Lecture 1 § Review of linear systems

1.2.13. Ange (om möjligt) villkor på $a, b, c \in \mathbb{R}$ så att systemen nedan får (i) entydig lösning, (ii) ingen lösning, (iii) oändligt många lösningar.

$$\text{(a)} \begin{cases} 3x - y + az = 3 \\ x + y - z = 2 \\ 2x - 2y + 3z = b \end{cases}, \quad \text{(b)} \begin{cases} x + ay = 0 \\ y + bz = 0 \\ cx + z = 0 \end{cases}.$$

Quick review: $A\vec{x} = \vec{b}$ $\quad A = (\quad)_{m \times n}$ $\quad \vec{x} = (\ )_{n \times 1}$ $\quad \vec{b} = (\ )_{m \times 1}$

If $\vec{b} = \vec{0}$ then $A\vec{x} = \vec{0}$ is homogeneous linear sym

otherwise non homogen.

Baby case $m = n = 1$.

$1x = b$ or $0x = 0$ or $0x = 1$

$x = b$ unique sol. $\quad$ ∞-many sol. $\quad$ no solution

(a) Row operation $\begin{cases} \cdot \text{ addition} \\ \text{non-zero scaler} \\ \text{change order of rows} \end{cases}$

$$\left(\begin{array}{ccc|c} 3 & -1 & a & 3 \\ 1 & 1 & -1 & 2 \\ 2 & -2 & 3 & b \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 3 & -1 & a & 3 \\ 2 & -2 & 3 & b \end{array}\right) \begin{array}{l} ② \\ ① \\ ③ \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -4 & a+3 & -3 \\ 0 & -4 & 5 & b-4 \end{array}\right) \begin{array}{l} ① \\ ② - 3 \times ① \\ ③ - 2 \times ① \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & \frac{a+3}{-4} & \frac{-3}{-4} \\ 0 & 0 & 2-a & b-1 \end{array}\right) \begin{array}{l} ① \\ ② \times \left(-\frac{1}{4}\right) \\ ③ - ② \end{array}$$

1) If $2-a=0$ & $b-1 \neq 0$ " $0z = b-1 \neq 0$ "

no solution.

2) If $2-a=0$ & $b-1=0$ " $0z=0$ " ∞-many solutions

3) If $2-a \neq 0$. $\Rightarrow$ $z = \frac{b-1}{2-a}$ & ... : unique solution

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & \frac{a+3}{-4} & \frac{-3}{-4} \\ 0 & 0 & 1 & \frac{b-1}{2-a} \end{array}\right) \begin{array}{l} ① \\ ② \\ ③ \times \left(\frac{1}{2-a}\right) \end{array} \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2+\frac{b-1}{2-a} \\ 0 & 1 & 0 & \frac{-3}{-4} + \frac{a+3}{4} \cdot \frac{b-1}{2-a} \\ 0 & 0 & 1 & \frac{b-1}{2-a} \end{array}\right) \begin{array}{l} ① + ③ \\ ② + \left(\frac{a+3}{4}\right) \times ③ \\ \end{array}$$

$x = 2 + \frac{b-1}{2-a}$

$y = \frac{-3}{-4} + \frac{a+3}{4} \cdot \frac{b-1}{2-a}$

$z = \frac{b-1}{2-a}$

2) again:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & \frac{a+3}{-4} & \frac{-3}{-4} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = x$

$x_2 = y$

$x_3 = z$

$\boxed{a = 2 \quad b = 1}$

$x_2 + \left(\frac{a+3}{-4}\right) x_3 = +\frac{3}{4}$

$x_2 + \frac{5}{-4} x_3 = \frac{3}{4}$

$x_2 = \frac{3}{4} + \frac{5}{4} x_3$

$x_1 + x_2 - x_3 = 2$

$x_1 = 2 + x_3 - \frac{3}{4} - \frac{5}{4} x_3$

$\boxed{x_3 = t}$

$$\begin{cases} x_1 = \frac{5}{4} - \frac{1}{4} t \\ x_2 = \frac{3}{4} + \frac{5}{4} t \\ x_3 = t \end{cases}$$

(b) $\begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & b & 0 \\ c & 0 & 1 & 0 \end{pmatrix}$

$\det A = 1 + abc$

$A\vec{x} = \vec{0}$

If $\det A \neq 0 \Rightarrow \exists A^{-1} \Rightarrow A^{-1}(A\vec{x}) = A^{-1}\vec{0}$

$\Rightarrow \vec{x} = \vec{0}$

If $\det A = 0 \Rightarrow \infty$ many solution.

$1 + abc \neq 0$

$\Leftrightarrow \cdots$

$1 + abc = 0$

$abc = -1$

3.2.4. Följande matriser är givna (se **Exempel 3.2.5**, sid 59):

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 \\ 5 & 1 \\ 2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{och} \quad D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Utför nedanstående multiplikationer i de fall de är definierade:
(a) AA, (b) AB, (c) CA, (d) CB, (e) $(AB)D$.
Gäller $AB = BA$?

Quick review:

$$A_{m\times n} = \begin{pmatrix} a_{11} & & a_{1n} \\ & & \\ a_{m1} & & a_{mn} \end{pmatrix} \begin{matrix} 1 \\ \vdots \\ i \\ m \end{matrix} \qquad B_{n\times k} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{matrix} 1 \\ \vdots \\ n \end{matrix}$$

$1 \cdots n$ $\qquad$ $1 \cdots j \cdots k$

$$C = A_{m\times n} \cdot B_{n\times k} = \begin{pmatrix} \bullet \end{pmatrix}_{m\times k}$$

match

$$c_{ij} = (a_{i1}, \cdots, a_{in}) \cdot \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

(a) $\underset{3\times 2}{A}\ \underset{3\times 2}{A} = ?$ we cannot do it.

(b) $$\begin{pmatrix} 1 & 0 & 2 \\ -3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 5 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 9 & -8 \end{pmatrix}_{2\times 2}$$

$$(1, 0, 2) \cdot \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} = 1\cdot 0 + 0\cdot 5 + 2\cdot 2 = 4$$

(c) $\underbrace{C_{3\times3} \; A_{2\times3}}$ = cannot compute.

(d) $C_{3\times3} \; B_{3\times2} = \begin{pmatrix} & \end{pmatrix}_{3\times2}$

(e) $(A \; B) \; D = \begin{pmatrix} 4 & 3 \\ 9 & -8 \end{pmatrix}_{2\times2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{2\times1} = \begin{pmatrix} 3 \\ -8 \end{pmatrix}_{2\times1}$

Associativity of multipl: $(A \; B) \; D = A \; (B \; D)$

We don't have commutativity $AB \neq BA$

- If $AB = BA \Rightarrow A_{m\times n} B_{n\times k} = \underbrace{B_{n\times k} A_{m\times n}}_{k=m}$

$\Rightarrow \begin{pmatrix} & \end{pmatrix}_{m\times m} = \begin{pmatrix} & \end{pmatrix}_{n\times n} \Rightarrow m = n = k$ A. B should be same size square matrices

this is a necessary condition.

3.6.4. (a) Bestäm inversen till matrisen

$$A = \begin{pmatrix} 1 & -2 & -3 & 1 \\ -1 & 3 & 5 & -1 \\ 2 & 1 & -1 & 3 \\ 3 & -1 & 1 & 4 \end{pmatrix}.$$

(b) Utnyttja resultatet i (a) till att lösa ekvationssystemet

$$\begin{cases} x_1 - 2x_2 - 3x_3 + x_4 = 7 \\ -x_1 + 3x_2 + 5x_3 - x_4 = -9 \\ 2x_1 + x_2 - x_3 + 3x_4 = 18 \\ 3x_1 - x_2 + x_3 + 4x_4 = 15 \end{cases}.$$

Method:

Row operation for the computation of inverse matrix

$$(A, I_n) = \left(\begin{array}{cccc|cccc} 1 & -2 & -3 & 1 & 1 & 0 & 0 & 0 \\ -1 & 3 & 5 & -1 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 3 & 0 & 0 & 1 & 0 \\ 3 & -1 & 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{row oper.}} \left(\begin{array}{cccc|cccc} 1 & -2 & -3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 1 & -2 & 0 & 1 & 0 \\ 0 & 5 & 10 & 1 & -3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ (2)+(1) \\ (3)-(1)\times 2 \\ (4)-(1)\times 3 \end{array}$$

$n = 4$

$$\ldots\ldots \sim (I_4,\ A^{-1})$$

4.2.3 Radoperationer och kofaktorutveckling

4.2.9. Beräkna följande determinanter (se **Exempel 4.5.3′**, sid 91 och **Exempel 4.7.3′**, sid 91)

(a) $\begin{vmatrix} 3 & -1 & 0 \\ 2 & 3 & 5 \\ -2 & 1 & 4 \end{vmatrix}$, (b) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 2 \\ 4 & 3 & 2 & -1 \end{vmatrix}$, (c) $\begin{vmatrix} 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 4 \\ 4 & 2 & 0 & 3 \\ 3 & 2 & 1 & 4 \end{vmatrix}$,

(b)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 6 \\ 0 & -5 & -10 & -17 \end{vmatrix} \begin{matrix} \\ \\ ③+① \\ ④-①\times 4 \end{matrix} = 1 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ -5 & -10 & -17 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{vmatrix} \begin{matrix} \\ ②-①\times 2 \\ ③+①\times 5 \end{matrix} = -2$$

Quick review:

— do reduction from <u>higher order det</u> to a lower order

size of square matrix.

- Scalar: $|\lambda \cdot A_{n\times n}| = \left|\begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \lambda a_{n1} & \cdots & \lambda a_{nn} \end{pmatrix}\right| = \lambda \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \lambda a_{21} & & \lambda a_{2n} \\ \vdots & & \vdots \\ \lambda a_{n1} & & \lambda a_{nn} \end{vmatrix} = \cdots = \lambda^n |A|$

  Scalar product

$$|\lambda \cdot A| = \lambda^n |A|$$

- change order of rows (columns) : give a sign (−1)

(a)

$$\begin{vmatrix} 3 & -1 & 0 \\ 2 & 3 & 5 \\ -2 & 1 & 4 \end{vmatrix}_{3\times 3} \underset{\text{row expen.}}{=} 3\,(-1)^{1+1} \begin{vmatrix} 3 & -1 & 0 \\ 2 & 3 & 5 \\ -2 & 1 & 4 \end{vmatrix} + (-1)\cdot(-1)^{1+2} \begin{vmatrix} 3 & -1 & 0 \\ 2 & 3 & 5 \\ -2 & 1 & 4 \end{vmatrix}$$

$$+\; 0\cdot(-1)^{1+3} \cdot \begin{vmatrix} 3 & -1 & 0 \\ 2 & 3 & 5 \\ -2 & 1 & 4 \end{vmatrix}$$

place of the entry 3

place of entry (-1)

place of 0

$$\therefore \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Cross-product of two "3-dim" vectors

row vectors here

$i = (1,0,0)$

$j = (0,1,0)$

$k = (0,0,1)$

$(a,b,c) \times (d,e,f) = $ 3-dim vector

pseudo determinant: i, j, k as symbols

formal computation

$$(a,b,c)\times(d,e,f) = \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} = i\underbrace{\begin{vmatrix} b & c \\ e & f \end{vmatrix}}_{\text{number}} - j\begin{vmatrix} a & c \\ d & f \end{vmatrix} + k\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

think i, j, k as standard basis of $\mathbb{R}^3$

$$= \left(\begin{vmatrix} b & c \\ e & f \end{vmatrix}, -\begin{vmatrix} a & c \\ d & f \end{vmatrix}, \begin{vmatrix} a & b \\ d & e \end{vmatrix} \right)$$

§ Vector spaces

5.2.1. Avgör vilka av följande mängder som är vektorrum över de reella talen (addition och multiplikation med skalär är de vanliga på respektive mängd).

(a) $M_a = \{$ alla polynom$\}$ ✓

(b) $M_b = \{$ alla polynom av grad exakt 3 $\}$ ✗

(c) $M_c = \{$ alla polynom av grad $\leq 3\}$ ✓

(d) $M_d = \{$ alla 2×2 matriser med reella elemnt $\}$ ✓

(e) $M_e = \{$ alla reella funktioner definierade på $[-1, 2]\}$

(f) $M_f = \mathbb{C}^0(0, 2) =$ alla reellvärda kontinuerliga funktioner med definitionsmängd $]0, 2[$

(g) $M_g = \{f(x) \in \mathbb{C}^0(0, 2)\colon f(1) = 1\}$

(h) $M_h = \{f(x) \in \mathbb{C}^0(0, 2)\colon f(1) = 0\}$

(i) $M_i = \mathbb{Z} =$ de hela talen

sets. We need to find $\ddot{\ }$ + $\ddot{\ }$ / define $\cdot\cdot$ $\cdot$ if possible.

what is a vector space over $\mathbb{R}$

Vect space $= (\text{set}, +, \cdot)$

axioms ...

0-vector $\vec{0} + \vec{x} = \vec{x} \quad \forall \vec{x} \in \text{set}$

distribution law $\lambda(\vec{x} + \vec{y}) = \lambda \cdot \vec{x} + \lambda \vec{y}$

$M_a = \{ f = f(x) \text{ is a one-varible polynom.} \}$

addition of numbers

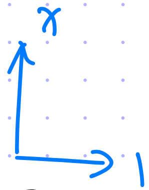
$f, g \in M_a$ "$+$" $\qquad f + g \overset{def}{=} f(x) + g(x)$

$\lambda f \overset{def}{=} \lambda \cdot f(x)$

It is a vet. space of ∞-dim.

$1, \quad x, \quad x^2 \notin \mathrm{Span}\{1, x\}$.

$\mathrm{Span}\{1, x, x^2\}$



dim 3

$\mathrm{span}\{1, x, \dots, x^{n-1}\}$ of dim n

. . . .

no control of n. $\quad n \to \infty$

$M_b = \{ f \text{ of deg exact } 3 \}$

$f(x) = 1 + x + x^3$

$g(x) = 1 + x + x^2 + x^3$

then $f + (-1) \cdot g = f(x) - g(x) = -x^2$ is NOT a poly of deg 3.

the set M_b is not "closed" under then the "$+$" or "$\cdot$"

$M_c := \{ \ldots \deg \le 3 \}$ fine it is a vec space

of dim $\underline{\underline{4}}$ with basis $1, x, x^2, x^3$

$M_e = \{ f : [-1, 2] \to \mathbb{R}$ Real fun. $\}$

add. of numbers

$+ \quad f, g \in M_e. \quad f + g \overset{\text{def}}{=} f(x) + g(x)$

$\lambda \cdot f \overset{\text{def}}{=} \lambda f(x)$

$(M_e, +, \cdot)$ is a vect space

$M_f = C^0(0, 2) = \{ f : (0, 2) \to \mathbb{R} \mid \underline{f \text{ is contin.}} \}$

is a vec. space of ∞-dim.

$M_g = C^0(0, 2) \cap \{ \underline{f(1) = 1} \}$

Not a vect. space

together

~~$\not\in$~~ M_g

$f_1 \in M_g \quad f_1(1) = 1 \quad (f_1 + f_2)(1)$

$f_2 \in M_g \quad f_2(1) = 1 \quad = f_1(1) + f_2(1) = 2$

$\neq 1$

$$M_h = C^0(0,2) \cap \{f(1)=0\}$$

$f_1 \in M_h \quad f_1(1)=0 \quad (f_1+f_2)(1) = f_1(1) + f_2(1) = 0+0 = 0$

$f_2 \in M_h \quad f_2(1)=0$

M_h is a vect. space.

M_h is a sub vect. space of M_f.

co-dim

co-dim

$M_i = \mathbb{Z}$ is NOT a vec. space over $\mathbb{R}$.

$\lambda = \frac{1}{2} \in \mathbb{R} \qquad \lambda \cdot 1 = \frac{1}{2} \notin \mathbb{Z}$

5.2.2. Definiera nya additions- och multiplikationsoperationer, $\mathcal{A}$ och $\mathcal{M}$ på $\mathbb{R}_+ = \{x \in \mathbb{R}: x > 0\}$ enligt följande: $a, b \in \mathbb{R}_+$, $\lambda \in \mathbb{R}$

$$a\mathcal{A}b = a \cdot b, \qquad \lambda\mathcal{M}a = a^\lambda.$$

Visa att $\mathbb{R}_+$ är ett linjärt rum om "+" och "·" ersätts med $\mathcal{A}$ och $\mathcal{M}$. Vilket blir det nya nollelementet?

log.

$$(\mathbb{R}_+, \mathcal{A}, \mathcal{M}) \xrightarrow{\text{linear is-morphism}} (\mathbb{R}, +, \cdot)$$

$$a \longmapsto \log a$$

$0 < a < 1 \quad \log a < 0$

$a = 1 \quad \log a = 0$

$a > 1 \quad \log a > 0$

$$\lambda\mathcal{M}(a\mathcal{A}b) = \lambda\mathcal{M}(a \cdot b) = (a \cdot b)^\lambda = a^\lambda \cdot b^\lambda = a^\lambda \mathcal{A} b^\lambda = (\lambda\mathcal{M}a)\mathcal{A}(\lambda\mathcal{M}b)$$

$$\boxed{\vec{0}\,\mathcal{A}\,\vec{b} = \vec{b} \quad \forall\, \vec{b} \in \mathbb{R}_+}$$

$$\vec{0} \cdot \vec{b} = \vec{b}$$

(now $\vec{b}$ is a one dim vect. write it as b)

$$\vec{0}\, b = b \quad \forall\, b \in \mathbb{R}_+ \Rightarrow \vec{0} = 1 \in \mathbb{R}_+$$