Notes on Book The Cauchy Schwarz Master Class

Zhihao hao@chopin.fm

September 24, 2018

1 Chapter 1

Problem 1.1. *Prove Cauchy's inequality.*

$$\sum_i^n a_i b_i \leq \sqrt{\sum_i^n a_i^2}\sqrt{\sum_i^n b_i^2} \tag{1}$$

Moreover, if you already know a proof of Cauchy's inequality, find another one!

Proof. We can prove it in algebra way.

Firstly, if the square of the inequality holds, the inequality will also hold. Since the right side is non-negative. From this view, we can transform the problem to prove the square of the inequality as follows.

$$(\sum_i^n a_i b_i)^2 \leq \sum_i^n a_i^2 \sum_i^n b_i^2$$

Let's define Δ as the subtraction of the left side from the right side,

$$\Delta = \sum_i^n a_i^2 \sum_i^n b_i^2 - (\sum_i^n a_i b_i)^2$$

which can be reduced to

$$\begin{aligned}\Delta &= \sum_{i,j} a_i^2 b_j^2 - \sum_{i,j} a_i b_i a_j b_j \\ &= \sum_{i,j} a_i b_j (a_i b_j - a_j b_i)\end{aligned}$$

Now if we add Δ with itself, since the indices i, j are interchangable, we will end up with the following:

$$\begin{aligned}
\Delta + \Delta &= \sum_{i,j} a_i b_j (a_i b_j - a_j b_i) + \sum_{i,j} a_j b_i (a_j b_i - a_i b_j) \\
&= \sum_{i,j} a_i b_j (a_i b_j - a_j b_i) - a_j b_i (a_i b_j - a_j b_i) \\
&= \sum_{i,j} (a_i b_j - a_j b_i)^2 \\
&\geq 0
\end{aligned}$$

We can also prove from the intuitive geometry or vector way, which is omitted here. ☐

Exercise 1.1. *For Book Exercise 1.1. Prove for each real sequence* $a_1, a_2, ..., a_n$ *one has:*

$$a_1 + a_2 + ... + a_n \leq \sqrt{n}(a_1^2 + a_2^2 + ... + a_n^2)^{\frac{1}{2}}$$

Proof. We can view the sum of the sequence as the inner product of vectors in $\mathbb{R}_n$: $(a_1, a_2, ..., a_n)$ and $(1, 1, ..., 1)$. Then it becomes obvious from Cauchy's inequality. ☐

Exercise 1.2. *For Book Exercise 1.1. Prove*

$$\sum_{k=1}^{n} a_k \leq (\sum_{k=1}^{n} |a_k|^{2/3})^{\frac{1}{2}} (\sum_{k=1}^{n} |a_k|^{4/3})^{\frac{1}{2}}$$

Proof. We can view the sum as the inner product of vectors in $\mathbb{R}_n$: $(a_1^{\frac{1}{3}}, a_2^{\frac{1}{3}}, ..., a_n^{\frac{1}{3}})$ and $(a_1^{\frac{2}{3}}, a_2^{\frac{2}{3}}, ..., a_n^{\frac{2}{3}})$. Then it becomes obvious from Cauchy's inequality. ☐

Exercise 1.3. *For Book Exercise 1.2. Prove*

$$1 \leq \left\{ \sum_{j=1}^{n} p_j a_j \right\} \left\{ \sum_{j=1}^{n} p_j b_j \right\}$$

Proof. We can expand the multipication.

$$\begin{aligned}
&\left\{ \sum_{j=1}^{n} p_j a_j \right\} \left\{ \sum_{j=1}^{n} p_j b_j \right\} \\
&= \sum_{i=1}^{n} p_i^2 a_i b_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} p_i p_j (a_i b_j + a_j b_i) \\
&\geq \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} \sum_{j=i+1}^{n} p_i p_j (a_i b_j + a_j b_i)
\end{aligned}$$

Since we know by the additive bound.

$$a_i b_j + a_j b_i \geq 2(a_i b_j * a_j b_i)^{\frac{1}{2}} = 2(a_i b_i * a_j b_j)^{\frac{1}{2}} \geq 2(1 * 1)^{\frac{1}{2}} = 2$$

The equation would be

$$\begin{aligned}
&\left\{ \sum_{j=1}^{n} p_j a_j \right\} \left\{ \sum_{j=1}^{n} p_j b_j \right\} \\
&\geq \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} \sum_{j=i+1}^{n} p_i p_j (a_i b_j + a_j b_i) \\
&\geq \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} \sum_{j=i+1}^{n} p_i p_j * 2 \\
&= \left(\sum_{i=1}^{n} p_i \right)^2 \\
&= 1
\end{aligned}$$

□

Exercise 1.4. *For Book Exercise 1.3 (Why Not Three or More?)*

Proof. We can apply the same trick(Cauchy's inequality) of Exercise 1 to solve Part (a). Granted Part(a) is solved, we can solve Part(b) based on Part(a). We only need to prove the following.

$$\sum_{k=1}^{n} a_k^4 \leq \left(\sum_{k=1}^{n} a_k^2 \right)^2$$

We can just expand the polynomial on the RHS and only keep the a_k^4 terms, then we can easily get the inequality.

Alternatively, We can also prove Part(b) and its more general form as follows, directly by Cauchy's inequality.

$$\left(\sum_{k=1}^{n} s_{1_k} s_{2_k} ... s_{m_k} \right)^2 \leq \sum_{k=1}^{n} s_{1_k}^2 \sum_{k=1}^{n} s_{2_k}^2 ... \sum_{k=1}^{n} s_{m_k}^2$$

Here is the proof.

$$
\begin{aligned}
&\left(\sum_{k=1}^{n} s_{1_k} s_{2_k} ... s_{m_k}\right)^2 \\
&= \left(\sum_{k=1}^{n} (s_{1_k} s_{2_k} ... s_{m-1_k}) * (s_{m_k})\right)^2 \\
&\leq \sum_{k=1}^{n} (s_{1_k} s_{2_k} ... s_{m-1_k})^2 \sum_{k=1}^{n} s_{m_k}^2 && \text{by Cauchy's inequality} \\
&\leq \left(\sum_{k=1}^{n} s_{1_k}^2 \sum_{k=1}^{n} s_{2_k}^2 ... \sum_{k=1}^{n} s_{m-1_k}^2\right) \sum_{k=1}^{n} s_{m_k}^2 && \text{by polynomial expansion} \\
&= \sum_{k=1}^{n} s_{1_k}^2 \sum_{k=1}^{n} s_{2_k}^2 ... \sum_{k=1}^{n} s_{m_k}^2
\end{aligned}
$$

□