Numerical Optimization — HW1

Deadline: October 21, 2025

1. Consider the standard-form linear program

$$\min\ c^\top x \quad \text{s.t.} \quad Ax = b,\ x \geq 0,$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. Let $B = [b_1, \ldots, b_m]$ be a basis consisting of m columns of A, and let p be an entering variable (index of a column chosen to enter the basis).

Assume that the column a_p of A can be represented as a linear combination of the basis columns, and all the coefficients in this representation are nonpositive.

Also assume that the right-hand-side vector b can be expressed as a linear combination of the basis columns, and all the coefficients in this representation are nonnegative.

Prove that the linear program is unbounded below.

Proof:

The current basic feasible solution is $x_B = B^{-1}b$, with $x_N = 0$.

$$x_B = B^{-1}b = \bar{b} \geq 0$$

The current objective value is $z_0 = c_B^\top x_B$.

Because p is the index of the entering non-basic variable, So the reduced cost $\bar{c}_p$ to be strictly negative:

$$\bar{c}_p = c_p - c_B^\top B^{-1} a_p < 0$$

The column a_p can be represented as a linear combination of the basis columns, $\mathbf{a}_p = B\bar{\mathbf{a}}_p$. All coefficients in this representation are non-positive:

$$\bar{\mathbf{a}}_p = B^{-1}a_p = \begin{pmatrix} \bar{a}_{1p} \\ \vdots \\ \bar{a}_{mp} \end{pmatrix} \leq 0$$

Let x^* be the current BFS: $x^* = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$, where $x_B = \bar{b}$ and $x_N = 0$.

We construct a feasible direction vector $d \in \mathbb{R}^n$ that, when followed from the current BFS x^*, remains feasible and reduces the objective function indefinitely.

Define the direction vector d as follows:

$$d_p = 1 \quad \text{(The entering non-basic variable index)}$$
$$d_j = 0, \quad \text{for all other non-basic indices } j \neq p$$
$$d_B = -B^{-1}a_p = -\bar{\mathbf{a}}_p \quad \text{(The components corresponding to basic variables)}$$

The vector d can be written as $d = \begin{pmatrix} d_B \\ d_N \end{pmatrix}$.

Then we want to prove $x(\theta) = x^* + \theta d$ is feasible

- $x(\theta) \geq 0$:
  - For non-basic indices $j \neq p$: $x_j(\theta) = x_j^* + \theta d_j = 0 + \theta \cdot 0 = 0 \geq 0$.
  - For index p: $x_p(\theta) = x_p^* + \theta d_p = 0 + \theta \cdot 1 = \theta \geq 0$.
  - For basic indices $i \in I_B$:

    $$x_i(\theta) = x_i^* + \theta d_i = \bar{b}_i + \theta(-\bar{a}_{ip})$$

    Since $\bar{b}_i \geq 0$ and, by assumption, $\bar{a}_{ip} \leq 0$, the term $d_i = -\bar{a}_{ip}$ is non-negative ($d_i \geq 0$). Thus, $x_i(\theta) = \bar{b}_i + \theta d_i \geq 0$ for all $\theta \geq 0$.
- $Ax(\theta) = b$:

  $$Ax(\theta) = A(x^* + \theta d) = Ax^* + \theta Ad$$

  Since x^* is a BFS, $Ax^* = b$. We must show $Ad = 0$:

  $$Ad = \sum_{j=1}^{n} a_j d_j = \sum_{i \in I_B} a_i d_i + a_p d_p$$

  $$Ad = Bd_B + a_p(1) = B(-\bar{\mathbf{a}}_p) + a_p = -B(B^{-1}a_p) + a_p = -a_p + a_p = 0$$

  Therefore, $Ax(\theta) = b + \theta \cdot 0 = b$.

So $x(\theta) = x^* + \theta d$ is feasible for all $\theta \geq 0$.

Then we want to preve the objective value associated with $x(\theta)$ is unbounded.

The objective value associated with $x(\theta)$ is:

$$z(\theta) = c^\top x(\theta) = c^\top(x^* + \theta d) = c^\top x^* + \theta c^\top d$$

The term $c^\top x^*$ is the current objective value z_0. The change in the objective value is determined by the term $c^\top d$:

$$c^\top d = \sum_{j=1}^{n} c_j d_j = \sum_{i \in I_B} c_i d_i + c_p d_p$$

$$c^\top d = c_B^\top d_B + c_p(1) = c_B^\top(-\bar{\mathbf{a}}_p) + c_p = c_p - c_B^\top \bar{\mathbf{a}}_p$$

This expression is exactly the reduced cost of the entering variable x_p:

$$c^\top d = \bar{c}_p$$

By the Simplex rule for entering variables, we chose p such that $\bar{c}_p < 0$.

Thus, $z(\theta) = z_0 + \theta \bar{c}_p$. Since $\bar{c}_p < 0$, as $\theta \to \infty$, the objective value $z(\theta) \to -\infty$.

Since a feasible ray $x(\theta)$ exists along which the objective function value decreases without bound, the linear program is **unbounded below**.

2. Write down the standard form of the following problem, and solve it using the simplex method (show the simplex table at each iteration).

$$\begin{aligned} \max \quad & 2x_1 + x_2 - x_3 \\ \text{s.t.} \quad & x_1 + x_2 + 2x_3 \leq 6 \\ & x_1 + 4x_2 - x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

The standard form is:

$$\begin{aligned} \max \quad & z = 2x_1 + x_2 - x_3 + 0s_1 + 0s_2 \\ \text{s.t.} \quad & x_1 + x_2 + 2x_3 + s_1 = 6 \\ & x_1 + 4x_2 - x_3 + s_2 = 4 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{aligned}$$

The initial basic feasible solution is $x_1 = x_2 = x_3 = 0$, $s_1 = 6$, $s_2 = 4$, with $z = 0$.

Initial Tableau (Iteration 0) The objective function (Row 0) is written as: $z - 2x_1 - x_2 + x_3 = 0$.

Basis	z	x_1	x_2	x_3	s_1	s_2	b	Ratio
z	1	$\mathbf{-2}$	-1	1	0	0	0	–
s_1	0	1	1	2	1	0	6	$6/1 = 6$
s_2	0	**1**	4	-1	0	1	4	$4/1 = \mathbf{4}$ ← Leaving

Pivot: Row s_2, Column x_1. s_2 leaves, x_1 enters.

Tableau 1 (Iteration 1)

Basis	z	x_1	x_2	x_3	s_1	s_2	b	Ratio
z	1	0	7	$\mathbf{-1}$	0	2	8	–
s_1	0	0	-3	**3**	1	-1	2	$2/3$ ← Leaving
x_1	0	1	4	-1	0	1	4	–

Pivot: Row s_1, Column x_3. s_1 leaves, x_3 enters.

Tableau 2 (Iteration 2)

Basis	z	x_1	x_2	x_3	s_1	s_2	b
z	1	0	6	0	$1/3$	$5/3$	$26/3$
x_3	0	0	-1	1	$1/3$	$-1/3$	$2/3$
x_1	0	1	3	0	$1/3$	$2/3$	$14/3$

Since all coefficients in the objective row (Row 0) are non-negative , the current solution is optimal.

The optimal basic feasible solution is:

- $x_1^* = 14/3$
- $x_3^* = 2/3$
- $x_2^* = 0$ (Non-basic)

The maximum objective value is $z^* = 26/3$.

3. Write down the equivalent linear programming formulation of the following problem. Given $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, m > n$

   (a) AVE (absolute value error) linear regression:

   $$\min_x \ ||Ax - b||_1$$

   (b) Robust linear regression:

   $$\min_x \ ||Ax - b||_\infty$$

   (c) Other equivalence:

   $$\begin{aligned} \min \quad & \max_{i=1,\dots,m} (c_i^T x + d_i) \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

   (a) Let $y = Ax - b$, and introduce $z \in \mathbb{R}^m$, $z \geq 0$.
   The equivalent LP is:

   $$\begin{aligned} \min_{x,z} \ & \sum_{i=1}^{m} z_i \\ \text{s.t.} \quad & (Ax - b)_i \leq z_i, \quad i = 1, \dots, m \\ & -(Ax - b)_i \leq z_i, \quad i = 1, \dots, m \\ & z_i \geq 0, \quad i = 1, \dots, m \end{aligned}$$

   (b) We introduce an auxiliary variable $t \in \mathbb{R}$. $t \geq |(Ax - b)_i|$ for all $i = 1, \dots, m$,
   The equivalent LP is:

   $$\begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & (Ax - b)_i \leq t, \quad i = 1, \dots, m \\ & -(Ax - b)_i \leq t, \quad i = 1, \dots, m \end{aligned}$$

   (c) We introduce an auxiliary variable $t \in \mathbb{R}$.
   The equivalent LP is:

   $$\begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & c_i^T x + d_i \leq t, \quad i = 1, \dots, m \\ & Ax \geq b \end{aligned}$$

4. (a) Show there exits 1-1 correspondence between the extreme points of the two problems.

$$S_1 = \{x \in \mathbb{R}^n : \; Ax \le b, x \ge 0\}$$
$$S_2 = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : \; Ax + y = b, x \ge 0, y \ge 0\}$$

(b) Does $P = \{x \in \mathbb{R}^2 \mid 0 \le x_1 \le 1\}$ have extreme points? What is its standard form? Does it have extreme points? Find an extreme point if there exits one, and explain why.

(a) For any $x \in S_1$, define $y = b - Ax$. Then $y \ge 0$ because $Ax \le b$. Hence $(x,y) \in S_2$. Conversely, for any $(x,y) \in S_2$, we have $Ax + y = b$ and $y \ge 0$, which implies $Ax \le b$, hence $x \in S_1$. Therefore, there is a one-to-one correspondence between $x \in S_1$ and $(x,y) \in S_2$ given by

$$x \longleftrightarrow (x, b - Ax).$$

Moreover, the extreme points correspond as well. Suppose x is an extreme point of S_1. If $(x, b - Ax) = \lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)$ with $(x_i, y_i) \in S_2$, then taking the x-components gives

$$x = \lambda x_1 + (1-\lambda)x_2.$$

Since x is extreme in S_1, we must have $x_1 = x_2 = x$, and then $y_1 = y_2 = b - Ax$. Thus $(x, b - Ax)$ is extreme in S_2. The converse follows symmetrically. Hence, there exists a one-to-one correspondence between the extreme points of S_1 and S_2 via $(x,y) = (x, b - Ax)$.

(b) P has no extreme point. Then we prove it:
Suppose it has an extreme point $a = (a_1, a_2)$, then $a_1 \in [0,1], a_2 \in \mathbb{R}$ Then we can definitely find two points $y_1 = (a_1, 0), y_2 = (a_1, 2a_2)$, and $\lambda = \frac{1}{2}$, because $a_2 \in \mathbb{R}$, so $2a_2 \in \mathbb{R}$ And

$$a = \tfrac{1}{2}(a_1, 0) + \tfrac{1}{2}(a_1, 2a_2),$$

which are two distinct feasible points, so x is not extreme.
Therefore, P has no extreme points.
The standard form of P is obtained by introducing a slack variable y_1:

$$P' = \{x' = [x_1, x_2, x_3, x_4] \in \mathbb{R}^4 \mid \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix} x' = \mathbf{1}, x' \ge 0\},$$

The corresponding feasible region P' has 2 extreme points: $(1,0,0,0)$ and $(0,1,0,0)$.
Because for point $x' = [x_1, 1 - x_1, x_3, x_4]$, we can always find two points $y_1' = [c, 1-c, x_3, x_4], y_2' = [2x_1 - c, 1 - 2x_1 + c, x_3, x_4]$ and $\lambda = \frac{1}{2}$,and because $x_1 \ge 0$, so $2x_1 \ge 0$

$$x_1' = \tfrac{1}{2}[c, 1-c, x_3, x_4] + \tfrac{1}{2}[2x_1 - c, 1 - 2x_1 + c, x_3, x_4],$$

Only when $x_1 = 0$ or $x_1 = 1$, $c = 2x_1 - c = x_1$, the point $x' = y_1' = y_2'$
Hence P' has 2 extreme points: $(1,0,0,0)$ and $(0,1,0,0)$.

5. Write down the dual problem of the following linear programming problem.

$$\begin{aligned} \min \quad & -2x_1 + 4x_2 - x_3 + x_4 \\ \text{s.t.} \quad & x_1 + 2x_2 + 4x_3 + x_4 \leq 20 \\ & -x_1 + x_2 \leq 3 \\ & x_1 \leq 4 \\ & x_3 - 5x_4 \leq 5 \\ & -x_3 + 2x_4 \leq 2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

$$\begin{aligned} \text{Lagrange} = \quad & (-2x_1 + 4x_2 - x_3 + x_4) \\ & + \lambda_1(x_1 + 2x_2 + 4x_3 + x_4 - 20) \\ & + \lambda_2(-x_1 + x_2 - 3) \\ & + \lambda_3(x_1 - 4) \\ & + \lambda_4(x_3 - 5x_4 - 5) \\ & + \lambda_5(-x_3 + 2x_4 - 2) \\ & + \lambda_6(-x_1) + \lambda_7(-x_2) + \lambda_8(-x_3) + \lambda_9(-x_4) \end{aligned}$$

$$\begin{aligned} L(\mathbf{x}, \boldsymbol{\lambda}) = \quad & (-2 + \lambda_1 - \lambda_2 + \lambda_3 - \lambda_6)x_1 \\ + \quad & (4 + 2\lambda_1 + \lambda_2 - \lambda_7)x_2 \\ + \quad & (-1 + 4\lambda_1 + \lambda_4 - \lambda_5 - \lambda_8)x_3 \\ + \quad & (1 + \lambda_1 - 5\lambda_4 + 2\lambda_5 - \lambda_9)x_4 \\ + \quad & (-20\lambda_1 - 3\lambda_2 - 4\lambda_3 - 5\lambda_4 - 2\lambda_5) \end{aligned}$$

So the dual problem is

$$\begin{aligned} \max \quad & -20\lambda_1 - 3\lambda_2 - 4\lambda_3 - 5\lambda_4 - 2\lambda_5 \\ \text{s.t.} \quad & \lambda_1 - \lambda_2 + \lambda_3 \leq 2 \\ & 2\lambda_1 + \lambda_2 \leq -4 \\ & 4\lambda_1 + \lambda_4 - \lambda_5 \leq 1 \\ & \lambda_1 - 5\lambda_4 + 2\lambda_5 \leq -1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0. \end{aligned}$$

6. Write down the primal-dual optimality conditions (primal feasibility, dual feasibility, and complementary slackness) for the following problem (i.e., the conditions shown on page 23 of the Duality lecture slides).

$$\begin{aligned}
\max \quad & 5x_1 - 2x_3 + x_4 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 \le 30 \\
& x_1 + x_2 \le 12 \\
& 2x_1 - x_2 \le 9 \\
& -x_3 + x_4 \le 2 \\
& x_3 + 2x_4 \le 10 \\
& x_1, x_2, x_3, x_4 \ge 0.
\end{aligned}$$

Dual Problem:

$$\begin{aligned}
\min \quad w = 30y_1 + 12y_2 + 9y_3 + 2y_4 + 10y_5 \\
\text{s.t.} \quad y_1 + y_2 + 2y_3 \ge 5 \\
y_1 + y_2 - y_3 \ge 0 \\
y_1 - y_4 + y_5 \ge -2 \\
y_1 + y_4 + 2y_5 \ge 1 \\
y_1, y_2, y_3, y_4, y_5 \ge 0
\end{aligned}$$

Primal-Dual Optimality Conditions

1. Primal Feasibility The primal solution $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ must satisfy:

$$\begin{aligned}
x_1^* + x_2^* + x_3^* + x_4^* &\le 30 \\
x_1^* + x_2^* &\le 12 \\
2x_1^* - x_2^* &\le 9 \\
-x_3^* + x_4^* &\le 2 \\
x_3^* + 2x_4^* &\le 10 \\
x_j^* &\ge 0, \quad j = 1, \dots, 4
\end{aligned}$$

2. Dual Feasibility The dual solution $\mathbf{y}^* = (y_1^*, y_2^*, y_3^*, y_4^*, y_5^*)$ must satisfy:

$$\begin{aligned}
y_1^* + y_2^* + 2y_3^* &\ge 5 \\
y_1^* + y_2^* - y_3^* &\ge 0 \\
y_1^* - y_4^* + y_5^* &\ge -2 \\
y_1^* + y_4^* + 2y_5^* &\ge 1 \\
y_i^* &\ge 0, \quad i = 1, \dots, 5
\end{aligned}$$

3. Complementarity

(a) **Primal Slack $\times$ Dual Variable $= 0$:**

$$
\begin{aligned}
y_1^*(30 - (x_1^* + x_2^* + x_3^* + x_4^*)) &= 0 \\
y_2^*(12 - (x_1^* + x_2^*)) &= 0 \\
y_3^*(9 - (2x_1^* - x_2^*)) &= 0 \\
y_4^*(2 - (-x_3^* + x_4^*)) &= 0 \\
y_5^*(10 - (x_3^* + 2x_4^*)) &= 0
\end{aligned}
$$

(b) **Dual Slack $\times$ Primal Variable $= 0$:**

$$
\begin{aligned}
x_1^*(y_1^* + y_2^* + 2y_3^* - 5) &= 0 \\
x_2^*(y_1^* + y_2^* - y_3^* - 0) &= 0 \\
x_3^*(y_1^* - y_4^* + y_5^* - (-2)) &= 0 \\
x_4^*(y_1^* + y_4^* + 2y_5^* - 1) &= 0
\end{aligned}
$$

7. **Urban Water Supply Network Scheduling**

A city operates a water distribution network consisting of:

- 5 reservoirs (supply nodes),
- 2 transfer stations (transit nodes),
- 8 residential areas (demand nodes),
- 19 pipelines connecting the nodes.

The water utility company must develop a supply plan that satisfies all residential water demands while minimizing the total transportation cost.

Data Description:

- **Node types:**
  - Nodes 1–5: reservoirs (supply nodes)
  - Nodes 6–7: transfer stations (transit nodes)
  - Nodes 8–15: residential areas (demand nodes)
- **Pipeline data (19 edges):** Each pipeline has a start node, an end node, a capacity (tons/hour), and a unit cost (CNY/ton).

Pipeline ID	Start Node	End Node	Capacity	Unit Cost
1	1	6	50	2
2	1	7	40	3
3	2	6	60	1
4	2	8	30	4
5	3	7	45	2
6	3	9	35	3
7	4	6	55	1
8	4	10	25	5
9	5	7	40	2
10	5	11	30	4
11	6	8	40	2
12	6	9	35	1
13	7	10	30	3
14	7	11	25	2
15	6	7	20	1
16	6	12	30	2
17	6	13	30	1
18	7	14	30	2
19	7	15	30	3

- **Supply and demand data:**
  - Reservoir supplies:

    Node 1: 40, Node 2: 50, Node 3: 45, Node 4: 40, Node 5: 35

– Residential demands:

$$\text{Node 8: } 25, \quad \text{Node 9: } 30, \quad \text{Node 10: } 20, \quad \text{Node 11: } 25,$$

$$\text{Node 12: } 15, \quad \text{Node 13: } 20, \quad \text{Node 14: } 25, \quad \text{Node 15: } 15$$

– Transfer stations: Node 6 and Node 7 have zero net supply/demand.

Task:

(a) Formulate this problem as a linear programming (LP) model.

(b) Clearly specify the decision variables, objective function, and constraints.

(c) Use excel solver to find the optimal solution and the minimum total cost.

(d) Submit the corresponding excel file along with your written solution.

Decision variables: $p_i \geq 0, \quad i = 1, 2, \ldots, 19,$ denote the flow through pipeline i.

Objective:

$$\min Z = 2p_1 + 3p_2 + 1p_3 + 4p_4 + 2p_5 + 3p_6 + 1p_7 + 5p_8 + 2p_9 + 4p_{10} + 2p_{11} + 1p_{12} + 3p_{13} + 2p_{14} + 1p_{15} + 2p_{16} + 1p_{17} + 2p_{18} + 3p_{19}$$

Subject to:

(Supply node 1) $p_1 + p_2 \leq 40,$

(Supply node 2) $p_3 + p_4 \leq 50,$

(Supply node 3) $p_5 + p_6 \leq 45,$

(Supply node 4) $p_7 + p_8 \leq 40,$

(Supply node 5) $p_9 + p_{10} \leq 35,$

(Transit node 6) $(p_1 + p_3 + p_7) - (p_{11} + p_{12} + p_{15} + p_{16} + p_{17}) \geq 0,$

(Transit node 7) $(p_2 + p_5 + p_9 + p_{15}) - (p_{13} + p_{14} + p_{18} + p_{19}) \geq 0,$

(Demand node 8) $p_4 + p_{11} \geq 25,$

(Demand node 9) $p_6 + p_{12} \geq 30,$

(Demand node 10) $p_8 + p_{13} \geq 20,$

(Demand node 11) $p_{10} + p_{14} \geq 25,$

(Demand node 12) $p_{16} \geq 15,$

(Demand node 13) $p_{17} \geq 20,$

(Demand node 14) $p_{18} \geq 25,$

(Demand node 15) $p_{19} \geq 15,$

(Positive) $p_i \geq 0, \quad i = 1, \ldots, 19.$

(Pipeline capacities) $p_i \leq \text{capacity}_i, \quad i = 1, \ldots, 19.$

Specification

For every variable p_i i =1,2,...,19 denotes the flow through each pipeline, the constraints denote that the flow out of supply nodes should less than its capabilities, and at the same time the flow in the residential area should satisfy the demand, and the flow out of transfer nodes should less than the flow in it, and the flow through pipeline should more than 0. So we have s seires inequality constraints. We want to minimize the total cost, so we need to $\min \mathrm{Obj}$, where the Obj function is the sum of each pipeline cost, which equals to Unit Cost $\times$ flow through this pipeline

Optimal Decision Variables (p_i^*)

The optimal values for the decision variables p_i ($i = 1$ to $i = 19$) are:

$$\mathbf{p}^* = (p_1^*, p_2^*, \ldots, p_{19}^*) = (5, 0, 50, 0, 45, 0, 35, 5, 10, 25, 25, 30, 15, 0, 0, 15, 20, 25, 15)$$

Minimum Total Cost (Z^*)

The minimum optimal value of the objective function (minimum total cost) is:

$$Z^* = \sum_{i=1}^{19} c_i p_i^* = 600$$