CGP DERIVED SEMINAR

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1. July 4: Wanmin Liu (Maps in the Homotopy Category of dg-Categories, part II)

Outline:

- recall the **Proposition 4.1.2** which links the dg-side (morphisms) to model-side (objects) (introduced by ***Tae-Su*** and proved by ***Gabriel*** on June 27).
- an exercise by using the Proposition.
- a corollary of the Proposition.

Recall that for any dg-category T, we also denote a $T-Mod$ by $(C(k))^T$.

Proposition 1.1. [To11, Prop 4.1.2] *Let T be any dg-category and M be a $C(k)$-model category satisfying technical conditions:*

(1) *M is cofibrantly generated, and the domain and codomain of the generating cofibrations are cofibrant objects in M.*
(2) *For any cofibrant object X in M, and any quasi-isomorphism $E \longrightarrow E'$ in $C(k)$, the induced morphism $E \otimes X \longrightarrow E' \otimes X$ is an equivalence.*
(3) *Infinite sums preserve weak equivalences in M.*

Then there exists a natural bijection of sets

$$[T, Int(M)] \simeq Iso(Ho(M^T))$$

between the set of morphisms from T to $Int(M)$ in $Ho(dg-cat)$ and the set of isomorphism classes of objects in $Ho(M^T)$.

Exercise 1.1. [To11, Exercise 4.1.6 or Exercise 21 in the preprint version] Let R be an associative and unital k-algebra, which is also considered as dg-category with a unique object and R as endomorphisms of this object. Show that there is a natural bijection between $[R, Int(C(k))]$ and the set of isomorphism classes of the derived category $D(R)$.

Proof. Let us recall some notions. We denote the associated dg-category of the R by BR (notations after [To11, Definition 3.2.3] or the preprint version page 31). More precisely, $Obj(BR)$ is the unique object, denoted by $*$. To make it as a dg-category, for any "two" object(s) $*$, and $*$, we need to define a chain complex of k-module. The natural choice is

$$BR(*, *) := R \in C(k), \quad i.e. \cdots \to 0 \to R \to 0 \to \cdots \in C(k),$$

where R is in the 0-th degree.

So $[R, Int(C(k))]$ really means $[BR, Int(C(k))]$ as in the category $Ho(dg{-}cat)$, where the dg-category structure on $Int(C(k))$ is induced from the dg-category $\underline{C(k)}$. More precisely, it is the full sub-dg-category of $\underline{C(k)}$ consisting of fibrant and cofibrant objects in $C(k)$, where you need to recall the standard model category structure on $C(k)$.

To use the Proposition 1.1 (Prop 1 of page 38 in preprint), we need to *assume* that R is flat over k (see also Exercise 4.1.1).

Now the Proposition gives us

$$[BR, Int(C(k)] \simeq Iso(Ho(C(k)^{BR})).$$

By the Definition 3.2.3, $D(BR) := Ho(BR - Mod) = Ho(C(k)^{BR})$.

To finish the proof, we only need to check

$$D(BR) \simeq D(R).$$

Then it follows that

$$[BR, Int(C(k)] \simeq Iso(Ho(C(k)^{BR})) = Iso(D(BR)) \simeq Iso(D(R)).$$

Now let us check the equivalence relation $D(BR) \simeq D(R)$.

What is $D(R)$? It is defined as $S^{-1}C(R)$, the localizing of $C(R)$ along S, where $C(R)$ is the category of chain complex of left R-modules, and S is the set of quasi-isomorphisms. (See ***Yoosik***'s lecture on Apr 4.)

What is $D(BR)$? It is defined as $W^{-1}(BR - Mod)$, the localizing of $BR - Mod$ along W, where $BR - Mod$ is the model category, with the model structure induced by the dg-structure of BR, and W is the set of weak equivalences, which is defined to be the set S of quasi-isomophisms. (See ***Morimichi***'s lecture on June 20.)

Claim. By forgetting the model structure on $BR - Mod$, we have an equivalence of categories $BR - Mod \simeq C(R)$.

Let us check in the object level. An object in $BR - Mod$ is a dg-functor $F : BR \longrightarrow \underline{C(k)}$, which maps the unique object $*$ to $F_* \in C(k)$, together with morphism in $\underline{C(k)}$

$$F_* \otimes_k BR(*,*) \longrightarrow F_*,$$

and for any "pair" of objects $(*,*) \in Obj(BR)^2$ a morphism in $C(k)$,

$$F_{*,*} : BR(*,*) \longrightarrow \underline{C(k)}(F_*, F_*) = \underline{Hom}^*(F_*, F_*),$$

satisfying the usual associativity (here we need R to be an associative k-algebra):

$$\begin{tikzcd} BR(*,*) \otimes_k BR(*,*) \arrow[r, "\mu_{*,*,*}"] \arrow[d, "F_{*,*}\otimes F_{*,*}"'] & BR(*,*) \arrow[d, "F_{*,*}"] \\ \underline{Hom}^*(F_*, F_*) \otimes \underline{Hom}^*(F_*, F_*) \arrow[r, "\mu'_{*,*,*}"] & \underline{Hom}^*(F_*, F_*), \end{tikzcd}$$

and unit conditions (here we need R to be a unital k-algebra):

$$\begin{tikzcd} k \arrow[r, "e_*"] \arrow[dr, "e'_{F_*}"'] & BR(*,*) \arrow[d, "F_{*,*}"] \\ & \underline{Hom}^*(F_*, F_*). \end{tikzcd}$$

We therefore associate each object $F \in Obj(BR - Mod)$ a complex of R-modules $F_* \otimes_k BR(*,*) \in Obj(C(R))$.

Let us check in the morphisms level. A morphism in $BR - Mod$ is a natural transformation between dg-functors $\alpha : F \longrightarrow F'$, which maps $F_* \longrightarrow F'_*$ with the commutative diagram

$$\begin{tikzcd} F_* \otimes_k BR(*,*) \arrow[r] \arrow[d, "\alpha_* \otimes_k BR(*,*)"'] & F_* \arrow[d, "\alpha_*"] \\ F'_* \otimes_k BR(*,*) \arrow[r] & F'_*. \end{tikzcd}$$

We therefore associate each morphism $\alpha \in Mor(BR - Mod)$ a morphism $\alpha_* \otimes_k BR(*,*) \in Mor(C(R))$. Moreover, α is defined to be a weak equivalence in $BR - Mod$ (i.e. $\alpha \in W$) if $\alpha_* \otimes_k BR(*,*)$ is a quasi-isomorphism in $C(R)$ (i.e. $\alpha_* \otimes_k BR(*,*) \in S$).

Therefore by localizing the equivalence of categories $BR - Mod \simeq C(R)$ along the same class of quasi-isomorphisms, we obtain the equivalence of categories $D(BR) \simeq D(R)$.

□

Corollary 1.1. [To11, Page 279 Corollary 1 or page 37 of preprint version] *Let T and T' be two dg-categories, one of them having cofibrant complexes of morphisms. Then, there exists a natural bijection between $[T, T']$ and the subset of $Iso(Ho(T \otimes (T')^{op} - Mod))$ consisting of $T \otimes (T')^{op}$-dg-modules F such that for any $x \in T$, there is $y \in T'$ such that $F_{x,-}$ and $\underline{h}_y$ are isomorphic in $Ho((T')^{op} - Mod)$.*

Proof. Let us denote the subset in the corollary by N. Let us denote the model category $(T')^{op} - Mod$ by M. We need to show the following commutative diagram:

$$\begin{tikzcd} {[T, T']} \arrow[r, dash, "\simeq"] \arrow[d, hook] & N \arrow[d, hook] \\ {[T, Int(M)]} \arrow[r, dash, "\simeq"] & Iso(Ho(T \otimes (T')^{op} - Mod)), \end{tikzcd}$$

i.e. we need to check that those conditions on N as a subset of $Iso(Ho(T \otimes (T')^{op} - Mod))$ exactly correspond to the left vertical inclusion, and the correspondence from left to right is given by the Proposition 4.1.2.

Firstly, let us check the bottom equivalence in the diagram. Recall notations

$$M = (C(k))^{(T')^{op}}, \quad T \otimes (T')^{op} - Mod = (C(k))^{T \otimes (T')^{op}}.$$

In ***Youngjin***'s lecture (Exercise 3.2.6), we know the equivalence of model categories

$$((C(k))^{(T')^{op}})^T \simeq (C(k))^{T \otimes (T')^{op}}.$$

The bottom equivalence follows by the Proposition 1.1.

Secondly, let us check the left inclusion in the diagram. Recall that the $C(k)$-enriched Yoneda embedding is a quasi-fully faithful dg-functor $\underline{h}$ (in ***Tae-Su***'s lecture on June 27):

$$\underline{h} : T' \longrightarrow Int(M),$$

sending an objects $y \in T'$ to an object $\underline{h}_y$ in the interior of M. Here

$$\underline{h}_y : (T')^{op} \longrightarrow \underline{C(k)}, \quad z \mapsto T'(z, y).$$

The interior $Int(M)$ is defined as the full subcategory of M consisting of fibrant and cofibrant objects of M. The quasi-fully faithful dg-functor means that the induced morphism of complexes of k-modules

$$\forall x, y \in T', \quad \underline{h}_{x,y} : T'(x, y) \longrightarrow \underline{Hom}(\underline{h}_x, \underline{h}_y) = Int(M)(\underline{h}_x, \underline{h}_y)$$

is an isomorphis. And the complex $\underline{Hom}(\underline{h}_x, \underline{h}_y) \in C(k)$ is defined from the $C(k)$-model structure on M by

$$Hom_{C(k)}(E, \underline{Hom}(\underline{h}_x, \underline{h}_y)) \simeq Hom_M(E \otimes \underline{h}_x, \underline{h}_y), \quad \forall E \in C(k).$$

Recall that $[T, T']$ and $[T, Int(M)]$ are morphisms in the category $Ho(dg\text{-}cat)$. We need to work in the category $dg\text{-}cat$ first. By *Tabuada's Theorem* (in ***Yong-Geun***'s lecture on June 12), there is a model category structure on $dg - cat$. By Lemma 4.1.3, we can assume that T is a cofibrant dg-category. Recall that all objects in the category $dg - cat$ are fibrant. By restriction to the sub-category $(dg - cat)^{cf}$ of cofibrant and fibrant objects, we hence get an equivalence of categories (here $\sim$ is the homotopic equivalence):

$$(dg - cat)^{cf} / \sim \longrightarrow Ho(dg - cat).$$

In the $(dg - cat)^{cf}$, by composing $\underline{h}$ we obtain

$$Hom_{(dg-cat)^{cf}}(T, T') \hookrightarrow Hom_{(dg-cat)^{cf}}(T, Int(M)),$$

which descends to (recall the notation $[T, T'] := Hom_{Ho(dg-cat)}(T, T')$)

$$[T, T'] \hookrightarrow [T, Int(M)],$$

whose image consists of morphisms in $Hom_{(dg-cat)^{cf}}(T, Int(M))$ factorizing in $Ho(dg - cat)$ through the quasi-essential image of $\underline{h}$.

Finally, let check the diagram is commutative, i.e. for an object F in the right-bottom and assume it comes from top-left by the left inclusion and bottom equivalence, then F is the form as in the definition of N. Let $F \in Obj(T \otimes (T')^{op} - Mod)$, i.e. a dg-functor

$$F : T \otimes (T')^{op} \longrightarrow \underline{C(k)}.$$

By the bottom equivalence of the diagram, we regard $F \in [T, Int(M)]$, i.e. $F = \tilde{F}/\sim$ for some lift $\tilde{F} \in Hom_{(dg-cat)^{cf}}(T, Int(M))$. To check the top equivalence of the diagram, we further regard $\tilde{F}$ in the essential image of $\underline{h}$. Therefore,

$$\tilde{F} \simeq \underline{h} \circ g, \quad \text{for some } g \in Hom_{(dg-cat)^{cf}}(T, T')$$

For any $x \in T$, we just take

$$y := g(x) \in T'.$$

Hence

$$\tilde{F}(x) \simeq \underline{h}(y) = \underline{h}_y \in M.$$

Since the lift $\tilde{F}$ is upto homotopy, we obtain $F_{x,-} \sim \tilde{F}(x) \in M$ and hence

$$F_{x,-} \simeq \underline{h}_y \in Ho(M).$$

□

References

[To11] B. Toën, *Lectures on DG-Categories*, Topics in Algebraic and Topological K-Theory, edited by P.F. Baum et al., Lecture Notes in Math. 2008, Springer-Verlag Berlin Heidelberg, 2011.