Undergraduate Research Opportunity Programme in Science

Financial Mathematics With Python

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Contents

Abstract

In this paper, we implemented the library packages for Python to solve problems in Financial Mathematics such as derivatives valuation and simulation as suggested by the book "*Python for Finance*". Those results correspond to the different approaches in pricing financial derivatives. The key effort is on the development of valuation scheme for the options, in particular, the strongly path-dependent options. More importantly, our model is open to extend for various assumptions about the market. We attempt to apply these valuation schemes onto the path-independent options with available closed-form formulain order to verify the valuation results from Monte Carlo simulations as well as the Finite Difference Model.

1 Introduction

The study upon using Python to carry out derivatives valuation has made much progress over the years as there have already been available libraries built for this purpose. The derivatives analytics library suggested in the book "*Python for Finance*" has its advantages on the coverage upon the various aspects of possible analysis for financial derivatives. Still, we can make justifiable modifications and extensions to improve on the accuracy and speed of computations for estimations.

2 Derivation of Black Scholes PDE

2.1 Basics

In the Black-Scholes World, we assume that the following two SDE hold:

$$dM_t = rM_t dt$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

We assume that the following It$\hat{o}'$s Lemma hold:

As $dS_t = \mu S_t dt + \sigma S_t dW_t$,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t$$

2.2 Delta-hedging Argument

Our first aim is to find ϕ_t

for $\Pi_t = V_t - \phi_t S_t$

such that

$d\Pi_t = dV_t - \phi_t dS_t$(Self-financing)

$d\Pi_t = r\Pi_t dt$(risk free)

From the equations, we can obtain that

$r\Pi_t dt = dV_t - \phi_t dS_t$

$r(V_t - \phi_t S_t)dt = dV_t - \phi_t dS_t$

$dV_t = r(V_t - \phi_t S_t)dt + \phi_t dS_t$

By It$\hat{o}'$s Lemma,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t$$

Hence we obtain two equations:

$$\phi_t = \frac{\partial V}{\partial S}$$

$$r(V_t - \phi_t S_t) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S}$$

$$r(V_t - \frac{\partial V}{\partial S} S_t) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S_t - rV_t = 0$$

At time t, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S - rV = 0$$

2.3 Replicating portfolio

Our aim is to find a_t and b_t such that

$\Pi_t = a_t S_t + b_t M_t$ can entirely replicate V_t

And also, the self-financing condition holds:

$$d\Pi_t = a_t dS_t + b_t dM_t$$

As $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $dM_t = rM_t dt$,

$$\begin{aligned} d\Pi_t &= a_t(\mu S_t dt + \sigma S_t dW_t) + b_t(rM_t dt) \\ &= (a_t \mu S_t + rb_t M_t)dt + (\sigma a_t S_t)dW_t \end{aligned}$$

By It$\hat{o}'$s Lemma,

$$\begin{aligned} dV_t &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S} dS_t \\ &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S})dt + \frac{\partial V}{\partial S}(\mu S_t dt + \sigma S_t dW_t) \\ &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t)dt + (\sigma S_t \frac{\partial V}{\partial S})dW_t \end{aligned}$$

As Π_t fully replicates V_t,

$$d\Pi_t = (a_t \mu S_t + rb_t M_t)dt + (\sigma a_t S_t)dW_t = dV_t$$

Also by It$\hat{o}'$s Lemma,

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t)dt + (\sigma S_t \frac{\partial V}{\partial S})dW_t$$

Hence we obtain that,

$$a_t = \frac{\partial V}{\partial S}$$

$$a_t \mu S_t + rb_t M_t = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t$$

$$\begin{aligned} \frac{\partial V}{\partial S}\mu S_t + rb_t M_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t \\ rb_t M_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} \\ ra_t S_t + rb_t M_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + ra_t S_t \\ rV_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S_t \end{aligned}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S_t - rV_t = 0$$

Hence,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S} S - rV = 0$$

3 Finite Difference Model for Numerical PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S} + r\frac{\partial V}{\partial S}S - rV = 0$$

In the Black-Scholes formula, we are able to approximate $\frac{\partial V}{\partial t}$ and $\frac{\partial V}{\partial S}$ when the diagram has been splited into tiny parts

and thus it will be easy to simply use the ratio of change in derivatives value with respect to change in time or change in underlying asset value.

3.1 One factor model

3.2 Feynman–Kac formula

3.3 Crank–Nicolson method

4 Variance Reduction Techniques for Monte Carlo Simulation

4.1 Control Variate

4.2 Stratified Sampling

4.3 Importance Sampling

5 European call

5.1 Closed-form formula

5.2 Numerical PDE

5.3 Monte Carlo Simulation

6 European put

6.1 Closed-form formula

6.2 Numerical PDE

6.3 Monte Carlo Simulation

7 Amercian call

7.1 Closed-form formula

7.2 Numerical PDE

7.3 Monte Carlo Simulation

8 American put

8.1 Closed-form formula

8.2 Numerical PDE

8.3 Monte Carlo Simulation

9 Barrier option

9.1 Closed-form formula

9.2 Numerical PDE

9.3 Monte Carlo Simulation