DeepSeek Chat

Solve the ODE y''(x)-8*y'(x)+16*y(x)=x*exp^{4x}, where y is a function of variable x.

To solve the second-order linear ordinary differential equation (ODE):

$$y''(x) - 8y'(x) + 16y(x) = xe^{4x},$$

we will follow these steps:

1. **Find the complementary solution $y_c(x)$ of the homogeneous equation.**
2. **Find a particular solution $y_p(x)$ of the nonhomogeneous equation.**
3. **Combine the complementary and particular solutions to get the general solution.**

Step 1: Complementary Solution $y_c(x)$

The homogeneous equation is:

$$y''(x) - 8y'(x) + 16y(x) = 0.$$

The characteristic equation is:

$$r^2 - 8r + 16 = 0.$$

Solve for r:

$$(r-4)^2 = 0 \implies r = 4 \text{ (double root)}.$$

Since there is a repeated root $r = 4$, the complementary solution is:

$$y_c(x) = (C_1 + C_2 x)e^{4x},$$

where C_1 and C_2 are constants.

Step 2: Particular Solution $y_p(x)$

The nonhomogeneous term is xe^{4x}. Since e^{4x} is already part of the complementary solution, we multiply by x^2 to avoid duplication. Assume a particular solution of the form:

$$y_p(x) = (Ax^3 + Bx^2)e^{4x},$$

where A and B are constants to be determined.

Compute $y_p'(x)$ and $y_p''(x)$:

1. First derivative :

$$y_p'(x) = \frac{d}{dx}\left((Ax^3 + Bx^2)e^{4x}\right).$$

Using the product rule:

$$y_p'(x) = (3Ax^2 + 2Bx)e^{4x} + (Ax^3 + Bx^2)(4e^{4x}).$$

Simplify :

$$y_p'(x) = (3Ax^2 + 2Bx + 4Ax^3 + 4Bx^2)e^{4x}.$$

2. Second derivative :

$$y_p''(x) = \frac{d}{dx}\left(y_p'(x)\right).$$

Again, using the product rule:

$$y_p''(x) = (6Ax + 2B + 12Ax^2 + 8Bx)e^{4x} + (3Ax^2 + 2Bx + 4Ax^3 + 4Bx^2)(4e^{4x}).$$

Simplify :

$$y_p''(x) = (6Ax + 2B + 12Ax^2 + 8Bx + 12Ax^2 + 8Bx + 16Ax^3 + 16Bx^2)e^{4x}.$$

Combine like terms:

$$y_p''(x) = (16Ax^3 + (12A + 12A + 16B)x^2 + (6A + 8B + 8B)x + 2B)e^{4x}.$$

Simplify further :

$$y_p''(x) = (16Ax^3 + (24A + 16B)x^2 + (6A + 16B)x + 2B)e^{4x}.$$

Substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into the ODE:

The ODE is:

$$y''(x) - 8y'(x) + 16y(x) = xe^{4x}.$$

Substitute :

$$(16Ax^3 + (24A + 16B)x^2 + (6A + 16B)x + 2B)e^{4x} - 8\left((3Ax^2 + 2Bx + 4Ax^3 + 4Bx^2)e^{4x}\right) + 16\left((Ax^3 + Bx^2)e^{4x}\right) = xe^{4x}.$$

Simplify :

$$(16Ax^3 + 24Ax^2 + 16Bx^2 + 6Ax + 16Bx + 2B)e^{4x} - 8(4Ax^3 + (3A + 4B)x^2 + 2Bx)e^{4x} + 16(Ax^3 + Bx^2)e^{4x} = xe^{4x}.$$

Expand and combine like terms:

$$(16Ax^3 + 24Ax^2 + 16Bx^2 + 6Ax + 16Bx + 2B - 32Ax^3 - 24Ax^2 - 32Bx^2 - 16Bx + 16Ax^3 + 16Bx^2)e^{4x} = xe^{4x}.$$

Simplify further:

$$((16A - 32A + 16A)x^3 + (24A - 24A + 16B - 32B + 16B)x^2 + (6A - 16B + 16B)x + 2B)e^{4x} = xe^{4x}.$$

This reduces to:

$$(0x^3 + 0x^2 + 6Ax + 2B)e^{4x} = xe^{4x}.$$

Thus:

$$6Ax + 2B = x.$$

Equate coefficients:

1. For x:

$$6A = 1 \implies A = \frac{1}{6}.$$

2. For the constant term:

$$2B = 0 \implies B = 0.$$

Particular solution:

$$y_p(x) = \left(\frac{1}{6}x^3 + 0 \cdot x^2\right)e^{4x} = \frac{1}{6}x^3e^{4x}.$$

Step 3: General Solution

Combine the complementary and particular solutions:

$$y(x) = y_c(x) + y_p(x).$$

Thus:

$$y(x) = (C_1 + C_2x)e^{4x} + \frac{1}{6}x^3e^{4x}.$$

Final Answer:

$$y(x) = (C_1 + C_2x)e^{4x} + \frac{1}{6}x^3e^{4x},$$

where C_1 and C_2 are constants .