# ODE, SageMath and General AI

WANMIN LIU
wanminliu@gmail.com

19th May 2024

**Abstract**

In this brief notes, we rigorously solve an ordinary differential equation using mathematical logic. Additionally, we present a solution obtained through the mathematical software SageMath. We also test the ODE on various General AI systems, finding that they are unable to solve this particular ODE. This example highlights the rigor and beauty of mathematics, demonstrates the power of SageMath, and reveals the current limitations of general artificial intelligence in solving complex mathematical problems.

## 1 THE ODE PROBLEM AND SOLUTIONS.

We assume that the reader has basic knowledge of calculus.

**The ODE Problem.** Solve the ordinary differential equation

$$y''(x) - 8y'(x) + 16y(x) = xe^{4x} \tag{1.1}$$

with the conditions $y'(0) = 1$ and $y'(1) = e^4$.

*Solution.* We begin with the corresponding homogeneous equation

$$y''(x) - 8y'(x) + 16y(x) = 0. \tag{1.2}$$

Its characteristic equation is given by

$$r^2 - 8r + 16 = 0. \tag{1.3}$$

The solution of (1.3) is given by $r = 4$ with multiplicity 2. By the standard theory of ODE (see for example [1, Ch. 19]), the solution of (1.2) is given by

$$y_c(x) = e^{4x}\left(Ax + B\right), \tag{1.4}$$

where $A$ and $B$ are constants.

The next step is to find a particular solution of (1.1). We observe that the right side of (1.1) is a solution for the homogeneous equation (1.2) by setting $A = 1$ and $B = 0$ in (1.4). This feature of the right side of (1.1) is what makes the equation intriguing, presenting challenges specifically for AI solvers.

So a guess of the solution of the form

$$y_p(x) = k_1x^2e^{4x} + k_2xe^{4x} + k_3e^{4x} \tag{1.5}$$

will not work, because $k_2 x e^{4x}$ and $k_3 e^{4x}$ are already solutions of homogeneous equation, and a quick computation can check that $y_p(x) = k_1 x^2 e^{4x}$ is not a solution of (1.1).

We must use new idea. The method is to let the solution be of the form

$$y(x) = e^{4x} g(x), \tag{1.6}$$

and we can convert the original ODE to a new ODE of the function $g(x)$.

By the chain rule and (1.6), we have

$$y'(x) = e^{4x} \left(4g(x) + g'(x)\right), \tag{1.7}$$

and

$$y''(x) = e^{4x} \left(16g(x) + 8g'(x) + g''(x)\right). \tag{1.8}$$

Put (1.8) and (1.7) into left side of (1.1), we obtain

$$\begin{aligned}
&y''(x) - 8y'(x) + 16y(x) \\
&= e^{4x} \left(16g(x) + 8g'(x) + g''(x) - 32g(x) - 8g'(x) + 16g(x)\right) \\
&= e^{4x} g''(x).
\end{aligned}$$

Thus, from the original ODE (1.1), we derive a new ODE for the function $g(x)$.

$$g''(x) = x. \tag{1.9}$$

The general solution of (1.9) is given by

$$g(x) = \frac{x^3}{6} + c_1 x + c_2.$$

Therefore a special solution of the inhomogeneous ODE (1.1) is given by (we can take $c_1 = c_2 = 0$)

$$y_p(x) = e^{4x} \frac{x^3}{6},$$

and the general solution of the (1.1) is of the form

$$y(x) = y_c(x) + y_p(x) = e^{4x} \left(\frac{x^3}{6} + Ax + B\right), \tag{1.10}$$

where $A$ and $B$ are some constants.

By using the chain rule and taking derivative, we obtain

$$y'(x) = e^{4x} \left(4\frac{x^3}{6} + 4Ax + 4B + \frac{x^2}{2} + A\right). \tag{1.11}$$

So

$$y'(0) = (4B + A), \quad y'(1) = e^4 \left(\frac{4}{6} + 4A + 4B + \frac{1}{2} + A\right). \tag{1.12}$$

By comparing with the conditions $y'(0) = 1$ and $y'(1) = e^4$, we obtain that

$$4B + A = 1, \tag{1.13}$$

$$\frac{4}{6} + 4A + 4B + \frac{1}{2} + A = 1. \tag{1.14}$$

It follows that $\frac{4}{6} + 4A + \frac{1}{2} = 0$ and $A = -\frac{7}{24}$, $B = \frac{31}{96}$.

We conclude that the solution of the (1.1) with the given initial conditions is

$$y(x) = e^{4x}\left(\frac{x^3}{6} - \frac{7x}{24} + \frac{31}{96}\right). \tag{1.15}$$

□

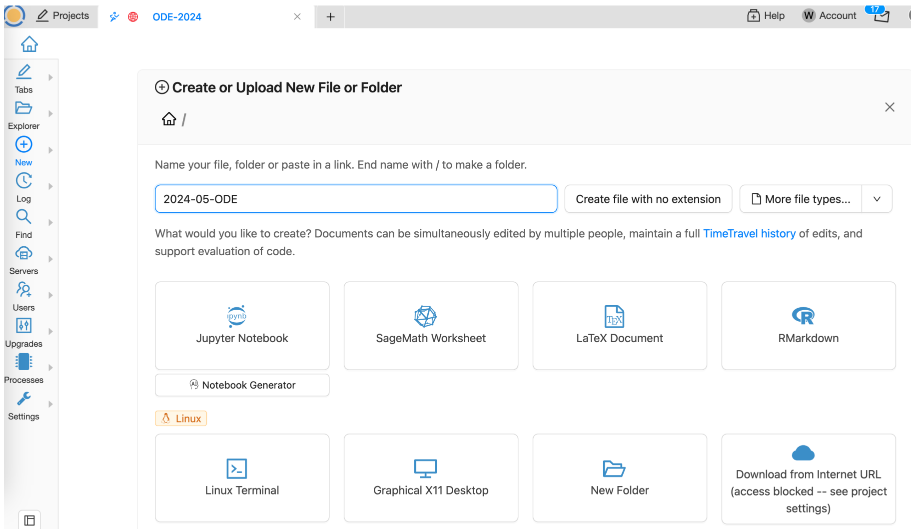
**Remark 1.1.** One can also use the Laplace transform ( [1, Ch. 20]) to solve the (1.1). The transformation (1.6) is simple from Laplace transform point-view.

## 2 SAGEMATH: AN APPLICATION

SageMath is a free open-source mathematics software system. The homepage is `https://www.sagemath.org/`. One can install it or use it online. For simplicity, we use the cloud version Sage on CoCalc

**Step 1.** Visit `https://cocalc.com/features/sage` and sign up for a free CoCalc account.

**Step 2.** After signing in, create a project `https://cocalc.com/projects`. We call the project **ODE-2024**. Then we can create a SageMath Worksheet.




**Step 3.** In the SageMath Worksheet, we write the following code.

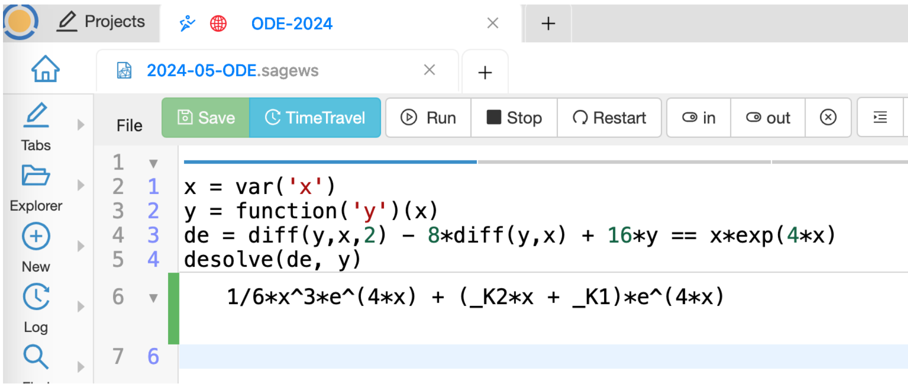
```
x = var('x')
y = function('y')(x)
de = diff(y,x,2) - 8*diff(y,x) + 16*y == x*exp(4*x)
desolve(de, y)
```

The first line of the code defines the variable $x$. The second line defines $y$ as a function of $x$. The third line defines the ODE, which is (1.1). The forth line solves the ODE by using the built-in function `desolve()`. More information of solving ordinary differential equations can be found at [2].

**Step 4.** We can run the code now by clicking button **Run**. It gives the solution of the general form, which is precisely (1.10):

```
1/6*x^3*e^(4*x) + (_K2*x + _K1)*e^(4*x)
```

So the SageMath solves the ODE and finds the general solution.




## 3 GENERAL ARTIFICIAL INTELLIGENCE.

The ODE serves as a good example to demonstrate that General AI cannot solve certain mathematical problems or, at the very least, have significant limitations. We have tested several General AI systems, including ChatGPT and Copilot. It is a good exercise to verify that the result from ChatGPT is incorrect, as stated on the next page:

**ChatGPT can make mistakes. Check important info.**

One reason is that General AI is based on large language models and statistics, rather than on mathematical logic.

There are two type of errors in the ChatGPT.

The *first error* is that the assumption of the form of particular solution $y_p(x) = Ax^2e^{4x} + Bxe^{4x} + Ce^{4x}$, as explained in (1.5).

The *second error* occurs in the computation of $y_p'(x)$. Specifically, it omits the term related to $C$ in the expression of $y_p'(x)$. Consequently, the computations of both $y_p'(x)$ and $y_p''(x)$ are completely incorrect.

We do not understand the mechanism why ChatGPT makes the second error. If it did not make this mistake, it would have the opportunity to recognize that there is no solution to this calculation. This would lead it back to the first error, allowing it to discover that the initial assumption is incorrect. Additionally, it could test more hypotheses to find the correct solution.

## 4 CONCLUSION.

This ODE example highlights the rigor and beauty of mathematics, demonstrates the power of SageMath, and reveals the current limitations of general artificial intelligence in solving complex mathematical problems.

## REFERENCES

[1] Robert Adams and Christopher Essex, *Calculus - a complete course.* Pearson, 10 ed.

[2] The Sage Development Team, 'Solving ordinary differential equations - Symbolic Calculus'. https://doc.sagemath.org/html/en/reference/calculus/sage/calculus/desolvers.html#sage.calculus.desolvers.desolve.

ChatGPT 3.5

Solve the ODE y''(x)-8*y'(x)+16*y(x)=x*exp^{4x}, where y is a function of variable x.

To solve the given ordinary differential equation (ODE), we'll first find the complementary solution, then the particular solution.

The characteristic equation corresponding to the homogeneous part of the ODE is:
$r^2 - 8r + 16 = 0.$

Solving this quadratic equation, we get a repeated root $r = 4$. Therefore, the complementary solution is:
$y_c(x) = (c_1 + c_2 x)e^{4x}.$

Now, for the particular solution, we can use the method of undetermined coefficients. We assume the particular solution has the form $y_p(x) = Ax^2 e^{4x} + Bxe^{4x} + Ce^{4x}$.

Differentiating $y_p(x)$ twice and substituting into the original ODE, we find the values of $A$, $B$, and $C$ that satisfy the equation.

Let's proceed with finding $A$, $B$, and $C$:
$y_p(x) = Ax^2 e^{4x} + Bxe^{4x} + Ce^{4x}$
$y_p'(x) = (2Ax + B)e^{4x} + 4Ax^2 e^{4x} + Be^{4x}$
$y_p''(x) = (2A + 8Ax + 4B)e^{4x} + 8Axe^{4x}$

Substituting these into the original ODE:

$$(2A + 8Ax + 4B)e^{4x} + 8Axe^{4x} - 8((2Ax + B)e^{4x} + 4Ax^2 e^{4x} + Be^{4x}) + 16(Ax^2 e^{4x} + Bxe^{4x} + Ce^{4x}) = xe^{4x}$$

Grouping like terms:

$$(2A + 8Ax + 4B - 8B)e^{4x} + (8Ax - 8Ax)e^{4x} + (16Ax^2 - 32Ax^2 + 16Bx)e^{4x} + (16C)e^{4x} = xe^{4x}$$

Simplify and equate coefficients:

$$(-4B)e^{4x} + (16Bx)e^{4x} + (16C)e^{4x} = xe^{4x}$$

Comparing coefficients:

$$-4B = 0 \implies B = 0$$
$$16B = 1 \implies B = \frac{1}{16}$$
$$16C = 0 \implies C = 0$$

So, we have $B = \frac{1}{16}$, and $A$ and $C$ are arbitrary constants.

Thus, the particular solution is:
$y_p(x) = \frac{1}{16}xe^{4x}.$

Now, the general solution is the sum of the complementary and particular solutions:
$y(x) = y_c(x) + y_p(x) = (c_1 + c_2 x)e^{4x} + \frac{1}{16}xe^{4x}.$

This is the general solution to the given ODE.

Message ChatGPT