Lecture 15
ANOVA

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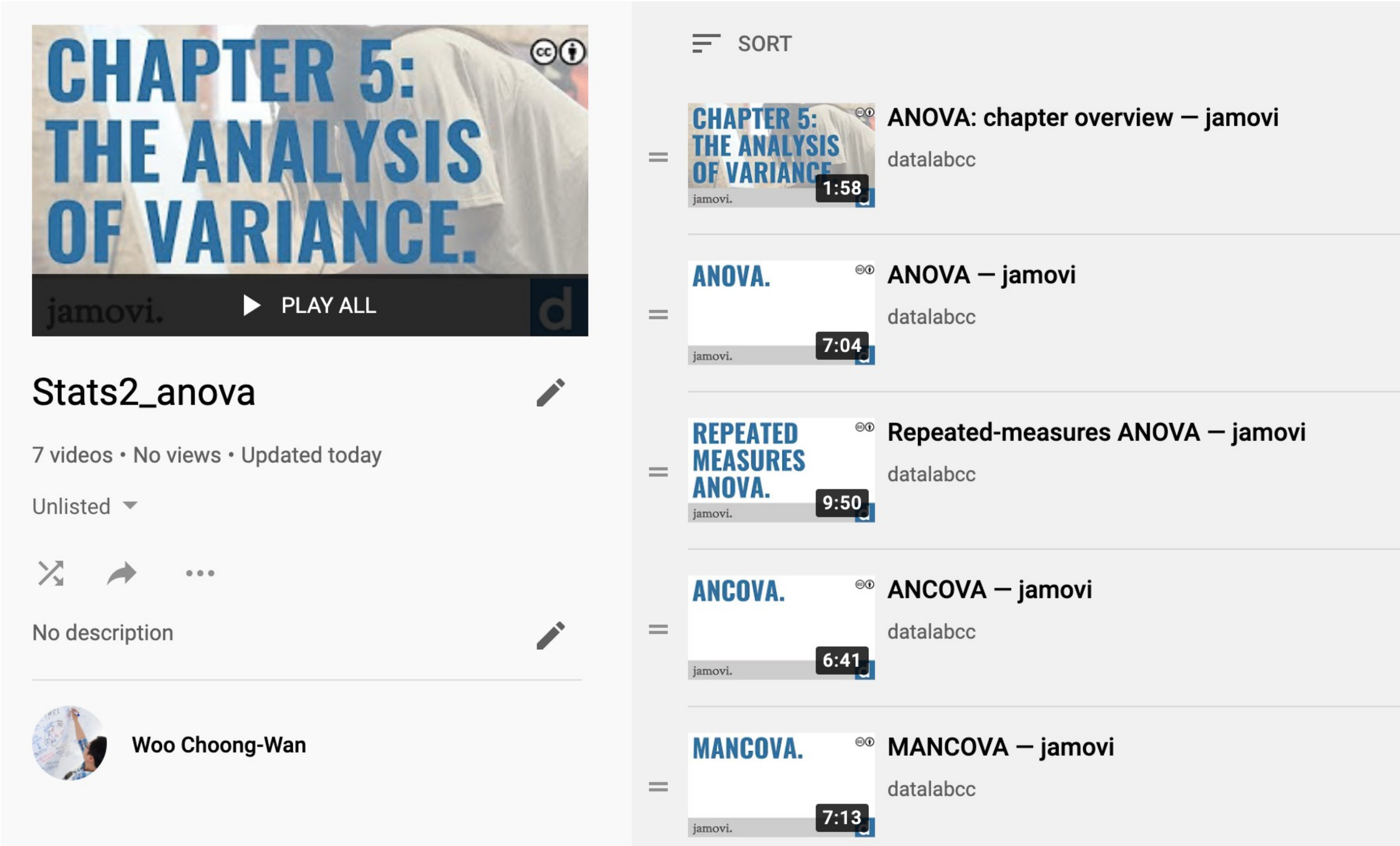
What we learned...

- Sampling distribution
- Standard error of the mean
- Confidence interval
- One-sample t-test
- Paired t-test
- Independent samples t-test
- Resampling (bootstrap, permutation tests)
- Binomial test
- Chi-square test

ANOVA

https://www.youtube.com/playlist?list=PLXCuLG6zw7mLmslikvkrA4H0i5vNv4tBp



Before going deep into ANOVA

I recommend you watching the previous lecture on ANOVA that I re-edited for you

Why?

ANOVA table

Lecture 23

Let's look at the data

Analysis of Variance Table

Source	Sum of Squares	DF	Mean Square	F-ratio	P-value
Method	29882	3	9960.64	7.0636	0.0011
Error	39484	28	1410.14		
Total	69366	31			

- $y_{ij} = \underbrace{\bar{\bar{y}}}_{\mu} + \underbrace{(\bar{y}_j - \bar{\bar{y}})}_{\tau_j} + \underbrace{(y_{ij} - \bar{y}_j)}_{\varepsilon_{ij}}$

y_{ij}: Observations

	Alcohol	AB Soap	Soap	Water
	51	70	84	74
	5	164	51	135
	19	88	110	102
	18	111	67	124
	58	73	119	105
	50	119	108	139
	82	20	207	170
	17	95	102	87
Treatment Means	37.5	92.5	106	117

$\bar{y}_j$

=

$\bar{\bar{y}}$: Grand mean

Alcohol	AB Soap	Soap	Water
88.25	88.25	88.25	88.25
88.25	88.25	88.25	88.25
88.25	88.25	88.25	88.25
88.25	88.25	88.25	88.25
88.25	88.25	88.25	88.25
88.25	88.25	88.25	88.25
88.25	88.25	88.25	88.25
88.25	88.25	88.25	88.25

+

$\bar{y}_j - \bar{\bar{y}}$: Treatment effect

Alcohol	AB Soap	Soap	Water
−50.75	4.25	17.75	28.75
−50.75	4.25	17.75	28.75
−50.75	4.25	17.75	28.75
−50.75	4.25	17.75	28.75
−50.75	4.25	17.75	28.75
−50.75	4.25	17.75	28.75
−50.75	4.25	17.75	28.75
−50.75	4.25	17.75	28.75

+

$y_{ij} - \bar{y}_j$: Residual

Alcohol	AB Soap	Soap	Water
13.5	−22.5	−22	−43
−32.5	71.5	−55	18
−18.5	−4.5	4	−15
−19.5	18.5	−39	7
20.5	−19.5	13	−12
12.5	26.5	2	22
44.5	−72.5	101	53
−20.5	2.5	−4	−30

Sum of Squares of all these values

$SS_T = \sum\sum(\bar{y}_j - \bar{\bar{y}})^2$

Treatment Sum of Squares

Sum of Squares of all these values

$SS_E = \sum\sum(y_{ij} - \bar{y}_j)^2$

Error Sum of Squares

- $MS_T = \frac{SS_T}{k-1}$, $MS_E = \frac{SS_E}{N-k}$, $F_{k-1,N-k} = \frac{MS_T}{MS_E}$

Tooth Growth

Modules

Model

Components

supp
dose

Model Terms

supp
dose
supp ✻ dose

ANOVA

ANOVA

	Sum of Squares	df	Mean Square	F	p	η²p
supp	205	1	205.4	15.57	<.001	0.224
dose	2426	2	1213.2	92.00	<.001	0.773
supp ✻ dose	108	2	54.2	4.11	0.022	0.132
Residuals	712	54	13.2			

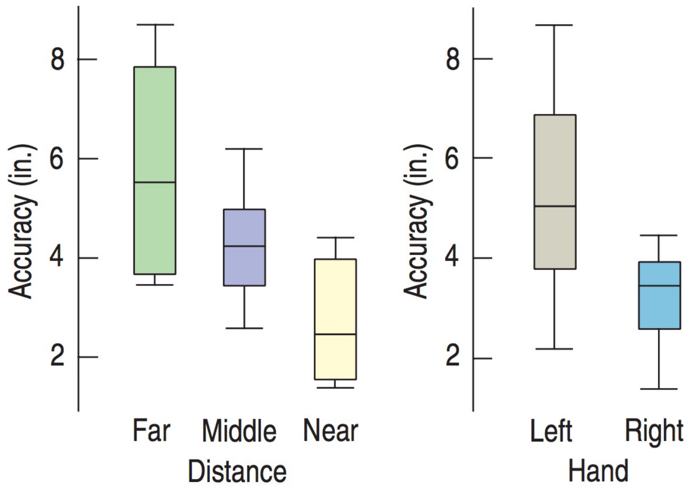
Lecture 24
Multifactor Analysis of Variance

Previous lecture video

https://youtu.be/jSjRH63zNAM

Example: Dart throwing

- How accurately can you throw a dart?
- It may depend on which hand you use and how far from the target you are.
- How can we measure the effect of each variable?



Two-factor ANOVA model

- Previously, $y_{ij} = \mu + \tau_j + \varepsilon_{ij}$

- Now we have two factors, *Hand* and *Distance*
    - $y_{ijk} = \mu + \tau_j + \gamma_k + \varepsilon_{ijk}$
    - *i*: *i*-th observation
    - *j*: level j of factor A
    - *k*: level k of factor B
    - τ_j and γ_k : difference between the mean response of that treatment and the grand mean (μ, or $\bar{\bar{y}}$)

Analysis of Variance for Accuracy

Source	df	Sum of Squares	Mean Square	F-Ratio	Prob
Distance	2	51.0439	25.5219	28.561	≤0.0001
Hand	1	39.6900	39.6900	44.416	≤0.0001
Error	32	28.5950	0.893594		
Total	35	119.329			

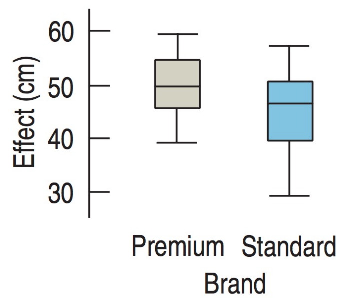
- In the example case (τ: Hand effect, γ: Distance effect)
- H_0: $\tau_1 = \tau_2$ and $\gamma_1 = \gamma_2 = \gamma_3$

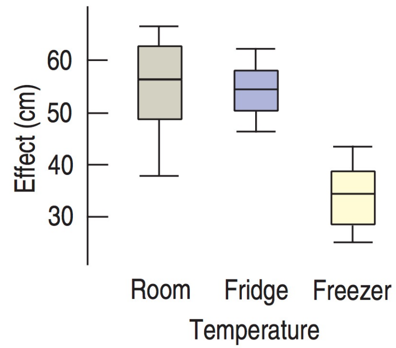
How two-factor ANOVA works

- Another example: tennis balls' bounce with different brands and temperature

Brand	Temperature	Bounce Height
Standard	Freezer	37
Standard	Fridge	59
Standard	Room	59
Premium	Freezer	45
Premium	Fridge	60
Premium	Room	63
Standard	Freezer	37
Standard	Fridge	58
Standard	Room	60
Premium	Freezer	39
Premium	Fridge	64
Premium	Room	62
Standard	Freezer	41
Standard	Fridge	60
Standard	Room	61
Premium	Freezer	37
Premium	Fridge	63
Premium	Room	61

- Side-by-side boxplots for each factor





Brand		Temperature: Room	Fridge	Freezer
Brand	Premium	63	60	45
		62	64	39
		61	63	37
	Standard	59	59	37
		60	58	37
		61	60	41

$$\bar{y}_{\text{Premium}} = 54.89$$
$$\bar{y}_{\text{Standard}} = 52.44$$

$$\bar{y}_{\text{Room}} = 61$$
$$\bar{y}_{\text{Fridge}} = 60.67$$
$$\bar{y}_{\text{Freezer}} = 39.33$$

$$\bar{\bar{y}} = 53.67$$

How two-factor ANOVA works

- $y_{ijk} = \mu + Brand\ effect_j + Temp\ effect_k + Error_{ijk}$
- $y_{ijk} = \bar{\bar{y}} + (\bar{y}_j - \bar{\bar{y}}) + (\bar{y}_k - \bar{\bar{y}}) + (y_{ijk} - \bar{y}_j - \bar{y}_k + \bar{\bar{y}})$

$y_{\text{Premium}} = 54.89$
$y_{\text{Standard}} = 52.44$

$y_{\text{Room}} = 61$
$y_{\text{Fridge}} = 60.67$
$y_{\text{Freezer}} = 39.33$

$\bar{\bar{y}} = 53.67$

y_{ijk}: Observations

Brand		Temperature: Room	Fridge	Freezer
Premium		63	60	45
		62	64	39
		61	63	37
Standard		59	59	37
		60	58	37
		61	60	41

=

$\bar{\bar{y}}$: Grand mean

Brand		Temperature: Room	Fridge	Freezer
Premium		53.67	53.67	53.67
		53.67	53.67	53.67
		53.67	53.67	53.67
Standard		53.67	53.67	53.67
		53.67	53.67	53.67
		53.67	53.67	53.67

+

Brand effect$_j$: $(\bar{y}_j - \bar{\bar{y}})$

Brand		Temperature: Room	Fridge	Freezer
Premium		1.22	1.22	1.22
		1.22	1.22	1.22
		1.22	1.22	1.22
Standard		−1.22	−1.22	−1.22
		−1.22	−1.22	−1.22
		−1.22	−1.22	−1.22

Sum of Squares of all these values

$$SS_{T1} = \sum\sum(\bar{y}_j - \bar{\bar{y}})^2$$

Treatment Sum of Squares

+

Temperature effect$_k$: $(\bar{y}_k - \bar{\bar{y}})$

Brand		Temperature: Room	Fridge	Freezer
Premium		7.33	7.00	−14.33
		7.33	7.00	−14.33
		7.33	7.00	−14.33
Standard		7.33	7.00	−14.33
		7.33	7.00	−14.33
		7.33	7.00	−14.33

Sum of Squares of all these values

$$SS_{T2} = \sum\sum(\bar{y}_k - \bar{\bar{y}})^2$$

Treatment Sum of Squares

+

Error$_{ijk}$

Brand		Temperature: Room	Fridge	Freezer
Premium		0.78	−1.89	4.44
		−0.22	2.11	−1.56
		−1.22	1.11	−3.56
Standard		−0.78	−0.44	−1.11
		0.22	−1.44	−1.11
		1.22	0.56	2.89

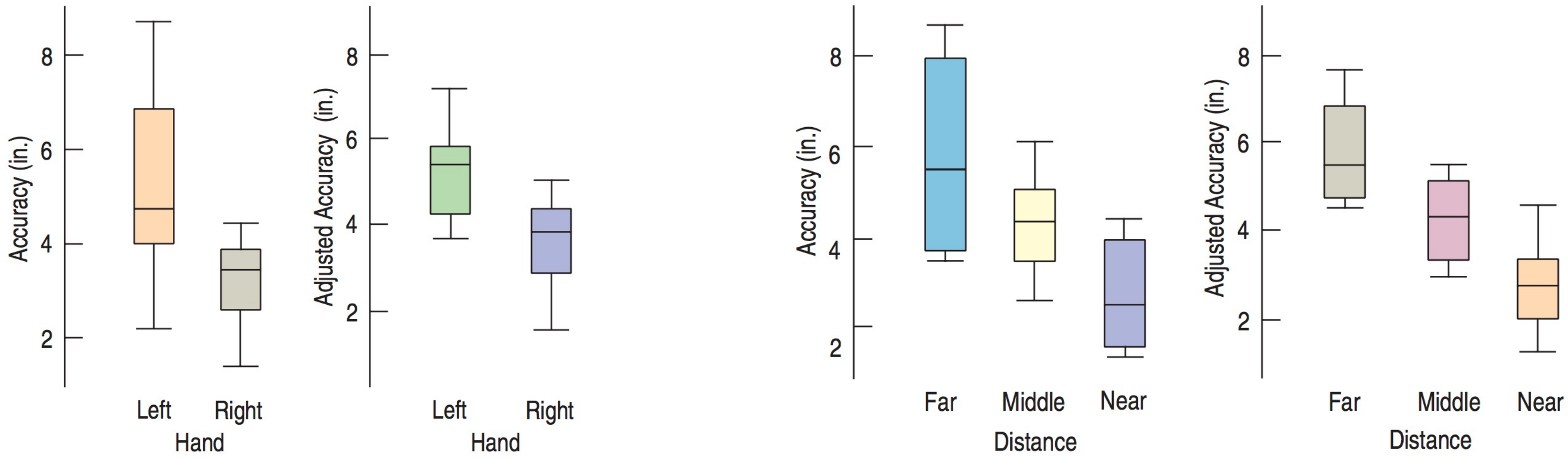
Sum of Squares of all these values

$$SS_E = \sum\sum(y_{ijk} - \bar{y}_j - \bar{y}_k + \bar{\bar{y}})^2$$

Error Sum of Squares

Partial boxplot

- Plot the data: but we cannot easily see the effects of *one* factor only.
    - **Partial boxplot**: plot the residuals for each level of the other factor (e.g., $y_{ijk} - \bar{y}_j + \bar{\bar{y}}$).

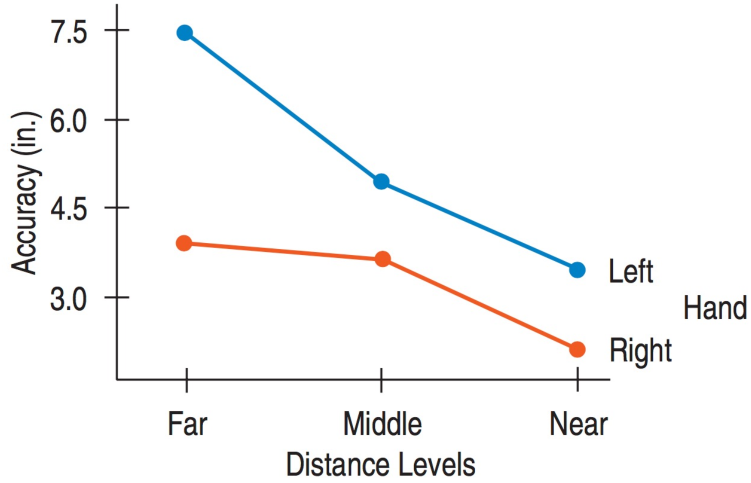


Additivity vs. interaction

- $y_{ijk} = \mu + Hand_j + Distance_k + Error_{ijk}$
- **Additivity:** the model above assumes that the effects of two factor levels are additive (we can just add the effects of the two factor levels together).

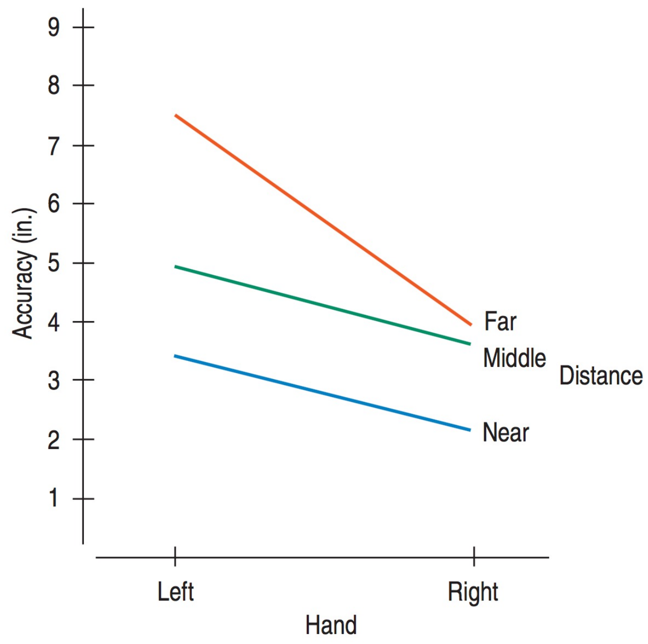
- **Interaction**: when the effects of one factor change for different levels of another factor

- **Additivity vs. interaction**: draw an interaction plot!
  - Interaction plot: plot the averages of the observations at each level of the factors
  - Parallel: additive ("additive enough condition")
    No parallel: interaction



Interactions

- Interaction plot for the first example
- From *middle* to *far*, there was a larger effect of hand.
  - $y_{ijk} = \mu + \tau_j + \gamma_k + \omega_{jk} + \varepsilon_{ijk}$
  - ω_{jk}: interaction term ($DF_{interaction} = DF_j \times DF_k$)



ANOVA Table for Accuracy

Source	DF	Sum of Squares	Mean Square	F-ratio	P-value
Distance	2	51.044	25.522	41.977	<0.0001
Hand	1	39.690	39.690	65.28	<0.0001
Distance × Hand	2	10.355	5.178	8.516	0.0012
Error	30	18.240	0.608		
Total	35	119.329			

Analysis of Variance for Accuracy

Source	df	Sum of Squares	Mean Square	F-Ratio	Prob
Distance	2	51.0439	25.5219	28.561	≤0.0001
Hand	1	39.6900	39.6900	44.416	≤0.0001
Error	32	28.5950	0.893594		
Total	35	119.329			

ANOVA table without the interaction term

How two-factor ANOVA works (with interaction)

$y_{\mathrm{Premium}} = 54.89$
$y_{\mathrm{Standard}} = 52.44$

$y_{\mathrm{Room}} = 61$
$y_{\mathrm{Fridge}} = 60.67$
$y_{\mathrm{Freezer}} = 39.33$

$\bar{\bar{y}} = 53.67$

- $y_{ijk} = \mu + Brand\ effect_j + Temp\ effect_k + Interaction_{jk} + \ Error_{ijk}$
- $y_{ijk} = \bar{\bar{y}} + (\bar{y}_j - \bar{\bar{y}}) + (\bar{y}_k - \bar{\bar{y}}) + (\bar{y}_{jk} - \bar{y}_j - \bar{y}_k + \bar{\bar{y}}) + residual$

y_{ijk}: Observations

Brand		Temperature: Room	Fridge	Freezer
Premium		63	60	45
		62	64	39
		61	63	37
Standard		59	59	37
		60	58	37
		61	60	41

=

$\bar{\bar{y}}$: Grand mean

Brand		Temperature: Room	Fridge	Freezer
Premium		53.67	53.67	53.67
		53.67	53.67	53.67
		53.67	53.67	53.67
Standard		53.67	53.67	53.67
		53.67	53.67	53.67
		53.67	53.67	53.67

+

Brand effect$_j$: $(\bar{y}_j - \bar{\bar{y}})$

Brand		Temperature: Room	Fridge	Freezer
Premium		1.22	1.22	1.22
		1.22	1.22	1.22
		1.22	1.22	1.22
Standard		−1.22	−1.22	−1.22
		−1.22	−1.22	−1.22
		−1.22	−1.22	−1.22

Sum of Squares of all these values

$SS_{T1} = \sum\sum(\bar{y}_j - \bar{\bar{y}})^2$

Treatment Sum of Squares

+

Temperature effect$_k$: $(\bar{y}_k - \bar{\bar{y}})$

Brand		Temperature: Room	Fridge	Freezer
Premium		7.33	7.00	−14.33
		7.33	7.00	−14.33
		7.33	7.00	−14.33
Standard		7.33	7.00	−14.33
		7.33	7.00	−14.33
		7.33	7.00	−14.33

Sum of Squares of all these values

$SS_{T2} = \sum\sum(\bar{y}_k - \bar{\bar{y}})^2$

Treatment Sum of Squares

+

Interaction$_{jk}$: $(\bar{y}_{jk} - \bar{y}_j - \bar{y}_k + \bar{\bar{y}})$

Brand		Temperature: Room	Fridge	Freezer
Premium		-0.22	0.44	-0.22
		-0.22	0.44	-0.22
		-0.22	0.44	-0.22
Standard		0.23	-0.44	0.23
		0.23	-0.44	0.23
		0.23	-0.44	0.23

Sum of Squares of all these values

$SS_{Int} = \sum\sum(\bar{y}_{jk} - \bar{y}_j - \bar{y}_k + \bar{\bar{y}})^2$

Treatment Sum of Squares

Eta-squared (η^2)

- Effect size measure
- Similar to R-squared, but in the ANOVA case

$$\eta^2 = \frac{\text{SS(Between groups)}}{\text{SS(Total)}}$$

- Example data:

Table 1
Results of hypothetical study comparing reading ability in 50 boys and 50 girls.

Source of variation	SS	df	MS	*F*	*p*
Between groups (gender)	300.00	1	300.00	30.00	.000
Within groups	980.00	98	10.00		
Total	1280.00	99			

Note: SS, sum of squares; MS, mean squares.

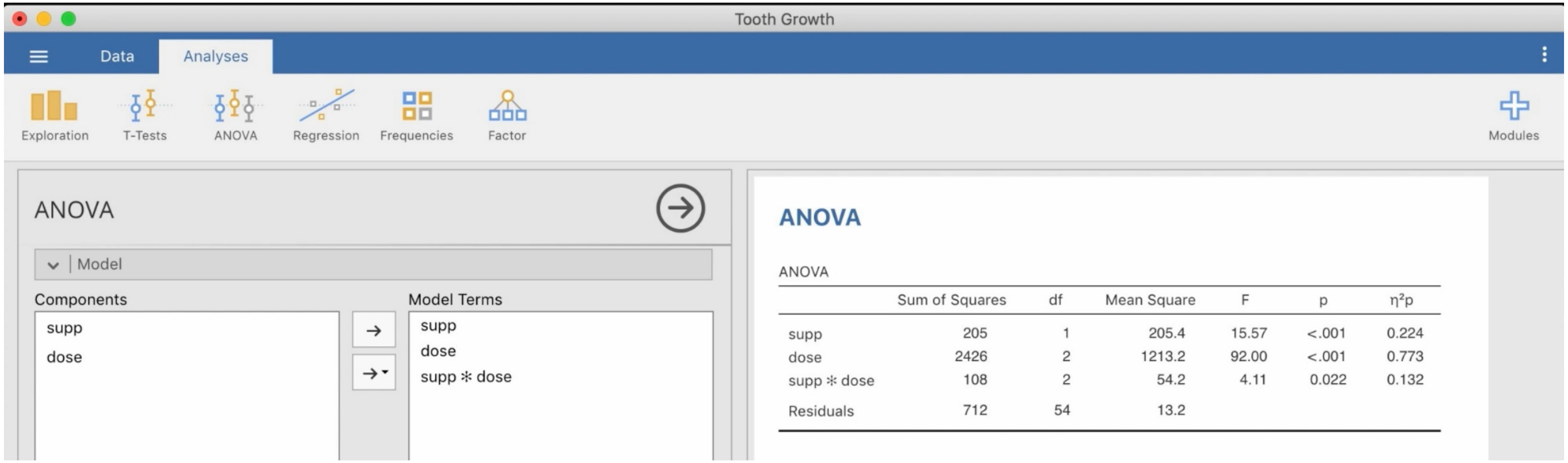
$$\eta^2 = \text{SS(Between groups)/SS(Total)} = 300.00/1280.00 = .234$$

- Partial eta-squared (if there are A and B factors):

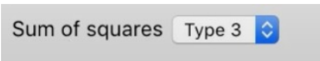
$$\eta^2 = \frac{\text{SS}(A)}{[\text{SS}(A) + \text{SS(Within groups)}]}$$

Richardson, 2011

Partial eta-squared ($\eta^2 p$) in the example case



	Sum of Squares	df	Mean Square	F	p	η²p
supp	205	1	205.4	15.57	<.001	0.224
dose	2426	2	1213.2	92.00	<.001	0.773
supp ✻ dose	108	2	54.2	4.11	0.022	0.132
Residuals	712	54	13.2			



- Type 3: SS calculated from $y_{ijk} = \bar{\bar{y}} + (y_j - \bar{\bar{y}}) + (y_k - \bar{\bar{y}}) + (y_{jk} - y_j - y_k + \bar{\bar{y}}) + residual$
- Type 2: SS calculated from $y_{ijk} = \bar{\bar{y}} + (y_j - \bar{\bar{y}}) + (y_k - \bar{\bar{y}}) + residual$
- Type 3 is good when there is an interaction
- Type 2 is good when there is no interaction
- Type 1 uses a sequential model (not appropriate most of the time)

Partial eta-squared ($\eta^2 p$)

- for supp = 205/(205+712) = 0.224
- for dose = 2426/(2426+712) = 0.773
- for supp*dose = 108/(108+712) = 0.132