

AN EXACT INEQUALITY INVOLVING LUXEMBURG NORM AND CONJUGATE-ORLICZ NORM IN $L^{p(x)}(\Omega)$

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Let $L^{p(x)}(\Omega)$ be the variable exponent Lebesgue space where $p : \Omega \rightarrow [1, \infty]$. Denote by $\|\cdot\|_{p(x)}$ and $\|\cdot\|_{p(x)}^o$ the Luxemburg norm and the conjugate-Orlicz norm in $L^{p(x)}(\Omega)$ respectively. The authors prove an exact inequality $\|\cdot\|_{p(x)} \leq \|\cdot\|_{p(x)}^o \leq d_{(p_-, p_+)} \|\cdot\|_{p(x)}$ where $d_{(p_-, p_+)}$ is a constant depending on $p_- = \text{ess inf}_\Omega p(x)$ and $p_+ = \text{ess sup}_\Omega p(x)$. When $1 < p_- < p_+ < \infty$,

$$d_{(p_-, p_+)} = \left(\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} \right)^{\frac{p_+ - 1}{p_+ - p_-}} \left(\frac{p_+^{p_+}}{(p_+ - 1)^{p_+ - 1}} \right)^{\frac{p_- - 1}{p_+ - p_-}} + \left(\frac{p_-^{p_-}}{(p_- - 1)^{p_- - 1}} \frac{(p_+ - 1)^{p_+ - 1}}{p_+^{p_+}} \right)^{\frac{1}{p_+ - p_-}},$$

when $p_- = 1$ or $p_+ = \infty$, $d_{(p_-, p_+)}$ is the corresponding limit.

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§1. INTRODUCTION

Let (Ω, Σ, μ) be a complete σ -finite measure space and $p : \Omega \rightarrow [1, \infty]$ a measurable function. There are many researches about the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ (see [1–4, 6–9]). By $S(\Omega)$ we denote the set of all measurable functions $u : \Omega \rightarrow \mathbb{R}$. For convenience, we introduce the notations

$$\Omega_1 = \{x \in \Omega : p(x) = 1\}, \quad \Omega_+ = \{x \in \Omega : p(x) \in (1, \infty)\}, \quad \Omega_\infty = \{x \in \Omega : p(x) = \infty\}.$$

It is usually to define the space $L^{p(x)}(\Omega)$ and norm $\|\cdot\|_{\rho_p}$ by introducing a convex modular ρ_p on $S(\Omega)$ (see [1,6,9]), that is

$$\rho_p(u) = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} d\mu + \text{ess sup}_{\Omega_\infty} |u(x)|, \quad (1.1)$$

$$L^{p(x)}(\Omega) = \{u \in S(\Omega) : \exists \lambda > 0 \text{ such that } \rho_p(u/\lambda) < \infty\}, \quad (1.2)$$

$$\|u\|_{\rho_p} = \inf\{\lambda > 0 : \rho_p(u/\lambda) \leq 1\}. \quad (1.3)$$

Then $(L^{p(x)}(\Omega), \|\cdot\|_{\rho_p})$ is a Banach space. For $D \subset \Omega$, we denote

$$p_-(D) = \text{ess inf}_D p(x), \quad p_+(D) = \text{ess sup}_D p(x).$$

Use p_- and p_+ instead of $p_-(\Omega)$ and $p_+(\Omega)$ respectively. Denote by $p^o(x)$ the conjugate number to $p(x)$, that is

$$p^o(x) = \begin{cases} \frac{p(x)}{p(x)-1}, & x \in \Omega_+, \\ \infty, & x \in \Omega_1, \\ 1, & x \in \Omega_\infty. \end{cases} \quad (1.4)$$

Kováčik and Rákosník [6] have introduced an equivalent norm $\|\cdot\|_{\rho_p}^o$:

$$\|u\|_{\rho_p}^o = \sup \left\{ \left| \int_{\Omega} u(x)v(x)dx \right| : v \in L^{p^o(x)}(\Omega), \rho_{p^o}(v) \leq 1 \right\},$$

and proved the following inequality (see [6, Theorem 2.3])

$$c_{p(x)}^{-1} \|u\|_{\rho_p} \leq \|u\|_{\rho_p}^o \leq r_{p(x)} \|u\|_{\rho_p}, \quad (1.5)$$

where

$$c_{p(x)} = \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_+}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty,$$

$$r_{p(x)} = c_{p(x)} + \frac{1}{p_-(\Omega_+)} - \frac{1}{p_+(\Omega_+)}$$

with the convention $\frac{1}{\infty} = 0$. Especially when $\Omega_1 = \Omega_\infty = \emptyset$, we have $c_{p(x)} = 1$ and $r_{p(x)} = 1 + \frac{1}{p_-} - \frac{1}{p_+}$. In this case the inequality (1.5) comes

$$\|u\|_{\rho_p} \leq \|u\|_{\rho_p}^o \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|u\|_{\rho_p}. \quad (1.6)$$

It is easy to see that the above constant $r_{p(x)}$, as an embedding constant of $(L^{p(x)}(\Omega), \|\cdot\|_{\rho_p}) \hookrightarrow (L^{p(x)}(\Omega), \|\cdot\|_{\rho_p}^o)$, is not the best. In this paper, we find the best embedding constant $d_{p(x)} = d_{(p_-, p_+)}$, which depends on p_- and p_+ . When $1 < p_- < p_+ < \infty$,

$$\begin{aligned} d_{(p_-, p_+)} &= \left(\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} \right)^{\frac{p_+ - 1}{p_+ - p_-}} \left(\frac{p_+^{p_+}}{(p_+ - 1)^{p_+ - 1}} \right)^{\frac{p_- - 1}{p_+ - p_-}} \\ &\quad + \left(\frac{p_-^{p_-}}{(p_- - 1)^{p_- - 1}} \frac{(p_+ - 1)^{p_+ - 1}}{p_+^{p_+}} \right)^{\frac{1}{p_+ - p_-}}. \end{aligned} \quad (1.7)$$

In this case, it is easy to verify $d_{(p_-, p_+)} < 1 + \frac{1}{p_-} - \frac{1}{p_+} = r_{p(x)}$ (see the Remark 3.1 below). In addition, when $\Omega_\infty \neq \emptyset$ the convex modular ρ_p defined by (1.1) is not generated by the integral of an Orlicz function $\Phi(x, \cdot)$, so the space $(L^{p(x)}(\Omega), \|\cdot\|_{\rho_p})$ is not a generalized Orlicz space in the sense of [7], but a Musielak-Nakano space. In this paper we introduce a norm $\|\cdot\|_{p(x)}$ in $L^{p(x)}(\Omega)$ such that $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a generalized Orlicz space. Let us consider a function defined by formula $\Phi(x, t) = t^{p(x)}$, $\forall t \geq 0$, $\forall x \in \Omega$, with the convention

$$t^\infty = \begin{cases} 0, & 0 \leq t \leq 1, \\ \infty, & t > 1. \end{cases} \quad (1.8)$$

Then for every $x \in \Omega$, $\Phi(x, \cdot)$ is an Orlicz function (i.e., a nonnegative lower semi-continuous convex function vanishing at zero but not identically 0 or ∞ on $(0, \infty)$). We define the Luxemburg norm in $L^{p(x)}(\Omega)$ by

$$\|u\|_{p(x)} = \inf \left\{ \lambda > 0, \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} d\mu \leq 1 \right\}. \quad (1.9)$$

It is easy to verify that the two norms $\|u\|_{p(x)}$ and $\|u\|_{\rho_p}$ are equivalent. Moreover when $\Omega_\infty = \emptyset$, they are same. It is also easy to see that when $\Omega = \Omega_\infty$ the norm $\|u\|_{p(x)}$ is just the same with the usual L^∞ -norm $\|u\|_\infty = \text{ess sup}_\Omega |u(x)|$. We define the conjugate-Orlicz norm $\|u\|_{p(x)}^o$ in $L^{p(x)}(\Omega)$ by

$$\|u\|_{p(x)}^o = \sup \left\{ \left| \int_{\Omega} u(x)v(x) d\mu \right| : v \in L^{p^o(x)}(\Omega), \int_{\Omega} |v(x)|^{p^o(x)} d\mu \leq 1 \right\}. \quad (1.10)$$

It is clear that $\|u\|_{p(x)}^o = \|u\|_{\rho_p}^o$ if $\Omega_1 = \Omega_\infty = \emptyset$.

When $1 < p_- < p_+ < \infty$, the constant $d_{p(x)} = d_{(p_-, p_+)}$ is defined by (1.7). When $p_- = 1$ or $p_+ = \infty$, $d_{p(x)}$ can be defined in the limit sense of (1.7), that is

$$d_{p(x)} = \begin{cases} d_{(p_-, p_+)} \text{ defined by (1.7),} & \text{if } 1 < p_- < p_+ < \infty, \\ d_{(1, p_+)} = \lim_{p_- \rightarrow 1^+} d_{(p_-, p_+)} = 1 + \left(\frac{(p_+ - 1)^{p_+ - 1}}{p_+^{p_+}} \right)^{\frac{1}{p_+ - 1}}, & \text{if } 1 = p_- < p_+ < \infty, \\ d_{(p_-, \infty)} = \lim_{p_+ \rightarrow \infty} d_{(p_-, p_+)} = \frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} + 1, & \text{if } 1 < p_- < p_+ = \infty, \\ d_{(1, \infty)} = \lim_{\substack{p_- \rightarrow 1^+ \\ p_+ \rightarrow \infty}} d_{(p_-, p_+)} = 2. & \text{if } 1 = p_- \text{ and } p_+ = \infty, \\ \lim_{q \rightarrow p_-} d_{(p_-, q)} = \lim_{r \rightarrow p_+} d_{(r, p_+)} = 1, & \text{if } p_- = p_+. \end{cases} \quad (1.11)$$

The main result of this paper is the following theorem.

Theorem 1.1. The inequality

$$\|u\|_{p(x)} \leq \|u\|_{p(x)}^o \leq d_{p(x)} \|u\|_{p(x)}, \quad \forall u \in L^{p(x)}(\Omega) \quad (1.12)$$

holds and is exact. In other words we have

$$\sup_{u \in L^{p(x)}(\Omega) \setminus \{0\}} \frac{\|u\|_{p(x)}}{\|u\|_{p(x)}^o} = 1, \quad (1.13)$$

$$\sup_{u \in L^{p(x)}(\Omega) \setminus \{0\}} \frac{\|u\|_{p(x)}^o}{\|u\|_{p(x)}} = d_{p(x)}. \quad (1.14)$$

Remark 1.1. When $p_- = p_+$, that is, $p(x) \equiv p \in [1, \infty]$ a.e., $(L^{p(x)}(\Omega), \|u\|_{p(x)})$ is the usual Lebesgue space $(L^p(\Omega), \|u\|_p)$. It is well known that $\|u\|_p = \|u\|_p^o$ in this case. This is a special case of Theorem 1.1. In order to prove Theorem 1.1, we need only to deal with the case $p_- < p_+$.

Remark 1.2. In the inequality (1.5), $c_{p(x)}$ = 1, 2 or 3, the constants $c_{p(x)}$ and $r_{p(x)}$ are discontinuous in (p_-, p_+) . While in our inequality (1.12), $d_{p(x)} = d_{(p_-, p_+)}$ is continuous in (p_-, p_+) . This reflects the rationality of using the norm $\|u\|_{p(x)}$ defined by (1.9) with the convention (1.8).

The paper is organized as follows. In Section 2 we prove the conclusion (1.13). In Section 3 we prove $\|u\|_{p(x)}^o \leq d_{p(x)} \|u\|_{p(x)}$. In Section 4 we prove the exactness of $d_{p(x)}$.

§2. INEQUALITY $\|u\|_{p(x)} \leq \|u\|_{p(x)}^o$ AND ITS EXACTNESS

The aim of this section is to prove (1.13). First let us prove the following lemma.

Lemma 2.1.

$$\|u\|_{p(x)} \leq \|u\|_{p(x)}^o, \quad \forall u \in L^{p(x)}(\Omega). \quad (2.1)$$

Proof. Let $u \in L^{p(x)}(\Omega)$ and $\|u\|_{p(x)} = 1$. We can assume $u(x) \geq 0$ for $x \in \Omega$ because the norms of $u(x)$ and $|u(x)|$ are equal. Let $c = \|u\|_{L^\infty(\Omega_\infty)}$ (when $\Omega_\infty = \emptyset$, we assume $c = 0$). By the definition (1.9) of $\|u\|_{p(x)}$, $c \leq 1$.

Case (i) $c < 1$.

In this case we have $\int_{\Omega} |u(x)|^{p(x)} d\mu = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} d\mu = 1$. Let

$$v(x) = \begin{cases} |u(x)|^{p(x)-1}, & \text{if } x \in \Omega \setminus \Omega_\infty, \\ 0, & \text{if } x \in \Omega_\infty. \end{cases}$$

Then $\int_{\Omega} |v(x)|^{p^o(x)} d\mu = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} d\mu = 1$. By the definition (1.10) of $\|u\|_{p(x)}^o$, we have

$$\|u\|_{p(x)}^o \geq \int_{\Omega} u(x)v(x) d\mu = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} d\mu = 1 = \|u\|_{p(x)}.$$

Case (ii) $c = 1$

Given any $\varepsilon > 0$, there exists $D \subset \Omega_\infty$ such that $\mu(D) > 0$ and $u(x) \geq 1 - \varepsilon$ for $x \in D$. Let

$$v(x) = \begin{cases} \frac{1}{\mu(D)}, & \text{if } x \in D, \\ 0, & \text{if } x \in \Omega \setminus D. \end{cases}$$

Then

$$\int_{\Omega} |v(x)|^{p^o(x)} d\mu = \int_D \frac{1}{\mu(D)} d\mu = 1.$$

We have

$$\|u\|_{p(x)}^o \geq \int_{\Omega} u(x)v(x) d\mu = \int_D u(x) \frac{1}{\mu(D)} d\mu \geq 1 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\|u\|_{p(x)}^o \geq 1 = \|u\|_{p(x)}$. The proof is complete.

Let us turn to the proof of (1.13). The inequality (2.1) means

$$\sup_{u \in L^{p(x)}(\Omega) \setminus \{0\}} \frac{\|u\|_{p(x)}}{\|u\|_{p(x)}^o} \leq 1.$$

In order to prove (1.13), it suffices to prove

$$\sup_{u \in L^{p(x)}(\Omega) \setminus \{0\}} \frac{\|u\|_{p(x)}}{\|u\|_{p(x)}^o} \geq 1. \quad (2.2)$$

When $\Omega_1 \neq \emptyset$ or $\Omega_{\infty} \neq \emptyset$, by the Remark 1.1 it is easy to see that there exists $u \in L^{p(x)}(\Omega)$ such that $\|u\|_{p(x)} = \|u\|_{p(x)}^o = 1$. Thus (2.2) holds. We now assume $\Omega_1 = \Omega_{\infty} = \emptyset$, i.e. $\Omega = \Omega_+$. Given any $\varepsilon > 0$, we can find $D \subset \Omega_+$ such that $\mu(D) > 0$ and $p_+(D) - p_-(D)$ small enough satisfying

$$\frac{1}{\bar{p}} + \frac{1}{\bar{q}} \leq 1 + \varepsilon, \quad \forall \bar{p} \in [p_-(D), p_+(D)], \quad \forall \bar{q} \in [(p^o)_-(D), (p^o)_+(D)].$$

Take $u \in L^{p(x)}(\Omega)$ such that $\|u\|_{p(x)} = 1$ and $\text{supp } u \subset D$. Then $\int_D |u(x)|^{p(x)} d\mu = 1$. For any $v \in L^{p^o(x)}(\Omega)$ with $\int_{\Omega} |v(x)|^{p^o(x)} d\mu \leq 1$ there holds

$$\begin{aligned} \left| \int_{\Omega} u(x)v(x) d\mu \right| &= \left| \int_D u(x)v(x) d\mu \right| \leq \int_D \left(\frac{1}{p(x)} |u(x)|^{p(x)} + \frac{1}{p^o(x)} |v(x)|^{p^o(x)} \right) d\mu \\ &= \frac{1}{\bar{p}} \int_D |u(x)|^{p(x)} d\mu + \frac{1}{\bar{q}} \int_D |v(x)|^{p^o(x)} d\mu \leq \frac{1}{\bar{p}} + \frac{1}{\bar{q}} \leq 1 + \varepsilon, \end{aligned}$$

where \bar{p} and \bar{q} are some constants in $[p_-(D), p_+(D)]$ and in $[(p^o)_-(D), (p^o)_+(D)]$ respectively. It implies that $\|u\|_{p(x)}^o \leq 1 + \varepsilon$, and thus $\frac{\|u\|_{p(x)}}{\|u\|_{p(x)}^o} \geq \frac{1}{1+\varepsilon}$. So the inequality (2.2) holds. This proves (1.13).

§3. PROOF OF INEQUALITY $\|u\|_{p(x)}^o \leq d_{p(x)} \|u\|_{p(x)}$

In this section we will prove

$$\|u\|_{p(x)}^o \leq d_{p(x)} \|u\|_{p(x)}, \quad \forall u \in L^{p(x)}(\Omega). \quad (3.1)$$

First, let us give some preliminaries. Denote

$$\Phi(x, t) = \frac{(p(x) - 1)^{p(x)-1}}{p(x)^{p(x)}} t^{p(x)}, \quad \forall x \in \Omega, \quad \forall t \geq 0. \quad (3.2)$$

Here we appoint that

$$\begin{aligned}\Phi(x, t) &= t, \quad \text{if } x \in \Omega_1, \\ \Phi(x, t) &= 0, \quad \text{if } x \in \Omega_\infty \text{ and } 0 \leq t \leq 1, \\ \Phi(x, t) &= \infty, \quad \text{if } x \in \Omega_\infty \text{ and } t > 1.\end{aligned}$$

Denote by $\Phi^*(x, \cdot)$ the complementary function to the function $\Phi(x, \cdot)$, that is

$$\Phi^*(x, s) = \sup_{t \geq 0} \{ts - \Phi(x, t)\}, \quad \forall x \in \Omega, \forall s \geq 0. \quad (3.3)$$

It is easy to see that

$$\Phi^*(x, s) = s^{p^o(x)}, \quad \forall x \in \Omega, \forall s \geq 0. \quad (3.4)$$

Set

$$L^\Phi(\Omega) = \left\{ u \in S(\Omega) : \exists \lambda > 0 \text{ such that } \int_{\Omega} \Phi\left(x, \left|\frac{u(x)}{\lambda}\right|\right) d\mu < \infty \right\}.$$

In $L^\Phi(\Omega)$, the Luxemburg norm $\|\cdot\|_\Phi$, Orlicz norm $\|\cdot\|_\Phi^o$, Amemiya norm $\|\cdot\|_\Phi^A$ are defined (see [7]) respectively by

$$\|u\|_\Phi = \inf\{\lambda > 0 : \int_{\Omega} \Phi(x, |u(x)/\lambda|) d\mu \leq 1\}, \quad (3.5)$$

$$\|u\|_\Phi^o = \sup \left\{ \left| \int_{\Omega} u(x)v(x) d\mu \right| : v \in L^{\Phi^*}(\Omega), \int_{\Omega} \Phi^*(x, |v(x)|) d\mu \leq 1 \right\}, \quad (3.6)$$

$$\|u\|_\Phi^A = \inf \left\{ \frac{1}{k} \left(\int_{\Omega} \Phi(x, |ku(x)|) d\mu + 1 \right) : k > 0 \right\}. \quad (3.7)$$

We can get $L^{p(x)}(\Omega) \subset L^\Phi(\Omega)$ because of $\frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}} \leq 1$. For any $u \in L^{p(x)}(\Omega)$, from (3.4), (3.6) and (1.10) we know

$$\|u\|_{p(x)}^o = \|u\|_\Phi^o. \quad (3.8)$$

By [3], we know

$$\|u\|_\Phi^o = \|u\|_\Phi^A. \quad (3.9)$$

(When $\Phi(x, t) = \Phi(t)$ is a general Orlicz function, the equality (3.9) has been proved by Hudzik and Maligranda in [5]). In order to prove the inequality (3.1) it suffices to prove the inequality

$$\|u\|_\Phi^A \leq d_{p(x)} \|u\|_{p(x)}, \quad \forall u \in L^{p(x)}(\Omega). \quad (3.10)$$

We prove (3.10) in lemma-size steps. We assume hereafter $p_- < p_+$ because of Remark 1.1.

Lemma 3.1. (3.10) holds when $1 < p_- < p_+ < \infty$.

Proof. Clearly $\Omega = \Omega_+$. Let $u \in L^{p(x)}(\Omega)$ and $\|u\|_{p(x)} = 1$. It suffices to prove

$$\|u\|_\Phi^A \leq d_{p(x)} = d_{(p_-, p_+)}. \quad (3.11)$$

For $k > 0$ let

$$G(k) = \frac{1}{k} \left(\int_{\Omega} \frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}} k^{p(x)} |u(x)|^{p(x)} d\mu + 1 \right). \quad (3.12)$$

By the definition (3.7) of $\|u\|_{\Phi}^A$, we have $\|u\|_{\Phi}^A = \inf_{k>0} G(k)$. For every $k > 0$, define $f_k : [p_-, p_+] \rightarrow \mathbb{R}$ by

$$f_k(p) = \frac{(p-1)^{p-1}}{p^p} k^p \quad \text{for } p \in [p_-, p_+]. \quad (3.13)$$

We can obtain

$$\sup_{p_- \leq p \leq p_+} f_k(p) = \max\{f_k(p_-), f_k(p_+)\} = f_k(p_*), \quad (3.14)$$

where $p_* = p_-$ or $p_* = p_+$ as following. Denote

$$\bar{k} = \left(\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} \frac{p_+^{p_+}}{(p_+ - 1)^{p_+ - 1}} \right)^{\frac{1}{p_+ - p_-}}. \quad (3.15)$$

Note that it follows from (3.15) that

$$\frac{p_+}{p_+ - 1} < \bar{k} < \frac{p_-}{p_- - 1}. \quad (3.16)$$

It is easy to see that, when $0 < k < \bar{k}$, we have $f_k(p_-) > f_k(p_+)$ and then $p_* = p_-$; when $k > \bar{k}$, we have $f_k(p_-) < f_k(p_+)$ and then $p_* = p_+$; when $k = \bar{k}$, we have $f_k(p_-) = f_k(p_+)$ and then $p_* = p_-$ or p_+ . Thus we have

$$G(k) \leq \frac{1}{k}(f_k(p_*) + 1), \quad (3.17)$$

$$\|u\|_{\Phi}^A = \inf_{k>0} G(k) \leq \inf_{k>0} \frac{1}{k}(f_k(p_*) + 1). \quad (3.18)$$

For every fixed $p_* > 1$, consider the function of k

$$g_{p_*}(k) = \frac{1}{k}(f_k(p_*) + 1) = \frac{1}{k} \left(\frac{(p_* - 1)^{p_* - 1}}{p_*^{p_*}} k^{p_*} + 1 \right), \quad \forall k > 0. \quad (3.19)$$

Denote $k^* = \frac{p_*}{p_* - 1}$. Then $g'_{p_*}(k^*) = 0$. We can obtain that $g_{p_*}(k)$ is decreasing in $(0, k^*]$ and increasing in $[k^*, \infty)$. When $k \in (0, \bar{k}]$,

$$p_* = p_-, \quad k^* = \frac{p_-}{p_- - 1} > \bar{k},$$

$g_{p_-}(k)$ is decreasing in $(0, \bar{k}]$. Thus

$$\inf_{0 < k \leq \bar{k}} \frac{1}{k}(f_k(p_*) + 1) = \inf_{0 < k \leq \bar{k}} g_{p_-}(k) = g_{p_-}(\bar{k}) = d_{(p_-, p_+)}. \quad (3.20)$$

When $k \in [\bar{k}, \infty)$,

$$p_* = p_+, \quad k^* = \frac{p_+}{p_+ - 1} < \bar{k},$$

$g_{p_+}(k)$ is increasing in $[\bar{k}, \infty)$. Thus

$$\inf_{k \geq \bar{k}} \frac{1}{k}(f_k(p_*) + 1) = \inf_{k \geq \bar{k}} g_{p_+}(k) = g_{p_+}(\bar{k}) = d_{(p_-, p_+)}. \quad (3.21)$$

By (3.20), (3.21) and (3.18) we obtain

$$\|u\|_{\Phi}^A = \inf_{k>0} G(k) \leq d_{(p_-, p_+)}.$$

The proof is complete.

Remark 3.1. By the proof of Lemma 3.1, we can get $d_{(p_-, p_+)} = g_{p_-}(\bar{k}) = g_{p_+}(\bar{k})$, where \bar{k} is defined by (3.15), $g_{p_*}(k)$ is defined by (3.19). Because $g_{p_-}(k)$ is decreasing in $(0, \bar{k}]$ and $\frac{p_+}{p_+-1} < \bar{k}$, we obtain

$$\begin{aligned} d_{(p_-, p_+)} &= g_{p_-}(\bar{k}) < g_{p_-}\left(\frac{p_+}{p_+-1}\right) \\ &= 1 + \frac{1}{p_-} \left(\frac{p_- - 1}{p_-} \frac{p_+}{p_+-1} \right)^{p_- - 1} - \frac{1}{p_+} \\ &< 1 + \frac{1}{p_-} - \frac{1}{p_+} = r_{p(x)}. \end{aligned}$$

Lemma 3.2. (3.10) holds when $\Omega_{\infty} = \emptyset$.

Proof. Actually, the proof of Lemma 3.1 is also valid for the case $1 = p_- < p_+ < \infty$. Here we only point out that when $p_- = 1$,

$$\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} = 1, \quad \bar{k} = \left(\frac{p_+^{p_+}}{(p_+ - 1)^{p_+ - 1}} \right)^{\frac{1}{p_+ - 1}}$$

and $d_{p(x)} = d_{(1, p_+)}$. When $\Omega_{\infty} = \emptyset$ and $p_+(\Omega_+) = \infty$, we still use the symbol as in the proof of Lemma 3.1. In this case,

$$f_k(p) = \frac{(p - 1)^{p - 1}}{p^p} k^p, \quad \forall p \in [p_-, \infty).$$

We have

$$\sup_{p \in [p_-, \infty)} f_k(p) = \begin{cases} \infty, & \text{if } k > 1, \\ f_k(p_-), & \text{if } k \leq 1. \end{cases}$$

Then for $u \in L^{p(x)}(\Omega)$ with $\|u\|_{p(x)} = 1$, we have

$$\begin{aligned} \|u\|_{\Phi}^A &= \inf_{k>0} G(k) \leq \inf_{k>0} \frac{1}{k} \left(\sup_{p \in [p_-, \infty)} f_k(p) + 1 \right) \\ &= \inf_{0 < k \leq 1} \frac{1}{k} (f_k(p_-) + 1) = \inf_{0 < k \leq 1} g_{p_-}(k) = g_{p_-}(1) \\ &= f_1(p_-) + 1 = \frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} + 1 = d_{(p_-, \infty)}. \end{aligned}$$

The proof is complete.

Lemma 3.3. (3.10) holds when $\Omega_{\infty} \neq \emptyset$.

Proof. Suppose that $u \in L^{p(x)}(\Omega)$ and $\|u\|_{p(x)} = 1$. In this case, $\|u\|_{L^{\infty}(\Omega_{\infty})} \leq 1$ and

$$\int_{\Omega} |u(x)|^{p(x)} d\mu = \int_{\Omega \setminus \Omega_{\infty}} |u(x)|^{p(x)} d\mu \leq 1.$$

Moreover,

$$\begin{aligned}
 \|u\|_{\Phi}^A &= \inf_{k>0} \frac{1}{k} \left(\int_{\Omega} \Phi(x, |ku(x)|) d\mu + 1 \right) \\
 &\leq \inf_{0 < k \leq 1} \frac{1}{k} \left(\int_{\Omega \setminus \Omega_{\infty}} \frac{(p(x) - 1)^{p(x)-1}}{p(x)^{p(x)}} k^{p(x)} |u(x)|^{p(x)} d\mu + 1 \right) \\
 &\leq \inf_{0 < k \leq 1} \frac{1}{k} \left(\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} k^{p_-} + 1 \right) \\
 &= \frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} + 1 = d_{(p_-, \infty)}.
 \end{aligned} \tag{3.22}$$

The proof is complete.

By the above lemmas, (3.1) is true.

§4. THE EXACTNESS OF $d_{p(x)}$

In this section, we will prove (1.14). The inequality (3.1) means that

$$\sup_{u \in L^{p(x)}(\Omega) \setminus \{0\}} \frac{\|u\|_{p(x)}^o}{\|u\|_{p(x)}} \leq d_{p(x)}.$$

To prove (1.14), it suffices to prove

$$\sup_{u \in L^{p(x)}(\Omega) \setminus \{0\}} \frac{\|u\|_{p(x)}^o}{\|u\|_{p(x)}} \geq d_{p(x)}. \tag{4.1}$$

We use the following lemmas to complete the proof.

Lemma 4.1. (4.1) holds for $1 < p_- < p_+ < \infty$.

Proof. Let \bar{k} be as in (3.15). Set

$$a = \left(\frac{p_- - 1}{p_-} \bar{k} \right)^{p_-}, \quad b = \left(\frac{p_+ - 1}{p_+} \bar{k} \right)^{p_+}.$$

Thus $a < 1$ and $b > 1$. Letting

$$\lambda_1 = \frac{b-1}{b-a}, \quad \lambda_2 = \frac{1-a}{b-a},$$

we have

$$\lambda_1, \lambda_2 > 0, \quad \lambda_1 + \lambda_2 = 1, \quad \lambda_1 a + \lambda_2 b = 1. \tag{4.2}$$

Let $k_0 > \bar{k}$ be large enough such that

$$\frac{(p(x) - 1)^{p(x)-1}}{p(x)^{p(x)}} k_0^{p(x)} > \frac{2}{\lambda_2}, \quad \forall x \in \Omega. \tag{4.3}$$

Given any $\varepsilon > 0$, we can find $A_- \subset \Omega$ and $A_+ \subset \Omega$ such that

$$\mu(A_-) > 0, \quad \mu(A_+) > 0, \quad A_- \cap A_+ = \emptyset,$$

$$\left| \frac{(p(x) - 1)^{p(x)-1}}{p(x)^{p(x)}} k^{p(x)} - \frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} k^{p_-} \right| < \frac{\varepsilon}{2}, \quad \forall x \in A_-, \quad \forall \frac{1}{2} \leq k \leq k_0, \quad (4.4)$$

$$\left| \frac{(p(x) - 1)^{p(x)-1}}{p(x)^{p(x)}} k^{p(x)} - \frac{(p_+ - 1)^{p_+ - 1}}{p_+^{p_+}} k^{p_+} \right| < \frac{\varepsilon}{2}, \quad \forall x \in A_+, \quad \forall \frac{1}{2} \leq k \leq k_0. \quad (4.5)$$

Take $u \in L^{p(x)}(\Omega)$ such that $\text{supp } u \subset (A_- \cup A_+)$ and

$$\int_{A_-} |u(x)|^{p(x)} d\mu = \lambda_1, \quad \int_{A_+} |u(x)|^{p(x)} d\mu = \lambda_2. \quad (4.6)$$

Thus we have

$$\int_{\Omega} |u(x)|^{p(x)} d\mu = \lambda_1 + \lambda_2 = 1, \quad \|u\|_{p(x)} = 1.$$

From Section 3 we already know

$$\|u\|_{p(x)}^o = \inf_{k>0} G(k) = \inf_{k>0} \frac{1}{k} \left(\int_{\Omega} \frac{(p(x) - 1)^{p(x)-1}}{p(x)^{p(x)}} k^{p(x)} |u(x)|^{p(x)} d\mu + 1 \right).$$

Since $\|u\|_{p(x)}^o \leq d_{(p_-, p_+)} \leq 2$ and $G(k) > 2$ for $k \in (0, \frac{1}{2}) \cup (k_0, \infty)$, we have

$$\|u\|_{p(x)}^o = \inf_{\frac{1}{2} \leq k \leq k_0} G(k).$$

By (4.4) and (4.5) we obtain

$$\begin{aligned} \|u\|_{p(x)}^o &= \inf_{\frac{1}{2} \leq k \leq k_0} \frac{1}{k} \left(\int_{\Omega} \frac{(p(x) - 1)^{p(x)-1}}{p(x)^{p(x)}} k^{p(x)} |u(x)|^{p(x)} d\mu + 1 \right) \\ &= \inf_{\frac{1}{2} \leq k \leq k_0} \frac{1}{k} \left\{ \int_{A_-} \left[\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} k^{p_-} + \left(\frac{(p(x) - 1)^{p(x)-1}}{p(x)^{p(x)}} k^{p(x)} \right. \right. \right. \\ &\quad \left. \left. \left. - \frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} k^{p_-} \right) \right] |u(x)|^{p(x)} d\mu + \int_{A_+} \left[\frac{(p_+ - 1)^{p_+ - 1}}{p_+^{p_+}} k^{p_+} \right. \\ &\quad \left. \left. + \left(\frac{(p(x) - 1)^{p(x)-1}}{p(x)^{p(x)}} k^{p(x)} - \frac{(p_+ - 1)^{p_+ - 1}}{p_+^{p_+}} k^{p_+} \right) \right] |u(x)|^{p(x)} d\mu + 1 \right\} \\ &\geq \inf_{\frac{1}{2} \leq k \leq k_0} \frac{1}{k} \left(\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} k^{p_-} \lambda_1 + \frac{(p_+ - 1)^{p_+ - 1}}{p_+^{p_+}} k^{p_+} \lambda_2 + 1 - \frac{\varepsilon}{2} \right) \\ &\geq \inf_{\frac{1}{2} \leq k \leq k_0} \frac{1}{k} \left(\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} k^{p_-} \lambda_1 + \frac{(p_+ - 1)^{p_+ - 1}}{p_+^{p_+}} k^{p_+} \lambda_2 + 1 \right) - \varepsilon. \end{aligned} \quad (4.7)$$

Define

$$H(k) = \frac{1}{k} \left(\frac{(p_- - 1)^{(p_- - 1)}}{p_-^{p_-}} k^{p_-} \lambda_1 + \frac{(p_+ - 1)^{(p_+ - 1)}}{p_+^{p_+}} k^{p_+} \lambda_2 + 1 \right), \quad \forall k \in [\frac{1}{2}, k_0].$$

It is easy to verify that \bar{k} is the minimum point of $H(k)$ in $[\frac{1}{2}, k_0]$ and

$$\begin{aligned} \inf_{\frac{1}{2} \leq k \leq k_0} H(k) &= H(\bar{k}) \\ &= \lambda_1 \frac{1}{\bar{k}} \left(\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} \bar{k}^{p_-} + 1 \right) + \lambda_2 \frac{1}{\bar{k}} \left(\frac{(p_+ - 1)^{p_+ - 1}}{p_+^{p_+}} \bar{k}^{p_+} + 1 \right) \\ &= \lambda_1 d_{(p_-, p_+)} + \lambda_2 d_{(p_-, p_+)} = d_{(p_-, p_+)}. \end{aligned} \quad (4.8)$$

By (4.7) and (4.8) we obtain

$$\|u\|_{p(x)}^o \geq \inf_{\frac{1}{2} \leq k \leq k_0} H(k) - \varepsilon \geq d_{(p_-, p_+)} - \varepsilon.$$

It means

$$\sup_{u \in L^{p(x)}(\Omega) \setminus \{0\}} \frac{\|u\|_{p(x)}^o}{\|u\|_{p(x)}} \geq d_{(p_-, p_+)} - \varepsilon.$$

Thus (4.1) holds since $\varepsilon > 0$ is arbitrary.

Lemma 4.2. (4.1) holds for $1 = p_- < p_+ < \infty$.

Proof. The proof is similar to that of Lemma 4.1. Please refer to the proof of Lemma 3.2 and notice that $a = 0$.

Lemma 4.3 (4.1) holds when $\Omega_\infty = \emptyset$ and $p_+(\Omega_+) = \infty$.

Proof. $\forall \varepsilon > 0$, for every natural number n , denote

$$\Omega^{(n)} = \{x \in \Omega : p(x) \leq n\}.$$

Thus $p_+(\Omega^{(n)}) \leq n < \infty$ on $\Omega^{(n)}$. Using (4.1) for $L^{p(x)}(\Omega^{(n)})$, there exists a $u \in L^{p(x)}(\Omega^{(n)})$ such that $\int_{\Omega^{(n)}} |u(x)|^{p(x)} d\mu = 1$ and $\|u\|_{p(x)|\Omega^{(n)}}^o \geq d_{(p_-, p_+)} - \varepsilon$. Let

$$\tilde{u}(x) = \begin{cases} u(x), & \text{if } x \in \Omega^{(n)}, \\ 0, & \text{if } x \in \Omega \setminus \Omega^{(n)}. \end{cases}$$

Then $\tilde{u}(x) \in L^{p(x)}(\Omega)$ and $\int_{\Omega} |\tilde{u}(x)|^{p(x)} d\mu = 1$. Thus $\|\tilde{u}(x)\|_{p(x)} = 1$. Clearly,

$$\|u\|_{p(x)|\Omega^{(n)}}^o = \|\tilde{u}(x)\|_{p(x)}.$$

It means

$$\sup_{u \in L^{p(x)}(\Omega) \setminus \{0\}} \frac{\|u\|_{p(x)}^o}{\|u\|_{p(x)}} \geq d_{(p_-, n)} - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\sup_{u \in L^{p(x)}(\Omega) \setminus \{0\}} \frac{\|u\|_{p(x)}^o}{\|u\|_{p(x)}} \geq d_{(p_-, n)}, \quad \forall n.$$

Thus

$$\sup_{u \in L^{p(x)}(\Omega) \setminus \{0\}} \frac{\|u\|_{p(x)}^o}{\|u\|_{p(x)}} \geq \lim_{n \rightarrow \infty} d_{(p_-, n)} = d_{(p_-, \infty)}.$$

The proof is complete.

Lemma 4.4. (4.1) holds when $\Omega_\infty \neq \emptyset$.

Proof. We assume $\Omega \setminus \Omega_\infty \neq \emptyset$ since when $\Omega = \Omega_\infty$ (4.1) is obviously true (see Remark 1.1). Given any $\varepsilon > 0$, let $A_- \subset \Omega \setminus \Omega_\infty$ such that $\mu(A_-) > 0$ and

$$\left| \frac{(p(x) - 1)^{p(x)-1}}{p(x)^{p(x)}} k^{p(x)} - \frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} k^{p_-} \right| \leq \frac{\varepsilon}{2}, \quad \forall x \in A_-, \quad \forall k \in \left[\frac{1}{2}, 1 \right].$$

Let

$$u(x) = \begin{cases} 1, & \text{if } x \in \Omega_\infty, \\ a, & \text{if } x \in A_-, \\ 0, & \text{if } x \in \Omega \setminus (A_- \cup \Omega_\infty), \end{cases}$$

where the positive constant a satisfies $\int_{A_-} a^{p(x)} d\mu = 1$. Then we have $\|u\|_{p(x)} = 1$ and

$$\begin{aligned} \|u\|_{p(x)}^o &= \inf_{k>0} \frac{1}{k} \left(\int_{\Omega \setminus \Omega_\infty} \Phi(x, |ku(x)|) d\mu + \int_{\Omega_\infty} \Phi(x, |ku(x)|) d\mu + 1 \right) \\ &= \inf_{\frac{1}{2} \leq k \leq 1} \frac{1}{k} \left(\int_{A_-} \frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}} k^{p(x)} |u(x)|^{p(x)} d\mu + 1 \right) \\ &\geq \inf_{\frac{1}{2} \leq k \leq 1} \frac{1}{k} \left(\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} k^{p_-} + 1 - \frac{\varepsilon}{2} \right) \\ &\geq \inf_{\frac{1}{2} \leq k \leq 1} \frac{1}{k} \left(\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} k^{p_-} + 1 \right) - \varepsilon \\ &\geq \frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} + 1 - \varepsilon = d_{(p_-, \infty)} - \varepsilon. \end{aligned} \quad (4.9)$$

Thus (4.1) holds.

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