Infinitely many solutions for two noncooperative $p(x)$-Laplacian elliptic systems *

Wanmin Liu †

Department of Mathematics, Lanzhou University, Lanzhou 730000, P. R. China

Abstract

The author deals with two noncooperative elliptic systems involving $p(x)$-Laplacian in a smooth bounded domain and in $\mathbb{R}^N$ respectively. With some symmetry assumptions and growth conditions on nonlinearities, the existences of infinitely many solutions are obtained by using a limit index theory developed by Li (Nonlinear Anal.: TMA, 25(1995) 1371) in variable exponent Sobolev spaces.

Keywords: variable exponent Sobolev spaces; noncooperative elliptic systems; limit index theory; $p(x)$-Laplacian

Mathematics Subject Classification(2000): 35J50; 35B38

1 Introduction

The theory of variable exponent Lebesgue and Sobolev spaces has been developed by several researchers in recent years. These spaces are natural generalization of the classical Lebesgue space $L^p(\Omega)$ and the Sobolev space $W^{k,p}(\Omega)$. Although the study of these spaces can go back to [21] and [20] as special cases of Musielak-Orlicz spaces, the first paper systematically investing these spaces appeared in 1991 by Kováčik and Rákosník [17]. These spaces have been independently rediscovered by several researchers based on different background. We refer to Samko [24], Fan and Zhao [12], Acerbi and Mingione [1]. We also refer to three survey papers of these areas by Harjulehto and Hästö [15], by Diening, Hästö and Nekvinda [4] and by Samko [25]. Many applications have been found such as variational integrals with non-standard growth conditions in nonlinear elasticity theory by Zhikov [31], models in electrorheological fluids by Růžička [23], and models in image restoration by Chen, Levine and Rao [3].

In this paper, we consider the following two noncooperative elliptic systems involving $p(x)$-Laplacian in a smooth bounded domain Ω and in $\mathbb{R}^N$ respectively:

*Project supported by National Natural Science Foundation of China (10671084)

†This is the English translation of the author's master's thesis, originally published in Chinese on 2007-05-21, Lanzhou University (China), with DOI: 10.7666/d.Y1089691. E-mail: wanminliu@gmail.com

$$\begin{cases} \triangle_{p(x)}u = F_s(x,u,v) & \text{in } \Omega, \\ -\triangle_{p(x)}v = F_t(x,u,v) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0; \end{cases} \tag{1.1}$$

$$\begin{cases} \triangle_{p(x)}u - |u|^{p(x)-2}u = G_s(|x|,u,v) & \text{in } \mathbb{R}^N, \\ -\triangle_{p(x)}v + |v|^{p(x)-2}v = G_t(|x|,u,v) & \text{in } \mathbb{R}^N; \end{cases} \tag{1.2}$$

where $\triangle_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian, $F(x,s,t) \in C^1(\bar{\Omega}\times\mathbb{R}^2,\mathbb{R})$, $G(r,s,t) \in C^1([0,\infty)\times\mathbb{R}^2,\mathbb{R})$, $F_s = \frac{\partial F}{\partial s}$ and similar to F_t, G_s, G_t.

Many results about $p(x)$-Laplacian equations with Dirichlet boundary conditions ([11, 5, 6]), Neumann boundary conditions ([19]) and in $\mathbb{R}^N$ cases ([8, 28]) have been obtained by variational approach and sub-supersolution method. Acerbi and Mingione [1] have obtained the local $C^{1,\alpha}$ regularity of minimizers of the integral functional with $p(x)$-growth conditions under the assumption that $p(x)$ is Hölder continuous. The global regularity results have also been obtained by Fan [7]. There are some results about elliptic systems. Hamidi [14] considered the following system

$$\begin{cases} -\triangle_{p(x)}u = F_s(x,u,v) & \text{in } \Omega, \\ -\triangle_{q(x)}v = F_t(x,u,v) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0; \end{cases} \tag{1.3}$$

and obtained the existence of solution since the integral functional of (1.3) is coercive and satisfies mountain pass geometry under some assumptions on F. The author also gave the multiplicity results by using the Fountain theorem when some symmetry condition on F is assumed. Zhang [29] considered the system

$$\begin{cases} -\triangle_{p(x)}u = f(v) & \text{in } \Omega, \\ -\triangle_{p(x)}v = g(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0; \end{cases} \tag{1.4}$$

on a bounded radial symmetric domain with $p(x)$ radial symmetric, and proved the existence of a positive solution under some assumptions by sub-supersolution method. He also considered the existence of solutions for weighted $p(r)$-Laplacian system boundary value problems via Leray-Schauder degree in [30].

The main difficulties we meet here are that the corresponding integral functionals of (1.1) and (1.2) are strongly indefinite. In addition to the nonlinearity of $p(x)$-Laplacian operator, we also lose a compact embedding theorem in $\mathbb{R}^N$ case. Thanks to a limit index theory developed by Li [18] and the principle of symmetric criticality due to Palais [22] in $\mathbb{R}^N$ case, we can obtain the existence of infinitely many solutions of problems (1.1) and (1.2) in the spaces $W_0^{1,p(x)}(\Omega)$ and $W^{1,p(x)}(\mathbb{R}^N)$ respectively under some nature assumptions on the nonlinearities.

Here are the assumptions. We denote by c or c_i the generic positive constants which may be different throughout the paper.

(P1) $p(x) \in C(\bar{\Omega})$ and $1 < \inf_\Omega p(x) := p_- \leq p_+ := \sup_\Omega p(x) < \infty$.
(P2) $p(x) = p(|x|) := p(r) \in C^{0,1}(\mathbb{R}^N)$ with $1 < \inf_{\mathbb{R}^N} p(x) := p_- \leq p_+ := \sup_{\mathbb{R}^N} p(x) < N$.
(F1) $F \in C^1(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$.
(F2) $|F_s(x,s,t)| + |F_t(x,s,t)| \leq c_1 + c_2(|s|^{r(x)-1} + |t|^{r(x)-1})$ where $r(x) \in C(\bar{\Omega})$, $2 \leq r(x) < p^*(x)$.

$$p^*(x) := \begin{cases} \frac{Np(x)}{(N-p(x))}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

(F3) $\exists M > 0$ and $\mu > p_+$ such that

$$0 < \mu F(x,s,t) \leq sF_s(x,s,t) + tF_t(x,s,t), \text{ for all } (x,s,t) \in \bar{\Omega} \times \mathbb{R}^2 \quad \text{with } s^2 + t^2 \geq M^2.$$

In this case, $F(x,s,t) \geq c_1(|s|^\mu + |t|^\mu) - c_2$.
(F4) $sF_s(x,s,t) \geq 0$, for all $(x,s,t) \in \bar{\Omega} \times \mathbb{R}^2$.
(F5) $F(x,-s,-t) = F(x,s,t)$, for all $(x,s,t) \in \bar{\Omega} \times \mathbb{R}^2$.
(G1) $G \in C^1([0,\ \infty) \times \mathbb{R}^2, \mathbb{R})$.
(G2) For some $p(x) \ll q(x) \ll p^*(x)$,

$$|G_s(|x|,s,t)| + |G_t(|x|,s,t)| \leq c_1(|s|^{p(x)-1} + |t|^{p(x)-1}) + c_2(|s|^{q(x)-1} + |t|^{q(x)-1}).$$

The symbol $\alpha(x) \ll \beta(x)$ means $\inf_{\bar{\Omega}}(\beta(x) - \alpha(x)) > 0$.
(G3) $\exists M > 0$ and $\mu > p_+$ such that

$$0 < \mu G(r,s,t) \leq sG_s(r,s,t) + tG_t(r,s,t), \text{ for all } (r,s,t) \in [0,\ \infty) \times \mathbb{R}^2 \quad \text{with } s^2 + t^2 \geq M^2.$$

(G4) $|G_s(|x|,s,t)| + |G_t(|x|,s,t)| = o(|s|^{p(x)-1}) + o(|t|^{p(x)-1})$ uniformly on $\mathbb{R}^N$ as $s^2 + t^2 \to 0$.
(G5) $sG_s(r,s,t) \geq 0$, for all $(r,s,t) \in [0,\ \infty) \times \mathbb{R}^2$.
(G6) $G(r,-s,-t) = G(r,s,t)$, for all $(r,s,t) \in [0,\ \infty) \times \mathbb{R}^2$.

The following are the main results.

Theorem 1.1. Suppose that (P1) and (F1)-(F5) are satisfied. Let $\Phi(u,v)$ be the integral functional of (1.1). Then problem (1.1) possesses a sequence of weak solutions $\{\pm(u_n, v_n)\}$ in $W_0^{1,p(x)}(\Omega) \times W_0^{1,p(x)}(\Omega)$ such that $\Phi(u_n, v_n) \to +\infty$ as $n \to \infty$.

Theorem 1.2. Suppose that (P2) and (G1)-(G6) are satisfied. Let $\Psi(u,v)$ be the integral functional of (1.2). Then problem (1.2) possesses a sequence of radial weak solutions $\{\pm(u_n, v_n)\}$ in $W^{1,p(x)}(\mathbb{R}^N) \times W^{1,p(x)}(\mathbb{R}^N)$ such that $\Psi(u_n, v_n) \to +\infty$ as $n \to \infty$. In addition, if $N = 4$ or $N \geq 6$, the problem (1.2) possesses infinitely many nonradial weak solutions.

Remark 1.3. The definitions of weak solution of (1.1) and (1.2) are Definition 3.1 and Definition 4.1 respectively.

Remark 1.4. When $p(x) \equiv p$ (a constant), the corresponding results have been obtained by Li in [18] and by Huang and Li in [16]. The aim of the present paper is to generalize their results to general cases.

The paper is organized as follows. In Section 2.1 we do some preliminaries of the space $W_0^{1,p(x)}(\Omega)$ and $W^{1,p(x)}(\mathbb{R}^N)$, review some basic properties of $p(x)$-Laplacian operator. In Section 2.2 we recall a limit index theory due to Li. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2.

2 Preliminaries

2.1 Variable exponent Sobolev spaces and $p(x)$-Laplacian operator

Let Ω be an open subset of $\mathbb{R}^N$. In this subsection, without further assumption, Ω could be $\mathbb{R}^N$. On the basic properties of the space $W^{1,p(x)}(\Omega)$ we refer to [17, 12]. In the following we display some facts which we will use later.

Denote by $S(\Omega)$ the set of all measurable real functions defined on Ω, and elements in $S(\Omega)$ that equal to each other almost everywhere are considered as one element. Denote $L_+^\infty(\Omega) = \{p \in L^\infty(\Omega) : \operatorname{ess\,inf}_\Omega p(x) := p_- \geq 1\}$.

For $p \in L_+^\infty(\Omega)$, define

$$L^{p(x)}(\Omega) = \{u \in S(\Omega) : \int_\Omega |u|^{p(x)} \mathrm{d}x < \infty\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |u/\lambda|^{p(x)} \mathrm{d}x \leq 1\};$$

and define

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Define

$$W_r^{1,p(x)}(\mathbb{R}^N) = \{u \in W^{1,p(x)}(\mathbb{R}^N) : \text{ u is radially symmetric}\}.$$

Hereafter, we always assume that $p(x)$ is continuous and $p_- > 1$.

Proposition 2.1. ([17, 12]) The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega)$ and $W_r^{1,p(x)}(\mathbb{R}^N)$ all are separable and reflexive Banach spaces.

Proposition 2.2. ([17, 12, 9]) The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p^o(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p^o(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p^o(x)}(\Omega)$, the Hölder inequality holds:

$$\int_\Omega |uv| \mathrm{d}x \leq 2|u|_{p(x)}|v|_{p^o(x)}. \tag{2.1}$$

Remark 2.3. In the right of (2.1), the constants 2 is suitable, but not the best. The best constant is given in [9] denoted by $d_{(p_-,p_+)}$ which only depends on p_- and p_+ when $p(x)$ is given and $d_{(p_-,p_+)}$ is smaller than $\frac{1}{p_-}+\frac{1}{p_+}$.

Proposition 2.4. ([12] Theorem 2.7) Suppose that Ω is a bounded domain. In $W_0^{1,p(x)}(\Omega)$ the Poincaré inequality holds, that is, there exists a positive constant c such that

$$|u|_{p(x)} \leq c|\nabla u|_{p(x)}, \qquad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

So $|\nabla u|_{p(x)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.

Remark 2.5. When Ω is a bounded domain, we denote by $\|u\| := |\nabla u|_{p(x)}$ as the equivalent norm in $W_0^{1,p(x)}(\Omega)$ in Section 3. In Section 4 we will use the following equivalent norm on $W^{1,p(x)}(\mathbb{R}^N)$ also with the symbol $\|u\|$:

$$\|u\| := \inf\{\lambda > 0 : \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})/\lambda^{p(x)} \mathrm{d}x \leq 1\}. \tag{2.2}$$

Proposition 2.6. ([12] Theorem 1.3) Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}\mathrm{d}x$. For $u, u_k \in L^{p(x)}(\Omega)$, we have (1) $|u|_{p(x)} < 1\ (= 1; > 1) \iff \rho(u) < 1\ (= 1; > 1)$;
(2) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$; $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;
(3) $\lim_{k\to\infty} |u_k|_{p(x)} = 0\ (= \infty) \iff \lim_{k\to\infty} \rho(u_k) = 0\ (= \infty)$.

Proposition 2.7. Let X be $W_0^{1,p(x)}(\Omega)$ or $W^{1,p(x)}(\mathbb{R}^N)$ with the norm $\|\cdot\|$ as in Remark 2.5. Set $I(u) = \int_\Omega |\nabla u(x)|^{p(x)}\mathrm{d}x$ when Ω is bounded or $I(u) = \int_{\mathbb{R}^N}(|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)})\mathrm{d}x$ in the $\mathbb{R}^N$ case respectively. If $u, u_k \in X$ then the similar conclusions of Proposition 2.6 hold for $\|\cdot\|$ and $I(\cdot)$.

Proposition 2.8. ([17, 12]) Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, and

$$|F(x,t)| \leq a(x) + b|t|^{\frac{p_1(x)}{p_2(x)}}, \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

where $a \in L^{p_2(x)}(\Omega)$, b is a positive constant, $p_1, p_2 \in L_+^\infty(\Omega)$. Denote by N_F the Nemytsky operator defined by F , i.e.

$$(N_F(u))(x) = F(x, u(x)),$$

then $N_F : L^{p_1(x)}(\Omega) \to L^{p_2(x)}(\Omega)$ is a continuous and bounded map.

Proposition 2.9. ([10] Theorem 1.1.) If p: $\Omega \to \mathbb{R}$ is Lipschitz continuous and $p_+ < N$, then for $q \in L_+^\infty(\Omega)$ with $p(x) \leq q(x) \leq p^*(x)$, there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Proposition 2.10. ([5] Proposition 2.4.) Assume that the boundary of Ω possesses the cone property and $p \in C(\bar{\Omega})$. If $q \in C(\bar{\Omega})$ and $1 \leq q(x) < p^*(x)$ for $x \in \bar{\Omega}$, then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Proposition 2.11.([13] Theorem 3.1.) Suppose that $p:\ \mathbb{R}^N \to \mathbb{R}$ is a uniformly continuous and radially symmetric function satisfying $1 < p_- \le p_+ < N$. Then, for any measurable function $q:\ \mathbb{R}^N \to \mathbb{R}$ with

$$p(x) \ll q(x) \ll p^*(x), \qquad \text{for all } x \in \mathbb{R}^N,$$

there is a compact embedding

$$W_r^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N).$$

Definition 2.12. On the space $L^{p(x)}(\Omega) \cap L^{q(x)}(\Omega)$, we define the norm $|u|_{p(x)\wedge q(x)} = |u|_{p(x)} + |u|_{q(x)}$. On the space $(L^{p(x)}(\Omega))^2 := L^{p(x)}(\Omega) \times L^{p(x)}(\Omega)$, we define the norm $|(u,v)|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega (|u|^{p(x)} + |v|^{p(x)})/\lambda^{p(x)} \mathrm{d}x \le 1\}$. On the space $(L^{p(x)}(\Omega))^2 \cap (L^{q(x)}(\Omega))^2$, we define the norm $|(u,v)|_{p(x)\wedge q(x)} = |(u,v)|_{p(x)} + |(u,v)|_{q(x)}$. On the space $L^{p(x)}(\Omega) + L^{q(x)}(\Omega)$, we define the norm $|u|_{p(x)\vee q(x)} = \inf\{|v|_{p(x)} + |w|_{q(x)} : v \in L^{p(x)}(\Omega), w \in L^{p(x)}(\Omega), u = v + w\}$.

Similar to Proposition 2.8, we have the following proposition.

Proposition 2.13. (1) Assume $1 \le p(x), r(x) < \infty$, $f \in C(\Omega \times \mathbb{R}^2)$ and

$$f(x,s,t) \le c_1(|s|^{\frac{p(x)}{r(x)}} + |t|^{\frac{p(x)}{r(x)}}).$$

Then for every $(u,v) \in (L^{p(x)}(\Omega))^2$, $f(\cdot,u,v) \in L^{r(x)}(\Omega)$ and the operator

$$T_1 : (L^{p(x)}(\Omega))^2 \to L^{r(x)}(\Omega) : (u,v) \mapsto f(x,u,v)$$

is continuous.
(2) Assume $1 \le p(x), r(x), q(x), s(x) < \infty$, $f \in C(\Omega \times \mathbb{R}^2)$ and

$$f(x,s,t) \le c_2(|s|^{\frac{p(x)}{r(x)}} + |t|^{\frac{p(x)}{r(x)}}) + c_3(|s|^{\frac{q(x)}{s(x)}} + |t|^{\frac{q(x)}{s(x)}}).$$

Then for every $(u,v) \in (L^{p(x)}(\Omega))^2 \cap (L^{q(x)}(\Omega))^2$, $f(\cdot,u,v) \in L^{r(x)}(\Omega) + L^{s(x)}(\Omega)$ and the operator

$$T_2 : (L^{p(x)}(\Omega))^2 \cap (L^{q(x)}(\Omega))^2 \to L^{r(x)}(\Omega) + L^{s(x)}(\Omega) : (u,v) \mapsto f(x,u,v)$$

is continuous. $\Box$

Now we display some basic properties of $p(x)$-Laplacian operators. Let X be $W_0^{1,p(x)}(\Omega)$ or $W^{1,p(x)}(\mathbb{R}^N)$. Consider the following two functionals:

$$J_1(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x, \qquad \text{for all } u \in X = W_0^{1,p(x)}(\Omega);$$

$$J_2(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\mathrm{d}x, \quad \text{for all } u \in X = W^{1,p(x)}(\mathbb{R}^N).$$

We know that $J_1,\ J_2 \in C^1(X, \mathbb{R})$, and the $p(x)$-Laplacian operator is the derivative operator of J_1 in the weak sense. We denote $L = J_1'$: $X \to X^*$ and $T = J_2'$: $X \to X^*$, then

$$\langle Lu, \tilde{u}\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \tilde{u}\mathrm{d}x, \quad \text{for all } u, \tilde{u} \in X, \tag{2.3}$$

$$\langle Tu, \tilde{u}\rangle = \int_{\mathbb{R}^N} (|\nabla u|^{p(x)-2}\nabla u \nabla \tilde{u} + |u|^{p(x)-2}u\tilde{u})\mathrm{d}x, \quad \text{for all } u, \tilde{u} \in X. \tag{2.4}$$

Proposition 2.14. ([11, 8])
(1) $L, T : X \to X^*$ are two continuous, bounded and strictly monotone operators.
(2) $L, T : X \to X^*$ are two mappings of type (S_+), where the mapping L of type (S_+) means

$u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle L(u_n) - L(u), u_n - u\rangle \leq 0$, then $u_n \to u$ in X, and similar to T.
(3) $L, T : X \to X^*$ are two homeomorphisms.

2.2 A limit index theory due to Li

In this section, we will recall a limit index theory developed by Li [18]. Suppose Z is a $\mathscr{G}$-Banach space, where $\mathscr{G}$ is a topological group. For the definition of index i we refer to [26] Definition 5.9.

Definition 2.15. An index i is said to satisfy the d-dimension property if there is a positive integer d such that

$$i(V^{dk} \cap S_1) = k$$

for all dk-dimensional subspaces $V^{dk} \in \Sigma := \{A \subset Z :\ A \text{ is closed and } gA = A \text{ for all } g \in \mathscr{G}\}$ such that $V^{dk} \cap \mathrm{Fix}\,\mathscr{G} = \{0\}$, where S_1 is the unit sphere in Z.

Proposition 2.16. ([18] Lemma 2.3) Suppose $Z = W_1 \oplus W_2$ and $\dim W_1 = kd$, where W_j is a $\mathscr{G}$-invariant subspace, $j = 1, 2$. Let i be an index satisfying the d-dimension property. If $W_1 \cap \mathrm{Fix}\,\mathscr{G} = \{0\}$, $A \in \Sigma$ and $i(A) > k$, then $A \cap W_2 \neq \emptyset$.

Suppose U and V are $\mathscr{G}$-invariant closed subspaces of Z such that $Z = U \oplus V$, where V is infinite dimension and $V = \overline{\cup_{j=1}^\infty V_j}$. Here V_j is a dn_j-dimensional $\mathscr{G}$-invariant subspace of V, and $V_1 \subset V_2 \subset \ldots$ for $j = 1,\ 2,\ \ldots$. Let $Z_j = U \oplus V_j$, and let $A_j = A \cap Z_j$ for all $A \in \Sigma$.

Definition 2.17. ([18] Definition 2.4) Let i be an index satisfying the d-dimension property. A limit index i^∞ with respect to $\{Z_j\}$ induced by i is a mapping

$$i^\infty : \Sigma \to \mathbb{Z} \cup \{-\infty,\ +\infty\}$$

given by

$$i^\infty(A) = \limsup_{j\to\infty}(i(A_j) - n_j).$$

Proposition 2.18. ([18] Proposition 2.5) Let $A,\ B \in \Sigma$. Then i^∞ satisfies:
(1) $A = \emptyset \iff i^\infty(A) = -\infty$;
(2) (Monotonicity) $A \subset B \Rightarrow i^\infty(A) \leq i^\infty(B)$;
(3) (Subadditivity) $i^\infty(A \cup B) \leq i^\infty(A) + i(B)$;
(4) If $V \cap \mathrm{Fix}\,\mathscr{G} = \{0\}$, then $i^\infty(S_\rho \cap V) = 0$, where $S_\rho = \{z \in Z, \|z\| = \rho\}$;
(5) If Y_0 and $\tilde{Y_0}$ are $\mathscr{G}$-invariant closed subspaces of V such that $V = Y_0 \oplus \tilde{Y_0}$, $\tilde{Y_0} \subset V_{j_0}$ for some j_0 and $\dim \tilde{Y_0} = dm$, then $i^\infty(S_\rho \cap Y_0) \geq -m$.

Definition 2.19. Let Z be a Banach space which has a decomposition $Z = \overline{\cup_{j=1}^\infty Z_j}$ where $Z_1 \subset Z_2 \cdots$, $\dim Z_j = dn_j$. A functional $f \in C^1(Z, \mathbb{R})$ is said to satisfy the $(PS)_c^*$ condition with respect to $\{Z_n\}$ at the level $c \in \mathbb{R}$ if any sequence $\{z_{n_k}\}$, $z_{n_k} \in Z_{n_k}$ such that

$$f(z_{n_k}) \to c \text{ and } \|(f_{n_k})'(z_{n_k})\| \to 0 \text{ as } n_k \to \infty$$

possesses a subsequence which converges in Z to a critical point of f, where $f_{n_k} := f|_{Z_{n_k}}$.

Theorem 2.20. ([18] Corollary 4.4, [16] Theorem 2.7) Assume that
(B1) $f \in C^1(Z, \mathbb{R})$ is $\mathscr{G}$-invariant;
(B2) there are $\mathscr{G}$-invariant closed subspaces U and V such that V is infinite dimension and
$Z = U \oplus V$;
(B3) there is a sequence of $\mathscr{G}$-invariant finite-dimensional subspaces

$$V_1 \subset V_2 \cdots \subset V_j \subset \cdots,\ \dim V_j = dn_j,$$

such that $V = \overline{\cup_{j=1}^\infty V_j}$;
(B4) there is an index i on Z satisfying the d-dimension property;
(B5) there are $\mathscr{G}$-invariant subspaces Y_0, $\tilde{Y_0}$, Y_1 of V such that $V = Y_0 \oplus \tilde{Y_0}$, $Y_1, \tilde{Y_0} \subset V_{j_0}$ for some
j_0 and $\dim \tilde{Y_0} = dm \leq dk = \dim Y_1$;
(B6) there are α and β, $\alpha < \beta$ such that f satisfies $(PS)_c^*$ with respect to $Z_n := U \oplus V_n$, for all $c \in [\alpha, \beta]$;
(B7)

$$\begin{cases} (a) & \text{either } \mathrm{Fix}\,\mathscr{G} \subset U \oplus Y_1 \text{ or } \mathrm{Fix}\,\mathscr{G} \cap V = \{0\}, \\ (b) & \text{there is } \rho > 0 \text{ such that } f(z) \geq \alpha, \quad \text{for all } z \in Y_0 \cap S_\rho, \\ (c) & f(z) \leq \beta, \quad \text{for all } z \in U \oplus Y_1. \end{cases}$$

If i^∞ is the limit index induced by i, then the numbers

$$d_j = \sup_{i^\infty(A) \geq j} \inf_{z \in A} f(z)$$

are critical values of f and $\alpha \leq d_{-m} \leq d_{-m-1} \leq \cdots \leq d_{-k+1} \leq \beta$. Moreover, if $d = d_l = \cdots = d_{l+r}$, $r > 0$, then $i(K_c) \geq r+1$, where $K_c = \{z \in Z; f'(z) = 0, f(z) = d\}$.

Proof. By Proposition 2.18(5), $i^\infty(S_\rho \cap Y_0) \geq -m$ thus $\alpha \leq d_{-m}$. It is obvious that $d_{-m} \leq d_{-m-1} \leq \cdots \leq d_{-k+1}$. Let us turn to prove $d_{-k+1} \leq \beta$. Let $V_j \ominus Y_1$ be a fixed $\mathscr{G}$-invariant complementary subspace of Y_1 in V_j, $j \geq j_0$. It is easy to obtain that $(V_j \ominus Y_1) \cap \operatorname{Fix}\mathscr{G} = \{0\}$ since of (B7)(a). Suppose $A \in \Sigma$ and $i^\infty(A) \geq -k+1$, there must be some j such that $i(A_j) - n_j > -k$, that is $i(A_j) > n_j - k$. On the other hand, we have $\dim(V_j \ominus Y_1) = d(n_j - k)$. By Proposition 2.16 we get $A_j \cap (U \oplus Y_1) \neq \emptyset$. Then $A \cap (U \oplus Y_1) \neq \emptyset$. By the definition of d_{-k+1} and (B7)(c), we get $d_{-k+1} \leq \beta$. The proof that d_j are critical values of f is the Theorem 4.1 in [18]. $\square$

Remark 2.21. In [18] Corollary 4.4 and [16] Theorem 2.7, this theorem is stated incorrectly , but the proof they gave there is essentially correct.

3 The bounded case

In this section, we always assume that (P1) is satisfied and use X to denote $W_0^{1,p(x)}(\Omega)$ with the norm $\|u\| = |\nabla u|_{p(x)}$ as in Remark 2.5. The integral functional of (1.1) is

$$\Phi(u,v) = -\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x + \int_\Omega \frac{1}{p(x)} |\nabla v|^{p(x)} \mathrm{d}x - \mathcal{F}(u,v),$$

where

$$\mathcal{F}(u,v) := \int_\Omega F(x,u,v)\mathrm{d}x, \quad u,\ v \in X.$$

Definition 3.1. $(u,v) \in X \times X$ is called a weak solution of (1.1) if

$$-\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \tilde{u} \mathrm{d}x + \int_\Omega |\nabla v|^{p(x)-2} \nabla v \nabla \tilde{v} \mathrm{d}x = \int_\Omega F_s(x,u,v)\tilde{u}\mathrm{d}x + \int_\Omega F_t(x,u,v)\tilde{v}\mathrm{d}x,$$
$$\text{for all } (\tilde{u},\tilde{v}) \in X \times X \tag{3.1}$$

For simplicity, using the operator L defined in (2.3), we rewrite (3.1) as

$$\langle(-Lu, Lv), (\tilde{u},\tilde{v})\rangle = \langle \mathcal{F}'(u,v), (\tilde{u},\tilde{v})\rangle, \qquad \text{for all } (\tilde{u},\tilde{v}) \in X \times X,$$

where

$$\langle(-Lu, Lv), (\tilde{u},\tilde{v})\rangle := \langle -Lu, \tilde{u}\rangle + \langle Lv, \tilde{v}\rangle,$$

and

$$\langle \mathcal{F}'(u,v), (\tilde{u},\tilde{v})\rangle := \int_\Omega F_s(x,u,v)\tilde{u}\mathrm{d}x + \int_\Omega F_t(x,u,v)\tilde{v}\mathrm{d}x.$$

Lemma 3.2. Suppose F satisfies (F1) and (F2), then
(1) $\Phi,\ \mathcal{F} \in C^1(X \times X, \mathbb{R})$ and

$$\langle \Phi'(u,v), (\tilde{u},\tilde{v})\rangle = \langle(-Lu, Lv), (\tilde{u},\tilde{v})\rangle - \langle \mathcal{F}'(u,v), (\tilde{u},\tilde{v})\rangle. \tag{3.2}$$

In particular, each critical point of Φ is a weak solution of (1.1).
(2) $\mathcal{F}' : X \times X \to X^* \times X^*$ is completely continuous.

Proof. The proof of (1) is routine. The proof of (2) relies on Proposition 2.10 and we omit it. □

As X is a separable and reflexive Banach space, there exist $\{e_j\}_{j=1}^{\infty} \subset X$ and $\{f_i\}_{i=1}^{\infty} \subset X^*$ such that

$$X = \overline{\mathrm{span}\{e_j | j = 1, 2, \cdots\}}, \quad X^* = \overline{\mathrm{span}\{f_i | i = 1, 2, \cdots\}}^{W^*}, \text{ and } \langle f_i, e_j \rangle = \delta_{ij}.$$

For convenience, we write $X_n = \mathrm{span}\{e_1, \cdots, e_n\}$, $X_n^{\perp} = \overline{\mathrm{span}\{e_{n+1}, \cdots\}}$. Now set $E = X \times X$, $E_n = X_n \times X_n$. Define a group of $\mathscr{G} = \{\iota, \tau\} \cong \mathbb{Z}_2$ by setting

$$\tau(u, v) = (-u, -v), \ \iota(u, v) = (u, v). \tag{3.3}$$

Let

$$\Sigma = \{A \subset E : \ A \text{ is closed and } (u, v) \in A \Rightarrow (-u, -v) \in A\}. \tag{3.4}$$

An index γ on Σ is defined by

$$\gamma(A) = \begin{cases} 0 \text{ if } A = \emptyset, \\ \min\{m \in \mathbb{Z}_+ : \exists h \in C(A, \mathbb{R}^m \backslash \{0\}) \text{ such that } h(-u, -v) = -h(u, v)\} , \\ +\infty \text{ if such } h \text{ dose not exist.} \end{cases} \tag{3.5}$$

Then γ is an index satisfying 1-dimension property by Borsuk-Ulam theorem (see [26] Proposition II 5.2.). We can obtain a limit index γ^{∞} with respect to $\{E_n\}$ from γ.

Lemma 3.3. Assume that F satisfies (F1) and (F2). Then any bounded sequence $\{(u_{n_k}, v_{n_k})\}$ such that

$$(u_{n_k}, v_{n_k}) \in E_{n_k}, \Phi(u_{n_k}, v_{n_k}) \to c, \|(\Phi_{n_k})'(u_{n_k}, v_{n_k})\| \to 0 \text{ as } n_k \to \infty \tag{3.6}$$

possesses a subsequence which converges in E to a critical point of Φ, where $\Phi_{n_k} := \Phi|_{E_{n_k}}$.

Proof. Since E is reflexive, going if necessary to a subsequence, we can assume that $u_{n_k} \rightharpoonup u$ and $v_{n_k} \rightharpoonup v$. Observing that $E = \overline{\cup_{n=1}^{\infty} E_n}$, we can choose $(\bar{u}_{n_k}, \bar{v}_{n_k}) \in E_{n_k}$ such that $\bar{u}_{n_k} \to u$ and $\bar{v}_{n_k} \to v$. Hence

$$\begin{aligned} & \lim_{n_k \to \infty} \langle \Phi'(u_{n_k}, v_{n_k}), (u_{n_k} - u, 0) \rangle \\ = & \lim_{n_k \to \infty} \langle \Phi'(u_{n_k}, v_{n_k}), (u_{n_k} - \bar{u}_{n_k}, 0) \rangle + \lim_{n_k \to \infty} \langle \Phi'(u_{n_k}, v_{n_k}), (\bar{u}_{n_k} - u, 0) \rangle \\ = & \lim_{n_k \to \infty} \langle (\Phi_{n_k})'(u_{n_k}, v_{n_k}), (u_{n_k} - \bar{u}_{n_k}, 0) \rangle = 0. \end{aligned} \tag{3.7}$$

Substituting (3.2) into (3.7) and noticing that $\mathcal{F}'$ is completely continuous, we obtain

$$\lim_{n_k \to \infty} \langle L u_{n_k}, u_{n_k} - u \rangle = 0. \tag{3.8}$$

By computing the limit of $\langle \Phi'(u_{n_k}, v_{n_k}), (0, v_{n_k} - v)\rangle$ in the similar way using $\bar{v}_{n_k}$, we obtain

$$\lim_{n_k\to\infty} \langle L v_{n_k}, v_{n_k} - v\rangle = 0. \tag{3.9}$$

From (3.8) and (3.9), we conclude that $u_{n_k} \to u$ and $v_{n_k} \to v$ since L is of type (S_+).

It remains to show that (u, v) is a critical point of Φ. Taking arbitrarily $(\bar{u}_j, \bar{v}_j) \in E_j$, then for $n_k \geq j$ we have

$$\langle \Phi'(u, v), (\bar{u}_j, \bar{v}_j)\rangle = \langle \Phi'(u, v) - \Phi'(u_{n_k}, v_{n_k}), (\bar{u}_j, \bar{v}_j)\rangle + \langle (\Phi_{n_k})'(u_{n_k}, v_{n_k}), (\bar{u}_j, \bar{v}_j)\rangle. \tag{3.10}$$

Taking $n_k \to \infty$ in the right side of (3.10), we obtain $\langle \Phi'(u, v), (\bar{u}_j, \bar{v}_j)\rangle = 0$. Hence $\Phi'(u, v) = 0$. □

Lemma 3.4. Suppose that F satisfies (F1)-(F4). Then the functional Φ satisfies $(PS)_c^*$ with respect to $\{E_n\}$ for each c.

Proof. By Lemma 3.3, we only need to prove that each sequence satisfying (3.6) is bounded. We can assume that $\|u_{n_k}\| \geq 1$ and $\|v_{n_k}\| \geq 1$. From Proposition 2.7 and (F4), we have

$$\|u_{n_k}\| \geq \langle -(\Phi_{n_k})'(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\rangle = \langle L u_{n_k}, u_{n_k}\rangle + \int_\Omega F_s(x, u_{n_k}, v_{n_k}) u_{n_k} \mathrm{d}x \geq \|u_{n_k}\|^{p_-}. \tag{3.11}$$

So $\|u_{n_k}\|$ is bounded. On the other hand, from (F3), Proposition 2.7 and Hölder inequality, we have

$$\begin{aligned}
c_1 &\geq \Phi(u_{n_k}, v_{n_k}) \\
&= -\int_\Omega \frac{1}{p(x)} |\nabla u_{n_k}|^{p(x)} \mathrm{d}x + \int_\Omega \frac{1}{p(x)} |\nabla v_{n_k}|^{p(x)} \mathrm{d}x - \mathcal{F}(u_{n_k}, v_{n_k}) \\
&\geq -\int_\Omega \frac{1}{p(x)} |\nabla u_{n_k}|^{p(x)} \mathrm{d}x + \int_\Omega \frac{1}{p(x)} |\nabla v_{n_k}|^{p(x)} \mathrm{d}x \\
&\quad -\frac{1}{\mu} \int_\Omega (u_{n_k} F_s(x, u_{n_k}, v_{n_k}) + v_{n_k} F_t(x, u_{n_k}, v_{n_k})) \mathrm{d}x \\
&= -\int_\Omega \frac{1}{p(x)} |\nabla u_{n_k}|^{p(x)} \mathrm{d}x + \int_\Omega \frac{1}{p(x)} |\nabla v_{n_k}|^{p(x)} \mathrm{d}x - \frac{1}{\mu} \langle \mathcal{F}'(u_{n_k}, v_{n_k}), (u_{n_k}, v_{n_k})\rangle \\
&= -\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{\mu}\right) |\nabla u_{n_k}|^{p(x)} \mathrm{d}x + \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{\mu}\right) |\nabla v_{n_k}|^{p(x)} \mathrm{d}x \\
&\quad +\frac{1}{\mu} \langle (\Phi_{n_k})'(u_{n_k}, v_{n_k}), (u_{n_k}, v_{n_k})\rangle \\
&\geq -\left(\frac{1}{p_-} - \frac{1}{\mu}\right) \|u_{n_k}\|^{p_+} + \left(\frac{1}{p_+} - \frac{1}{\mu}\right) \|v_{n_k}\|^{p_-} \\
&\quad -\frac{2}{\mu} \|(\Phi_{n_k})'(u_{n_k}, v_{n_k})\| \left(\|u_{n_k}\| + \|v_{n_k}\|\right).
\end{aligned} \tag{3.12}$$

So $\|v_{n_k}\|$ is bounded. Thus $\{(u_{n_k}, v_{n_k})\}$ is a bounded sequence in E. □

Proposition 3.5. ([8] Lemma 3.3) Assume that $X = \overline{\text{span}\{e_j|j=1,2,\cdots\}}$, $X_m^\perp = \overline{\text{span}\{e_{m+1},\cdots\}}$, $f : X \to \mathbb{R}$ is a weakly-strongly continuous and $f(0) = 0$. Then

$$\delta_m := \sup_{u\in X_m^\perp,\ \|u\|=1} |f(u)| \to 0, \text{ as } m \to \infty.$$

Proof of Theorem 1.1. Note that Φ is invariant with respect to the action of $\mathscr{G}$. We shall verify that Φ satisfies the hypotheses of Theorem 2.20. Set $E = U \oplus V$, where $U = X \times \{0\}$ and $V = \{0\} \times X$. Set $Y_0 = \{0\} \times X_m^\perp$ and $Y_1 = \{0\} \times X_k$ where m and k are to be determined. Then Y_0 and Y_1 are $\mathscr{G}$-invariant and $\text{codim}_V\, Y_0 = m$, $\dim Y_1 = k$, $\text{Fix}\,\mathscr{G} = \{(0,0)\}$. So $\text{Fix}\,\mathscr{G} \cap V = \{(0,0)\}$ and (B7)(a) of Theorem 2.20 is satisfied. It remains to verify (b) and (c) of (B7).

First, we verify (b) of (B7). By (F3), we have

$$\Phi(u,0) = -\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x - \mathcal{F}(u,0) \le -\frac{1}{p_+}\int_\Omega |\nabla u|^{p(x)}\mathrm{d}x - c_1\int_\Omega |u|^\mu \mathrm{d}x + c_2.$$

Therefore $\sup_{u\in X}\Phi(u,0) < +\infty$. Choose α such that $\alpha > \sup_{u\in X}\Phi(u,0)$.

If $(0,v) \in Y_0 \cap S_\rho$ (where $\rho > 1$ is to be determined), we have $v \in X_m^\perp$ and $\|v\| = \rho$. Define $f : X \to \mathbb{R}, f(v) = |v|_{r(x)}$. Since the embedding $X \hookrightarrow L^{r(x)}(\Omega)$ is compact by Proposition 2.10, f is weakly-strongly continuous. By Proposition 3.5, we have $\delta_m \to 0$ as $m \to \infty$. By (F2) we obtain

$$\begin{aligned}\Phi(0,v) &= \int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\mathrm{d}x - \mathcal{F}(0,v)\\ &\ge \frac{1}{p_+}\int_\Omega |\nabla v|^{p(x)}\mathrm{d}x - c_3\int_\Omega |v|^{r(x)}\mathrm{d}x - c_4\\ &\ge \frac{1}{p_+}\|v\|^{p_-} - c_3|v|_{r(x)}^{r_+} - c_4\\ &\ge \frac{1}{p_+}\rho^{p_-} - c_3\delta_m^{r_+}\rho^{r_+} - c_4.\end{aligned}$$

Setting $\rho = (\frac{c_3 p_+ r_+ \delta_m^{r_+}}{p_-})^{\frac{1}{p_- - r_+}}$, we have

$$\Phi|_{Y_0\cap S_\rho} \ge (r_+ - p_-)(p_+ r_+)^{\frac{r_+}{p_- - r_+}}\left(\frac{c_3}{p_-}\right)^{\frac{p_-}{p_- - r_+}}\delta_m^{\frac{p_- r_+}{p_- - r_+}} - c_4 \to +\infty \text{ as } m \to \infty.$$

Next, we verify (c) of (B7). For each $(u,v) \in U \oplus Y_1$ and $\|u\| > 1$, $\|v\| > 1$,

$$\begin{aligned}\Phi(u,v) &\le -\frac{1}{p_+}\|u\|^{p_-} + \frac{1}{p_-}\|v\|^{p_+} - c_5\int_\Omega (|u|^\mu + |v|^\mu)\mathrm{d}x + c_6\\ &\le \frac{1}{p_-}\|v\|^{p_+} - c_5\int_\Omega |v|^\mu \mathrm{d}x + c_6.\end{aligned}$$

Since all norms are equivalent in the finite dimension space Y_1, we get

$$\Phi(u,v) \le \frac{1}{p_-}\|v\|^{p_+} - c_7\|v\|^\mu + c_8.$$

Then we have $\sup \Phi|_{U\oplus Y_1} < +\infty$ since $\mu > p_+$. Thus we can choose $k > m$ and $\beta > \alpha$ such that $\Phi|_{U\oplus Y_1} \leq \beta$.

So

$$d_j = \sup_{\gamma^\infty(A)\geq j} \inf_{z\in A} \Phi(z), \quad -k+1 \leq j \leq -m,$$

are critical values of Φ and $\alpha \leq d_j \leq \beta$. Since α can be chosen arbitrarily large, Φ has a sequence of critical values $d_n \to +\infty$. □

4 The $\mathbb{R}^N$ case

In this section, we always assume that (P2) is satisfied and denote X by $W^{1,p(x)}(\mathbb{R}^N)$ with the norm $\|u\|$ defined by (2.2) and denote X_r by $W_r^{1,p(x)}(\mathbb{R}^N)$ with the same norm. The integral functional of (1.2) is

$$\Psi(u,v) = -\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\mathrm{d}x + \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla v|^{p(x)} + |v|^{p(x)})\mathrm{d}x - \mathcal{G}(u,v),$$

where

$$\mathcal{G}(u,v) := \int_{\mathbb{R}^N} G(|x|,u,v)\mathrm{d}x, \quad u,\ v \in X.$$

Definition 4.1. $(u,v) \in X \times X$ is called a weak solution of (1.2) if

$$\begin{aligned} -\int_{\mathbb{R}^N}(|\nabla u|^{p(x)-2}\nabla u\nabla\tilde{u} + |u|^{p(x)-2}u\tilde{u})\mathrm{d}x + \int_{\mathbb{R}^N}(|\nabla v|^{p(x)-2}\nabla v\nabla\tilde{v} + |v|^{p(x)-2}v\tilde{v})\mathrm{d}x \\ = \int_{\mathbb{R}^N} G_s(|x|,u,v)\tilde{u}\mathrm{d}x + \int_{\mathbb{R}^N} G_t(|x|,u,v)\tilde{v}\mathrm{d}x, \quad \text{for all } (\tilde{u},\tilde{v}) \in X \times X. \end{aligned} \tag{4.1}$$

Denote

$$\langle (Tu,Tv),(\tilde{u},\tilde{v})\rangle := \langle Tu,\tilde{u}\rangle + \langle Tv,\tilde{v}\rangle,$$

where T is defined as (2.4) and denote

$$\langle \mathcal{G}'(u,v),(\tilde{u},\tilde{v})\rangle := \int_{\mathbb{R}^N} G_s(|x|,u,v)\tilde{u}\mathrm{d}x + \int_{\mathbb{R}^N} G_t(|x|,u,v)\tilde{v}\mathrm{d}x.$$

Then (4.1) can be rewritten as

$$\langle (-Tu,Tv),(\tilde{u},\tilde{v})\rangle = \langle \mathcal{G}'(u,v),(\tilde{u},\tilde{v})\rangle, \quad \text{for all } (\tilde{u},\tilde{v}) \in X \times X.$$

Proposition 4.2. ([22] Principle of symmetric criticality) If u is a critical point of $\Psi|_{X_r\times X_r}$, then u is also a critical point of $\Psi|_{X\times X}$ and thus a radially symmetric solution of problem (1.2).

By the principle of symmetric criticality, to solve problem (1.2), we shall to find the critical points of Ψ restricted on $X_r \times X_r$ using the limit index theory.

Lemma 4.3. Suppose G satisfies (G1)-(G4). Then
(1) $\Psi, \mathcal{G} \in C^1(X_r \times X_r, \mathbb{R})$ and

$$\langle \Psi'(u,v), (\tilde{u}, \tilde{v})\rangle = \langle(-Tu, Tv), (\tilde{u}, \tilde{v})\rangle - \langle \mathcal{G}'(u,v), (\tilde{u}, \tilde{v})\rangle, \quad \text{for all } (\tilde{u}, \tilde{v}) \in X_r \times X_r. \tag{4.2}$$

In particular, each critical point of Ψ is a weak solution of the problem (1.2).
(2) $\mathcal{G}' : X_r \times X_r \to X_r^* \times X_r^*$ is completely continuous.

Proof. (1) is obvious. Now we shall prove $\mathcal{G}'$ is continuous. Suppose $(u_n, v_n) \to (u, v) \in X_r \times X_r$. By Proposition 2.9, we have $(u_n, v_n) \to (u, v) \in (L^{p(x)}(\mathbb{R}^N))^2 \cap (L^{q(x)}(\mathbb{R}^N))^2$. It follows from (G2) and Proposition 2.13(2) that

$$G_s(|x|, u_n, v_n) \to G_s(|x|, u, v) \text{ in } L^{p^o(x)}(\mathbb{R}^N) + L^{q^o(x)}(\mathbb{R}^N),$$

$$G_t(|x|, u_n, v_n) \to G_t(|x|, u, v) \text{ in } L^{p^o(x)}(\mathbb{R}^N) + L^{q^o(x)}(\mathbb{R}^N).$$

For all $(\tilde{u}, \tilde{v}) \in X_r \times X_r$, we obtain, by Hölder inequality (2.1),

$$\begin{aligned}
& |\langle \mathcal{G}'(u_n, v_n), (\tilde{u}, \tilde{v})\rangle - \langle \mathcal{G}'(u,v), (\tilde{u}, \tilde{v})\rangle| \\
\leq\ & \int_{\mathbb{R}^N} |G_s(|x|, u_n, v_n) - G_s(|x|, u, v)||\tilde{u}|\mathrm{d}x + \int_{\mathbb{R}^N} |G_t(|x|, u_n, v_n) - G_t(|x|, u, v)||\tilde{v}|\mathrm{d}x \\
\leq\ & 2|G_s(|x|, u_n, v_n) - G_s(|x|, u, v)|_{p^o(x)\vee q^o(x)}|\tilde{u}|_{p(x)\wedge q(x)} \\
& +2|G_t(|x|, u_n, v_n) - G_t(|x|, u, v)|_{p^o(x)\vee q^o(x)}|\tilde{v}|_{p(x)\wedge q(x)},
\end{aligned}$$

where $1/p(x) + 1/p^o(x) = 1$, $1/q(x) + 1/q^o(x) = 1$. Thus $\|\mathcal{G}'(u_n, v_n) - \mathcal{G}'(u, v)\|_{X_r^* \times X_r^*} \to 0$ as $n \to \infty$.

Now let us prove that $\mathcal{G}'$ is completely continuous. For any $\varepsilon > 0$, using (G2) and (G4), we obtain $C_\varepsilon > 0$ such that

$$|G_s(|x|, s, t)| + |G_t(|x|, s, t)| \leq \varepsilon(|s|^{p(x)-1} + |t|^{p(x)-1}) + C_\varepsilon(|s|^{q(x)-1} + |t|^{q(x)-1}).$$

Assume that $(u_n, v_n) \rightharpoonup (u, v)$ in $X_r \times X_r$. Since $X_r \hookrightarrow L^{q(x)}(\mathbb{R}^N)$ is compact by Proposition 2.11, we have $(u_n, v_n) \to (u, v)$ in $(L^{q(x)}(\mathbb{R}^N))^2$. By Proposition 2.13(1) we have

$$G_s(|x|, u_n, v_n) - \varepsilon(|u_n|^{p(x)-1} + |v_n|^{p(x)-1}) \to G_s(|x|, u, v) - \varepsilon(|u|^{p(x)-1} + |v|^{p(x)-1}) \text{ in } (L^{q^o(x)}(\mathbb{R}^N))^2,$$

$$G_t(|x|, u_n, v_n) - \varepsilon(|u_n|^{p(x)-1} + |v_n|^{p(x)-1}) \to G_t(|x|, u, v) - \varepsilon(|u|^{p(x)-1} + |v|^{p(x)-1}) \text{ in } (L^{q^o(x)}(\mathbb{R}^N))^2.$$

So we obtain

$$\begin{aligned}
& \|G_s(|x|, u_n, v_n) - G_s(|x|, u, v)\| + \|G_t(|x|, u_n, v_n) - G_t(|x|, u, v)\| \\
=\ & \sup_{\|\tilde{u}\| \leq 1} \int_{\mathbb{R}^N} |G_s(|x|, u_n, v_n) - G_s(|x|, u, v)||\tilde{u}|\mathrm{d}x \\
& + \sup_{\|\tilde{v}\| \leq 1} \int_{\mathbb{R}^N} |G_t(|x|, u_n, v_n) - G_t(|x|, u, v)||\tilde{v}|\mathrm{d}x < c\varepsilon.
\end{aligned}$$

Therefore $\mathcal{G}'$ is completely continuous. $\square$

Since X_r is a separable and reflexive Banach space, there exist $\{e_j\}_{j=1}^{\infty} \subset X_r$ such that $(X_r)_n := \mathrm{span}\{e_1, \cdots, e_n\}$ and $(X_r)_n^{\perp} = \overline{\mathrm{span}\{e_{n+1}, \cdots\}}$. Now set $E = X_r \times X_r$ and $E_n = (X_r)_n \times (X_r)_n$. As we have done in (3.3), (3.4) and (3.5), we can obtain a limit index γ^{∞} with respect to $\{E_n\}$.

Lemma 4.4. Suppose that G satisfied (G1)-(G5). Then Ψ satisfies $(PS)_c^*$ condition with respect to $\{E_n\}$ for each c.

Proof. Lemma 3.3 is also suitable here if we replace Φ and L by Ψ and T respectively. Thus we only need to prove each sequence satisfying

$$\{(u_{n_k}, v_{n_k})\} \in E_{n_k}, \Psi(u_{n_k}, v_{n_k}) \to c, \|(\Psi_{n_k})'(u_{n_k}, v_{n_k})\| \to 0 \text{ as } n_k \to \infty,$$

is bounded where $\Psi_{n_k} := \Psi|_{E_{n_k}}$. By (G5) and Proposition 2.7, similar to (3.11), we have

$$\|u_{n_k}\| \geq \langle -(\Psi_{n_k})'(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\rangle \geq \|u_{n_k}\|^{p_-}.$$

So $\|u_{n_k}\|$ is bounded in X_r. On the other hand, by (G3), similar to (3.12), we have

$$c_1 \geq -\left(\frac{1}{p_-} - \frac{1}{\mu}\right)\|u_{n_k}\|^{p_+} + \left(\frac{1}{p_+} - \frac{1}{\mu}\right)\|v_{n_k}\|^{p_-} - \frac{2}{\mu}\|(\Psi_{n_k})'(u_{n_k}, v_{n_k})\|\left(\|u_{n_k}\| + \|v_{n_k}\|\right).$$

So $\|v_{n_k}\|$ is bounded in X_r. Thus $\{(u_{n_k}, v_{n_k})\}$ is a bounded sequence in E. $\square$

Proof of Theorem 1.2. We shall find the critical points of Ψ in E by using Theorem 2.20. By the assumption (G6), Ψ is invariant with respect to $\mathscr{G}$. Set $E = U \oplus V$, where $U = X_r \times \{0\}$ and $V = \{0\} \times X_r$. Set $Y_0 = \{0\} \times (X_r)_m^{\perp}$ and $Y_1 = \{0\} \times (X_r)_k$ where m and k are to be determined. Then Y_0 and Y_1 are $\mathscr{G}$-invariant and $\mathrm{codim}_V Y_0 = m$, $\dim Y_1 = k$, $\mathrm{Fix}\,\mathscr{G} = \{(0,0)\}$. So $\mathrm{Fix}\,\mathscr{G} \cap V = \{(0,0)\}$ and (B7)(a) of Theorem 2.20 is satisfied. It remains to verify (b) and (c) of (B7).

First, we verify (b) of (B7). After integrating, we obtain from (G2)-(G4) the existence of two positive constants c_1 and $c_2 < 1/p_+$ such that

$$G(|x|, s, 0) \geq c_1|s|^{\mu} - c_2|s|^{p(x)}, \quad \text{for all } x \in \mathbb{R}^N, s \in \mathbb{R}.$$

Hence, for all $u \in X_r$, we have

$$\begin{aligned} \Psi(u,0) &= -\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\mathrm{d}x - \mathcal{G}(u,0) \\ &\leq -\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\mathrm{d}x - c_1\int_{\mathbb{R}^N} |u|^{\mu}\mathrm{d}x + c_2\int_{\mathbb{R}^N} |u|^{p(x)}\mathrm{d}x < \infty. \end{aligned}$$

Then we can choose α such that $\alpha > \sup_{u \in X_r} \Psi(u,0)$.

If $(0,v) \in Y_0 \cap S_{\rho}$ (where $\rho > 1$ is to be determined), we have $v \in (X_r)_m^{\perp}$ and $\|v\| = \rho$. Define $f : X_r \to \mathbb{R}, f(v) = |v|_{q(x)}$. Since the compact embedding $X_r \hookrightarrow L^{q(x)}(\Omega)$, f is weakly-strongly continuous. By Proposition 3.5, $\delta_m \to 0$ as $m \to \infty$. Then by (G2), (G3),

$$\begin{aligned}
\Psi(0,v) &= \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla v|^{p(x)} + |v|^{p(x)})\mathrm{d}x - \mathcal{G}(0,v) \\
&\geq \frac{1}{p_+}\int_{\mathbb{R}^N}(|\nabla v|^{p(x)} + |v|^{p(x)})\mathrm{d}x - c_3\int_{\mathbb{R}^N}|v|^{q(x)}\mathrm{d}x - c_4 \\
&\geq \frac{1}{p_+}\|v\|^{p_-} - c_3|v|_{q(x)}^{q_+} - c_4 \\
&\geq \frac{1}{p_+}\rho^{p_-} - c_3\delta_m^{q_+}\rho^{q_+} - c_4.
\end{aligned}$$

Setting $\rho = (\frac{c_3 p_+ q_+ \delta_m^{q_+}}{p_-})^{\frac{1}{p_- - q_+}}$, we have

$$\Psi|_{Y_0 \cap S_\rho} \geq (q_+ - p_-)(p_+ q_+)^{\frac{q_+}{p_- - q_+}}\left(\frac{c_3}{p_-}\right)^{\frac{p_-}{p_- - q_+}}\delta_m^{\frac{p_- q_+}{p_- - q_+}} - c_4 \to +\infty \text{ as } m \to \infty.$$

Next, we verify (c) of (B7). For each $(u,v) \in U \oplus Y_1$, and $\|u\| > 1$, $\|v\| > 1$,

$$\begin{aligned}
\Psi(u,v) &\leq -\frac{1}{p_+}\|u\|^{p_-} + \frac{1}{p_-}\|v\|^{p_+} - c_5\int_\Omega(|u|^\mu + |v|^\mu)\mathrm{d}x + c_6 \\
&\leq \frac{1}{p_-}\|v\|^{p_+} - c_5\int_\Omega |v|^\mu \mathrm{d}x + c_6.
\end{aligned}$$

Since all norms are equivalent in the finite dimension space Y_1, we get

$$\Psi(u,v) \leq \frac{1}{p_-}\|v\|^{p_+} - c_7\|v\|^\mu + c_8.$$

Then we have $\sup \Psi|_{U \oplus Y_1} < +\infty$ since $\mu > p_+$. Thus we can choose $k > m$ and $\beta > \alpha$ such that $\Psi|_{U\oplus Y_1} \leq \beta$.

So

$$d_j = \sup_{\gamma^\infty(A) \geq j} \inf_{z \in A} \Psi(z), \quad -k+1 \leq j \leq -m,$$

are critical values of Ψ and $\alpha \leq d_j \leq \beta$. Since α can be chosen arbitrarily large, Ψ has a sequence of critical values $d_n \to +\infty$.

If $N = 4$ or $N \geq 6$, using the Bartsch-Willem's famous nonradial solutions result in [2] (see also [27] Theorem 1.31), the problem (1.2) possesses infinitely many nonradial solutions. □

Acknowledgement. The author would like to thank Professor Xian-Ling Fan for his valuable suggestions and comments on this paper.

References

[1] E. Acerbi and G. Mingione: Regularity results for a class of functionals with non-standard growth, *Arch. Ration. Mech. Anal.*, **156**(2001), 121-140.

[2] T. Bartsch and M. Willem: Infinitely many nonradial solutions of a Euclidean scalar field equation, *J. Funct. Anal.*, **117**(1993), 447-460.

[3] Y. M. Chen, S. Levine and M. Rao: Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.*, **66(4)**(2006), 1383-1406.

[4] L. Diening, P. Hästö and A.Nekvinda: Open problems in variable exponent Lebesgue and Sobolev spaces, *FSDONA04 Proceedings*, (P. Drábek and J. Rákosník (eds.); Milovy, Czech Republic, 2004), 38-58, Academy of Sciences of the Czech Republic, Prague, 2005.

[5] X. L. Fan: Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients, *J. Math. Anal. Appl.*, **312**(2005), 464-477.

[6] X. L. Fan: On the sub-supersolution method for $p(x)$-Laplacian equations, *J. Math. Anal. Appl.*, (2006), doi:10.1016/j.jmaa.2006.07.093

[7] X. L. Fan: Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form, preprint.

[8] X. L. Fan and X. Y. Han: Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, *Nonlinear Analysis: TMA*, **59**(2004), 173-188.

[9] X. L. Fan and W. M. Liu: An exact inequality involving Luxemburg norm and conjugate-Orlicz norm in $L^{p(x)}(\Omega)$, *Chinese Annals of Mathematics, Ser. A*, **27(2)**(2006), 177-188.(Chinese) The English translation is published in: *Chinese Journal of Contemporary Mathematics*, **27(2)**(2006), 147-158.

[10] X. L. Fan, J. S. Shen and D. Zhao: Sobolev embedding theorems for spaces $W^{k,p(x)}$, *J. Math. Anal. Appl.* **262**(2001), 749-760.

[11] X. L. Fan and Q. H. Zhang: Existence of solutionas for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Analysis: TMA*, **52**(2003), 1843-1852.

[12] X. L. Fan and D. Zhao: On the space $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263**(2001), 423-446.

[13] X. L. Fan, Y. Z. Zhao and D. Zhao: Compact imbedding theorems with symmetry of Strauss-Lions type for the space $W^{1,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **255**(2001), 333-348.

[14] A. Hamidi: Existence results to elliptic systems with nonstandard growth conditions, *J. Math. Anal. Appl.*, **300**(2004), 30-42.

[15] P. Harjulehto and P. Hästö: An overview of variable exponent Lebesgue and Sobolev spaces, *Future Trends in Geometric Function Theory*, (D. Herron (ed.), RNC Workshop, Jyväskylä, 2003), 85-94.

[16] D. W. Huang and Y. Q. Li: Multipliciy of solutions for a noncooperative p-Laplacian elliptic system in $\mathbb{R}^N$, *J. Differential Equations*, **215**(2005), 206-223.

[17] O. Kováčik and J. Rákosník: On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czech. Math. J.*, **41(116)**(1991), 592-618.

[18] Y. Q. Li: A limit index theory and its applications, *Nonlinear Analysis: TMA*, **25(12)**(1995), 1371-1389.

[19] M. Mihailescu: Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator, *Nonlinear Analysis: TMA*, (2006), doi:10.1016/j.na.2006.07.027

[20] J. Musielak: Orlicz spaces and modular spaces, *Lecture Notes in Math.*, Vol. **1034**, Springer-Verlag, Berlin, 1983.

[21] W. Orlicz: Über konjugierte Exponentenfolgen, *Studia Math.*, **3**(1931), 200-211.

[22] R. Palais: The principle of symmetric criticality, *Comm. Math. Phys.*, **69**(1979), 19-30.

[23] M. Růžička: Electrorheological Fluids: Modeling and Mathematical Theory, *Lecture Notes in Math.*, Vol. **1748**, Springer-Verlag, Berlin, 2000.

[24] S. Samko: Convolution type operators in $L^{p(x)}$, *Integr. Transform. and Special Funct.*, **7(1-2)**(1998), 123-144.

[25] S. Samko: On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators, *Integral Transforms and Special Functions*, **16**(2005), 461-482.

[26] M. Sturwe: Variational methods, 2nd edition, Springer-Verlag, 1996.

[27] M. Willem: Minimax theorems, *Progress in Nonlinear Differential Equations and Their Applications*, Vol. **24**, Birkhäuser, 1996.

[28] Q. H. Zhang: Existence of radial solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, *J. Math. Anal. Appl.*, **315**(2006), 506-516.

[29] Q. H. Zhang: Existence of positive solutions for elliptic systems with nonstandard $p(x)$-growth conditions via sub-supersolution method, *Nonlinear Analysis: TMA*, (2006), doi:10. 1016/j.na.2006.06.017.

[30] Q. H. Zhang: Existence of solutions for weighted $p(r)$-Laplacian system boundary value problems, *J. Math. Anal. Appl.*, (2006), doi:10.1016/j.jmaa.2006.03.087.

[31] V. V. Zhikov: Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izvestiya*, **29**(1987), 33-36.