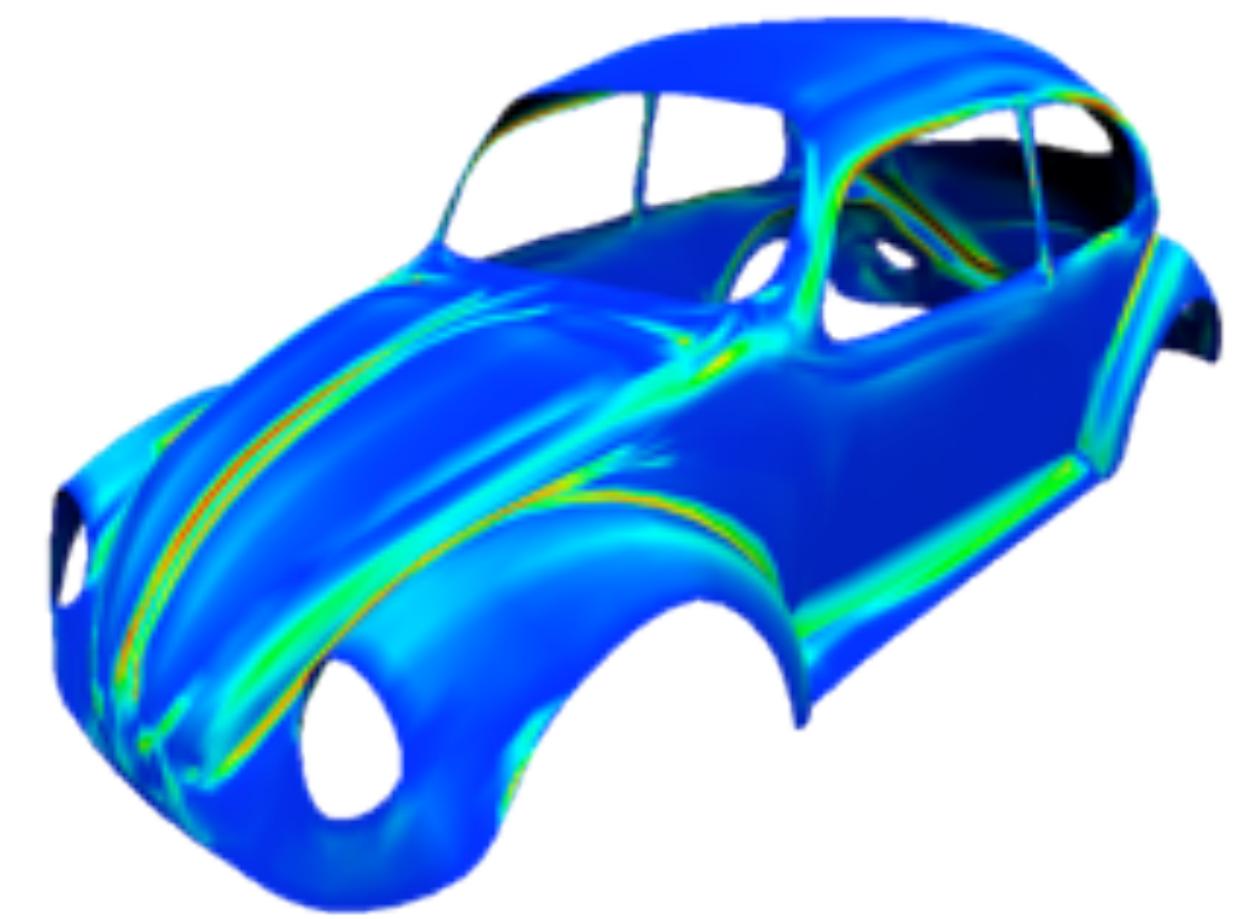
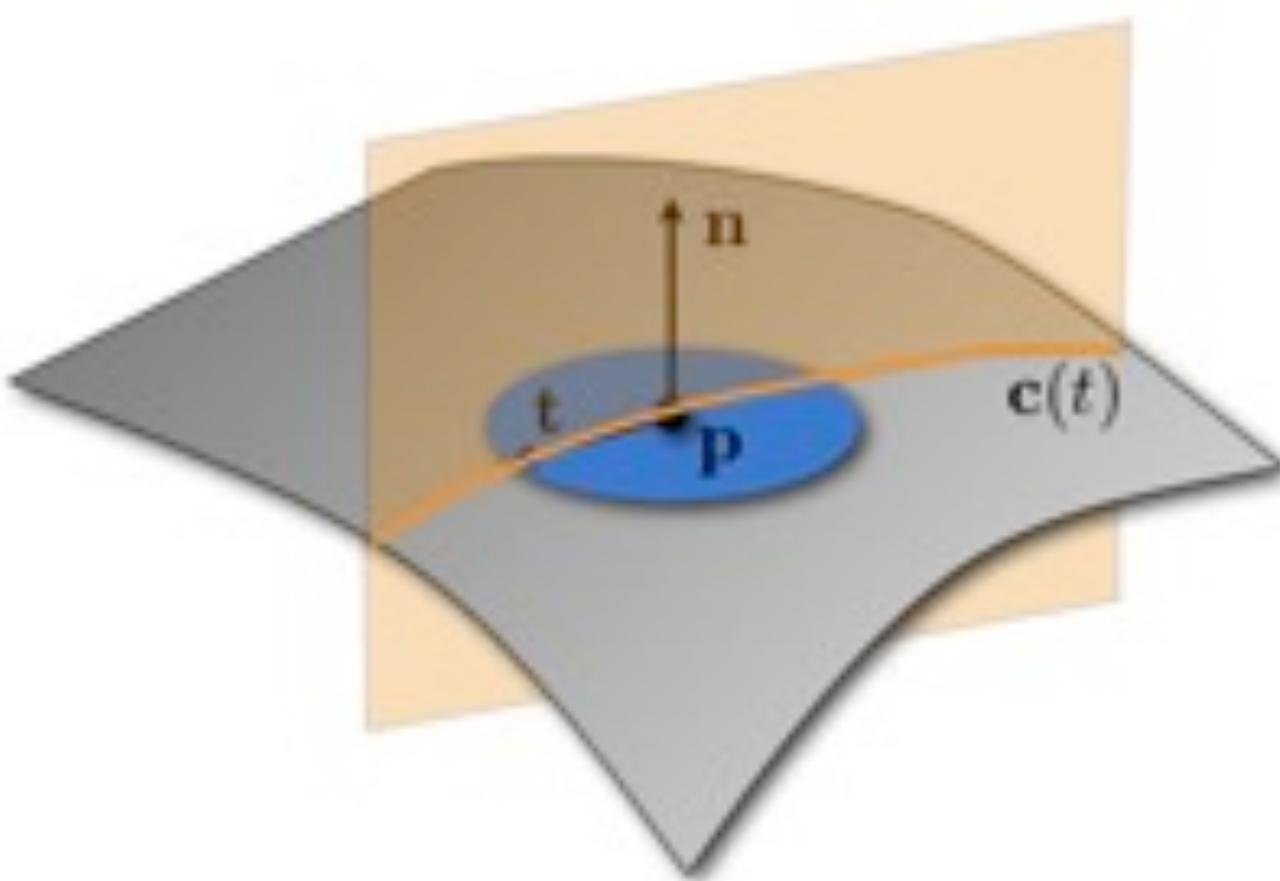




Differential Geometry



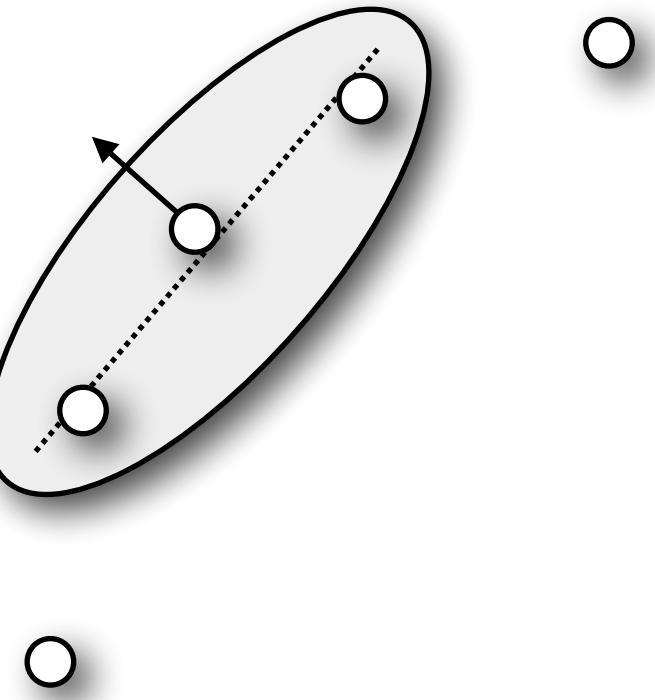
Last Time



- Explicit Reconstruction
 - Fast Marching Method
- Implicit Reconstruction
 - SDF from point clouds
 - (SDF from range scans)
 - Fast Marching Cubes

Normal Estimation

- Find normal \mathbf{n}_i for each sample point \mathbf{p}_i
 1. Examine local neighborhood for each point
 - Set of k nearest neighbors
 2. Compute best approximating tangent plane
 - Covariance analysis
 3. Determine normal orientation
 - MST propagation



Implicit Reconstruction

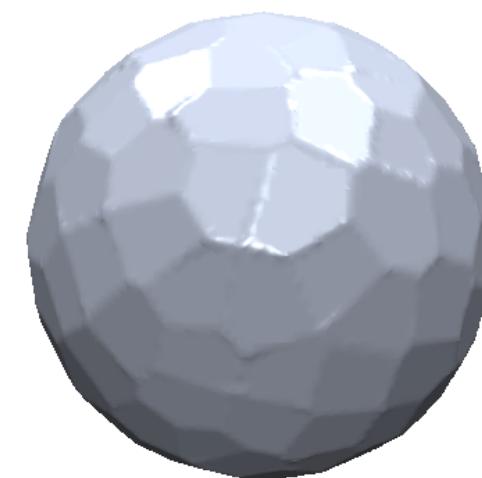
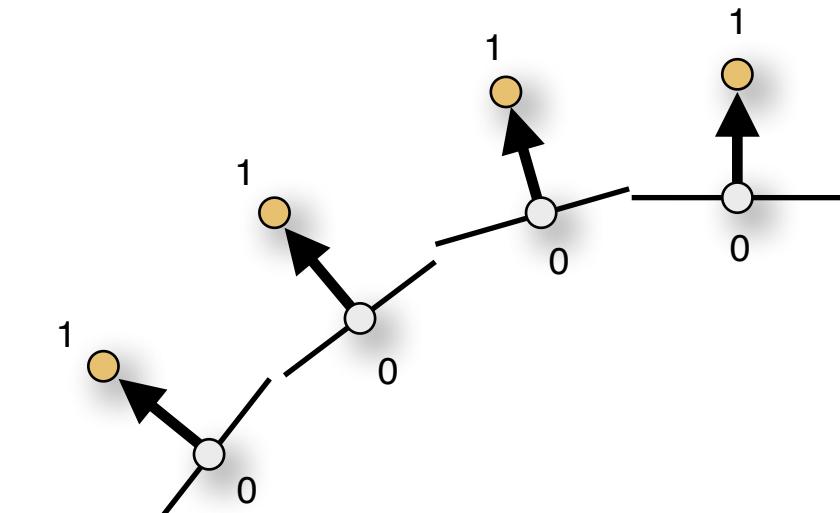
- Scattered data interpolation problem

- On-surface constraints $\text{dist}(\mathbf{p}_i) = 0$
- Off-surface constraints $\text{dist}(\mathbf{p}_i + \mathbf{n}_i) = 1$

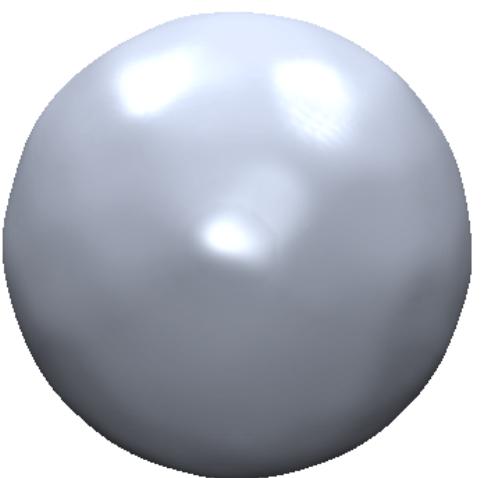
- Radial basis functions (RBFs)

$$\text{dist}(\mathbf{x}) = \sum_i w_i \cdot \varphi(\|\mathbf{x} - \mathbf{c}_i\|)$$

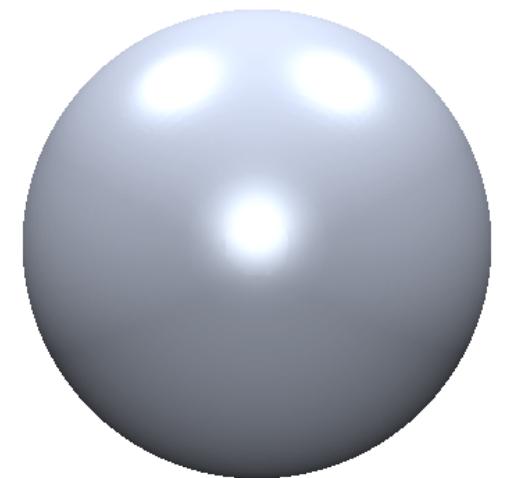
- Solve symmetric linear system for weights w_i



Hoppe '92



Compact RBF Wendland C^2

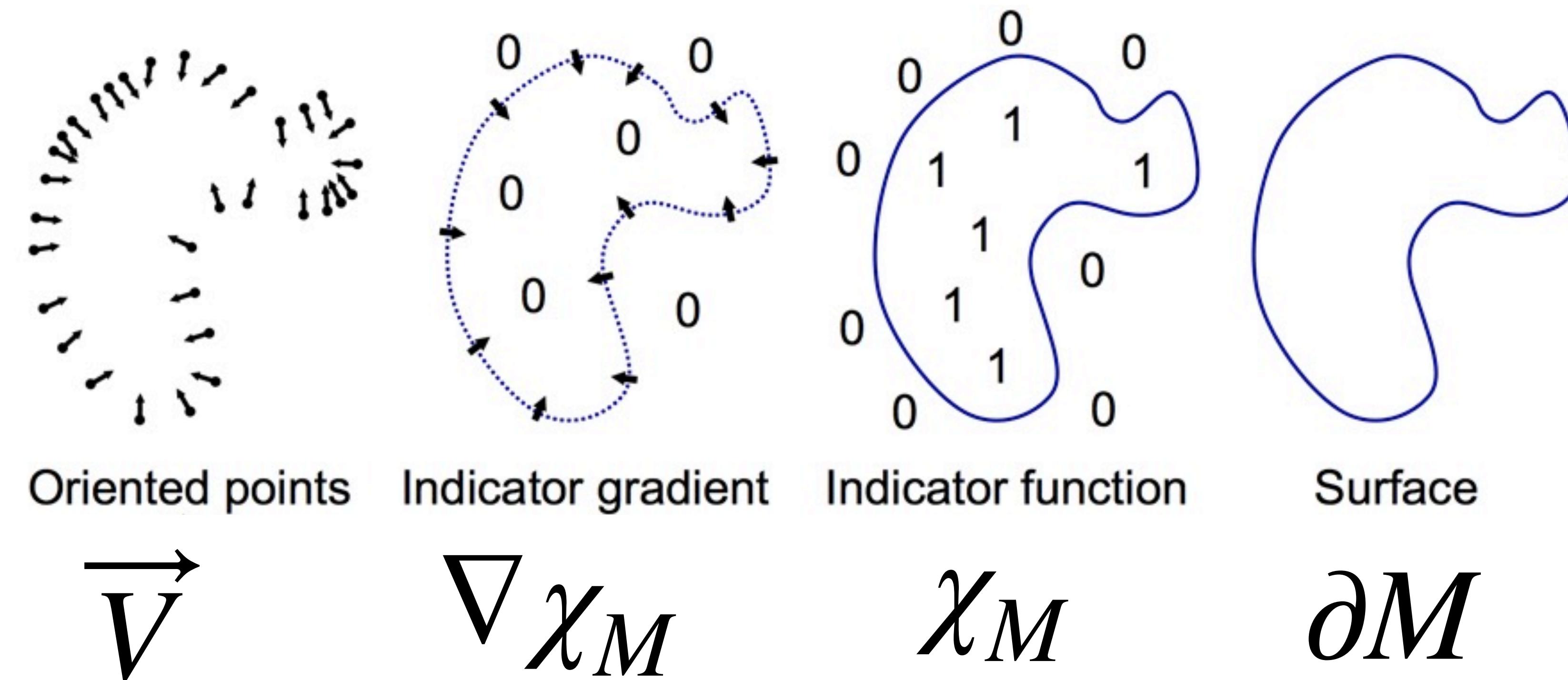


Global RBF Triharmonic

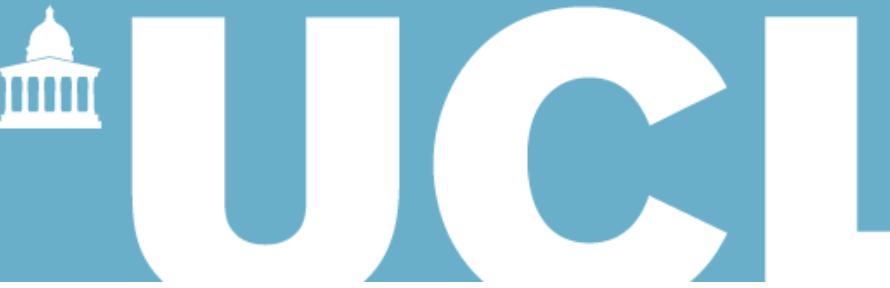
Poisson Surface Reconstruction



- relationship between normal field and gradient of indicator function



Outline



- **Differential Geometry**
- Discrete Differential Geometry
- Mesh Quality Measures

Motivation



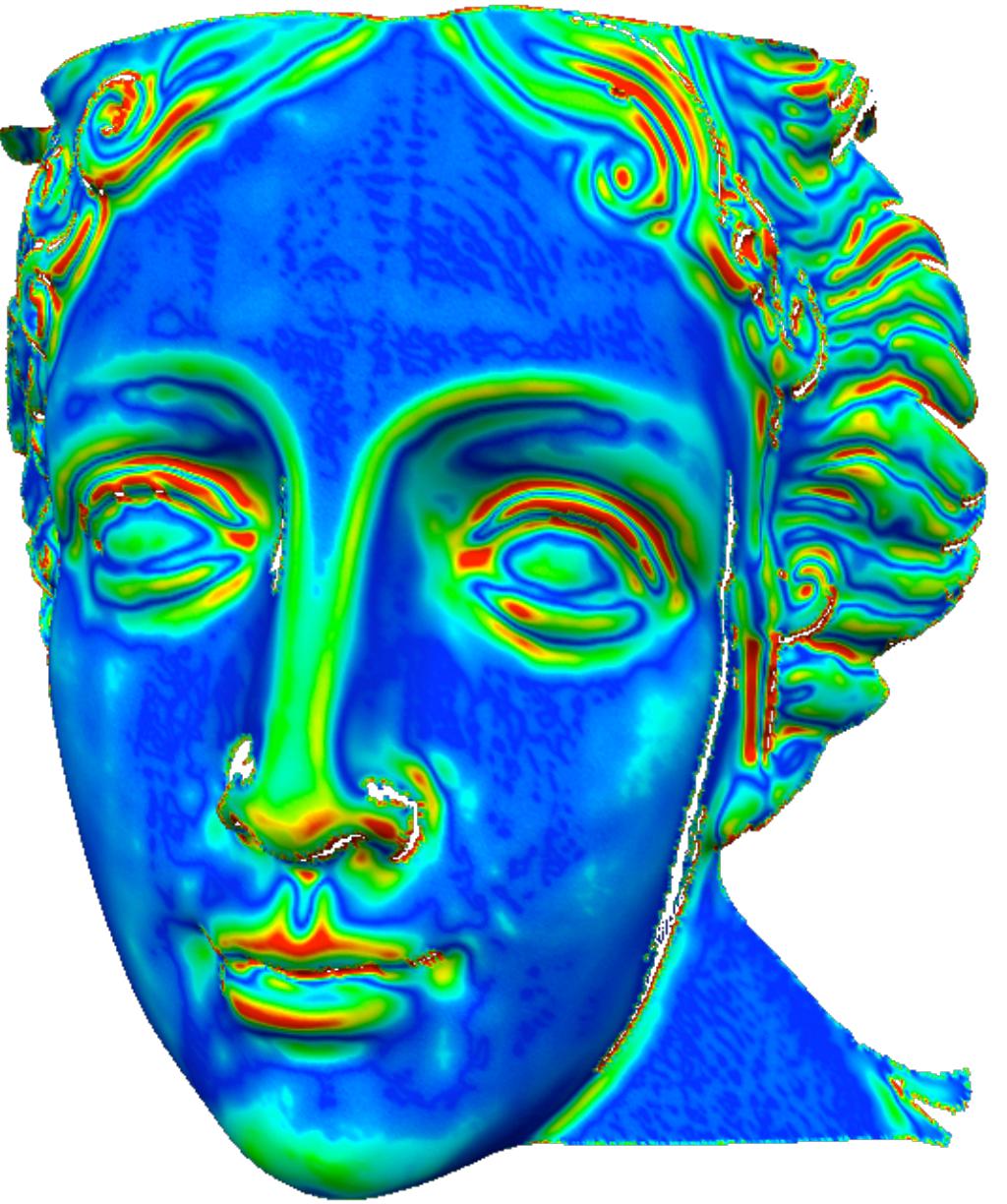
Motivation



- We need differential geometry to compute

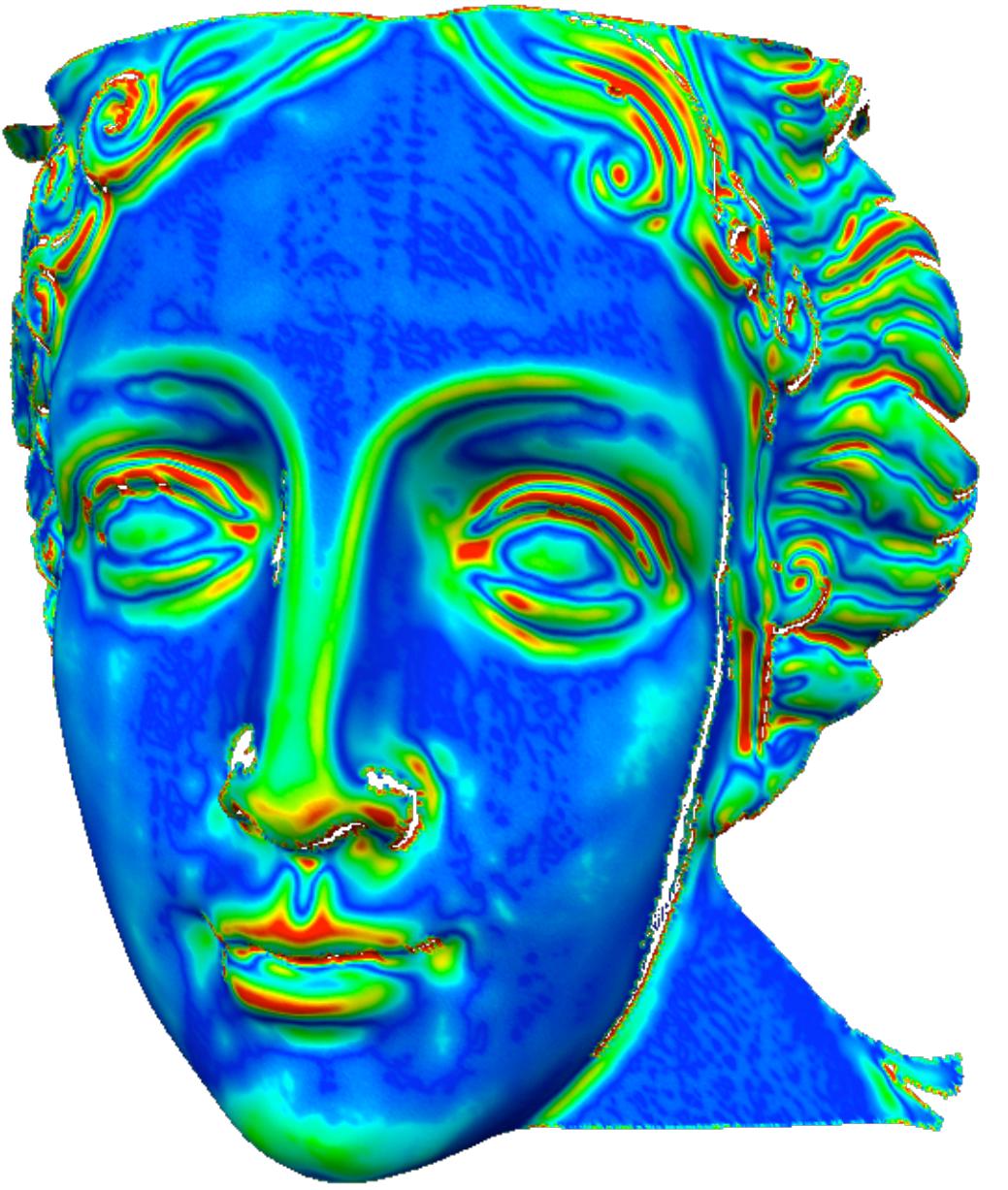
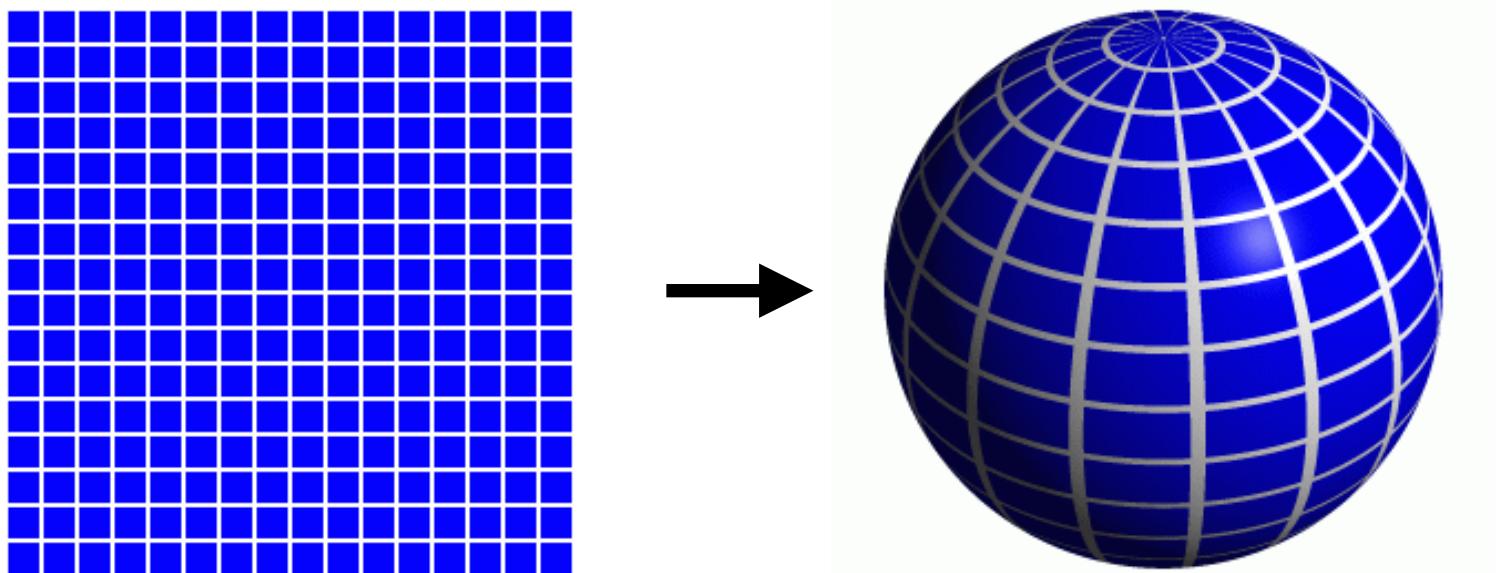
Motivation

- We need differential geometry to compute
 - surface curvature



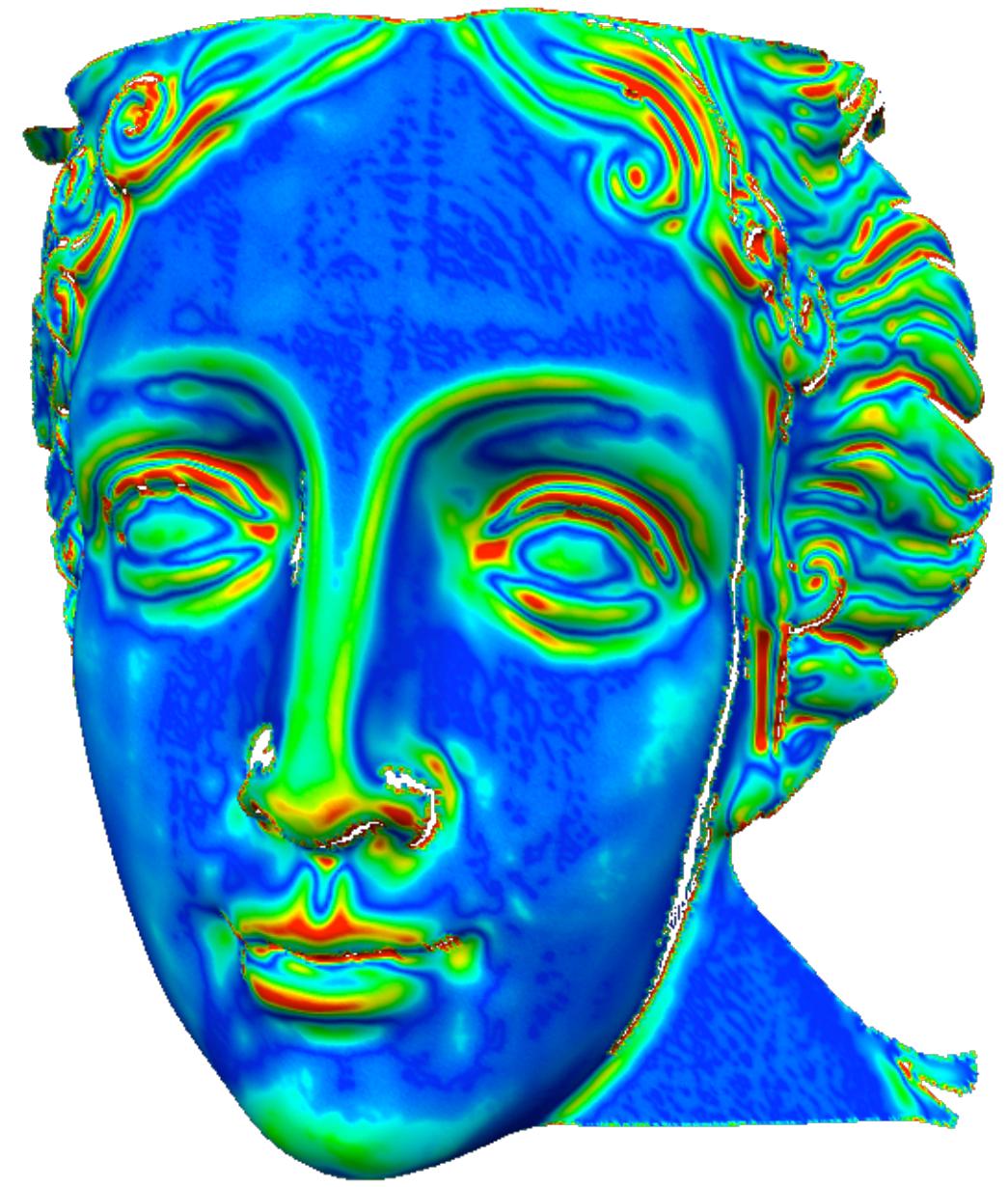
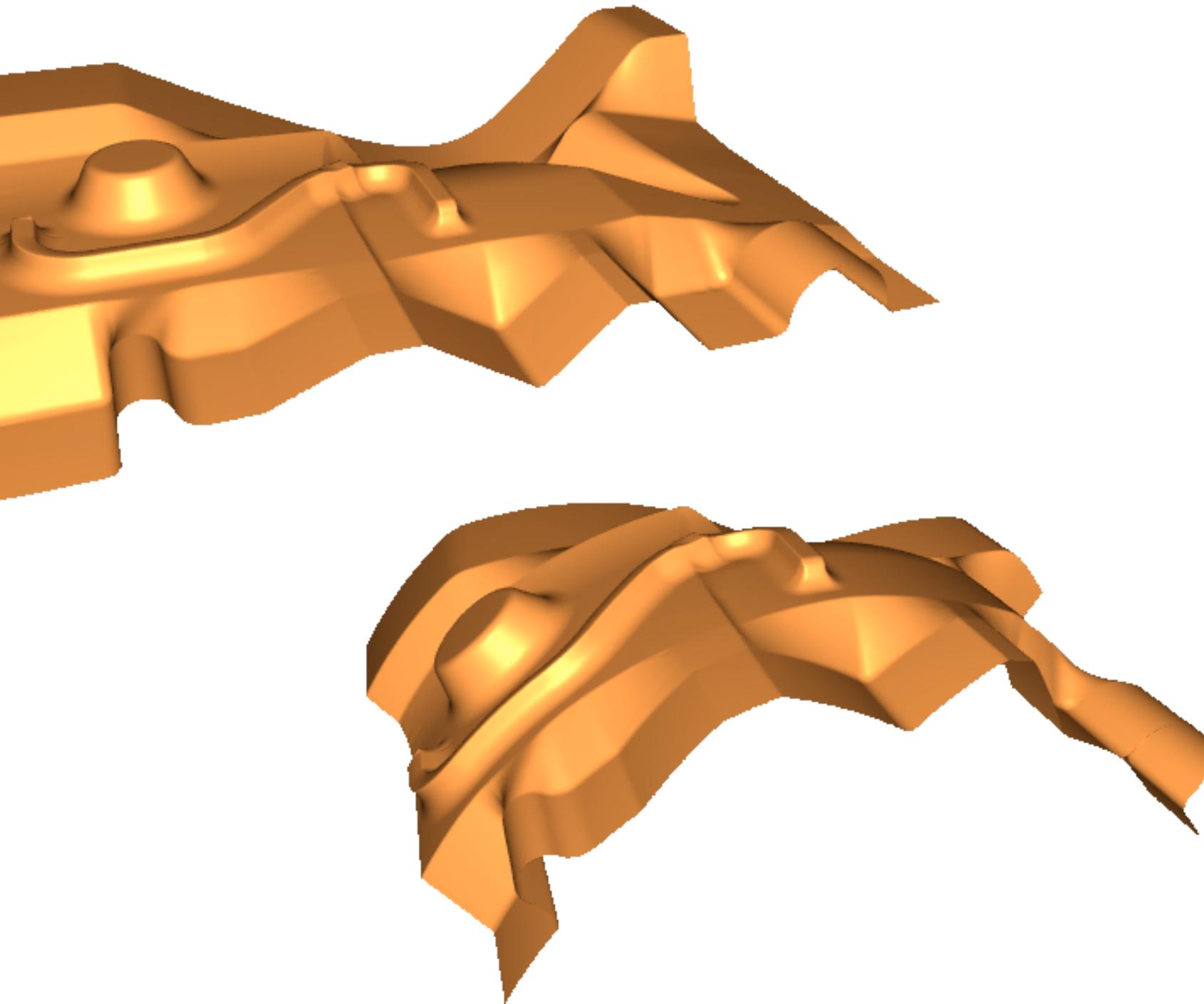
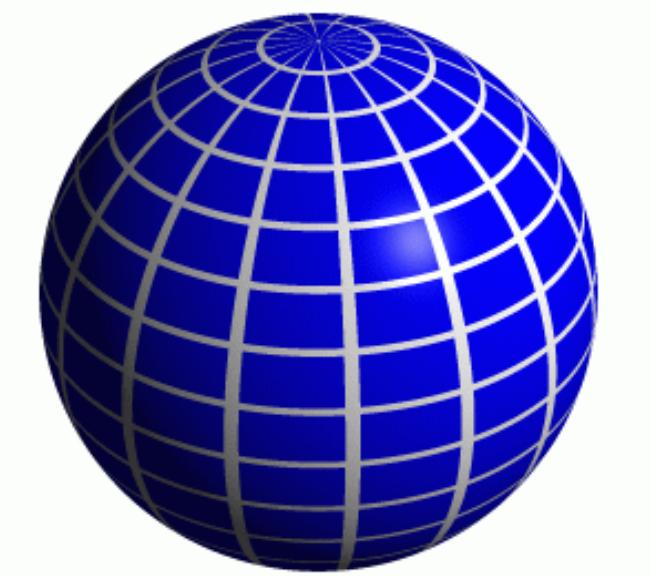
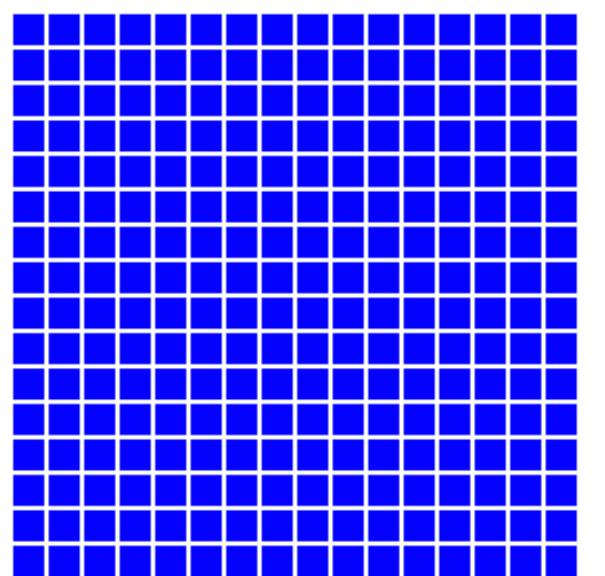
Motivation

- We need differential geometry to compute
 - surface curvature
 - parameterization distortion



Motivation

- We need differential geometry to compute
 - surface curvature
 - parameterization distortion
 - deformation energies



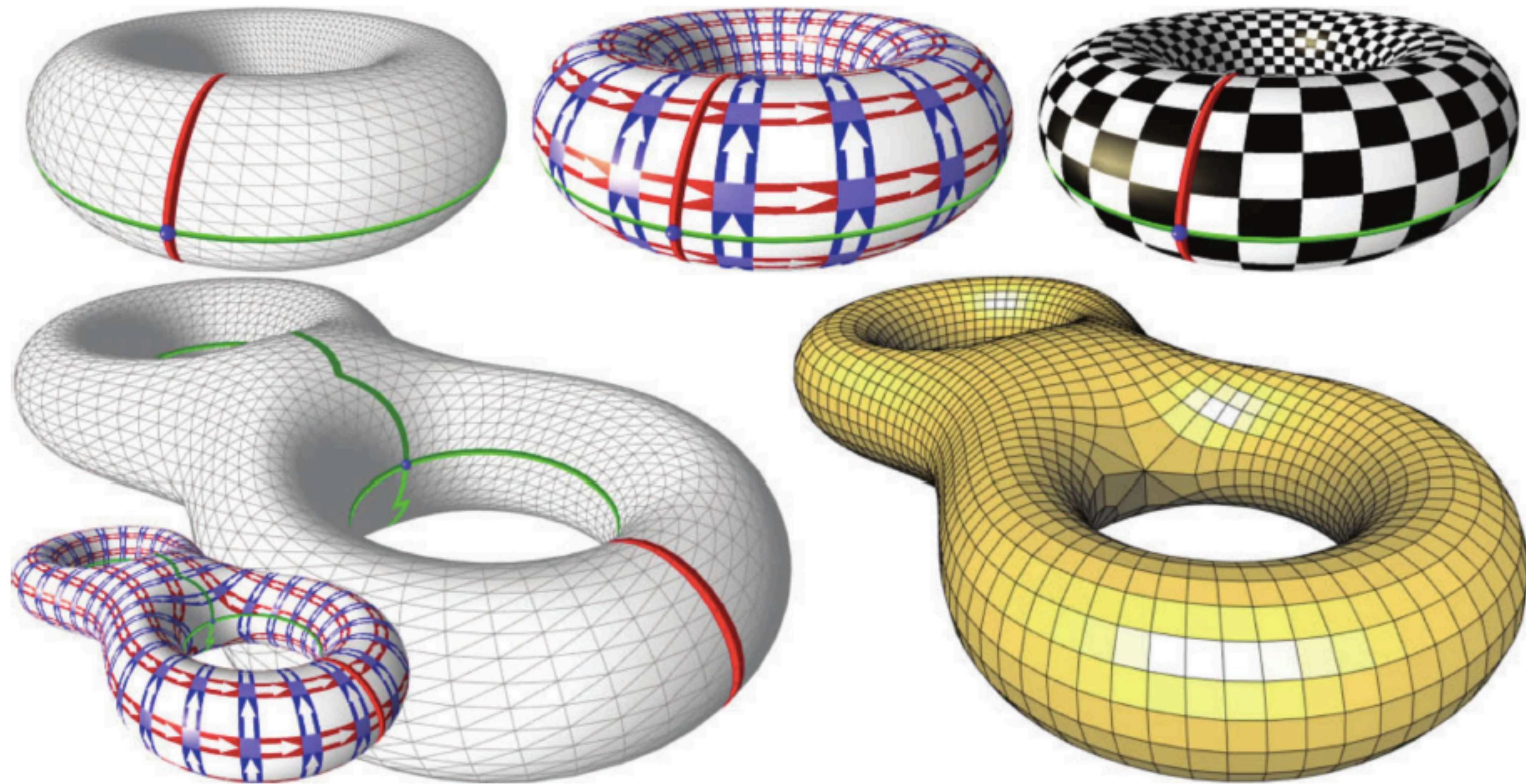
Texture Mapping



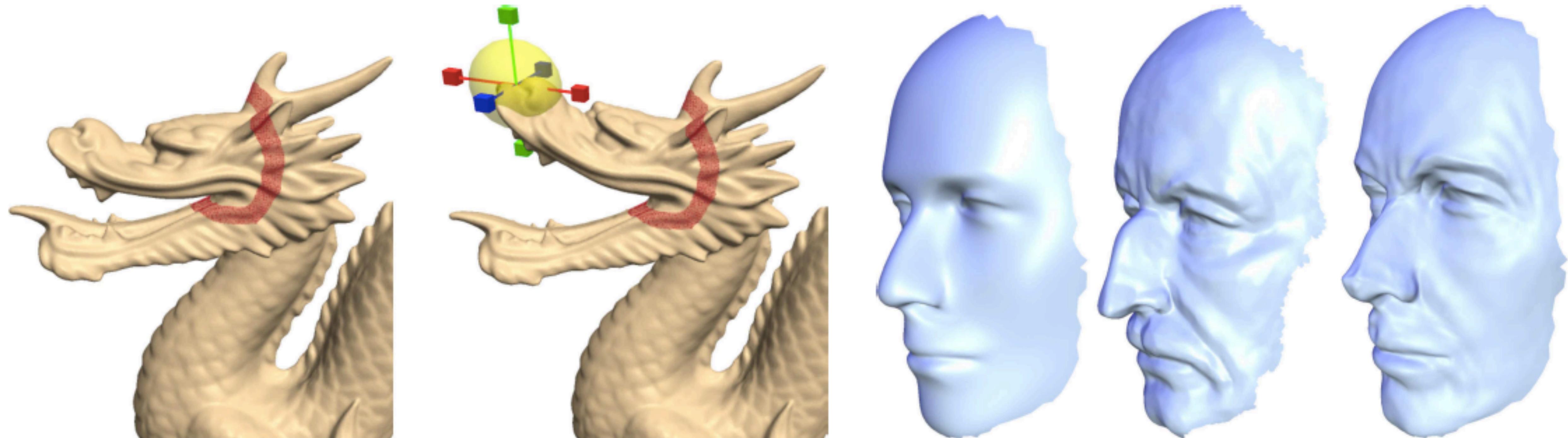
Texture Mapping



Parameterization + Meshing



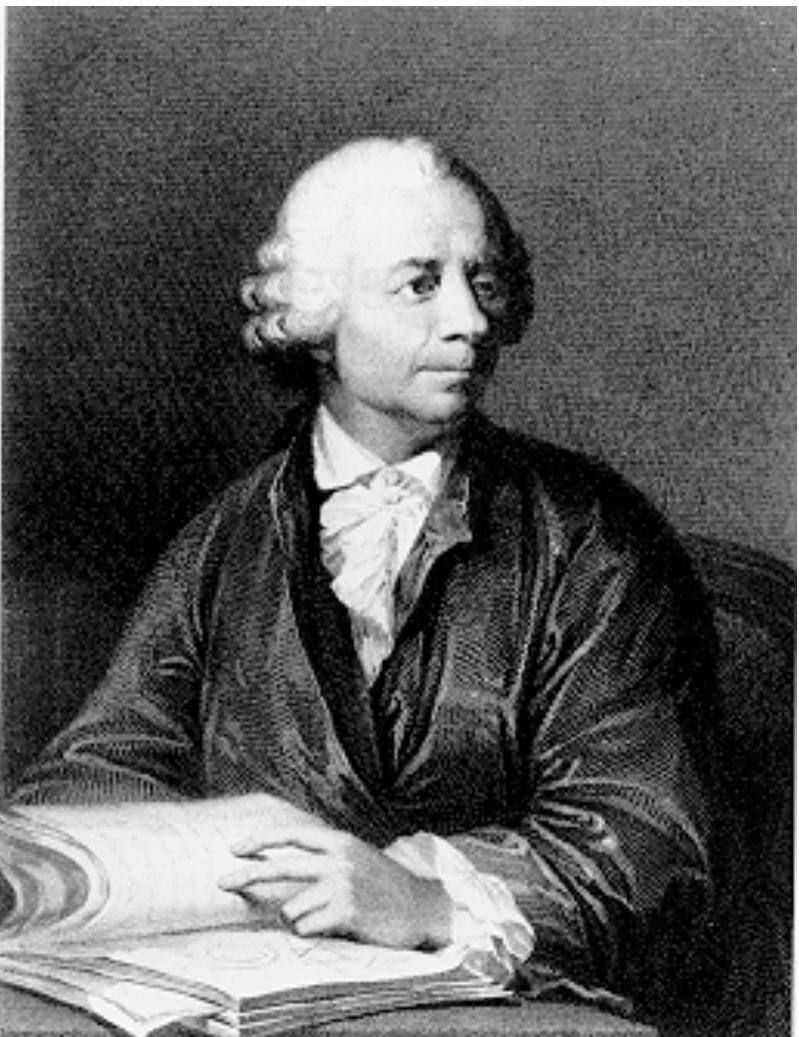
Surface Deformation



Differential Geometry



- Manfredo P. do Carmo: *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976
- Andrew Pressley: *Elementary Differential Geometry*, Springer, 2010



Leonard Euler (1707 - 1783)



Carl Friedrich Gauss (1777 - 1855)

Overview — Curves



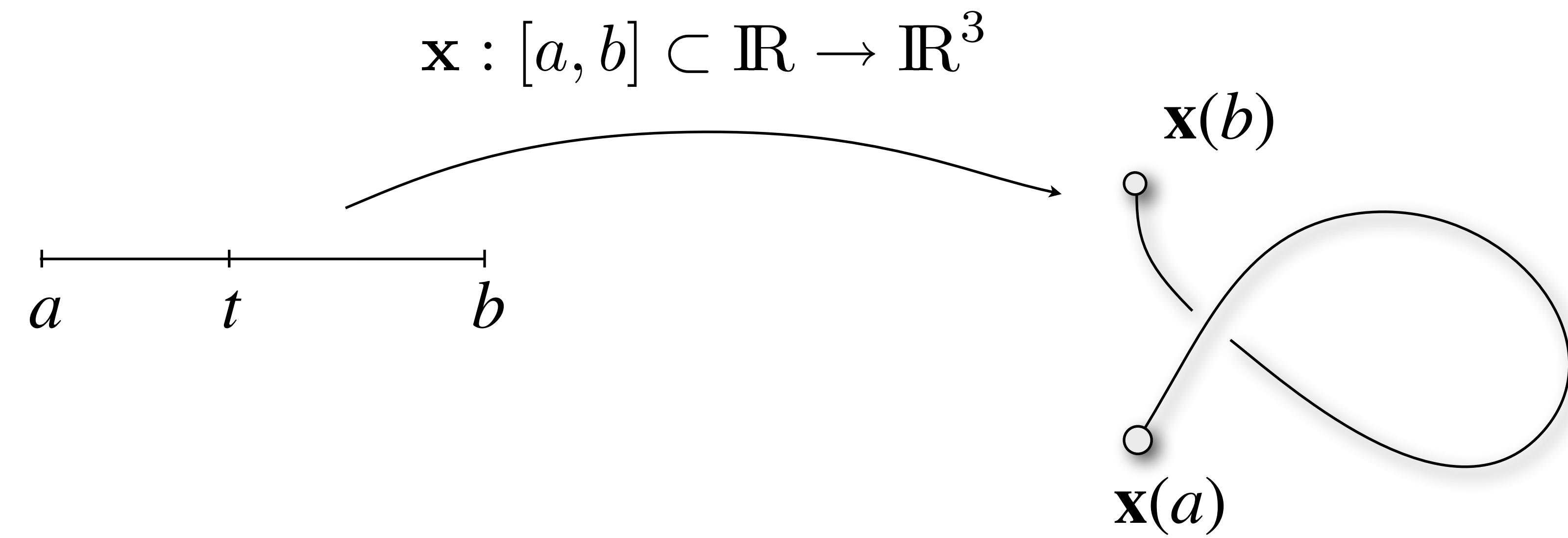
- **Parameterized** curves + **trace** of a curve
- **Ambiguity** due to parameterization
- Instantaneous **velocity** and **acceleration**
- constant **speed** versus constant **velocity**
- measuring curve length
- constant speed: velocity orthogonal to acceleration
- no self-intersection; regular parameterization

Overview (cont.)

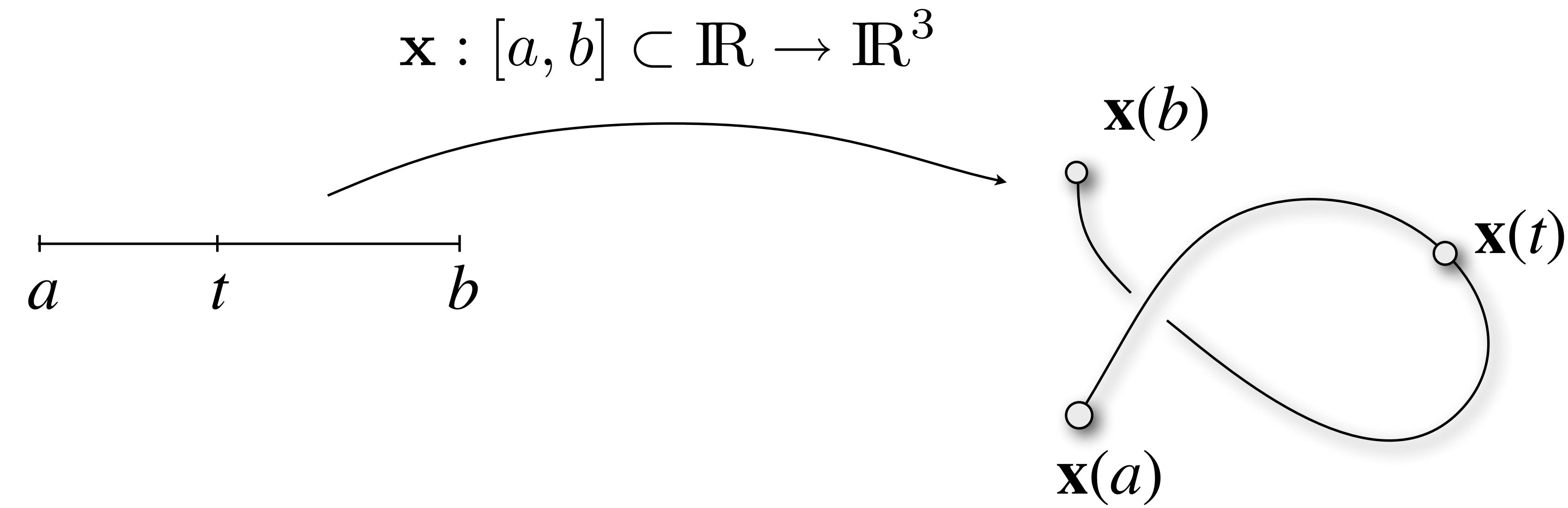


- radius of curvature
- Frenet frame; osculating plane
- tangent, normal, binormal
- — — — end curves

Parametric Curves



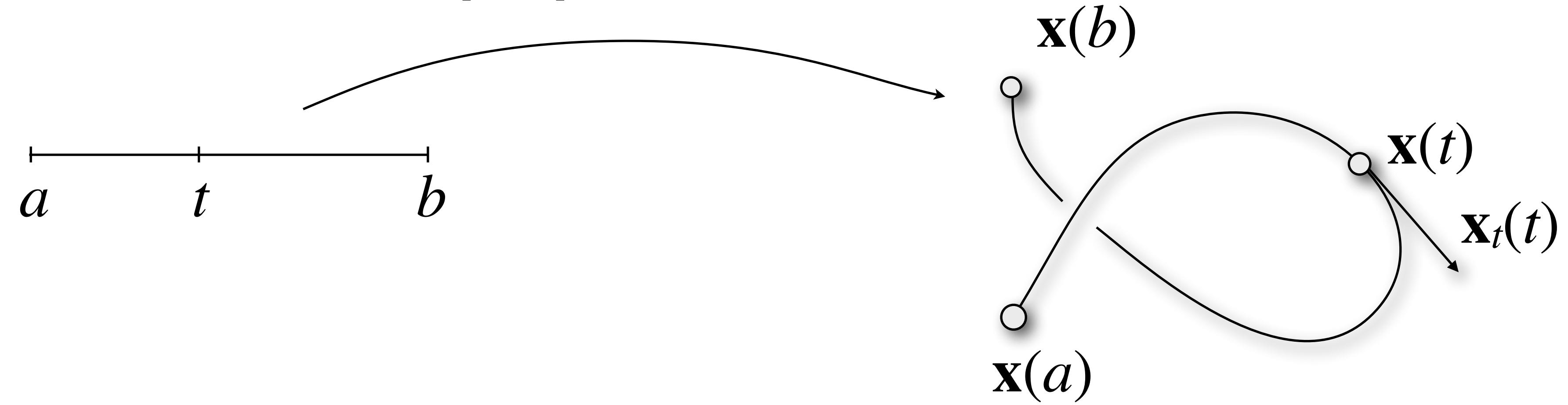
Parametric Curves



$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Parametric Curves

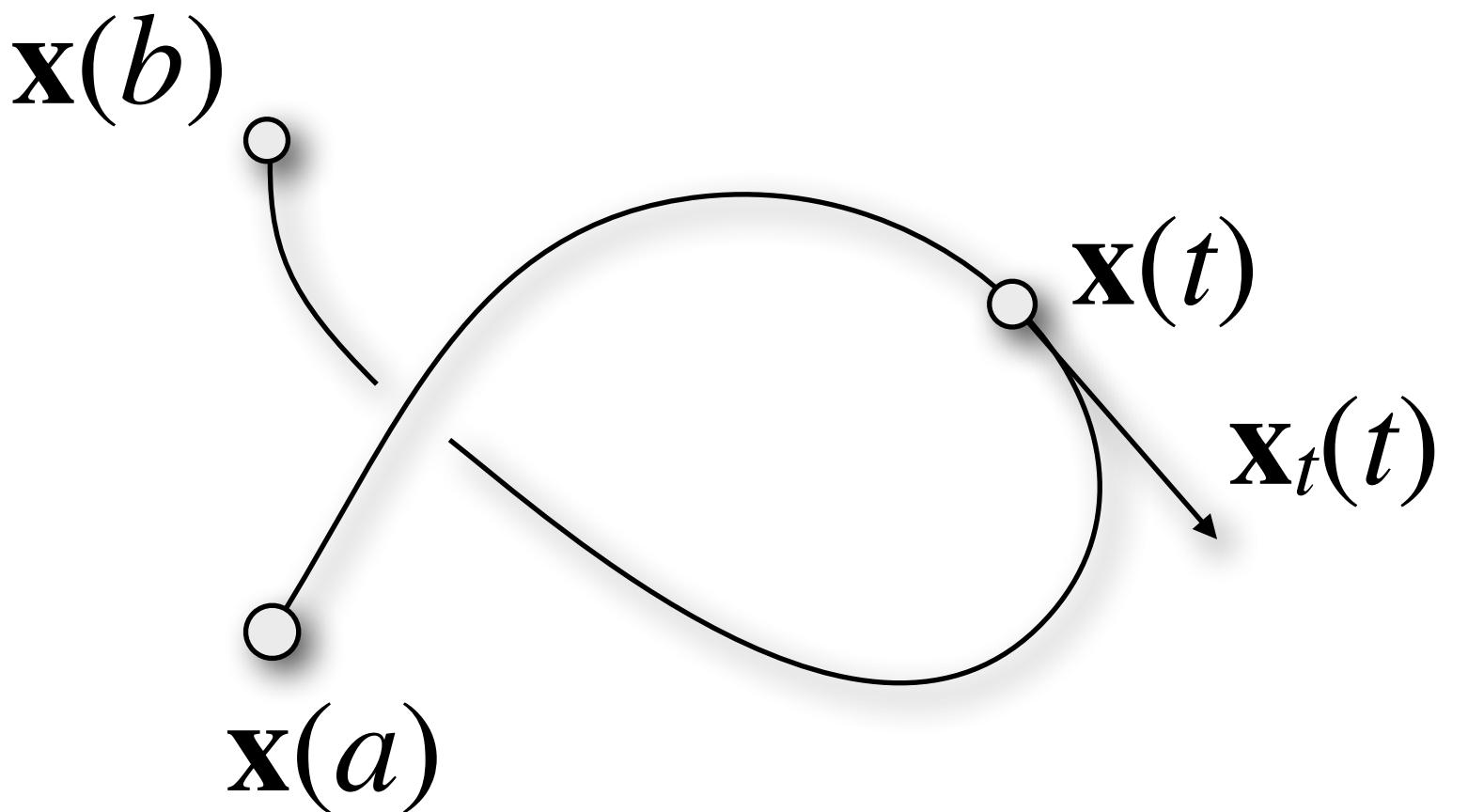
$$\mathbf{x} : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^3$$



$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \quad \mathbf{x}_t(t) := \frac{d\mathbf{x}(t)}{dt} = \begin{pmatrix} dx(t)/dt \\ dy(t)/dt \\ dz(t)/dt \end{pmatrix}$$

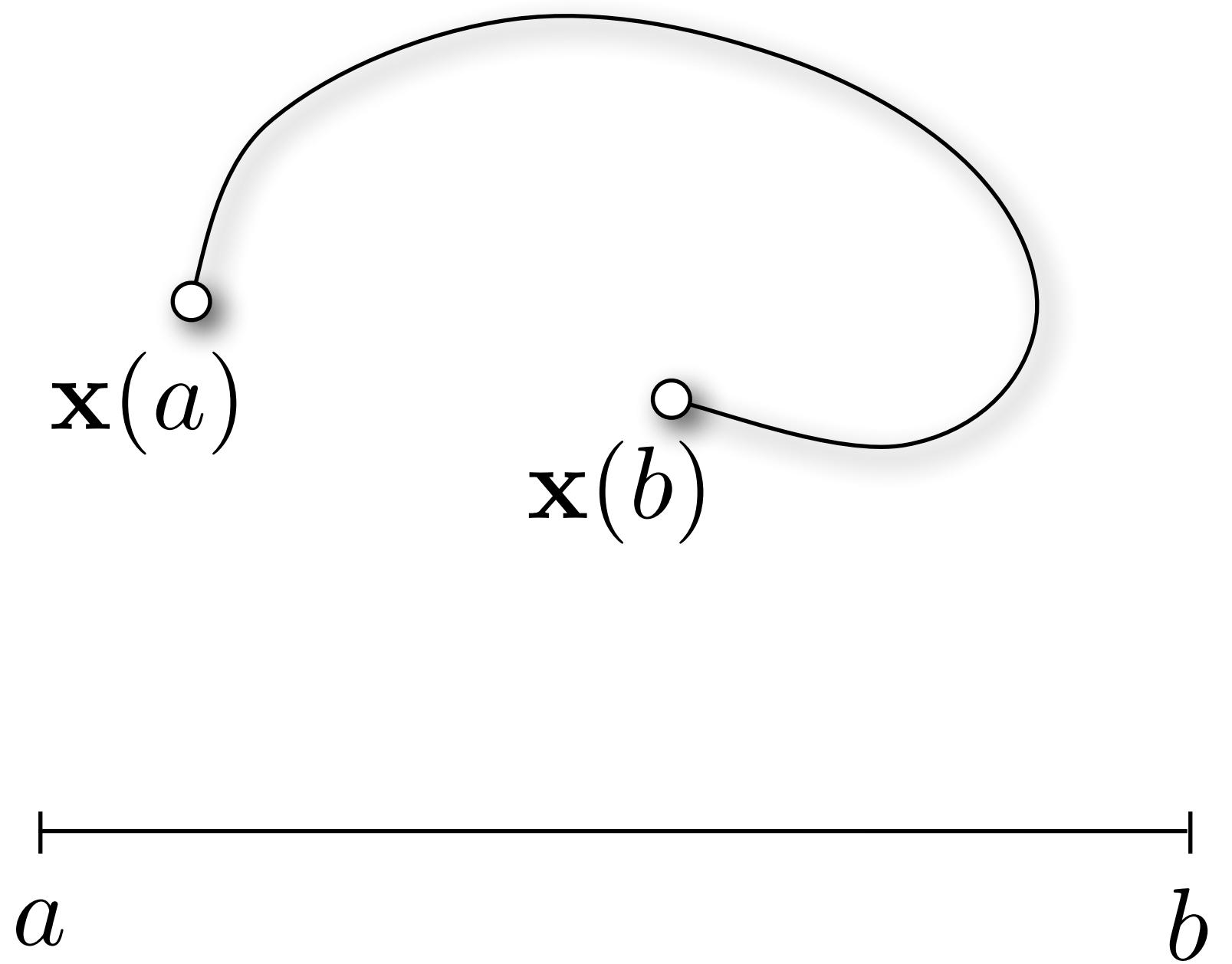
Parametric Curves

- A parametric curve $\mathbf{x}(t)$ is
 - *simple* $\mathbf{x}(t)$ is injective (no self-intersections)
 - *differentiable* $\mathbf{x}_t(t)$ is defined for all $t \in [a, b]$
 - *regular* $\mathbf{x}_t(t) \neq 0$ for all $t \in [a, b]$



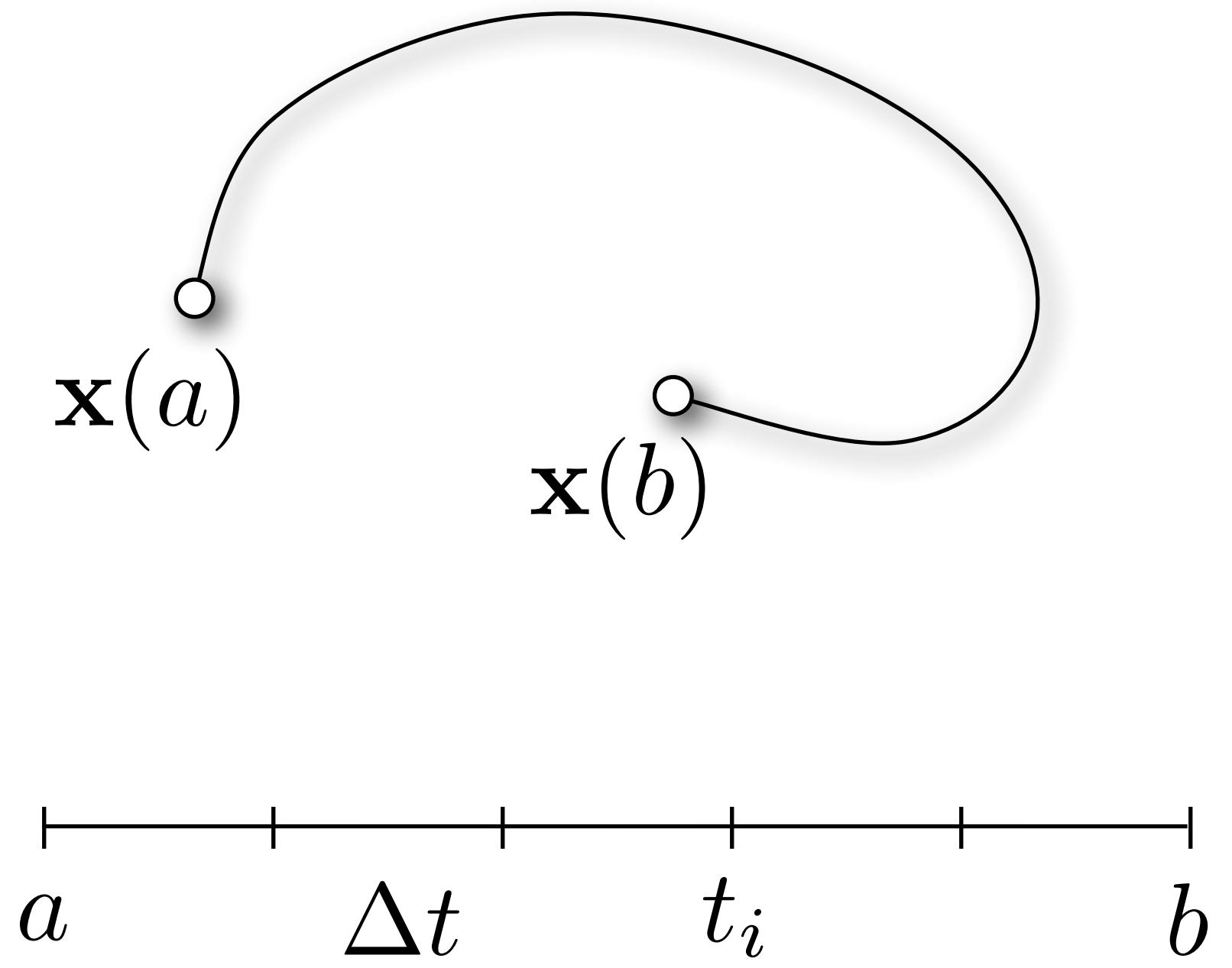
Length of a Curve

- Let $t_i = a + i\Delta t$ and $\mathbf{x}_i = \mathbf{x}(t_i)$



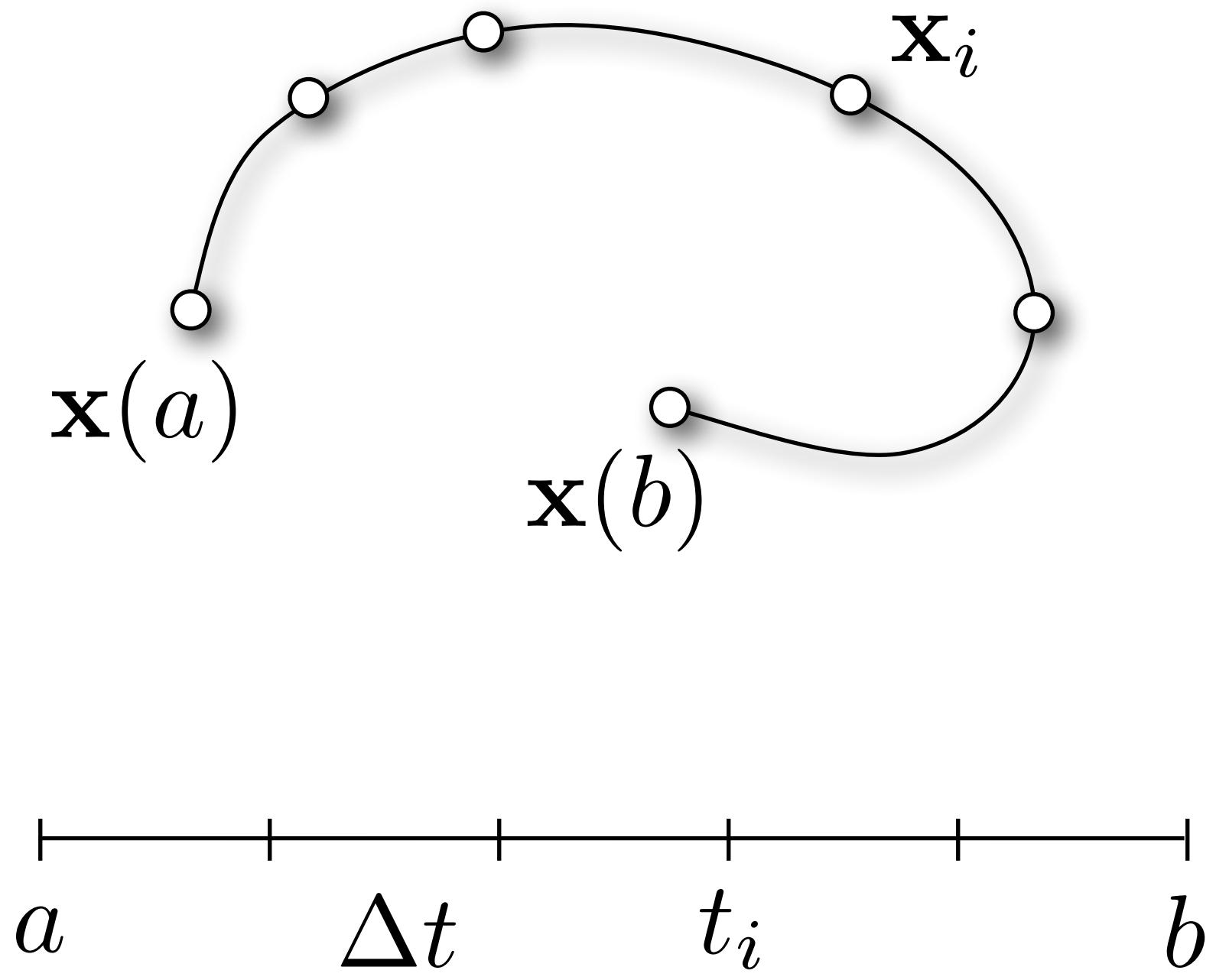
Length of a Curve

- Let $t_i = a + i\Delta t$ and $\mathbf{x}_i = \mathbf{x}(t_i)$



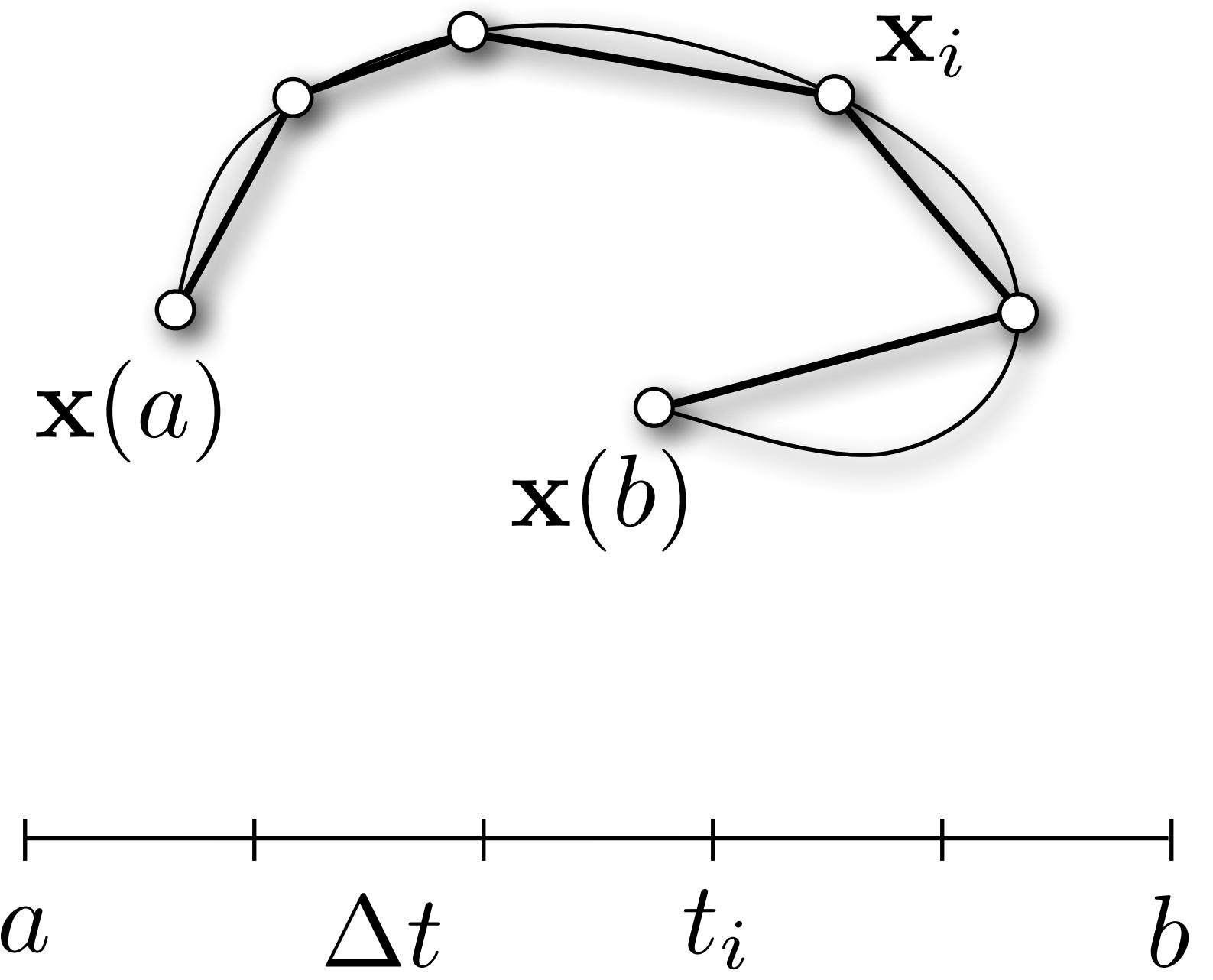
Length of a Curve

- Let $t_i = a + i\Delta t$ and $\mathbf{x}_i = \mathbf{x}(t_i)$



Length of a Curve

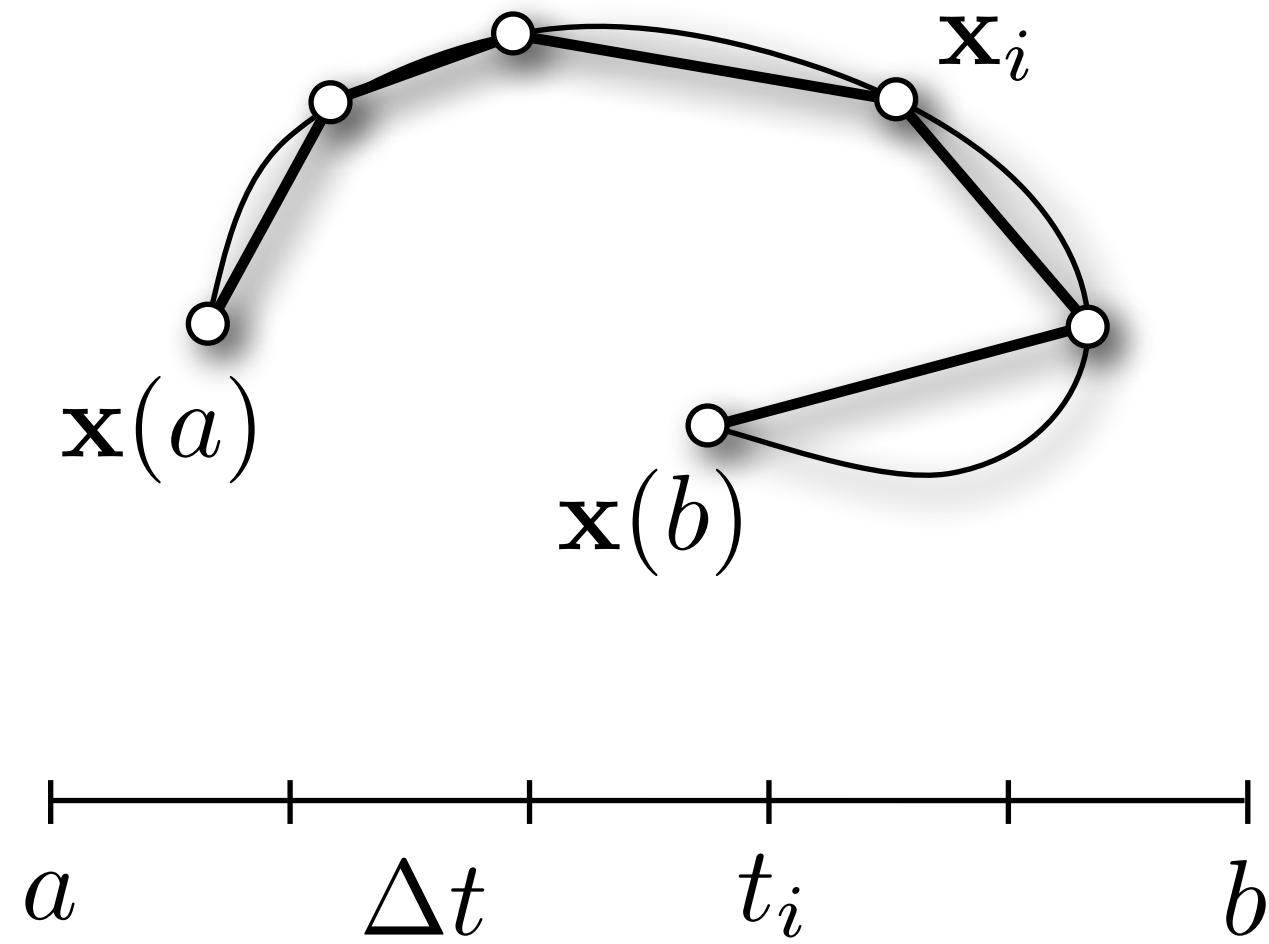
- Let $t_i = a + i\Delta t$ and $\mathbf{x}_i = \mathbf{x}(t_i)$



Length of a Curve

- Polyline *chord length*

$$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \left\| \frac{\Delta \mathbf{x}_i}{\Delta t} \right\| \Delta t, \quad \Delta \mathbf{x}_i := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$$



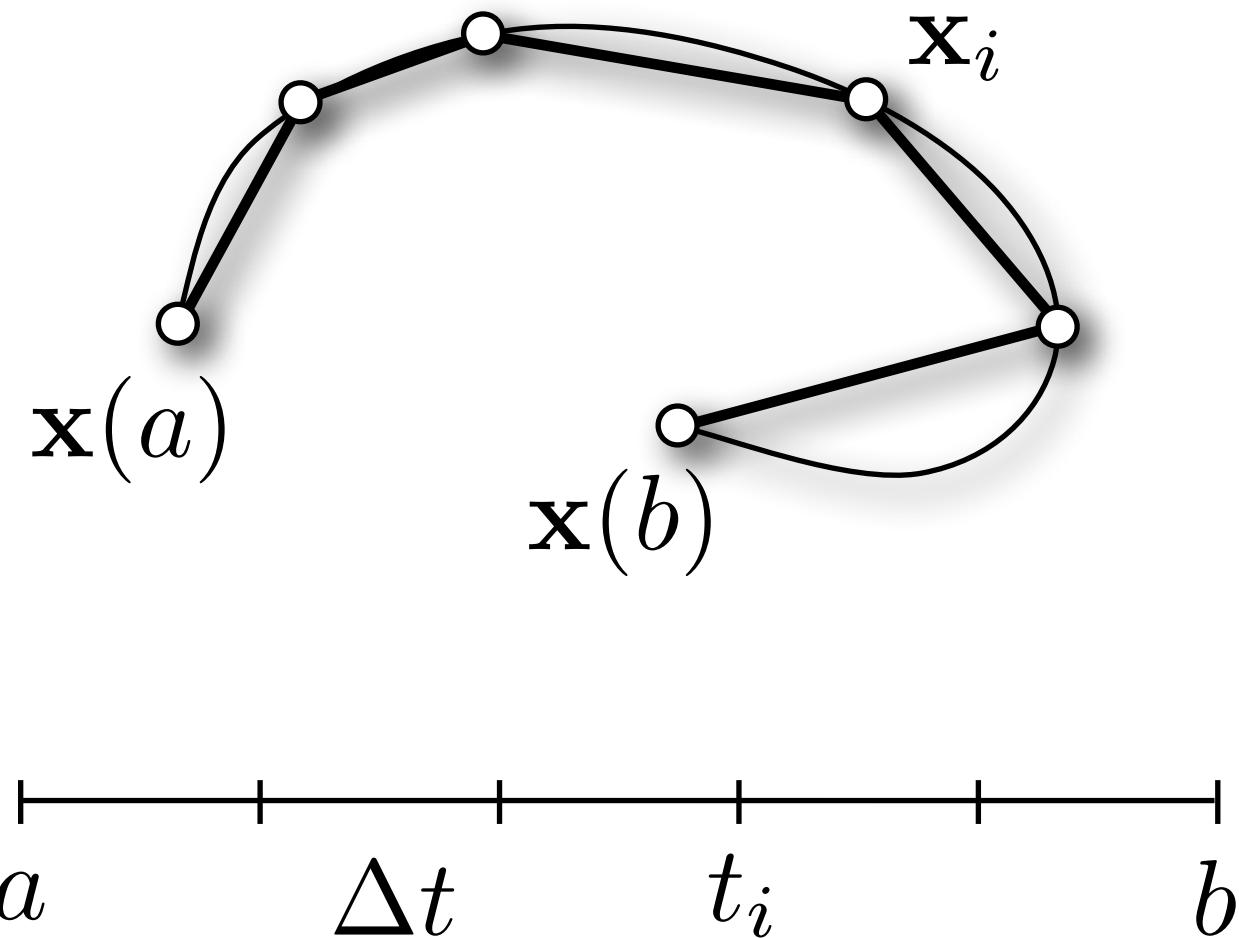
Length of a Curve

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- Curve *arc length* ($\Delta t \rightarrow 0$)

$$s = s(t) = \int_a^t \|\mathbf{x}_t\| dt$$



Re-Parameterization



Re-Parameterization



- Mapping of parameter domain

$$u : [a, b] \rightarrow [c, d]$$

Re-Parameterization



- Mapping of parameter domain

$$u : [a, b] \rightarrow [c, d]$$

- Re-parameterization w.r.t. $u(t)$

$$[c, d] \rightarrow \mathbb{R}^3, \quad t \mapsto \mathbf{x}(u(t))$$

Re-Parameterization



- Mapping of parameter domain

$$u : [a, b] \rightarrow [c, d]$$

- Re-parameterization w.r.t. $u(t)$

$$[c, d] \rightarrow \mathbb{R}^3, \quad t \mapsto \mathbf{x}(u(t))$$

- Derivative (chain rule)

$$\frac{d\mathbf{x}(u(t))}{dt} = \frac{d\mathbf{x}}{du} \frac{du}{dt} = \mathbf{x}_u(u(t)) u_t(t)$$

Re-Parameterization



- Example

$$\mathbf{f} : \left[0, \frac{1}{2}\right] \rightarrow \mathbb{R}^2 , \quad t \mapsto (4t, 2t)$$

$$\phi : \left[0, \frac{1}{2}\right] \rightarrow [0, 1] , \quad t \mapsto 2t$$

$$\mathbf{g} : [0, 1] \rightarrow \mathbb{R}^2 , \quad t \mapsto (2t, t)$$

Re-Parameterization



- Example

$$\mathbf{f} : \left[0, \frac{1}{2}\right] \rightarrow \mathbb{R}^2 , \quad t \mapsto (4t, 2t)$$

$$\phi : \left[0, \frac{1}{2}\right] \rightarrow [0, 1] , \quad t \mapsto 2t$$

$$\mathbf{g} : [0, 1] \rightarrow \mathbb{R}^2 , \quad t \mapsto (2t, t)$$

$$\Rightarrow \mathbf{g}(\phi(t)) = \mathbf{f}(t)$$

Arc Length Parameterization



Arc Length Parameterization



- Mapping of parameter domain:

$$t \mapsto s(t) = \int_a^t \|\mathbf{x}_t\| dt$$

Arc Length Parameterization



- Mapping of parameter domain:

$$t \mapsto s(t) = \int_a^t \|\mathbf{x}_t\| dt$$

- Parameter s for $\mathbf{x}(s)$ equals length from $\mathbf{x}(a)$ to $\mathbf{x}(s)$

$$\mathbf{x}(s) = \mathbf{x}(s(t)) \quad ds = \|\mathbf{x}_t\| dt$$

Arc Length Parameterization



- Mapping of parameter domain:

$$t \mapsto s(t) = \int_a^t \|\mathbf{x}_t\| dt$$

- Parameter s for $\mathbf{x}(s)$ equals length from $\mathbf{x}(a)$ to $\mathbf{x}(s)$

$$\mathbf{x}(s) = \mathbf{x}(s(t)) \quad ds = \|\mathbf{x}_t\| dt$$

- Special properties of resulting curve

$$\|\mathbf{x}_s(s)\| = 1, \quad \mathbf{x}_s(s) \cdot \mathbf{x}_{ss}(s) = 0$$

The Frenet Frame



- Taylor expansion

$$\mathbf{x}(t + h) = \mathbf{x}(t) + \mathbf{x}_t(t) h + \frac{1}{2}\mathbf{x}_{tt}(t) h^2 + \frac{1}{6}\mathbf{x}_{ttt}(t) h^3 + \dots$$

- Define local frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ (*Frenet frame*)

$$\mathbf{t} = \frac{\mathbf{x}_t}{\|\mathbf{x}_t\|}$$

tangent

$$\mathbf{n} = \mathbf{b} \times \mathbf{t}$$

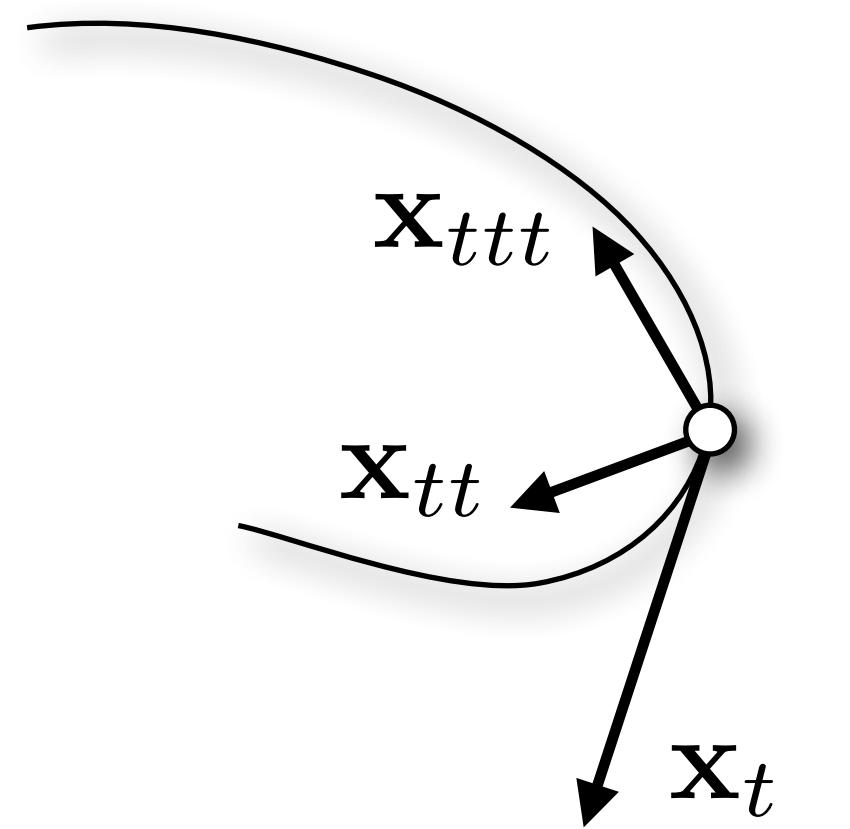
principal normal

$$\mathbf{b} = \frac{\mathbf{x}_t \times \mathbf{x}_{tt}}{\|\mathbf{x}_t \times \mathbf{x}_{tt}\|}$$

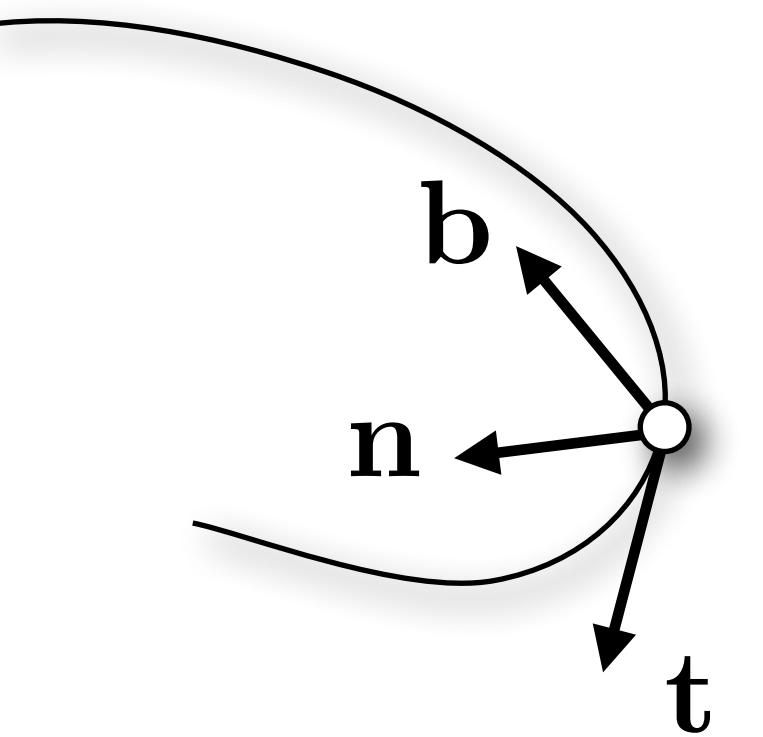
binormal

The Frenet Frame

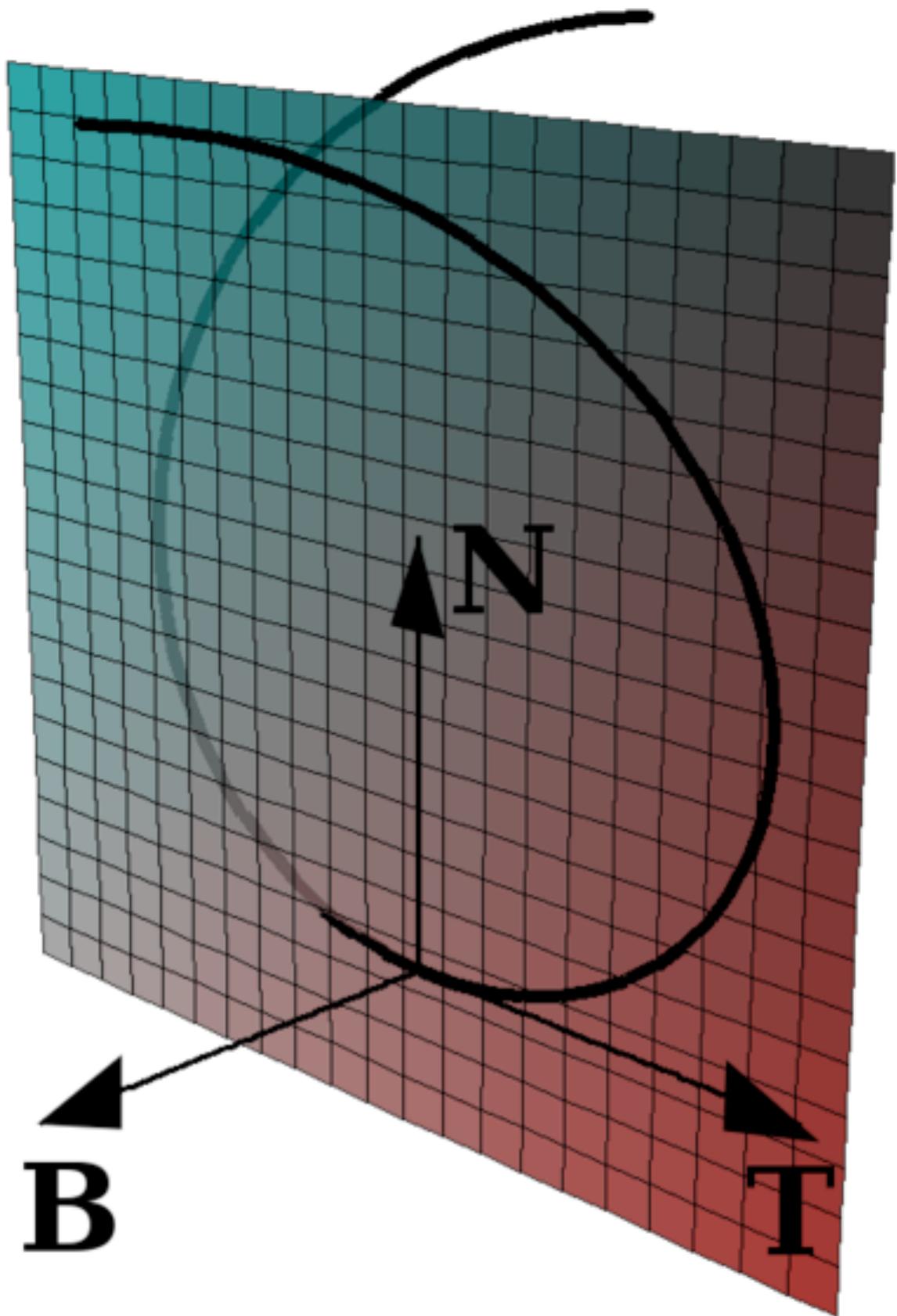
- Orthonormalization of local frame



local affine frame



Frenet frame



The Frenet Frame



The Frenet Frame



- *Frenet-Serret*: Derivatives w.r.t. arc length s

$$\mathbf{t}_s = +\kappa \mathbf{n}$$

$$\mathbf{n}_s = -\kappa \mathbf{t} + \tau \mathbf{b}$$

$$\mathbf{b}_s = -\tau \mathbf{n}$$

The Frenet Frame



- *Frenet-Serret*: Derivatives w.r.t. arc length s

$$\begin{aligned}\mathbf{t}_s &= +\kappa \mathbf{n} \\ \mathbf{n}_s &= -\kappa \mathbf{t} \quad +\tau \mathbf{b} \\ \mathbf{b}_s &= -\tau \mathbf{n}\end{aligned}$$

- Curvature (*deviation from straight line*)

$$\kappa = \|\mathbf{x}_{ss}\|$$

The Frenet Frame



- *Frenet-Serret:* Derivatives w.r.t. arc length s

$$\begin{aligned}\mathbf{t}_s &= +\kappa \mathbf{n} \\ \mathbf{n}_s &= -\kappa \mathbf{t} \quad +\tau \mathbf{b} \\ \mathbf{b}_s &= -\tau \mathbf{n}\end{aligned}$$

- Curvature (*deviation from straight line*)

$$\kappa = \|\mathbf{x}_{ss}\|$$

- Torsion (*deviation from planarity*)

$$\tau = \frac{1}{\kappa^2} \det([\mathbf{x}_s, \mathbf{x}_{ss}, \mathbf{x}_{sss}])$$

Curvature and Torsion



- Planes defined by x and two vectors:
 - *osculating plane*: vectors t and n
 - *normal plane*: vectors n and b
 - *rectifying plane*: vectors t and b
- Osculating circle
 - second order contact with curve
 - center $c = x + (1/\kappa)n$
 - radius $1/\kappa$

Curvature and Torsion



Curvature and Torsion



- Curvature: Deviation from straight line

Curvature and Torsion



- Curvature: Deviation from straight line
- Torsion: Deviation from planarity

Curvature and Torsion



- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization

Curvature and Torsion



- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization
 - intrinsic properties of the curve

Curvature and Torsion



- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization
 - intrinsic properties of the curve
- Euclidean invariants

Curvature and Torsion



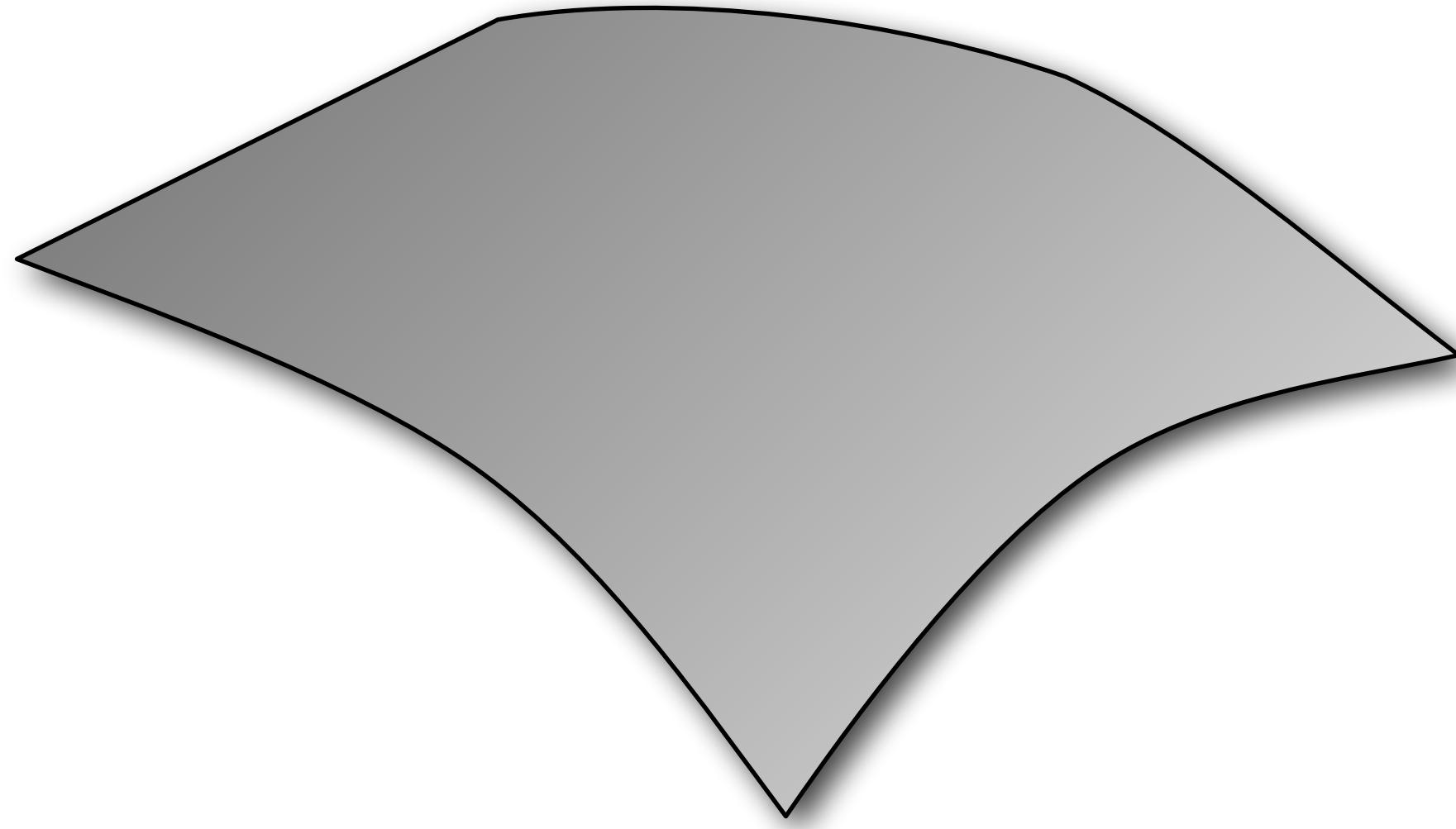
- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization
 - intrinsic properties of the curve
- Euclidean invariants
 - invariant under rigid motion

Curvature and Torsion



- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization
 - intrinsic properties of the curve
- Euclidean invariants
 - invariant under rigid motion
- Define curve uniquely up to rigid motion

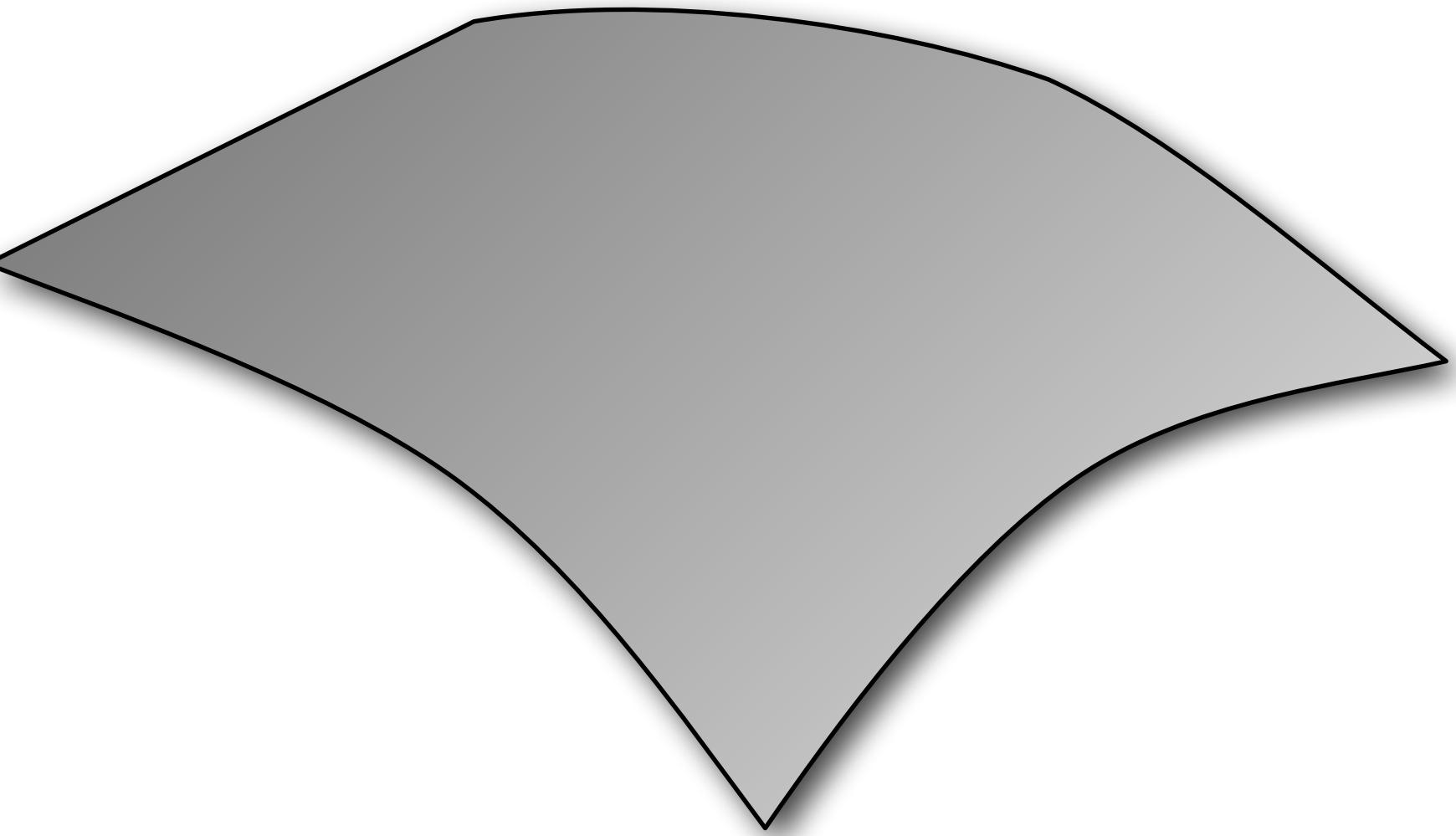
Parametric Surfaces



Parametric Surfaces

- Continuous surface

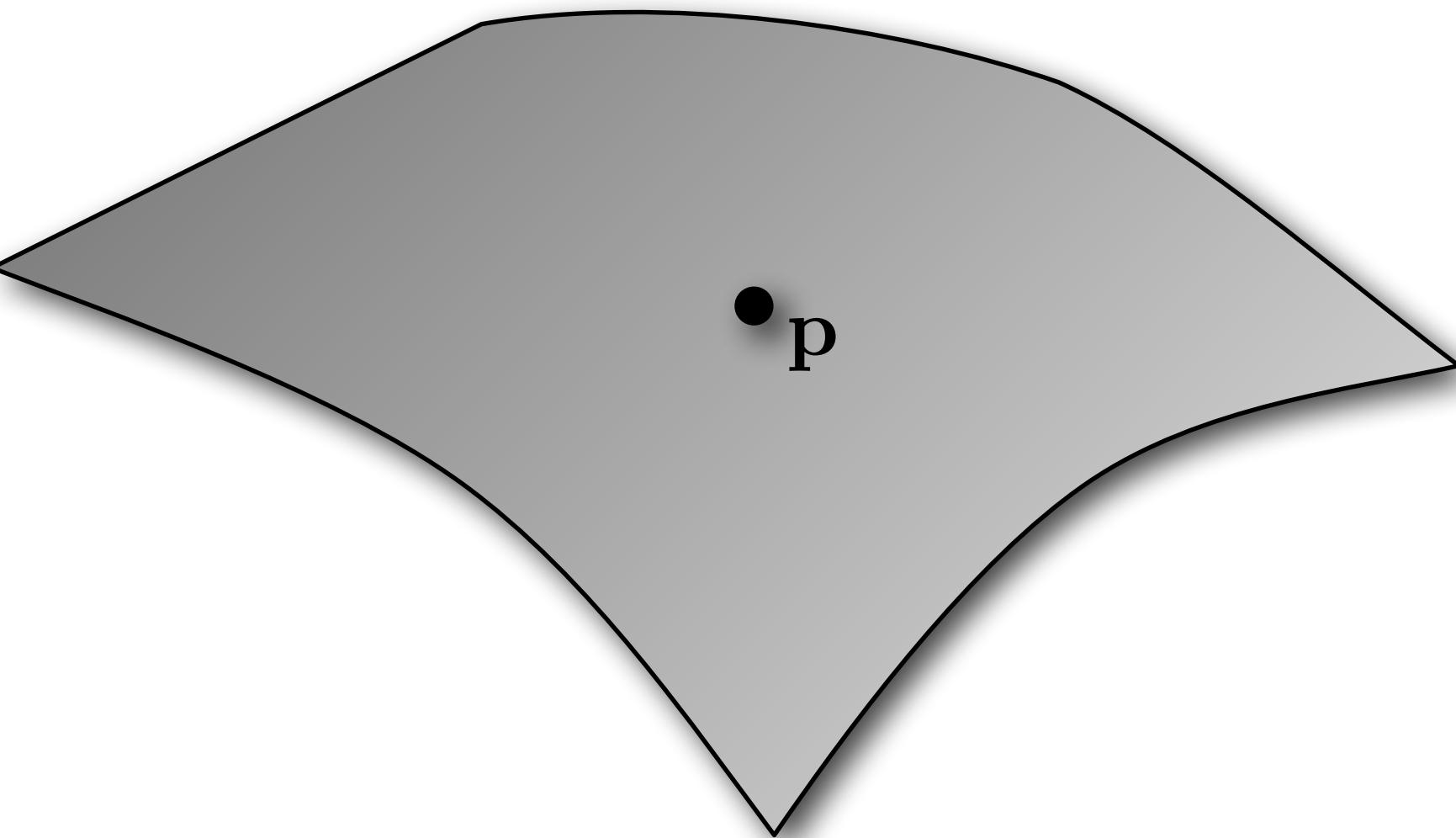
$$\mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$



Parametric Surfaces

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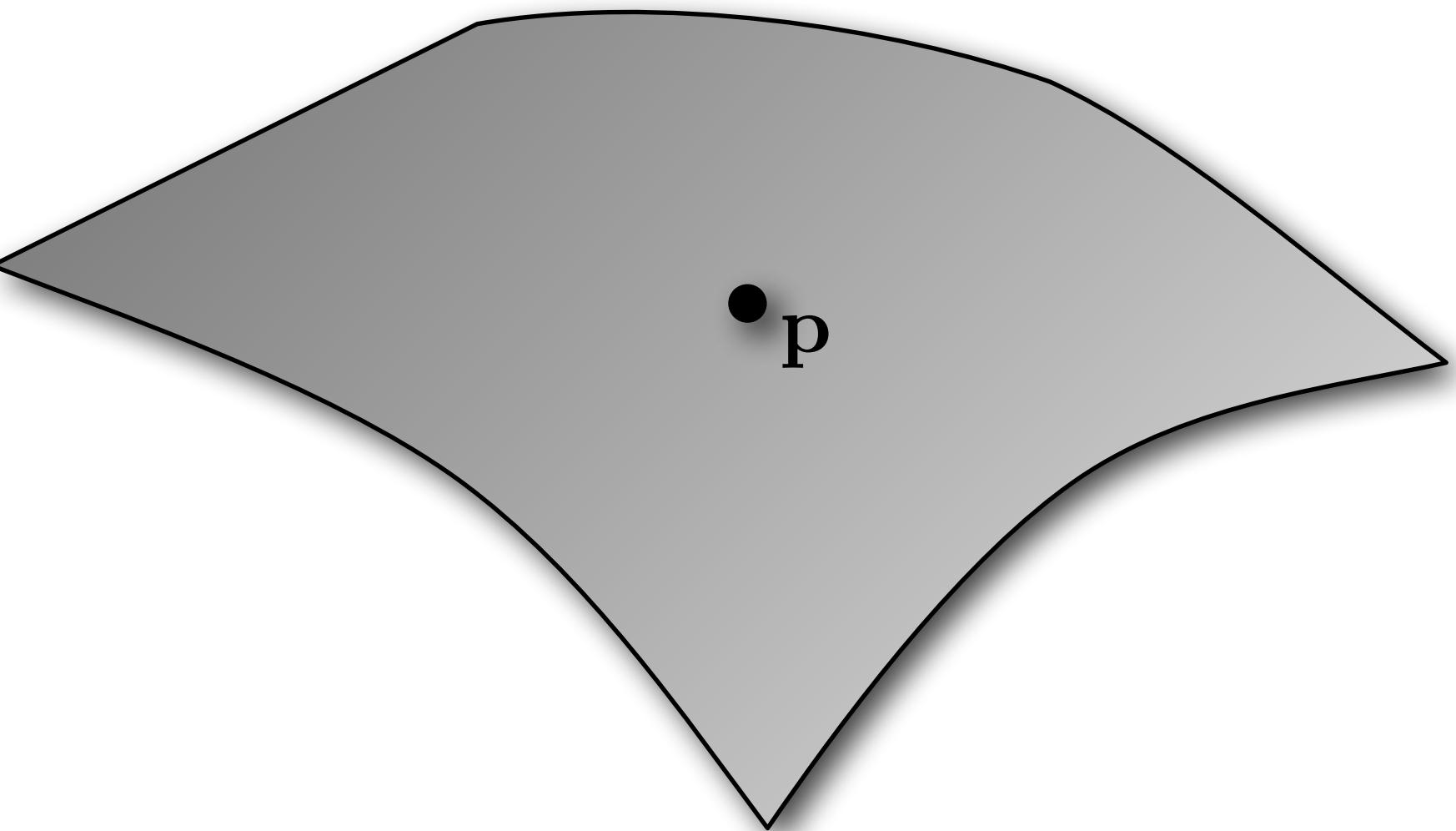
Parametric Surfaces

- Continuous surface

$$\mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

- Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$



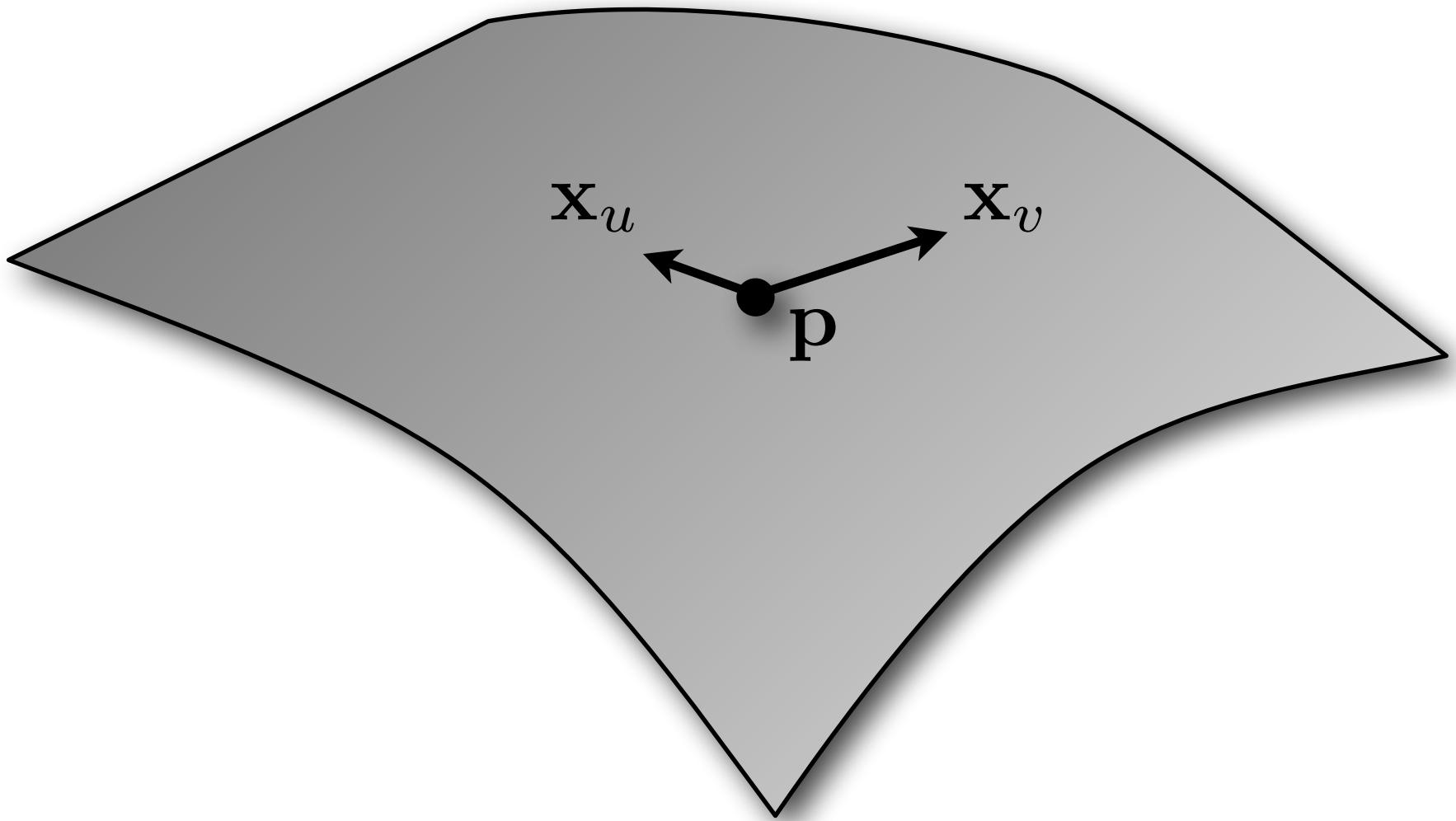
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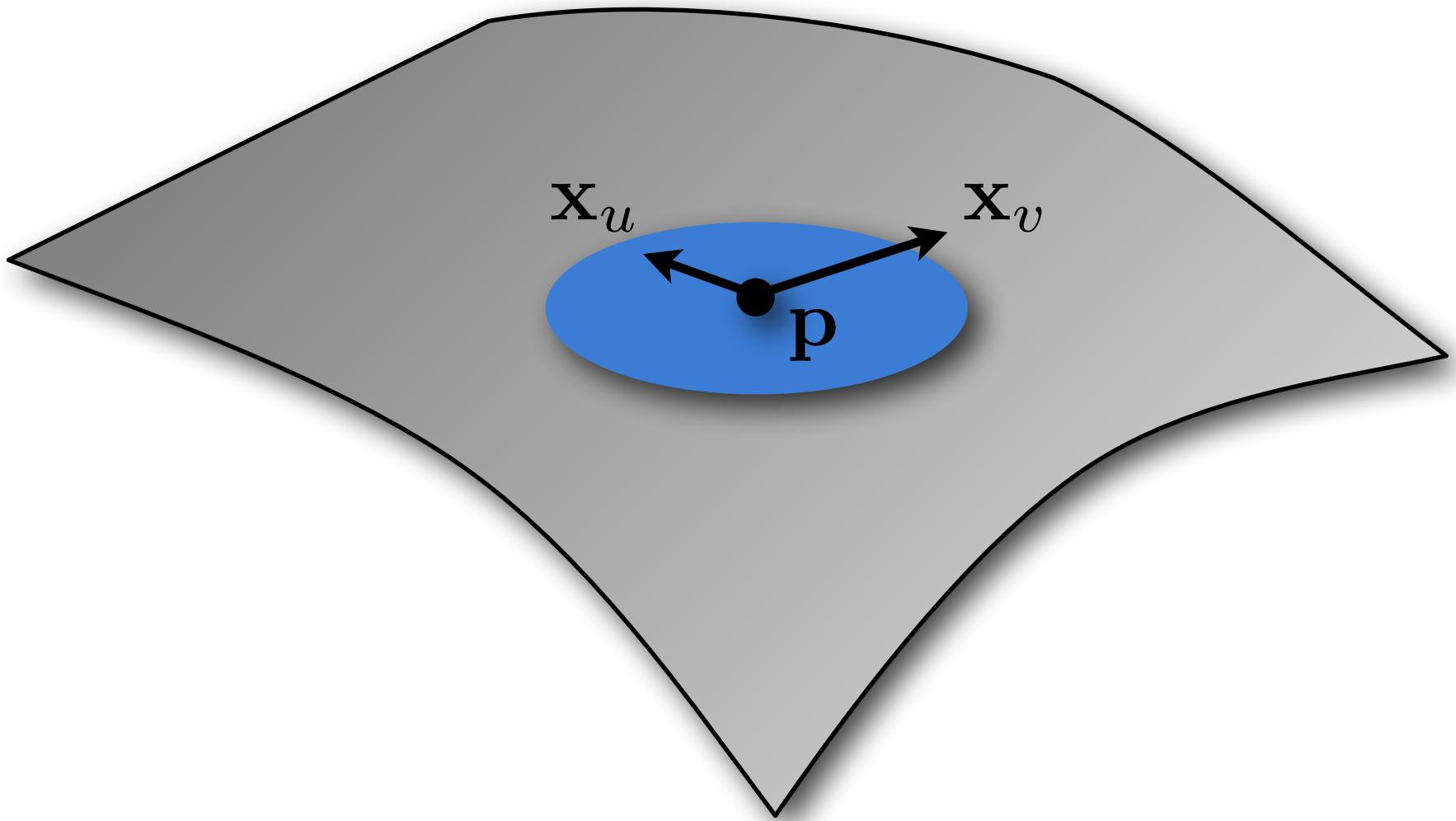
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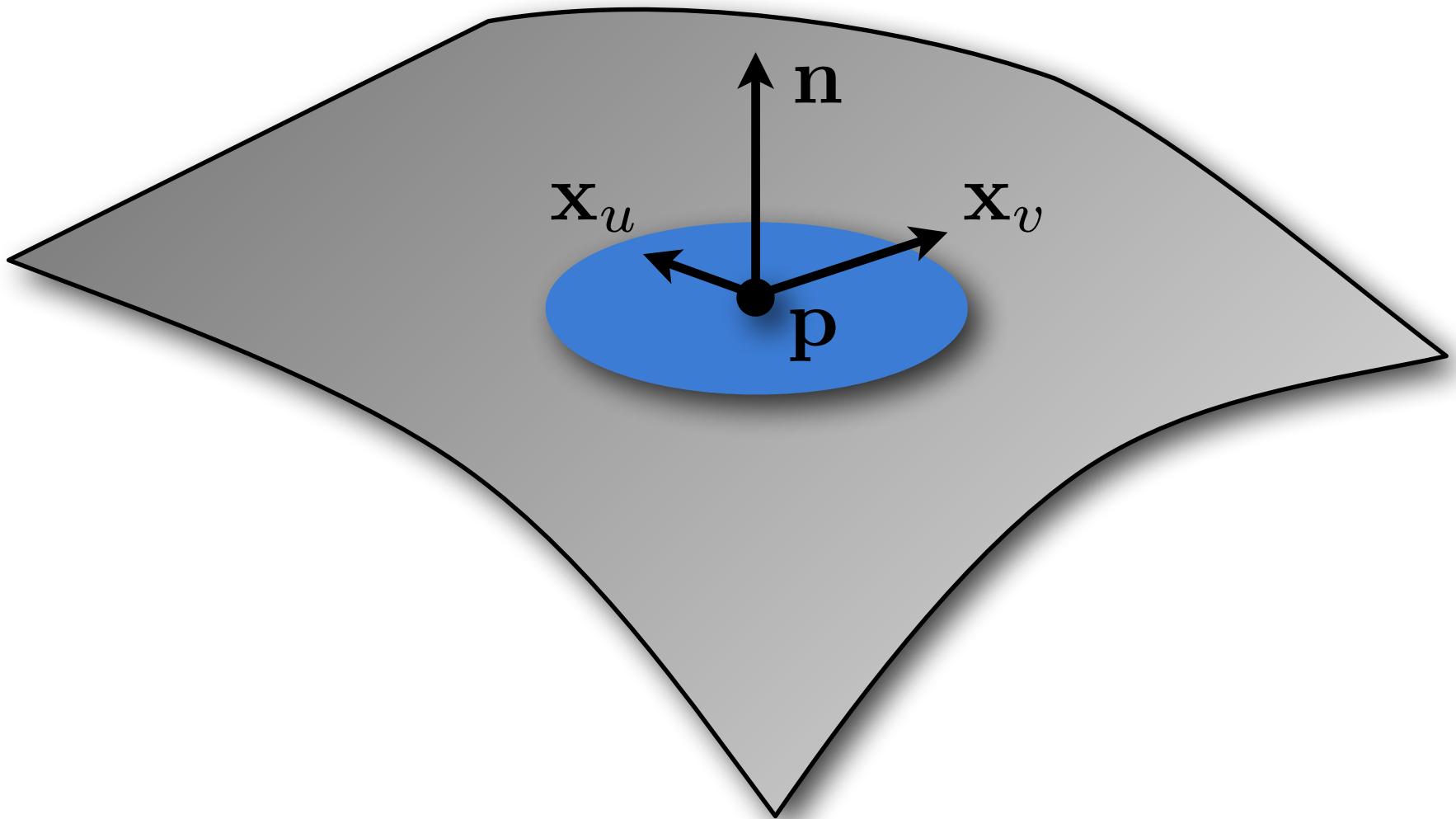
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Parametric Surfaces

- Continuous surface

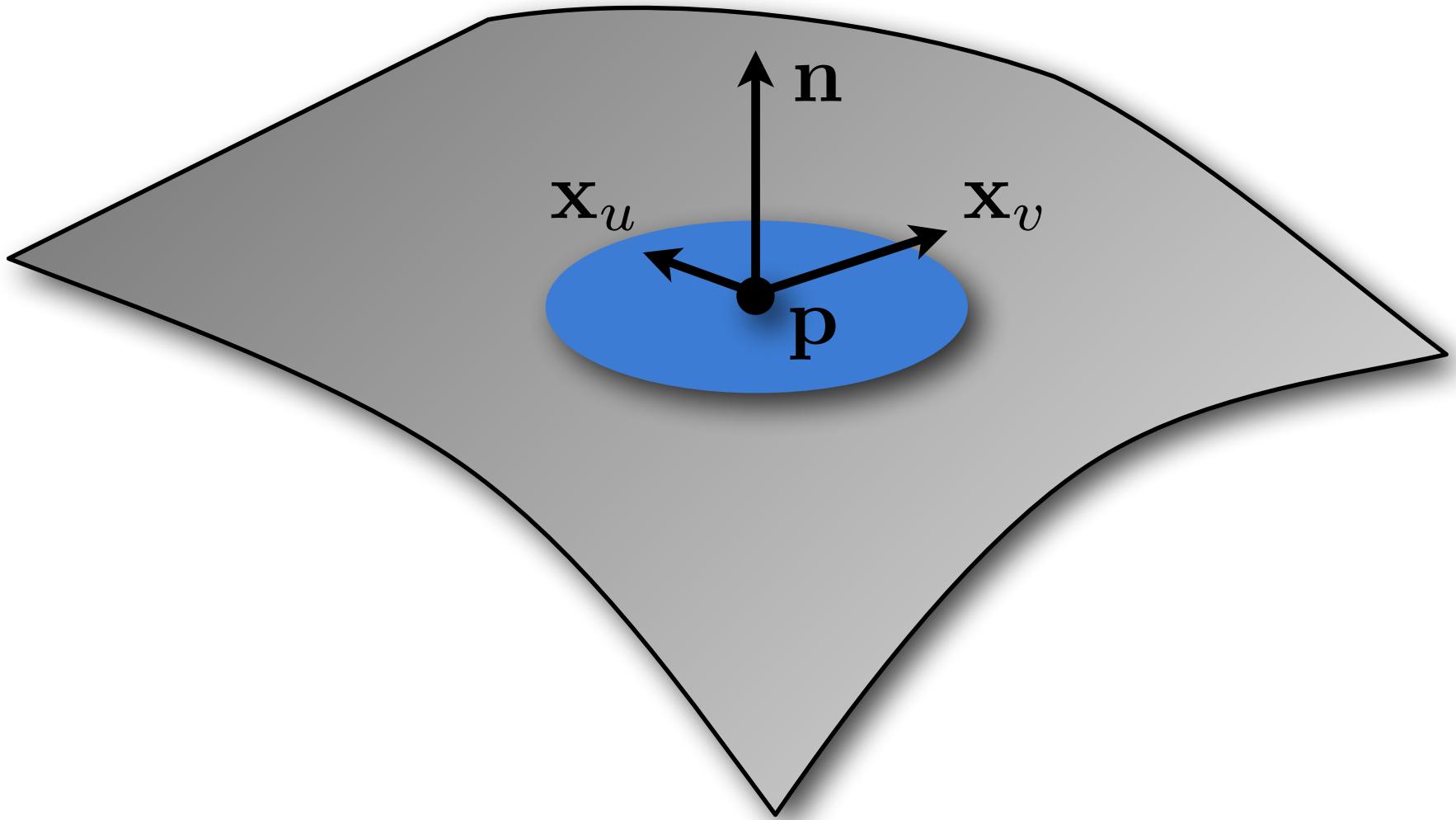
$$\mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

- Normal vector

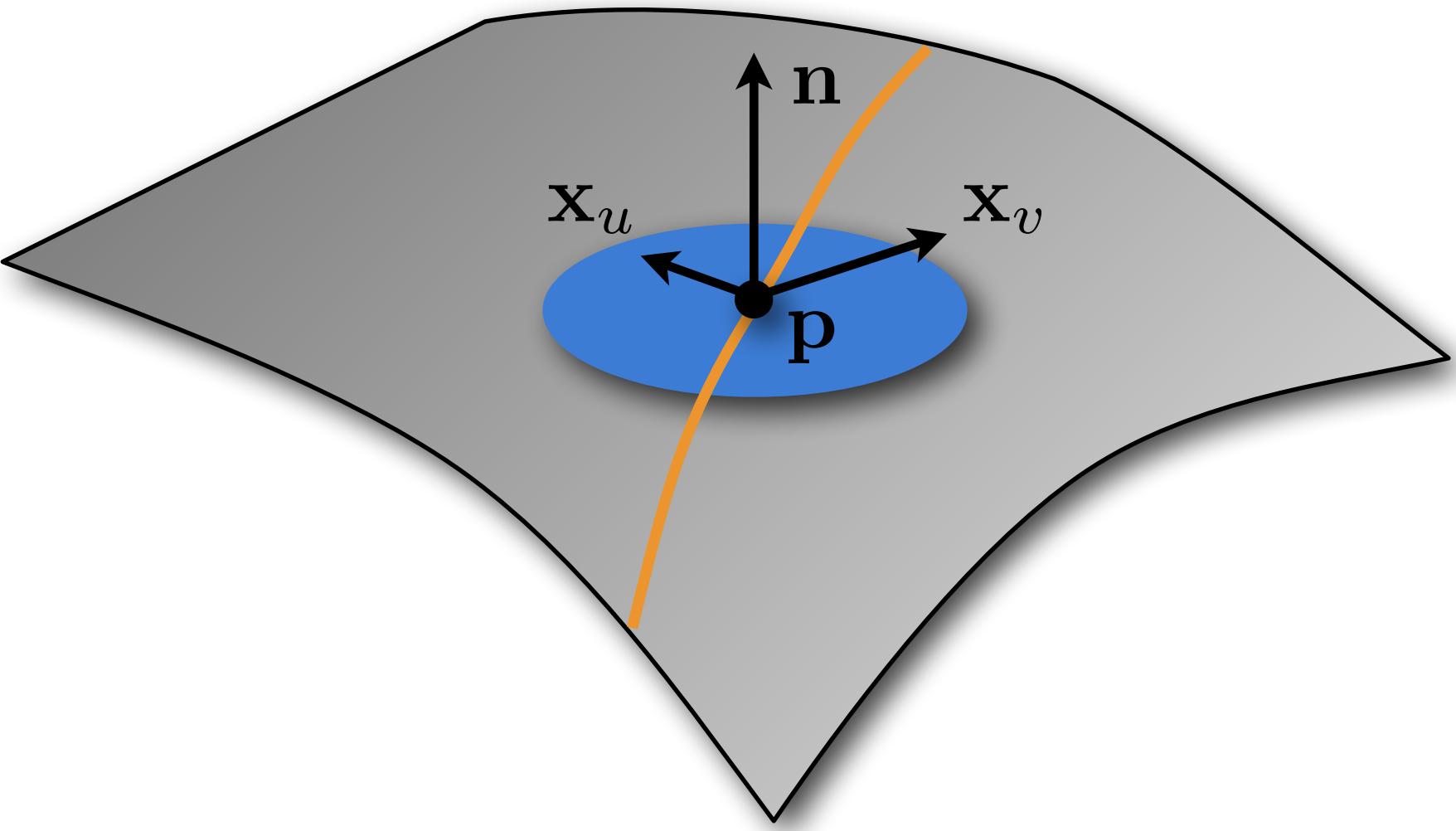
$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

- Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq 0$$

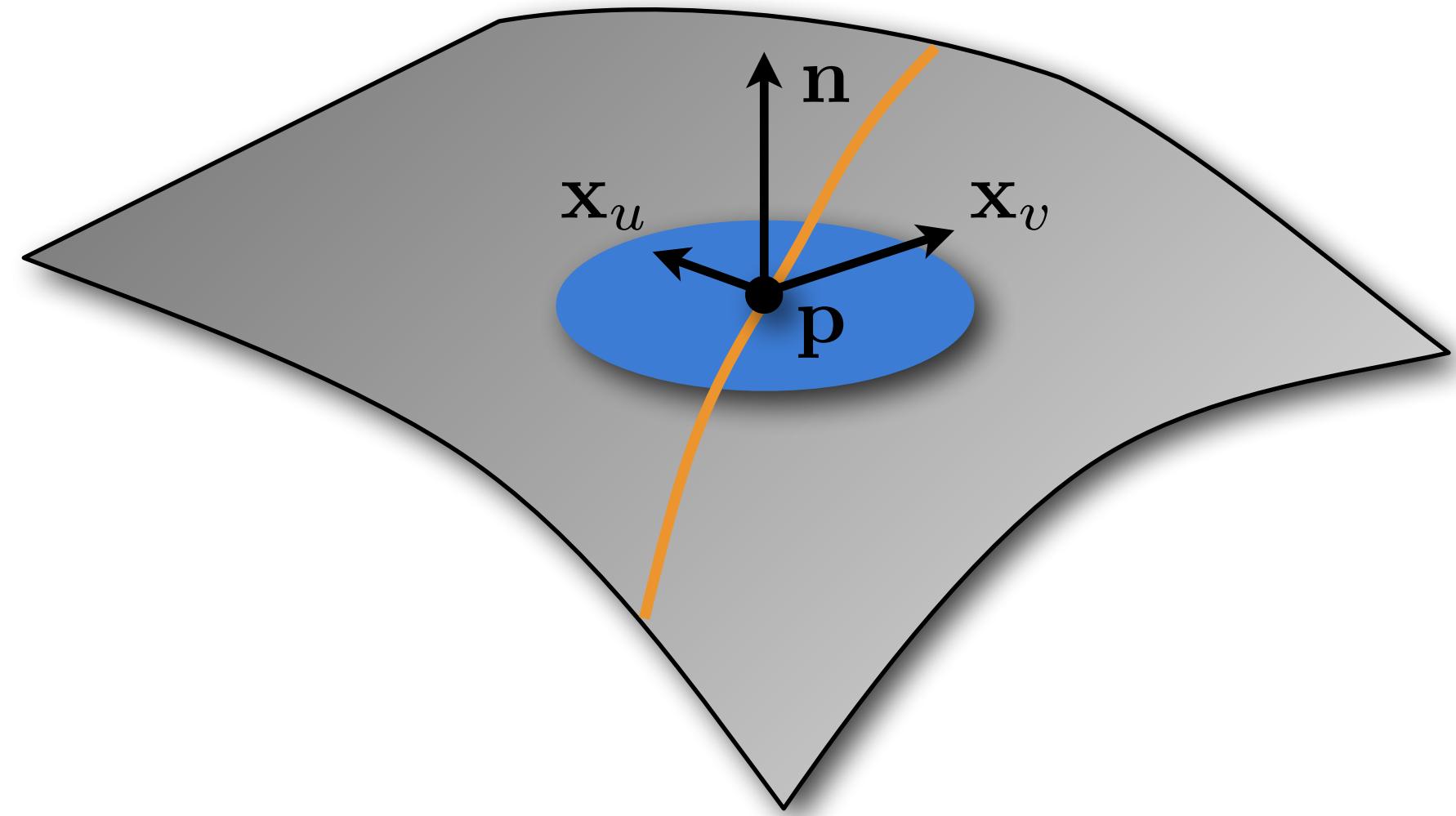


Angles on Surface



Angles on Surface

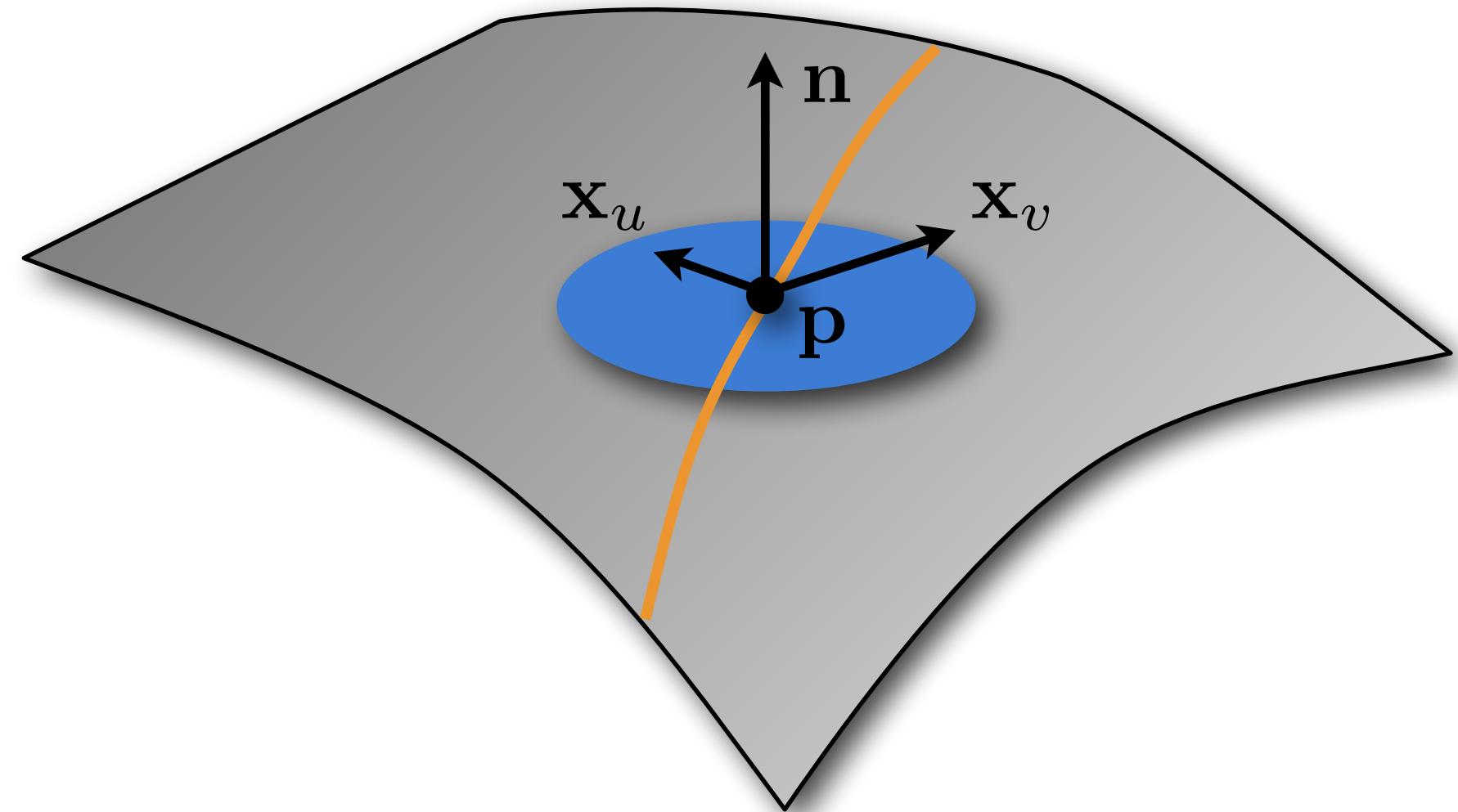
- Curve $[u(t), v(t)]$ in uv -plane defines curve on the surface $\mathbf{x}(u, v)$



Angles on Surface

- Curve $[u(t), v(t)]$ in uv -plane defines curve on the surface $\mathbf{x}(u, v)$

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

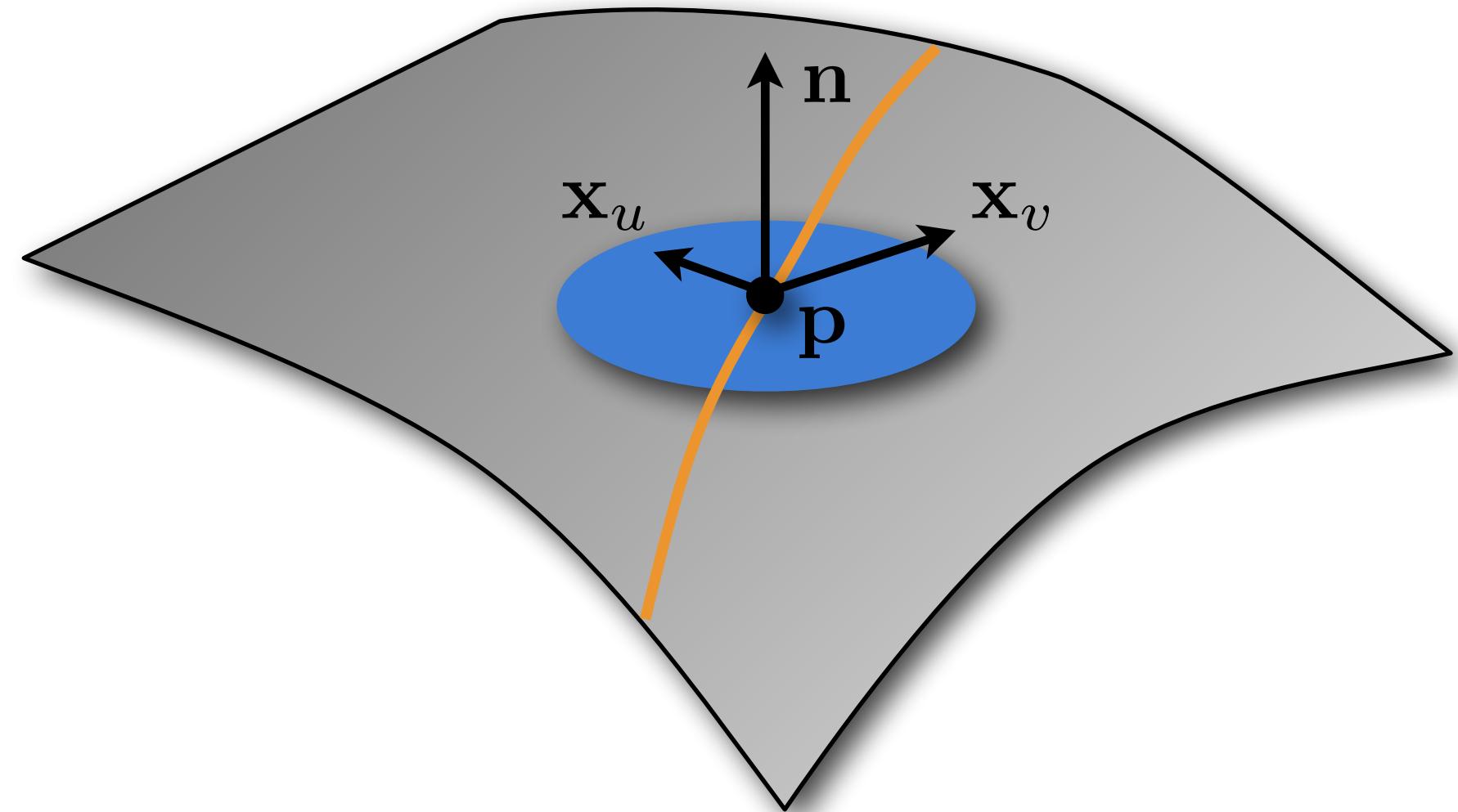


Angles on Surface

- Curve $[u(t), v(t)]$ in uv -plane defines curve on the surface $\mathbf{x}(u, v)$

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

- Two curves \mathbf{c}_1 and \mathbf{c}_2 intersecting at p

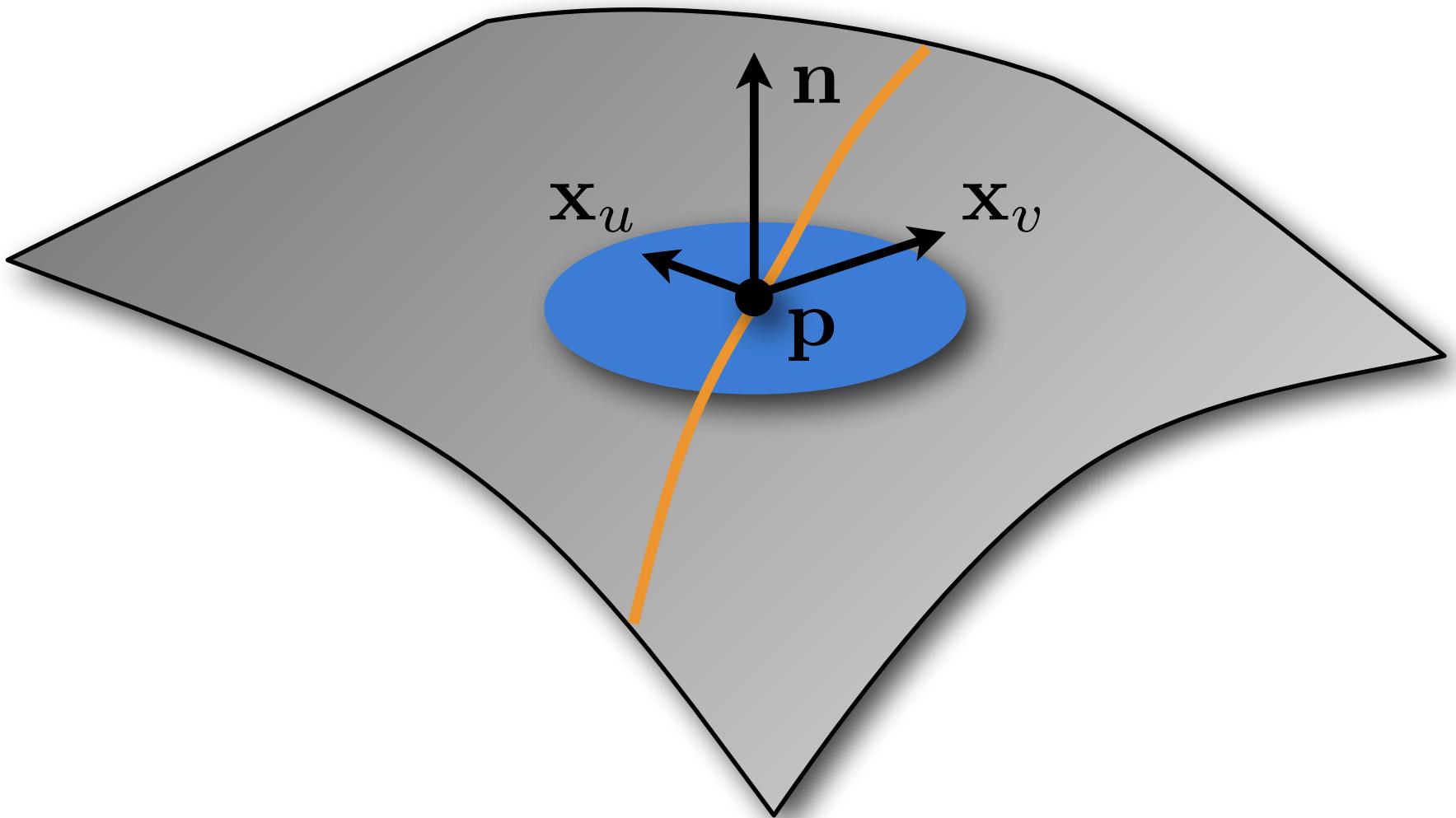


Angles on Surface

- Curve $[u(t), v(t)]$ in uv -plane defines curve on the surface $\mathbf{x}(u, v)$

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

- Two curves \mathbf{c}_1 and \mathbf{c}_2 intersecting at \mathbf{p}
 - Angle of intersection?

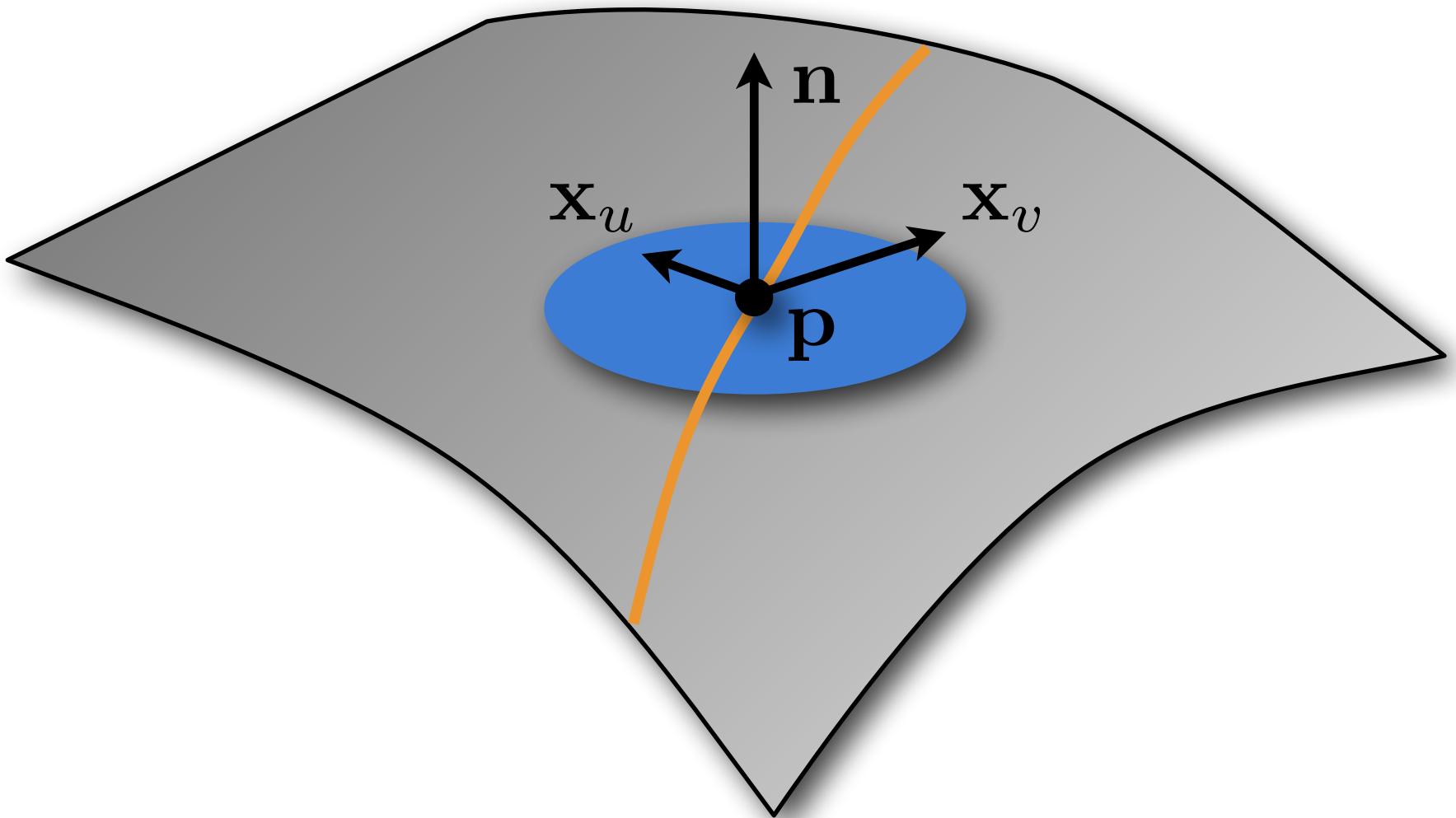


Angles on Surface

- Curve $[u(t), v(t)]$ in uv -plane defines curve on the surface $\mathbf{x}(u, v)$

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

- Two curves \mathbf{c}_1 and \mathbf{c}_2 intersecting at \mathbf{p}
 - Angle of intersection?
 - Two tangents \mathbf{t}_1 and \mathbf{t}_2



Angles on Surface

- Curve $[u(t), v(t)]$ in uv -plane defines curve on the surface $\mathbf{x}(u, v)$

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

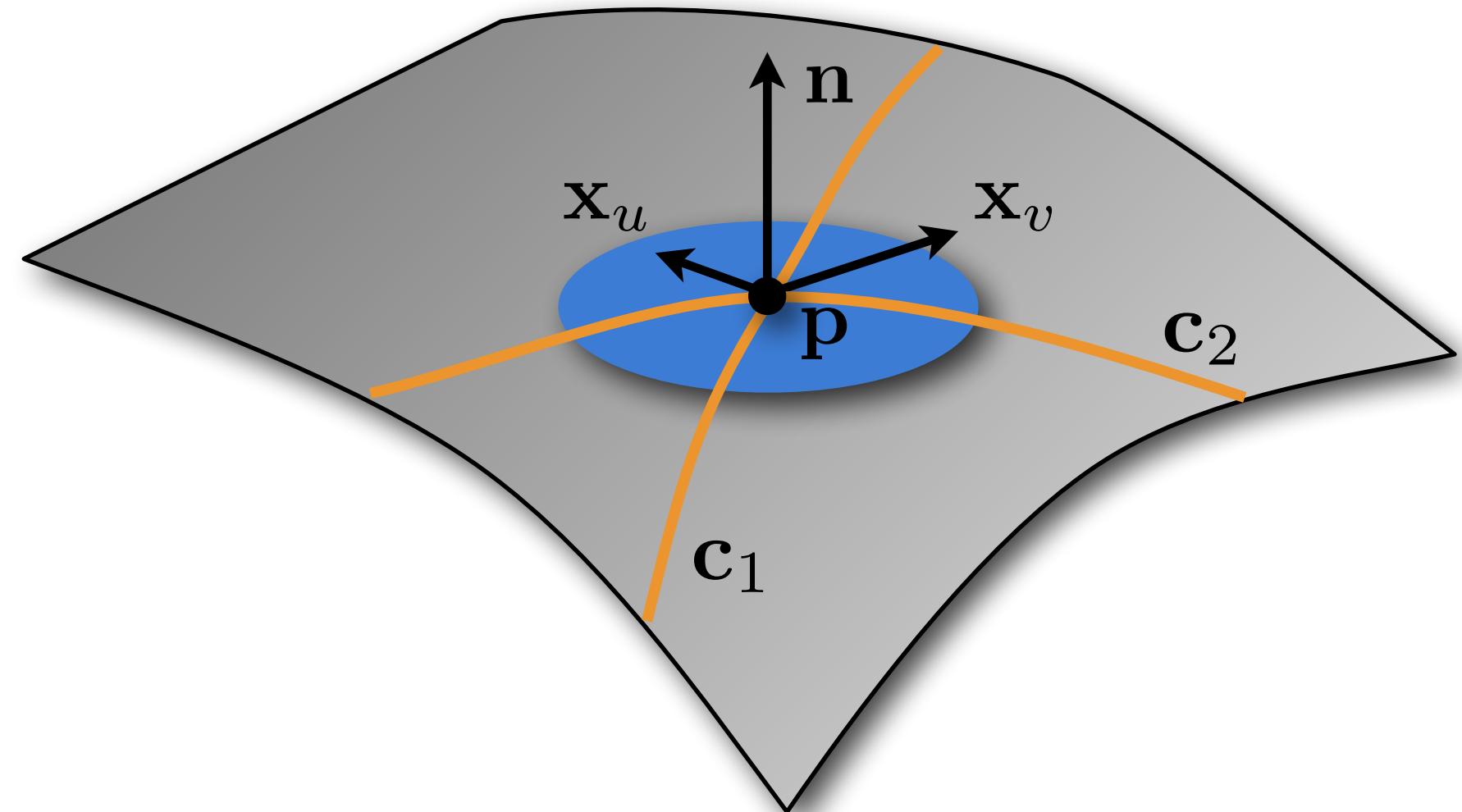
- Two curves \mathbf{c}_1 and \mathbf{c}_2 intersecting at \mathbf{p}

- Angle of intersection?
 - Two tangents \mathbf{t}_1 and \mathbf{t}_2

$$\mathbf{t}_i = \alpha_i \mathbf{x}_u + \beta_i \mathbf{x}_v$$

- Compute inner product

$$\mathbf{t}_1^T \mathbf{t}_2 = \cos \theta \|\mathbf{t}_1\| \|\mathbf{t}_2\|$$



Angles on Surface



- Curve $[u(t), v(t)]$ in uv -plane defines curve on the surface $\mathbf{x}(u, v)$

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- Two curves \mathbf{c}_1 and \mathbf{c}_2 intersecting at \mathbf{p}

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Angles on Surface



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Angles on Surface



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First Fundamental Form



- **First fundamental form**

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

- Defines inner product on tangent space

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

First Fundamental Form



- First fundamental form I allows to measure
(with respect to surface metric)

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$$\mathbf{t}_1^T \mathbf{t}_2 = \langle (\alpha_1, \beta_1), (\alpha_1, \beta_1) \rangle$$

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$$\begin{aligned} ds^2 &= \langle (du, dv), (du, dv) \rangle \\ &= Edu^2 + 2Fdudv + Gdv^2 \end{aligned}$$

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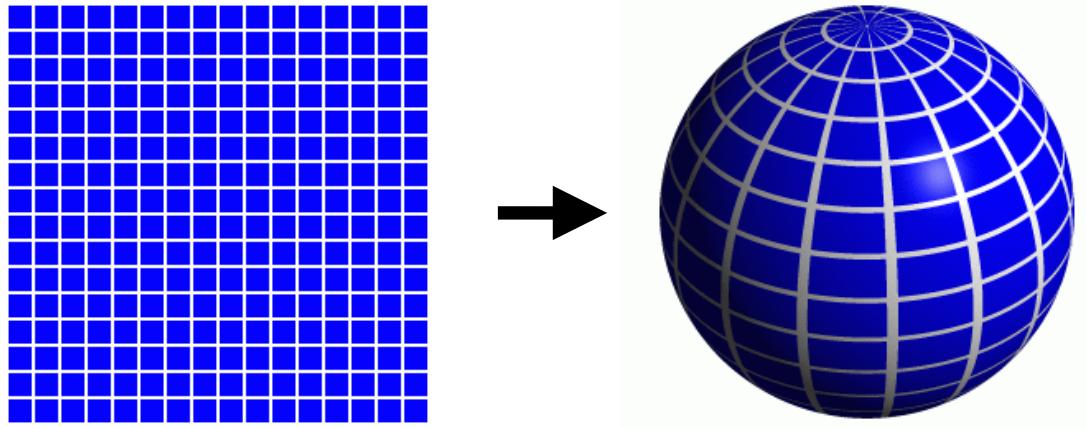
- Area

$$\begin{aligned} dA &= \|\mathbf{x}_u \times \mathbf{x}_v\| du dv \\ &= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - (\mathbf{x}_u^T \mathbf{x}_v)^2} du dv \\ &= \sqrt{EG - F^2} du dv \end{aligned}$$

Example: Sphere

- Spherical parameterization

$$\mathbf{x}(u, v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u, v) \in [0, 2\pi) \times [0, \pi)$$

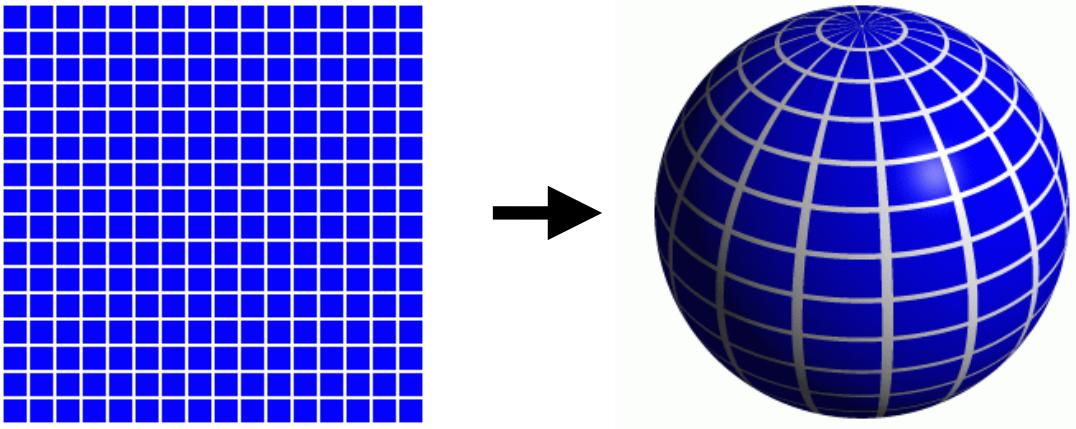


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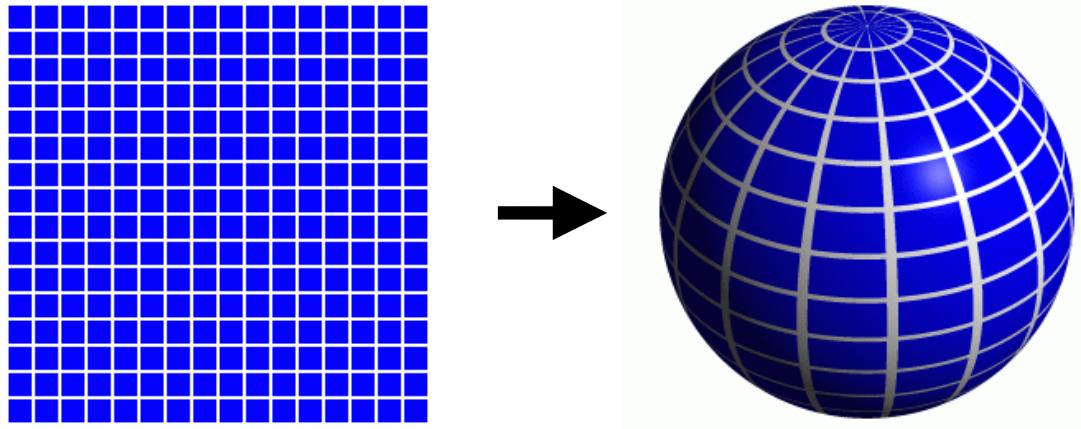
- Tangent vectors



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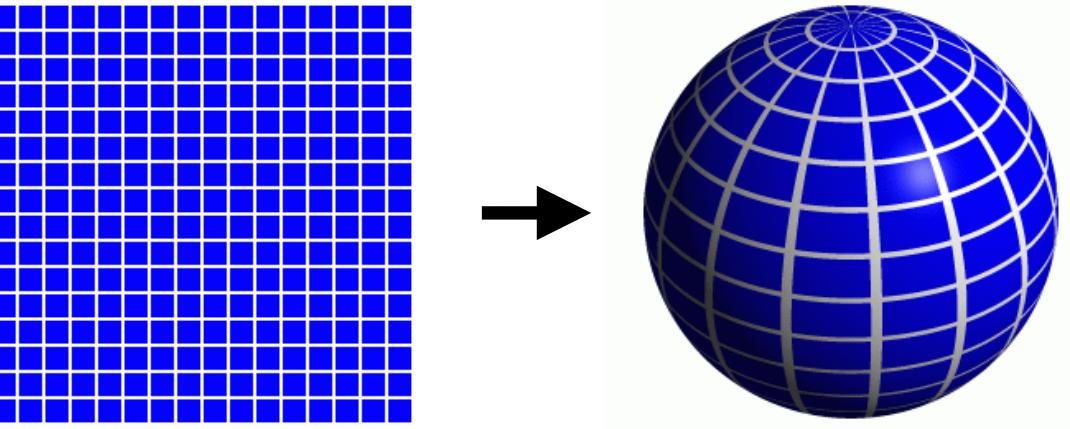
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$$\mathbf{x}_u(u, v) = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix}$$

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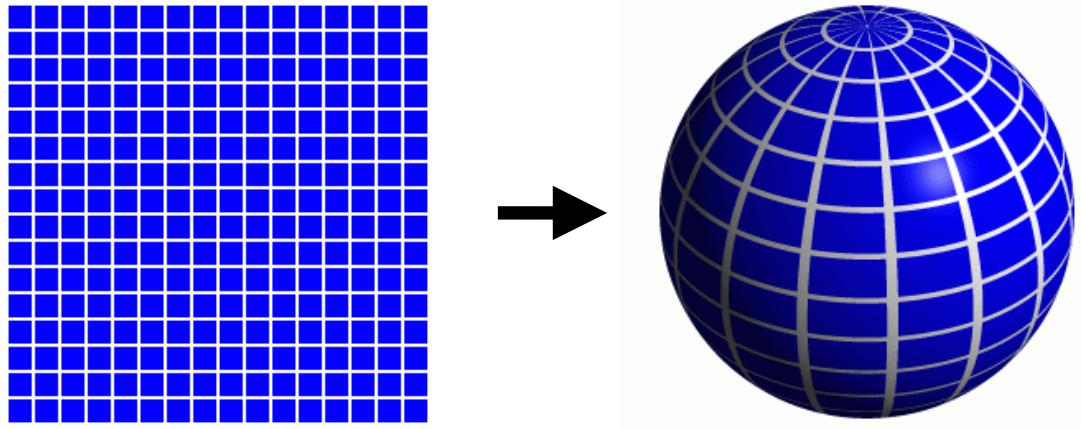
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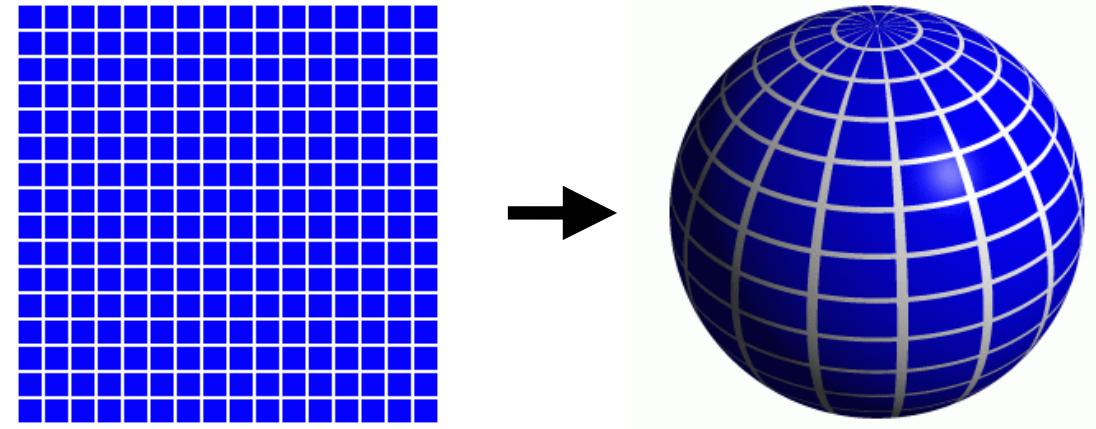
- First fundamental form

$$\mathbf{I} = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Example: Sphere



- Length of equator $\mathbf{x}(t, \pi / 2)$

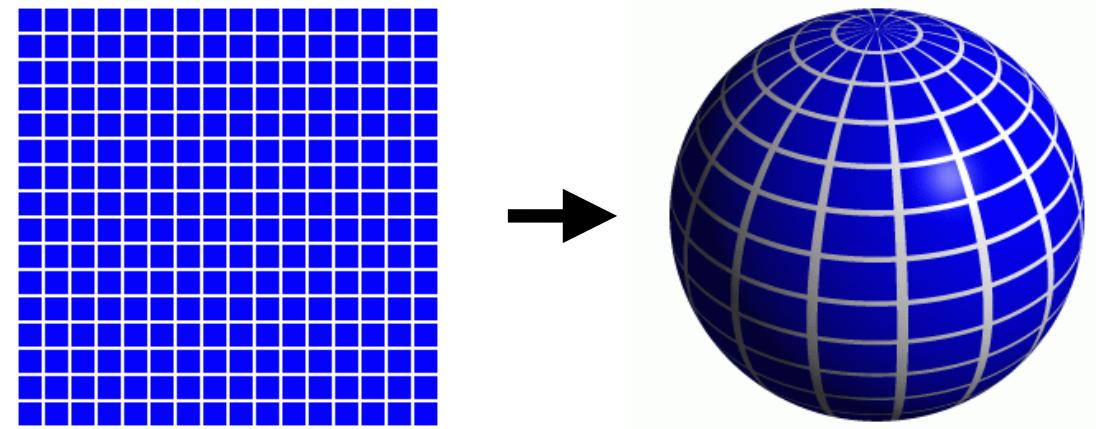


$$\int_0^{2\pi} 1 \, ds = \int_0^{2\pi} \sqrt{E (u_t)^2 + 2Fu_tv_t + G (v_t)^2} \, dt$$

Example: Sphere



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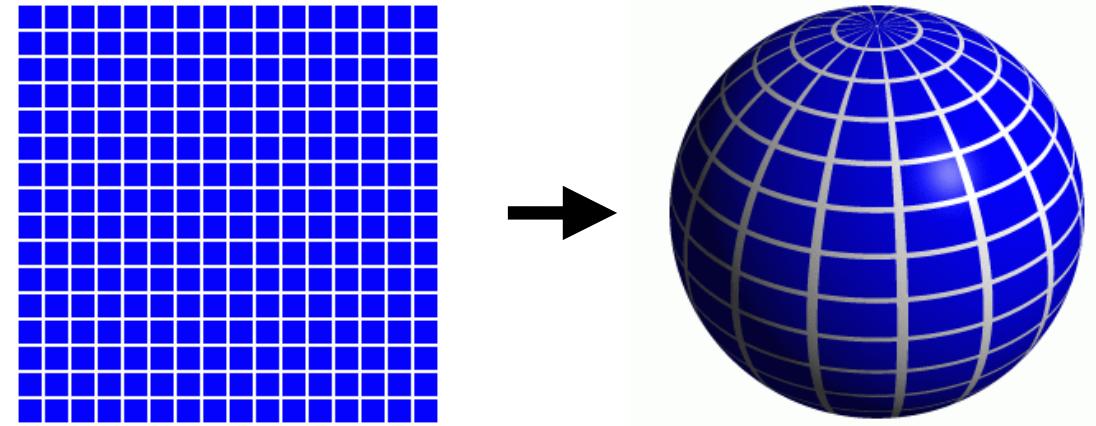
$$\int_0^{2\pi} 1 \, ds = \int_0^{2\pi} \sqrt{E(u_t)^2 + 2Fu_tv_t + G(v_t)^2} \, dt$$

$$= \int_0^{2\pi} \sin v \, dt$$

Example: Sphere



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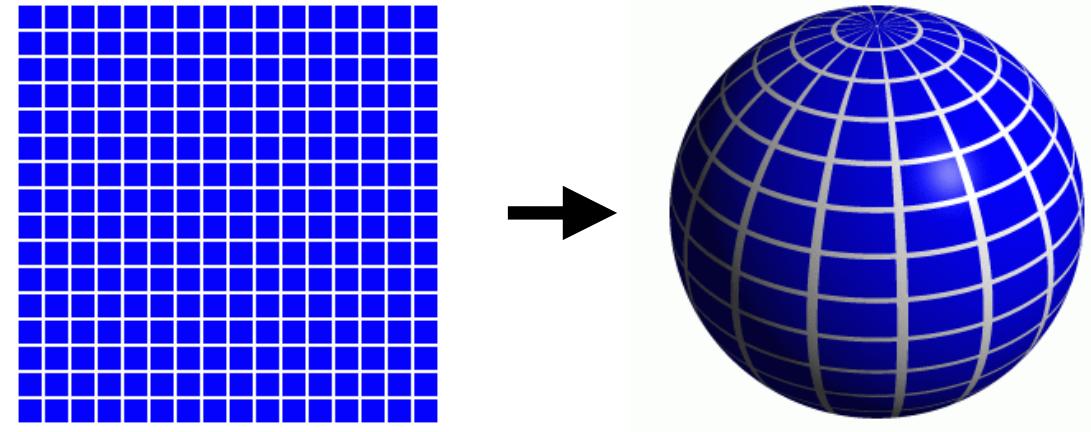


$$\begin{aligned} \int_0^{2\pi} 1 \, ds &= \int_0^{2\pi} \sqrt{E(u_t)^2 + 2Fu_tv_t + G(v_t)^2} \, dt \\ &= \int_0^{2\pi} \sin v \, dt \\ &= 2\pi \sin v = 2\pi \end{aligned}$$

Example: Sphere



- Area of a sphere

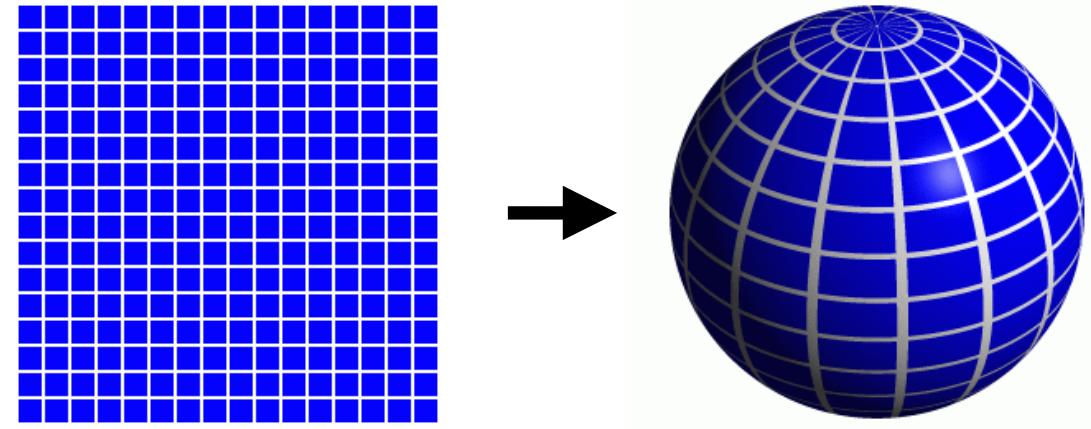


$$\int_0^\pi \int_0^{2\pi} 1 \, dA = \int_0^\pi \int_0^{2\pi} \sqrt{EG - F^2} \, du \, dv$$

Example: Sphere



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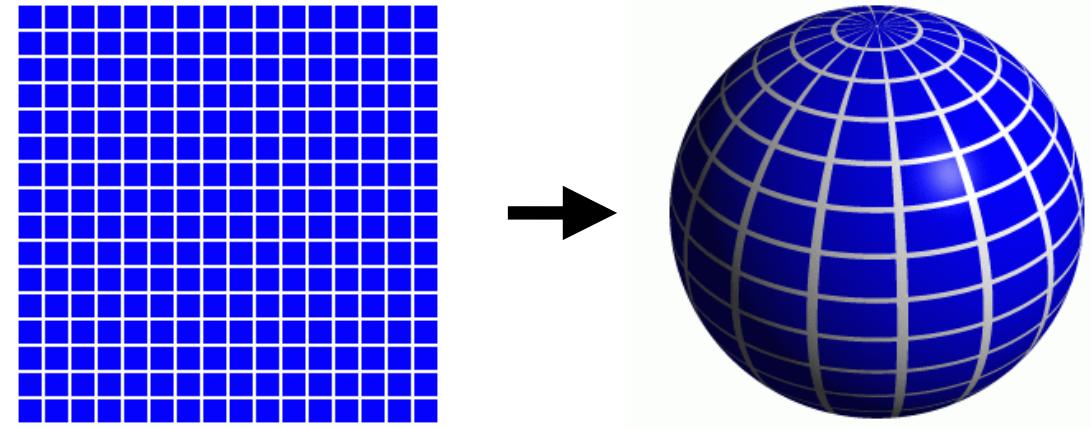


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Example: Sphere



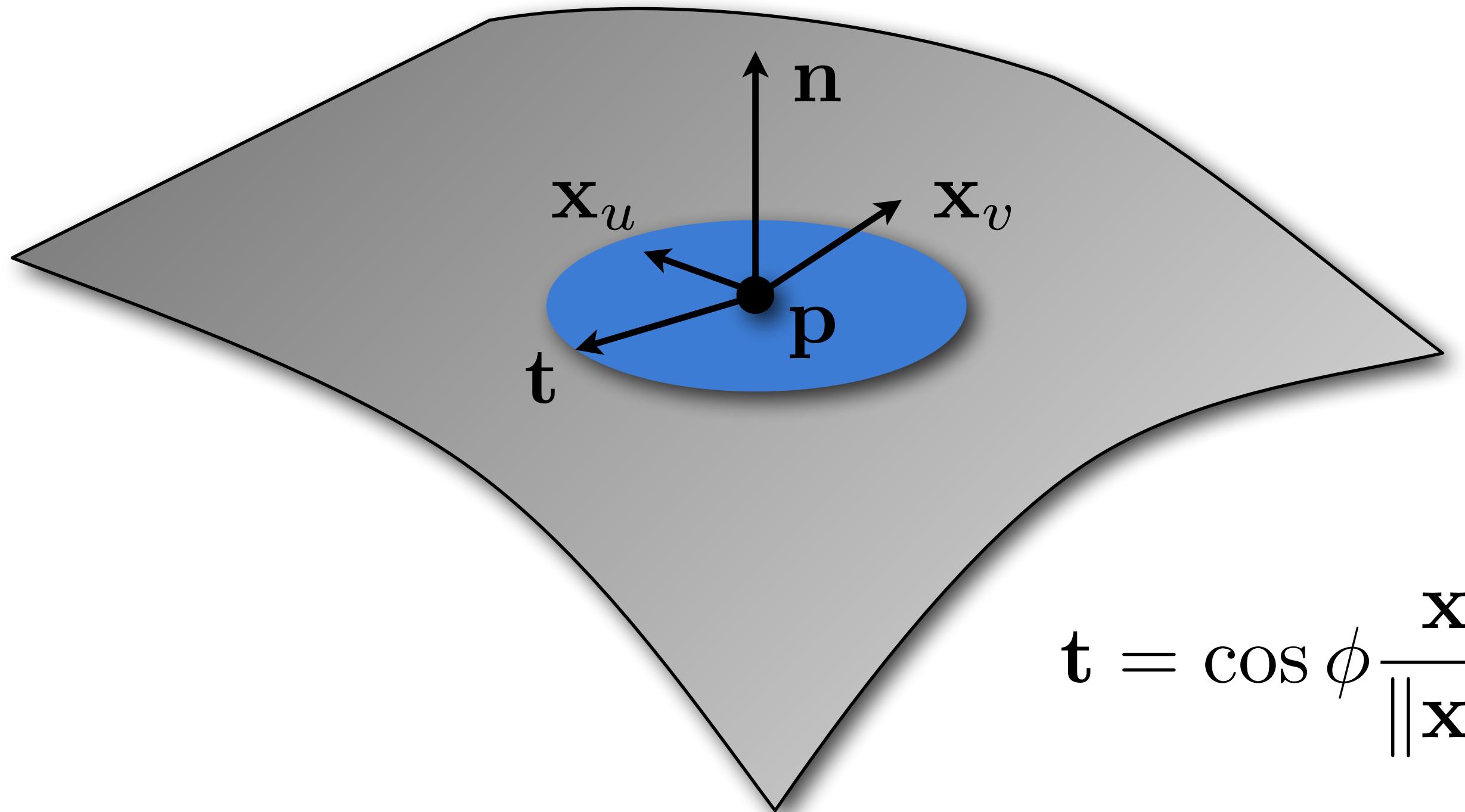
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Normal Curvature

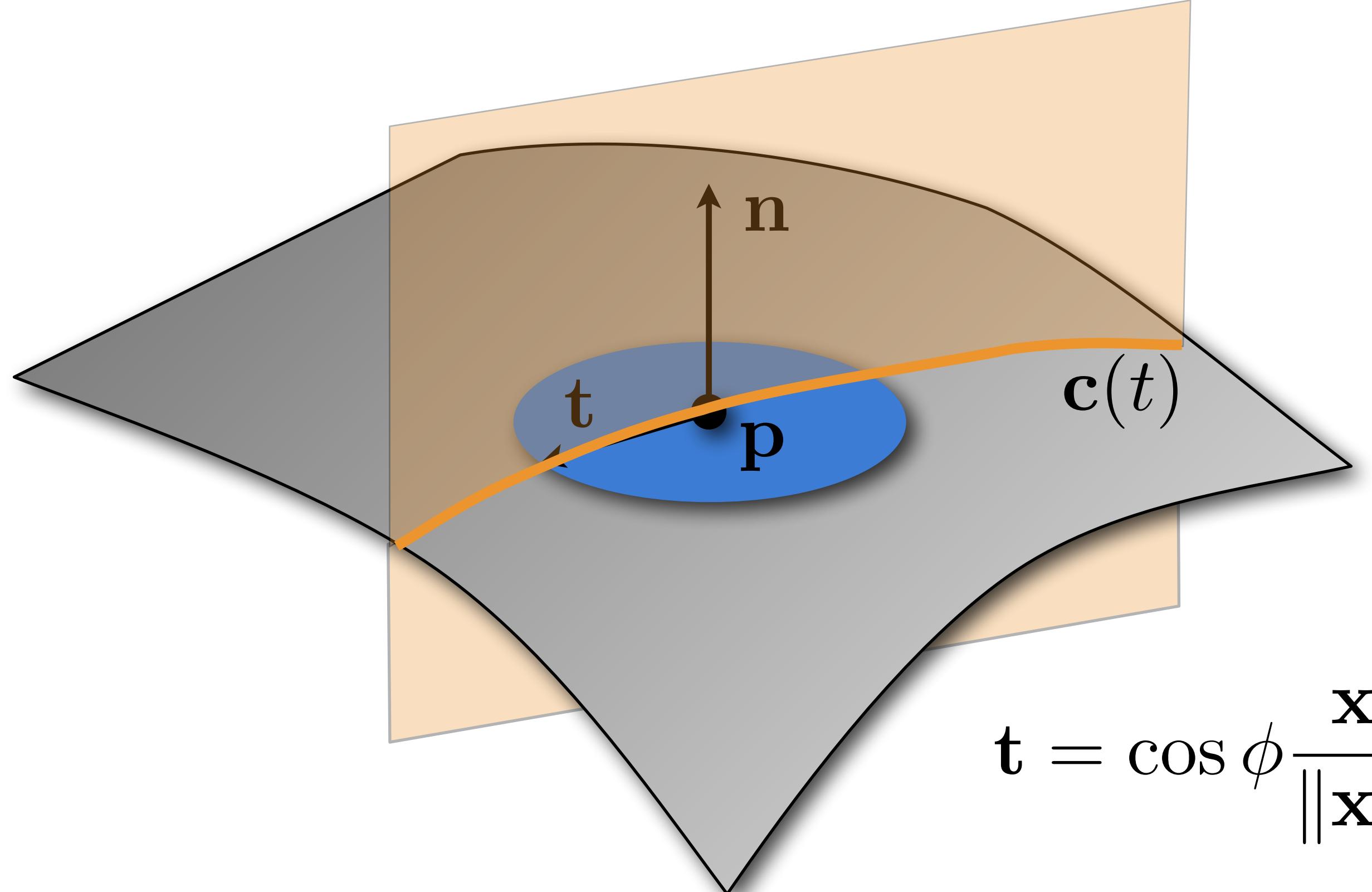
- Tangent vector t ...



$$\mathbf{t} = \cos \phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin \phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature

- .. defines intersection plane, yielding curve $c(t)$



$$\mathbf{t} = \cos \phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin \phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature



Normal Curvature



- Normal curvature $\kappa_n(t)$ is defined as curvature of the normal curve $c(t)$ at point $p = x(u, v)$.

Normal Curvature



- Normal curvature $\kappa_n(t)$ is defined as curvature of the normal curve $c(t)$ at point $p = \mathbf{x}(u, v)$.
- With **second fundamental form**

$$\text{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{pmatrix}$$

normal curvature can be computed as

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \text{II} \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2} \quad \begin{aligned} \mathbf{t} &= a\mathbf{x}_u + b\mathbf{x}_v \\ \bar{\mathbf{t}} &= (a, b) \end{aligned}$$

Surface Curvature(s)



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- *Principal curvatures*

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 - Maximum curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$

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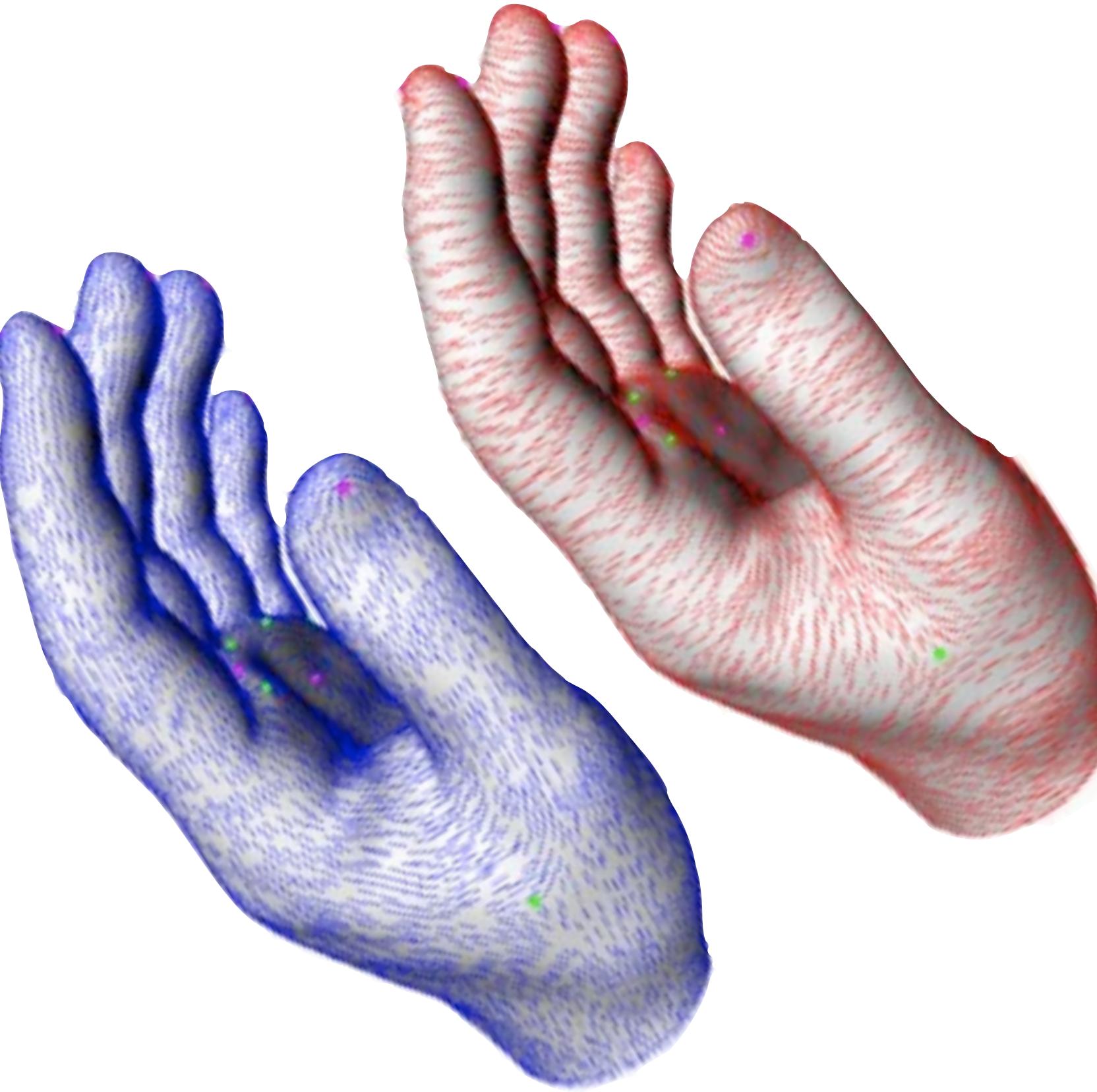
Surface Curvature(s)



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Surface Curvature(s)



- *Principal curvatures*

- Maximum curvature
- Minimum curvature
- Euler theorem:
- Corresponding *principal directions* $\mathbf{e}_1, \mathbf{e}_2$ are orthogonal

$$\kappa_1 = \max_{\phi} \kappa_n(\phi)$$

$$\kappa_2 = \min_{\phi} \kappa_n(\phi)$$

$$\kappa_n(\phi) = \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi$$

- Special curvatures

- Mean curvature
- Gaussian curvature

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

$$K = \kappa_1 \cdot \kappa_2$$

Curvature of Surfaces



Curvature of Surfaces



- Mean curvature

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

Curvature of Surfaces



- Mean curvature

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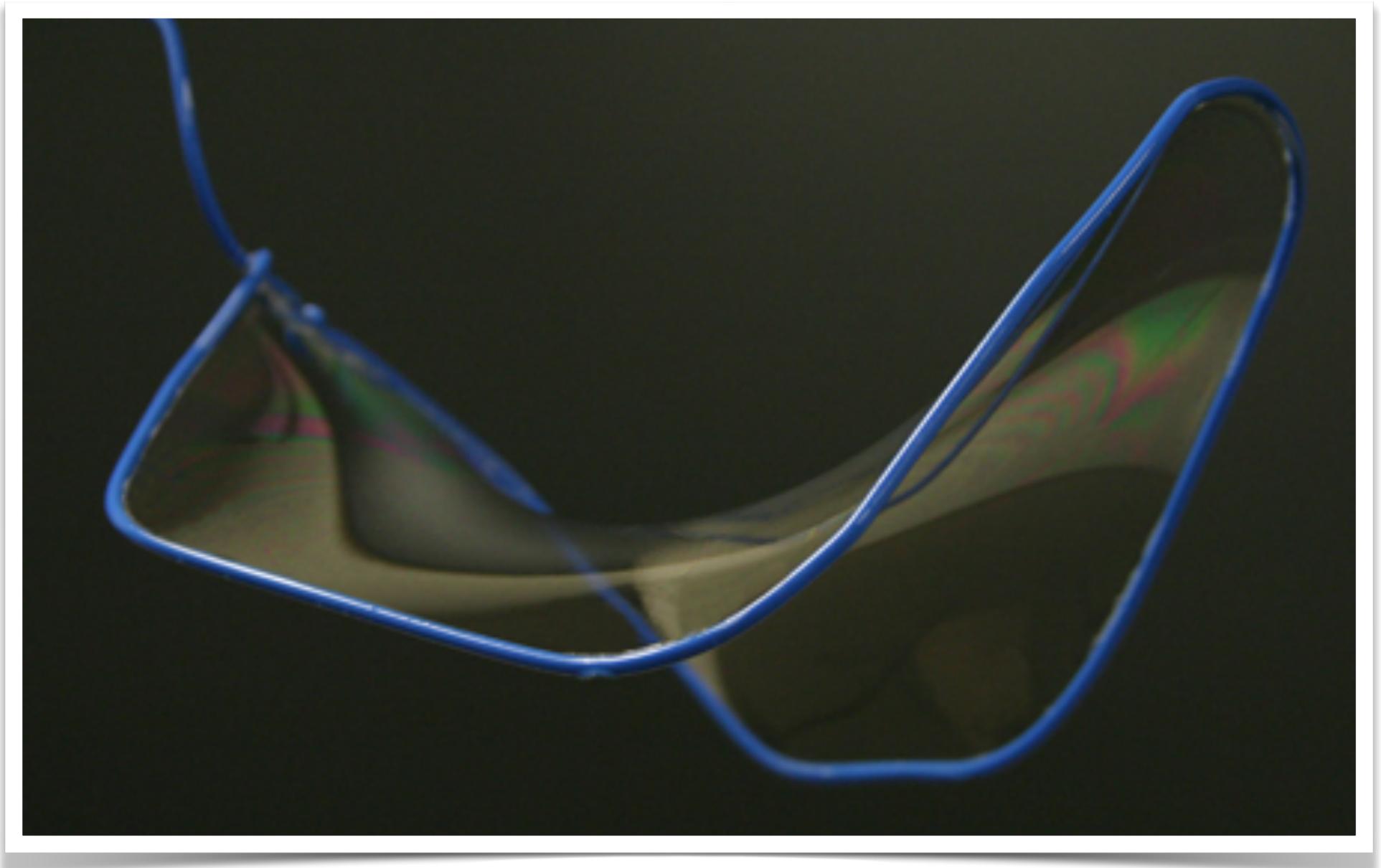
- $H = 0$ everywhere \rightarrow minimal surface

Curvature of Surfaces

- Mean curvature

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

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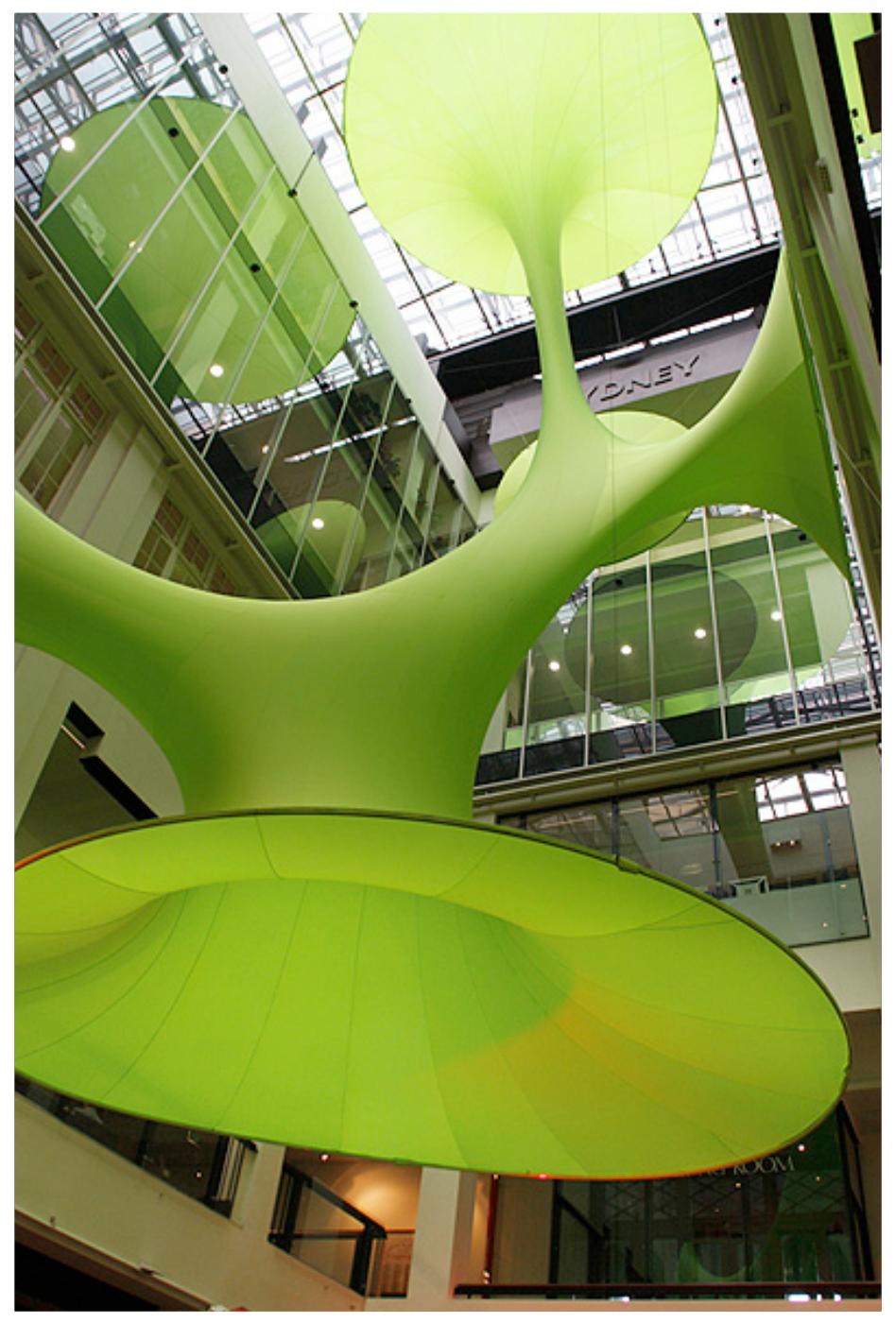
soap films

Curvature of Surfaces

- Mean curvature
 - $H = 0$ everywhere \rightarrow minimal surface



Green Void, Sydney
Architects: LAVA

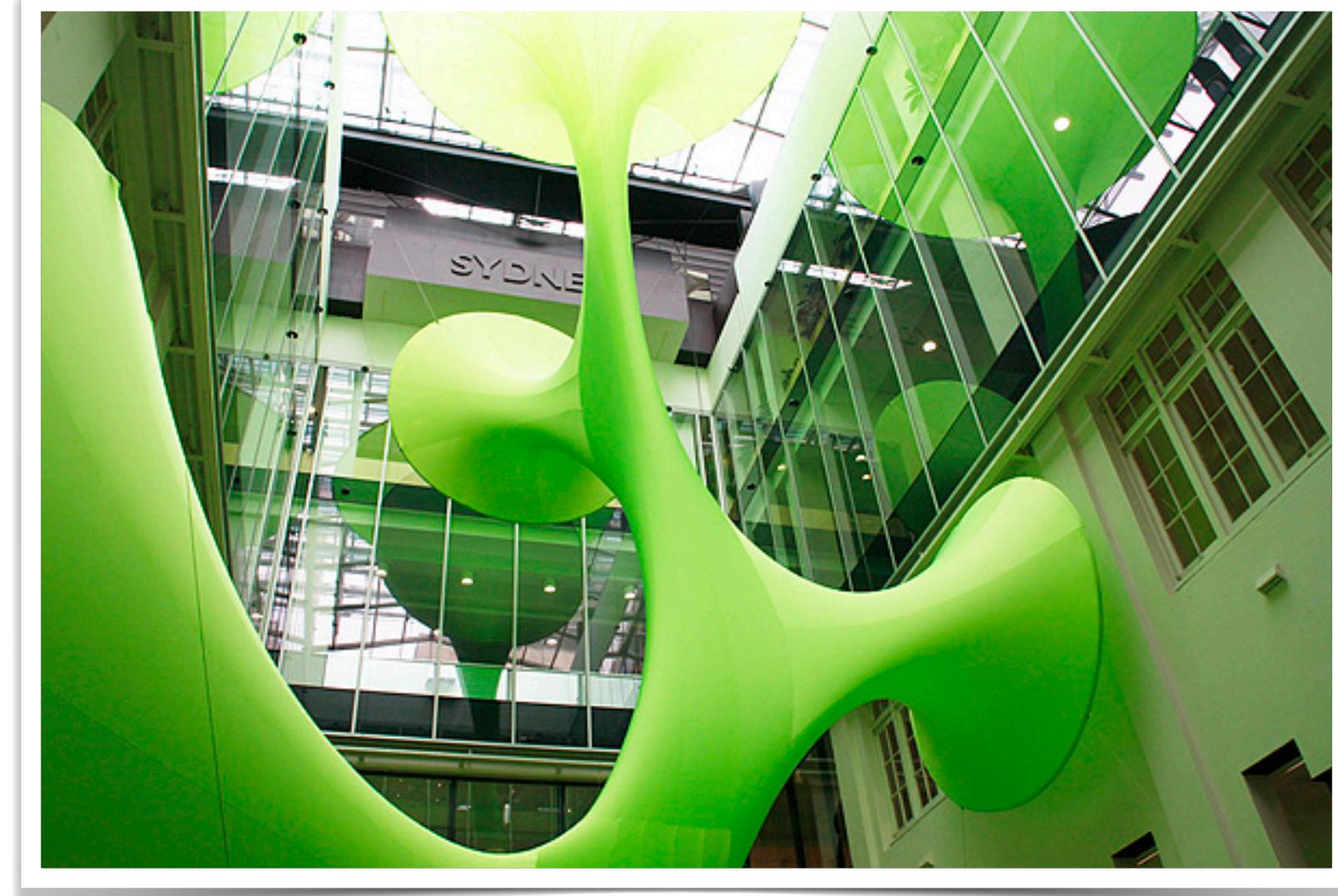


Curvature of Surfaces

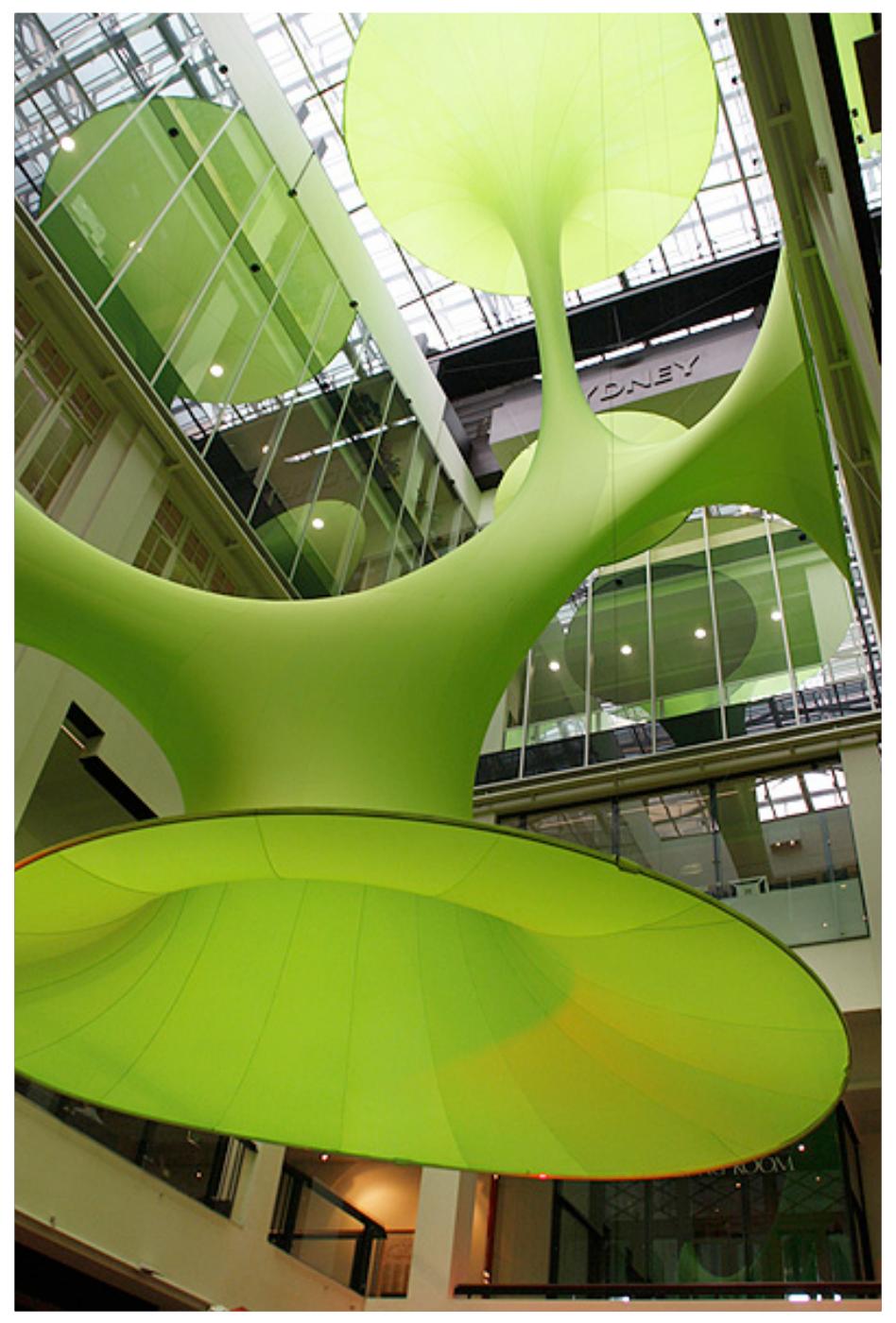
- Mean curvature

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

- $H = 0$ everywhere \rightarrow minimal surface



Green Void, Sydney
Architects: LAVA



Curvature of Surfaces



Curvature of Surfaces



- Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2$$

Curvature of Surfaces



- Gaussian curvature $K = \kappa_1 \cdot \kappa_2$
 - $K = 0$ everywhere \rightarrow developable surface

Curvature of Surfaces

- Gaussian curvature
$$K = \kappa_1 \cdot \kappa_2$$
 - $K = 0$ everywhere \rightarrow developable surface



Disney Concert Hall, L.A.
Architects: Gehry Partners



Timber Fabric
IBOIS, EPFL

Intrinsic Geometry



Intrinsic Geometry



- Properties of the surface that only depend on the first fundamental form

Intrinsic Geometry



- Properties of the surface that only depend on the first fundamental form
 - length

Intrinsic Geometry



- Properties of the surface that only depend on the first fundamental form
 - length
 - angles

Intrinsic Geometry



- Properties of the surface that only depend on the first fundamental form
 - length
 - angles
 - Gaussian curvature (Theorema Egregium)

Intrinsic Geometry



- Properties of the surface that only depend on the first fundamental form
 - length
 - angles
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$$K = \lim_{r \rightarrow 0} \frac{6\pi r - 3C(r)}{\pi r^3}$$

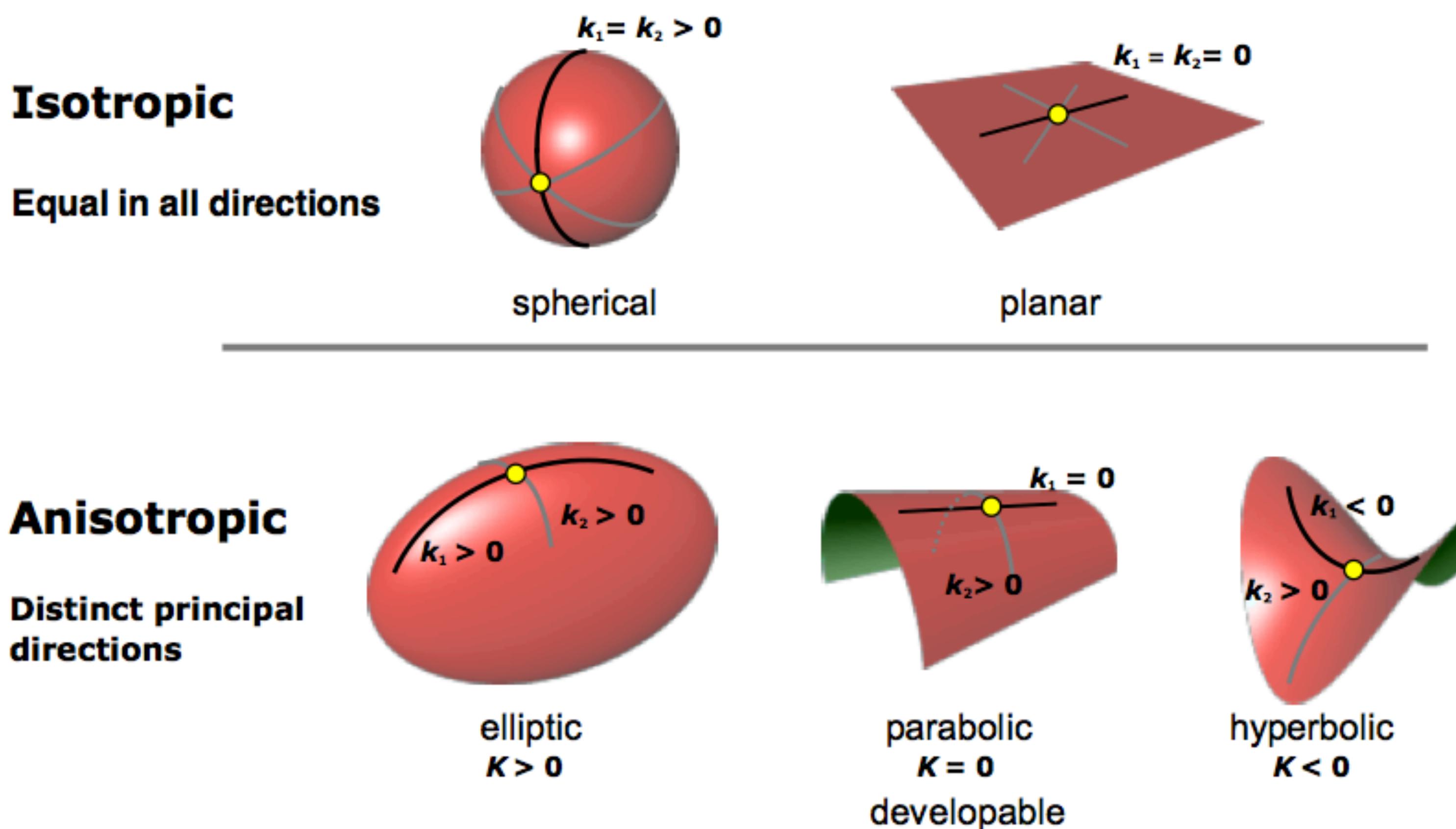
Classification



- A point x on the surface is called
 - *elliptic*, if $K > 0$
 - *hyperbolic*, if $K < 0$
 - *parabolic*, if $K = 0$
 - *umbilic*, if $\kappa_1 = \kappa_2$

Classification

- A point x on the surface is called



Gauss-Bonnet Theorem



- For **any** closed manifold surface with Euler characteristic $\chi = 2 - 2g$

$$\int K = 2\pi\chi$$

Gauss-Bonnet Theorem



- For **any** closed manifold surface with Euler characteristic $\chi = 2-2g$

$$\int K = 2\pi\chi$$

$$\int K(\text{Hand}) = \int K(\text{Cow}) = \int K(\text{Sphere}) = 4\pi$$

Gauss-Bonnet Theorem



Gauss-Bonnet Theorem



- Sphere

Gauss-Bonnet Theorem



- Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

Gauss-Bonnet Theorem



- Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1 \kappa_2 = 1/r^2$$

Gauss-Bonnet Theorem



- Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1 \kappa_2 = 1/r^2$$

$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$

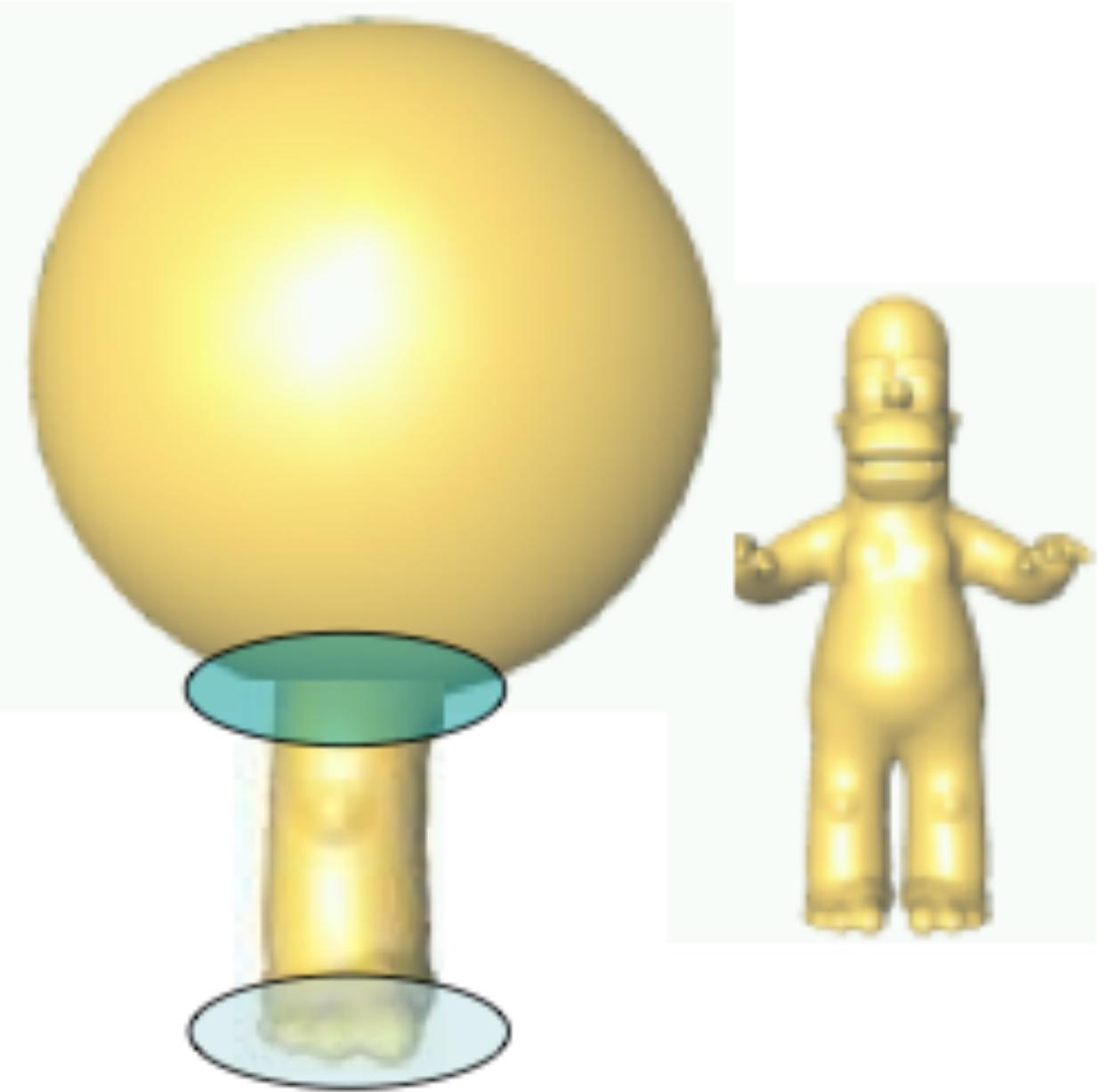
Gauss-Bonnet Theorem

- Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1 \kappa_2 = 1/r^2$$

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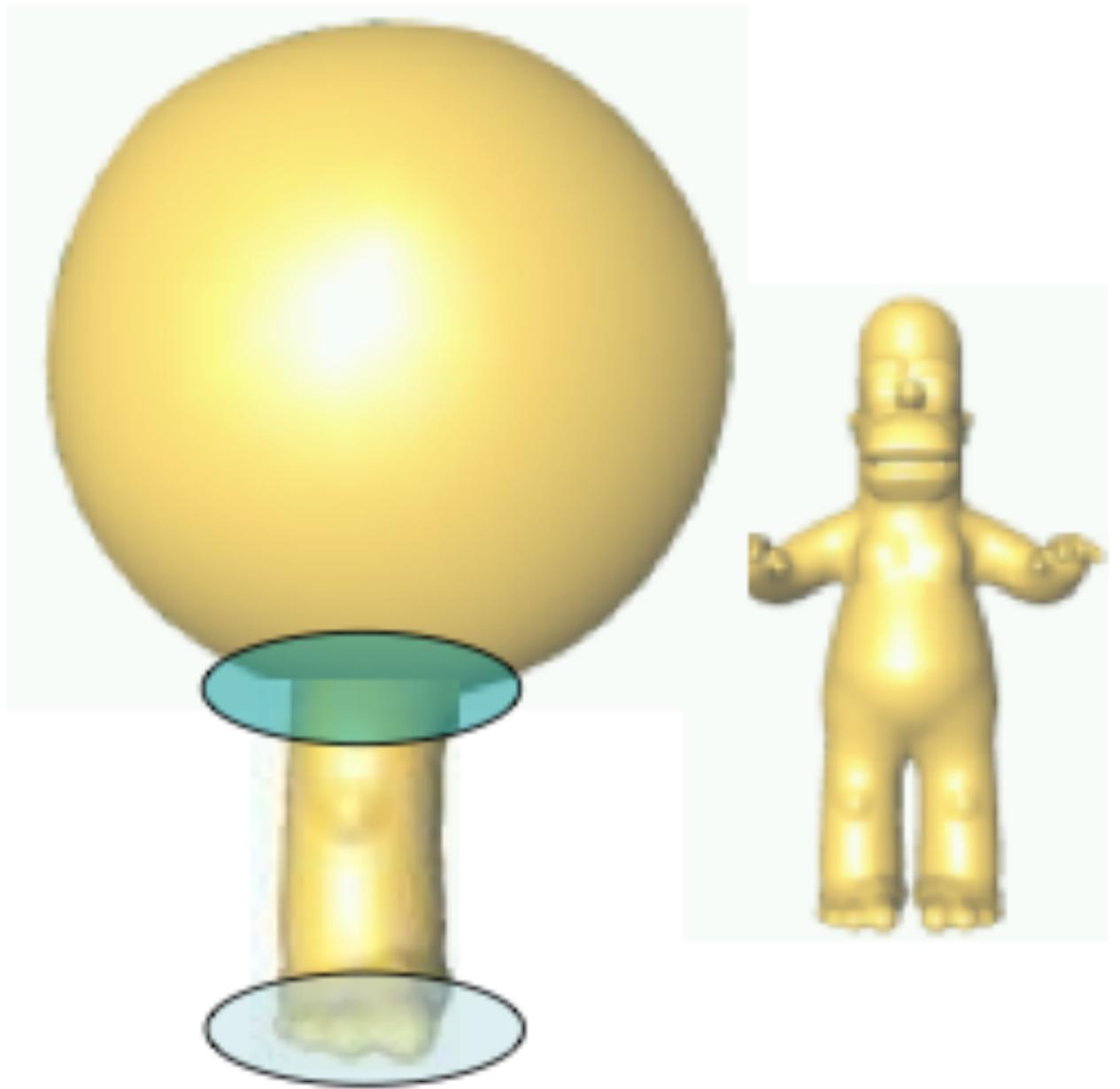
Gauss-Bonnet Theorem

- Sphere

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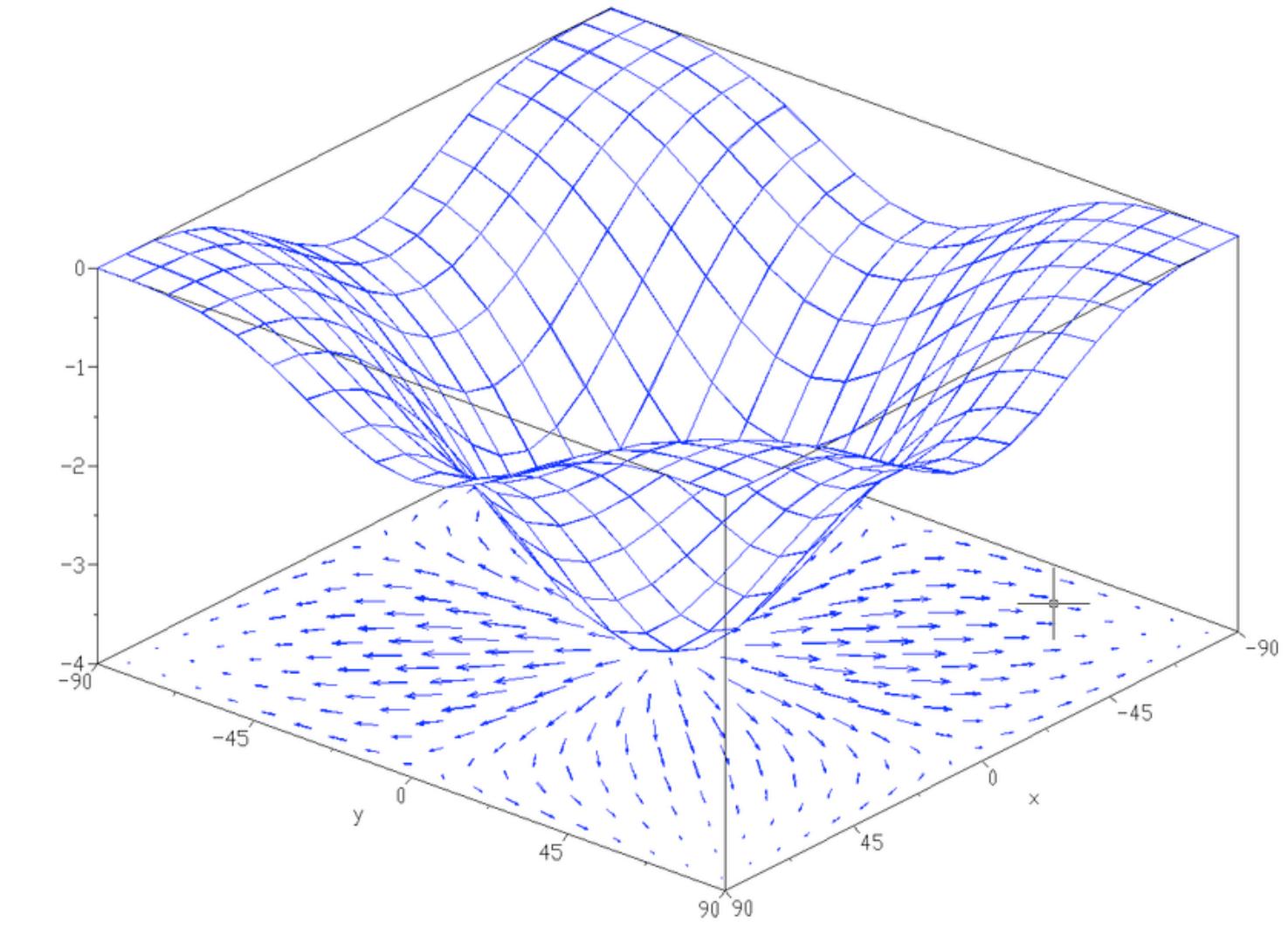
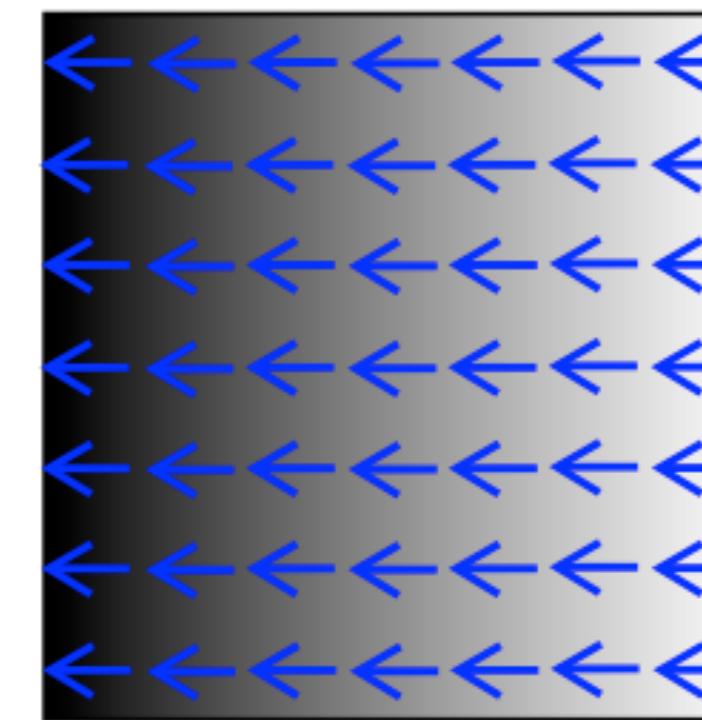
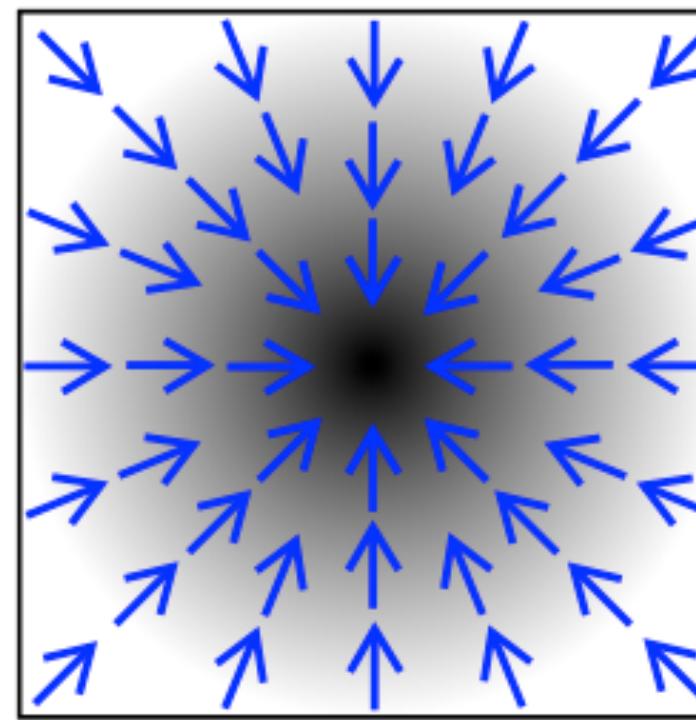
– when sphere is deformed new positive and negative curvature cancel out!

Differential Operators

- Gradient

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

- points in the direction of steepest ascent



Differential Operators



- Divergence

$$\operatorname{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$

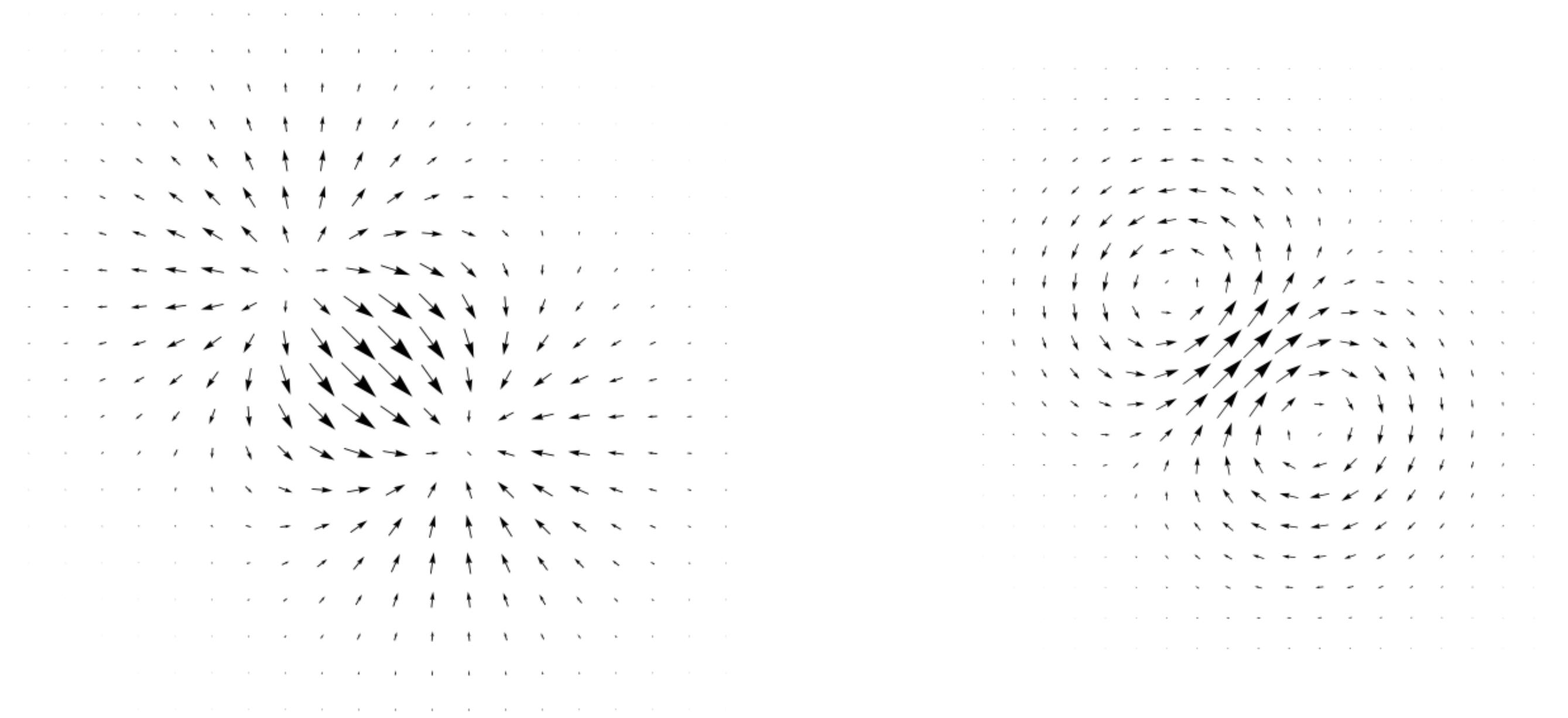
- volume density of outward flux of vector field
- magnitude of source or sink at given point
- Example: Incompressible fluid
 - velocity field is divergence-free

Differential Operators



- Divergence

$$\operatorname{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$

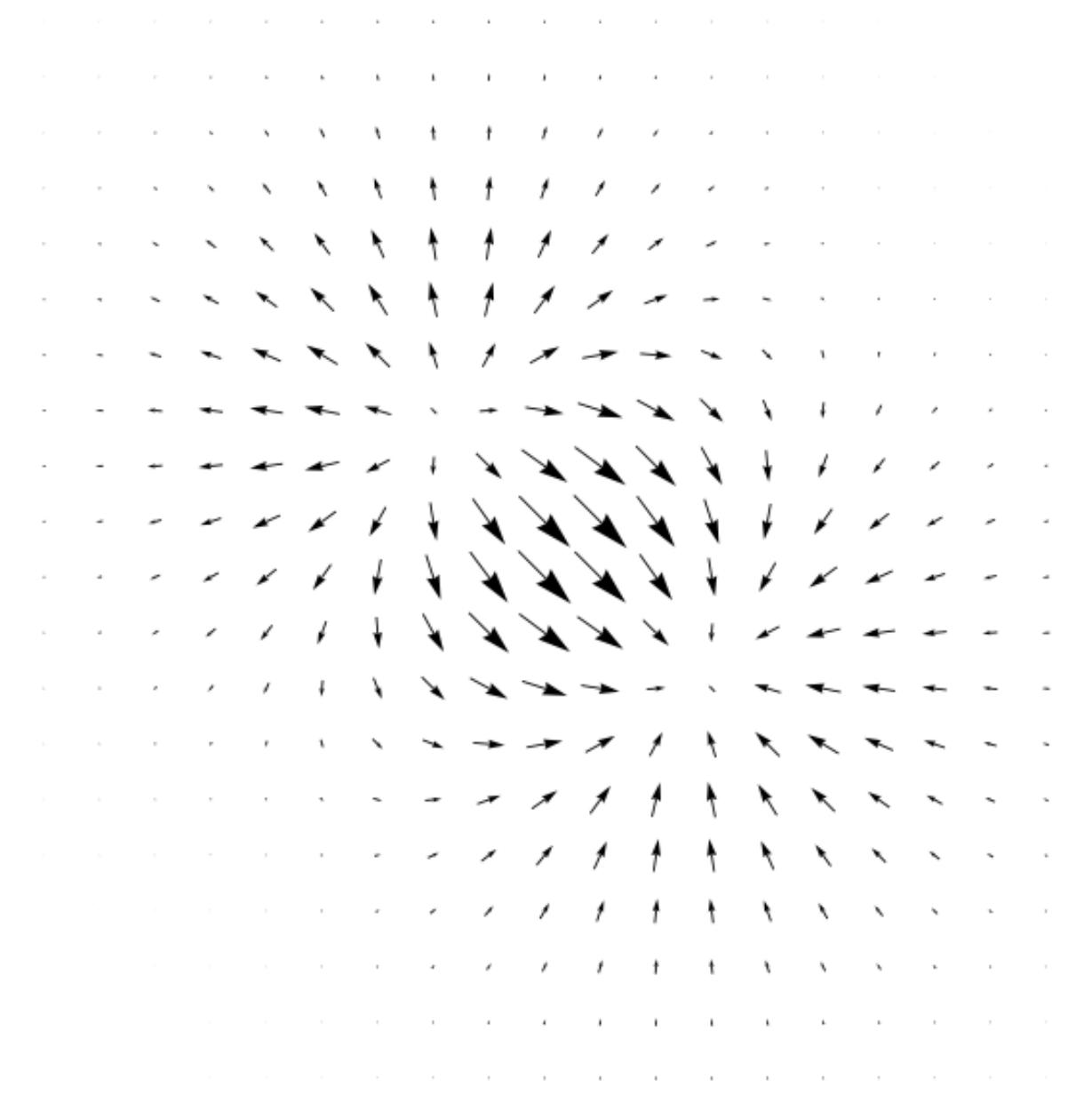


Differential Operators

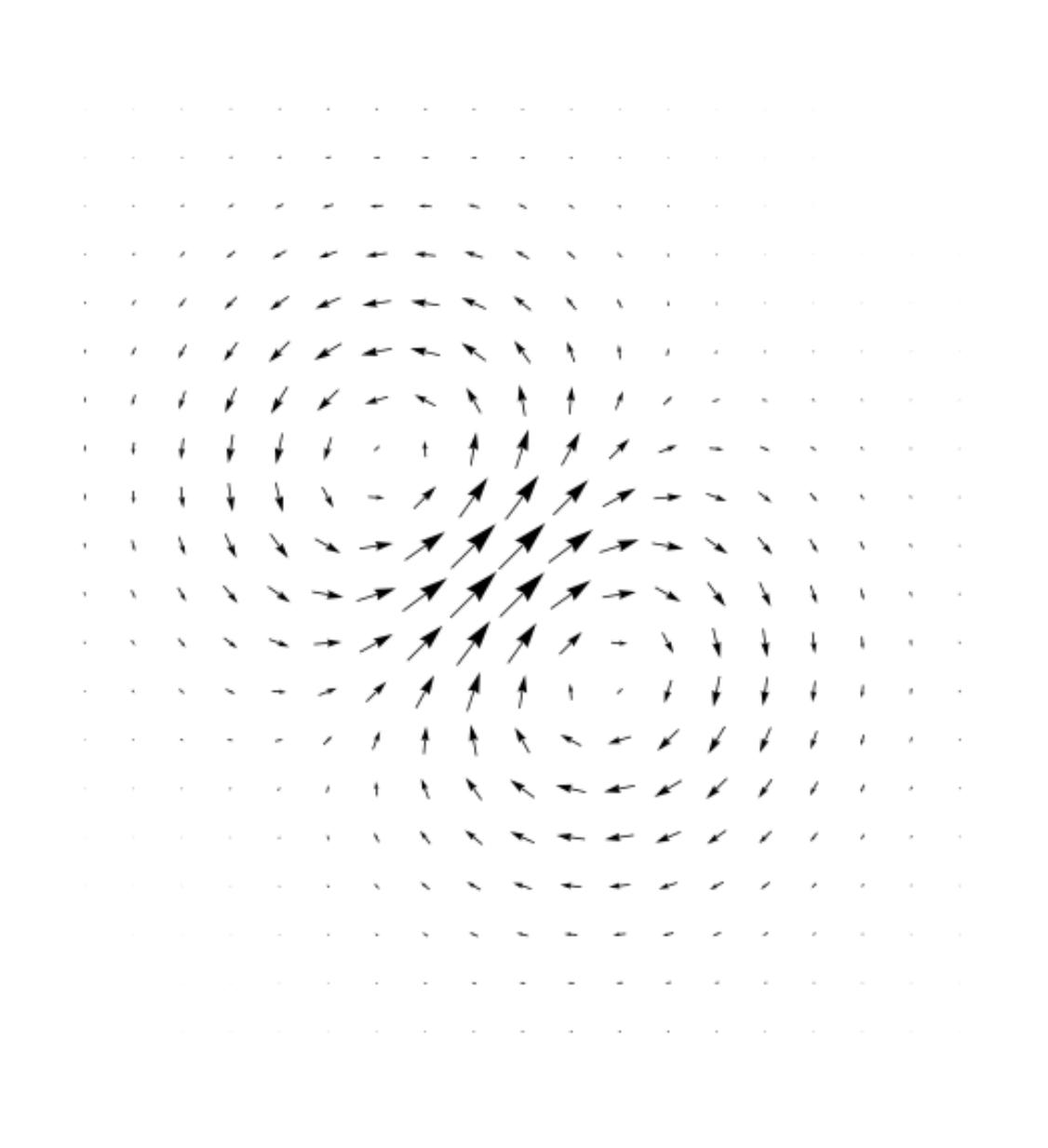


- Divergence

$$\operatorname{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$



high divergence



low divergence

Literature



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- M. Do Carmo: ***Differential Geometry of Curves and Surfaces***, Prentice Hall, 1976.
- A. Pressley: ***Elementary Differential Geometry***, Springer, 2010