ODE, SAGEMATH AND GENERAL AI

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Abstract

In this notes, we solve an ODE by mathematical logic, and by using the mathematical software SageMath. We also test the ODE on different types of the General AI and they cannot solve this particular ODE. Through this example, we can appreciate the rigor and beauty of mathematics, and we can also know the power of the mathematics software SageMath and the limitations of general artificial intelligence.

1 THE ODE PROBLEM AND MATHMATICAL SOLVING.

The ODE Problem. Solve the ODE

$$y''(x) - 8y'(x) + 16y(x) = xe^{4x} \tag{1.1}$$

with the initial conditions $y'(0) = 1$ and $y'(1) = e^4$.

Remark 1.1. If we solve the corresponding homogeneous equation of (1.1), we could find that the right hand side of (1.1) is precisely a solution of the homogeneous equation. So if we use the general AI, such as ChatGPT, it will not solve this ODE.

Solution. We begin with the homogeneous equation

$$y''(x) - 8y'(x) + 16y(x) = 0. \tag{1.2}$$

The characteristic equation is given by

$$r^2 - 8r + 16 = 0.$$

The solution is given by $r = 4$ with multiplicity 2. So the solution of (1.2) is given by

$$y_c(x) = e^{4x}(Ax + B). \tag{1.3}$$

However the right side of (1.1) is of the special type of a solution for the homogeneous equation (1.2).

We must use new idea. The method is to let the solution be of the form

$$y(x) = e^{4x}g(x), \tag{1.4}$$

and we can convert the original ODE to a new ODE of the function $g(x)$.

By the chain rule and (1.4), we have

$$y'(x) = e^{4x}\left(4g(x) + g'(x)\right), \tag{1.5}$$

and

$$y''(x) = e^{4x}\left(16g(x) + 8g'(x) + g''(x)\right). \tag{1.6}$$

Put (1.6) and (1.5) into left side of (1.1), we obtain

$$\begin{aligned} &y''(x) - 8y'(x) + 16y(x) \\ &= e^{4x}\left(16g(x) + 8g'(x) + g''(x) - 32g(x) - 8g'(x) + 16g(x)\right) \\ &= e^{4x}g''(x). \end{aligned}$$

So by the (1.1), we obtain

$$g''(x) = x. \tag{1.7}$$

It follows that $g(x) = \frac{x^3}{6} + c_1x + c_2$. Therefore a special solution of the inhomogeneous (1.1) is given by (we can take $c_1 = c_2 = 0$)

$$y_p(x) = e^{4x}\frac{x^3}{6},$$

and the general solution of the (1.1) is of the form

$$y(x) = y_c(x) + y_p(x) = e^{4x}\left(\frac{x^3}{6} + Ax + B\right), \tag{1.8}$$

where A and B are some constants. By using the chain rule and taking derivative, we obtain

$$y'(x) = e^{4x}\left(4\frac{x^3}{6} + 4Ax + 4B + \frac{x^2}{2} + A\right). \tag{1.9}$$

So

$$y'(0) = (4B + A), \quad y'(1) = e^4\left(\frac{4}{6} + 4A + 4B + \frac{1}{2} + A\right). \tag{1.10}$$

By comparing with the initial conditions $y'(0) = 1$ and $y'(1) = e^4$, we obtain

$$4B + A = 1, \tag{1.11}$$

$$\frac{4}{6} + 4A + 4B + \frac{1}{2} + A = 1. \tag{1.12}$$

So

$$\frac{4}{6} + 4A + \frac{1}{2} = 0.$$

It follows that

$$A = -\frac{7}{24}, \quad B = \frac{31}{96}.$$

We conclude that the solution of the (1.1) with the given initial conditions is

$$y(x) = e^{4x}\left(\frac{x^3}{6} - \frac{7x}{24} + \frac{31}{96}\right). \tag{1.13}$$

2 SAGEMATH: AN APPLICATION

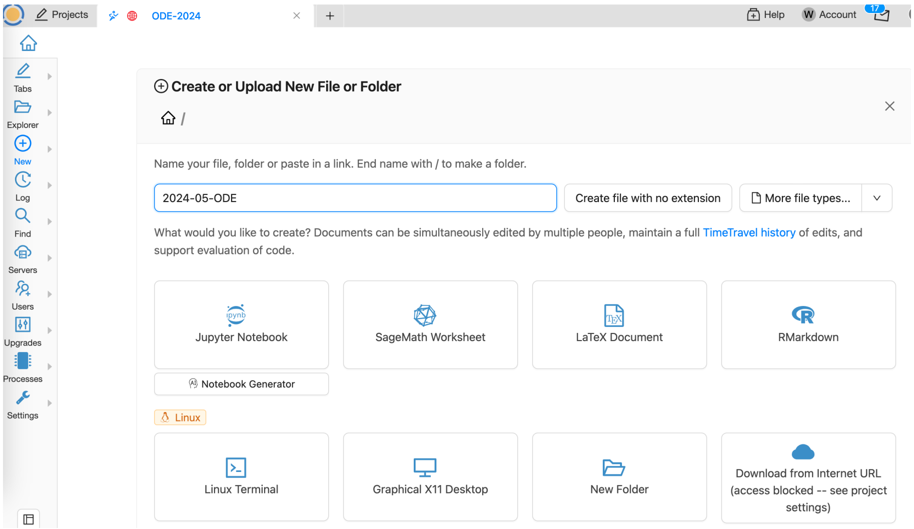
SageMath is a free open-source mathematics software system. The homepage is `https://www.sagemath.org/`.

One can install it or use it online Sage on CoCalc.

We use it to solve the ODE (1.1). For simplicity, we use the cloud version of SageMath on CoCalc.

Step 1. Visit `https://cocalc.com/features/sage` and sign up for a free CoCalc account.

Step 2. After signing in, create a project `https://cocalc.com/projects`. We call the project **ODE-2024**. Then we can create a SageMath Worksheet.



Step 3. In the SageMath Worksheet, we write the following code.

```
x = var('x')
y = function('y')(x)
de = diff(y,x,2) - 8*diff(y,x) + 16*y == x*exp(4*x)
desolve(de, y)
```

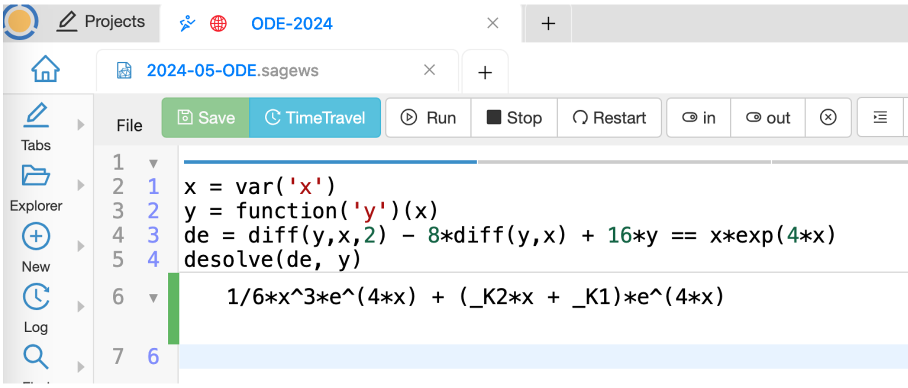
The first line defines the variable x. The second line defines y as a function of x. The third line defines the ODE, which is (1.1). The forth line solves the ODE by using the built-in function `desolve()`

More information of solving ordinary differential equations can be found at `https://doc.sagemath.org/html/en/reference/calculus/sage/calculus/desolvers.html`

Step 4. We can run the code now by clicking button **Run**. It gives the solution of the general form, which is precisely (1.8):

```
1/6*x^3*e^(4*x) + (_K2*x + _K1)*e^(4*x)
```

So the SageMath solves the ODE and finds the general solution.



3 GENERAL AI.

The ODE is a good example to show that the General AI cannot solve the mathematical problem, or at least, have many limitations. We have tested several General AI, such as ChatGPT, Copilot. It is a good exercise to check the result from the ChatGPT is wrong in the next page, as stated that:

ChatGPT can make mistakes. Check important info.

One reason is that the General AI is based on large language model and statistic, but not based on mathematical logic.

4 CONCLUSION.

Through this example, we can appreciate the rigor and beauty of mathematics, and we can also know the power of the mathematics software SageMath and the limitations of general artificial intelligence.

ChatGPT 3.5

Solve the ODE y''(x)-8*y'(x)+16*y(x)=x*exp^{4x}, where y is a function of variable x.

To solve the given ordinary differential equation (ODE), we'll first find the complementary solution, then the particular solution.

The characteristic equation corresponding to the homogeneous part of the ODE is:
$r^2 - 8r + 16 = 0$.

Solving this quadratic equation, we get a repeated root $r = 4$. Therefore, the complementary solution is:
$y_c(x) = (c_1 + c_2 x)e^{4x}$.

Now, for the particular solution, we can use the method of undetermined coefficients. We assume the particular solution has the form $y_p(x) = Ax^2e^{4x} + Bxe^{4x} + Ce^{4x}$.

Differentiating $y_p(x)$ twice and substituting into the original ODE, we find the values of A, B, and C that satisfy the equation.

Let's proceed with finding A, B, and C:
$y_p(x) = Ax^2e^{4x} + Bxe^{4x} + Ce^{4x}$
$y_p'(x) = (2Ax + B)e^{4x} + 4Ax^2e^{4x} + Be^{4x}$
$y_p''(x) = (2A + 8Ax + 4B)e^{4x} + 8Axe^{4x}$

Substituting these into the original ODE:

$(2A + 8Ax + 4B)e^{4x} + 8Axe^{4x} - 8((2Ax + B)e^{4x} + 4Ax^2e^{4x} + Be^{4x}) + 16(Ax^2e^{4x} + Bxe^{4x} + Ce^{4x}) = xe^{4x}$

Grouping like terms:

$(2A + 8Ax + 4B - 8B)e^{4x} + (8Ax - 8Ax)e^{4x} + (16Ax^2 - 32Ax^2 + 16Bx)e^{4x} + (16C)e^{4x} = xe^{4x}$

Simplify and equate coefficients:

$(-4B)e^{4x} + (16Bx)e^{4x} + (16C)e^{4x} = xe^{4x}$

Comparing coefficients:

$-4B = 0 \implies B = 0$
$16B = 1 \implies B = \frac{1}{16}$
$16C = 0 \implies C = 0$

So, we have $B = \frac{1}{16}$, and A and C are arbitrary constants.

Thus, the particular solution is:
$y_p(x) = \frac{1}{16}xe^{4x}$.

Now, the general solution is the sum of the complementary and particular solutions:
$y(x) = y_c(x) + y_p(x) = (c_1 + c_2 x)e^{4x} + \frac{1}{16}xe^{4x}$.

This is the general solution to the given ODE.

Message ChatGPT

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