SI152: Numerical Optimization

Lecture 2: Introduction to Linear Programming

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Outline

- Objective: linear
- Constraint: linear equalities or/and inequalities
- Variables: real value.

Example:

$$\begin{aligned} \min \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 3 \\ & 2x_1 + x_2 \geq 3 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b, \quad \text{with } c = \begin{bmatrix} 3 & 1 \end{bmatrix}, x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ & x \geq 0 \end{aligned}$$

Notice: vectors are all column vectors and inequalities are componentwise.

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & a_i^T x \geq b_i, \quad & i \in M_1 \\
& a_i^T x \leq b_i, \quad & i \in M_2 \\
& a_i^T x = b_i, \quad & i \in M_3 \\
& x_j \geq 0, \quad & j \in N_1 \\
& x_j \leq 0, \quad & j \in N_2
\end{aligned}
$$

- $c, a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$.
- This form could be simplified.

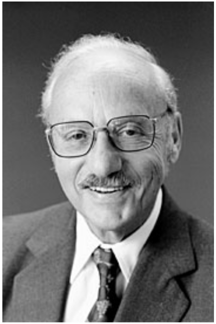
- The diet problem was one of the first optimization problems studied in the 1930s and 1940s.
- The search for diet solutions started with Jerry Cornfield, who formulated "The Diet Problem" for the Army during World War II (1941–1945), in search of a low-cost diet that would meet the nutritional needs of a soldier
- The economist George Stigler (1982 Nobel Prize for contribution in Economic Theory of Regulation), endeavored optimization techniques to establish the cheapest diet delivering enough energy, proteins, vitamins, and minerals. According to Buttriss et al., this diet should be composed by the available list of 77 US foods of which the costs and nutrient composition were measured

George Joseph Stigler (1911 – 1991), Chicago School of Economics

- Nutrients: These are the various dietary requirements (e.g., vitamins, minerals, proteins, fats, carbohydrates) that must be met. m different nutrients.
- Foods: These are the different food items available, each providing various amounts of the nutrients. n different food items.
- Matrix Representation:
  - a_{ij}: the amount of nutrient i found in food j.
  - c_j: the cost of one unit of food j .
  - b_i: be the minimum required amount of nutrient i.
- Variables: x_j, the amount of food j to be consumed.
- Goal: minimize the total cost while meeting all nutritional requirements.

$$
\begin{aligned}
\min \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m, \\
& x_1 \geq 0, \ldots, x_n \geq 0.
\end{aligned}
$$

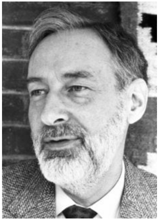
- For the duration of World War II, the Air Force and other parts of the army were hiring mathematicians to solve the important diet problem and to plan affordable meals. Among the researchers involved in solving this problem was George Dantzig.
- George Dantzig proposed a new algorithm he had developed. It took him until 1947, being the first to deliver the correct mathematical result. His work was inspired by earlier efforts in mathematical optimization, especially contributions by John von Neumann.
- RAND Corporation, a think tank involved in military research, played a significant role in advancing the use of linear programming and related optimization techniques, applying them to Cold War-era problems, such as the optimal deployment of military assets and defense logistics.



George Bernard Dantzig
(1914 - 2005)

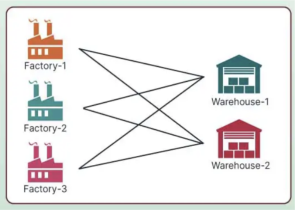
- In the Soviet Union, Leonid Kantorovich is considered the father of linear programming. In 1939, he independently developed a method for optimizing resource allocation in the context of industrial production. His research was deeply connected to the challenges of planned economies, where the allocation of resources was centrally managed.
- However, Kantorovich's work was initially met with skepticism in the Soviet Union.
- The post-World War II period saw greater interest in mathematical optimization in the USSR, especially as the Soviet Union faced complex problems of economic planning and industrial management.
- In 1975, Leonid Kantorovich and the American economist Tjalling Koopmans were awarded the Nobel Prize in Economic Sciences for their contributions to the theory of optimal allocation of resources.

Leonid V. Kantorovich (1912-1986)



Tjalling C. Koopmans (1910-1985)

- m suppliers (warehouse): each supplier i with a certain supply capacity a_i.
- n consumers (factory): each consumer j with a certain demand b_j
- The cost to transport a unit of the product from supplier i to consumer j is denoted by c_{ij}.
- Variables: x_{ij}, the quantity of the product from supplier i to consumer j



j i

$$\begin{aligned}
\min \quad & \sum_{ij} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = a_i, \ i = 1, ..., m \\
& \sum_{i=1}^{m} x_{ij} = b_j, \ j = 1, ..., n \\
& x_{ij} \geq 0, i = 1, ..., m; j = 1, ..., n
\end{aligned}$$

Goal: to find the sparse solution of a linear system

$$\min_x \ \|x\|_1, \qquad \text{s.t.} \quad Ax = b.$$

Recast:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} z_i \\ \text{s.t.} \quad & Ax = b \\ & x_i \le z_i, \ i = 1, ..., n \\ & -x_i \le z_i, i = 1, ..., n \end{aligned}$$

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} (x_i^+ + x_i^-) \\ \text{s.t.} \quad & Ax^+ - Ax^- = b \\ & x^+ \ge 0, x^- \ge 0 \end{aligned}$$

Given $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, m > n.$

- AVE (absolute value error) linear regression:

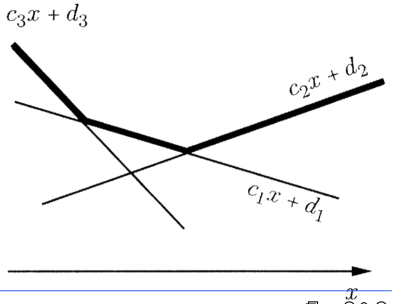
$$\min_x \|Ax - b\|_1.$$

- Robust linear regression:

$$\min_x \|Ax - b\|_\infty.$$

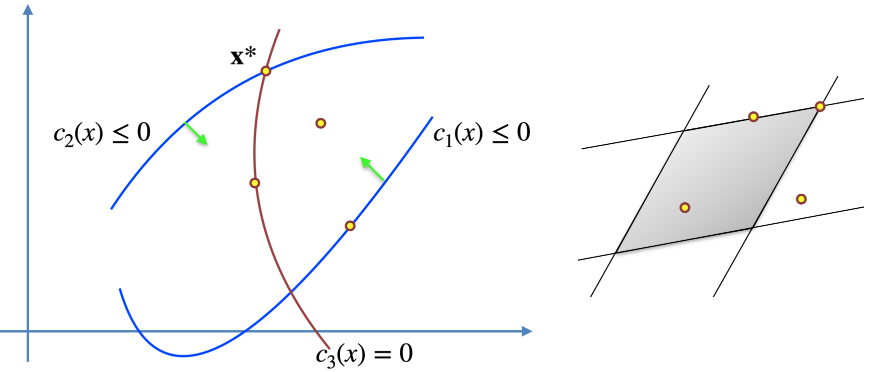
- Other equivalence:

$$\begin{aligned} \min \quad & \max_{i=1,\dots,m} (c_i^T x + d_i) \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$



Outline

At a given point x^*, constraints can be classified as active constraints, inactive constraints, violated constraints



Definition 1 (Polyhedron)

A polyhedron is a set that can be described in the form $\{x \in \mathbb{R}^n \mid Ax \geq b\}$, where $A \in \mathbb{R}^{m \times n}$ matrix and $b \in \mathbb{R}^n$.

Definition 2 (Bounded set)

A set $S \subset \mathbb{R}^n$ is bounded if there exists a constant K such that $|x| \leq K$ for all $x \in S$.

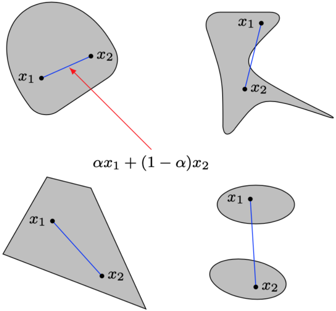
Definition 3

Let a be a nonzero vector in $\mathbb{R}^n$ and let b be a scaler.

- The set $\{x \in \mathbb{R}^n \mid a^T x = b\}$ is called a hyperplane.
- The set $\{x \in \mathbb{R}^n \mid a^T x \geq b\}$ is called a halfspace.

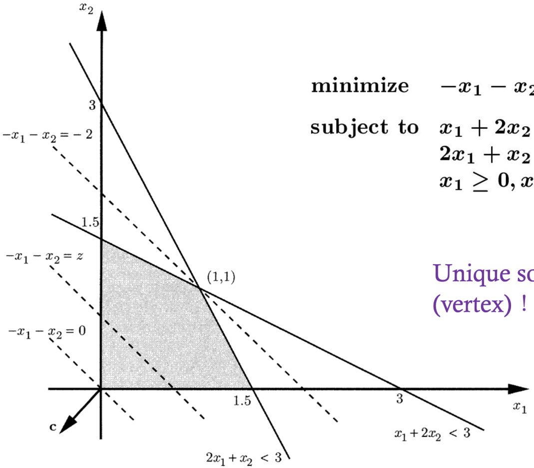
Definition 4

A set $S \subset \mathbb{R}^n$ is convex if for any $x, y \in S$, and any $\lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \in S$.



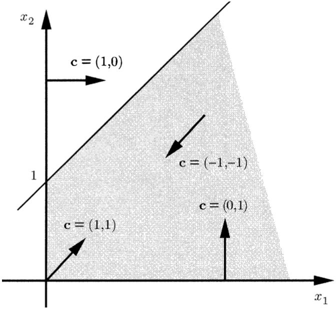
Sets that are convex (left) and not convex (right)

- The intersection of convex sets is still a convex set.



$$
\begin{array}{ll}
\text{minimize} & -x_1 - x_2 \\
\text{subject to} & x_1 + 2x_2 \leq 3 \\
 & 2x_1 + x_2 \leq 3 \\
 & x_1 \geq 0, x_2 \geq 0.
\end{array}
$$

Unique solution (vertex) !



$$\begin{array}{ll} \text{minimize} & c_1 x_1 + c_2 x_2 \\ \text{subject to} & -x_1 + x_2 \le 1 \\ & x_1 \ge 0, x_2 \ge 0 \end{array}$$

c	x_*	solution
$(1, 1)$	$(0, 0)$	unique vertex
$(0, 1)$	$(x_1, 0), x_1 \ge 0$	an edge
$(1, 0)$	$(0, x_2), x_2 \in [0, 1]$	an edge
$(-1, -1)$	unbounded solution set	unbounded (below)

Feasible region of LP: polyhedron in $\mathbb{R}^n$

- solution exists: unique (vertex solution) or infinitely many (an edge, facet)
- unbounded: unbounded optimal solution ($-\infty$ for minimization and $+\infty$ for maximization)
- infeasible: doesn't have optimal solution

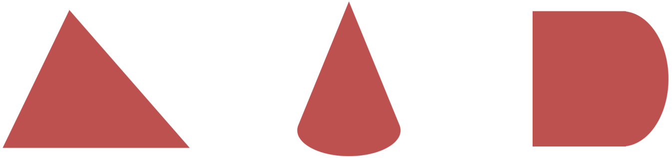
Geometric observation $\Longleftrightarrow$ Algebraic characterization

$\Downarrow$

Algorithm design & Analysis, proof

Definition 5 (Extreme points)

Let P be a convex set. A vector $x \in P$ is an extreme point of P if we cannot find two vectors $y, z \in P$, both different from x, and a scalar $\lambda \in [0, 1]$, such that $x = \lambda y + (1 - \lambda) z$.



Definition 6 (Vertex)

Let P be a polyhedron. A vector $x \in P$ is a vertex of P if there exists some c such that $c^T x < c^T y$ for all y satisfying $y \in P$ and $y \neq x$.

Definition 2.9 *Consider a polyhedron P defined by linear equality and inequality constraints, and let $\mathbf{x}^*$ be an element of $\Re^n$.*

(a) *The vector $\mathbf{x}^*$ is a* **basic solution** *if:*

- *(i) All equality constraints are active;*
- *(ii) Out of the constraints that are active at $\mathbf{x}^*$, there are n of them that are linearly independent.*

(b) *If $\mathbf{x}^*$ is a basic solution that satisfies all of the constraints, we say that it is a* **basic feasible solution.**

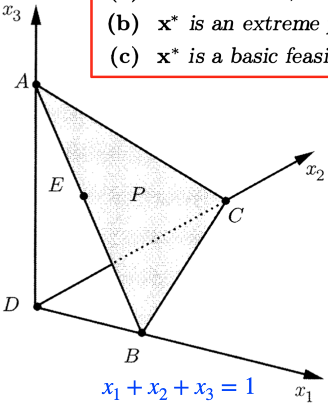
Vertex $\Longleftrightarrow$ Extreme points $\Longleftrightarrow$ Basic feasible solution

Theorem 2.3 *Let P be a nonempty polyhedron and let $\mathbf{x}^* \in P$. Then, the following are equivalent:*

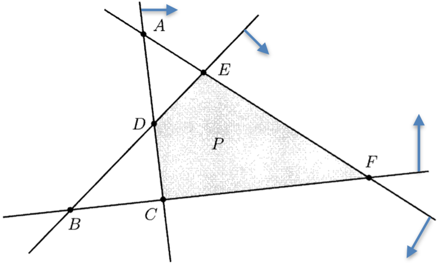
(a) *$\mathbf{x}^*$ is a vertex;*

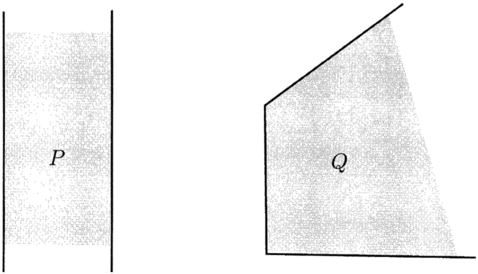
(b) *$\mathbf{x}^*$ is an extreme point;*

(c) *$\mathbf{x}^*$ is a basic feasible solution.*



$$x_1 + x_2 + x_3 = 1$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$





Theorem 2.6 *Suppose that the polyhedron* $P = \{\mathbf{x} \in \Re^n \mid \mathbf{a}_i'\mathbf{x} \geq b_i,\ i = 1, \ldots, m\}$ *is nonempty. Then, the following are equivalent:*

(a) *The polyhedron* P *has at least one extreme point.*

(b) *The polyhedron* P *does not contain a line.*

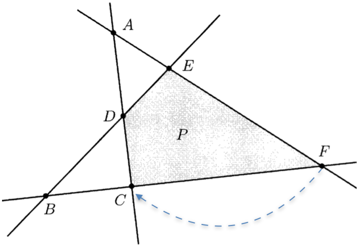
(c) *There exist* n *vectors out of the family* $\mathbf{a}_1, \ldots, \mathbf{a}_m$*, which are linearly independent.*

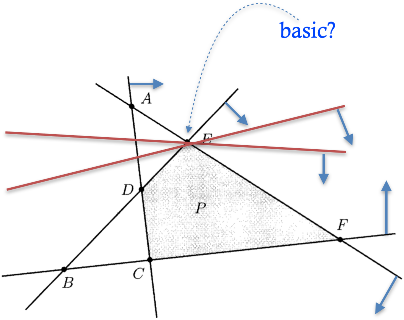
basis

- How many basic feasible solutions does a polyhedron in $\mathbb{R}^n$ have at most? Consider

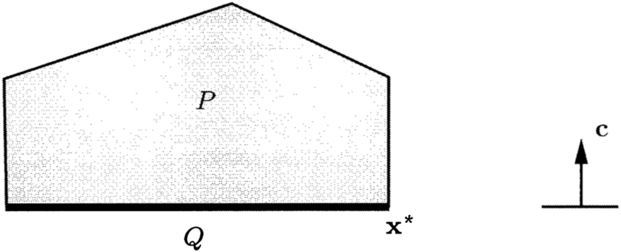
$$\{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1,\ i = 1, ..., n\}$$

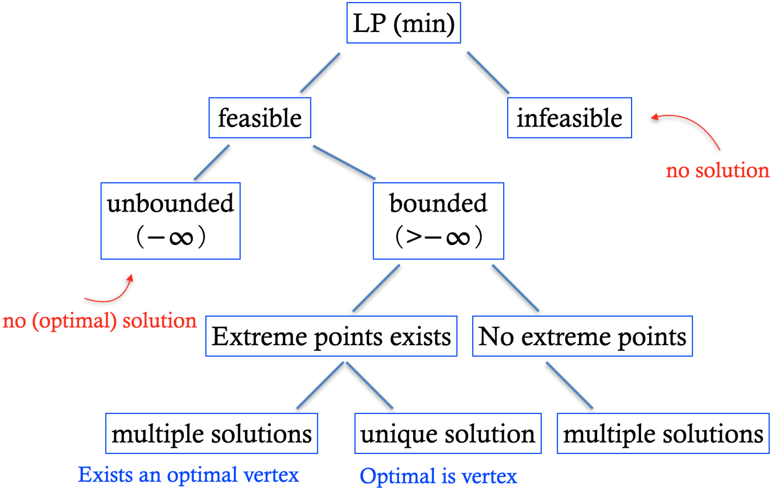
- Adjacent basic solutions: two basic solutions sharing $n-1$ basis.
- Degenerate basic feasible solutions: more than n constraints are active





Theorem 2.7 *Consider the linear programming problem of minimizing* $\mathbf{c}'\mathbf{x}$ *over a polyhedron* P. *Suppose that* P *has at least one extreme point and that there exists an optimal solution. Then, there exists an optimal solution which is an extreme point of* P.




LP (min)
feasible
infeasible
no solution
unbounded ($-\infty$)
bounded ($>-\infty$)
no (optimal) solution
Extreme points exists
No extreme points
multiple solutions
unique solution
multiple solutions
Exists an optimal vertex
Optimal is vertex

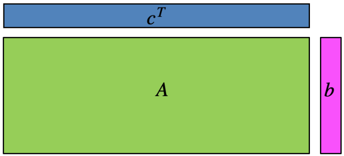
Outline

A form that is convenient for theoretical analysis and algorithm design.

$$\begin{aligned} \min \quad & c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ & \quad \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\ & x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \end{aligned}$$

Matrix-vector form:

$$\begin{aligned} \min \quad & c^Tx \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$



where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ are given.

- minimization, equality, nonnegativity

$$
\begin{array}{llll}
\min & c^T x & & \\
\text{s.t.} & a_i^T x \geq b_i, & i \in M_1 & \Longrightarrow \; s_i = a_i^T x - b_i \\
& a_i^T x \leq b_i, & i \in M_2 & \Longrightarrow \; s_i = b_i - a_i^T x \\
& a_i^T x = b_i, & i \in M_3 & \\
& x_j \geq 0, & j \in N_1 & \\
& x_j \leq 0, & j \in N_2 & \Longrightarrow \; w_i = -x_i \\
& x_j \text{ is free}, & j \in N_3 & \Longrightarrow \; x_i = u_i - v_i, u_i \geq 0, v_i \geq 0
\end{array}
$$

- s_i is called slack variable or surplus variable

BS, BFS, degenerate, adjacent BFS are convenient to characterize!

BFS of standard form: $Ax = b, x \geq 0$

Full rank assumption

$m < n$, and the rows are linearly independent.

- Now we can check the columns, let B be the matrix of m linearly independent columns, and the rest is N.
- The solution of $Bx_B = b, x_N = 0$ is a basic solution.
- If the solution of $Bx_B = b, x_N = 0$ is nonnegative, it is a basic feasible solution.
- B is called the basic matrix; N is the nonbasic matrix.
- The variables in x_B are the basic variables.
- The variables in x_N are the nonbasic variables.

Example

basic variables

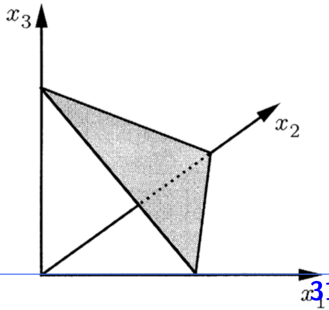
basis

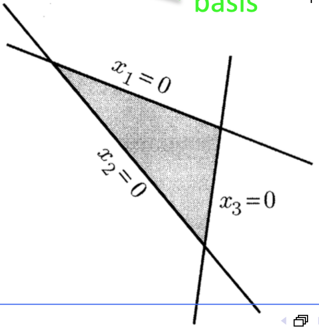
$$\begin{array}{ccccccc|c}
c_1 & c_2 & c_3 & c_4 & \cdots & \cdots & c_n & \\
x_1 & x_2 & x_3 & x_4 & \cdots & \cdots & x_n & \\
\end{array}$$

$$\left[\begin{array}{ccccccc|c}
a_{11} & a_{12} & a_{13} & a_{14} & \cdots & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & a_{23} & a_{24} & \cdots & \cdots & a_{2n} & b_2 \\
a_{31} & a_{32} & a_{33} & a_{34} & \cdots & \cdots & a_{3n} & b_3 \\
a_{41} & a_{42} & a_{43} & a_{44} & \cdots & \cdots & a_{4n} & b_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \\
a_{m1} & a_{m2} & a_{m3} & a_{m4} & \cdots & \cdots & a_{mn} & b_m
\end{array}\right]$$

basis

$$x_1 + x_2 + x_3 = 1,$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

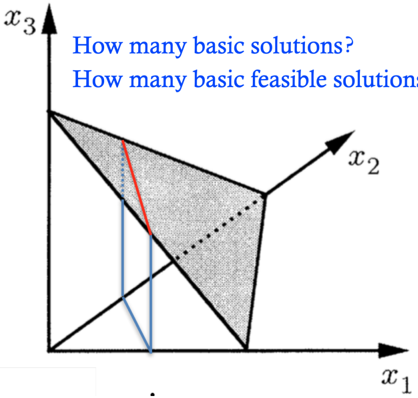




$$x_1 + x_2 + x_3 = 1$$
$$2x_1 + 3x_2 = 1$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$B = [a_1, a_3]$ and $B = [a_3, a_1]$

the same basis?



- There are at most $\binom{n}{m} = \frac{n!}{m!(n-m)!}$
- Is there a 1-to-1 correspondence between the basis and the BFS?
- Degenerate BFS: there exists $x_i = 0$ in x_B
- Is degeneracy purely a geometrical issue?

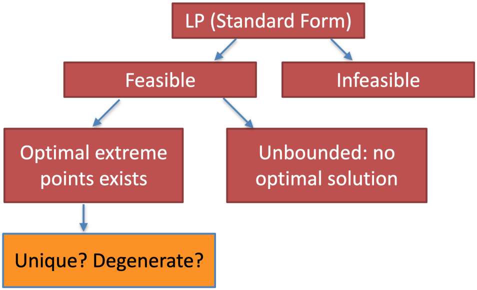
$$\{x \mid x_1 = x_2, x_2 \geq 0\} = \{x \mid x_1 = x_2, x_1 \geq 0, x_2 \geq 0\}$$

Theorem 7

Consider the standard form $\{x \mid Ax = b, x \geq 0\}$, $A \in \mathbb{R}^{m\times n}$, *rank*$(A) = m$. *If the problems is feasible, then there exists a basic feasible solution.*

Theorem 8

Consider the standard form $\{x \mid Ax = b, x \geq 0\}$, $A \in \mathbb{R}^{m\times n}$, *rank*$(A) = m$. *If the problems is feasible, then there exists an optimal basic feasible solution.*



1. Show there exits 1-1 correspondence between the extreme points of the two problems

$$S_1 = \{x \in \mathbb{R}^n : Ax \le b, x \ge 0\}$$
$$S_2 = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : Ax + y = b, x \ge 0, y \ge 0\}$$

2. Does $P = \{x \in \mathbb{R}^2 \mid \; 0 \le x_1 \le 1\}$ have extreme points? What is its standard form? Does it have extreme points? Find an extreme point if there exits one, and explain why.