}\sum_{i=1}^{n} Y_i(\lambda)$;

Algorithm 1: General Control Variate Algorithm

Here we put down the algorithm for Monte Carlo simulations of option prices using underlying prices as control variates. Denote the underlying prices by U_i, payoff function by f, option payoffs by P_i and the option premium by $\bar{P}(\lambda)$.

Data: Scenarios for simulations of U and P
Result: Estimation for $\mathrm{E}(P)$
initialization;
assign N to be the number of simulations to do;
for *i = 1 ... N* **do**
    $generate(U_i)$;
    $P_i = f(U_i)$;
end
Assign $\lambda^* = \frac{\mathrm{Cov}(U,P)}{\mathrm{Var}(U)} = \frac{\sum_{i=1}^{n}(U_i-\bar{U})(P_i-\bar{P})}{\sum_{i=1}^{n}(U_i-\bar{U})^2}$;
for *i = 1 ... N* **do**
    $generate(U_i)$;
    $P_i = f(U_i)$;
    set $P_i(\lambda) = P_i + \lambda^*(U_i - \mathrm{E}[U])$;
end
$\bar{P}(\lambda) = e^{-rT}\frac{1}{n}\sum_{i=1}^{n} P_i(\lambda)$;

Algorithm 2: General Control Variate Algorithm

4.2 Stratified Sampling

Stratified sampling refers broadly to any sampling mechanism that constrains the fraction of observations drawn from specific subsets (or strata) of the sample space. Our goal is to estimate $\mathrm{E}[Y]$, by dividing the sample space into n parts, with $A_1, \dots, A_n$ being disjoint subsets of the real line for which $\mathrm{P}(Y \in \cup_i A_i) = 1$.
By Bayes$'$ Theorem, $\mathrm{E}[Y] = \sum_{i=1}^{n} \mathrm{P}(Y \in A_i)\mathrm{E}[Y|Y \in A_i] = \sum_{i=1}^{n} p_i \mathrm{E}[Y|Y \in A_i]$
We shall exercise proportional sampling which is the simplest case, ensuring $p_i = \mathrm{P}(Y \in A_i)$, the fraction of observations drawn from stratum A_i exactly matches theoretical probability.
Unbiased estimator of $\mathrm{E}(Y)$ is given by $\hat{Y} = \sum_{i=1}^{K}(p_i \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}) = \frac{1}{n} \sum_{i=1}^{K} \sum_{j=1}^{n_i} Y_{ij}$

$$\mathrm{E}[\hat{Y}] = \mathrm{E}[\sum_{i=1}^{K}(p_i \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij})] = \sum_{i=1}^{K} \mathrm{P}(Y \in A_i)\mathrm{E}[Y|Y \in A_i] = \mathrm{E}[Y]$$

We shall generate a stratification variable X which take values in the union of the disjoint sets A_is. As supposedly the values of X are high correlated to the values of Y, in many cases, Y is a function of X. Let X be the discrete path of underlying asset prices, with Y being the European call payoff discounted to time 0. If X has h discrete prices on one path, $\Omega \in \mathbb{R}^h$ is the sample space for the X_i's, where $\Omega = \cup_i A_i$ is the union of disjoint sets.
For the purpose of stratified sampling, $\mathbb{P}(X \in A_i)$ and $Y|X \in A_i$ should be easy to generate.
In order to further simplify the trouble of dividing Ω into disjoint subsets, we are going to use the stock prices at time $\frac{T}{2}$ as the condition to determine the distributions of A_i. The stock prices at time $\frac{T}{2}$ will be computed based on the Geometric Brownian Motion of the underlying stock prices. Given that we are to produce n equally probable subsets of Ω, the standard normal distribution can be sliced into n pieces to provide sources of generation. With the underlying asset prices at time $\frac{T}{2}$, we can generate k prices at time T and calculate the respective option payoffs. The average payoffs for each stratum A_i is then computed and discounted back to time $\frac{T}{2}$, at which we calculate the average payoffs across the stratum and discount to time 0.

Data: N the number of total simulations and f the payoff function
Result: $\bar{P}$, Estimation for the option premium
initialization;
$k = \frac{N}{n}$ is the number of simulations for each stratum;
for *i = 1 . . . n* **do**
  z_{α_i} is the quantile of standard normal distribution at probability $\frac{i}{n}$;
  $S_{\frac{T}{2},i} = S_0 \exp\{(r - \frac{1}{2}\sigma^2)\frac{T}{2} + \sigma\sqrt{\frac{T}{2}} z_{\alpha_i}\}$;
  for *j = 1 . . . k* **do**
    $S_{T,i,j} = S_{\frac{T}{2},i} \exp\{(r - \frac{1}{2}\sigma^2)\frac{T}{2} + \sigma\sqrt{\frac{T}{2}} z_{i,j}\}$;
    $P_{T,i,j} = f(S_{T,i,j})$;
  end
  $\bar{P}_{\frac{T}{2},i} = \frac{1}{k} e^{-r\frac{T}{2}} \sum_{j=1}^{k} P_{T,i,j}$;
end
$\bar{P} = \frac{1}{n} \sum_{i=1}^{n} \bar{P}_{\frac{T}{2},i}$;

Algorithm 3: General Proportional Sampling Algorithm

4.3 Importance Sampling

Importance sampling is a method to reduce variance by changing the probability measure. The word 'Importance' comes from the aim to give more weight to 'important' outcomes by changing measures.

We are going to estimate $\alpha = \mathrm{E}[h(X)] = \int h(x)f(x)dx$.

In Monte Carlo simulation, $\hat{\alpha} = \hat{\alpha}(n) = \frac{1}{n}\sum_{i=1}^{n} h(X_i)$

By using a change of measure based on the following assumption: $\forall x \in \mathbb{R}^d, f(x) > 0 \implies g(x) > 0$

Now the estimate becomes $\alpha = \mathrm{E}[h(X)] = \mathrm{E}[h(X)\frac{f(X)}{g(X)}] = \int h(x)\frac{f(x)}{g(x)}dx$

And for simulation, $\hat{\alpha}_G = \frac{1}{n}\sum_{i=1}^{n} h(X_i)\frac{f(X_i)}{g(X_i)}$ We are choosing g to make $X \in A$ more likely after changing measure.

In order to reduce variance, we need to implicitly find the g which gives the minimum possible variance. According to the variance formula $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$, we have

$$\begin{aligned}
\mathrm{Var}^G[h(X)\frac{f(X)}{g(X)}] &= \mathrm{E}^G[h(X)^2\frac{f(X)^2}{g(X)^2}] - (\mathrm{E}^G[h(X)\frac{f(X)}{g(X)}])^2 \\
&= \int \frac{h(x)^2 f(x)^2}{g(x)}dx - (\int \frac{h(x)f(x)}{g(x)}g(x)dx)^2
\end{aligned}$$

Here if $h(x) = \frac{g(x)f(x)}{\mathrm{E}(g(x))}$, the variance of estimate will become 0.

Consider the scenario in which we simulate discrete price paths $S(t_i), \forall i = 0, 1, \ldots, m$, which is assumed to be a Markov Chain with homogeneous property. Let the continuous transition probability be $f_i(S(t_{i-1}), S(t_i))$.

We shall use likelihood ratio $\prod_{i=1}^{m} \frac{f_i(S(t_{i-1}),S(t_i))}{g_i(S(t_{i-1}),S(t_i))}$ as risk-neutral measure.

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}(X) &= \int xg(x)dx \\
&= \mathbb{E}^{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}}X) \\
&= \mathbb{E}^{\mathbb{P}}(e^{\lambda W_T - \frac{1}{2}\lambda^2 T}X)
\end{aligned}$$

Our aim is to make (where $A > 0$)

$$\mathbb{E}^{\mathbb{P}}((S_T - K)^+) = \mathbb{E}^{\mathbb{Q}}(\frac{d\mathbb{P}}{d\mathbb{Q}}(S_T - K)^+) = \mathbb{E}^{\mathbb{Q}}(e^{((\mu - \frac{\sigma^2}{2} + \sigma A)T + \sigma W_T^{\mathbb{Q}}})(S_T - K)^+)$$

Hence $\frac{d\mathbb{Q}}{d\mathbb{P}} = exp\{AW_T - \frac{1}{2}A^2 T\}$

5 European call options

5.1 Closed-form formula

We can derive the closed-form formula for the European call options using risk-neutral pricing.
Payoff function of the European call option is : $c_T = (S_T - K)^+$
By the risk-neutral pricing formula,

$$\begin{aligned} e^{-\mu t} c_t &= \mathbb{E}^Q[e^{-\mu T} c_T | \mathcal{F}_t] \\ &= \mathbb{E}_t^Q[e^{-\mu T}(S_T - K)^+] \end{aligned} \tag{3}$$

By the lognormal property, $S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi}$

$$\begin{aligned} c_t &= e^{\mu t}\mathbb{E}_t^Q[e^{-rT}(S_T - K)^+] \\ &= e^{\mu t}\mathbb{E}_t^Q[e^{-rT}(S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} - K)^+] \\ &= e^{-\mu\tau}\mathbb{E}^Q[(S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} - K)^+] \\ &= e^{-\mu\tau}\int_{-\infty}^{\infty}(S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} - K)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= e^{-\mu\tau}\int_{-d_2}^{\infty}(S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} - K) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= e^{-\mu\tau}\int_{-d_2}^{\infty} S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi - e^{-r\tau}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} K\, d\phi \\ &= e^{(\mu - \frac{1}{2}\sigma^2)\tau - \mu\tau} S_t \int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi^2 - 2\sigma\sqrt{\tau}\phi)}\, d\phi - K e^{-r\tau}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= S_t\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi^2 - 2\sigma\sqrt{\tau}\phi + \sigma^2\tau)}\, d\phi - K e^{-r\tau}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= S_t\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi - \sigma\sqrt{\tau}\phi)^2}\, d\phi - K e^{-r\tau}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= S_t\int_{-d_2 - \sigma\sqrt{\tau}}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy - K e^{-r\tau}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= S_t\int_{-\infty}^{d_2 + \sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy - K e^{-r\tau}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= S_t N(d_1) - K e^{-r\tau} N(d_2) \end{aligned} \tag{4}$$

where,

$$d_2 = \frac{\ln\frac{S_t}{K} + (\mu - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_1 = d_2 + \sigma\sqrt{\tau}$$

Also, N is the cumulative density function of standard normal function.

5.2 Numerical PDE

5.3 Monte Carlo Simulation

With the following settings: $\sigma = 0.25$, $\mu = 0.05$, $T = 1$, $S_0 = 100$, when we compare the results from Monte Carlo simulation with the option prices computed using the closed form formula, it is observable that a simulation with 500000 data points still has variation of error within range from -0.02 to 0.02. However, this comparison has ensured that our valuation scheme using Monte Carlo simulation and the variance reduction techniques is on the right way, and may be applied onto other options.
A tabulation of the resultant option prices are shown below:

Closed-form formula	Ordinary Monte Carlo	Control Variate
10.0022021172	10.005252032814727	10.005252032814751
8.02638469385	8.0166707887876427	8.016670788787664
6.37924904693	6.3659464432937911	6.3659464432937733
5.02541348179	5.0148870431746415	5.0148870431746184
3.92690420603	3.9241698581683577	3.9241698581683728
3.04592058431	3.044679765401427	3.0446797654014213
2.34679877596	2.3310307686205207	2.3310307686205225
1.79723400902	1.8055531375480696	1.8055531375480751
1.36889248498	1.3630119824610754	1.3630119824610734
1.03756650489	1.0254470920297398	1.0254470920297361
0.783018613011	0.77819086371547286	0.77819086371547308
0.588637155719	0.5924848566970754	0.59248485669707973
0.440997057228	0.44342591182216823	0.4434259118221664
0.329392108384	0.32449718396190719	0.32449718396190735
0.245381782063	0.2462392632801686	0.24623926328016907
0.182377553986	0.17995020687354496	0.1799502068735449
0.13528073067	0.13478417666883458	0.13478417666883413
0.100175092579	0.099449778570450814	0.099449778570450523
0.0740722950356	0.074452813719303179	0.074452813719303304
0.0547050187389	0.051631534936688268	0.051631534936688074

6 European put options

6.1 Closed-form formula

We can derive the closed-form formula for the European put options using risk-neutral pricing.
Payoff function of the European put option is : $p_T = (K - S_T)^+$
By the risk-neutral pricing formula,

$$\begin{aligned} e^{-\mu t} p_t &= \mathbb{E}^Q[e^{-\mu T} p_T | \mathcal{F}_t] \\ &= \mathbb{E}_t^Q[e^{-\mu T}(K - S_T)^+] \end{aligned} \tag{5}$$

By the lognormal property, $\quad S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi}$

$$\begin{aligned} p_t &= e^{\mu t}\mathbb{E}_t^Q[e^{-rT}(K - S_T)^+] \\ &= e^{\mu t}\mathbb{E}_t^Q[e^{-rT}(K - S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi})^+] \\ &= e^{-\mu\tau}\mathbb{E}^Q[(K - S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi})^+] \\ &= e^{-\mu\tau}\int_{-\infty}^{\infty}(K - S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi})^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= e^{-\mu\tau}\int_{-\infty}^{-d_2}(K - S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi}) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= e^{-r\tau}\int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} K\, d\phi - e^{-\mu\tau}\int_{-\infty}^{-d_2} S_t e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\phi} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi \\ &= K e^{-r\tau}\int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi - e^{(\mu - \frac{1}{2}\sigma^2)\tau - \mu\tau} S_t \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi^2 - 2\sigma\sqrt{\tau}\phi)}\, d\phi \\ &= K e^{-r\tau}\int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi - S_t \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi^2 - 2\sigma\sqrt{\tau}\phi + \sigma^2\tau)}\, d\phi \\ &= K e^{-r\tau}\int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi - S_t \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\phi - \sigma\sqrt{\tau}\phi)^2}\, d\phi \\ &= K e^{-r\tau}\int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi - S_t \int_{-\infty}^{-d_2 - \sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy \\ &= K e^{-r\tau}\int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}\, d\phi - S_t \int_{-\infty}^{-d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy \\ &= K e^{-r\tau} N(-d_2) - S_t N(-d_1) \end{aligned} \tag{6}$$

where,

$$d_2 = \frac{\ln \frac{S_t}{K} + (\mu - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_1 = d_2 + \sigma\sqrt{\tau}$$

Also, N is the cumulative density function of standard normal function.

6.2 Numerical PDE

6.3 Monte Carlo Simulation

With the following settings: $\sigma = 0.25$, $\mu = 0.05$, $T = 1$, $S_0 = 100$, when we compare the results from Monte Carlo simulation with the option prices computed using the closed form formula, it is observable that a simulation with 500000 data points still has variation of error within range from -0.02 to 0.02. However, this comparison has ensured that our valuation scheme using Monte Carlo simulation and the variance reduction techniques is on the right way, and may be applied onto other options.
A tabulation of the resultant option prices are shown below:

Closed-form formula	Ordinary Monte Carlo	Control Variate
9.88129168973	9.89121100448	9.89121100448
12.6616213889	12.6718073072	12.6718073072
15.7706328645	15.7786013449	15.7786013449
19.1729444219	19.1488969419	19.1488969419
22.8305822686	22.8396769379	22.8396769379
26.7057457694	26.7349375768	26.7349375768
30.7627710836	30.7487677853	30.7487677853
34.9693534391	34.9407041949	34.9407041949
39.2971590376	39.3339188168	39.3339188168
43.72198018	43.6672929866	43.6672929866
48.2235794106	48.2561306956	48.2561306956
52.7853450758	52.7364667849	52.7364667849
57.3938520998	57.3635053884	57.3635053884
62.0383942735	62.0538619165	62.0538619165
66.7105310697	66.7099933687	66.7099933687
71.4036739641	71.3460094069	71.3460094069
76.1127242633	76.1619698093	76.1619698093
80.8337657477	80.8390739688	80.8390739688
85.5638100727	85.5359768969	85.5359768969
90.3005899189	90.2678714265	90.2678714265

7 Barrier option

7.1 Joint distribution of Minimum and Terminal Value

We define the minimum value $m_T = \min\limits_{0 \le t \le T} \ln \dfrac{S_t}{S_0}$ and the terminal value $x_T = \ln \frac{S_T}{S_0}$.

Corollary 1.

$$\mathbb{P}(X_T \ge x, m_T \ge m) = N(\frac{\mu T - x}{\sigma\sqrt{T}}) - e^{\frac{2\mu m}{\sigma^2}} N(\frac{2m - x + \mu T}{\sigma\sqrt{T}})$$

Corollary 2.

$$\mathbb{P}(m_T \ge m) = N(\frac{\mu T - m}{\sigma\sqrt{T}}) - e^{\frac{2\mu m}{\sigma^2}} N(\frac{m + \mu T}{\sigma\sqrt{T}})$$

Details of the proofs for the two corollaries are omitted here.

7.2 Closed-form formula

The payoff function of a down-and-out call option can express as $(S_T - K)\mathbb{1}_{\mathbb{F}}$, where $\mathbb{1}_{\mathbb{F}}$ is the indicator variable for the payoff to be non-zero, as $\mathbb{F} = \{S_T \geq K, \min_{0\leq t\leq T} S_t \geq B\}$

For risk-neutral valuation, we split the payoff in order to apply the Change of Numeraire Theorem where appropriate. Let $V_T^{(1)} = S_T\mathbb{1}_{\mathbb{F}}$, $V_T^{(2)} = K\mathbb{1}_{\mathbb{F}}$, with the assumption that there is no dividend yield, we can express the option price as such:

$$\begin{aligned}
c_{do}(S, B, K) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T - K)\mathbb{1}_{\mathbb{F}}] \\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}(S_T\mathbb{1}_{\mathbb{F}}) - e^{-rT}\mathbb{E}^{\mathbb{Q}}(K\mathbb{1}_{\mathbb{F}}) \\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}(V_T^{(1)}) - e^{-rT}\mathbb{E}^{\mathbb{Q}}(V_T^{(2)}) \\
&= \mathbb{E}^{\mathbb{Q}}[\frac{V_T^{(1)}}{M_T}] - \mathbb{E}^{\mathbb{Q}}[\frac{V_T^{(2)}}{M_T}] = \frac{V_0^{(1)}}{M_0} - \frac{V_0^{(2)}}{M_0} = V_0^{(1)} - V_0^{(2)}
\end{aligned}$$

We can now use the stock measure $\mathbb{Q}^S$ on $V_0^{(1)}$ and the risk-neutral measure $\mathbb{Q}$ on $V_0^{(2)}$.
By the Change of Numeraire Theorem, $\frac{V_0^{(1)}}{S_0} = \mathbb{E}^{\mathbb{Q}^S}[\frac{V_T^{(1)}}{S_T}]$, $\frac{V_0^{(2)}}{M_0} = \mathbb{E}^{\mathbb{Q}^S}[\frac{V_T^{(2)}}{M_T}]$

$$V_0^{(1)} = S_0\mathbb{E}^{\mathbb{Q}^S}[\frac{V_T^{(1)}}{S_T}] = S_0\mathbb{E}^{\mathbb{Q}^S}[\frac{S_T\mathbb{1}_{\mathbb{F}}}{S_T}] = S_0\mathbb{E}^{\mathbb{Q}^S}(\mathbb{1}_{\mathbb{F}}) = S_0\mathbb{Q}^S(\mathbb{F})$$

$$V_0^{(2)} = M_0\mathbb{E}^{\mathbb{Q}}[\frac{V_T^{(2)}}{M_T}] = M_0\mathbb{E}^{\mathbb{Q}}[\frac{K\mathbb{1}_{\mathbb{F}}}{M_T}] = \frac{K}{M_T}\mathbb{E}^{\mathbb{Q}}(\mathbb{1}_{\mathbb{F}}) = e^{-rT}K\mathbb{Q}(\mathbb{F})$$

For the case that $K > B$,

$$\begin{aligned}
\mathbb{F} &= \{S_T \geq K, \min_{0\leq t\leq T} S_t \geq B\} = \{\ln S_T \geq \ln K, \min_{0\leq t\leq T} \ln S_t \geq lnB\} \\
&= \{\ln S_T - \ln S_0 \geq \ln K - \ln S_0, \min_{0\leq t\leq T} \ln S_t - \ln S_0 \geq \ln B - \ln S_0\} \\
&= \{\ln \frac{S_T}{S_0} \geq \ln \frac{K}{S_0}, \min_{0\leq t\leq T} \ln \frac{S_t}{S_0} \geq \ln \frac{B}{S_0}\}
\end{aligned}$$

We can now let $x_T = \ln \frac{S_T}{S_0}$, $m_T = \min_{0\leq t\leq T} \ln \frac{S_t}{S_0}$, $x = \ln(\frac{K}{S_0})$, $m = \ln(\frac{B}{S_0})$
and apply Corollary 1 to solve for $\mathbb{Q}^S(\mathbb{F})$ with $\mu = r + \frac{\sigma^2}{2}$.

$$\begin{aligned}
V_0^{(1)} &= S_0\mathbb{Q}^S(\mathbb{F}) = S_0\mathbb{Q}^S(x_T \geq x, m_T \geq m) \\
&= S_0\{N(\frac{(r + \frac{\sigma^2}{2})T - \ln(\frac{K}{S_0})}{\sigma\sqrt{T}}) - e^{\frac{2(r+\frac{\sigma^2}{2})}{\sigma^2}\ln(\frac{B}{S_0})}N(\frac{2\ln(\frac{B}{S_0}) - \ln(\frac{K}{S_0}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}})\} \\
&= S_0\{N(\frac{(r + \frac{\sigma^2}{2})T + \ln(\frac{S_0}{K})}{\sigma\sqrt{T}}) - (\frac{B}{S_0})^{\frac{2r}{\sigma^2}+1}N(\frac{\ln(\frac{B^2}{S_0K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}})\} \\
&= S_0\{N(d_1) - (\frac{B}{S_0})^{\frac{2r}{\sigma^2}+1}N(d_2)\}
\end{aligned}$$

Similarly, we can express $V_0^{(2)}$ as $e^{-rT}K\mathbb{Q}(\mathbb{F})$, and apply Corollary 1 with to solve for $\mathbb{Q}(\mathbb{F})$ with $\mu = r - \frac{\sigma^2}{2}$.

$$\begin{aligned}
V_0^{(2)} &= e^{-rT}K\mathbb{Q}(\mathbb{F}) = e^{-rT}K\mathbb{Q}(x_T \geq x, m_T \geq m) \\
&= e^{-rT}K\{N(\frac{(r-\frac{\sigma^2}{2})T - \ln(\frac{K}{S_0})}{\sigma\sqrt{T}}) - e^{\frac{2(r-\frac{\sigma^2}{2})}{\sigma^2}\ln(\frac{B}{S_0})}N(\frac{2\ln\frac{B}{S_0} - \ln\frac{K}{S_0} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}})\} \\
&= e^{-rT}K\{N(\frac{(r-\frac{\sigma^2}{2})T + \ln(\frac{S_0}{K})}{\sigma\sqrt{T}}) - (\frac{B}{S_0})^{\frac{2r}{\sigma^2}-1}N(\frac{\ln(\frac{B^2}{S_0K}) + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}})\} \\
&= e^{-rT}K\{N(d_3) - (\frac{B}{S_0})^{\frac{2r}{\sigma^2}-1}N(d_4)\}
\end{aligned}$$

For the case that $K > B$, the situation is simplied as the condition $S_T \geq K$ is redundant. The new condition $\mathbb{F}' = \{\min_{0\leq t\leq T} S_t \geq B\}$, with the payoff function being $(S_T - K)\mathbb{1}_{\{\min_{0\leq t\leq T} S_t \geq B\}}$.

We shall apply Corollary 2 to solve for $\mathbb{Q}^S(\mathbb{F}')$ in this case with $\mu = r + \frac{\sigma^2}{2}$, $m_T = \min_{0\leq t\leq T} \ln\frac{S_t}{S_0}$.

$$\begin{aligned}
V_0^{(1)} &= S_0\mathbb{Q}^S(\mathbb{F}') = S_0\mathbb{Q}^S(m_T \geq m) \\
&= S_0\{N(\frac{(r+\frac{\sigma^2}{2})T - \ln(\frac{B}{S_0})}{\sigma\sqrt{T}}) - e^{\frac{2(r+\frac{\sigma^2}{2})}{\sigma^2}\ln(\frac{B}{S_0})}N(\frac{\ln\frac{B}{S_0} + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}})\} \\
&= S_0\{N(\frac{(r+\frac{\sigma^2}{2})T + \ln(\frac{S_0}{B})}{\sigma\sqrt{T}}) - (\frac{B}{S_0})^{\frac{2r}{\sigma^2}+1}N(\frac{\ln(\frac{B}{S_0}) + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}})\} \\
&= S_0\{N(d_5) - (\frac{B}{S_0})^{\frac{2r}{\sigma^2}+1}N(d_6)\}
\end{aligned}$$

We can also apply Corollary 2 to solve for $\mathbb{Q}(\mathbb{F}')$ in this case with $\mu = r - \frac{\sigma^2}{2}$.

$$\begin{aligned}
V_0^{(2)} &= e^{-rT}K\mathbb{Q}(\mathbb{F}') \\
&= e^{-rT}K\mathbb{Q}(m_T \geq m) \\
&= e^{-rT}K\{N(\frac{(r-\frac{\sigma^2}{2})T - \ln(\frac{B}{S_0})}{\sigma\sqrt{T}}) - e^{\frac{2(r-\frac{\sigma^2}{2})}{\sigma^2}\ln(\frac{B}{S_0})}N(\frac{2\ln\frac{B}{S_0} - \ln\frac{K}{S_0} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}})\} \\
&= e^{-rT}K\{N(\frac{(r-\frac{\sigma^2}{2})T + \ln(\frac{S_0}{B})}{\sigma\sqrt{T}}) - (\frac{B}{S_0})^{\frac{2r}{\sigma^2}-1}N(\frac{\ln(\frac{B}{S_0}) + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}})\} \\
&= e^{-rT}K\{N(d_7) - (\frac{B}{S_0})^{\frac{2r}{\sigma^2}-1}N(d_8)\}
\end{aligned}$$

Conclusively,

$$c_{do}(S_0, B, K) = \begin{cases} c(S_0, K) - (\frac{B}{S_0})^{\frac{2r}{\sigma^2}+1}c(\frac{B^2}{S_0}, K), \text{if } K > B \\ S_0\{N(d_5) - (\frac{B}{S_0})^{\frac{2r}{\sigma^2}+1}N(d_6)\} - e^{-rT}K\{N(d_7) - (\frac{B}{S_0})^{\frac{2r}{\sigma^2}-1}N(d_8)\}, \text{if } K \leq B \end{cases}$$

where $c(S_0, K, T)$ denotes the European call option premium with initial stock price S_0, strike price K.

7.3 Numerical PDE

7.4 Monte Carlo Simulation