전자기학 (Tay T. Han)

October 5
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) TA. 연습시간.

Vector

* Scalar Triple Product

$$A\cdot(B\times C) = B\cdot(C\times A) = C\cdot(A\times B)$$

* Vector " "

$$A\times(B\times C) = B\times(A\times C) = C\times(A\times B)$$

* Orthogonal Coordinate system.

unit vectors. $\hat{n}_i, \hat{n}_j, \hat{n}_k$

$$\hat{n}_i \times \hat{n}_j = \hat{n}_k$$

$$\hat{n}_i \cdot \hat{n}_j = \begin{cases} 0 & (i\neq j) \\ 1 & (i=j) \end{cases}$$

δ_{ij} Kronecker's delta

1. Rectangular / Cartesian Coordinate system.

$\hat{x}, \hat{y}, \hat{z}$ ≠ unit vectors.

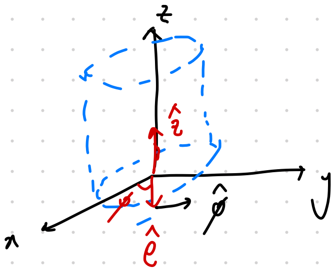
$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

area in x-y plane — $S_{xy} = \underset{\text{convention}}{S_z} = \hat{z}\,dx\,dy$

differential area → ∴ $dS_z = \hat{z}\,dx\,dy$, $dS_x = \hat{x}\,dy\,dz$, $dS_y = \hat{y}\,dx\,dz$

differential volume → $dv = dx\,dy\,dz$

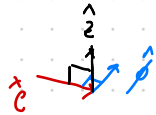
2. Cylinderical Coordinate system.



$\hat{\rho}$ & $\hat{\phi}$: orthogonal

(ρ, ϕ, z) $\quad A = \hat{\rho} A_\rho + \hat{\phi} A_\phi + \hat{z} A_z$

$\hat{\rho}, \hat{\phi}, \hat{z}$



$$\begin{cases} \hat{\rho} \times \hat{\phi} = \hat{z} \\ \hat{\phi} \times \hat{z} = \hat{\rho} \\ \hat{z} \times \hat{\rho} = \hat{\phi} \end{cases}$$

$\hat{z}$ is constant, $\hat{\phi}$, $\hat{\rho}$ changes depending on the position.

→ Cartesian.

$$\begin{cases} x = \rho \cos\phi \\ y = \rho \sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi \\ \hat{y} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi \\ \hat{z} = \hat{z} \end{cases}$$

differential length

$\hat{\rho}$: $d\rho$, $\hat{\rho}\, d\rho$

$\hat{\phi}$: $\rho\, d\phi$, $\hat{\phi}\, \rho\, d\phi$

$\hat{z}$: dz, $\hat{z}\, dz$

differential area

$\underbrace{dS_\rho}_{}$ $= \hat{\rho}\,(\rho\, d\phi)\, dz$

↳ normal to the ρ direction

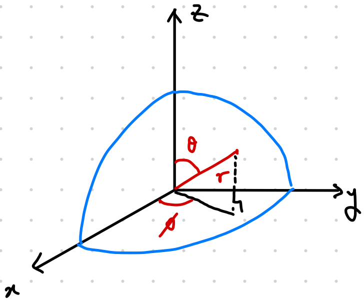
$dS_\phi = \hat{\phi}\, dz\, d\rho$

$dS_z = \hat{z}\, d\rho\, (\rho\, d\phi) = \hat{z}\, \rho\, d\rho\, d\phi$

differential volume

$dv = \rho\, d\rho\, d\phi\, dz$

3. Spherical Coordinate System.



r, θ, ϕ

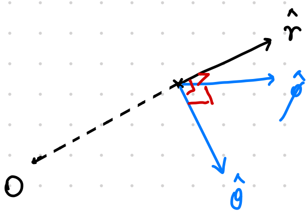
$$\begin{cases} \rho = r\sin\theta \\ \phi = \phi \\ z = r\cos\theta \end{cases}$$

: cylinderic - spherical coordinate.

$$\begin{cases} x = \rho\cos\phi \\ \quad = r\sin\theta\cos\phi \\ y = \rho\sin\phi \\ \quad = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

: rectangular - spherical coordinate

$\hat{r}, \hat{\theta}, \hat{\phi}$: orthonormal.



$\hat{r} \times \hat{\theta} = \hat{\phi}$ $\qquad$ $\hat{r} \times \hat{r} = 0$

$\hat{\theta} \times \hat{\phi} = \hat{r}$ $\qquad$ $\hat{r} \cdot \hat{r} = 1$

$\hat{\phi} \times \hat{r} = \hat{\theta}$

$$\hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$$

$$= (\hat{r}\sin\theta + \hat{\theta}\cos\theta)\cos\phi - \hat{\phi}\sin\phi$$

* Differential Length

$\hat{r}$ $\qquad$ $\hat{r}\,dr$

$\hat{\theta}$ $\qquad$ $\hat{\theta}\,r\,d\theta$

$\hat{\phi}$ $\qquad$ $\hat{\phi}\,r\sin\theta\,d\phi$

* Differential Area

$$dS_r = \hat{r}\,r\,d\theta \cdot r\sin\theta\,d\phi$$

$$= \hat{r}\,r^2\sin\theta\,d\theta\,d\phi$$

$$dS_\theta = \hat{\theta}\,dr \cdot r\sin\theta\,d\phi = \hat{\theta}\,r\sin\theta\,dr\,d\phi$$

$$dS_\phi = \hat{\phi}\,dr \cdot r\,d\theta = \hat{\phi}\,r\,dr\,d\theta$$

* Differential Volume.

$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$S = \iint dS_r = \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta \, d\theta \, d\phi = 2\pi r^2 (-\cos\theta)\Big]_0^{\pi} = 4\pi r^2$$

$$\int \mathbb{F} \, dv = \int (\hat{x} F_x + \hat{y} F_y + \hat{z} F_z) \, dv$$

$$\int V d\mathbb{l} = \int V (\hat{x} dl_x + \hat{y} dl_y + \hat{z} dl_z)$$

$$\oint_C \mathbb{F} d\mathbb{l}$$

$$\oint_S \mathbb{A} d\mathbb{s}$$

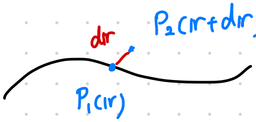
cf:

∇f Gradient

$\nabla \cdot \mathbb{A}$ Divergence

$\nabla \times \mathbb{A}$ Curl

> Gradient of a scalar field.



$f(\mathbb{r}) = C$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad (\because \text{Chain Rule})$$

Let's say $\mathbb{G} = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$

Then, $df = \mathbb{G} \, d\mathbb{r}$

$= G \, dr \cos\theta$

$\therefore$ $G = \dfrac{df}{dr}$ at $\theta = 0$

$\rightarrow$ if it follows the direction of f,

- $df = 0$
- $\mathbb{G}$ is normal to the surface
- G becomes maximum when $\theta = 0$

$$G \rightarrow \nabla f = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\underbrace{f(r)}_{f(x,y,z)}$$

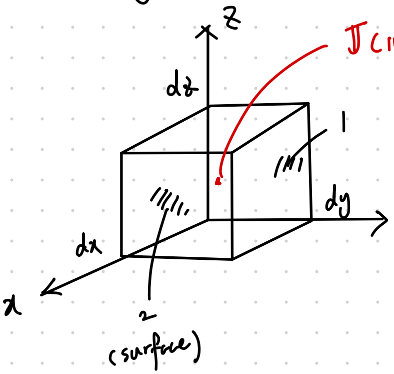
$$\left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}\right) f(\rho,\phi,z)$$

$$\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) f(r,\theta,\phi)$$

> Divergence in vector field.

Later becomes current density!

$J(r) = J(x,y,z)$



$$\text{outflow}\Big|_2 = J_x\left(x+\frac{1}{2}dx,\ y,\ z\right) dy\, dz$$

$$\approx \left(J_x(x,y,z) + \frac{\partial J_x}{\partial x}\cdot\frac{1}{2}dx\right) dy\, dz$$

$$\text{outflow}\Big|_1 = J_x\left(x-\frac{1}{2}dx,\ y,\ z\right) dy\, dz$$

$$\approx \left(J_x(x,y,z) - \frac{\partial J_x}{\partial x}\cdot\frac{1}{2}dx\right) dy\, dz$$

$\vdots$

x: net flow $= \frac{\partial J_x}{\partial x} dx\,dy\,dz$ (−)

y: net flow $= \frac{\partial J_y}{\partial y} dx\,dy\,dz$

z: net flow $= \frac{\partial J_z}{\partial z} dx\,dy\,dz$

$$\oint_S J\, ds = \left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right)\underbrace{dx\,dy\,dz}_{= dv}$$

$$\lim_{\Delta v \to 0}\frac{\oint_S J\, ds}{\Delta v} = \underline{\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}}$$

⇒ "Divergence"

Noted as $\nabla \cdot J$

$$\nabla \cdot \mathbb{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$ (Rectangular coordinate)

$$= \left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}\right)(\hat{\rho}A_\rho + \hat{\phi}A_\phi + \hat{z}A_z)$$

$$\left(\neq \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}\right)$$

↳ ($\because$ unit vectors are not constant in cylinderic coordinate sys.)

$$= \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \underline{\frac{A_\rho}{\rho}} + \frac{\partial A_z}{\partial z}$$ (Cylinderic coordinate sys.)

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin^2\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

* Divergence Theorem

$$\oint_v A\,ds = \int_v \nabla \cdot \mathbb{A}\, dv$$

> Curl of a vector field

$$\nabla \times \mathbb{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \times (\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$ (Cartesian)

$$= \left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}\right) \times (\hat{\rho}A_\rho + \hat{\phi}A_\phi + \hat{z}A_z)$$

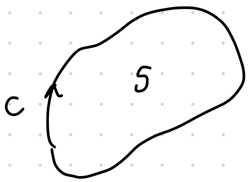
$$= \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

(Cylinderical Coordinate sys.)

$$\nabla \times \mathbb{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{r} & \hat{\theta} r & \hat{\phi} r\sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & (r\sin\theta)A_\phi \end{vmatrix}$$

(Spherical Coordinate sys.)

* Stokes theorem (Important for Magnetic field. Curl)



$$\int_S \nabla \times \mathbb{A} \cdot d\mathbb{S} = \oint_C \mathbb{A} \cdot d\mathbb{l}$$

* Laplacian Operator

$$\nabla \cdot \nabla f = \nabla^2 f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x,y,z)$$ (Cartesian)

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$ (Cylinderical)

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$ (Spherercal)

$$\left.\begin{aligned} \nabla \times \nabla f &= 0 \\ \nabla \cdot \nabla \times \mathbb{A} &= 0 \end{aligned}\right\}$$ Null identity

cf; In Maxwell eq.

* $\nabla \times \mathbb{E} = 0$

$\therefore$ By Null identity... ∇f exists that $\mathbb{E} = \nabla f$

we call this f. "potential"

* $\nabla \cdot \mathbb{B} = 0$

$\therefore$ By Null identity, $\mathbb{B} = \nabla \times \mathbb{A}$

we call this $\mathbb{A}$, "vector magnetic potential".