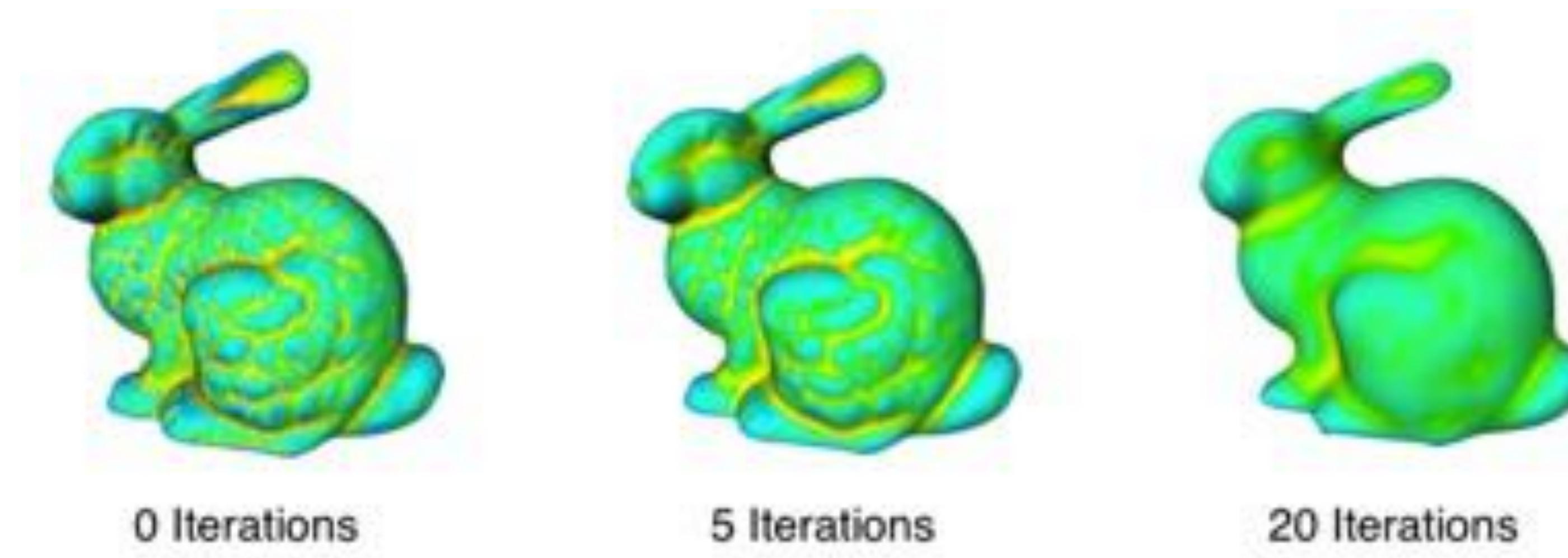


Mesh Smoothing



Outline



- Differential Geometry
- Discrete Differential Geometry
- Mesh Quality Measures

Differential Geometry: Surfaces



- Normals
- First/Second Fundamental forms
- Normal curvature
- Principal curvatures
- mean/Gauss Curvatures

Gauss-Bonnet Theorem



- For any closed manifold surface with Euler characteristic $\chi = 2 - 2g$

$$\int_{\Omega \in S} K(u, v) dudv = 2\pi\chi$$

$$\int K(\text{hand}) = \int K(\text{cow}) = \int K(\text{sphere}) = 4\pi$$

Gauss-Bonnet Theorem

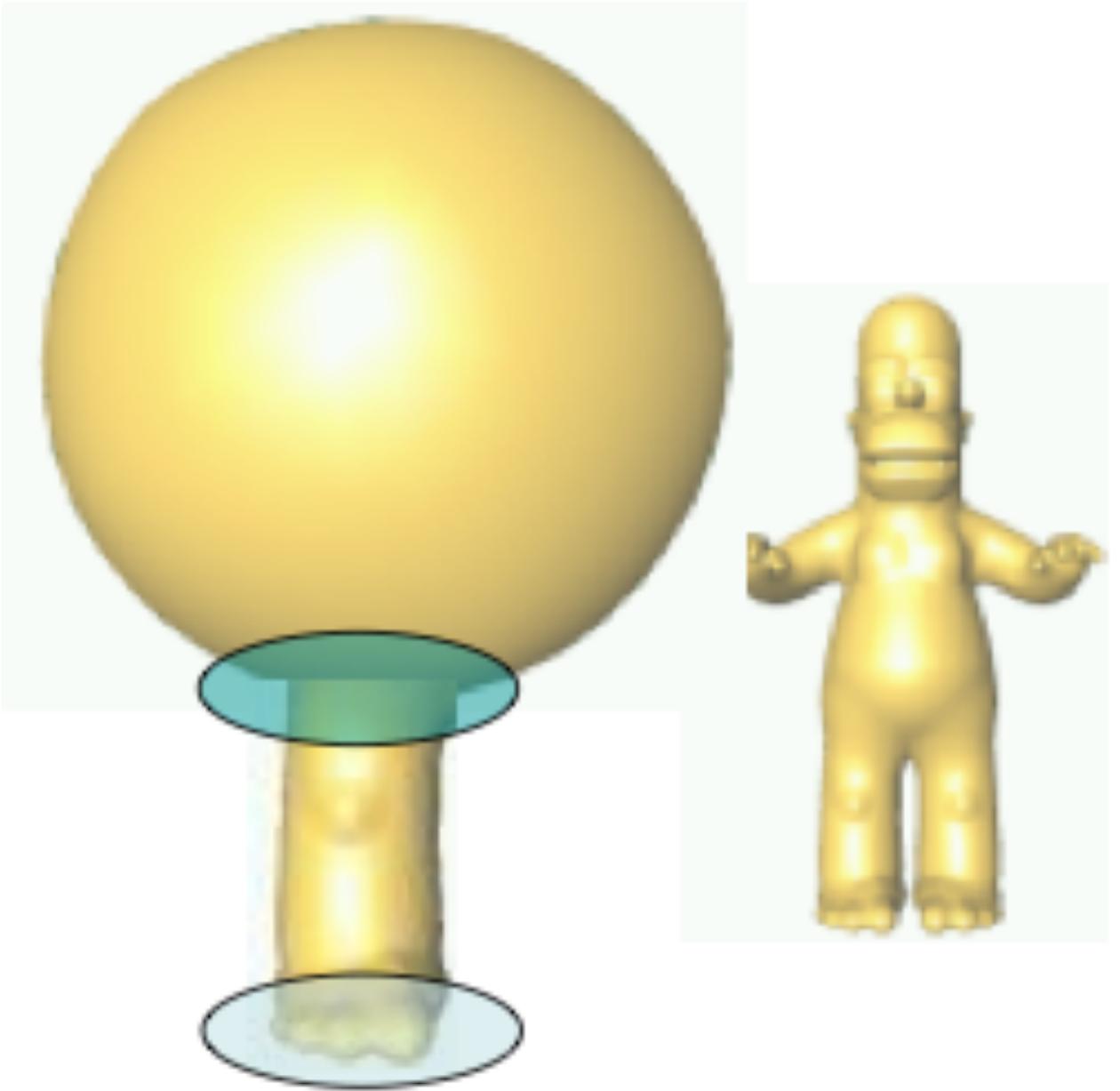
- Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

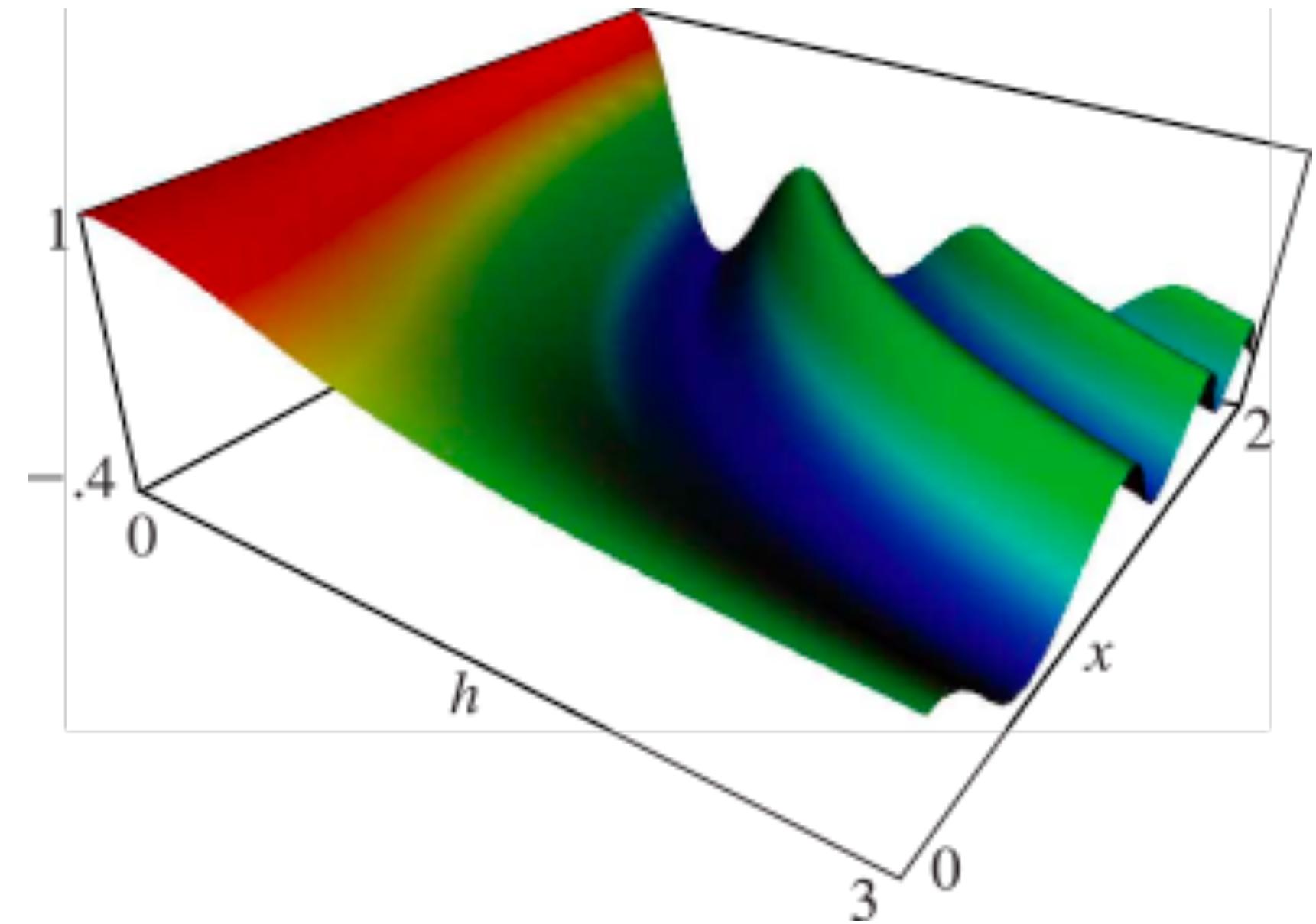
$$K = \kappa_1 \kappa_2 = 1/r^2$$

$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$

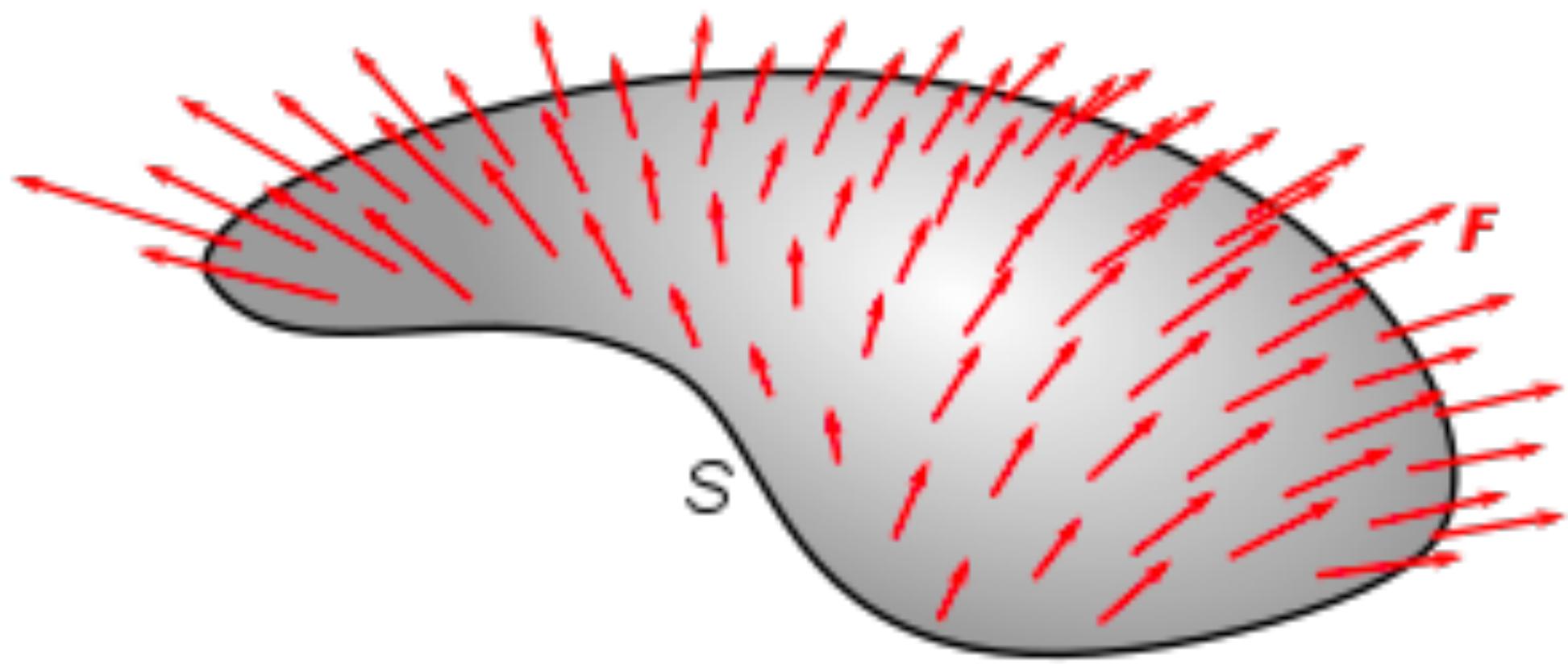
- when sphere is deformed new positive and negative curvature cancel out!



Functions on Surfaces



scalar values on surfaces



vector values on surfaces

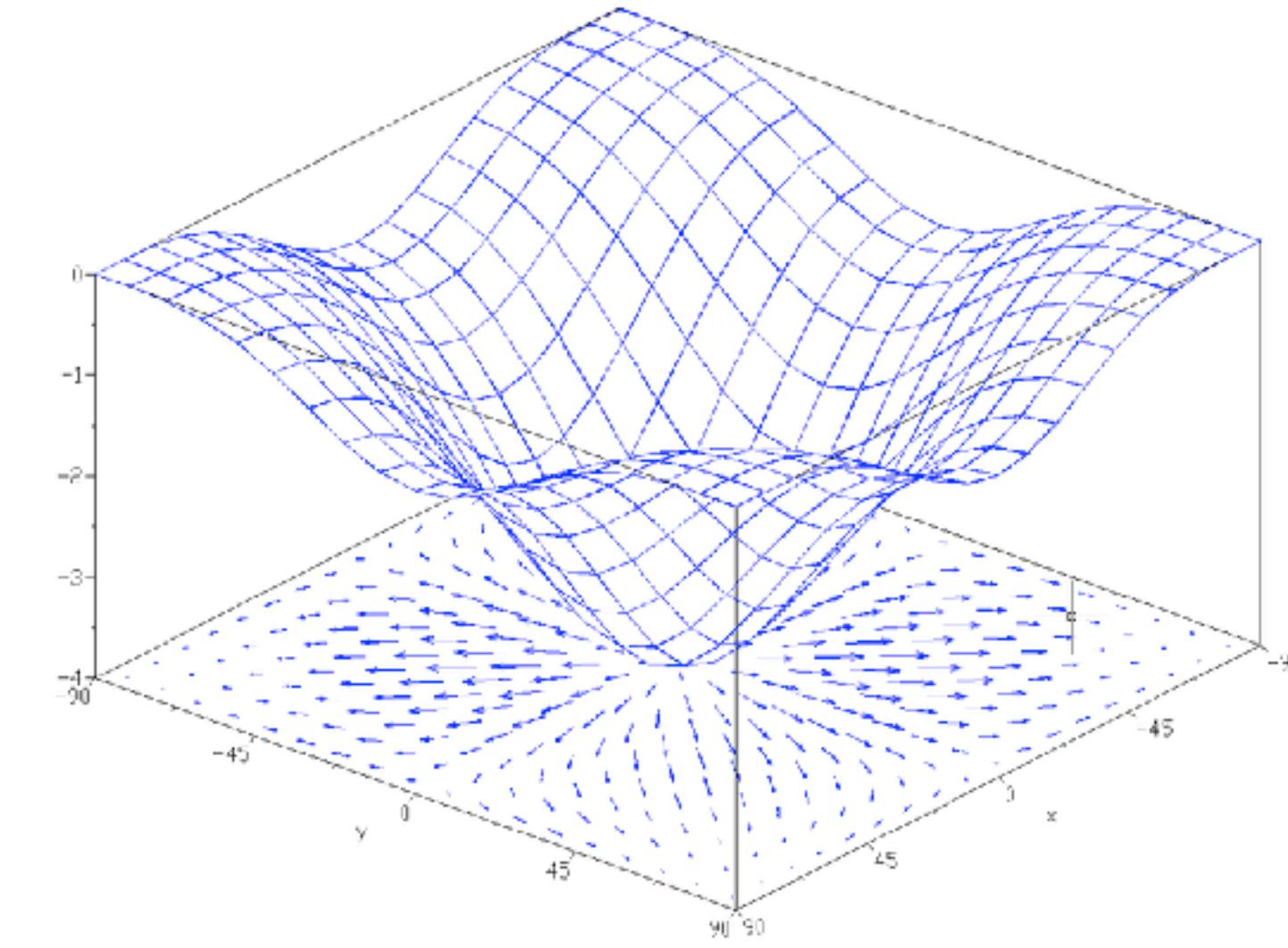
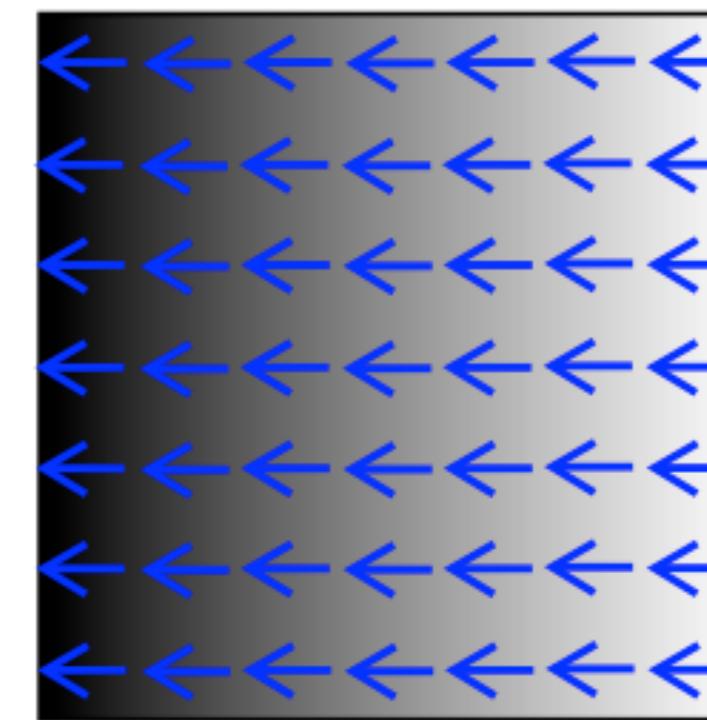
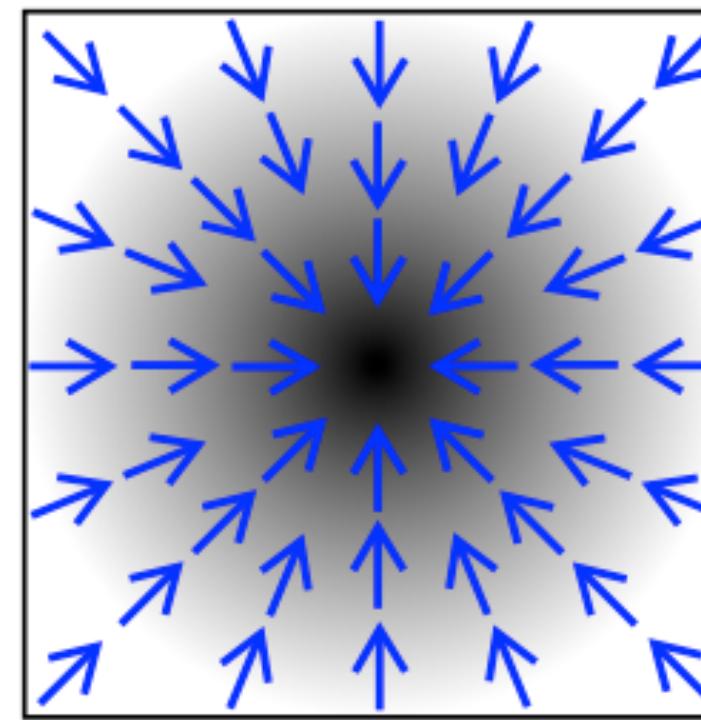
Differential Operators

- Gradient (vector)

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

$$\operatorname{div} g := \left\langle \left(\frac{\partial}{\partial x_1}, \dots \right), g \right\rangle = \nabla \cdot g$$

- points in the direction of steepest ascent



Differential Operators



$$\operatorname{div} g := \left\langle \left(\frac{\partial}{\partial x_1}, \dots \right), g \right\rangle = \nabla \cdot g$$

$$\operatorname{div} \mathbf{G} := \left\langle \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), (G_x, G_y, G_z) \right\rangle = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$$

Differential Operators



- Divergence

$$\operatorname{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$

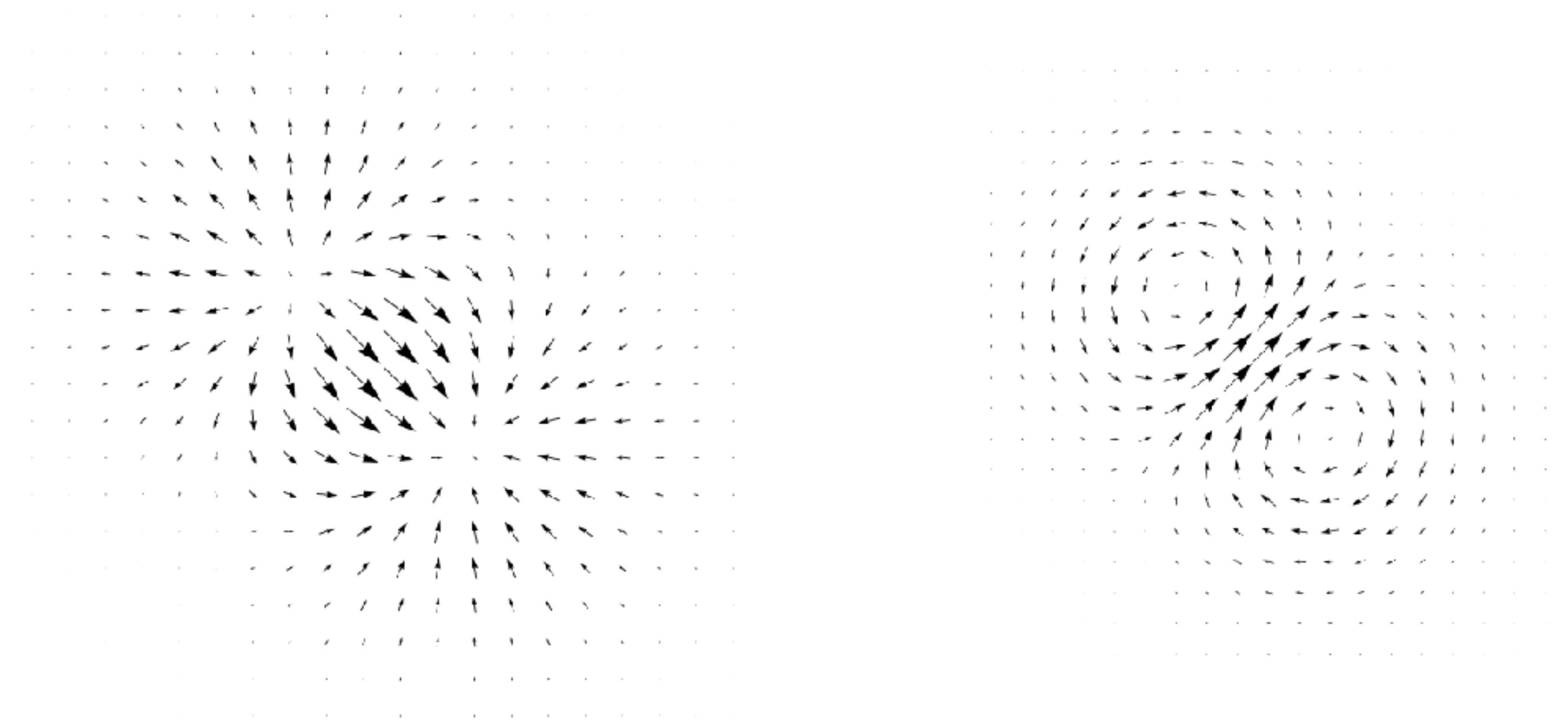
- volume density of outward flux of vector field
- magnitude of source or sink at given point
- Example: Incompressible fluid
 - velocity field is divergence-free

Differential Operators



- Divergence

$$\operatorname{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$



high divergence

low divergence

Laplace Operator



$$\operatorname{div} \nabla f$$

$$\operatorname{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

$$\operatorname{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2} = \Delta f$$

Laplace Operator



$$\Delta f = \operatorname{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

Diagram illustrating the components of the Laplace operator:

- Laplace operator
- function in Euclidean space
- gradient operator
- divergence operator
- 2nd partial derivatives
- Cartesian coordinates

Arrows point from each label to its corresponding term in the equation.

Laplace-Beltrami Operator



Extension of Laplace to **functions on manifolds**

$$\Delta_S f = \operatorname{div}_S \nabla_S f$$

Diagram illustrating the components of the Laplace-Beltrami operator:

- Laplace-Beltrami
- gradient operator
- function on manifold S
- divergence operator

The diagram shows arrows pointing from the words "Laplace-Beltrami", "gradient operator", and "divergence operator" to the corresponding terms in the equation $\Delta_S f = \operatorname{div}_S \nabla_S f$. Additionally, an arrow points from the word "function on manifold S " to the variable f .

Laplace-Beltrami Operator



Extension of Laplace to **functions on manifolds**

$$\Delta_S \mathbf{x} = \operatorname{div}_S \nabla_S \mathbf{x} = -2H \mathbf{n}$$

A diagram illustrating the components of the Laplace-Beltrami operator. The central equation is $\Delta_S \mathbf{x} = \operatorname{div}_S \nabla_S \mathbf{x} = -2H \mathbf{n}$. Arrows point from each term to its corresponding definition:

- An arrow points from "Laplace-Beltrami" to $\Delta_S \mathbf{x}$.
- An arrow points from "gradient operator" to $\nabla_S \mathbf{x}$.
- An arrow points from "mean curvature" to $-2H$.
- An arrow points from "surface normal" to \mathbf{n} .
- An arrow points from "coordinate function" to div_S .
- An arrow points from "divergence operator" to div_S .

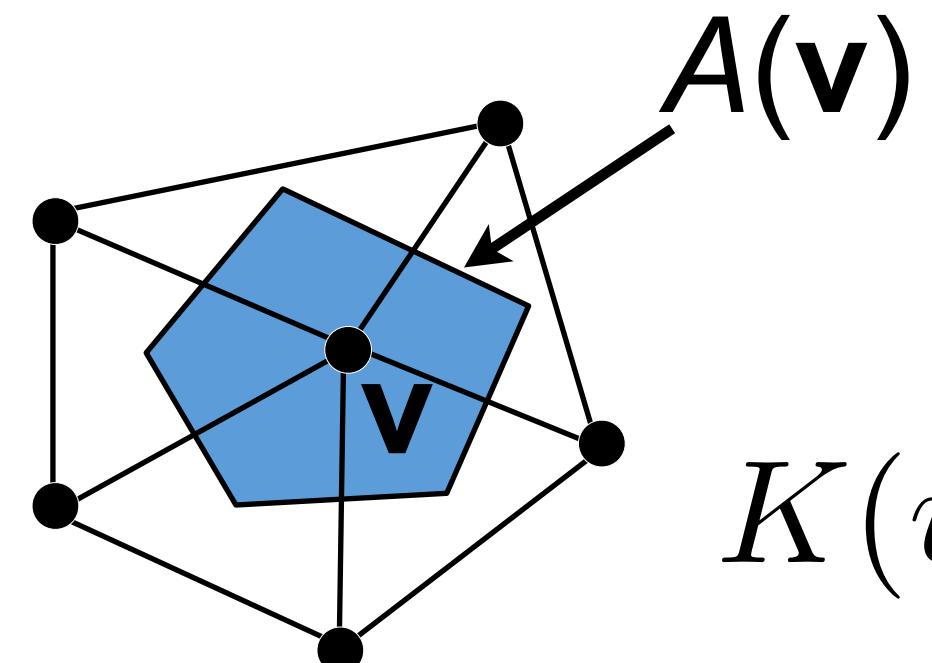
Outline



- Differential Geometry
- Discrete Differential Geometry
- Mesh Quality Measures

Discrete Curvatures

- How to discretize curvatures on a mesh?
 - Zero curvature within triangles
 - Infinite curvature at edges / vertices
 - Point-wise definition does not make sense
- Approximate differential properties at point \mathbf{v} as average over local neighborhood $A(\mathbf{v})$
 - \mathbf{v} is a mesh vertex
 - $A(\mathbf{v})$ within one-ring neighborhood



$$K(\mathbf{v}) \approx \frac{1}{A(\mathbf{v})} \int_{A(\mathbf{v})} K(\mathbf{x}) \, dA$$

Area Discretizations



Voronoi Cells



Discrete Curvatures

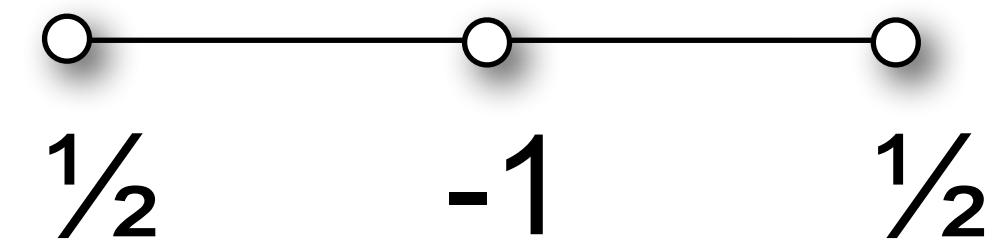


- Which curvatures to discretize?
 - Discretize Laplace-Beltrami operator
 - Laplace-Beltrami gives us mean curvature H
 - Discretize Gaussian curvature K

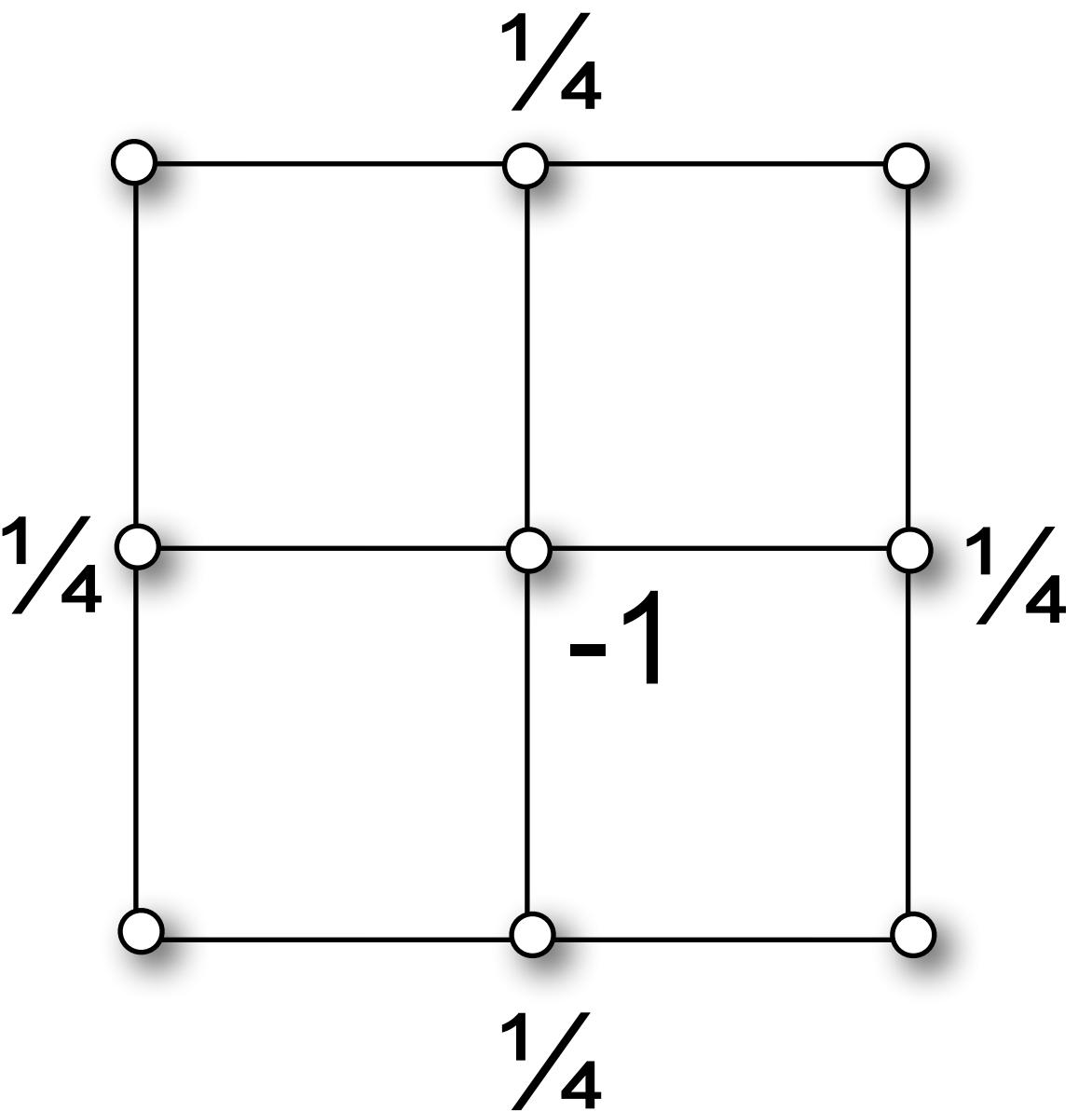
Laplace Operator on Meshes?



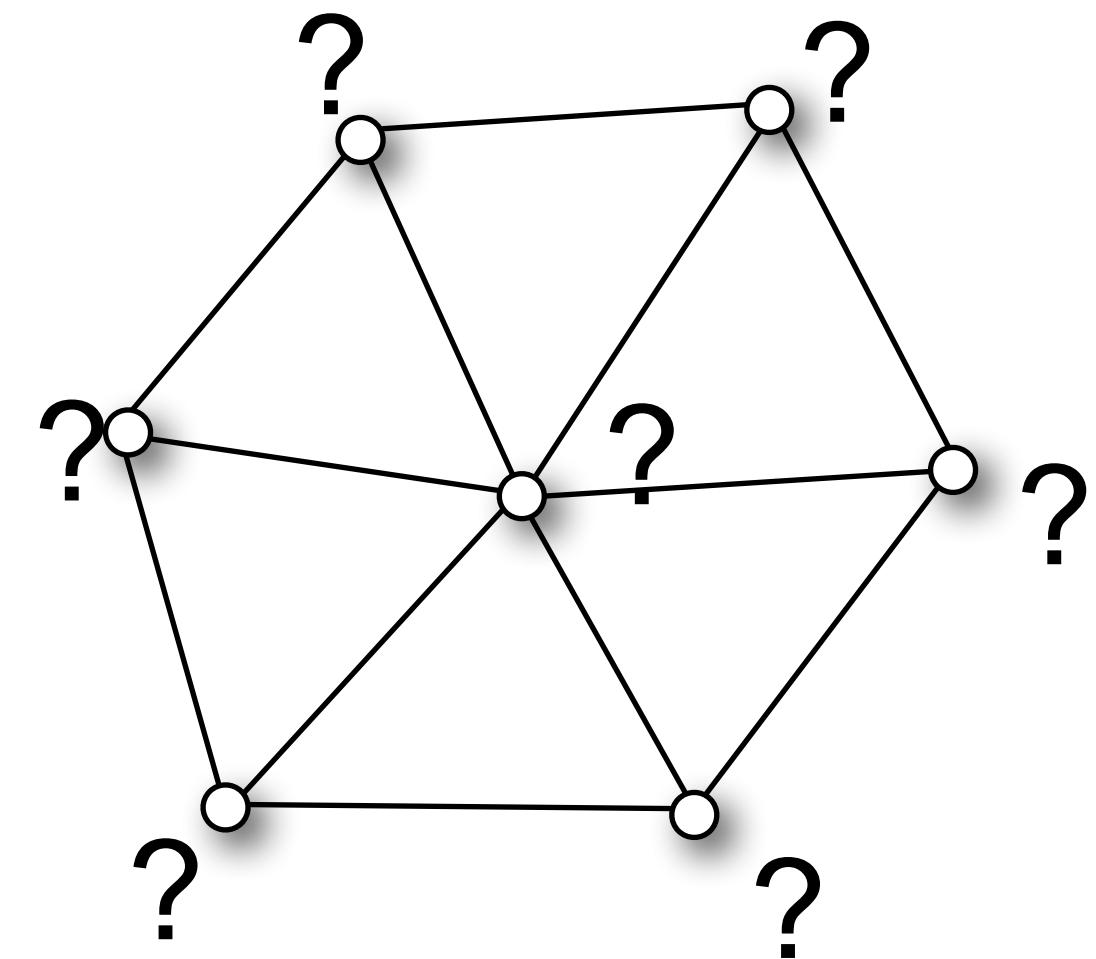
- Extend finite differences to meshes?
 - What weights per vertex / edge?



1D grid

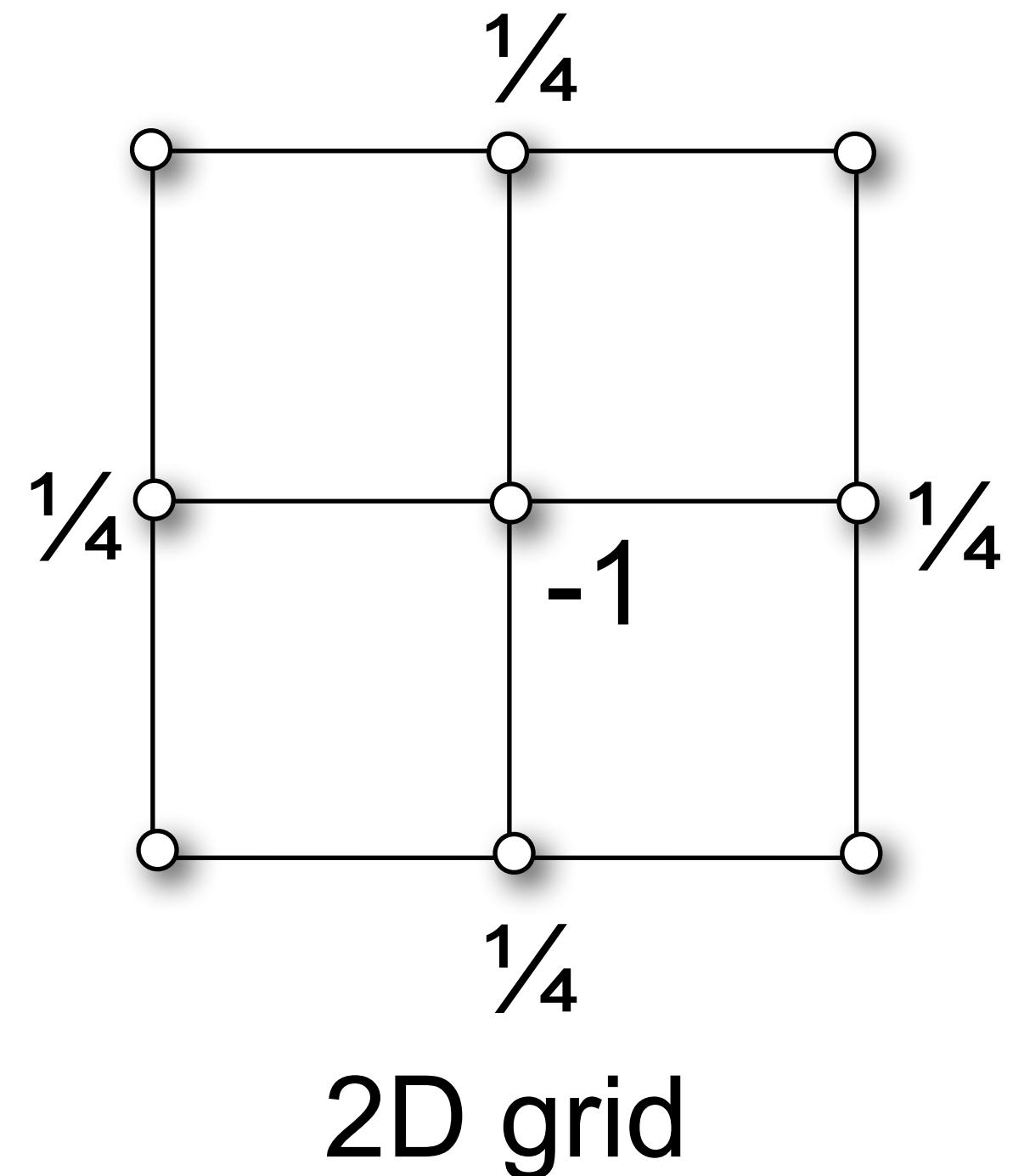


2D grid



2D/3D mesh

Matrix Version of Neighborhood

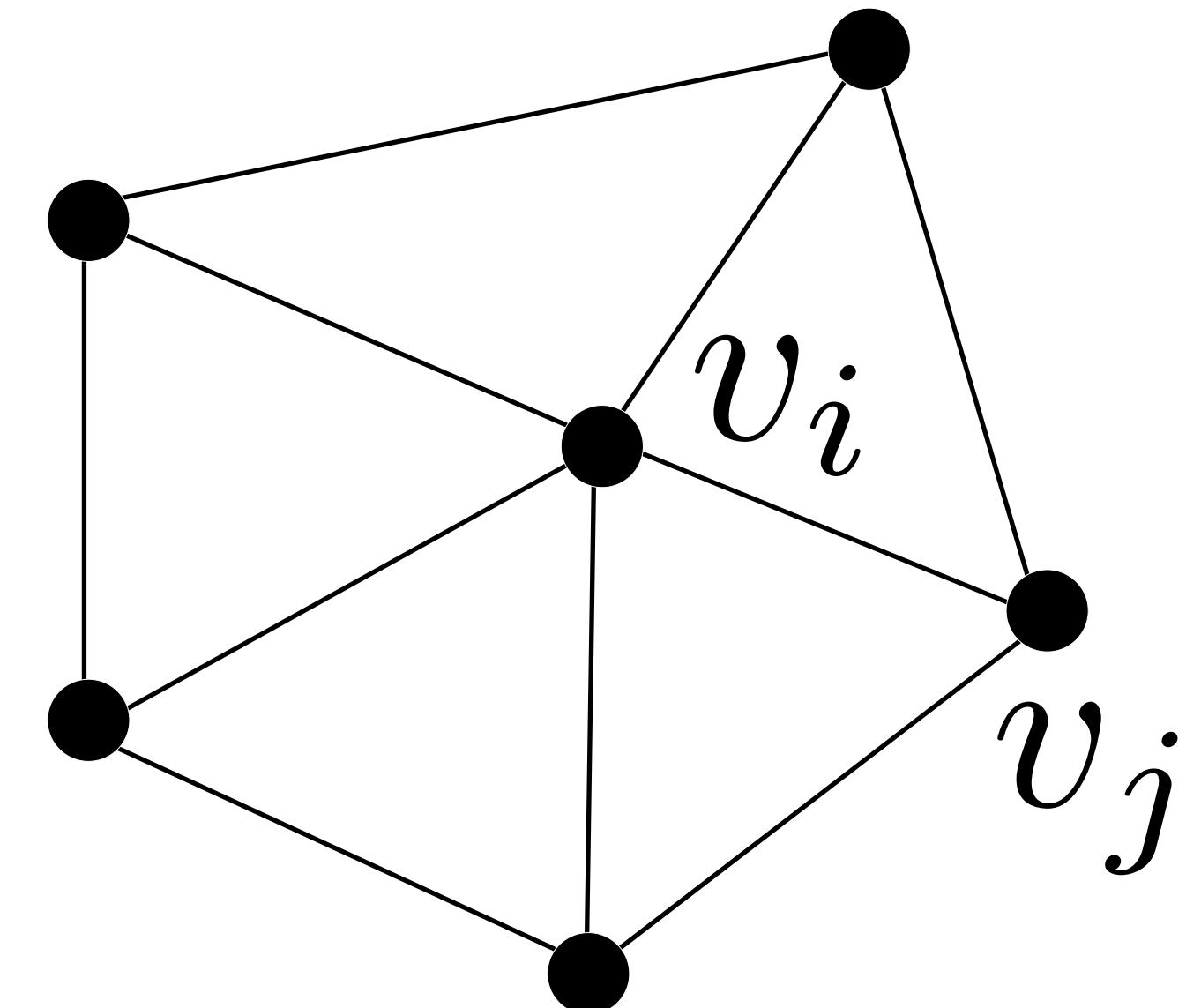


Uniform Laplacian Matrix

- Uniform discretization

$$\Delta_{\text{uni}} f(v_i) := \frac{1}{|\mathcal{N}_1(v_i)|} \sum_{v_j \in \mathcal{N}_1(v_i)} (f(v_j) - f(v_i))$$

- Properties
 - depends only on connectivity
 - simple and efficient
 - **bad approximation for irregular triangulations**
 - can give non-zero H for planar meshes
 - tangential drift for mesh smoothing

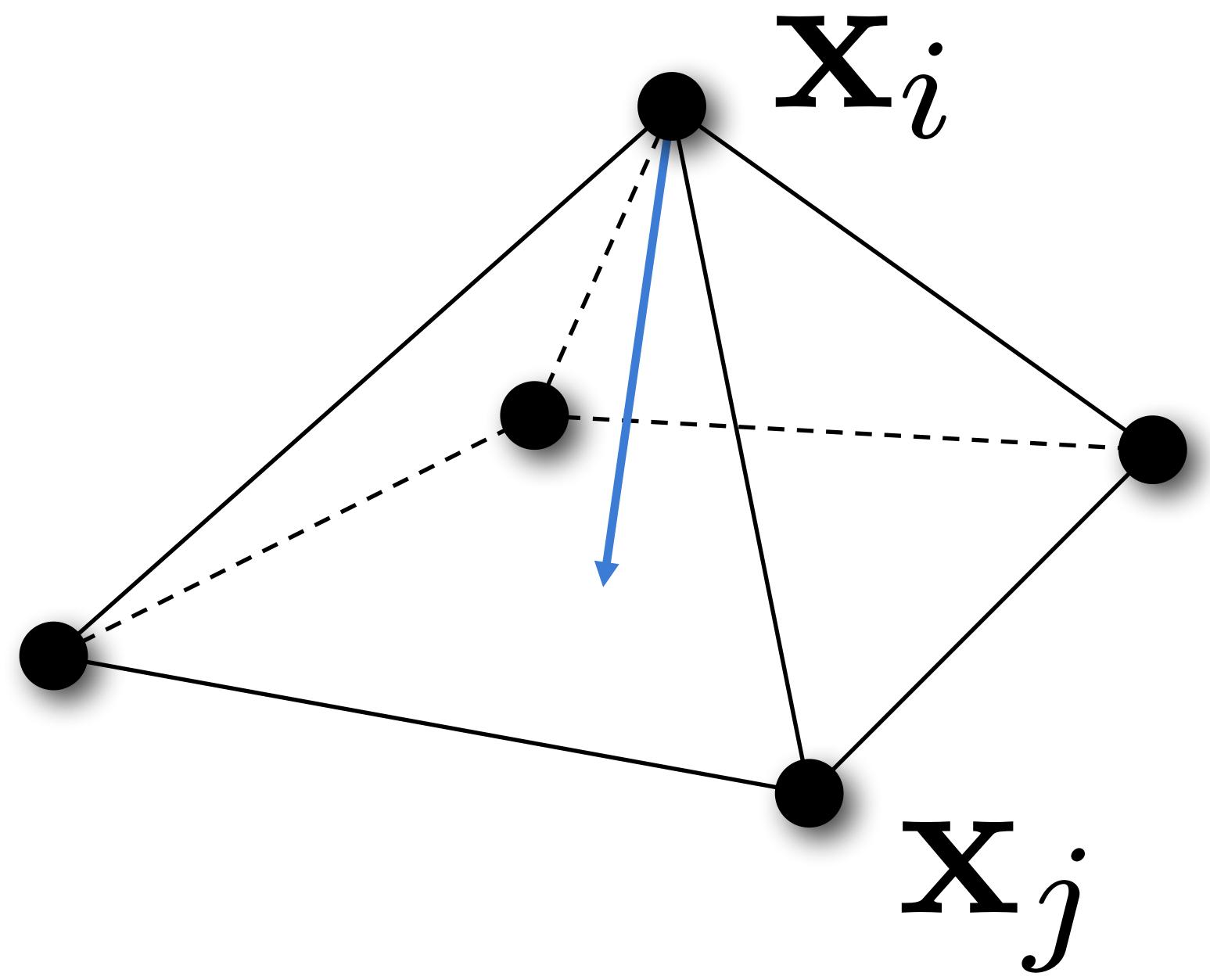


Uniform Laplacian Matrix

- Uniform discretization

$$\Delta_{\text{uni}} \mathbf{x}_i := \frac{1}{|\mathcal{N}_1(v_i)|} \sum_{v_j \in \mathcal{N}_1(v_i)} (\mathbf{x}_j - \mathbf{x}_i) \approx -2H\mathbf{n}$$

- Properties
 - depends only on connectivity
 - simple and efficient
 - **bad approximation for irregular triangulations**
 - can give non-zero H for planar meshes
 - tangential drift for mesh smoothing

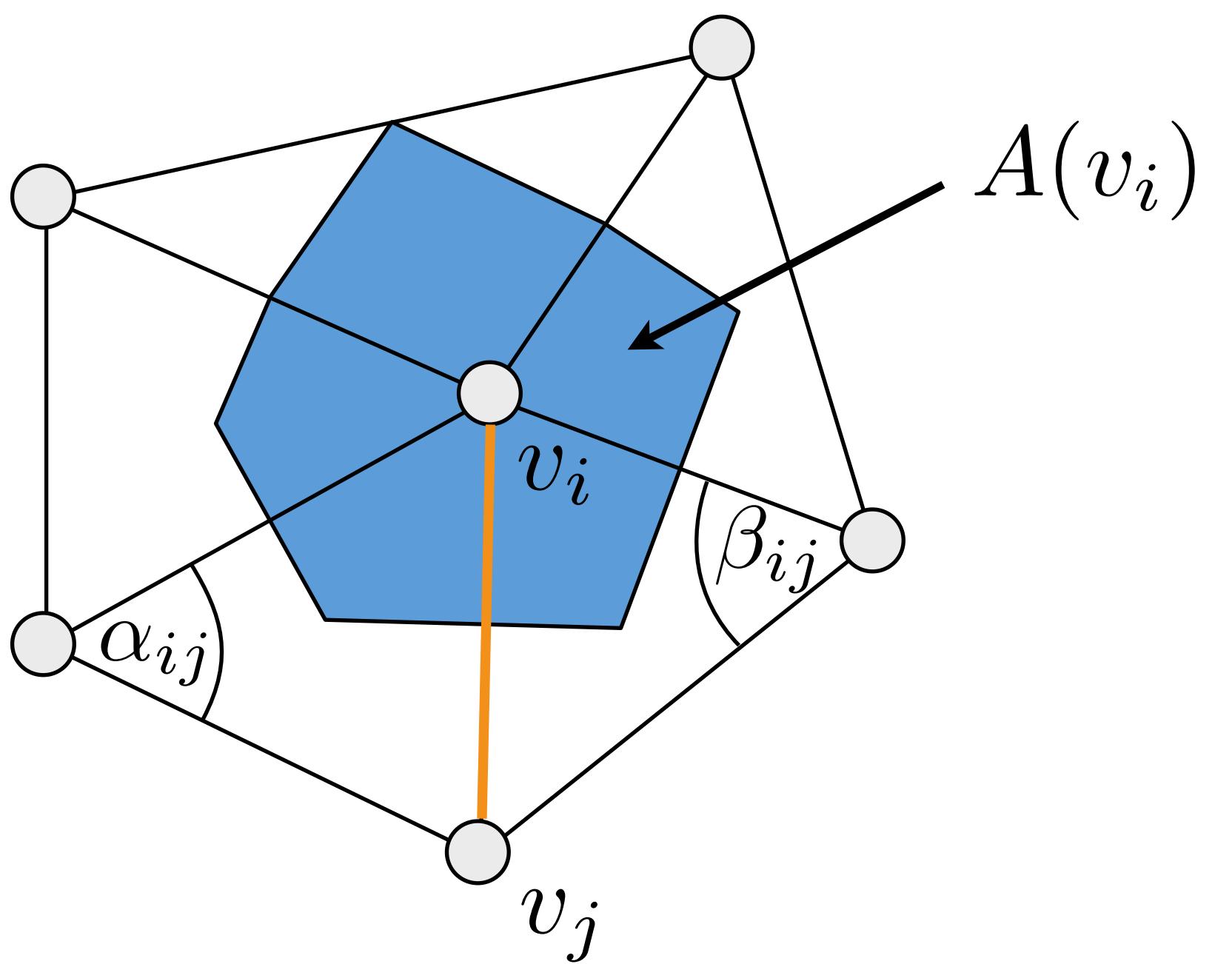


Discrete Laplace-Beltrami



- Cotangent discretization

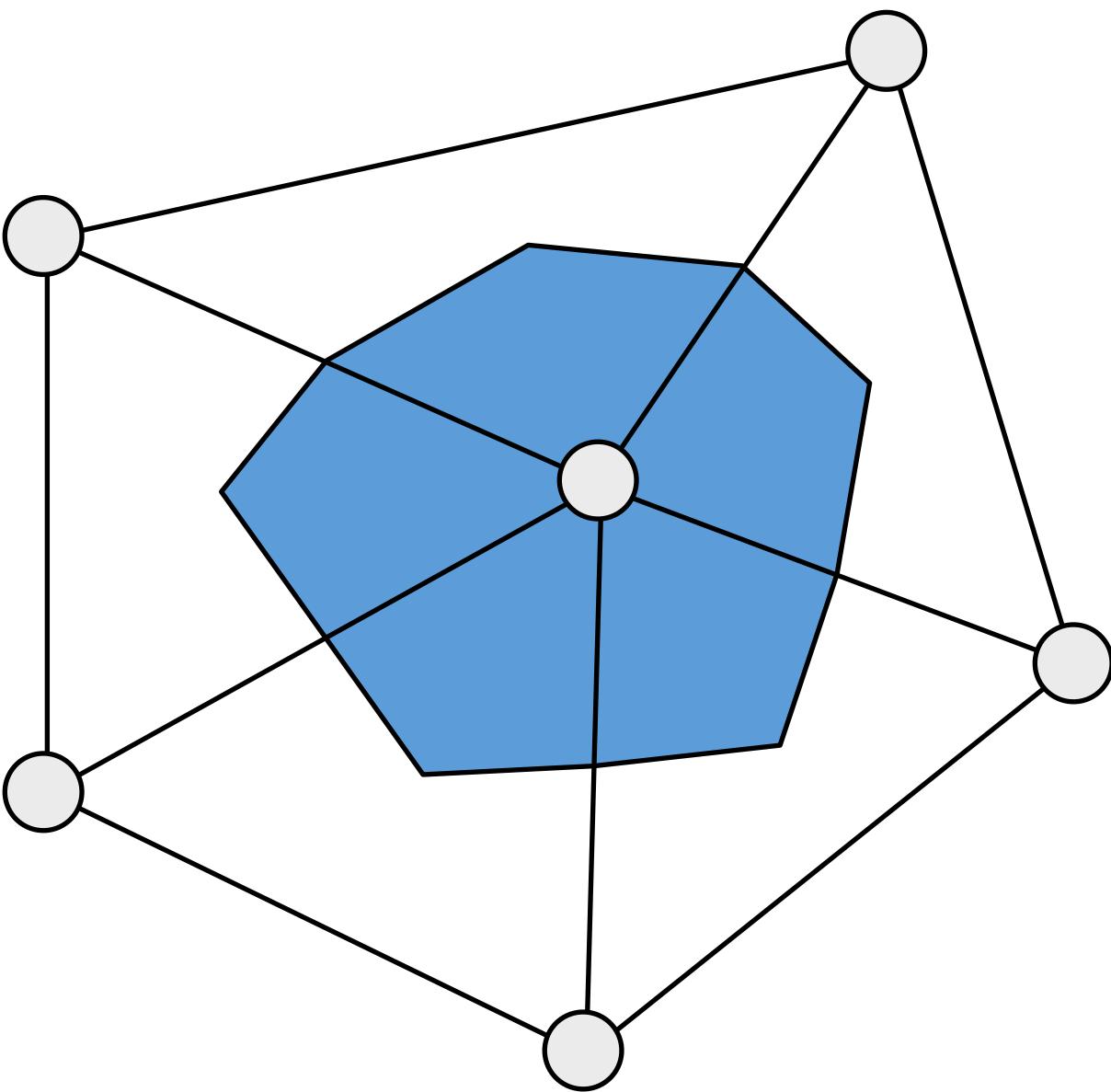
$$\Delta_S f(v_i) := \frac{1}{2A(v_i)} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot \alpha_{ij} + \cot \beta_{ij}) (f(v_j) - f(v_i))$$



$$\Delta_S \mathbf{x} = -2H\mathbf{n}$$

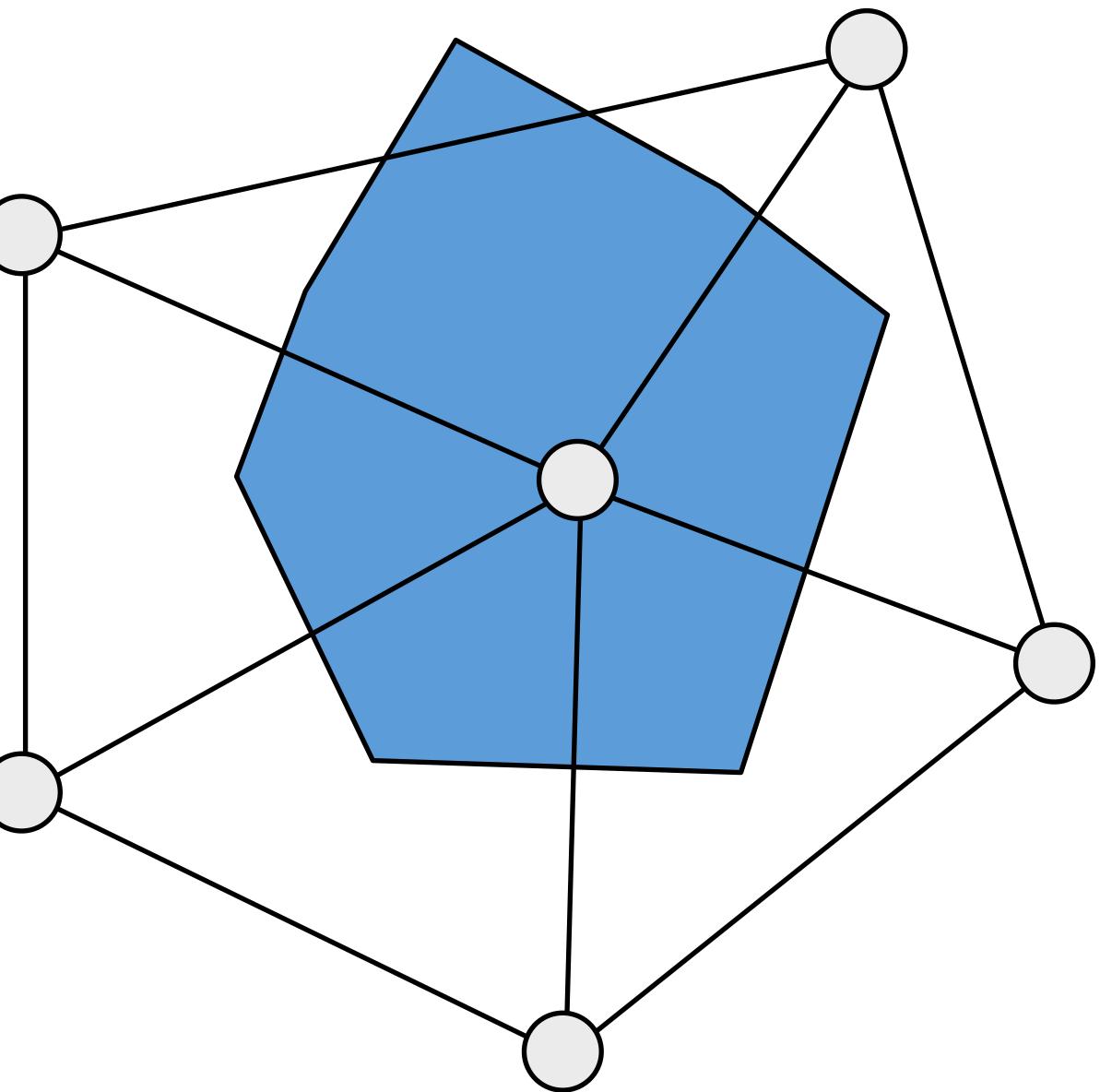
#1: Barycentric Cells

- Connect edge midpoints and triangle barycenters
 - Simple to compute
 - Area is $1/3$ of triangle areas



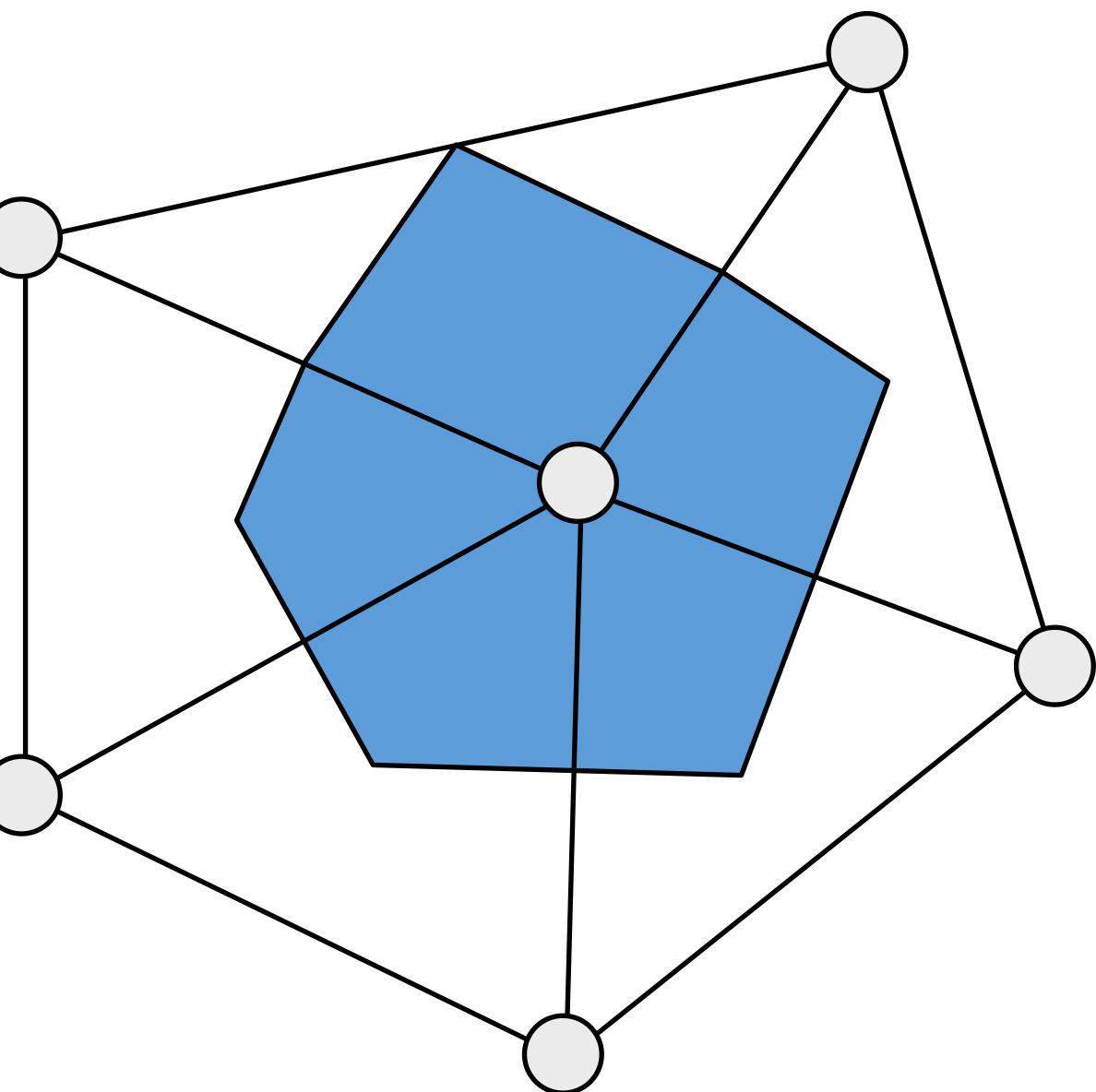
#2: Voronoi Cells

- Connect edge midpoints and
 - Circumcenters for all triangles
 - What about obtuse triangles?



#3: Mixed Cells

- Connect edge midpoints and
 - Circumcenters for non-obtuse triangles
 - Midpoint of opposite edge for obtuse triangles
 - Better approximation, more complex to compute...



Discrete Laplace-Beltrami



- Cotangent discretization

$$\Delta_S f(v) := \frac{1}{2A(v)} \sum_{v_i \in \mathcal{N}_1(v)} (\cot \alpha_i + \cot \beta_i) (f(v_i) - f(v))$$

- Problems
 - weights can become negative (when?)
 - depends on triangulation
- Still the most widely used discretization

Discrete Curvatures

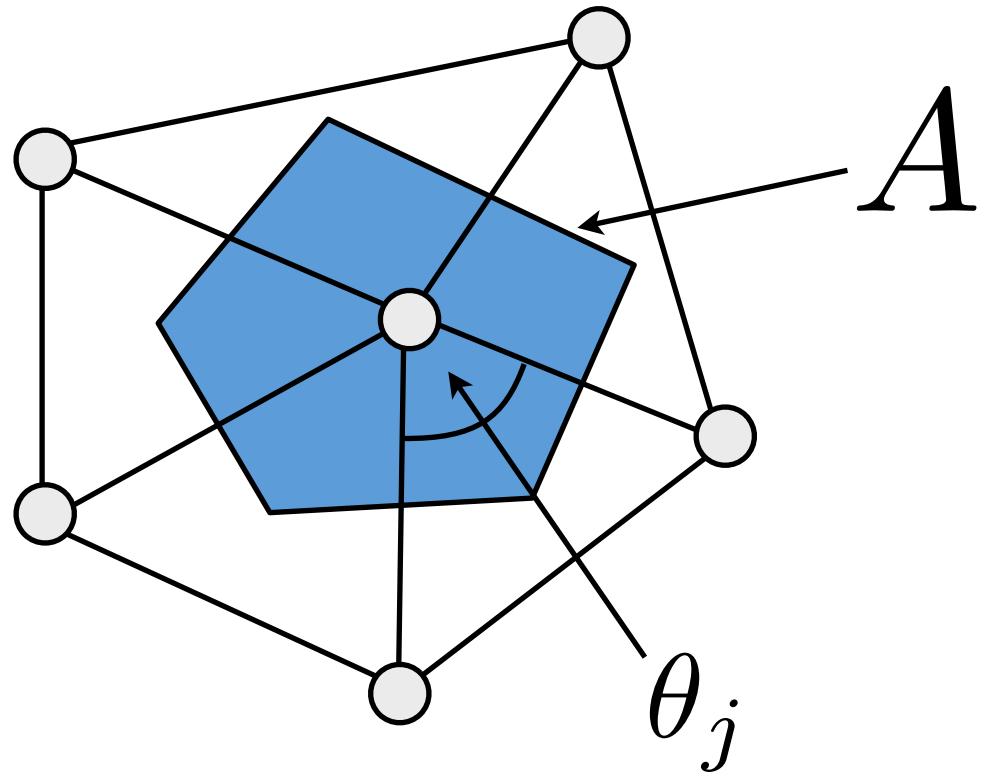
- Mean curvature (absolute value)

$$H = \frac{1}{2} \|\Delta_S \mathbf{x}\|$$

- Gaussian curvature

$$K = (2\pi - \sum_j \theta_j)/A$$

$$\Delta_S \mathbf{x} = -2H \mathbf{n}$$



Literature



- Taubin: *A signal processing approach to fair surface design*, SIGGRAPH 1996.
- Desbrun et al.: *Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow*, SIGGRAPH 1999.
- Meyer et al.: *Discrete Differential-Geometry Operators for Triangulated 2-Manifolds*, VisMath 2002.
- Wardetzky, Mathur, Kaelberer, Grinspun: Discrete Laplace Operators: No free lunch, SGP 2007

Outline



- Differential Geometry
- Discrete Differential Geometry
- **Mesh Quality Measures**

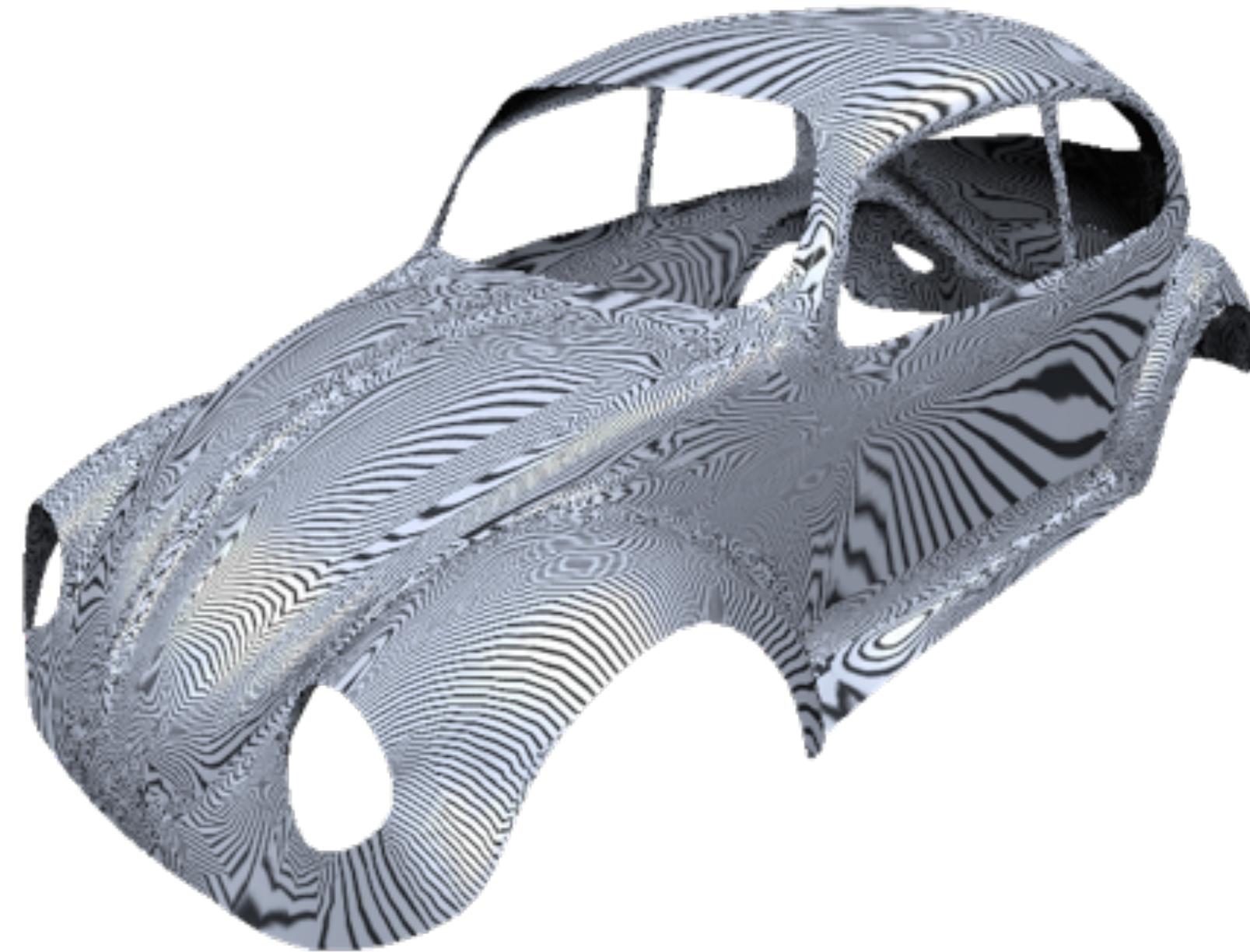
Mesh Quality

- Visual inspection of “sensitive” attributes
 - Specular shading



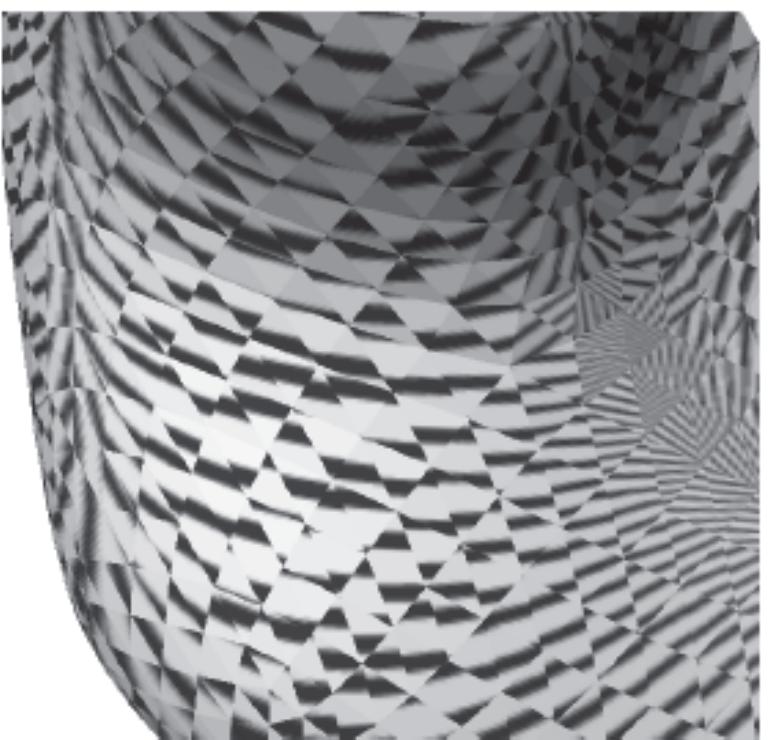
Mesh Quality

- Visual inspection of “sensitive” attributes
 - Specular shading
 - Reflection lines

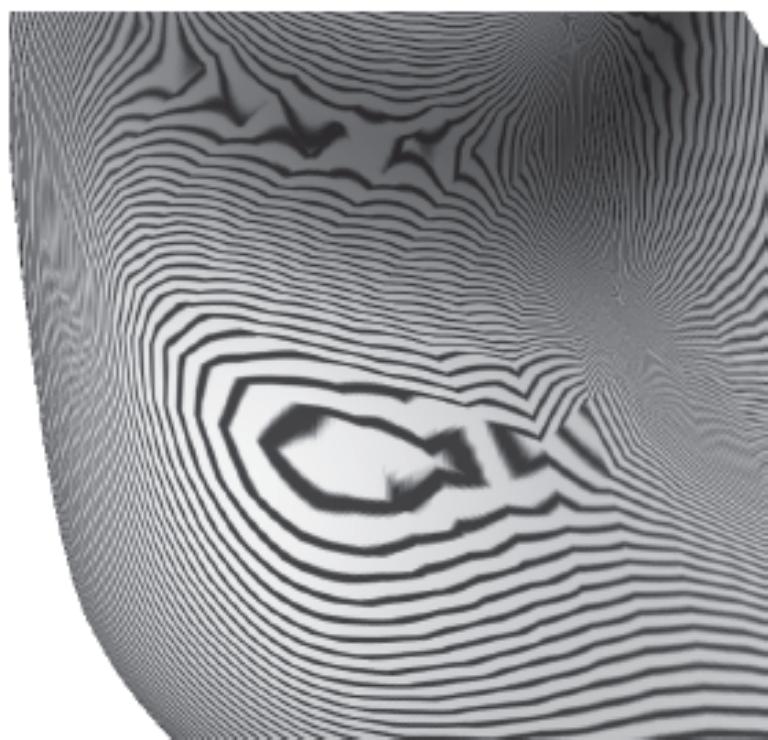


Mesh Quality

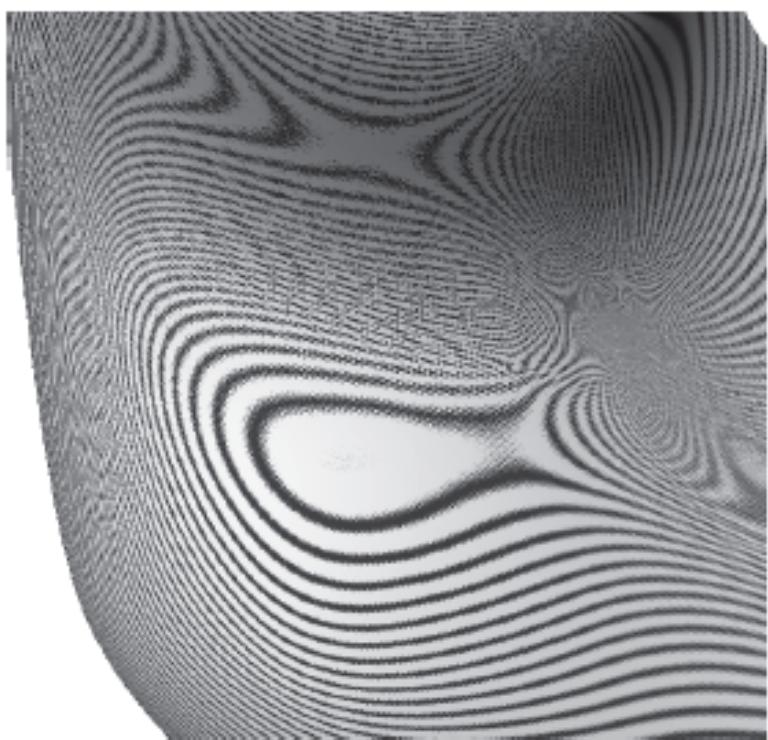
- Visual inspection of “sensitive” attributes
 - Specular shading
 - Reflection lines
 - differentiability one order lower than surface
 - can be efficiently computed using graphics hardware



C^0



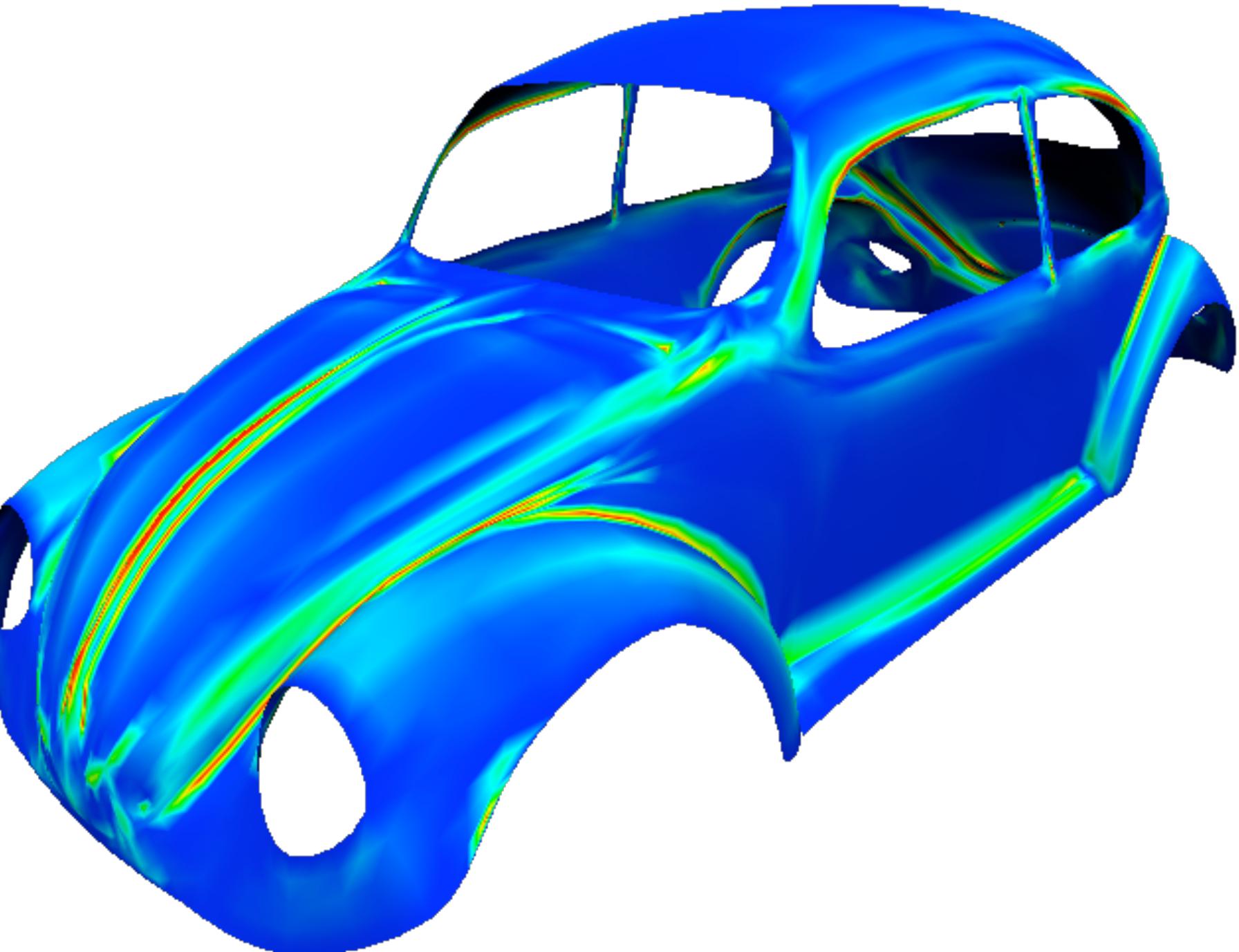
C^1



C^2

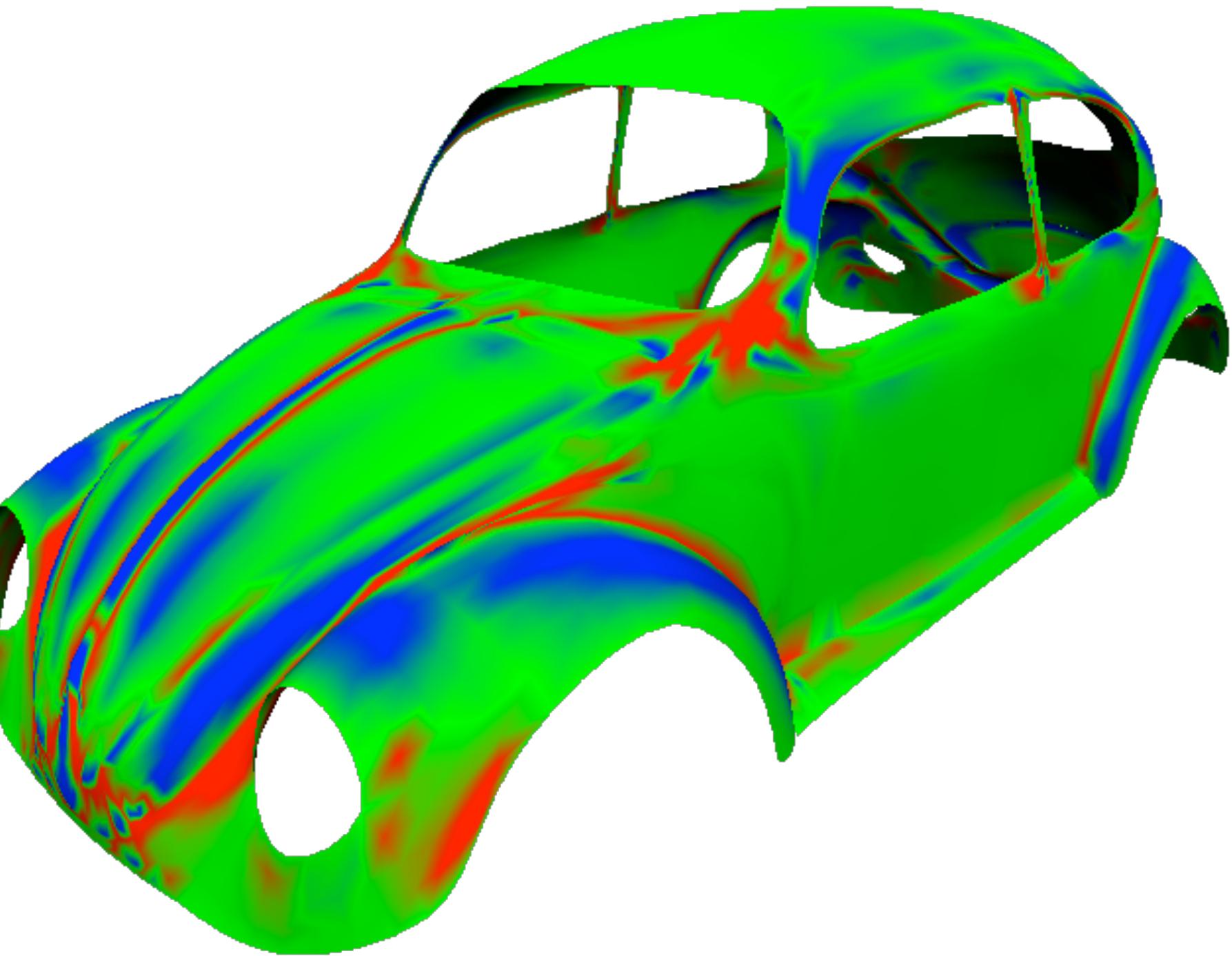
Mesh Quality

- Visual inspection of “sensitive” attributes
 - Specular shading
 - Reflection lines
 - Curvature
 - Mean curvature



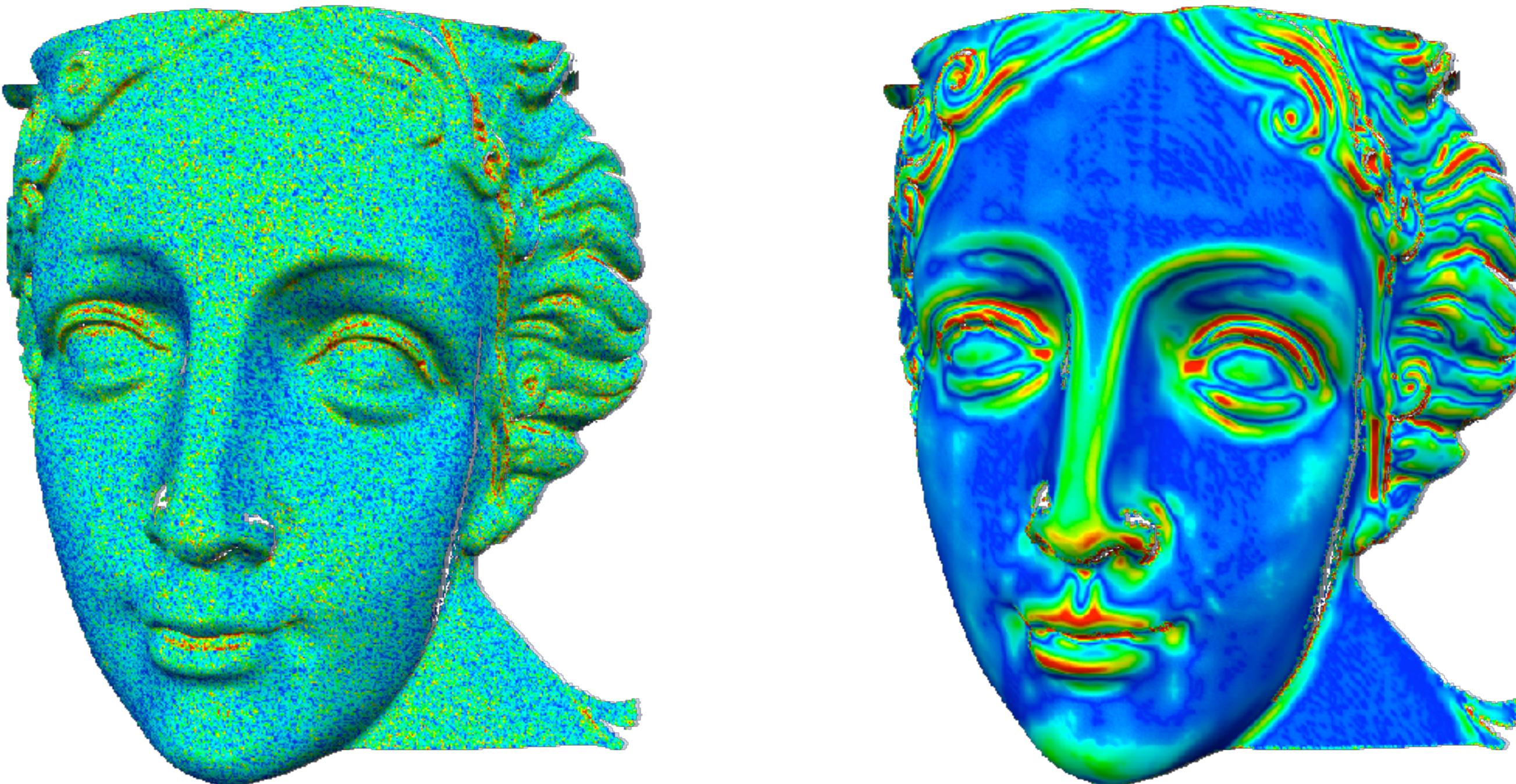
Mesh Quality

- Visual inspection of “sensitive” attributes
 - Specular shading
 - Reflection lines
 - Curvature
 - Mean curvature
 - Gauss curvature



Mesh Quality Criteria

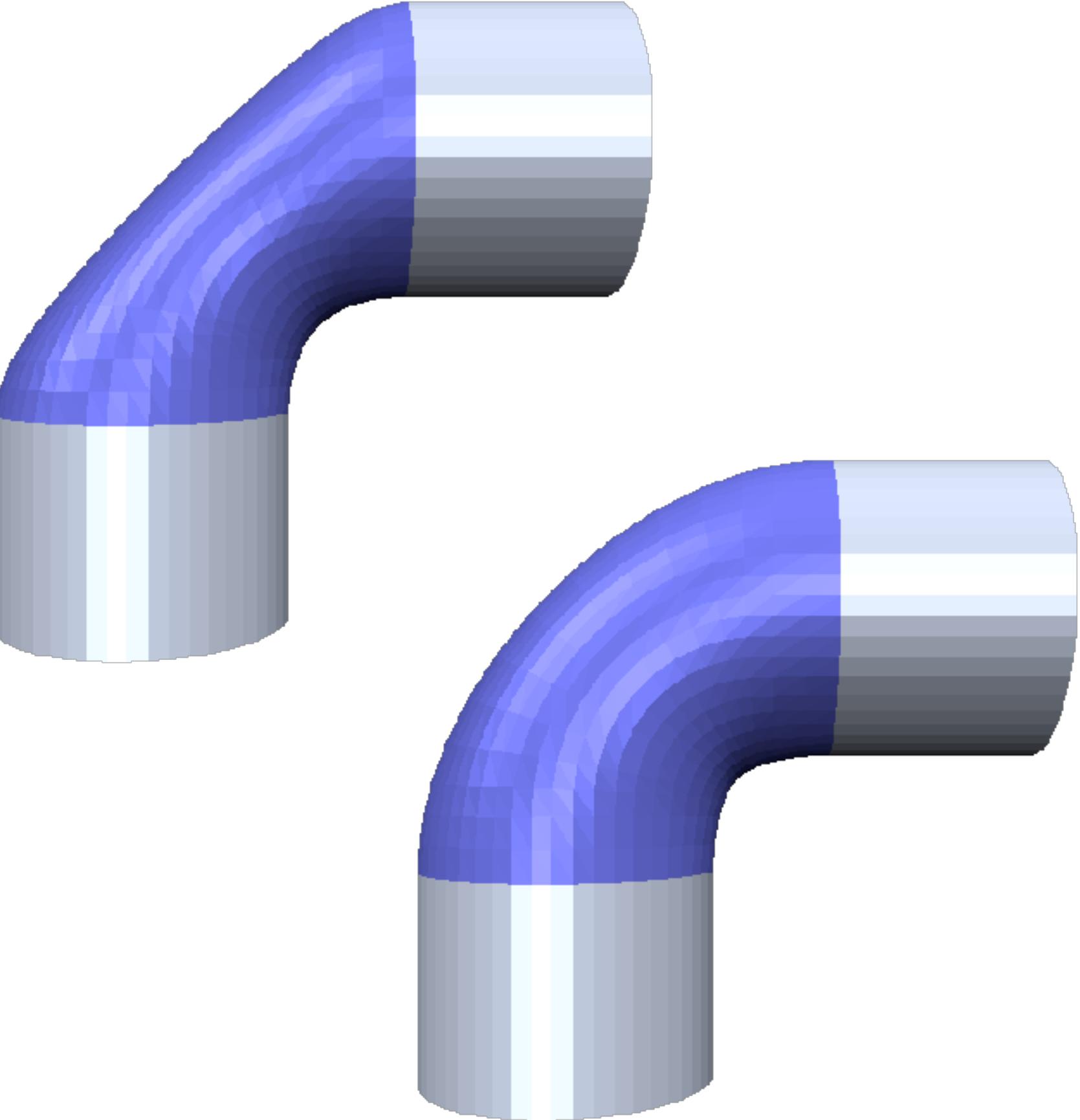
- Smoothness
 - Low geometric noise



Mesh Quality Criteria

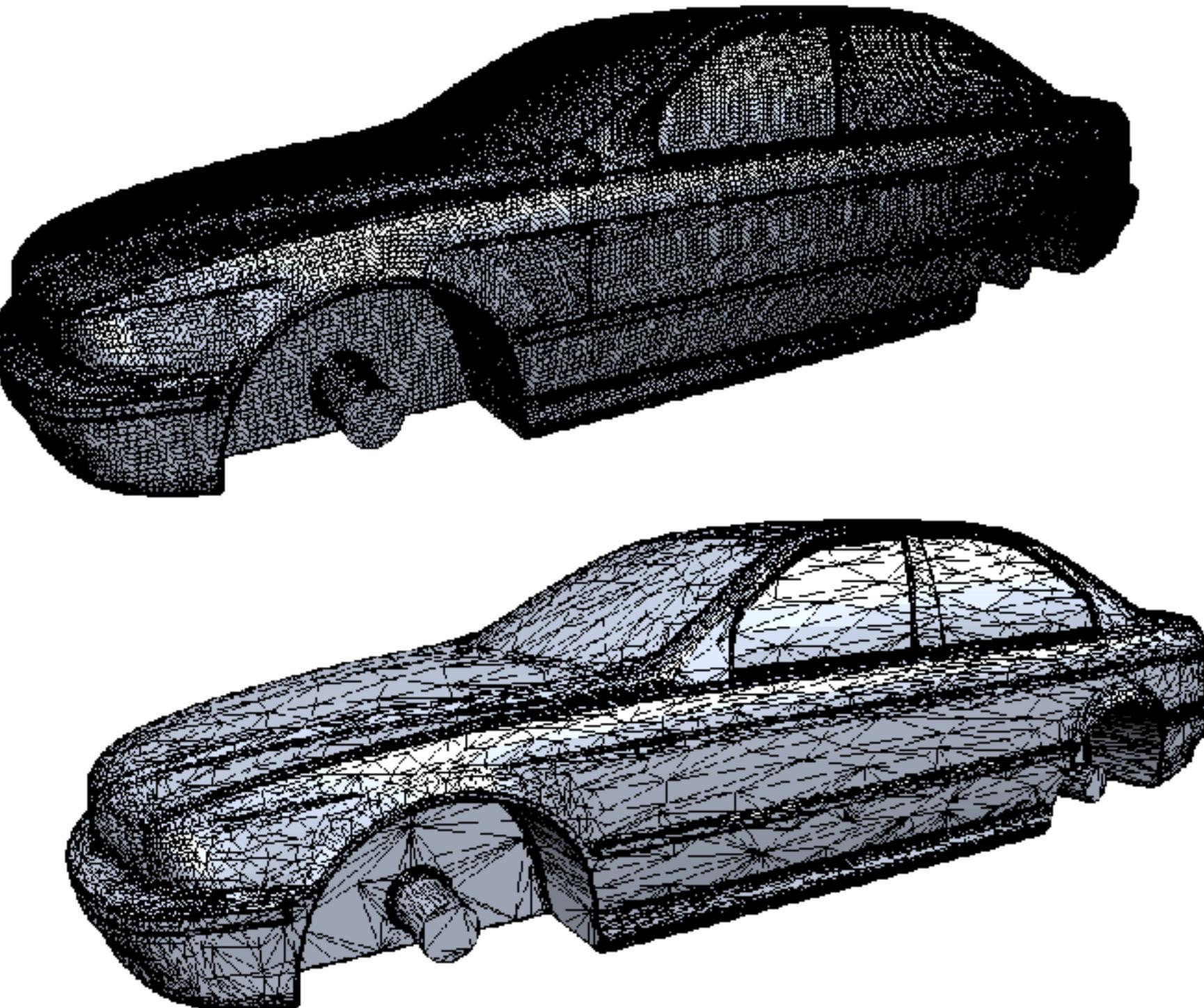


- Smoothness
 - Low geometric noise
- Fairness
 - Simplest shape



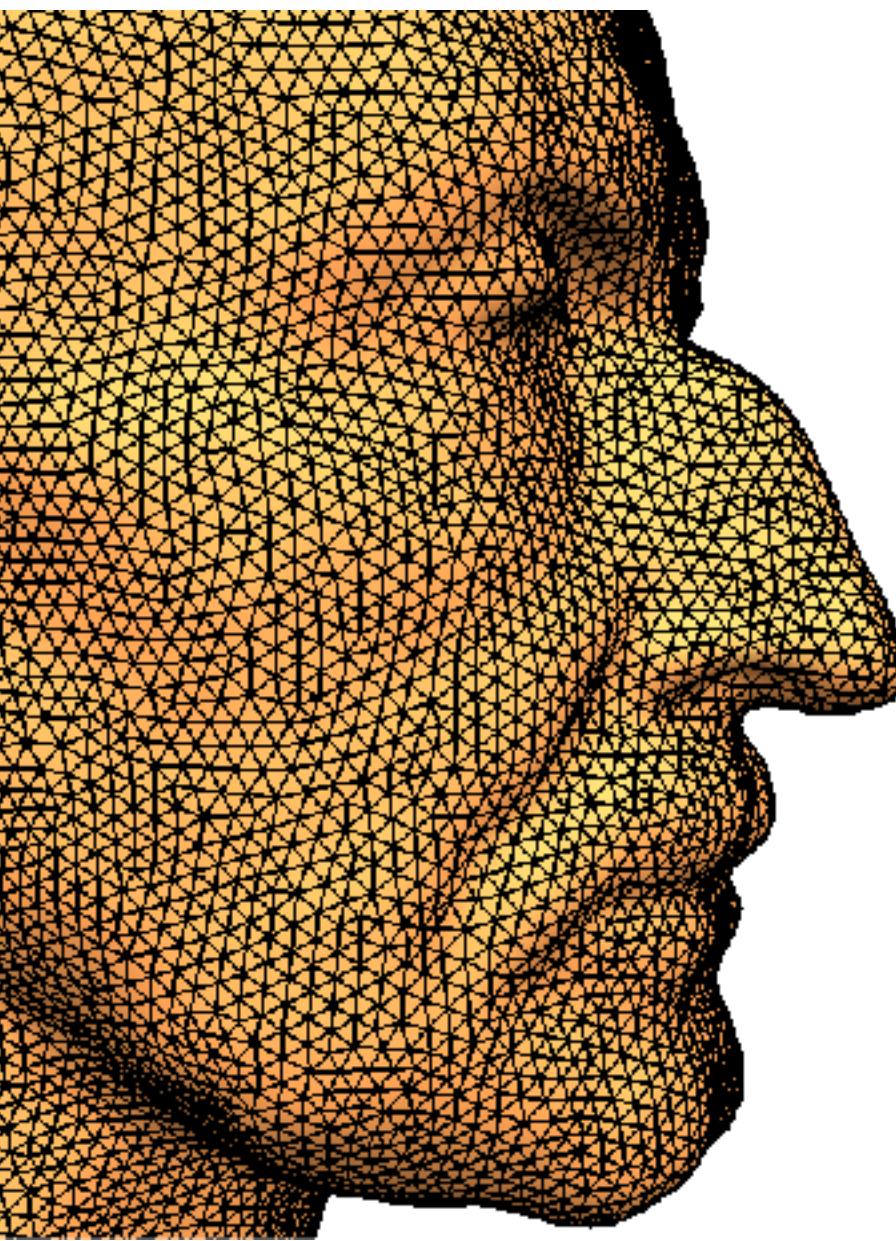
Mesh Quality Criteria

- Smoothness
 - Low geometric noise
- Fairness
 - Simplest shape
- Adaptive tessellation
 - Low complexity



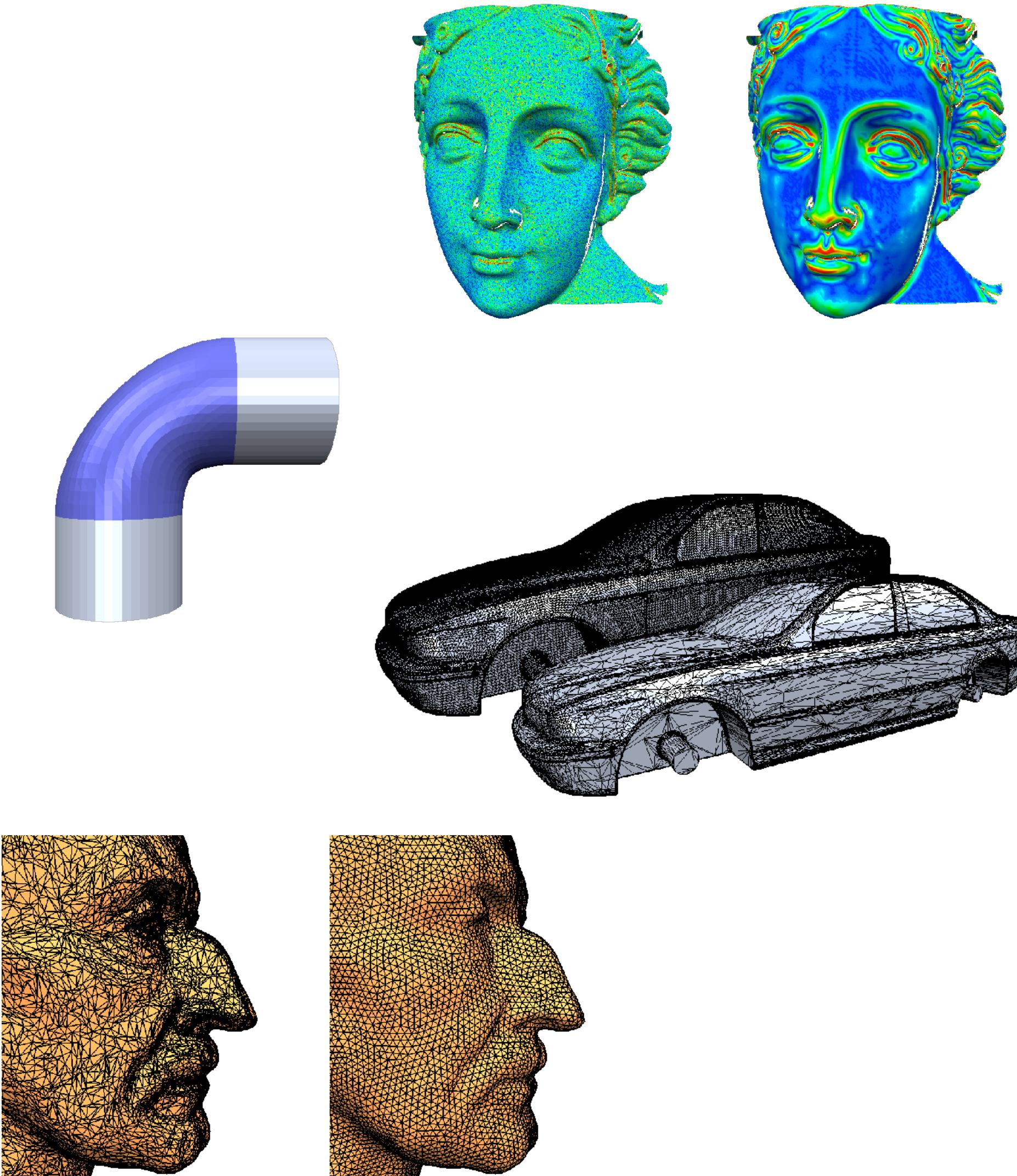
Mesh Quality Criteria

- Smoothness
 - Low geometric noise
- Fairness
 - Simplest shape
- Adaptive tessellation
 - Low complexity
- Triangle shape
 - Numerical robustness



Mesh Optimization

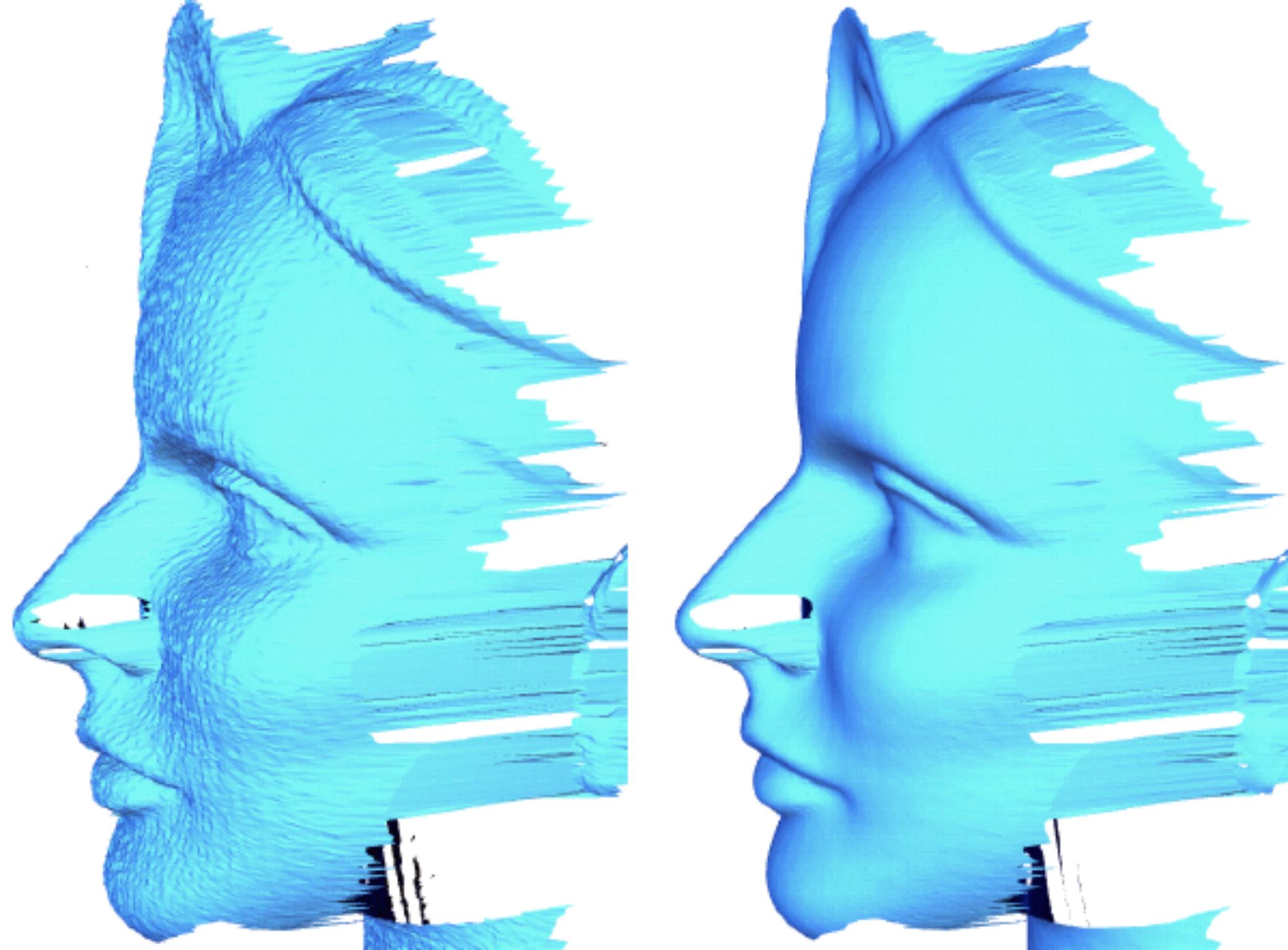
- Smoothness
 - Smoothing
- Fairness
 - Fairing
- Adaptive tessellation
 - Decimation
- Triangle shape
 - Remeshing



#1: Filter Scanning Noise

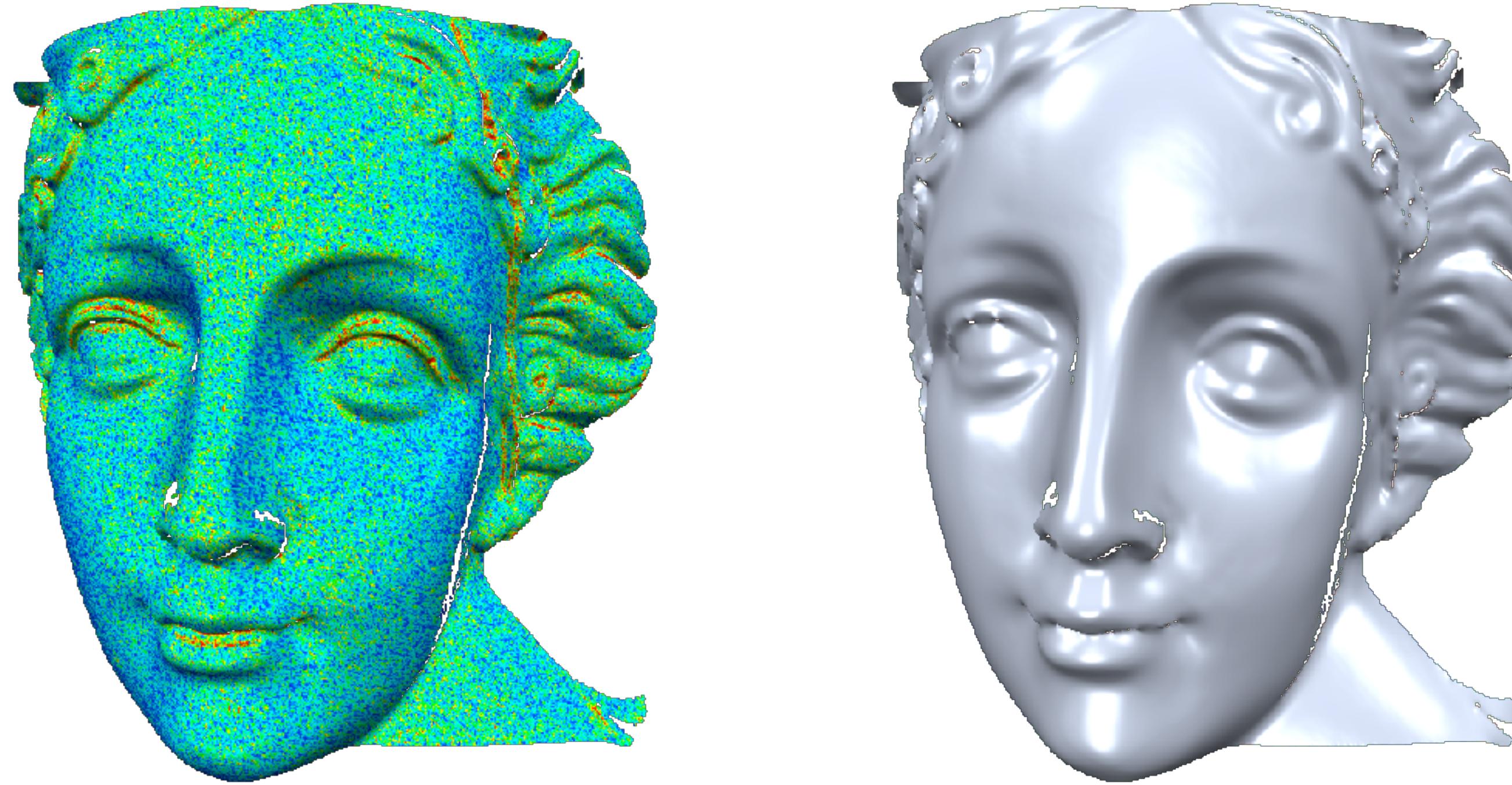


- Filter out high frequency noise

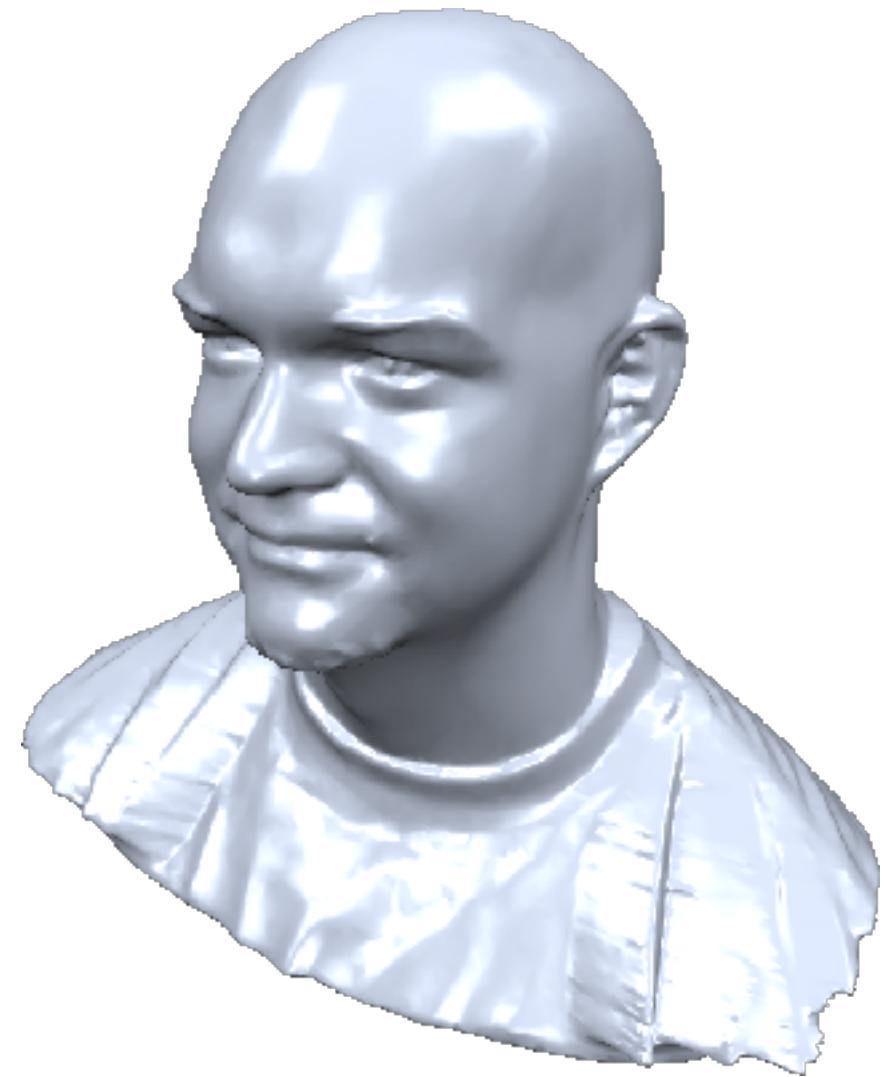


Desbrun, Meyer, Schroeder, Barr
Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow
SIGGRAPH 99

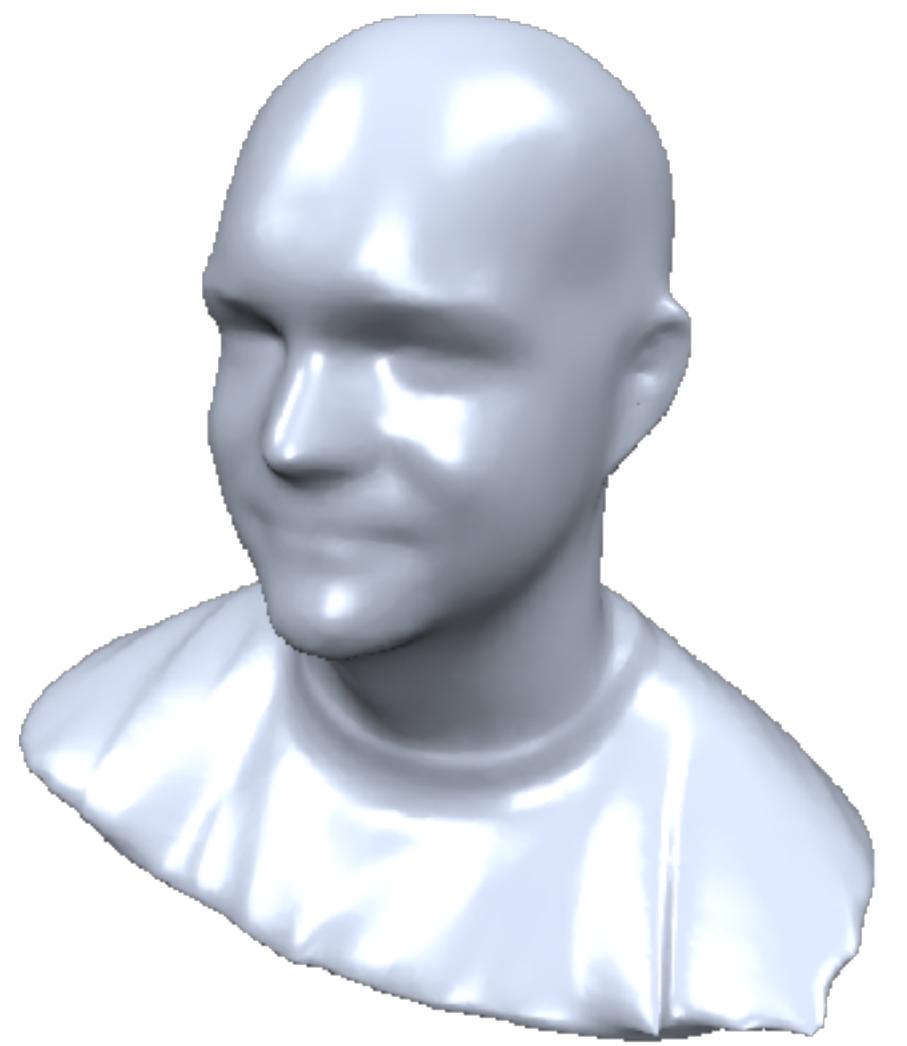
Mesh Smoothing (*reduce noise*)



#2: Feature Modification



Original



Low-Pass



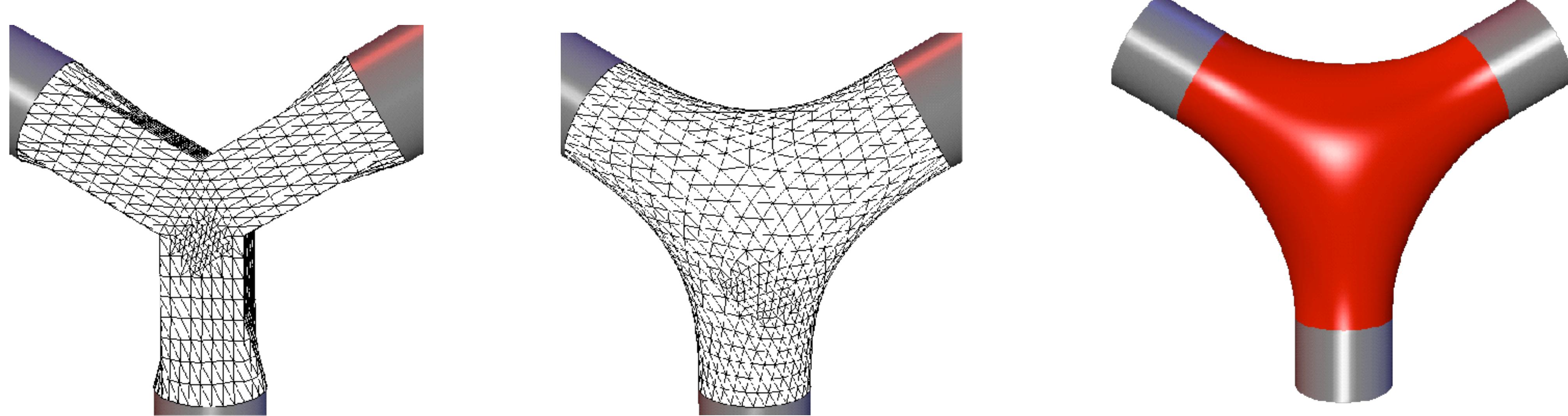
Exaggerate

Kim, Rosignac: *Geofilter: Geometric Selection of Mesh Filter Parameters*,
Eurographics 05

#3: Fair Surface Design

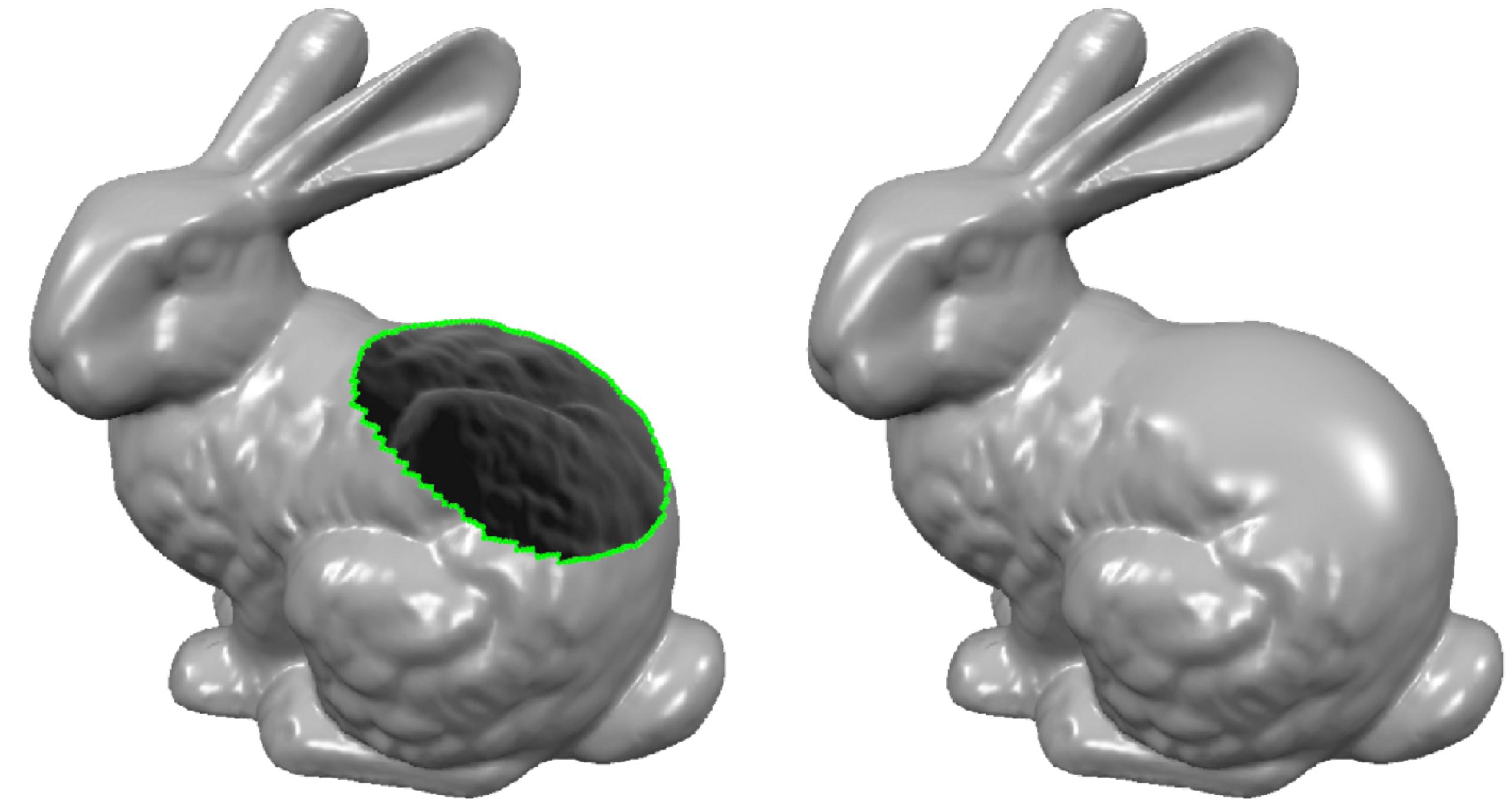


- Fair Surface Design



Schneider, Kobbelt: *Geometric fairing of irregular meshes for free-form surface design*,
CAGD 18(4), 2001

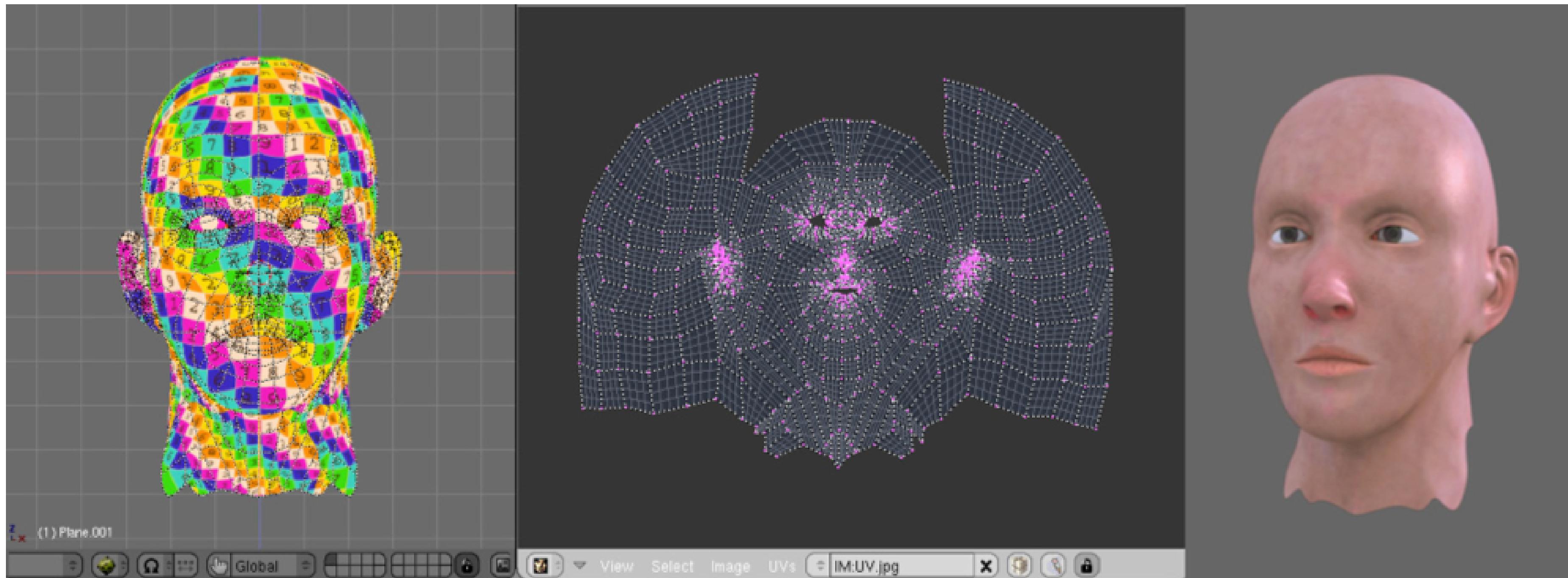
#4: Hole Filling (*energy minimization*)



#5: Mesh Parameterization

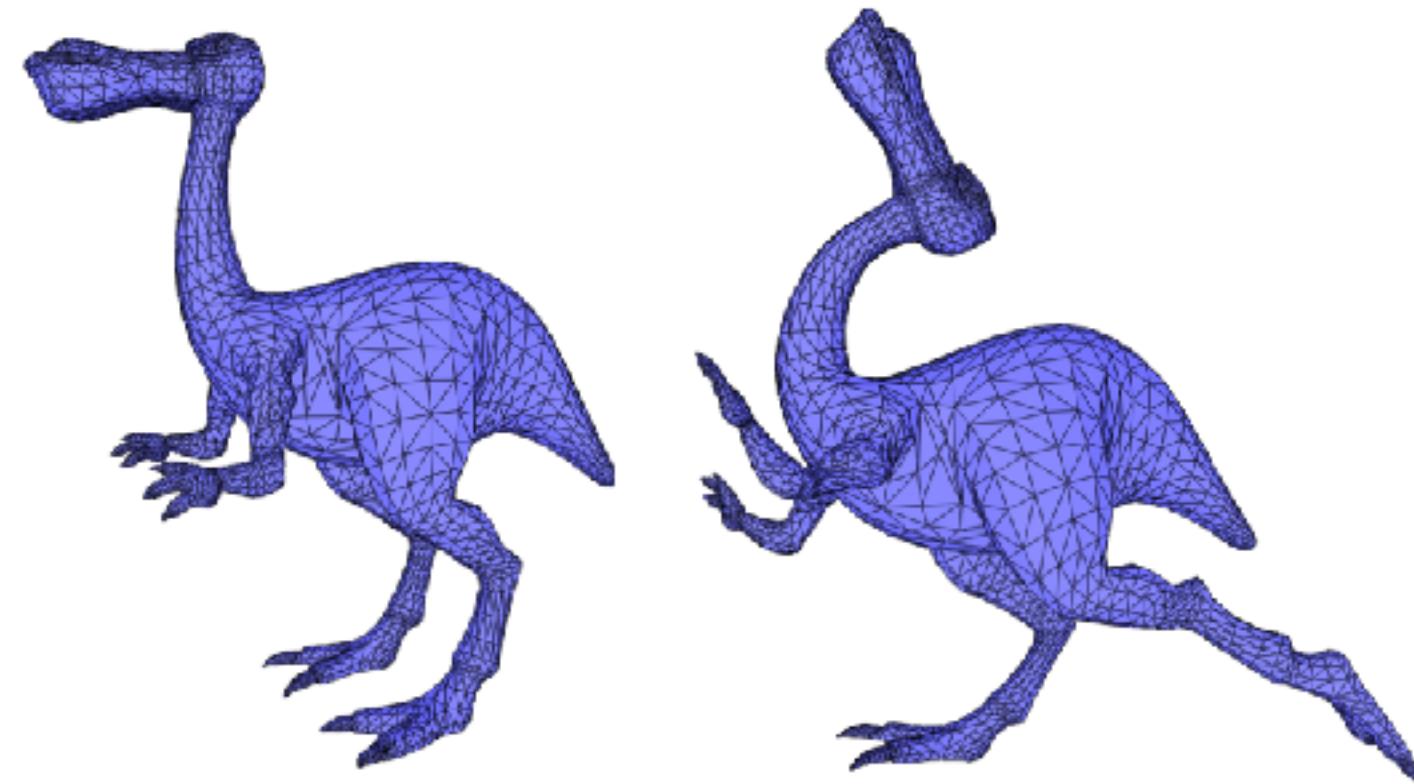


$$\mathbf{x}(u, v) \quad (u, v)$$

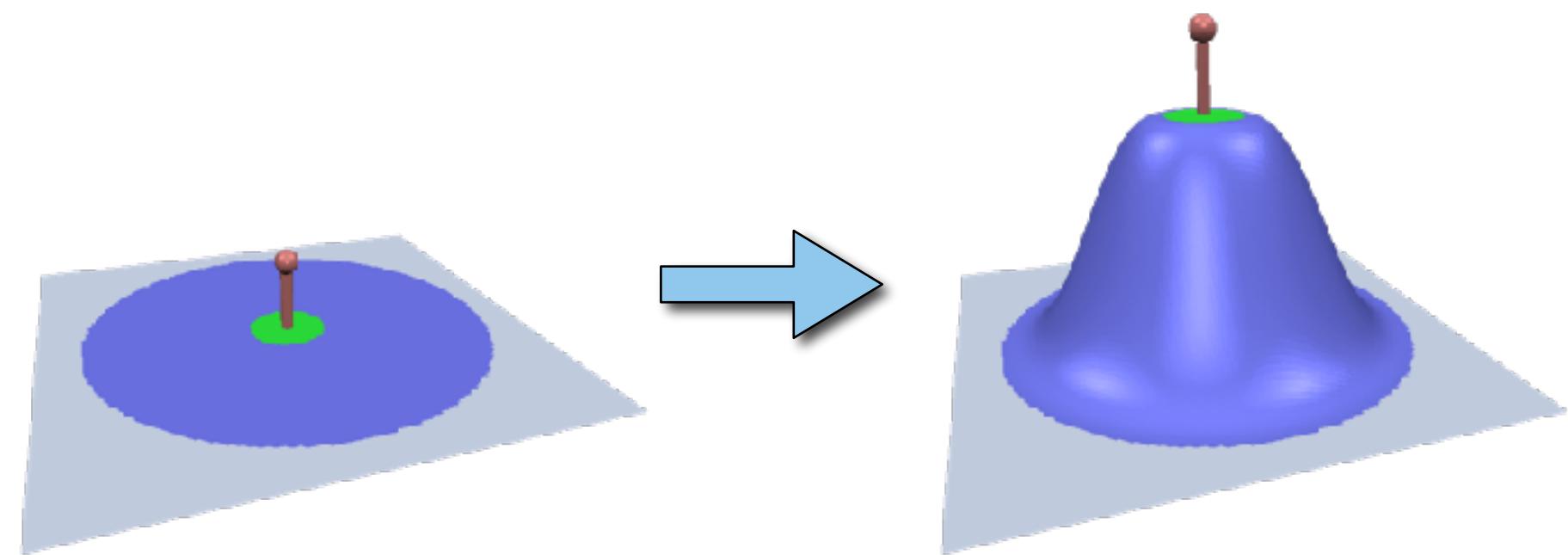


Levy et al.: Least squares conformal maps for automatic texture atlas generation, SIGGRAPH 2002.

#6: Surface Deformation



Olga Sorkine and Marc Alexa. As-rigid-as-possible surface modeling, SGP 2007



Botsch, Kobbelt: *An intuitive framework for real-time freeform modeling*, SIGGRAPH 04

Laplace Operator on Surfaces



- Uniform Laplace operator
 - depends on mesh connectivity only
 - approximates second derivative on surfaces
- Cotan operator (Laplace Beltrami)

$$\Delta = M^{-1}C$$

$$M = M^T \quad C = C^T$$

$$\Delta^T \neq \Delta$$

$$c_{ij} = \frac{1}{2} (\cot(\alpha_{ij}) + \cot(\beta_{ij}))$$

$$c_{ii} = - \sum_{j \in N_1(i)} c_{ij}$$

$$\sum_j c_{ij} = 0$$

Common Problem Types



- Laplace equation (special case of Poisson equation)

$$\Delta f = 0 \quad M^{-1}Cf = 0 \rightarrow Cf = 0 \quad \text{subject to boundary conditions}$$
$$f(x) = f_0 \quad x \in \partial B$$

Common Problem Types



- Laplace equation (special case of Poisson equation)

$$\Delta f = 0 \quad M^{-1}Cf = 0 \rightarrow Cf = 0 \quad \text{subject to boundary conditions}$$
$$f(x) = f_0 \quad x \in \partial B$$

- Poisson equation

$$\Delta f = g \quad M^{-1}Cf = g \rightarrow Cf = Mg$$

Common Problem Types



- Laplace equation (special case of Poisson equation)

$$\Delta f = 0 \quad M^{-1}Cf = 0 \rightarrow Cf = 0 \quad \text{subject to boundary conditions}$$
$$f(x) = f_0 \quad x \in \partial B$$

- Poisson equation

$$\Delta f = g \quad M^{-1}Cf = g \rightarrow Cf = Mg$$

- Differential equation

$$f_t - \frac{\partial f}{\partial t} = \Delta f$$

Common Problem Types



- Laplace equation (special case of Poisson equation)

$$\Delta f = 0 \quad M^{-1}Cf = 0 \rightarrow Cf = 0 \quad \text{subject to boundary conditions}$$
$$f(x) = f_0 \quad x \in \partial B$$

- Poisson equation

$$\Delta f = g \quad M^{-1}Cf = g \rightarrow Cf = Mg$$

- Differential equation

$$f_t = \frac{\partial f}{\partial t} = \Delta f$$

- Eigen/spectral analysis

$$\Delta \phi_i = \lambda_i \phi_i \quad \left(M^{-\frac{1}{2}} C M^{-\frac{1}{2}} \right) M^{\frac{1}{2}} \phi_i = \lambda_i M^{\frac{1}{2}} \phi_i$$
$$\left(M^{-\frac{1}{2}} C M^{-\frac{1}{2}} \right) \xi_i = \lambda_i \xi_i \quad \rightarrow \phi_i = M^{-\frac{1}{2}} \xi_i$$

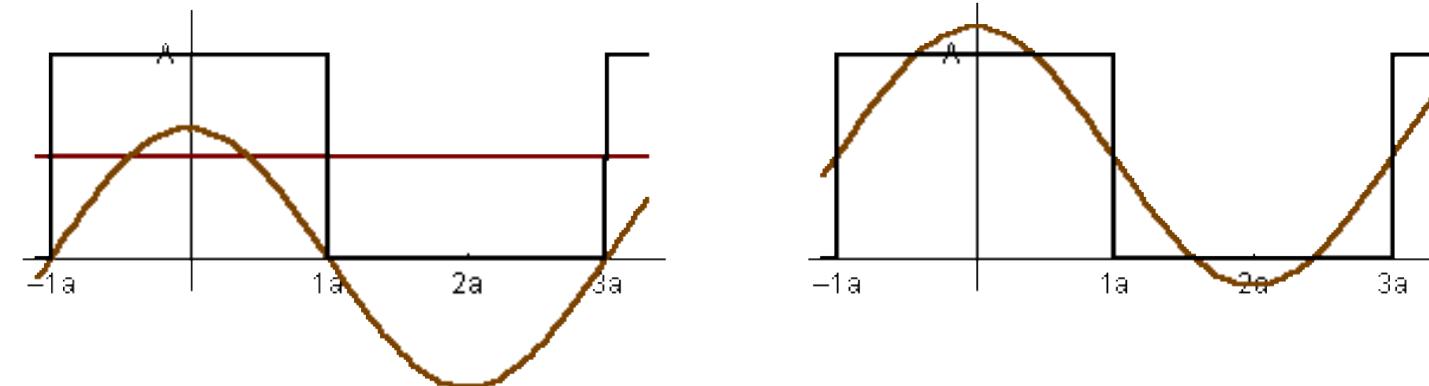
Outline



- **Spectral Analysis**
- Diffusion Flow
- Energy Minimization

Fourier Transform

- Represent a function as a weighted sum of sines and cosines



Joseph Fourier 1768 - 1830

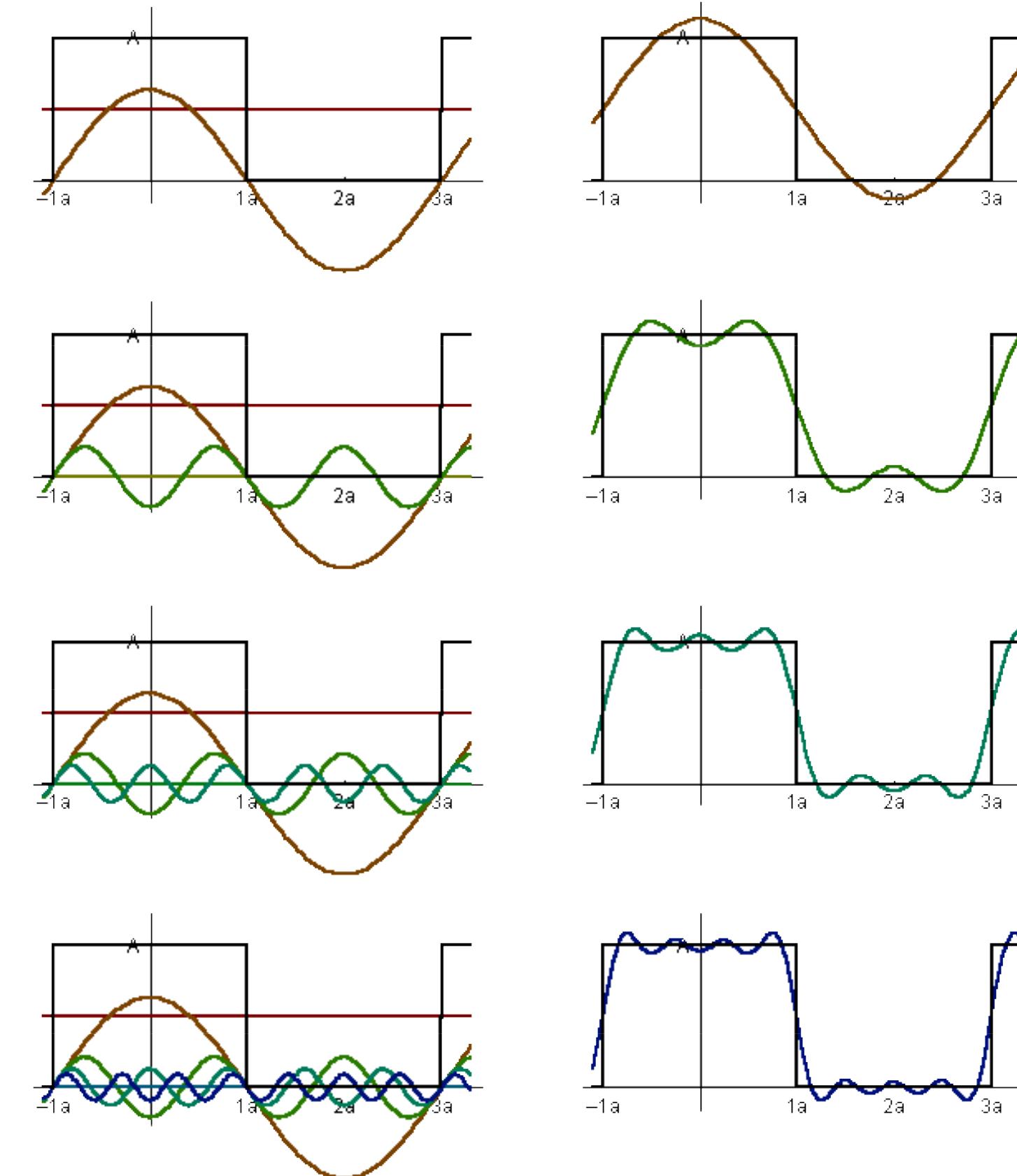
$$f(x) = a_0 + a_1 \cos(x)$$

Fourier Transform

- Represent a function as a weighted sum of sines and cosines



Joseph Fourier 1768 - 1830

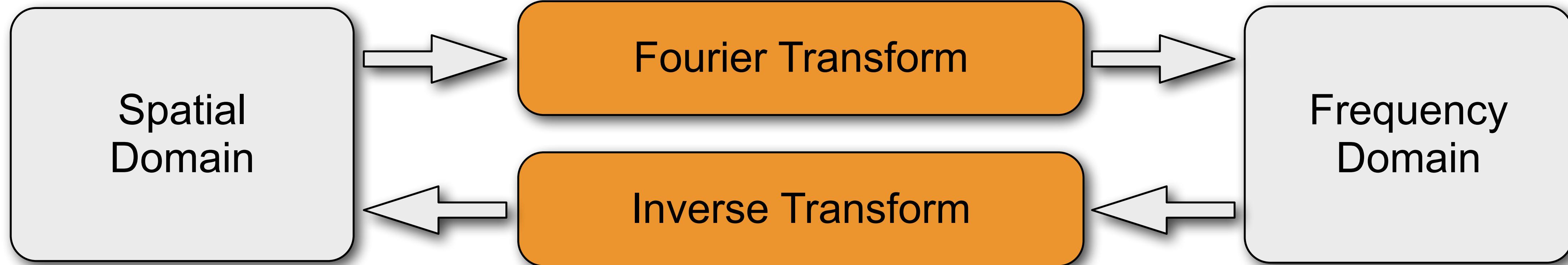


$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x) + a_3 \cos(5x) + a_4 \cos(7x) + \dots$$

Fourier Transform



$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \omega x} dx$$



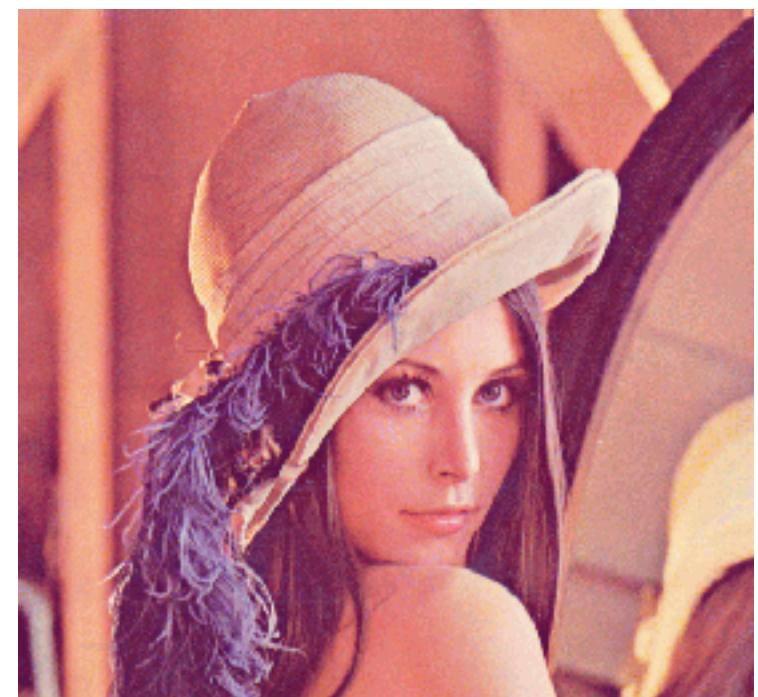
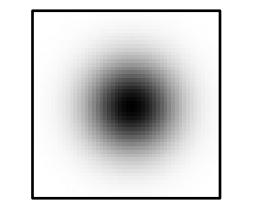
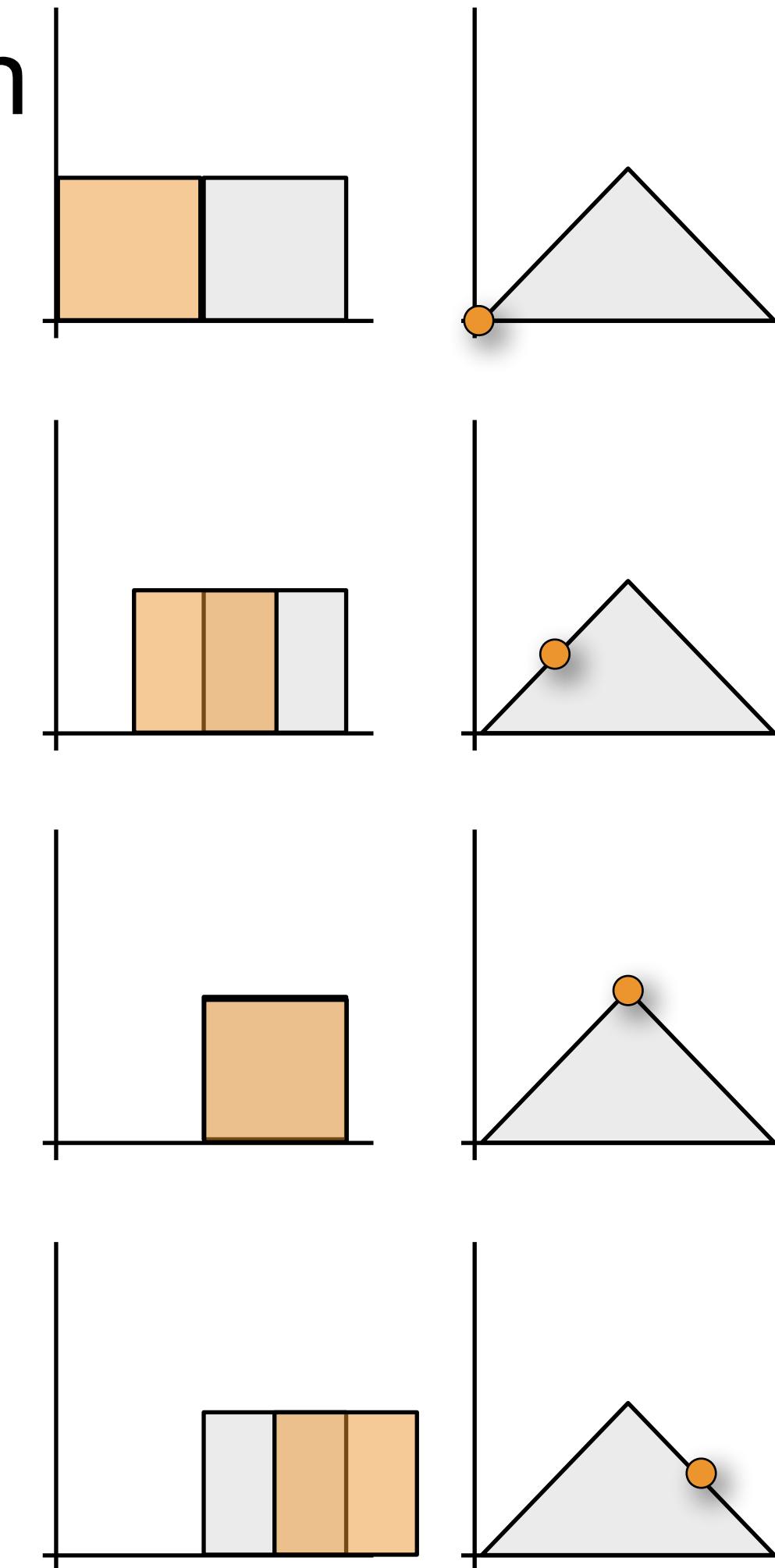
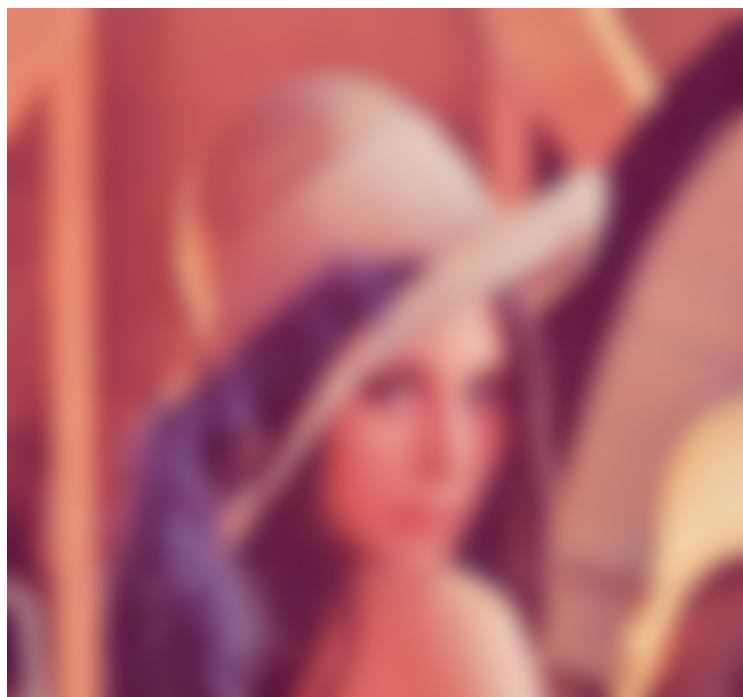
$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{2\pi i \omega x} d\omega$$

Convolution

- Smooth signal by convolution with a kernel function

$$h(x) = f * g := \int f(y) \cdot g(x - y) dy$$

- Example: Gaussian blurring

 $*$  $=$ 

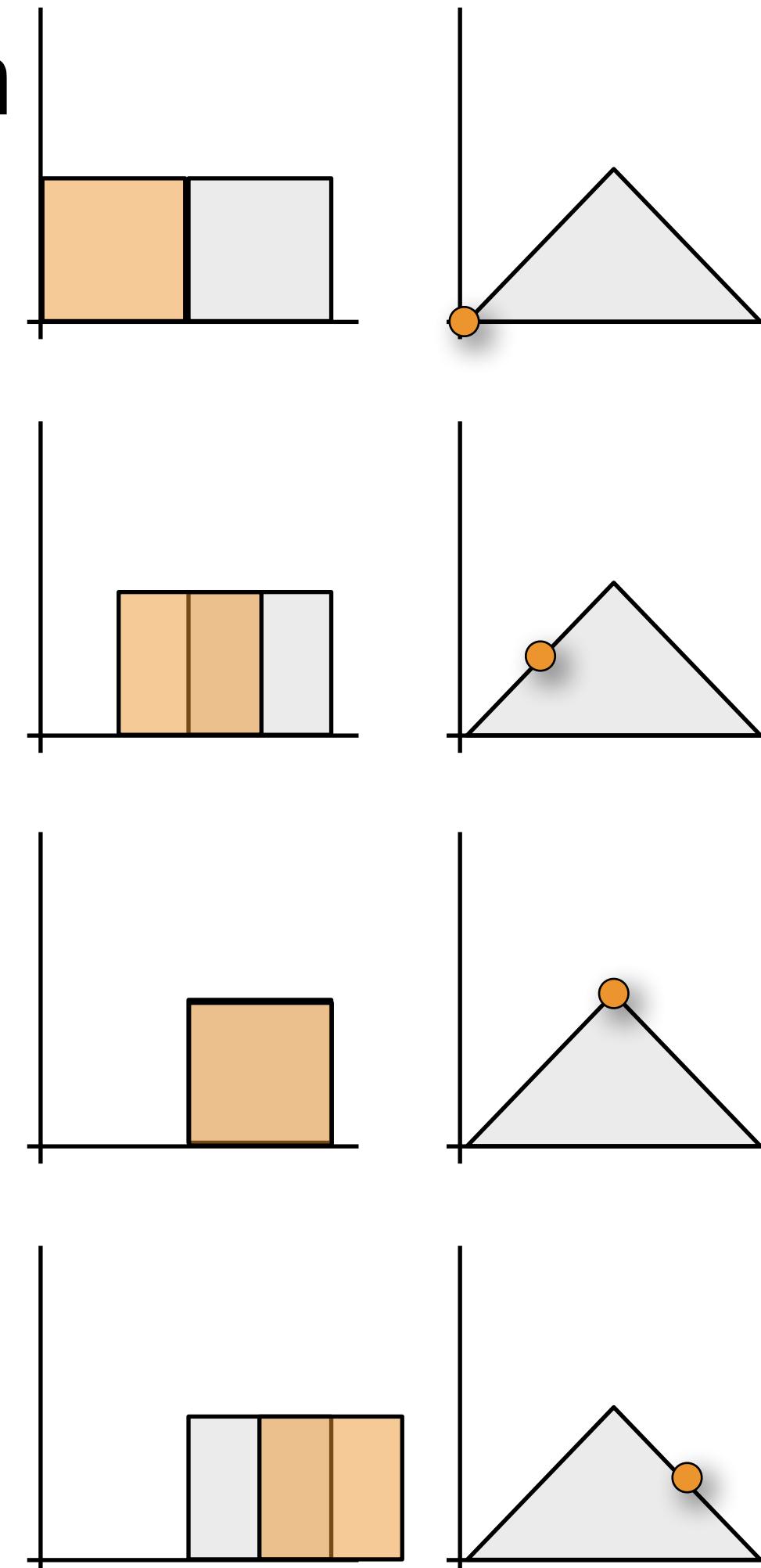
Convolution

- Smooth signal by convolution with a kernel function

$$h(x) = f * g := \int f(y) \cdot g(x - y) dy$$

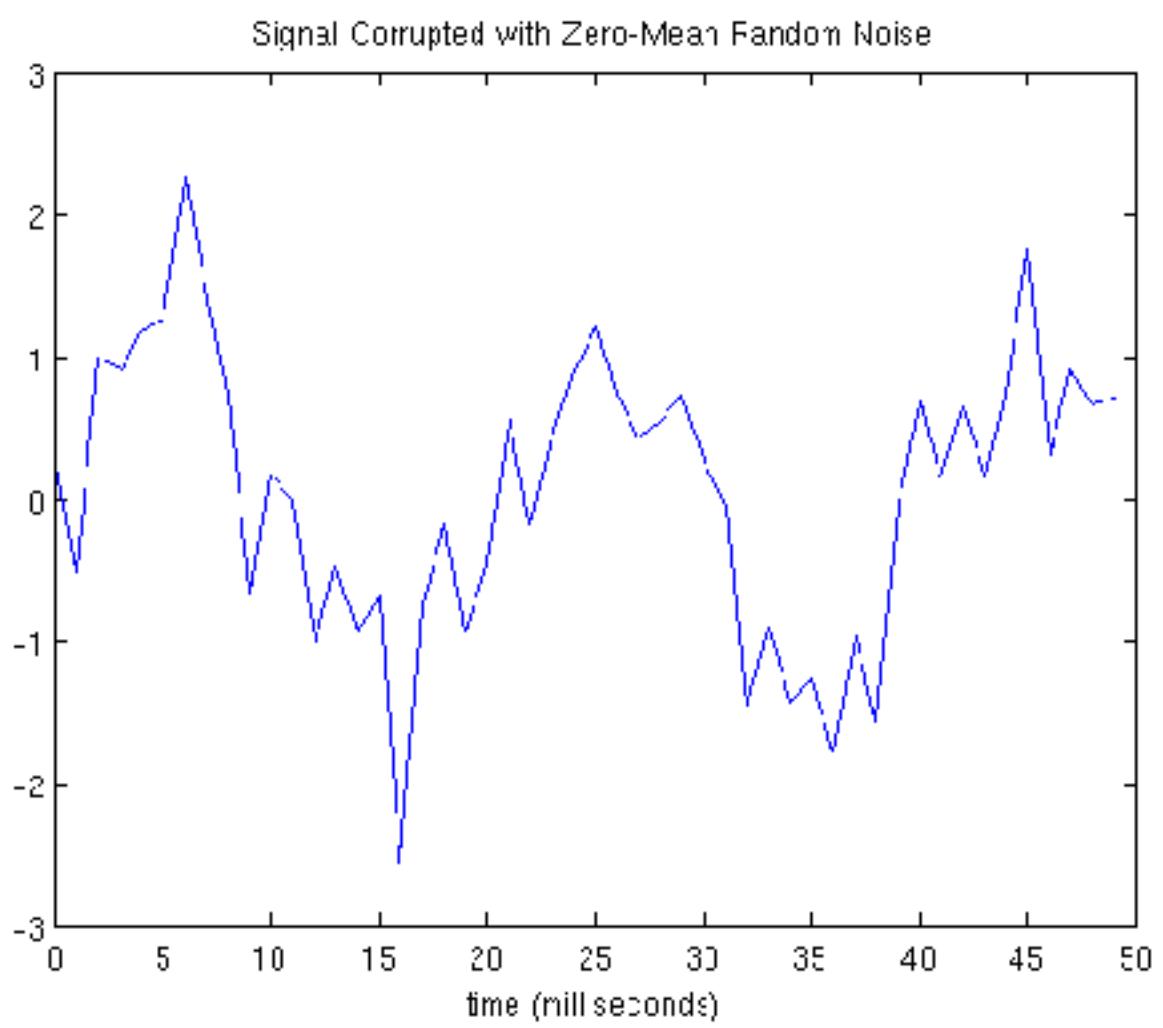
- Convolution in spatial domain \Leftrightarrow
Multiplication in frequency domain

$$H(\omega) = F(\omega) \cdot G(\omega)$$

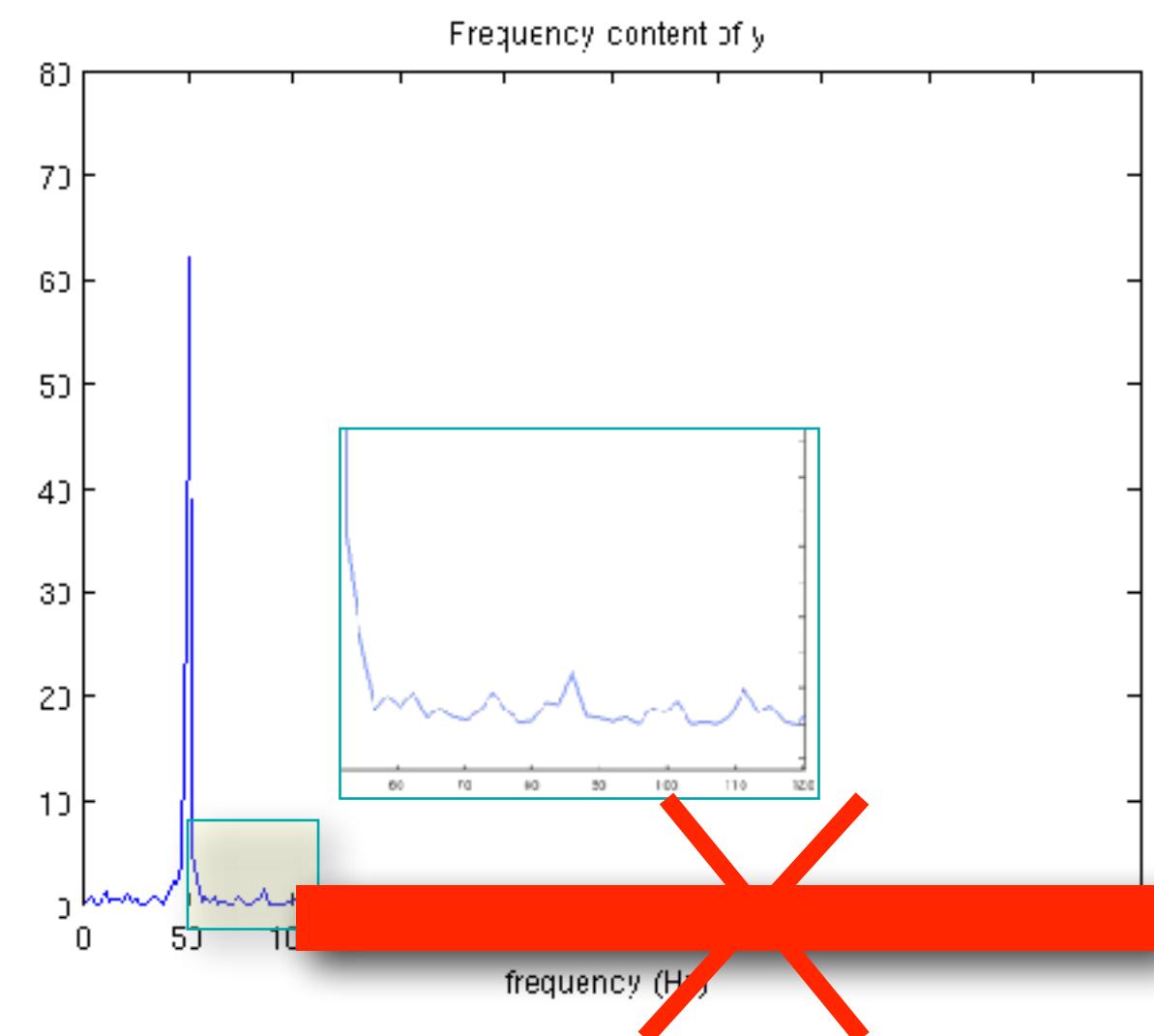


Fourier Analysis

- Low-pass filter discards high frequencies



spatial domain



frequency domain

LOW PASS

Fourier Transform



- Spatial domain $f(x) \rightarrow$ Frequency domain $F(\omega)$

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \omega x} dx$$

- Multiply by low-pass filter $G(\omega)$

$$F(\omega) \leftarrow F(\omega) \cdot G(\omega)$$

- Frequency domain $F(\omega) \rightarrow$ Spatial domain $f(x)$

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{2\pi i \omega x} d\omega$$

Fourier Transform



- Consider L^2 function space with inner product

$$\langle f, g \rangle := \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx$$

- Complex “waves” build an orthonormal basis

$$e_\omega(x) := e^{-2\pi i \omega x} = \cos(2\pi \omega x) - i \sin(2\pi \omega x)$$

- Fourier transform is a change of basis

$$f(x) = \int_{-\infty}^{\infty} \langle f, e_\omega \rangle e_\omega \, d\omega$$

$$f(x) = \sum_{\omega=-\infty}^{\infty} \langle f, e_\omega \rangle e_\omega \, d\omega$$

Fourier Analysis on Meshes?



- Only applicable to parametric patches $f(u, v)$
- Generalize frequency concept to meshes!
- Complex waves are Eigenfunctions of Laplace

$$\Delta(e^{2\pi i \omega x}) = \frac{d^2}{dx^2} e^{2\pi i \omega x} = -(2\pi\omega)^2 e^{2\pi i \omega x}$$

→ *Use Eigenfunctions of discrete Laplace-Beltrami*

Discrete Laplace-Beltrami

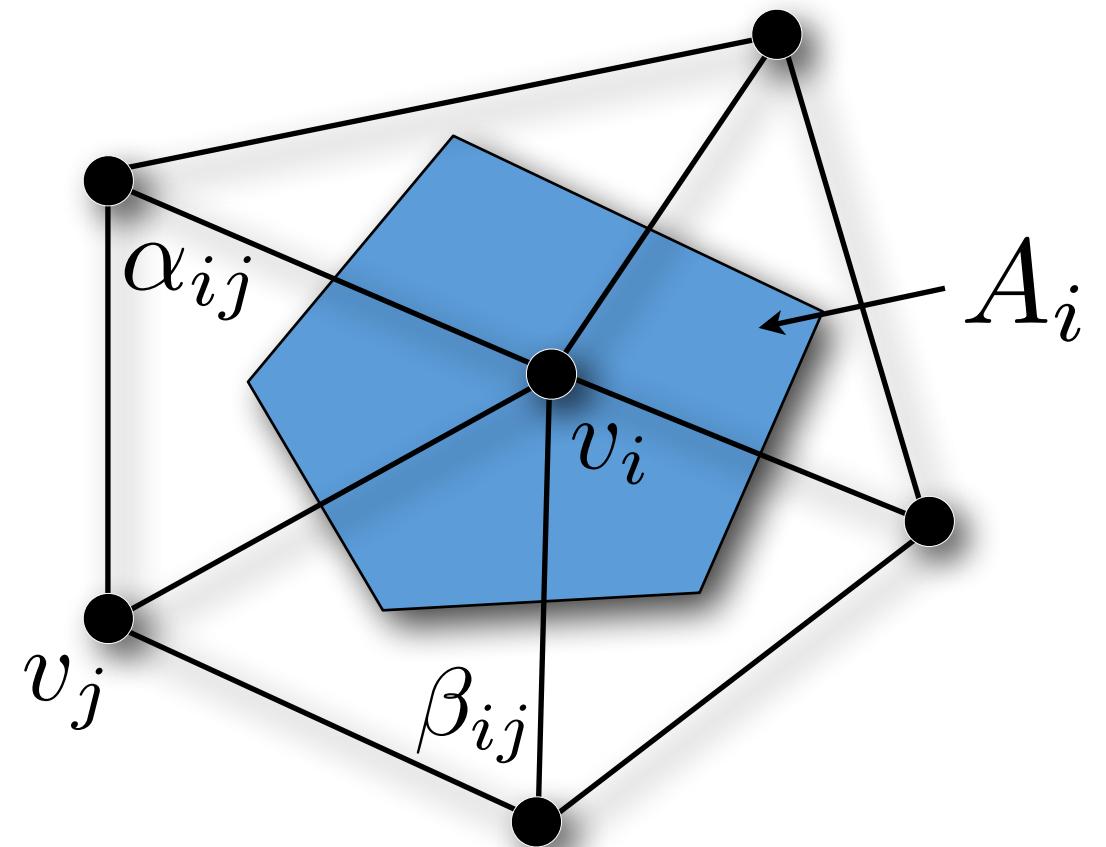


- Function values sampled at mesh vertices

$$\mathbf{f} = [f_1, f_2, \dots, f_n] \in \mathbb{R}^n$$

- Discrete Laplace-Beltrami (per vertex)

$$\Delta_S f(v_i) := \frac{1}{2A_i} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij}) (f(v_j) - f(v_i))$$



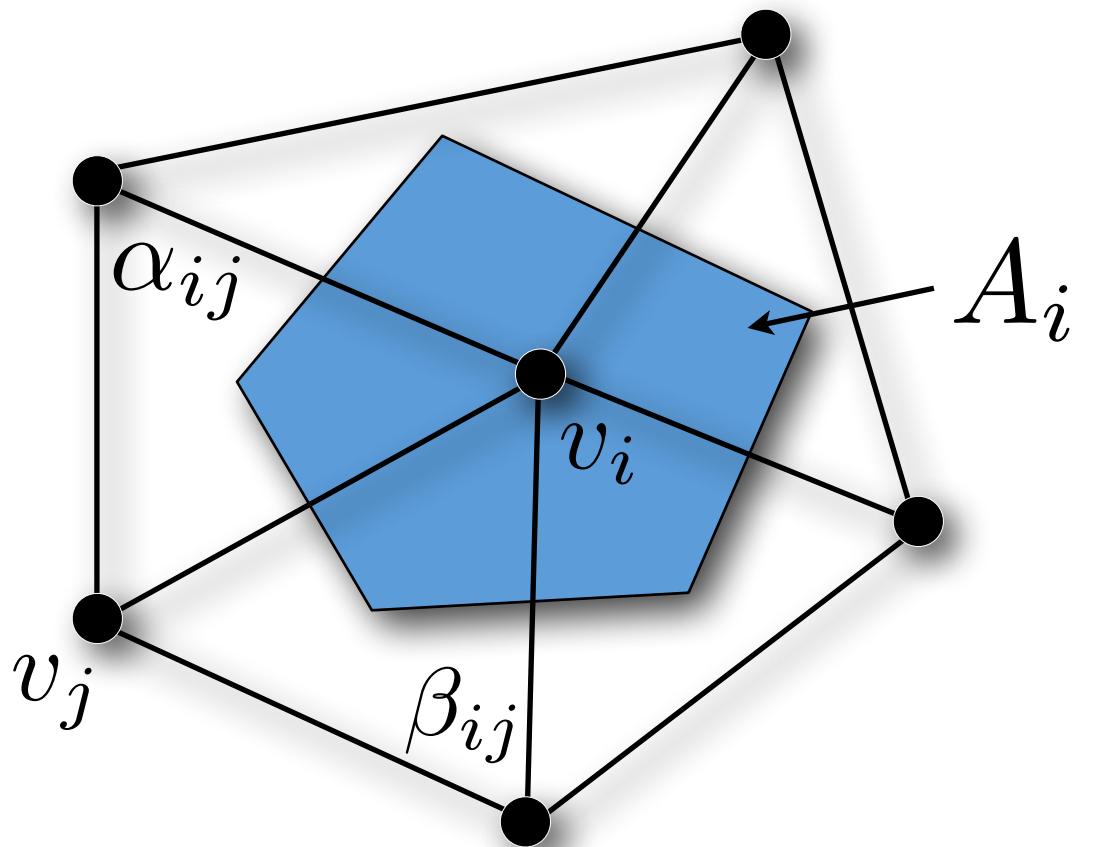
Discrete Laplace-Beltrami



- Discrete Laplace operator (per mesh)
 - Sparse matrix

$$\mathbf{L} = \mathbf{M}^{-1} \mathbf{C} \in \mathbb{R}^{n \times n}$$

$$\begin{pmatrix} \vdots \\ \Delta_S f(v_i) \\ \vdots \end{pmatrix} = \mathbf{L} \cdot \begin{pmatrix} \vdots \\ f(v_i) \\ \vdots \end{pmatrix}$$



Discrete Laplace-Beltrami



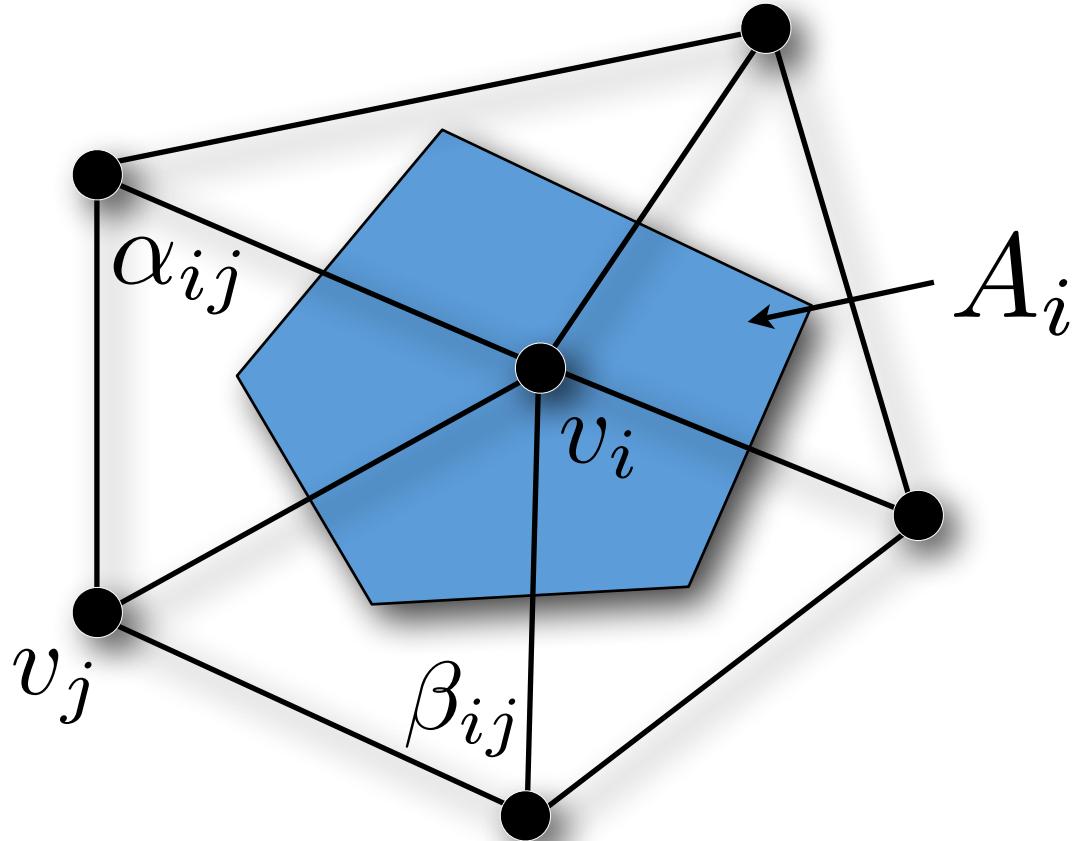
- Discrete Laplace operator (per mesh)
 - Sparse matrix

$$\mathbf{L} = \mathbf{M}^{-1} \mathbf{C} \in \mathbb{R}^{n \times n}$$

$$\mathbf{C}_{ij} = \begin{cases} \cot\alpha_{ij} + \cot\beta_{ij}, & i \neq j, j \in \mathcal{N}_1(v_i) \\ -\sum_{v_j \in \mathcal{N}_1(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij}) & i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{M}^{-1} = \text{diag} \left(\dots, \frac{1}{2A_i}, \dots \right)$$

- Note fraction 1/2 can either be in C or in M matrix



Discrete Laplace-Beltrami



- Discrete function sampled at mesh vertices

$$\mathbf{f} = [f_1, f_2, \dots, f_n] \in \mathbb{R}^n$$

- Discrete Laplace-Beltrami (per vertex)

$$\Delta_S f(v_i) := \frac{1}{2A_i} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij}) (f(v_j) - f(v_i))$$

- Discrete Laplace-Beltrami matrix \mathbf{L}

- Eigenvectors are “natural vibrations”
 - Eigenvalues are “natural frequencies”

$$\Delta \mathbf{e}_i = \lambda_i \mathbf{e}_i$$

Discrete Laplace-Beltrami



- Discrete Laplace-Beltrami matrix L
 - Eigenvectors are “natural vibrations”
 - Eigenvalues are “natural frequencies”

Spectral Analysis

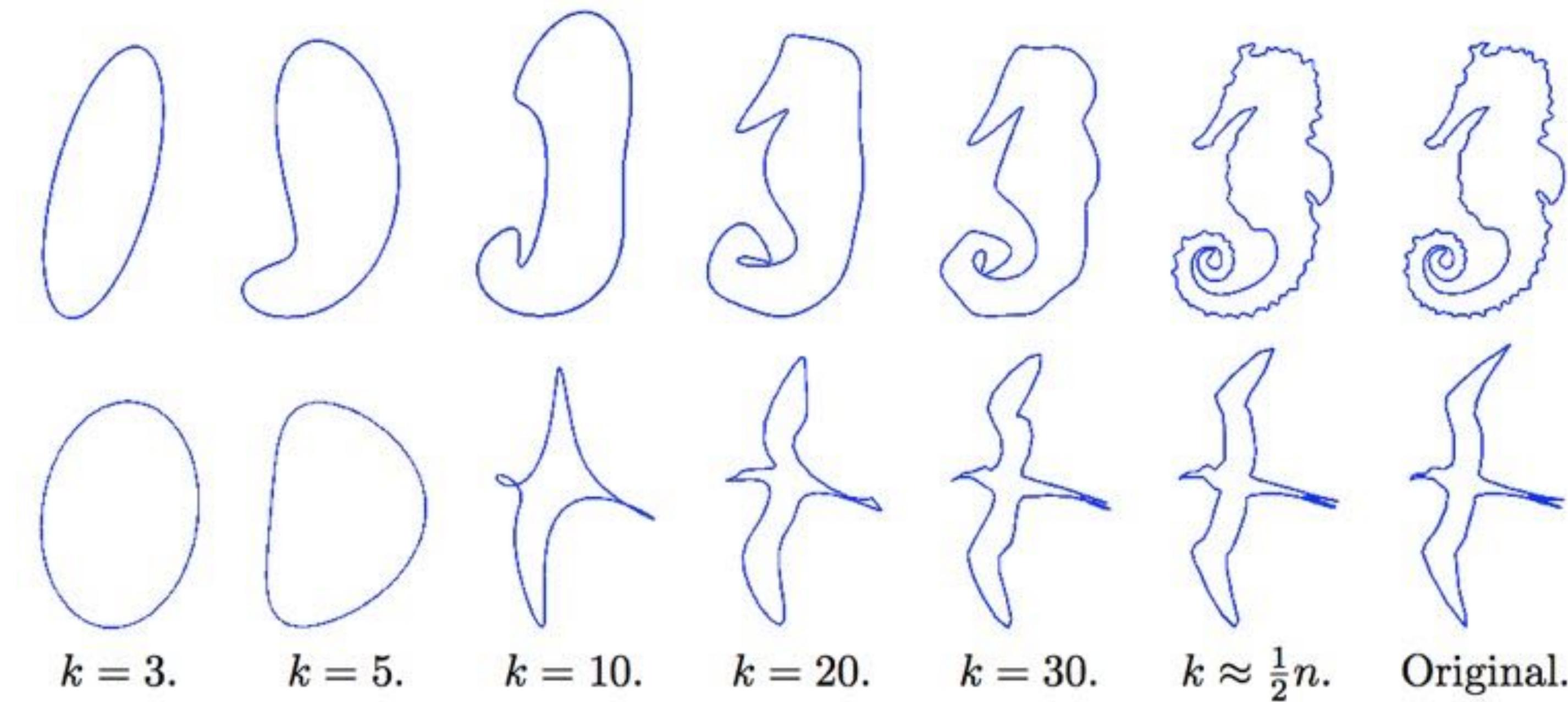


1. Setup Laplace-Beltrami matrix $\mathbf{L} = \Delta$
2. Compute k smallest eigenvectors $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$
3. Reconstruct mesh from those (component-wise)

$$\begin{aligned}\mathbf{x} &:= [x_1, \dots, x_n] & \mathbf{y} &:= [y_1, \dots, y_n] & \mathbf{z} &:= [z_1, \dots, z_n] \\ \mathbf{x} &\leftarrow \sum_{i=1}^k (\mathbf{x}^T \mathbf{e}_i) \mathbf{e}_i & \mathbf{y} &\leftarrow \sum_{i=1}^k (\mathbf{y}^T \mathbf{e}_i) \mathbf{e}_i & \mathbf{z} &\leftarrow \sum_{i=1}^k (\mathbf{z}^T \mathbf{e}_i) \mathbf{e}_i\end{aligned}$$

Spectral Analysis

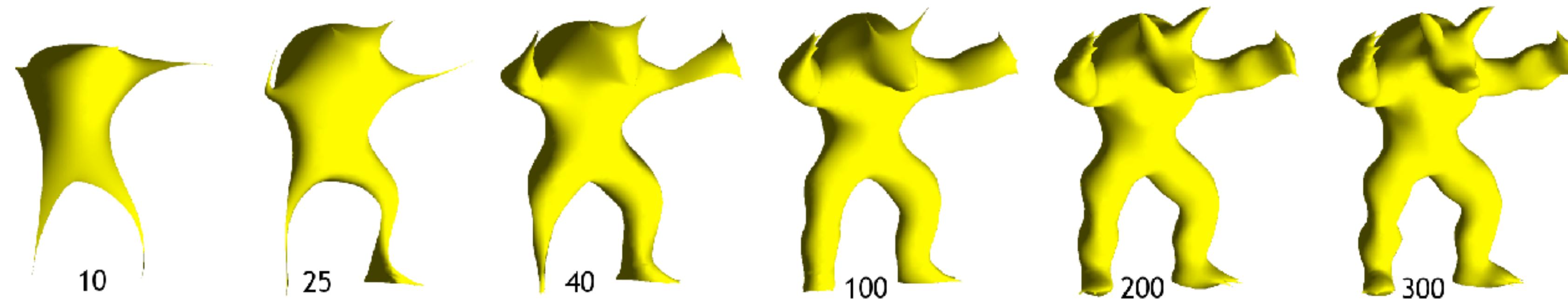
1. Setup Laplace-Beltrami matrix \mathbf{L}
2. Compute k smallest eigenvectors $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$
3. Reconstruct mesh from those



Spectral Analysis

1. Setup Laplace-Beltrami matrix \mathbf{L}
2. Compute k smallest eigenvectors $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$
3. Reconstruct mesh from those

Too complex for
large meshes!



Bruno Levy: *Laplace-Beltrami Eigenfunctions: Towards an algorithm that understands geometry*, Shape Modeling and Applications, 2006

Review: Curves



Differential Geometry of Curves

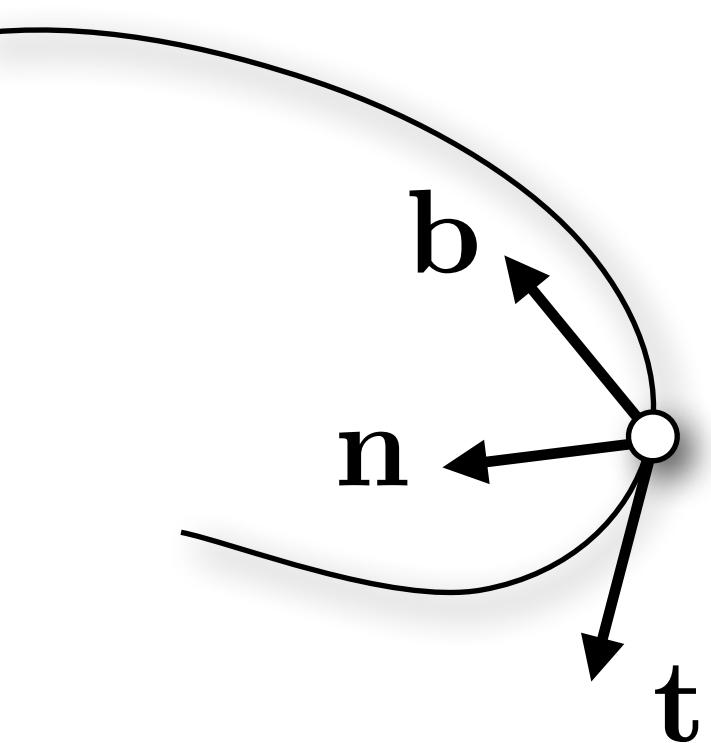
- chord length and arc length
- arc-length parameterization
- Frenet frame

$$\mathbf{t} = \frac{\mathbf{x}_t}{\|\mathbf{x}_t\|}$$

$$\mathbf{n} = \mathbf{b} \times \mathbf{t}$$

$$\mathbf{b} = \frac{\mathbf{x}_t \times \mathbf{x}_{tt}}{\|\mathbf{x}_t \times \mathbf{x}_{tt}\|}$$

$$s = s(t) = \int_a^t \|\mathbf{x}_t\| dt$$



Frenet frame

Review: Curves



Differential Geometry of Curves

- **Frenet-Serret:** Derivatives w.r.t. arc length s

$$\begin{aligned}\mathbf{t}_s &= +\kappa \mathbf{n} \\ \mathbf{n}_s &= -\kappa \mathbf{t} \quad +\tau \mathbf{b} \\ \mathbf{b}_s &= -\tau \mathbf{n}\end{aligned}$$

- **Curvature** (*deviation from straight line*)

$$\kappa = \|\mathbf{x}_{ss}\|$$

- **Torsion** (*deviation from planarity*)

$$\tau = \frac{1}{\kappa^2} \det([\mathbf{x}_s, \mathbf{x}_{ss}, \mathbf{x}_{sss}])$$

Review: Surface



Differential Geometry of Surfaces

– First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_v^T \mathbf{x}_u & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} \quad \left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

- **Angle**

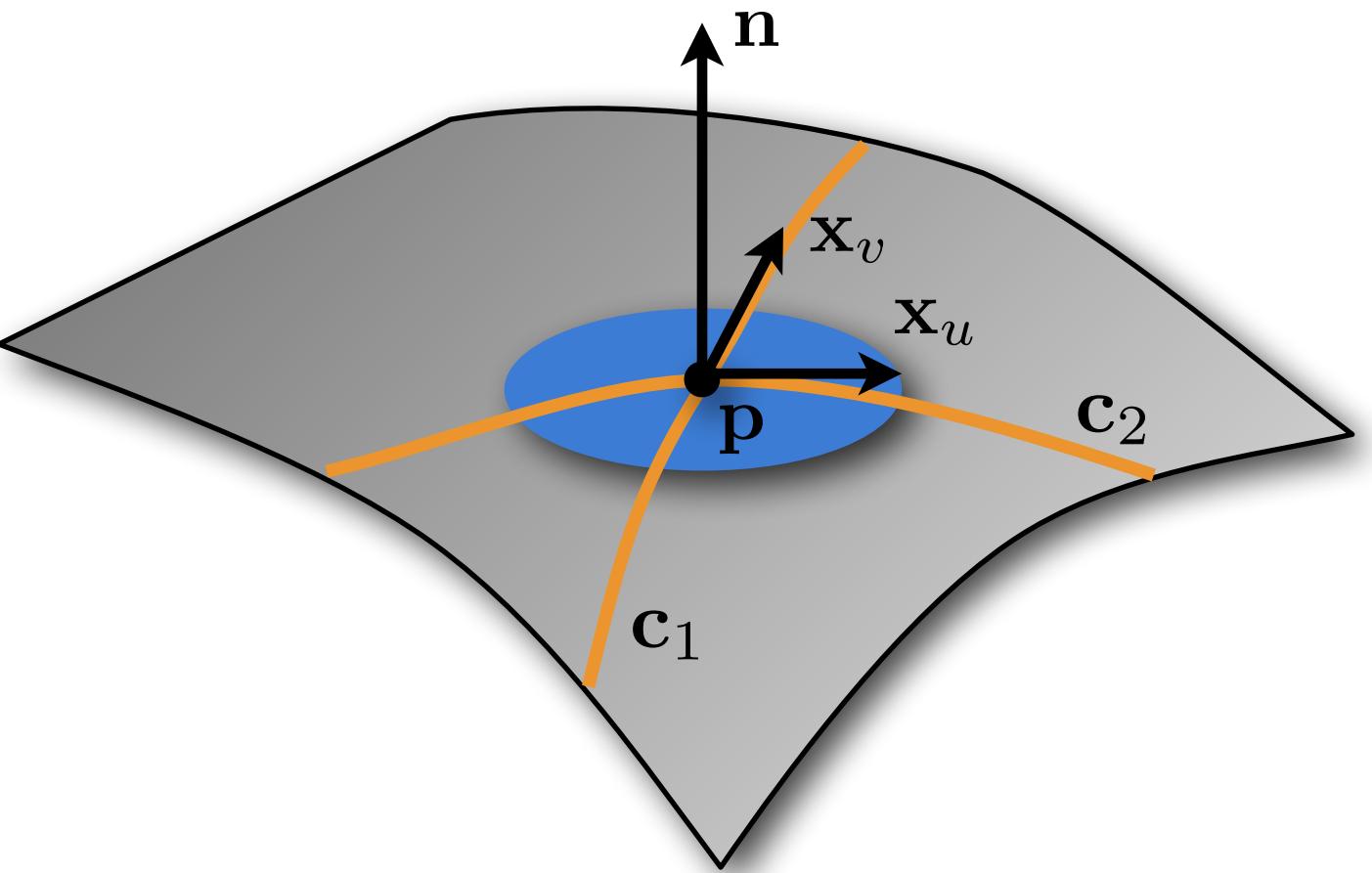
$$\mathbf{t}_1^T \mathbf{t}_2 = \langle (\alpha_1, \beta_1), (\alpha_1, \beta_1) \rangle$$

- **Length**

$$ds^2 = Edu^2 + 2Fdu dv + Gdv^2$$

- **Area**

$$dA = \sqrt{EG - F^2} du dv$$



Review: Surface



Differential Geometry of Surfaces

– Second fundamental form

$$\text{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{pmatrix}$$

- **normal curvature**

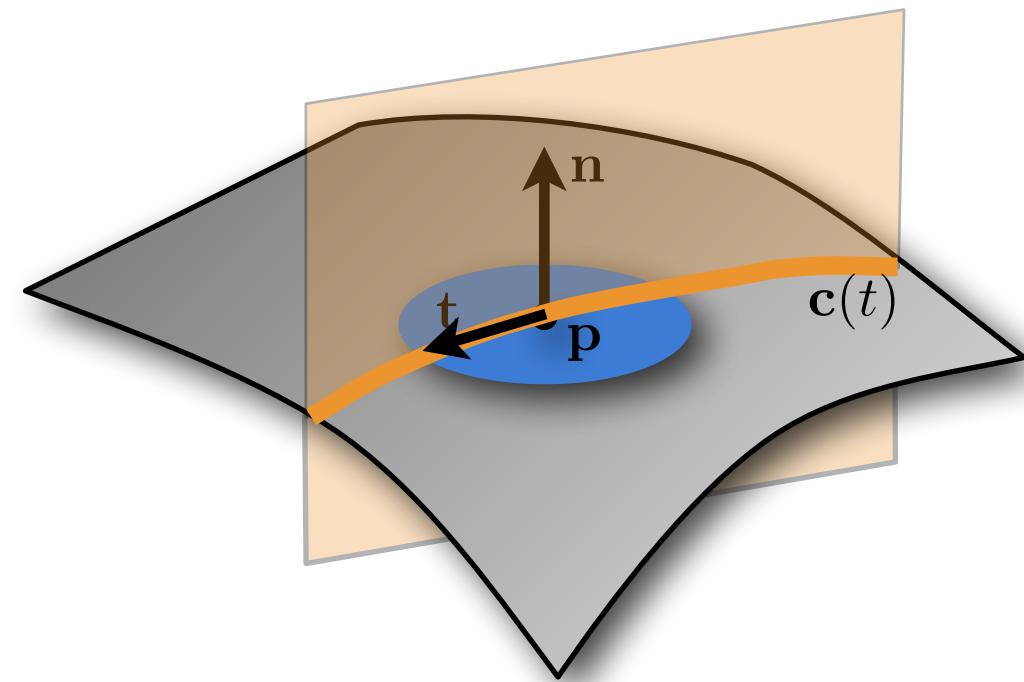
$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \text{II} \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}$$

- **principal curvatures**

$$\kappa_1 = \max_{\phi} \kappa_n(\phi) \quad \kappa_2 = \min_{\phi} \kappa_n(\phi)$$

- **mean and Gaussian curvature**

$$H = \frac{\kappa_1 + \kappa_2}{2} \quad K = \kappa_1 \cdot \kappa_2$$



Outline



- Spectral Analysis
- Diffusion Flow
- Energy Minimization

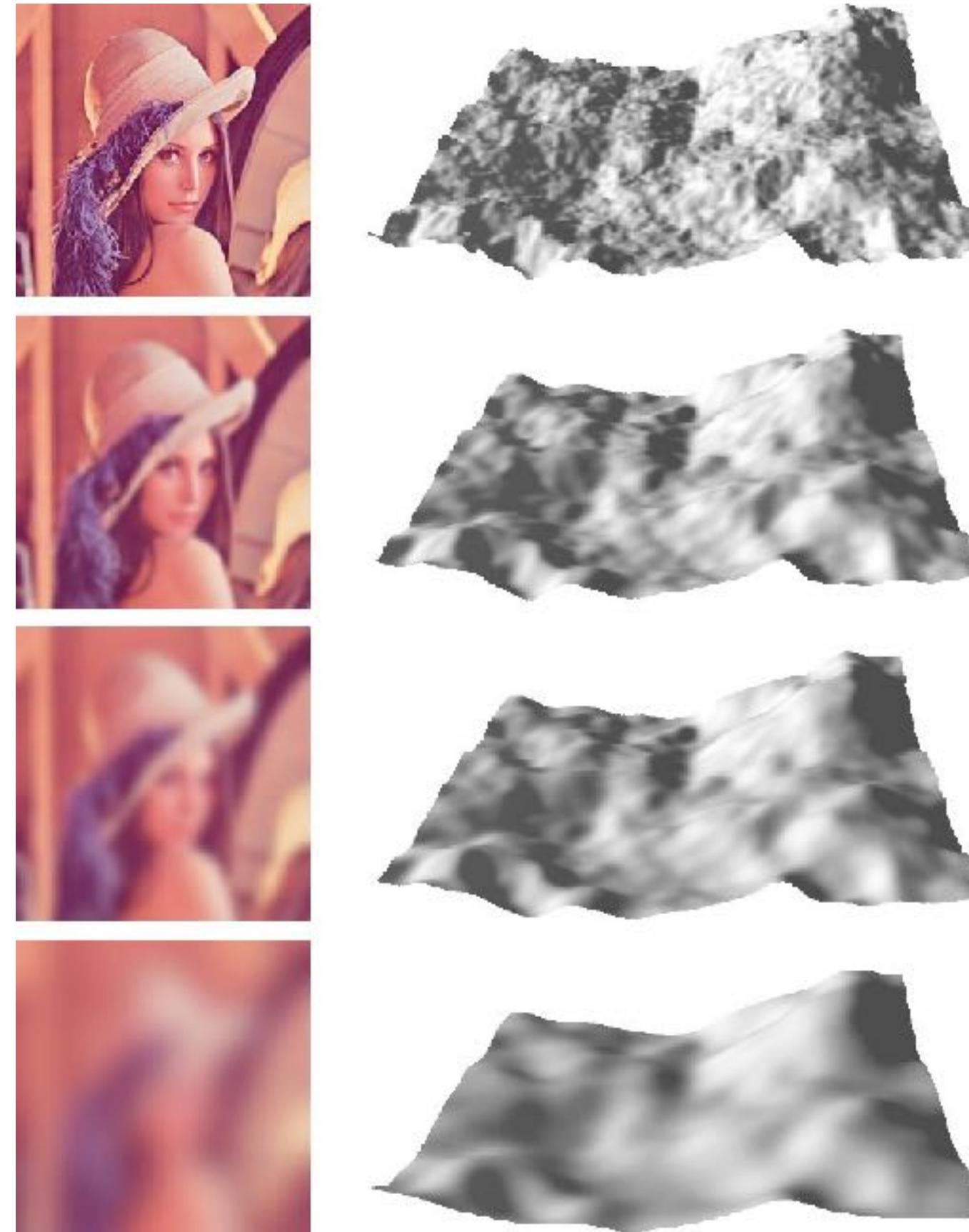
Diffusion Flow on Height Fields



diffusion constant

$$\frac{\partial f}{\partial t} = \lambda \Delta f$$

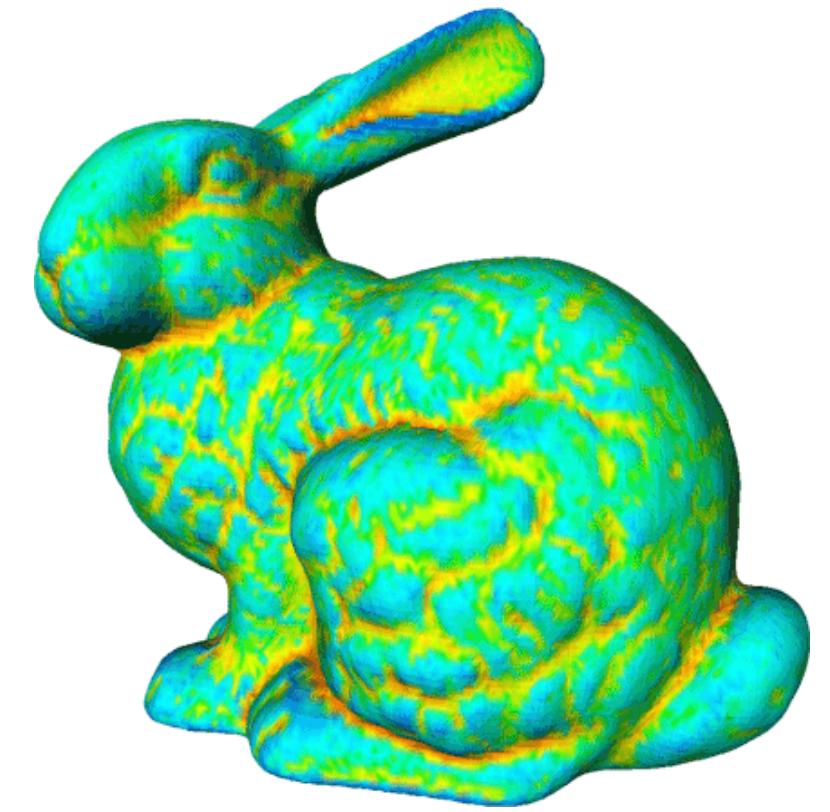
A diagram illustrating the diffusion equation. A downward-pointing arrow from the word "diffusion constant" points to the term λ in the equation. An upward-pointing arrow from the word "Laplace operator" points to the term Δ .



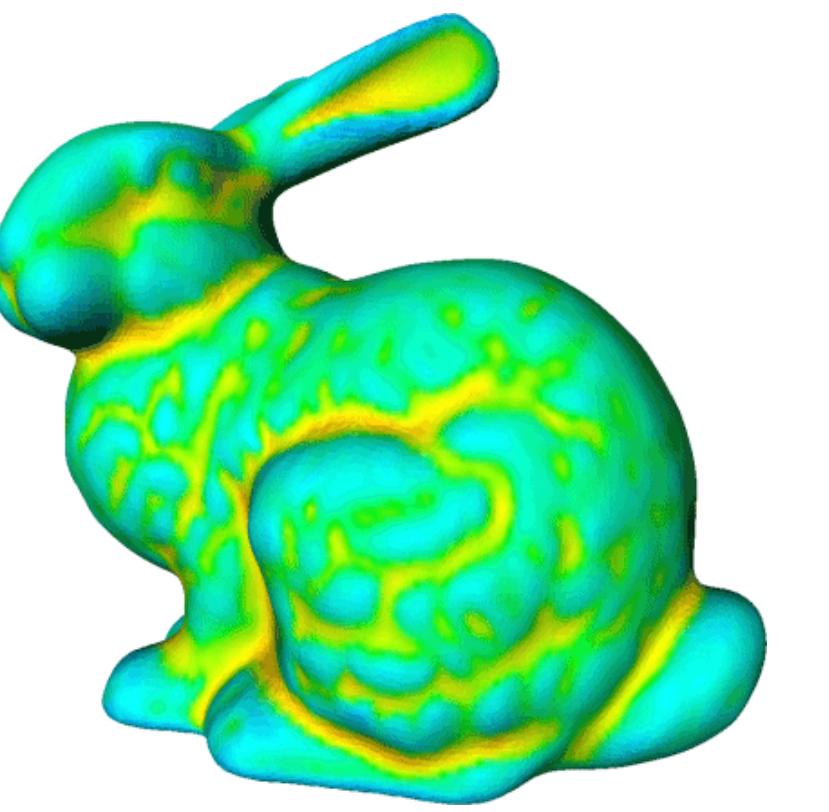
Diffusion Flow on Meshes



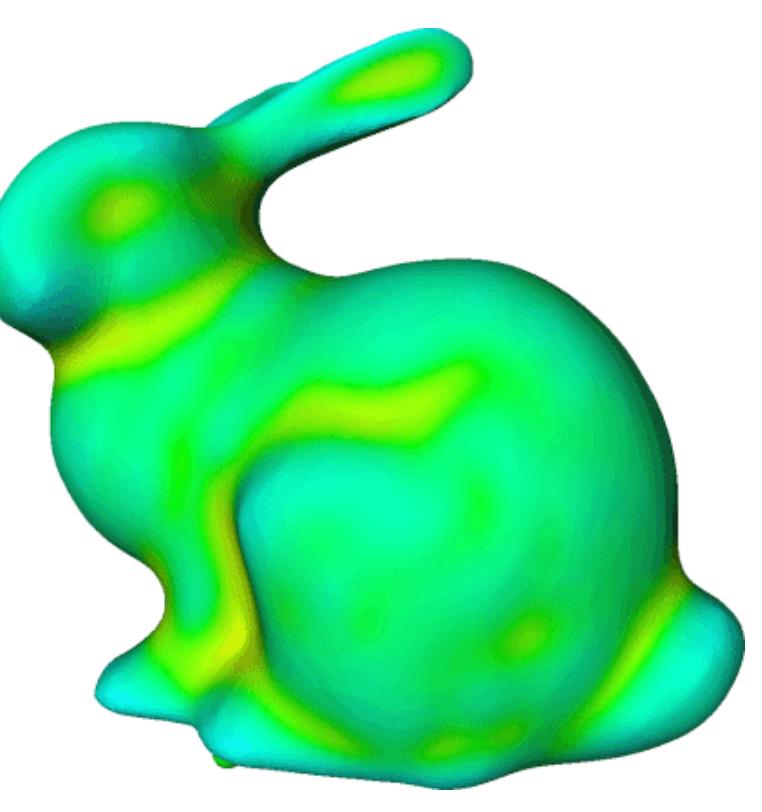
- Iterate $p_i \leftarrow p_i + \lambda \Delta p_i$



0 Iterations



5 Iterations



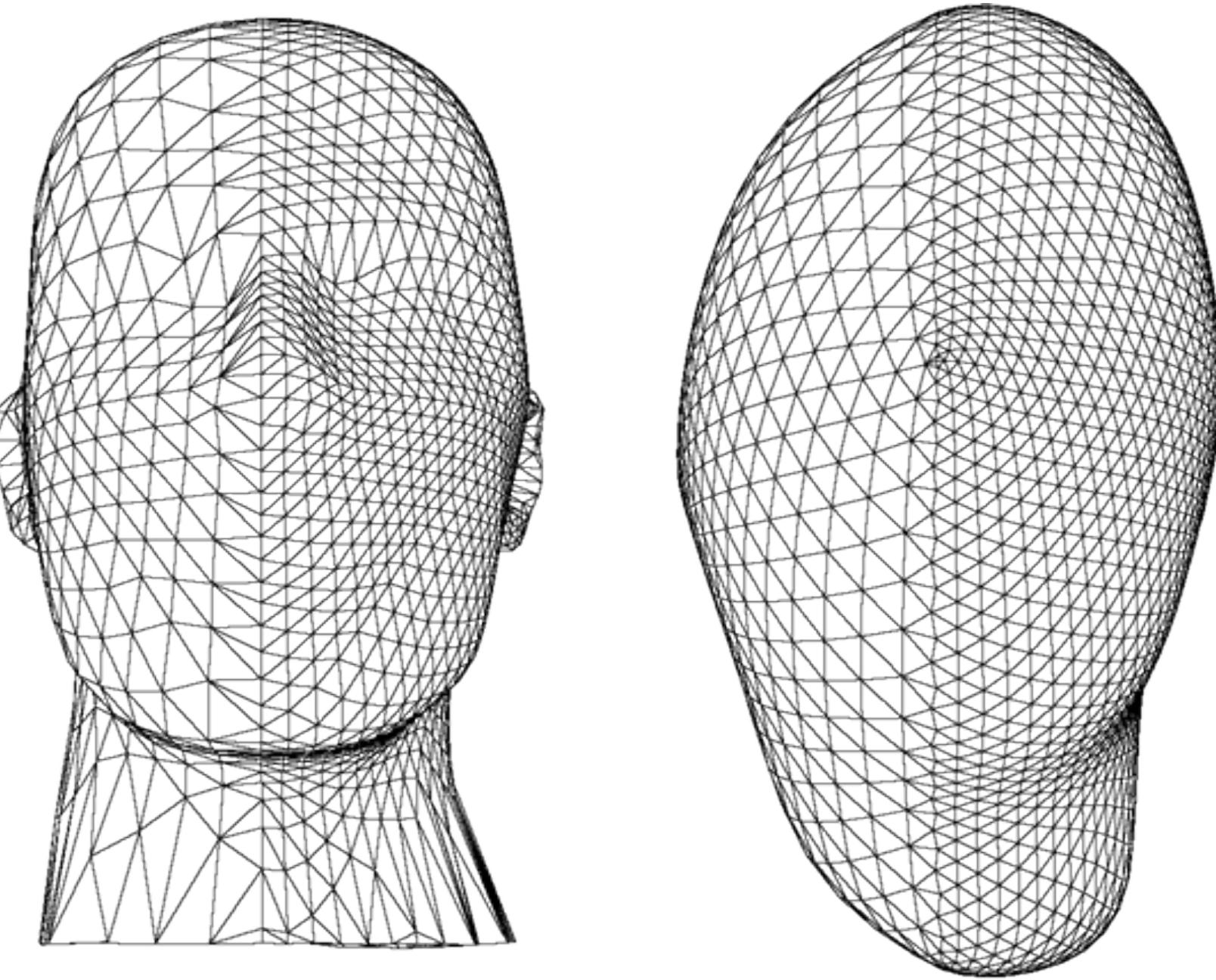
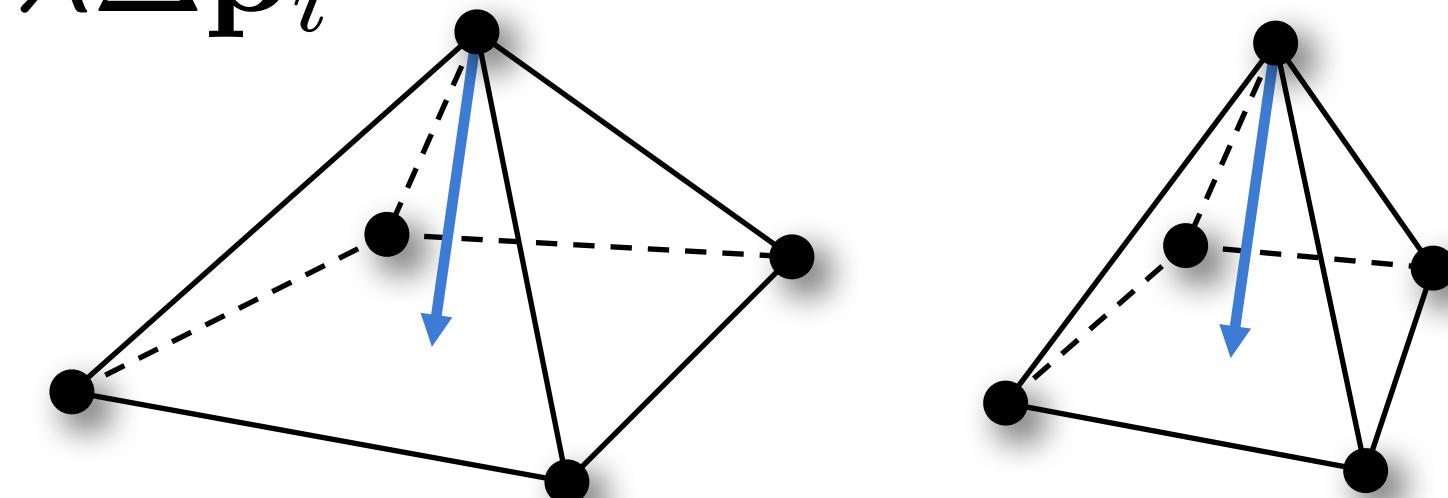
20 Iterations

Uniform Laplace Discretization



- Smoothes geometry and triangulation
- Can be non-zero even for planar triangulations
- Vertex drift can lead to distortions
- Might be desired for mesh regularization

$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda \Delta \mathbf{p}_i$$



Desbrun et al., Siggraph 1999

Curvature Flow

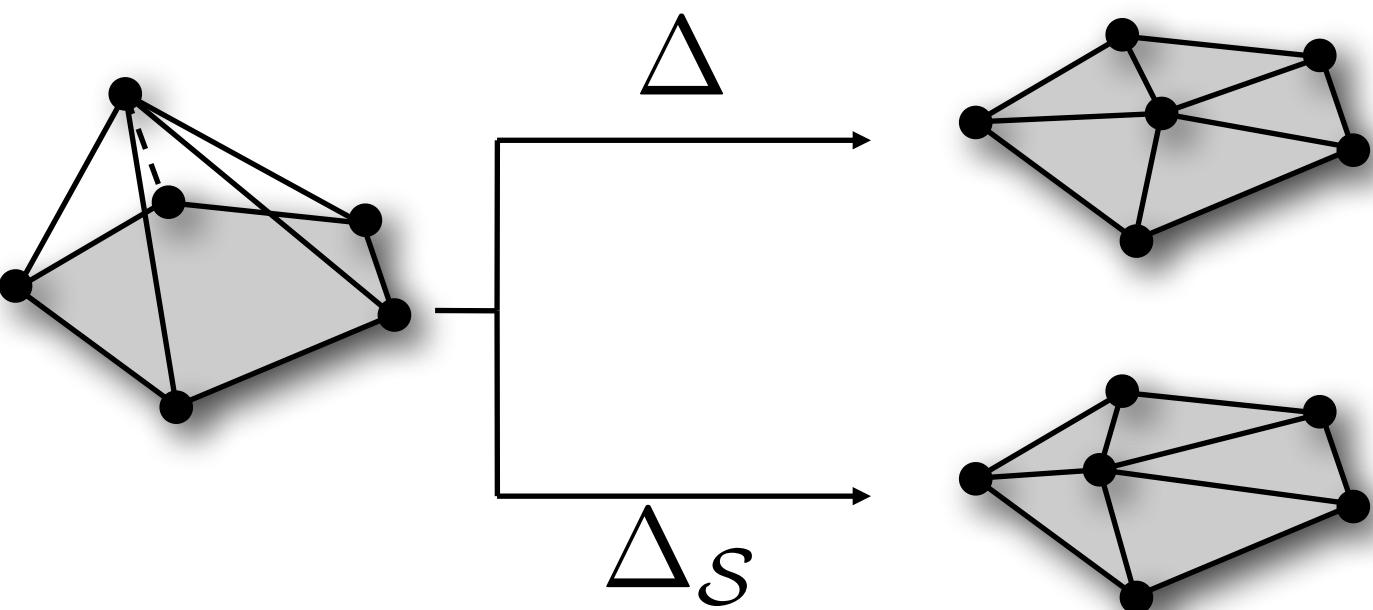
- Use diffusion flow with Laplace-Beltrami

$$\frac{\partial \mathbf{p}}{\partial t} = \lambda \Delta_S \mathbf{p}$$

- Laplace-Beltrami is parallel to surface normal

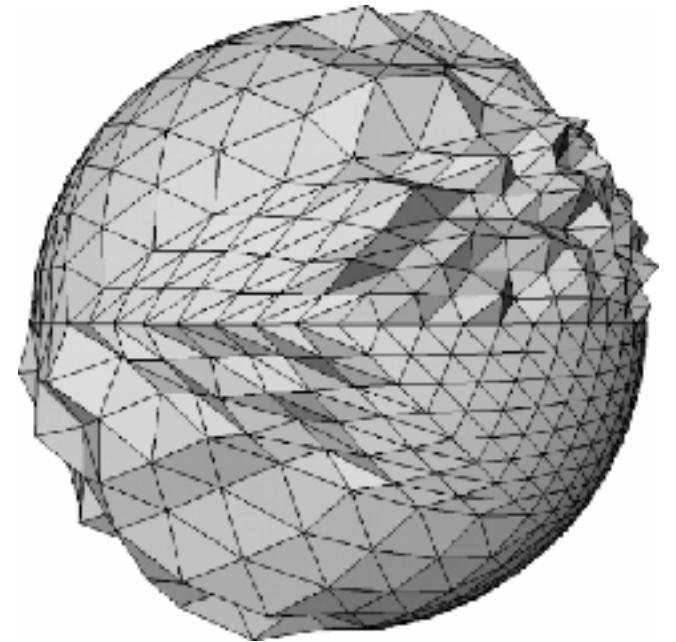
$$\frac{\partial \mathbf{p}}{\partial t} = -2\lambda H \mathbf{n}$$

→ Avoids vertex drift on surface

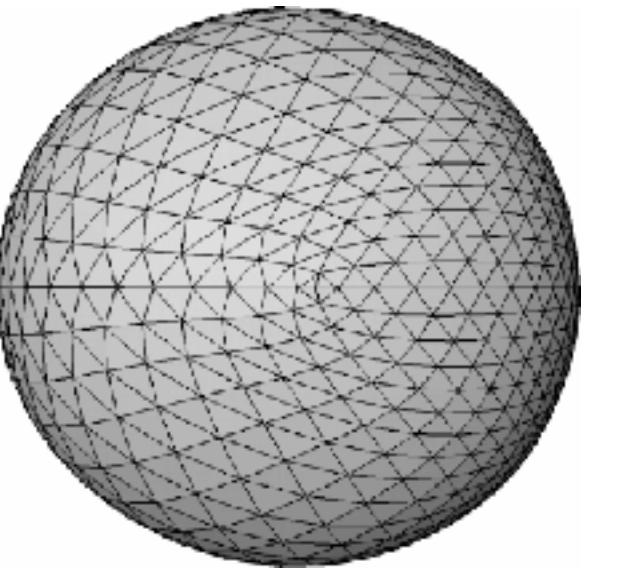


Comparison

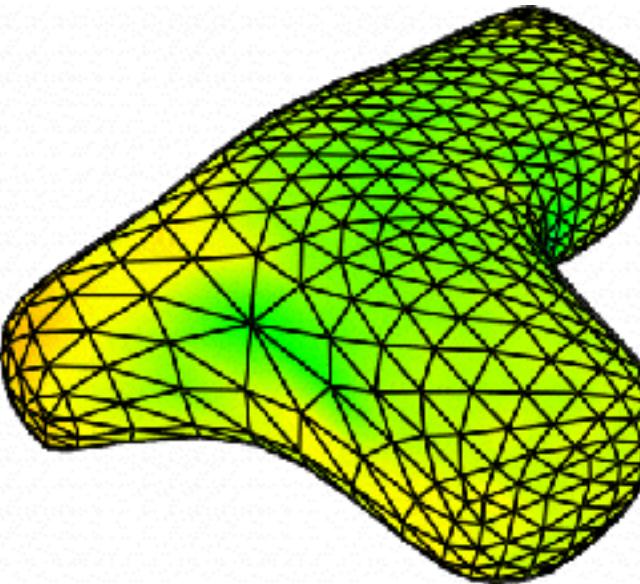
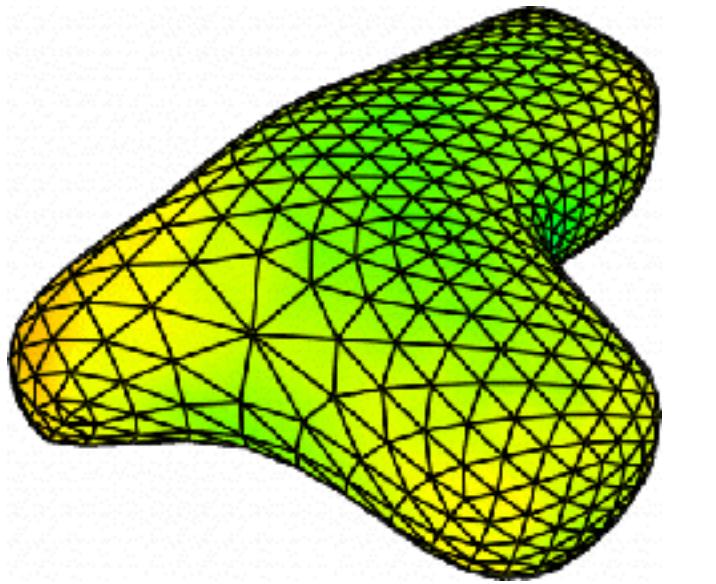
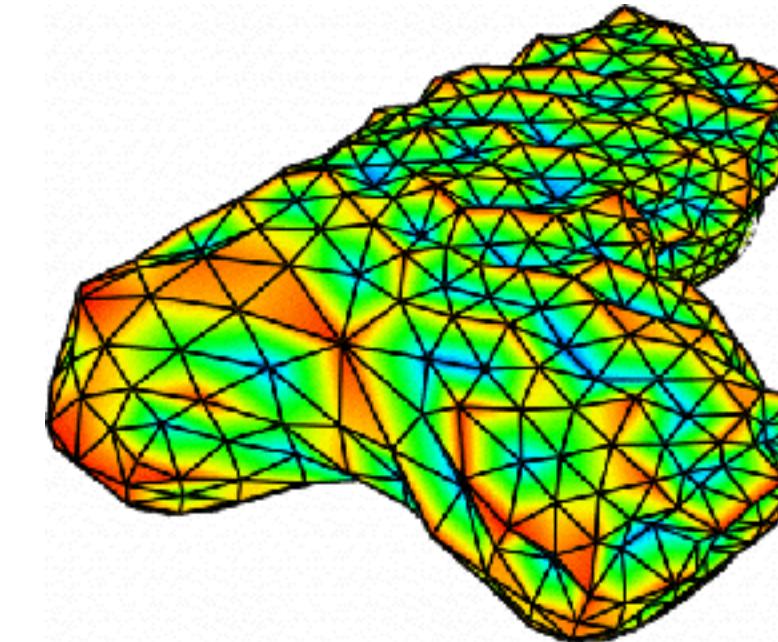
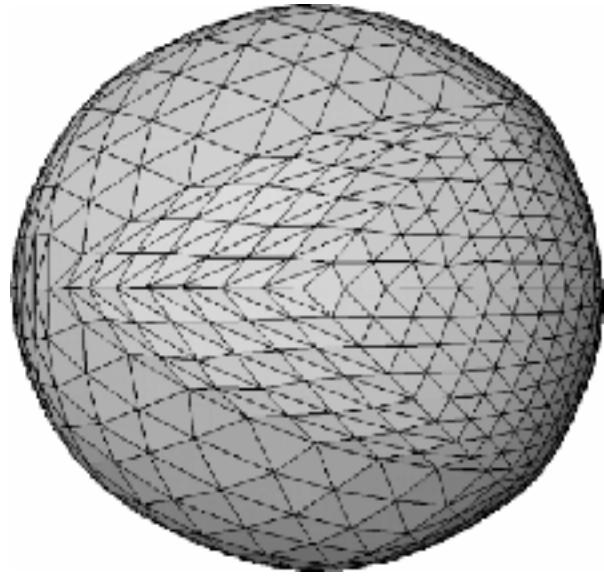
Original



Uniform Laplace



Laplace-Beltrami



Numerical Integration



- Write update $\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda \Delta \mathbf{p}_i^{(t)}$ in matrix notation

$$\mathbf{P}^{(t)} = \left(\mathbf{p}_1^{(t)}, \dots, \mathbf{p}_n^{(t)} \right)^T \in \mathbb{R}^{n \times 3}$$

$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda \Delta \mathbf{p}_i$$

- Corresponds to **explicit** integration

$$\mathbf{P}^{(t+1)} = (\mathbf{I} + \lambda \mathbf{L}) \mathbf{P}^{(t)}$$

Requires small λ for stability!

- **Implicit** integration is *unconditionally stable*

$$(\mathbf{I} - \lambda \mathbf{L}) \mathbf{P}^{(t+1)} = \mathbf{P}^{(t)}$$

Implementation



- Solve linear system each iteration

$$(\mathbf{I} - \lambda \mathbf{L}) \mathbf{P}^{(t+1)} = \mathbf{P}^{(t)}$$

- Matrix $L = M^{-1}L_w$ is not symmetric because of M

→ Symmetrize by multiplying M from left

$$(\mathbf{M} - \lambda \mathbf{L}_w) \mathbf{P}^{(t+1)} = \mathbf{M} \mathbf{P}^{(t)}$$

- Solve sparse symmetric positive definite system

→ Iterative conjugate gradients, sparse Cholesky

Outline



- Spectral Analysis
- Diffusion Flow
- Energy Minimization

Fairness



- Idea: Penalize “unaesthetic behavior”
- **Measure** fairness
 - Principle of the simplest shape
 - Physical interpretation
- **Minimize** some fairness functional
 - Surface area, curvature
 - Membrane energy, thin plate energy

Non-Linear Energies



- Membrane energy (surface area)

$$\int_S dA \rightarrow \min \quad \text{with} \quad \delta S = c$$

- Thin-plate surface (curvature)

$$\int_S \kappa_1^2 + \kappa_2^2 dA \rightarrow \min \quad \text{with} \quad \delta S = c, \quad n(\delta S) = d$$

- Too complex ... simplify energies

Membrane Surfaces



- Surface parameterization

$$\mathbf{x} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

- Membrane energy (surface area)

$$\min_{\mathbf{x}} \int_{\Omega} \|\mathbf{x}_u\|^2 + \|\mathbf{x}_v\|^2 du dv$$

Variational Calculus in 1D



- 1D membrane energy

$$L(f) = \int_a^b f'^2(x) \, dx \rightarrow \min$$

- Add test function u with $u(a) = u(b) = 0$

$$L(f + \lambda u) = \int_a^b (f' + \lambda u')^2 = \int_a^b f'^2 + 2\lambda f'u' + \lambda^2 u'^2$$

- If f minimizes L , the following has to vanish

$$\left. \frac{\partial L(f + \lambda u)}{\partial \lambda} \right|_{\lambda=0} = \int_a^b 2f'u' \stackrel{!}{=} 0$$

Variational Calculus in 1D



- Has to vanish for any u with $u(a) = u(b) = 0$

$$\int_a^b f'u' = \underbrace{[f'u]_a^b}_{=0} - \int_a^b f''u \stackrel{!}{=} 0 \quad \forall u$$

$$\int_0^1 x'y = [xy]_0^1 - \int_0^1 xy'$$

- Only possible if

$$f'' = \Delta f = 0$$

→ *Euler-Lagrange equation*

Bivariate Variational Calculus



- Find minimum of functional

$$\operatorname{argmin}_f \int_{\Omega} L(f_{uu}, f_{vv}, f_u, f_v, f, u, v)$$

- Euler-Lagrange PDE defines the minimizer f

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial u} \frac{\partial L}{\partial f_u} - \frac{\partial}{\partial v} \frac{\partial L}{\partial f_v} + \frac{\partial^2}{\partial u^2} \frac{\partial L}{\partial f_{uu}} + \frac{\partial^2}{\partial u \partial v} \frac{\partial L}{\partial f_{uv}} + \frac{\partial^2}{\partial v^2} \frac{\partial L}{\partial f_{vv}} = 0$$

- Again, subject to suitable boundary constraints

Membrane Surfaces



- Surface parameterization

$$\mathbf{x} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

- Membrane energy (surface area)

$$\min_{\mathbf{x}} \int_{\Omega} \|\mathbf{x}_u\|^2 + \|\mathbf{x}_v\|^2 du dv$$

- Variational calculus

$$\Delta \mathbf{x} = 0$$

Laplace-Beltrami Operator



- Extension of Laplace to functions on manifolds

$$\Delta_S \mathbf{x} = \operatorname{div}_S \nabla_S \mathbf{x} = -2H\mathbf{n}$$

gradient operator

mean curvature

Laplace-Beltrami

The diagram illustrates the components of the Laplace-Beltrami operator. It shows three arrows pointing from labels to a central equation. The top-left arrow points from 'Laplace-Beltrami' to the term $\Delta_S \mathbf{x}$. The top-right arrow points from 'gradient operator' to the term $\nabla_S \mathbf{x}$. The bottom-right arrow points from 'mean curvature' to the term $-2H\mathbf{n}$.

$$\nabla \mathbf{x} = [\partial \mathbf{x} / \partial u \quad \partial \mathbf{x} / \partial v]$$

$$\operatorname{div} \mathbf{y} := \nabla \cdot \mathbf{y} = \partial \mathbf{y}_u / \partial u + \partial \mathbf{y}_v / \partial v$$

$$\operatorname{div}(\nabla \mathbf{x}) = \Delta \mathbf{x}$$

Thin-Plate Surfaces



- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

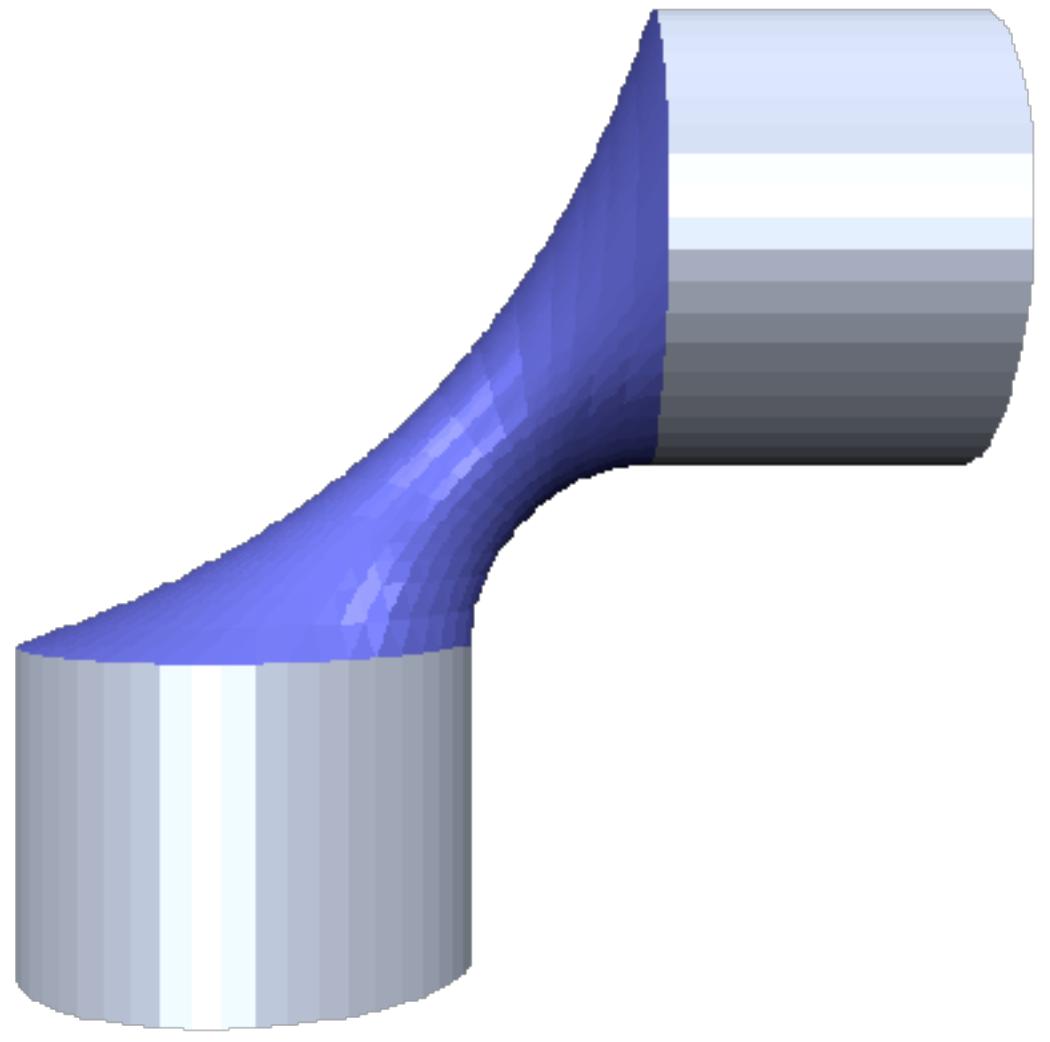
- Thin-plate energy (curvature)

$$\int_{\Omega} \|\mathbf{p}_{uu}\|^2 + 2\|\mathbf{p}_{uv}\|^2 + \|\mathbf{p}_{vv}\|^2 \, dudv \rightarrow \min$$

- Variational calculus

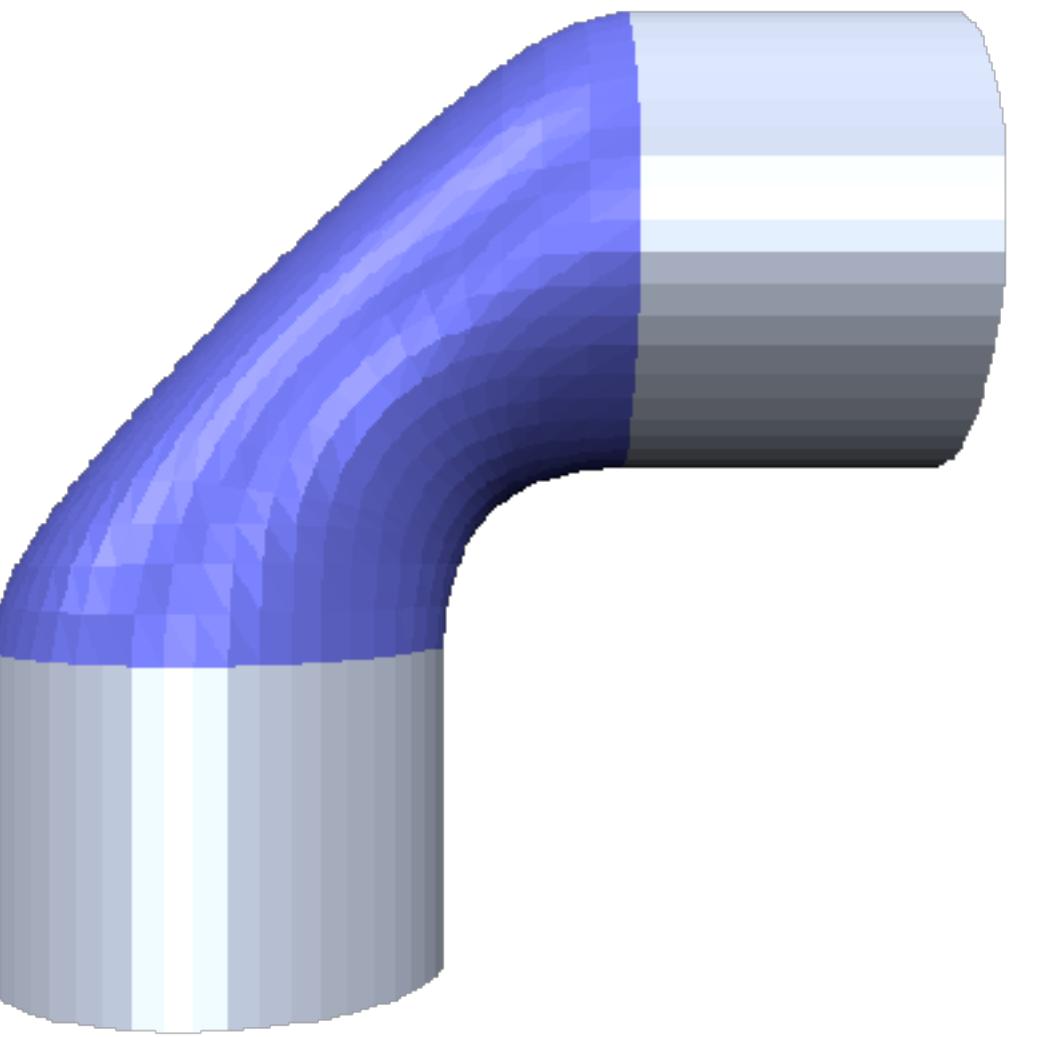
$$\Delta^2 \mathbf{p} = 0$$

Energy Functionals



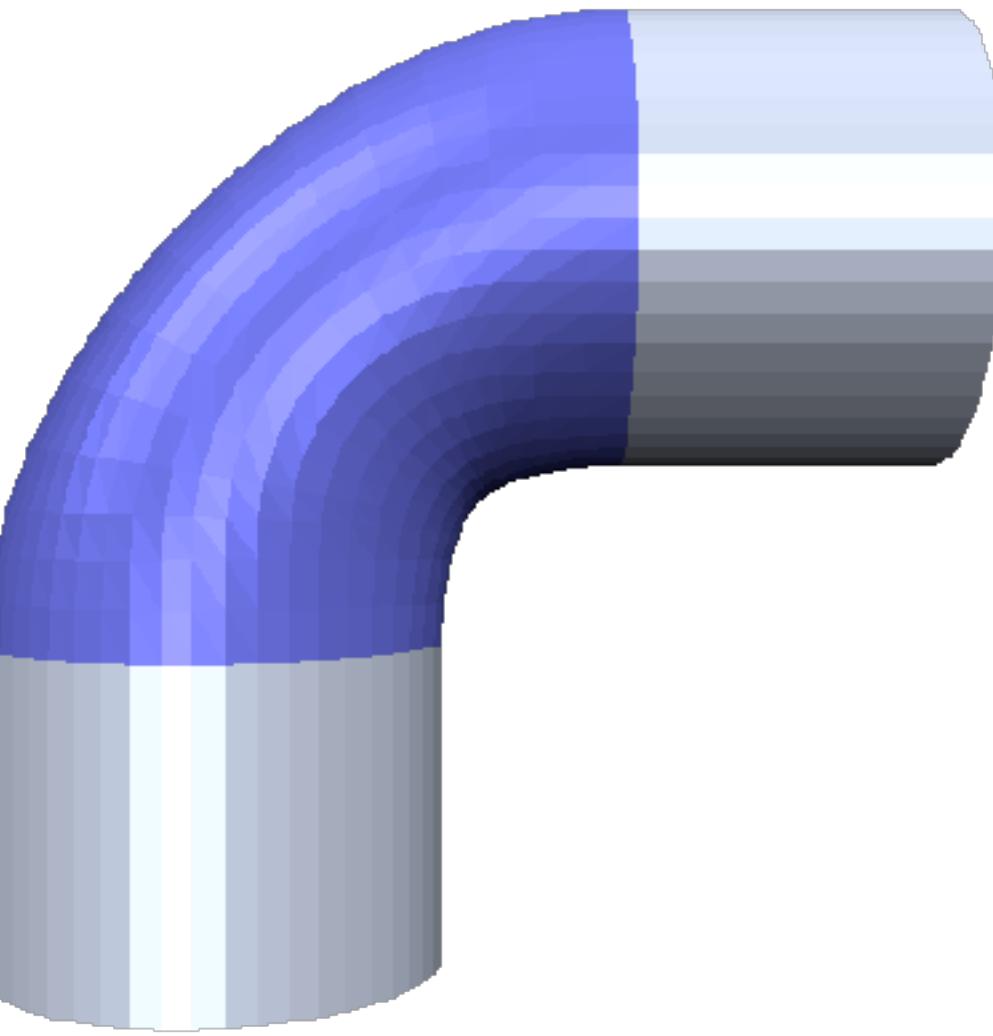
Membrane

$$\Delta_S p = 0$$



Thin Plate

$$\Delta_S^2 p = 0$$



$$\Delta_S^3 p = 0$$

- Minimizer surfaces satisfy Euler-Lagrange PDE

$$\Delta_{\mathcal{S}}^k \mathbf{p} = 0$$

- They are *stationary surfaces of Laplacian flows*

$$\frac{\partial \mathbf{p}}{\partial t} = \Delta_{\mathcal{S}}^k \mathbf{p}$$

- Explicit flow integration corresponds to iterative solution of linear system

References



- Book: Chapter 4 (Botsch et al.),
Chapter 5 (CohenOr et al.)
- Levy: *Laplace-Beltrami Eigenfunctions: Towards an algorithm that understands geometry*, Shape Modeling and Applications, 2006
- Taubin: *A signal processing approach to fair surface design*, SIGGRAPH 1996
- Desbrun, Meyer, Schroeder, Barr: *Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow*, SIGGRAPH 99