SI152: Numerical Optimization

Lecture 3: Simplex Method

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Outline

Outline

THE TOP10 ALGORITHMS FROM THE 20TH CENTURY

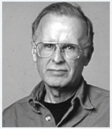
1946: The Metropolis Algorithm
1947: Simplex Method
1950: Krylov Subspace Method
1951: The Decompositional Approach to Matrix Computations
1957: The Fortran Optimizing Compiler
1959: QR Algorithm
1962: Quicksort
1965: Fast Fourier Transform
1977: Integer Relation Detection
1987: Fast Multipole Method

Dantzig

von Neumann

Hestenes

Householder

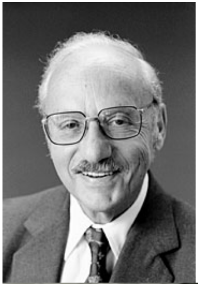


Backus

Hoare

Greengard

History of Simplex Method

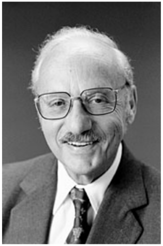


- George Bernard Dantzig

1914-2005
University of Maryland (BS)
University of Michigan (MS)
University of California, Berkeley (PhD)
mathematical adviser to the military (1946-1952),
a research mathematician at the RAND Corp. (1952-1960)
chair and professor of the Operations Research Center at UC-Berkeley (1960-1966).

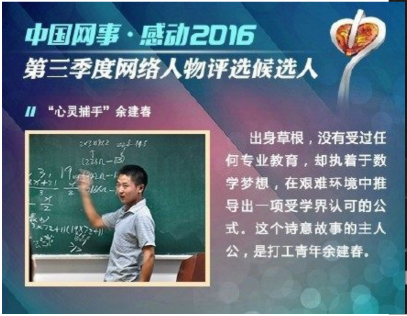
The recipients of the Dantzig Prize are:

- 1982: Michael J.D. Powell, R. Tyrell Rockafellar
- 1985: Ellis Johnson, Manfred Padberg
- 1988: Michael J. Todd
- 1991: Martin Grotschel, Arkady S. Nemirovskii
- 1994: Claude Lemarechal, Roger J.B. Wets
- 1997: Roger Fletcher, Stephen M. Robinson
- 2000: Yurii Nesterov
- 2003: Jong-Shi Pang, Alexander Schrijver
- 2006: Eva Tardos
- 2009: Gérard Cornuéjols
- 2012: Jorge Nocedal, Laurence Wolsey
- 2015: Dimitri P. Bertsekas
- 2018: Andrzej Piotr Ruszczyński, Alexander Shapiro
- 2021: Hedy Attouch, Michel Goemans
- 2024: Stephen Wright[5]



- George Bernard Dantzig

传奇永流传



Outline

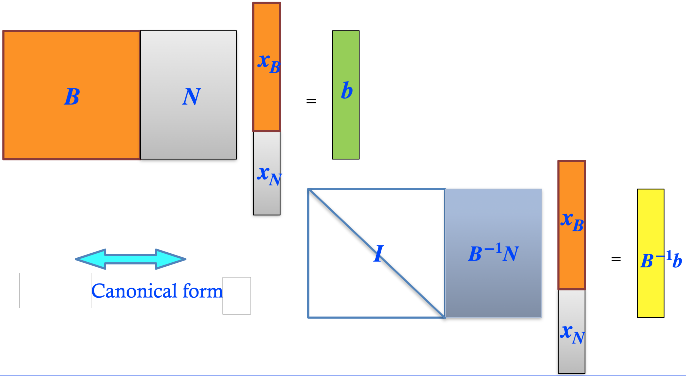
- Applicable form: standard form
- Basic idea: start from a BFS, move consecutively to the adjacent BFS, until hit the optimal solution, or determine the infeasibility.
- Iteration: How to move from a BFS to its adjacent BFS? (pivot)
- Termination: When is optimal? When is unbounded? (reduced cost)
- Initialization: How to get the first BFS?
- Degeneracy: How to avoid cycling? (Bland rule)

Current basis:

$$A = [B \ N], \ x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}, \ c = \begin{bmatrix} c_B \\ c_N \end{bmatrix}$$

By $Ax = b$, x_B is represented by x_N:

$$x_B = B^{-1}b - B^{-1}Nx_N$$



At x, the current objective: $f = c^T x = [c_B^T, c_N^T] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = c_B^T x_B + c_N^T x_N$

Eliminating by $x_B = B^{-1}b - B^{-1}Nx_N$, the objective is:

$$
\begin{aligned}
f &= c_B^T(B^{-1}b - B^{-1}Nx_N) + c_N^T x_N \\
&= \underbrace{c_B^T B^{-1} b}_{f_0} + \underbrace{(c_N^T - c_B^T B^{-1} N)}_{r_N^T\text{, reduced cost}} x_N \\
&= f_0 + r_N^T x_N
\end{aligned}
$$

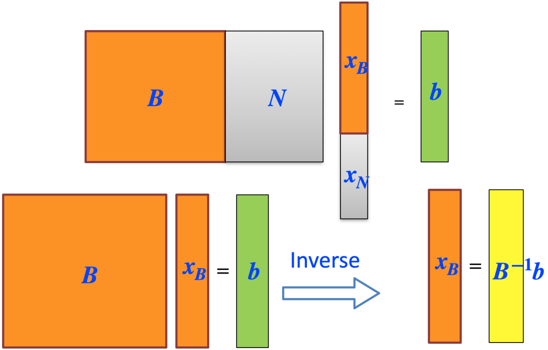
What is r_B?

$$
\begin{aligned}
r^T &= [r_B^T, r_N^T] \\
&= [\underbrace{c_B^T - c_B^T B^{-1} B}_{r_B^T = 0},\ \underbrace{c_N^T - c_B^T B^{-1} N}_{r_N^T}] \\
&= c^T - c_B^T B^{-1} A
\end{aligned}
$$

In particular, setting $x_N = 0$ to obtain the corresponding basic solution:

$$x_B = B^{-1}b, \quad x_N = 0$$

If $x_B \geq 0$, it is a BFS.



At the current BFS, the objective: $f_0 = [c_B^T, c_N^T] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = c_B^T B^{-1} b$

In other words,

$$f = \underbrace{c_B^T B^{-1} b}_{f_0} + \underbrace{(c_N^T - c_B^T B^{-1} N) x_N}_{0} = f_0$$

To change to another BFS, need set $x_N \neq 0$. Plugging in, new objective is:

$$f = \underbrace{c_B^T B^{-1} b}_{f_0} + \underbrace{(c_N^T - c_B^T B^{-1} N)}_{r_N^T \text{ reduced cost}} x_N$$

Question: what if $r \geq 0$? what if some $r_q < 0$?

Theorem 1 (Optimality Criterion)

At a nondegenerate BFS, if $c_N^T - c_B^T B^{-1} N \geq 0$, that is $r \geq 0$, then $x_B, x_N = 0$ is optimal.

What if some $r_q < 0$?

WLOG, let $B = [a_1, ..., a_m]$, $N = [a_{m+1}, ..., a_N]$, $q = m + 1$.
Increase x_q, objective decreases:

$$\min_{x_N} \; f_0 + r_N^T x_N = f_0 + r_{m+1}0 + r_q \boxed{x_q} + r_{q+1}0 + ... + r_n 0$$

Increase x_q, what about feasibility of constraints?
$Ax = b \iff x_B = B^{-1}b - B^{-1}Nx_N = \bar{b} - Yx_N$, now check $x \geq 0$.

$$\begin{array}{llll} x_1 & + y_{1,m+1}x_{m+1} + \ldots + y_{1,n}x_n & = \bar{b}_1 \\ \quad x_2 & + y_{2,m+1}x_{m+1} + \ldots + y_{2,n}x_n & = \bar{b}_2 \\ \quad\quad \ddots & \quad \vdots & \quad \vdots \\ \quad\quad\quad x_m & + y_{m,m+1}x_{m+1} + \ldots + y_{m,n}x_n & = \bar{b}_m \end{array}$$

nondegenerate, all positive

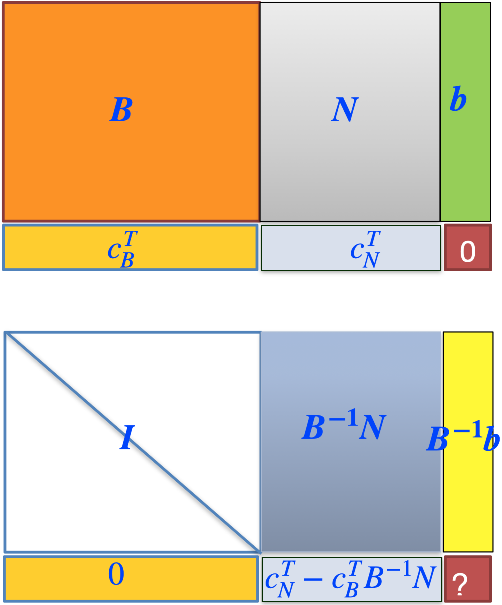
$$\begin{array}{l} (x_1 =)\ \bar{b}_1 - y_{1q}x_q \geq 0 \\ \quad \vdots \\ (x_m =)\ \bar{b}_m - y_{mq}x_q \geq 0 \\ x_{m+1} \geq 0, \ldots, x_n \geq 0 \end{array}$$

x_q can increase

Pivot

- x_p decreases to 0, $p = \arg\min_i \{ \frac{\bar{b}_i}{y_{iq}} \mid y_{iq} > 0, i = 1, ..., m \}$,
- x_q increases to $\frac{\bar{b}_p}{y_{pq}}$
- Pivot: x_q enter the basis and x_p leaves the basis, obtain a new BFS.
- From current BFS to an adjacent BFS, and the objective decreases.
- If $q \in \{ j \mid r_j < 0 \}$ and $y_q \leq 0$, then the problem is unbounded!

Tableau



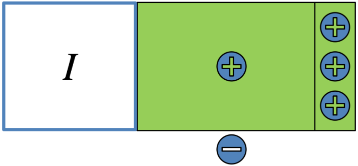
Initial Tableau

	x_1	$\cdots$	x_p	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_q	$\cdots$	x_n	$B^{-1}b$
	1	$\cdots$	0	$\cdots$	0	$y_{1,m+1}$	$y_{1,m+2}$	$\cdots$	y_{1q}	$\cdots$	y_{1n}	$\bar{b}_1$
		$\ddots$				$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	0	$\cdots$	1	$\cdots$	0	$y_{p,m+1}$	$y_{p,m+2}$	$\cdots$	y_{pq}	$\cdots$	y_{pn}	$\bar{b}_p$
				$\ddots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	0	$\cdots$	0	$\cdots$	1	$y_{m,m+1}$	$y_{m,m+2}$	$\cdots$	y_{mq}	$\cdots$	y_{mn}	$\bar{b}_m$
c^T	c_1	$\cdots$	c_p	$\cdots$	c_m	c_{m+1}	c_{m+2}	$\cdots$	c_q	$\cdots$	c_n	0

Simplex Method in Tableau

$$
\begin{array}{c|ccccccccccc|c}
\hline
 & x_1 & \cdots & x_p & \cdots & x_m & x_{m+1} & x_{m+2} & \cdots & x_q & \cdots & x_n & B^{-1}b \\
\hline
 & 1 & \cdots & 0 & \cdots & 0 & y_{1,m+1} & y_{1,m+2} & \cdots & y_{1q} & \cdots & y_{1n} & \bar{b}_1 \\
 & & \ddots & & & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
 & 0 & \cdots & 1 & \cdots & 0 & y_{p,m+1} & y_{p,m+2} & \cdots & y_{pq} & \cdots & y_{pn} & \bar{b}_p \\
 & & & & \ddots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
 & 0 & \cdots & 0 & \cdots & 1 & y_{m,m+1} & y_{m,m+2} & \cdots & y_{mq} & \cdots & y_{mn} & \bar{b}_m \\
r^{\mathrm{T}} & 0 & \cdots & 0 & \cdots & 0 & r_{m+1} & r_{m+2} & \cdots & r_q & \cdots & r_n & \boxed{-f} \\
\hline
\end{array}
$$

$1 \cdot x_1$				$+y_{1q}x_q$	$+\dots+$	$y_{1n}x_n$	$=\bar{b}_1$
	$1 \cdot x_p$			$+\boxed{y_{pq}}x_q$	$+\dots+$	$y_{pn}x_n$	$=\bar{b}_p$
		$\ddots$		$\dots$	$\dots$		
			$1 \cdot x_m$	$+y_{mq}x_q$	$+\dots+$	$y_{mn}x_n$	$=\bar{b}_m$
0	$\boxed{0}$	$\dots$	0	$\boxed{r_q}$	$\dots$	$\boxed{r_n}$	
0	$\boxed{-\frac{1}{y_{pq}}r_q}$	$\dots$	0	$\boxed{0}$	$\dots$	$\boxed{r_n - \frac{y_{pn}}{y_{pq}}r_q}$	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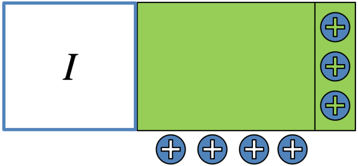
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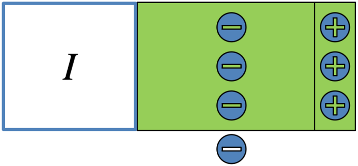
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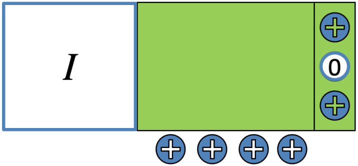
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0

$$
\begin{array}{ll}
\textbf{maximize} & 3x_1 + x_2 + 3x_3 \\
\textbf{subject to} & 2x_1 + x_2 + x_3 \le 2, \\
& x_1 + 2x_2 + 3x_3 \le 5, \\
& 2x_1 + 2x_2 + x_3 \le 6, \\
& x_1 \ge 0, x_2 \ge 0, x_3 \ge 0
\end{array}
$$

First, standard form

The initial tableau

$$
\begin{array}{c|cccccc|c}
 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{b} \\
\hline
 & \underline{2} & \boxed{1} & 1 & 1 & 0 & 0 & 2 \\
 & 1 & 2 & \underline{3} & 0 & 1 & 0 & 5 \\
 & 2 & 2 & 1 & 0 & 0 & 1 & 6 \\
\hline
\mathbf{c}^T/\mathbf{r}^T & -3 & -1 & -3 & 0 & 0 & 0 & 0
\end{array}
$$

pivot

Initial Tableau

x_1	x_2	x_3	x_4	x_5	x_6	
2	1	1	1	0	0	2
−3	0	$\boxed{1}$	−2	1	0	1
−2	0	−1	−2	0	1	2
−1	0	−2	1	0	0	2

pivot

x_1	x_2	x_3	x_4	x_5	x_6	
$\boxed{5}$	1	0	3	−1	0	1
−3	0	1	−2	1	0	1
−5	0	0	−4	1	1	3
−7	0	0	−3	2	0	4

pivot

0 ↓ -2 ↓ -4 ↓

x_1	x_2	x_3	x_4	x_5	x_6	
1	$1/5$	0	$3/5$	$-1/5$	0	$1/5$
0	$3/5$	1	$-1/5$	$2/5$	0	$8/5$
0	1	0	-1	0	1	4
0	$7/5$	0	$6/5$	$3/5$	0	$27/5$

Optimal solution:

$$x_1 = \frac{1}{5}, x_3 = \frac{8}{5}, x_6 = 4, x_2 = x_4 = x_5 = 0$$

Optimal value: $-\dfrac{27}{5}$ Original problem: $\dfrac{27}{5}$

Step 1: Create the first tableau associated with the initial BFS.

Step 2: If $r_j \geq 0, \forall j$, stop; current BFS is optimal.

Step 3: Choose $r_q = \min\{r_j \mid r_j < 0, j = 1, ..., n\}$.

Step 4: If $y_q \leq 0$, stop. The problem is infinite. Otherwise, choose p

$$\frac{\bar{b}_p}{y_{pq}} = \min\{\frac{\bar{b}_i}{y_{iq}} \mid y_{iq} > 0, i = 1, ..., m\}$$

Step 5: Pivot using y_{pq}. Update the tableau. Go to Step 1.

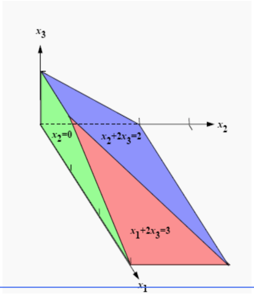
Outline

Theorem 2

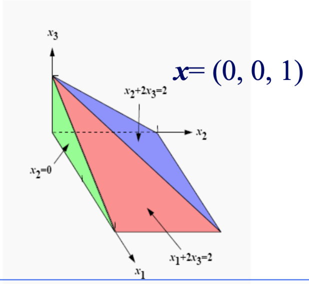
Assume the LP problem is nondegenerate, which means every BFS is nondegenerate. Starting from a BFS, the simplex method terminates finitely.

Why? and what if the problem is degenerate?

$$\begin{array}{llcccccc} \text{maximize} & x_1 & + & 2x_2 & + & 3x_3 & & \\ \text{subject to} & x_1 & & & + & 2x_3 & \leq & 3 \\ & & & x_2 & + & 2x_3 & \leq & 2 \\ & & & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$



$$\begin{array}{llcccccc} \text{maximize} & x_1 & + & 2x_2 & + & 3x_3 & & \\ \text{subject to} & x_1 & & & + & 2x_3 & \leq & 2 \\ & & & x_2 & + & 2x_3 & \leq & 2 \\ & & & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$



Degeneracy

- Unique basis for each BFS: nondegenerate. Multiple basis for a BFS: degenerate.
- BFS is dengenerate if and only the last column in the tableau has at least one 0.

Degenerate BFS $\Longrightarrow$ degenerate pivot $\Longrightarrow$ objective the same $\Longrightarrow$ cycling

nondegenerate pivot

0	2	1	-3	0	2	0	0	1
0	-1	0	3	1	-1	0	0	4
1	0	0	0	0	2	0	0	3
0	0	0	1	0	-1	1	0	2
0	1	0	-1	0	0	0	1	0
0	-5	0	-4	0	-1	0	0	-6

degenerate pivot

degenerate BFS

- Cycling: from a tableau, after some iterations, returns to the same tableau.
- First example: E. M. L. Beale (1955)* [1]

$$
\begin{aligned}
\text{minimize} \quad & -\tfrac{3}{4}x_4 + 20x_5 - \tfrac{1}{2}x_6 + 6x_7 \\
\text{subject to} \quad & x_1 + \tfrac{1}{4}x_4 - 8x_5 - x_6 + 9x_7 = 0 \\
& x_2 + \tfrac{1}{2}x_4 - 12x_5 - \tfrac{1}{2}x_6 + 3x_7 = 0 \\
& x_3 + x_6 = 1 \\
& x_1 \geq 0, \cdots, x_7 \geq 0
\end{aligned}
$$

- Rule for choose the entering basic variable: smallest reduced cost (smallest index for a tie)
- Rule for choose the leaving basic variable: smallest ratio (smallest index for a tie)

[1] Naval Research Logistics Quarterly Article, Cycling in the dual simplex algorithm, E. M. L. Beale, 1955

$$\begin{array}{ccccccc|c}
1 & 0 & 0 & \boxed{1/4} & -8 & -1 & 9 & 0 \\
0 & 1 & 0 & 1/2 & -12 & -1/2 & 3 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & -3/4 & 20 & -1/2 & 6 & 0
\end{array}$$

$$B = (a_1, a_2, a_3)$$

$$x = (0, 0, 1, 0, 0, 0, 0)^T$$

$$\begin{array}{ccccccc|c}
4 & 0 & 0 & 1 & -32 & -4 & 36 & 0 \\
-2 & 1 & 0 & 0 & \boxed{4} & 3/2 & -15 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
3 & 0 & 0 & 0 & -4 & -7/2 & 33 & 0
\end{array}$$

$$B = (a_4, a_2, a_3)$$

$$\begin{array}{ccccccc|c}
-12 & 8 & 0 & 1 & 0 & \boxed{8} & -84 & 0 \\
-1/2 & 1/4 & 0 & 0 & 1 & 3/8 & -15/4 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
1 & 1 & 0 & 0 & 0 & -2 & 18 & 0
\end{array}$$

$$B = (a_4, a_5, a_3)$$

$$\begin{array}{rrrrrrr|r} -3/2 & 1 & 0 & 1/8 & 0 & 1 & -21/2 & 0 \\ 1/16 & -1/8 & 0 & -3/64 & 1 & 0 & \boxed{3/16} & 0 \\ 3/2 & -1 & 1 & -1/8 & 0 & 0 & 21/2 & 1 \\ \hline -2 & 3 & 0 & 1/4 & 0 & 0 & -3 & 0 \end{array}$$

smallest reduced cost

$$B = (a_6, a_5, a_3)$$

$$\begin{array}{rrrrrrr|r} \boxed{2} & -6 & 0 & -5/2 & 56 & 1 & 0 & 0 \\ 1/3 & -2/3 & 0 & -1/4 & 16/3 & 0 & 1 & 0 \\ -2 & 6 & 1 & 5/2 & -56 & 0 & 0 & 1 \\ \hline -1 & 1 & 0 & -1/2 & 16 & 0 & 0 & 0 \end{array}$$

$$B = (a_6, a_7, a_3)$$

$$\begin{array}{rrrrrrr|r} 1 & -3 & 0 & -5/4 & 28 & 1/2 & 0 & 0 \\ 0 & \boxed{1/3} & 0 & 1/6 & -4 & -1/6 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & -2 & 0 & -7/4 & 44 & 1/2 & 0 & 0 \end{array}$$

$$B = (a_1, a_7, a_3)$$

$$
\begin{array}{ccccccc|c}
1 & 0 & 0 & 1/4 & -8 & -1 & 9 & 0 \\
0 & 1 & 0 & 1/2 & -12 & -1/2 & 3 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & -3/4 & 20 & -1/2 & 6 & 0
\end{array}
$$

$B = (a_1, a_2, a_3)$

The same tableau; all iterates are the same BFS, but different basis.

- In practice, degeneracy happens quite often, but cycling hardly happens.
- Techniques to avoid cycling: perturbation (Charnes, 1954), lexicography (Dantzig, Orden, Wolfe, 1954), Bland's rule.
- Bland's rule (Bland, 1977)—smallest index[2].
  - Rule for entering the basis: choose the smallest index with a negative (reduced) cost
  - Rule for leaving the basis: choose the smallest ratio (smallest index if there is a tie)

[2] New Finite Pivoting Rules for the Simplex Method. Mathematics of Operations Research. Vol. 2, No. 2 (May, 1977), pp. 103-107 (5 pages)

If we continue with Beale's example by applying Bland's rule: (starting from the 4th tableau)

$$\begin{array}{ccccccc|c}
-3/2 & 1 & 0 & 1/8 & 0 & 1 & -21/2 & 0 \\
1/16 & -1/8 & 0 & -3/64 & 1 & 0 & \boxed{3/16} & 0 \\
3/2 & -1 & 1 & -1/8 & 0 & 0 & 21/2 & 1 \\
\hline
-2 & 3 & 0 & 1/4 & 0 & 0 & -3 & 0
\end{array}$$

$$\begin{array}{ccccccc|c}
0 & -2 & 0 & -1 & 24 & 1 & -6 & 0 \\
1 & -2 & 0 & -3/4 & 16 & 0 & 3 & 0 \\
0 & \boxed{2} & 1 & 1 & -24 & 0 & 6 & 1 \\
\hline
0 & -1 & 0 & -5/4 & 32 & 0 & 3 & 0
\end{array}$$

$$
\begin{array}{ccccccc|c}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1/4 & -8 & 0 & 9 & 1 \\
0 & 1 & 1/2 & \boxed{1/2} & -12 & 0 & 3 & 1/2 \\
\hline
0 & 0 & 1/2 & -3/4 & 20 & 0 & 6 & 1/2
\end{array}
$$

$$
\begin{array}{ccccccc|c}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & -1/2 & 3/4 & 0 & -2 & 0 & 15/2 & 3/4 \\
0 & 2 & 1 & 1 & -24 & 0 & 6 & 1 \\
\hline
0 & 3/2 & 5/4 & 0 & 2 & 0 & 21/2 & 5/4
\end{array}
$$

From the last tableau, the optimal solution is

$$x^* = (3/4, 0, 0, 1, 0, 1, 0)^T, \quad z^* = -5/4$$

Outline

Complexity: time (how long it takes to run) + space (how much memory it requires).

- Average cost: expected running time or space of an algorithm, averaged over all possible inputs of a given size. Hard to analyze, but represents the "real" experience.
- Worst-Case cost: the maximum amount of time or space the algorithm will ever require. Possible to analyze, but doesn't represent the "real" experience.

Measures of size:

- # of constraints: m, and/or # of variables: n
- # of datum: mn
- # of nonzeros
- size (bytes)

Measures of time:

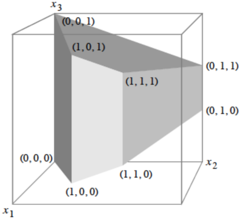
- # of iterations
- complexity per iteration

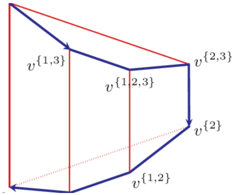
Klee-Minty Problem

$$\max \quad 2^{D-1}x_1 + 2^{D-2}x_2 + \cdots + 2x_{D-1} + x_D$$

subject to

$$\begin{aligned} x_1 &\le 5 \\ 4x_1 + x_2 &\le 25 \\ 8x_1 + 4x_2 + x_3 &\le 125 \\ &\vdots \\ 2^D x_1 + 2^{D-1}x_2 + \cdots + 4x_{D-1} + x_D &\le 5^D \\ x_1 \ge 0, \;\ldots,\; x_D &\ge 0. \end{aligned}$$





Klee-Minty Problem Implies:

- When solving a problem with n variables and constraints, the simplex method may require $2^n - 1$ pivots (thus visiting 2^n vertices of the twisted cube)
- If $n = 70$, $2^n = 1.2 \times 10^{21}$.
- If a computer can finish 1000 iterations per second, solving this problem needs 40,000,000,000 years. The age of our universe is about 13,800,000,000 years.
- Common problem often involve 10,000–100,000 variables.

Worst case analysis is just that: worst case.

- Sorting: $O(n \log n)$.
- Matrix-vector multiplication: $O(n^2)$.
- Gaussian elimination: $O(n^3)$.

The simplex method:

- Iteration: $O(2^n)$ (worst-case), $O(n^3)$ (average)
- Pivot: $O(mn)$ (naive), $O(m^2)$ (efficient)
- Total: $O(m^2 \cdot 2^n)$ (worst-case), $O(n^3 \cdot m^2)$ (average)

Is there a polynomial algorithm for LP?

上交大：EE本科，CS硕士
CMU：CS PhD
91--09年：BU、MIT、Minnesota、UIUC、Xerox PARC, NASA Ames Research Center, Intel Corporation, IBM Almaden Research Center, Akamai Technologies, Microsoft Research Redmond, Microsoft Research New England and Microsoft Research Asia……SCU

- 腾尚华

- Gödel Prize 2008
  Spielman, Daniel A., and Shang-Hua Teng. "Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time." Journal of the ACM (JACM) 51.3 (2004): 385-463.
- Gödel Prize 2015: "nearly-linear-time Laplacian solvers"
  Fulkerson Prize 2009

Outline

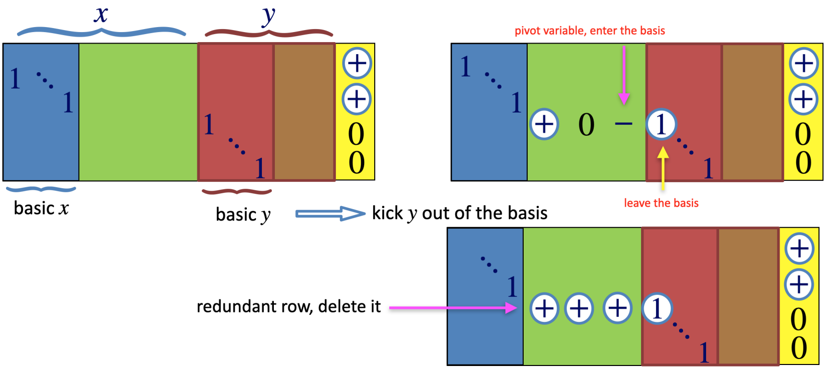
Adding auxiliary variables: detect whether $Ax = b, x \geq 0$ is feasible, if it is, remove the redundant constraint, find the initial BFS.

- Multiply both sides by -1 if necessary to make $b \geq 0$.
- Introduce auxiliary variables: $y_i, i = 1, ..., m$

$$\begin{aligned} \underset{x\in\mathbb{R}^n,\ y\in\mathbb{R}^m}{\textbf{minimize}} \quad & \sum_{i=1}^{m} y_i \\ \textbf{subject to} \quad & Ax + y = b \\ & x \geq 0, y \geq 0 \end{aligned}$$

  $x = 0, y = b$ is a BFS for this.
- Starting from $x = 0, y = b$ as the initial BFS, use the simplex method to find an optimal solution (x', y'), optimal value f', optimal basis B'.

- $f' > 0$, the original LP is infeasible.
- $f' = 0$, the original LP is feasible, and x' is a potential BFS.
  - no y_i is in the basis $\implies$ x' is BFS, B' is the basis.
  - some y_i is in the basis $\implies$ pivot, kick y_i out of the basis.
  - some y_i is in the basis with all nonnegative coefficients in this row, remove this redundant row.



$$\begin{aligned} 2x_1 + \quad x_2 + \quad 2x_3 &= 4, \\ 3x_1 + \quad 3x_2 + \quad x_3 &= 3, \\ x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 &\geq 0. \end{aligned}$$

Introduce $x_4 \geq 0$, $x_5 \geq 0$, objective is now: minimize $x_4 + x_5$

	x_1	x_2	x_3	x_4	x_5	b
	2	1	2	1	0	4
	3	3	1	0	1	3
c^T	0	0	0	1	1	0

Tableau for initialization

$x = (0, 0, 0, 4, 3)^T$

BFS

x_1	x_2	x_3	x_4	x_5	b
2	1	2	1	0	4
$\boxed{3}$	3	1	0	1	3
$\underline{-5}$	-4	-3	0	0	-7

x_1	x_2	x_3	x_4	x_5	b
0	-1	$\boxed{4/3}$	1	$-2/3$	2
1	1	$1/3$	0	$1/3$	1
0	1	$\underline{-4/3}$	0	$5/3$	-2

x_1	x_2	x_3	x_4	x_5	b
0	$-3/4$	1	$3/4$	$-1/2$	$3/2$
1	$5/4$	0	$-1/4$	$1/2$	$1/2$
0	0	0	1	1	0

Optimal value for the auxiliary problem is: 0.
BFS for the original problem is:

$$x_1 = 1/2, \quad x_2 = 0, \quad x_3 = 3/2.$$

Example

$$\begin{array}{ll} \text{minimize} & 4x_1 + x_2 + x_3 \\ \text{subject to} & 2x_1 + x_2 + 2x_3 = 4, \\ & 3x_1 + 3x_2 + x_3 = 3, \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array}$$

Phase I: auxiliary problem is

$$\begin{array}{ll} \text{minimize} & x_4 + x_5 \\ \text{subject to} & 2x_1 + x_2 + 2x_3 + x_4 = 4, \\ & 3x_1 + 3x_2 + x_3 + x_5 = 3, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{array}$$

Last tableau for the auxiliary problem:

x_1	x_2	x_3	x_4	x_5	b
0	$-3/4$	1	$3/4$	$-1/2$	$3/2$
1	$5/4$	0	$-1/4$	$1/2$	$1/2$
0	0	0	1	1	0

Original tableau for the original problem

	x_1	x_2	x_3	b
	0	$-3/4$	1	$3/2$
	1	$5/4$	0	$1/2$
c^T	4	1	1	0

	x_1	x_2	x_3	b
	0	$-3/4$	1	$3/2$
	1	$\boxed{5/4}$	0	$1/2$
r^T	0	$-13/4$	0	$-7/2$

	x_1	x_2	x_3	b
	$3/5$	0	1	$9/5$
	$4/5$	1	0	$2/5$
r^T	$13/5$	0	0	$-11/5$

Optimal solution for the original problem: $x_1 = 0$, $x_2 = 2/5$, $x_3 = 9/5$.

Two-Phase method: solve any LP.

- Phase I: initialization
  - Construct the standard form, and the auxiliary problem
  - Solving the auxiliary problem, determine whether it is feasible
  - If feasible, remove the redundant constraint and obtain the original tableau
- Phase II: starting from the initial BFS, solve the original problem for the optimal solution or detect infeasibility.

Other method: Big-M method
For sufficiently large $M > 0$, solve

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} + M\sum_{i=1}^{m} y_i \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0},\ \boldsymbol{y} \geq \boldsymbol{0} \end{aligned}$$

Outline

Given B, the tableau is

$$\begin{bmatrix} B^{-1}A & B^{-1}b \\ c^T - c_B^T B^{-1}A & -c_B^T B^{-1}b \end{bmatrix}$$

$$\lambda^T = c_B^T B^{-1},\ r_j = c_j - \lambda^T a_j,\ y_q = B^{-1}a_q,\ y_0 = B^{-1}b$$

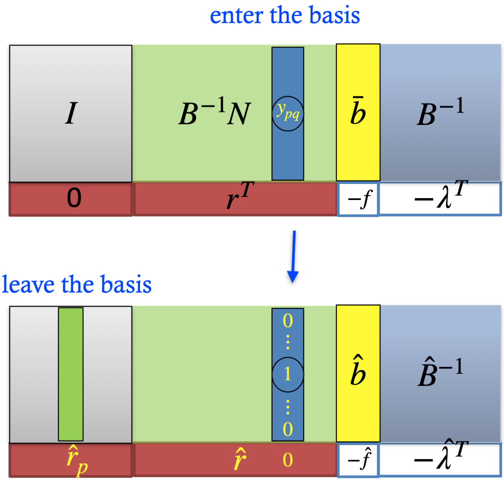
We call λ are the multipliers

For every iteration, the key computational cost is B^{-1}, which is used to update the basis.

B	N	b	I
c_B^T	c_N^T	0	0

I	$B^{-1}N$	$\bar{b}$	B^{-1}
c_B^T	c_N^T	0	0

I	$B^{-1}N$	$\bar{b}$	B^{-1}
0	$c_N^T - c_B^T B^{-1}N$	$-f$	$-c_B^T B^{-1}$



$$\hat{b}_{ij} = \begin{cases} b_{ij} - \dfrac{y_{iq}}{y_{pq}} b_{pj} & i \neq p \\ \dfrac{b_{pj}}{y_{pq}} & i = p \end{cases}$$

$$\hat{\lambda}^T = \lambda^T + \frac{r_q}{y_{pq}} \boldsymbol{u}_p, \quad \text{where } \boldsymbol{u}_p$$

is the pth row of $\boldsymbol{B}^{-1}$

Pivot y_{pq}, a_q enters the basis and a_p leaves the basis

index	$\boldsymbol{B}^{-1}$			$\boldsymbol{x}_B$	$\boldsymbol{y}_q$
i_1				$\bar{b}_1$	y_{1q}
$\vdots$				$\vdots$	$\vdots$
i_p				$\bar{b}_2$	$\boxed{y_{pq}}$
$\vdots$				$\vdots$	$\vdots$
i_m				$\bar{b}_m$	y_{mq}
$\boldsymbol{\lambda}^T$	λ_1	$\cdots$	λ_m	$\boxed{f}$	$\boxed{-r_q}$

1: Given BFS and B^{-1}. Compute $\bar{b} = B^{-1}b$, $\lambda^T = c_B^T B^{-1}$.

2: Compute $r_N^T = c_N^T - \lambda^T N$. If $r_N \geq 0$, terminate. Optimal solution found.

3: Choose q to satisfy $r_q = \min\{r_j \mid r_j < 0, j = 1, ..., n\}$.

4: Compute $y_q = B^{-1}a_q$. If $y_q = (y_{1q}, y_{2q}, ..., y_{mq})^T \leq 0$, problem is unbounded. Otherwise, choose p to satisfy

$$\frac{\bar{b}_p}{y_{pq}} = \min\left\{\frac{\bar{b}_i}{y_{iq}} \mid y_{iq} > 0,\ i = 1, \ldots, m\right\}$$

5: Update B^{-1}, $B^{-1}b$ and λ^T, go to Step 1.

a_1	a_2	a_3	a_4	a_5	a_6	b
2	1	1	1	0	0	2
1	2	3	0	1	0	5
2	2	1	0	0	1	6

$\boldsymbol{c} = (-3, -1, -1, 0, 0, 0)^T$

$\boldsymbol{\lambda}^T = (0,0,0)\boldsymbol{B}^{-1} = (0,0,0)$

$\boldsymbol{r}_N^T = \boldsymbol{c}_N^T - \boldsymbol{\lambda}^T \boldsymbol{N} = (-3, -1, -1)$

Index	B^{-1}			x_B	y_1
4	1	0	0	2	$\boxed{2}$
5	0	1	0	5	1
6	0	0	1	6	2
λ^T	0	0	0	0	3

$\boldsymbol{a}_1$ **enters the basis, compute** $\boldsymbol{y}_1$.
obtain $q = 1$

Pivot:

Index	B^{-1}			x_B	y_1
4	1	0	0	2	$\boxed{2}$
5	0	1	0	5	1
6	0	0	1	6	2
λ^T	0	0	0	0	3

Index	B^{-1}			x_B
1	$\frac{1}{2}$	0	0	1
5	$-\frac{1}{2}$	1	0	4
6	-1	0	1	4
λ^T	$-\frac{3}{2}$	0	0	-3

Compute: $r_2 = \frac{1}{2},\ r_3 = -\frac{3}{2},\ r_4 = \frac{3}{2},\ q = 3$

Compute: $y_3 = \boldsymbol{B}^{-1}\boldsymbol{a}_3 = (\frac{1}{2}, \frac{5}{2}, 0)^T$

Index	B^{-1}			x_B	y_3
1	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$
5	$-\frac{1}{2}$	1	0	4	$\boxed{\frac{5}{2}}$
6	-1	0	1	4	0
λ^T	$-\frac{3}{2}$	0	0	-3	$\frac{3}{2}$

Index	B^{-1}			x_B
1	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{1}{5}$
3	$-\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{8}{5}$
6	-1	0	1	4
λ^T	$-\frac{6}{5}$	$-\frac{3}{5}$	0	$-\frac{27}{5}$

Compute: $r_2 = \frac{7}{5},\ r_3 = \frac{6}{5},\ r_4 = \frac{3}{5}$
Optimal value: $\boldsymbol{z}^* = -27/5$
Optimal solutioin: $\boldsymbol{x}^* = (\frac{1}{5}, 0, \frac{8}{5}, 0, 0, 4)^T$

- Upper and lower bounded standard form

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}
\end{aligned}
$$

- BFS and simplex method can be defined accordingly.