Contractibility of space of stability conditions on $\mathbb{P}^2$ via global dimension function

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Part I. Motivation

$\mathcal{D}$: a triangulated category. For example,

- $\mathcal{D}(X) = \mathcal{D}^b(\mathrm{Coh}(X))$ for smooth projective variety X over $\mathbb{C}$;
- $\mathcal{D}(Q) = \mathcal{D}^b(\mathrm{Rep}(Q))$ for quiver Q.

$\mathrm{Stab}(\mathcal{D})$: Bridgeland stability manifold on $\mathcal{D}$.
It is a complex manifold.

$\mathrm{Stab}^\dagger(\mathcal{D})$: the principal connected component.

Conjecture

$\mathrm{Stab}^\dagger(\mathcal{D})$ is contractible. More ambitious, $\mathrm{Stab}(\mathcal{D})$ is contractible.

Known cases of the Conjecture for $\mathcal{D} = \mathcal{D}(X)$ or $\mathcal{D}(Q)$

- smooth projective curves [Okada 06, Bridgeland 07, Macrì 07].
- K3 surfaces with Picard rank one [Bayer-Bridgeland 17].
- local $\mathbb{P}^1$ [Ishii-Ueda-Uehara 10]; local $\mathbb{P}^2$ [Bayer-Macrì 11].
- $\mathbb{P}^2$ [Li 17].
- Abelian surfaces [Bridgeland 08] and Abelian threefolds with Picard rank one [Bayer-Macrì-Stellari 16].
- ADE Dynkin quiver [Qiu-Woolf 18] and some new classes of examples [August-Wemyss 19].
- Calabi–Yau-3 affine type A quiver [Qiu 16].
- acyclic triangular quiver [Dimitrov-Katzarkov 16].
- wild Kronecker quiver [Dimitrov-Katzarkov 19].
- ...

The proofs in each case are quite different.

New idea via gldim

Ikeda and the fourth-named author [Qiu 18, Ikeda-Qiu 18] introduce the global dimension function gldim on $\mathrm{Stab}(\mathcal{D})$,

$$\mathrm{gldim}\colon \mathrm{Stab}(\mathcal{D}) \to \mathbb{R}_{\geq 0} \cup \{+\infty\},$$

which sends $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$ to

$$\mathrm{gldim}\,\sigma = \mathrm{gldim}\,\mathcal{P} := \sup\{\phi_2 - \phi_1 \mid \mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) \neq 0\}.$$

Such a function is

- continuous,
- invariant under the natural left action by $\mathrm{Aut}(\mathcal{D})$ and the right action of $\mathbb{C}$,

and thus descends to a continuous function

$$\mathrm{gldim}\colon \mathrm{Aut}(\mathcal{D})\backslash \mathrm{Stab}(\mathcal{D})/\mathbb{C} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

Philosophy: stability manifold contracts along the values of the global dimension function.

(i) The infimum of gldim on $\mathrm{Stab}(\mathcal{D})$ (or the principal component of it) should be considered as the global dimension $\operatorname{gd}\mathcal{D}$ of the category $\mathcal{D}$.

(ii) If the subspace $\mathrm{gldim}^{-1}(\operatorname{gd}\mathcal{D})$ is non-empty, then it is contractible. Moreover, $\mathrm{gldim}^{-1}([\operatorname{gd}\mathcal{D}, x))$ contracts to $\mathrm{gldim}^{-1}(\operatorname{gd}\mathcal{D})$ for any real number $\operatorname{gd}\mathcal{D} < x$.

(iii) When $\mathrm{gldim}^{-1}(\operatorname{gd}\mathcal{D})$ is empty, $\mathrm{gldim}^{-1}(\operatorname{gd}\mathcal{D}, x)$ contracts to $\mathrm{gldim}^{-1}(\operatorname{gd}\mathcal{D}, y)$ for any real number $\operatorname{gd}\mathcal{D} < y < x$.

Note that for a Calabi–Yau category, the global dimension function is constant. If the global dimension function gldim is not constant, it sheds some lights on why $\mathrm{Stab}(\mathcal{D})$ should be contractible.

Main Theorem

The above philosophy is true for the projective plane $\mathbb{P}^2$.

Main Theorem

Consider the global dimension function

$$\mathrm{gldim}\colon \mathrm{Stab}^\dagger(\mathbb{P}^2) \to \mathbb{R}_{\geq 0}$$

on the principal component $\mathrm{Stab}^\dagger(\mathbb{P}^2)$ of the space of stability conditions on the bounded derived category $\mathcal{D}^b(\mathrm{Coh}\,\mathbb{P}^2)$ of coherent sheaves on $\mathbb{P}^2$. Then $\mathrm{gd}\,(\mathcal{D}(\mathbb{P}^2)) = 2$,

- $\mathrm{gldim}\,\mathrm{Stab}^\dagger(\mathbb{P}^2) = [2, \infty)$,
- the subspace $\mathrm{gldim}^{-1}[2, x)$ contracts to $\mathrm{gldim}^{-1}(2)$, for any $x \geq 2$,
- the subspace $\mathrm{gldim}^{-1}(2)$ is contractible and is contained in $\overline{\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)}$, where $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$ consists of geometric stability conditions.

Part II: What is $\mathrm{Stab}(\mathcal{D})$?

Definition

A *slicing* $\mathcal{P}$ is a collect. of subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for $\phi \in \mathbb{R}$ s.t.

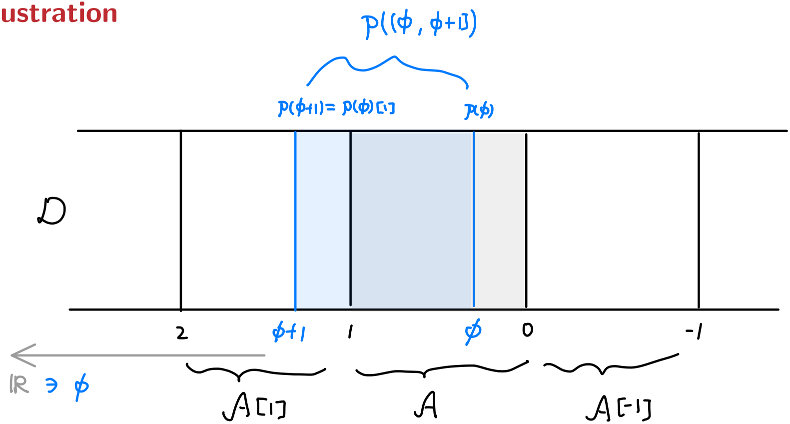
- $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$,
- if $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$, then $\mathrm{Hom}(A_1, A_2) = 0$,
- for all $E \in \mathcal{D}$ there are real numbers $\phi^+(E) := \phi_1 > \ldots > \phi_m =: \phi^-(E)$, and objects $E_i \in \mathcal{D}$ for $i = 1, \ldots, m$, and a collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{m-1} \arrow[r] \arrow[d] & E_m = E \arrow[d] \\
 & A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & A_{m-1} \arrow[ul, dashed] & A_m \arrow[ul, dashed]
\end{tikzcd}$$

where $A_i \in P(\phi_i)$.

Let $\mathcal{A} := \mathcal{P}((0, 1])$ to be the extension closure of the subcategories $\{\mathcal{P}(\phi) : \phi \in (0, 1]\}$, we get the heart of a bounded t-structure. So a slicing is $\mathbb{R}$-indexed refinement of $\mathbb{Z}$-indexed t-structure of $\mathcal{D}$.

Illustration



Definition

A *stability condition* $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of

- a group homomorphism $Z \colon K(\mathcal{D}) \to \mathbb{C}$ (called the *central charge*, where $K(\mathcal{D})$ is the Grothendieck group) and
- a slicing $\mathcal{P}$

satisfying

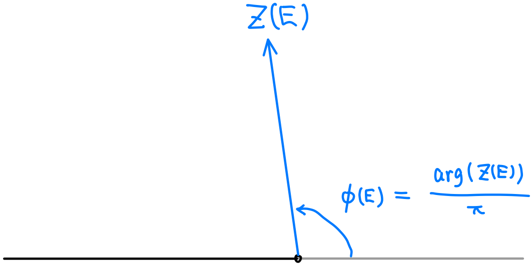
- (compatibility) $\forall 0 \neq E \in \mathcal{P}(\phi) \implies Z(E) \in \mathbb{R}_{>0} \cdot e^{\sqrt{-1}\pi\phi}$
- (support property) $C_\sigma := \inf\left\{ \frac{|Z((E))|}{\|E\|} \,:\, 0 \neq E \in \mathcal{P}(\phi), \phi \in \mathbb{R} \right\} > 0.$

There is an equivalent definition by using $\sigma = (Z, \mathcal{A})$ with $\mathcal{A} = \mathcal{P}((0,1])$. Denote $\phi(E) := \arg(Z(E))/\pi$, then

- (Positivity) $0 \neq E \in \mathcal{A} \implies \phi(E) \in (0,1]$,
- $0 \neq E \in \mathcal{A}$ is Z-semistable if any nonzero $F \subset E$ admits $\phi(F) \leq \phi(E)$,
- for $\phi \in (0,1]$, $\mathcal{P}(\phi) = \langle E \in \mathcal{A}$ is Z-semistable of phase $\phi \rangle$.

Examples

Let C be a smooth projective curve and let $\mathcal{A} := \mathrm{Coh}(C)$ and $Z(E) := -\deg(E) + \sqrt{-1}\,\mathrm{rank}\,(E)$ for $E \in \mathcal{A}$. Then $\sigma = (Z, \mathcal{A})$ is a Bridgeland stability condition.



$$\phi(E) = \frac{\arg(Z(E))}{\pi} \in (0,1] \quad \forall\, 0 \neq E \in \mathrm{Coh}\, C$$

Definition: $\mathrm{Stab}\,(\mathcal{D})$

Let $\mathrm{Stab}\,(\mathcal{D})$ be the *set* of all Bridgeland stability conditions on $\mathcal{D}$. It can be equipped with the *coarsest topology* s.t. for any $E \in \mathcal{D}$, the maps $(Z, \mathcal{P}) \mapsto Z(E)$, $(Z, \mathcal{P}) \mapsto \phi^{+}(E)$ and $(Z, \mathcal{P}) \mapsto \phi^{-}(E)$ are continuous.

Group actions

There are two natural group actions on $\mathrm{Stab}\,(\mathcal{D})$:

- a left action by $\mathrm{Aut}\,(\mathcal{D})$,
- a right action by the universal cover $\widetilde{\mathrm{GL}}^{+}(2, \mathbb{R})$ of $\mathrm{GL}^{+}(2, \mathbb{R})$.

Bridgeland's deformation theorem

The forgetful map $\mathcal{Z}\colon \mathrm{Stab}(\mathcal{D}) \to \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$ given by $(Z, \mathcal{P}) \mapsto Z$ is a local homeomorphism. In particular, assume that $K(\mathcal{D})$ is of finite rank, then $\mathrm{Stab}(X)$ is a complex manifold of dimension $\mathrm{rank}(K(\mathcal{D}))$.

Part III: What is $\mathrm{Stab}(\mathbb{P}^2)$?

It is a complex manifold of dimension 3. We don't know whether it is connected or not.

Conjecture [Li 17]: $\mathrm{Stab}(\mathbb{P}^2)$ is connected.

We know a connected component $\mathrm{Stab}^\dagger(\mathbb{P}^2)$ that containing geometric Bridgeland stability conditions. Recall that a $\sigma \in \mathrm{Stab}(X)$ is called *geometric* if all if all skyscraper sheaves are σ-stable of the same phase. Denote the set of all geometric stability conditions by $\mathrm{Stab}^{\mathrm{Geo}}(X)$.

Theorem [Li 17]: $\mathrm{Stab}^\dagger(\mathbb{P}^2) = \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2) \bigcup \mathrm{Stab}^{\mathrm{Alg}}(\mathbb{P}^2)$.

- We will give the definition of $\mathrm{Stab}^{\mathrm{Alg}}(\mathbb{P}^2)$ soon.
- The final goal is to compute the global dimension function gldim on $\mathrm{Stab}^\dagger(\mathbb{P}^2)$ (see Propositions A & B below) and to show that $\mathrm{Stab}^\dagger(\mathbb{P}^2)$ contracts along the values of gldim.

Geometric stability condition $\sigma_{s,q}$

Reduced Chern characters $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane

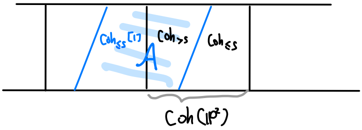
For $E \in \mathcal{D}$, we can identify $\mathrm{ch}_1(E)$ as a number, and we have $\mathrm{ch}(E) = (\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E)) \in \mathbb{R}^3$, and its reduced Chern character in $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. By abuse of notations, we also denote the reduced Chern character by E. For example,

- line bundle $\mathcal{O}(p)$ will be the point $(1, p, \frac{p^2}{2})$,
- the tangent bundle $T_{\mathbb{P}^2}$ will be the point $(1, \frac{3}{2}, \frac{3}{4})$.

Definition

Define $\sigma_{s,q} := (Z_{s,q}, \mathcal{A})$ with $\mathcal{A} = \mathrm{Coh}_{\#s} := \langle \mathrm{Coh}_{\leq s}[1], \mathrm{Coh}_{>s} \rangle$ and

$$Z_{s,q}(E) := (-\mathrm{ch}_2(E) + q \cdot \mathrm{ch}_0(E)) + i(\mathrm{ch}_1(E).H - s \cdot \mathrm{ch}_0(E)).$$



$\mathrm{Coh}_{>s} = \langle E \in \mathrm{Coh}(\mathbb{P}^2) \mid \mu(E) > s \rangle$ $\qquad \mu = \dfrac{\mathrm{ch}_1(-)H}{\mathrm{ch}_0(-)}$

$\mathrm{Coh}_{\leq s} = \langle E \in \mathrm{Coh}(\mathbb{P}^2) \mid \mu(E) \leq s \rangle$

A fractal curve: the Le Potier Curve $\mathrm{C_{LP}}$

Dyadic integers and exceptional bundles

An object $E \in \mathcal{D}$ is called exceptional if $\mathrm{Hom}(E, E[k]) = 0$ for $k \neq 0$; $= \mathbb{C}$ for $k = 0$. There is a one-to-one correspondence between the dyadic integers $\frac{p}{2^m}$ ($p \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$) and exceptional bundles $E(\frac{p}{2^m})$. For example, $E(p) = \mathcal{O}(p)$, $E(\frac{3}{2}) = T_{\mathbb{P}^2}$, $E(\frac{p}{2^m} + 1) = E(\frac{p}{2^m}) \otimes \mathcal{O}(1)$.

Three points E^+, e^l, e^r associated to $E = E(\frac{p}{2^m})$

In the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, we define

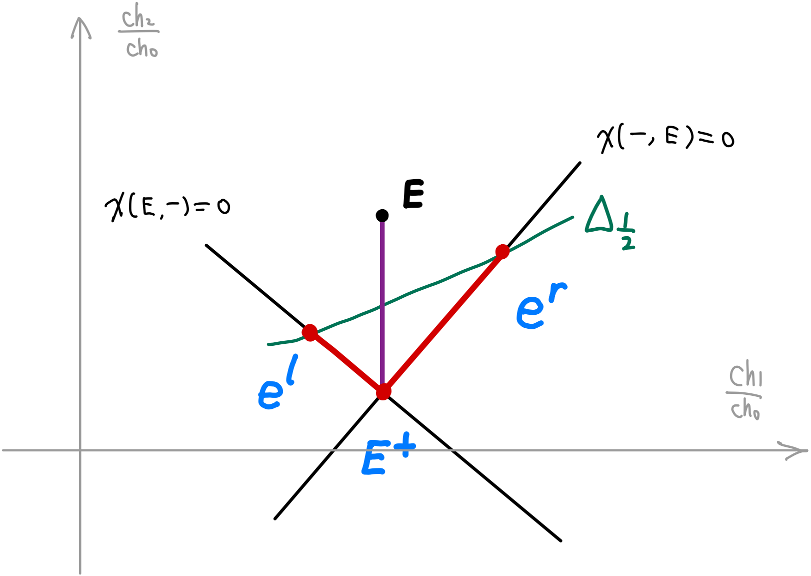
$$E^+ := \{\chi(E, -) = 0\} \cap \{\chi(-, E) = 0\},$$

$$e^l := \{\chi(E, -) = 0\} \cap \Delta_{\frac{1}{2}}, \quad e^r := \{\chi(-, E) = 0\} \cap \Delta_{\frac{1}{2}},$$

where Δ_a is the parabola $\frac{1}{2}\left(\mathrm{ch}_1/\mathrm{ch}_0\right)^2 - \left(\mathrm{ch}_2/\mathrm{ch}_0\right) = a$.

$$\mathrm{C_{LP}} := \coprod_{\{E = E(\frac{p}{2^m}) \mid p \in \mathbb{Z},\, m \in \mathbb{Z}_{\geq 0}\}} \left(\overline{E^+ e^l} \cup \overline{E^+ e^r}\right) \coprod \{\text{Cantor pieces of } \Delta_{\frac{1}{2}}\}.$$

Illustration in reduced char. plane



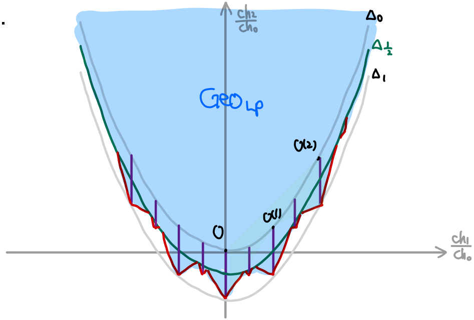
Geometric stability conditions $\mathsf{Stab}^{\mathsf{Geo}}(\mathbb{P}^2)$

Lemma [Bayer-Macrì 11]

The $\mathrm{GL}^+(2,\mathbb{R})$ acts freely on $\mathsf{Stab}^{\mathsf{Geo}}(\mathbb{P}^2)$ with quotient

$$\mathsf{Stab}^{\mathsf{Geo}}(\mathbb{P}^2)/\widetilde{\mathrm{GL}^+(2,\mathbb{R})} \cong \mathsf{Geo}_{\mathrm{LP}},$$

where the region $\mathsf{Geo}_{\mathrm{LP}} := \{(1,s,q) \in \{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}\text{-plane} \mid (1,s,q)$ is above C_{LP} and not on line segment $\overline{EE^+}$ for any exceptional bundle $E\}$.

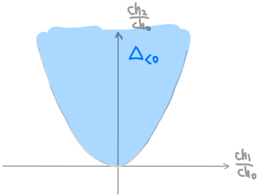


gldim on $\mathrm{Stab}^{\mathrm{Geo}}\,\mathbb{P}^2$

Proposition A

Let $\sigma = \sigma_{s,q}$ be a geometric stability condition in the region $\Delta_{<0}$ on the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. Then

$$\mathrm{gldim}\,\sigma_{s,q} = 2.$$



Proof.

The Serre duality is the isomorphism

$$\mathrm{Hom}(E, F) = \mathrm{Hom}(F, \mathbb{S}(E))^*, \quad \mathbb{S} := (-) \otimes \mathcal{O}_{\mathbb{P}^2}(-3)[2].$$

Now skyscraper sheaf $\mathcal{O}_x$ is σ-stable and $\mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_x[2]) = \mathrm{Hom}(\mathcal{O}_x[2], \mathcal{O}_x[2])^* = \mathbb{C}$. Since $\phi(\mathcal{O}_x[2]) - \phi(\mathcal{O}_x) = 2$, so $\mathrm{gldim}\,\sigma_{s,q} \geq 2$.
We need to show $\mathrm{gldim}\,\sigma_{s,q} \leq 2$. Let E, F be σ-stable with $0 < \phi(E) < \phi(F) - 2 \leq 1$, $\mathrm{ch}_0(E) \neq 0$, and σ is *left* to E, we could show $\mathrm{Hom}(E, F) = 0$. $\square$

Algebraic stability conditions $\mathrm{Stab}^{\mathrm{Alg}}(\mathbb{P}^2)$

Definition

We call an ordered set of exceptional objects $\mathcal{E} = \{E_1, E_2, E_3\}$ *exceptional triple* on $\mathcal{D}^b(\mathbb{P}^2)$ if $\mathcal{E}$ is a full strong exceptional collection of coherent sheaves on $\mathcal{D}^b(\mathbb{P}^2)$ i.e.

$$\mathrm{Hom}(E_i, E_j[k]) = 0 \text{ for any } i > j \text{ and for all } k \in \mathbb{Z};$$

$$\mathrm{Hom}(E_i, E_j[k]) = 0, \text{ for } k \neq 0 \text{ and for all } i, j.$$

The exceptional triples have been classified by Gorodentsev and Rudakov. Up to a cohomological shift, the exceptional triples are labeled by $\{\frac{p-1}{2^m}, \frac{p}{2^m}, \frac{p+1}{2^m}\}$ or their mutations $\{\frac{p}{2^m}, \frac{p+1}{2^m}, \frac{p-1}{2^m} + 3\}$, $\{\frac{p+1}{2^m} - 3, \frac{p-1}{2^m}, \frac{p}{2^m}\}$.

Proposition [Macrì 07]

Let $\mathcal{E}$ be an exceptional triple on $\mathcal{D}^b(\mathbb{P}^2)$. For any positive real numbers m_1, m_2, m_3 and real numbers ϕ_1, ϕ_2, ϕ_3 such that:

$$\phi_1 < \phi_2 < \phi_3, \text{ and } \phi_1 + 1 < \phi_3,$$

there is a unique stability condition $\sigma = (Z, \mathcal{P})$ such that

1. each E_j is stable with phase ϕ_j;
2. $Z(E_j) = m_j e^{i\pi\phi_j}$.

Definition

Given an exceptional triple $\mathcal{E} = \{E_1, E_2, E_3\}$ on $\mathcal{D}^b(\mathbb{P}^2)$, we write $\Theta_{\mathcal{E}}$ as the space of all stability conditions above. We denote

$$\mathrm{Stab}^{\mathrm{Alg}}(\mathbb{P}^2) := \bigcup_{\mathcal{E} \text{ exc triples}} \Theta_{\mathcal{E}}$$

and call the elements of it as the algebraic stability conditions.

Glue: $\Theta_{\mathcal{E}}^{\mathsf{Geo}} := \Theta_{\mathcal{E}} \cap \mathsf{Stab}^{\mathsf{Geo}}(\mathbb{P}^2)$

For an exceptional triple $\mathcal{E} = \{E_1, E_2, E_3\}$, Li [Li 17] shows

$$\Theta_{\mathcal{E}}^{\mathsf{Geo}} = \widetilde{\mathrm{GL}^+(2,\mathbb{R})} \cdot \mathsf{MZ}_{\mathcal{E}}.$$

Define some subsets of $\Theta_{\mathcal{E}}$ as follows.

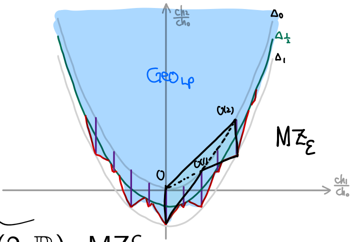
- $\Theta_{\mathcal{E}}^{\mathrm{Pure}} := \{\sigma \in \Theta_{\mathcal{E}} \mid \phi_2 - \phi_1 \geq 1 \text{ and } \phi_3 - \phi_2 \geq 1\}$;
- $\Theta_{\mathcal{E},E_3}^{\mathrm{left}} := \{\sigma \in \Theta_{\mathcal{E}} \mid \phi_2 - \phi_1 < 1 \text{ and } E_3(3) \text{ is not } \sigma\text{-stable}\}$;
- $\Theta_{\mathcal{E},E_1}^{\mathrm{right}} := \{\sigma \in \Theta_{\mathcal{E}} \mid \phi_3 - \phi_2 < 1 \text{ and } E_1(-3) \text{ is not } \sigma\text{-stable}\}$.

Lemma

Given exceptional triples $\mathcal{E}$ and $\mathcal{E}'$ on $\mathcal{D}^b(\mathbb{P}^2)$ with the same $E_3 = E_3' = E$, then $\Theta_{\mathcal{E},E_3}^{\mathrm{left}} = \Theta_{\mathcal{E}',E_3'}^{\mathrm{left}}$. We denote *this subspace* by $\Theta_{E_3}^{\mathrm{left}}$.

In a similar way, we define the subspace $\Theta_{E_1}^{\mathrm{right}} := \Theta_{\mathcal{E},E_1}^{\mathrm{right}}$.

Similarly, denote $\Theta^-_{E_3} = \Theta^-_{\mathcal{E},E_3} = \Theta_{\mathcal{E}}(\phi_2 - \phi_1 < 1) \setminus \Theta^{\mathrm{Geo}}_{\mathcal{E}}$;
$\Theta^+_{E_1} = \Theta^+_{\mathcal{E},E_1} = \Theta_{\mathcal{E}}(\phi_3 - \phi_2 < 1) \setminus \Theta^{\mathrm{Geo}}_{\mathcal{E}}$.

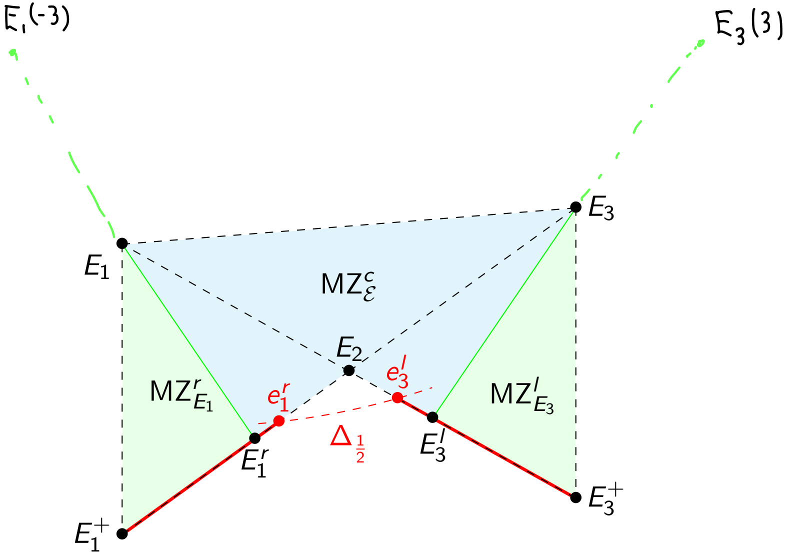


Lemma.

- $\Theta^{\mathrm{left}}_{E_3} = \Theta^-_{E_3} \coprod \widetilde{\mathrm{GL}^+(2,\mathbb{R})} \cdot \mathsf{MZ}^l_{E_3}$,
- $\Theta^{\mathrm{right}}_{E_1} = \Theta^+_{E_1} \coprod \widetilde{\mathrm{GL}^+(2,\mathbb{R})} \cdot \mathsf{MZ}^r_{E_1}$,
- $\Theta_{\mathcal{E}} \setminus (\Theta^{\mathrm{right}}_{E_1} \cup \Theta^{\mathrm{left}}_{E_3} \cup \Theta^{\mathrm{Pure}}_{\mathcal{E}}) = \widetilde{\mathrm{GL}^+(2,\mathbb{R})} \cdot \mathsf{MZ}^c_{\mathcal{E}}$,
- $\mathsf{MZ}_{\mathcal{E}} = \mathsf{MZ}^r_{E_1} \coprod \mathsf{MZ}^c_{\mathcal{E}} \coprod \mathsf{MZ}^l_{E_3}$.

Picture

line $\overline{E_2E_3} = \{\chi(-, E_1) = 0\}$



gldim on $\Theta_{\mathcal{E}}$

The Serre functor on $\mathcal{D}(\mathbb{P}^2)$ is given by $\mathbb{S} := (-) \otimes \mathcal{O}_{\mathbb{P}^2}(-3)[2]$. The right and left mutations of an object F with respect to an exceptional object E are defined by

$$\begin{aligned} \mathsf{R}_E(F) &:= \operatorname{Cone}\left(F \xrightarrow{\mathrm{ev}} E \otimes \operatorname{Hom}(F,E)^*\right)[-1], \\ \mathsf{L}_E(F) &:= \operatorname{Cone}\left(E \otimes \operatorname{Hom}(E,F) \xrightarrow{\mathrm{ev}} F\right). \end{aligned}$$

Proposition B

The value of the global dimension function $\operatorname{gldim}(\sigma) =$

$$\begin{cases} 2, & \text{when } \sigma \in \Theta_{\mathcal{E}} \setminus \left(\Theta_{E_1}^{\text{right}} \cup \Theta_{E_3}^{\text{left}} \cup \Theta_{\mathcal{E}}^{\text{Pure}}\right); \\ \phi(\mathsf{R}_{E_1}(\mathbb{S}E_1)) - \phi_1, & \text{when } \sigma \in \Theta_{E_1}^{\text{right}}; \\ \phi_3 - \phi(\mathsf{L}_{E_3}(\mathbb{S}^{-1}E_3)), & \text{when } \sigma \in \Theta_{E_3}^{\text{left}}; \\ \phi_3 - \phi_1, & \text{when } \sigma \in \Theta_{\mathcal{E}}^{\text{Pure}}. \end{cases}$$

$$\text{Proposition A} + \text{Proposition B} \implies \text{Main Theorem}.$$

Thank you!