Chapter 0 Problems

Hui

1 Problem 0.10

You cannot divide either side of the equation by $(a - b)$ because you assume $a = b$, meaning $a - b = 0$.

2 Problem 0.11

a) **Proof:** *Basis:* Let $n = 1$. Then, $S(1) = 1$ and $\frac{1}{2}(1)(1+1) = 1$, which means $S(n) = \frac{1}{2}n(n+1)$ is true for $n = 1$.
Induction step: Assume n is an arbitrary but fixed natural number such that $1 + 2 + ... + n = \frac{1}{2}n(n+1)$. Then, $1 + 2 + ... + n + (n+1) = \frac{1}{2}n(n+1) + (n+1) = \frac{1}{2}n^2 + \frac{3}{2}n + 1 = \frac{1}{2}(n^2 + 3n + 2) = \frac{1}{2}(n+1)((n+1)+1)$. Since $1 + 2 + ... + n + (n+1) = \frac{1}{2}(n+1)((n+1)+1)$, $1 + 2 + ... + n = \frac{1}{2}n(n+1)$ is true for $n+1$. Thus, $S(n) = \frac{1}{2}n(n+1)$ for all natural numbers. Q.E.D.

b) **Proof:** *Basis:* Let $n = 1$. Then, $C(1) = 1$ and $\frac{1}{4}(1)^2(1+1)^2 = 1$, which means $C(n) = \frac{1}{4}n^2(n+1)^2$ is true for $n = 1$.
Induction step: Assume n is an arbitrary but fixed natural number such that $1^3+2^3+...+n^3 = \frac{1}{4}n^2(n+1)^2$. Then, $1^3+2^3+...+n^3+(n+1)^3 = \frac{1}{4}n^2(n+1)^2+(n+1)^3 = (n+1)^2(\frac{1}{4}n^2+(n+1)) = \frac{1}{4}(n+1)^2(n^2+4n+4) = \frac{1}{4}(n+1)^2((n+1)+1)$. Since $1^3+2^3+...+n^3+(n+1)^3 = \frac{1}{4}(n+1)^2((n+1)+1)$, $1^3+2^3+...+n^3 = \frac{1}{4}n^2(n+1)^2$ is true for $n+1$. Thus, $C(n) = \frac{1}{4}n^2(n+1)^2$ for all natural numbers. Q.E.D.

Conclusion: Because $C(n) = \frac{1}{4}n^2(n+1)^2$ and $S(n) = \frac{1}{2}n(n+1)$ for all natural numbers, $C(n) = S^2(n)$ for all natural numbers.

3 Problem 0.12

You cannot conclude that all the horses in H_2 are the same color because the initially removed horse may be a different color than those in H_1 with k horses. This means that replacing that removed horse and then removing an alternative one may mean H_2 has one differently-colored horse.

4 Problem 0.13

Proof: *Basis:* Let graph F contain only nodes a and b. Then, both nodes clearly have equal degrees.
Induction step: Assume k is an arbitrary but fixed natural number greater than 2, that graph G contains k nodes, and that there exists two nodes contained in G, which we'll also call a and b, that both have n degrees (i.e. equal degrees). Now say we add an additional node c to graph G by adding a vertex from a to c and a vertex from b to c. Then, node a and node b both have $n+1$ degrees, which means our new graph of $k+1$ nodes still contains two nodes with equal degrees. Thus, we can conclude that every graph with two or more nodes contains two nodes with equal degrees.

5 Problem 0.14

Proof: *Basis:* Let graph F contain $n = 1$ nodes. Then, it can be said that F itself is vacuously a clique or anti-clique. Since F has 1 node, and $\frac{1}{2}\log_2 1 = 0$, F

6 Problem 0.15