Fourier-Mukai transforms of slope stable torsion-free sheaves on Weierstrass elliptic surfaces

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Abstract. On a Weierstraß elliptic surface X, we define a 'limit' of Bridgeland stability conditions, denoted as Z^l-stability, by varying the polarisation along a curve in the ample cone. We show that a slope stable torsion-free sheaf of positive (twisted) degree or a slope stable locally free sheaf is taken by a Fourier-Mukai transform to a Z^l-stable object, while a Z^l-semistable object of nonzero fiber degree can be modified so that its inverse Fourier-Mukai transform is a slope semistable torsion-free sheaf. As an application, on a Weierstraß elliptic surface of Picard rank two with a negative section, we show that a line bundle of fiber degree at least 2 is taken by the inverse Fourier-Mukai transform to a slope semistable locally free sheaf.

Slogan. Let X be a smooth projective variety over $\mathbb{C}$ and denote $D^b(X)$ by its bounded derived category of coherent sheaves. Let $\Phi \in \mathrm{Aut}(D^b(X))$. Let σ be a stability condition (eg Mumford, Gieseker, Pandharipande-Thomas, Bridgeland etc). Then Φ acts on σ, and could produce relation/symmetry of σ-invariants (eg DT, PT) on X. For example, the modularity of generating series of PT invariants on special elliptic Calabi-Yau 3fold is proved by Oberdieck and Shen.

Question. Given an elliptic fibration and the associated Fourier-Mukai transform Φ (an auto-equivalence of $D^b(X)$), what happens to a slope stable coherent sheaf under Φ? What is a notion of slope stability under Φ? The key premise is that we do not fix Chern character invariants.

A **Weierstraß elliptic surface** $p : X \to B$ is a flat morphism where X is a smooth projective surface and B is a smooth projective curve, where all the fibers are Gorenstein curves of arithmetic genus 1 and geometrically integral, and p admits a section $\sigma : B \to X$ whose image $\Theta = \sigma(B)$ does not meet any singular point of any fiber. There is a pair of **relative Fourier-Mukai transforms** $\Phi, \widehat{\Phi} : D^b(X) \xrightarrow{\sim} D^b(X)$ whose kernels are both sheaves on $X \times_B X$, satisfying

$$\widehat{\Phi}\Phi = \mathrm{id}_{D^b(X)}[-1] = \Phi\widehat{\Phi}.$$

In particular, the kernel of Φ is the relative Poincaré sheaf for the fibration p, which is a universal sheaf for the moduli problem that parametrises degree-zero, rank-one torsion-free sheaves on the fibers of p.

Let ω and B be two $\mathbb{R}$-divisors with ω ample. Define the twisted Chern character $\mathrm{ch}^B(E) = e^{-B}\mathrm{ch}(E)$.

Slope stability. For any $E \in \mathrm{Coh}(X)$, we define

$$\mu_{\omega,B}(E) = \begin{cases} \frac{\omega \mathrm{ch}_1^B(E)}{\mathrm{ch}_0^B(E)} & \text{if } \mathrm{ch}_0^B(E) \neq 0 \\ +\infty & \text{if } \mathrm{ch}_0^B(E) = 0 \end{cases}.$$

An object $E \in \mathrm{Coh}(X)$ is said to be $\mu_{\omega,B}$-stable or slope stable (resp. $\mu_{\omega,B}$-semistable or slope semistable) if, for every short exact sequence in $\mathrm{Coh}(X)$

$$0 \to M \to E \to N \to 0$$

where $M, N \neq 0$, we have $\mu_{\omega,B}(M) <$ (resp. $\leq$) $\mu_{\omega,B}(N)$. The B-field twist here is not essential but just for a computational reason.

Bridgeland stability. Define the following subcategories of $\mathrm{Coh}(X)$

$$\mathcal{T}_{\omega,B} = \langle F \in \mathrm{Coh}(X) : F \text{ is } \mu_{\omega,B}\text{-semistable, } \mu_{\omega,B}(F) > 0\rangle,$$
$$\mathcal{F}_{\omega,B} = \langle F \in \mathrm{Coh}(X) : F \text{ is } \mu_{\omega,B}\text{-semistable, } \mu_{\omega,B}(F) \leq 0\rangle.$$

Since the slope function $\mu_{\omega,B}$ has the Harder-Narasimhan property, the pair $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ is a torsion pair in $\mathrm{Coh}(X)$. The extension closure

$$\mathcal{B}_{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B}\rangle$$

in $D^b(X)$ is thus a tilt of the heart $\mathrm{Coh}(X)$, i.e. $\mathcal{B}_{\omega,B}$ is the heart of a bounded t-structure on $D^b(X)$ and is an abelian subcategory of $D^b(X)$. If we set

$$Z_{\omega,B}(F) = -\int_X e^{-i\omega}\mathrm{ch}^B(F) = -\mathrm{ch}_2^B(F) + \tfrac{\omega^2}{2}\mathrm{ch}_0(F) + i\omega\mathrm{ch}_1^B(F),$$

then the pair $(Z_{\omega,B}, \mathcal{B}_{\omega,B}) =: \sigma_{\omega,B}$ gives a Bridgeland stability condition on $D^b(X)$, as shown by Arcara-Bertram.

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KEY observation. We want to find a notion of stability condition which corresponds to slope stability under the FMT $\Phi[1]$. We fix a positive number m such that the divisor $\Theta + mf$ on X is ample. Denote $e := -\Theta^2$. Let α, u and v be real variables. We have a relation

$$Z_{\omega,0}(\Phi E[1]) = \mathrm{ch}_0(E)\left(-\mu_{\overline{\omega},\overline{B}}(E) + i(u \cdot * + v)\right). \tag{1}$$

provided that u and v are in a curve

$$m + \alpha = (m - \tfrac{e}{2})u^2 + uv + e, \tag{2}$$

where $\overline{\omega} = (\Theta + mf) + \alpha f$, $\overline{B} = \frac{e}{2}f$ before FMT, and $\omega = u(\Theta + mf) + vf$, $B = 0$ after FMT. In particular, to simplify the theory we take the B-field after FMT to be 0, which forces the B-field before FMT to be $\frac{e}{2}f$.

Limit Bridgeland stability. We define a 'limit Bridgeland stability' σ^l as

$$\sigma^l = (Z^l, \mathcal{B}^l) := \lim_{v \to \infty \text{ along } (2)} \sigma_{\omega,0}.$$

We have the relation between different hearts of t-structures in $D^b(X)$, where a wave type arrow with a pair $(\mathcal{T}, \mathcal{F})$ means that (i) such pair is a torsion pair in the source heart and (ii) the target heart is the tilt at such torsion pair, i.e. the target heart is $\langle \mathcal{F}[1], \mathcal{T}\rangle$. For $i = 0, 1$, $W_{i,\widehat{\Phi}} := \{F \in \mathrm{Coh}(X) : \widehat{\Phi}(F)[i] \cong E \text{ for some } E \in \mathrm{Coh}(X)\}$.

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathrm{Coh}(X) \arrow[r, shift left, "\Phi[1]\ \cong"] \arrow[ddr, shift left, "\textbf{Thm 1(A)}"] & \Phi(\mathrm{Coh}(X))[1] \arrow[l, shift left, "\widehat{\Phi}\ \cong"] & \\
& & \mathrm{Coh}(X) \arrow[ul, squiggly, "{(W_{0,\widehat{\Phi}}, W_{1,\widehat{\Phi}})}"'] \arrow[dl, squiggly, "{(\mathcal{T}^l, \mathcal{F}^l)}"] \arrow[d, squiggly, "{(\mathcal{T}_\omega, \mathcal{F}_\omega)}"] \\
\widehat{\Phi}\mathcal{B}^l \arrow[uu, squiggly, "{(\langle\widehat{\Phi}\mathcal{F}^l[1], W_{1,\Phi}\rangle, \widehat{\Phi}(W_{1,\widehat{\Phi}} \cap \mathcal{T}^l))}"] \arrow[r, shift left, "\Phi[1]\ \cong"] & \mathcal{B}^l \arrow[l, shift left, "\widehat{\Phi}\ \cong"] \arrow[uu, squiggly, "{(\langle\mathcal{F}^l[1], W_{0,\widehat{\Phi}}\rangle, W_{1,\widehat{\Phi}} \cap \mathcal{T}^l)}"'] \arrow[uul, shift left, "\textbf{Thm 1(B)}"] & \mathcal{B}_\omega \arrow[l, dashed, "\text{limit along curve (2) as } v \to \infty"]
\end{tikzcd}$$

Theorem 1. *(slope stability vs limit Bridgeland stability before and after the FMT $\Phi[1]$) Let $p : X \to B$ be a Weierstraß elliptic surface.*

(A) Suppose E is a $\mu_{\overline{\omega}}$-stable torsion-free sheaf on X and $\overline{B} = \frac{e}{2}f$.

- *(A1) If $\overline{\omega}\mathrm{ch}_1^{\overline{B}}(E) > 0$, then $\Phi E[1]$ is a Z^l-stable object in $\mathcal{B}^l$.*
- *(A2) If $\overline{\omega}\mathrm{ch}_1^{\overline{B}}(E) = 0$, then $\Phi E[1]$ is a Z^l-semistable object in $\mathcal{B}^l$, and the only $\mathcal{B}^l$-subobjects G of $\Phi E[1]$ where $\phi(G) = \phi(\Phi E[1])$ are objects in $\Phi(\mathrm{Coh}^{\leq 0}(X))$.*
- *(A3) If E is locally free, then $\Phi E[1]$ is a Z^l-stable object in $\mathcal{B}^l$.*

(B) Suppose $F \in \mathcal{B}^l$ is a Z^l-semistable object with $f\mathrm{ch}_1(F) \neq 0$, and F fits in the $\mathcal{B}^l$-short exact sequence

$$0 \to F' \to F \to F'' \to 0$$

where $F' \in \langle \mathcal{F}^l[1], W_{0,\widehat{\Phi}}\rangle$ and $F'' \in \langle W_{1,\widehat{\Phi}} \cap \mathcal{T}^l\rangle$. Then $\widehat{\Phi}F'$ is a $\mu_{\overline{\omega}}$-semistable torsion-free sheaf on X.

Note that the objects of $\Phi(\mathrm{Coh}^{\leq 0}(X))$ are precisely direct sums of semistable fiber sheaves of degree 0. The object $\widehat{\Phi}F''[1]$ must be a torsion sheaf.

As an application, we could produce slope semistable locally free sheaves by finding limit Bridgeland stable objects $(\mathcal{O}_X(a_L\Theta))$ first and then applying the inverse FMT $\widehat{\Phi}$ to them as Thm 1(B).

Theorem 2. *Let $p : X \to B$ be a Weierstraß elliptic surface such that X has Picard rank two and $e > 0$. Let $m > 0$ be such that $\Theta + m'f$ is ample for all $m' \geq m$. Then for any positive integer $a_L \geq 2$ and real number $\alpha > 0$ satisfying*

$$m + \alpha - e \neq \tfrac{e}{2}a_L(a_L - 1), \tag{3}$$

the line bundle $\mathcal{O}_X(a_L\Theta)$ is σ_ω-stable for any Bridgeland stability σ_ω lying on the curve (2) for $v > v_0$ for some fixed v_0. Moreover, the transform $\widehat{\Phi}\mathcal{O}_X(a_L\Theta)$ is a $\mu_{\overline{\omega}}$-semistable locally free sheaf where $\overline{\omega} = \Theta + (m + \alpha)f$.

The preprint is available at `wanminliu.github.io`