Let (1) $\lim_{x\to x_0} f(x) = A$ (2) $\lim_{x\to x_0} g(x) = B$ and $B \neq 0$.

https://wanminliu.github.io/KTH/

Show that (3) $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$.

Proof. We want use the definition of limit to show (3).

i.e. $\forall \varepsilon > 0$, we need to find δ (depending on ε)

so that for all x with $0 < |x - x_0| < \delta$,

we have $\left| \frac{f(x)}{g(x)} - \frac{A}{B} \right| < \varepsilon$. (☆)

We find such δ by using definition of limit for (1) and (2)

$$(1) \Leftrightarrow \left[\begin{array}{l} \forall \varepsilon_1 > 0, \ \exists\ \delta_1 = \delta_1(\varepsilon_1) \text{ s.t.} \\ |f(x) - A| < \varepsilon_1 \text{ for all } x: 0 < |x - x_0| < \delta_1 \end{array} \right] \quad (4)$$

$$(2) \Leftrightarrow \left[\begin{array}{l} \forall \varepsilon_2 > 0 \ \exists\ \delta_2 = \delta_2(\varepsilon_2) \text{ s.t.} \\ |g(x) - B| < \varepsilon_2 \text{ for all } x: 0 < |x - x_0| < \delta_2 \end{array} \right] \quad (5)$$

Now

$$\left| \frac{f(x)}{g(x)} - \frac{A}{B} \right| = \left| \frac{f(x)}{g(x)} - \frac{A}{g(x)} + \frac{A}{g(x)} - \frac{A}{B} \right|$$

$$\leq \left| \frac{f(x)}{g(x)} - \frac{A}{g(x)} \right| + \left| \frac{A}{g(x)} - \frac{A}{B} \right|$$

$$= \frac{1}{|g(x)|} |f(x) - A| + \frac{|A|}{|g(x)||B|} |g(x) - B| \quad (6)$$

Since $|g(x)|$ appears in the denominator of formulars in (6), we want to give a bound of $g(x)$.

Using (5)

By ~~(5)~~: we take $\varepsilon_2 = \frac{|B|}{2}$

then $\exists$ $\delta_2(\frac{|B|}{2})$, we call it δ_3 now,

so that $|g(x) - B| < \frac{|B|}{2}$ for all x : $0 < |x - x_0| < \delta_3$

~~The~~ We always have the triangle inequality

$$\big||g(x)| - |B|\big| \leqslant |g(x) - B|$$

so $$\big||g(x)| - |B|\big| \leqslant |g(x) - B| < \frac{|B|}{2}$$

i.e. $$-\frac{|B|}{2} < |g(x)| - |B| < \frac{|B|}{2}$$

i.e. $$\frac{|B|}{2} < |g(x)| < \frac{3}{2}|B|$$

Since $B \neq 0$ we know

$$\frac{2}{3|B|} < \frac{1}{|g(x)|} < \frac{2}{|B|} \qquad (7)$$

We use (7) for the inequality (6)

So $$\frac{1}{|g(x)|}|f(x) - A| + \frac{|A|}{|g(x)||B|}|g(x) - B|$$

$$< \frac{2}{|B|}|f(x) - A| + \frac{|A| \cdot 2}{|B|^2}|g(x) - B|$$

$$< \frac{2}{|B|}|f(x) - A| + \frac{2|A| + 1}{|B|^2}|g(x) - B| \qquad (8)$$

Now we show ⊛

$\forall \varepsilon > 0$,

By using (1) (or (4)), and taking

$$\varepsilon_1 = \frac{\varepsilon}{2} \cdot \frac{2}{|B|}$$

we find

$\delta_1(\varepsilon_1)$ — we call it δ_1

so that

(9) $|f(x) - A| < \varepsilon_1 = \frac{\varepsilon}{2} \cdot \frac{2}{|B|}$ for all x: $0<|x-x_0|<\delta_1$

By using (5) again, and taking

$$\varepsilon_2 = \frac{\varepsilon}{2} \cdot \frac{|B|^2}{2|A|+1}$$

we find $\delta_2(\varepsilon_2)$ — we call it δ_2

so that

(10) $|g(x) - B| < \varepsilon_2 = \frac{\varepsilon}{2} \frac{|B|^2}{2|A|+1}$ for all x: $0<|x-x_0|<\delta_2$

Now we take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$

Then for all x with $0<|x-x_0|<\delta$

we have

$$\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| \overset{(8)}{<} \frac{2}{|B|}\,|f(x)-A| + \frac{2|A|+1}{|B|^2}\,|g(x)-B|$$

$$\overset{(9)\text{ and }(10)}{<} \frac{2}{|B|}\cdot\frac{\varepsilon}{2}\cdot\frac{|B|}{2} + \frac{2|A|+1}{|B|^2}\cdot\frac{\varepsilon}{2}\cdot\frac{|B|^2}{2|A|+1}$$

$$= \varepsilon.$$

So we checked (✱), the definition of limit in (3)

Therefore (3) holds. $\square$

Remark ① The above is a rigious proof, where we use (5) twice — one time for the control the bound of $\frac{1}{|g(x)|}$

— the other is for $|g(x)-B|$

② The essential part is inequality (6), where on the right hand side, we know

$\frac{1}{|g(x)|}$ — bounded. $\frac{|A|}{|g(x)|\cdot|B|}$ — bounded

$|f(x)-A|$ — small $|g(x)-B|$ — small

③ The trick in (8) "$2|A|+1$" is because it is NOT zero so we can put it into ε_2.

You can use the number you prefer for the computation, for example, 2|A|+2023