Notes on Dynamic Macroeconomic Theory

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1 Dynamic Programming

1.1 Sequential problems

We want to choose an infinite sequence of "controls" $\{u_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t r(x_t, u_t), \tag{1}$$

subject to $x_{t+1} = g(x_t, u_t)$, with x_0 given. Dynamic programming seeks a time-invariant **policy function** h mapping the state x_t into the control u_t, such that the squence $\{u_s\}_{s=0}^{\infty}$ generated by iterating the two functions

$$\begin{aligned} u_t &= h(x_t) \\ x_{t+1} &= g(x_t, u_t), \end{aligned} \tag{2}$$

starting from initial condition x_0 at $t = 0$. A solution in the form of 2 is **recursive**. To find the policy function h we need to know the **value function** $V(x)$. In particular, define

$$V(x_0) = \max_{\{u_s\}_{s=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t), \tag{3}$$

where the maximization is subject to $x_{t+1} = g(x_t, u_t)$, with x_0 given. If we knew $V(x_0)$, then the policy function h could be computed by solving for each $x \in X$ the problem

$$\max_u \{r(x, u) + \beta V(\tilde{x})\}, \tag{4}$$

where the maximization is subject to $\tilde{x} = g(x, u)$ with x given, and $\tilde{x}$ denotes the state next period. Thus, we have changed the original problem for the problem of finding $V(x)$ and a function h that solves the continuum of maximum problem 4 –one maximum problem for each value of x.

Our task has become jointly to solve for $V(x)$, $h(x)$, which are linked by the **Bellman equation**

$$V(x) = \max_u \{r(x, u) + \beta V[g(x, u)]\}. \tag{5}$$

The maximizer is a policy function $h(x)$ that satisfies

$$V(x) = r[(x, h(x))] + \beta V\{g[x, h(x)]\}. \tag{6}$$

Under particular assumptions about r and g, it turns out that

1. The fucntional equation 5 has a unique strictly concave solution.

2. This solution is approached in the limit as $j \to \infty$ by iterations on

$$V_{j+1}(x) = \max_u \{r(x,u) + \beta V_j(\tilde{x})\}$$

subject to $\tilde(x) = g(x,u)$, x given, starting from any bounded and continuous initial V_0

3. There is a unique and time-invariant optimal policy of the form $u_t = h(x_t)$, where h is chosen to maximize the right side of 5.

4. Off corners, the limiting value function V is differentiable.

Since V is differentiable, the first order condition for 4 becomes

$$r_2(x,u) + \beta V'\{g(x,u)\} g_2(x,u) = 0. \tag{7}$$

If we also assume that the policy function h(x) is differentiable, differentiation of expression 6 yields

$$\begin{aligned} V'(x) =& r_1[x,h(x)] + r_2[x,h(x)]h'(x) \\ &+ \beta V'\{g[x,h(x)]\}\{g_1[x,h(x)] + g_2[x,h(x)]h'(x)\}. \end{aligned} \tag{8}$$

When the states and controls can be defined in such a way that only u appears in the transition equation $\tilde{x} = g(u)$: 8 becomes, after substituting 7 with $u = h(x)$,

$$V'(x) = r_1[x,h(x)]. \tag{9}$$

This is a version of a formula of **Benveniste and Scheinkman (1979)**.

1.1.1 Three computational methods

Value function iteration/ Iterating on the Bellman equation. Construct a sequence of value functions and associated policy functions by iterating on the following equation, starting from $V_0 = 0$, and continuing until V_j has converged.

$$V_{j+1}(x) = \max_u \{r(x,u) + \beta V_j(\tilde{x})\}, \tag{10}$$

subject to $\tilde{x} = g(x,u)$, x given.

Guess and verify. Guess and verify a solution V to 5. Not generally available.

Howard's improvement algorithm/ Policy function iteration.

1. Pick a feasible policy, $u = h_0(x)$, and compute the value associated with operating forever with that policy:

$$V_{h_j}(x) = \sum_{t=0}^{\infty} \beta^t r[x_t, h_j(x_t)],$$

where $x_{t+1} = g[x_t, h_j(x_t)]$, with $j = 0$.

2. Generate a new policy $u = h_{j+1}(x)$ that solves the two-period problem

$$\max_u \{r(x,u) + \beta V_{h_j}[g(x,u)]\},$$

for each x.

3. Iterate over j to convergence on steps 1 and 2.

The policy improvement algorithm often converges faster than does value function iteration. Each of our three methods for solving dynamic programming problems has its uses. Each is easier said than done, because it is typically impossible analytically to compute even one iteration on equation 10.

1.1.2 Cobb-Douglas transition, logarithmic preferences

A planner chooses sequences $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t ln(c_t)$$

subject to a given k_0 and a transition law

$$k_{t+1} + c_t = Ak_t^{\alpha}, \tag{11}$$

where $A > 0, \alpha \in (0,1), \beta \in (0,1)$.

Here we only show how to apply value function iteration. Start with $v_0(k) = 0$, and solve

$$\begin{aligned} \max_{c,\tilde{k}} \quad & ln(c) \\ \text{s.t.} \quad & c + \tilde{k} = Ak^{\alpha} \end{aligned}$$

The solution is evidently to set $c = Ak^{\alpha}, \tilde{k} = 0$, which produces an optimized value $v_1(k) = ln(c) = ln(Ak^{\alpha}) = lnA + \alpha lnk$. At the second step, we solve

$$\begin{aligned} \max_{c,\tilde{k}} \quad & ln(c) + \beta(lnA + \alpha ln\tilde{k}) \\ \text{s.t.} \quad & c + \tilde{k} = Ak^{\alpha} \end{aligned}$$

By Lagrangian $c = \frac{1}{1+\beta\alpha}Ak^{\alpha}, \tilde{k} = \frac{\beta\alpha}{1+\beta\alpha}Ak^{\alpha}, v_2(k) = ln\frac{A}{1+\beta\alpha} + \beta lnA + \alpha\beta ln\frac{\alpha\beta A}{1+\beta\alpha} + \alpha(1 + \alpha\beta)lnk$. Continuing and using the algebra of geometric series gives the results.

1.1.3 Euler equations

When the states and controls can be defined in such a way that only u appears in the transition equation, i.i., $\tilde{x} = g(u)$: the first-oder condition 7 in conjunction with the Benveniste-Scheinkman formula 9 implies

$$r_2(x_t, u_t) + \beta r_1(x_{t+1}, u_{t+1})g'(u_t) = 0, \qquad x_{t+1} = g(u_t). \tag{12}$$

The first equation is called an **Euler equation**.

1.1.4 A sample Euler equation

Consider the **Ramsey problem** of choosing $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize $\sum_{t=0}^{\infty} \beta^t u(c_t)$ subjecy to $c_t + k_{t+1} = f(k_t)$, where k_0 is given and the one-period utility function satisfies $u'(c) > 0, u''(c) < 0, \lim_{c_t \to 0} u'(c_t) = \infty$, and where $f'(k) > 0, f''(k) < 0$. Let the state be k and the control be $\tilde{k}$[1]. Substitute $c = f(k) - \tilde{k}$ into the utility function and express the Bellman equation as

$$v(k) = \max_{\tilde{k}} \left\{ u[f(k) - \tilde{k}] + \beta v(\tilde{k}) \right\}. \tag{13}$$

Application of the Benveniste-Scheinkman formula gives

$$v'(k) = u'[f(k) - \tilde{k}]f'(k). \tag{14}$$

[1]Note the transition equation in this problem is $\tilde{k} = \tilde{k}$, so it satisfies the requirement of apllying the Benveniste-Scheinkman formula

Notice that the first-order condition for the maximum problem on the 13 is $-u'[f'(k)-\tilde{k}]+\beta v'(\tilde{k})=0$, which, using 14, gives

$$u'[f(k)-\tilde{k}]=\beta u'[f(\tilde{k})-\hat{k}]f'(\tilde{k}), \tag{15}$$

where $\hat{k}$ denotes the two-period-ahead value of k. Equation 15 can be expressed as

$$1=\beta\frac{u'(c_{t+1})}{u'(c_t)}f'(k_{t+1}). \tag{16}$$

1.2 Stochastic control problems

Consider the problem of maximizing

$$E_0\sum_{t=0}^{\infty}\beta^t r(x_t,u_t), \qquad 0<\beta<1, \tag{17}$$

subject to

$$x_{t+1}=g(x_t,u_t,\epsilon_{t+1}), \tag{18}$$

with x_0 known, where ϵ_t is a sequence of independently and identically distributed random variables with cumulative probability distribution function $\text{prob}\{\epsilon_t\le e\}=F(e)$ for all t. At time t, x is assumed to be known, but $x_{t+1}, j\ge 1$ is not known at t. That is, ϵ_{t+1} is realized at $(t+1)$, after u_t has been chosen at t.

We continue to have a recursive structure. In particular, controls dated t affect returns $r(x_s,u_s)$ for $s\ge t$ but not earlier. This feature implies that dynamic programming methods remain appropriate.

The Bellman equation becomes

$$V(x)=\max_u\{r(x,u)+\beta E[V[g(x,u,\epsilon)]|x]\}, \tag{19}$$

where $E\{V[g(x,u,\epsilon)|x]\}=\int V[g(x,u,\epsilon)]dF(\epsilon)$. The solution can be computed by iterating on

$$V_{j+1}(x)=\max_u\{r(x,u)+\beta E[V_j[g(x,u,\epsilon)]|x]\}, \tag{20}$$

starting from any bounded continuous initial V_0.

The first-order necessary condition for the problem on the right side of 19 is

$$r_2(x,u)+\beta E\{V'[g(x,u,\epsilon)]g_2(x,u,\epsilon)|x\}=0.$$

Off corners, the value function satisfies

$$\begin{aligned}V'(x)=&r_1[x,h(x)]+r_2[x,h(x)]h'(x)\\&+\beta E\{V'\{g[x,h(x),\epsilon]\}\{g_1[x,h(x),\epsilon]\}+g_2[x,h(x),\epsilon]h'(x)|x\}.\end{aligned}$$

When the states and controls can be defined in such a way that x does not appear in the transition equation, the formula for $V'(x)$ becomes

$$V'(x)=r_1[x,h(x)].$$

Substituting it into the first-order necessary condition for the problem gives the **stochastic Euler equation**

$$r_2(x,u)+\beta E[r_1(\tilde{x},\tilde{u})g_2(x,u,\epsilon)|x]=0. \tag{21}$$

2 Practical Dynamic Programming

2.1 The curse of dimensionality

This chapter describes two popular methods for obtaining numerical approximations. The first method replaces the original problem with another problems that forces the state vector to live on a finite and discrete grid of points, then applies discrete-state dynamic programming to this problem. The "curse of dimensionality" impels us to keep the number of points in the discrete state space small. The second approach uses polynomials to approximate the value function.

2.2 Discrete-state dynamic programming

We introduce the method of discretization of state space by an example: An infinitely lived household likes to consume one good that can acquire by spending labor income or accumulated savings. The household has an endowment of labor at time t, s_t, that evolves according to an m-state Markov chain with transition matrix $\mathcal{P}$ and state space $[\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_m]$. If the realization of the process at t is $\bar{s}_i$, then the household receives labor income $w\bar{s}_i$.

The household can choose to hold an asset in discrete amounts $a_t \in \mathcal{A}$ where $\mathcal{A}$ is a grid $[a_1 < a_2 < \cdots < a_n]$. The asset bears a gross rate of return r.

Given initial values (a_0, s_0), the household choose a policy $\{a_{t+1}\}_{t=0}^{\infty}$ to maximize

$$E\sum_{t=0}^{\infty}\beta^t u(c_t), \tag{22}$$

subject to

$$\begin{aligned} c_t + a_{t+1} &= (r+1)a_t + ws_t \\ c_t &\geq 0 \\ a_{t+1} &\in \mathcal{A} \end{aligned} \tag{23}$$

We assume that $\beta(1+r) < 1$. Here $u(c)$ is strictly increasing, concave one-period utility function. Associated with this problem is the Bellman equation

$$v(a,s) = \max_{a'\in\mathcal{A}}\left\{u[(r+1)a + ws - a'] + \beta E v(a', s')|s\right\},$$

where a' is next period's value of asset holdings, and s' is next period's value of the shock. We seek a value function $v(a,s)$ and an associated policy function $a' = g(a,s)$ mapping this period's (a,s) pair into an optimal choice of assets to carry into next period. Let assets live on the grid $\mathcal{A} = [a_1 < a_2 < \cdots < a_n]$. Then we can express the Bellman equation as

$$v(a_i, \bar{s}_j) = \max_{a_h\in\mathcal{A}}\left\{u[(r+1)a_i + w\bar{s}_j - a_h] + \beta\sum_{l=1}^{m}\mathcal{P}_{jl}v(a_h, \bar{s}_l)\right\}, \tag{24}$$

for each $i \in [1, \cdots, n]$ and each $j \in [1, \cdots, m]$.

2.3 Bookkeeping

Let there be n states $[a_1, a_2, \cdots, a_n]$ for assets and two states $[s_1, s_2]$ for employment status. For $j = 1, 2$, define $n \times 1$ vectors $v_j, j = 1, 2$, whose ith rows are determined by

$v_j(i) = v(a_i, s_j), i = 1, \cdots, n$. For $j = 1, 2$, define two $n \times n$ matrices R_j whose (i, h) elements are

$$R_j(i, h) = u[(r + 1)a_i + ws_j - a_h], \quad i = 1, \cdots, n, \quad h = 1, \cdots, n.$$

Define an operator $T([v_1, v_2])$ that maps a pair of $n \times 1$ vectors $[v_1, v_2]$ into a pair of $n \times 1$ vectors $[tv_1, tv_2]$:

$$tv_j(i) = \max_h \{R_j(i, h) + \beta \mathcal{P}_{j1} v_1(h) + \beta \mathcal{P}_{j2} v_2(h)\}$$

for $j = 1, 2$. Here the "max" operator applied to an $(n \times m)$ matrix M returns an $(n \times 1)$ vector whose ith elements is the maximum of the ith row of the matrix M. This equation can be written as

$$\begin{bmatrix} tv_1 \\ tv_2 \end{bmatrix} = \max \left\{ \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \beta(\mathcal{P} \otimes \mathbf{1}) \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} \right\}, \tag{25}$$

where $\otimes$ is the Kronecker product.[2] The Bellman equation $[v_1, v_2] = T([v_1, v_2])$ can be solved by iterating to convergence on $[v_1, v_2]_{m+1} = T([v_1, v_2]_m)$.

2.4 Application of Howard improvement algorithm

It is also easy to implement the Howard improvement alogorithm in the present setting. At time t, the system resides in one of N predetermined positions, denoted x_i for $i = 1, 2, \cdots, N$. There exists a predetermined set $\mathcal{M}$ of $(N \times N)$ stochastic matrices P that are the objects of choice. Here $P_{ij} = \text{Prob}[x_{t+1} = x_j | x_t = x_i], i = 1, \cdots, N; j = 1, \cdots, N$.

The one-period return function is represented as c_P, a vector of length N, and is a function of P. The ith entry of c_P denotes the one-period return when the state of the system is x_i, and the transition matrix is P. The Bellman equation is

$$v_P(x_i) = \max_{P \in \mathcal{M}} \{c_P(x_i) + \beta \sum_{j=1}^{N} P_{ij} v_P(x_j)\}$$

or

$$v_P = \max_{P \in \mathcal{M}} \{c_P + \beta P v_P\}. \tag{26}$$

We can express this as

$$v_P = T v_P,$$

Define the operator

$$B = T - I,$$

so that

$$Bv = \max_{P \in \mathcal{M}} \{c_P + \beta P v\} - v. \tag{27}$$

In terms of the operator B, the Bellman equation is

$$Bv = 0. \tag{28}$$

The policy improvement algorithm consists of iterations on the following to steps.

[2] If A is an $m \times n$ matrix and B is a $p \times q$ matrix, the the Kronecker product $A \otimes B$ is the $mp \times nq$ block matrix

$$\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

1. For fixed P_n, solve
$$(I - \beta P_n)v_{P_n} = c_{P_n} \tag{29}$$
for v_{P_n}.

2. Find P_{n+1} such that
$$c_{P_{n+1}} + (\beta P_{n+1} - I)v_{P_n} = Bv_{P_n} \tag{30}$$

Step 2 amounts to finding a policy function (i.e., a stochastic matrix $P_{n+1} \in \mathcal{M}$) that solves a two-period problem with v_{P_n} as the terminal value function.

The policy improvement algorithm can be interpreted as a version of Newton's method.[3] Using equation 29 for $n+1$ to eliminate $c_{P_{n+1}}$ from equation 30 gives
$$(I - \beta P_{n+1})v_{P_{n+1}} + (\beta P_{n+1} - I)v_{P_n} = Bv_{P_n}$$
which implies
$$v_{P_{n+1}} = v_{P_n} + (I - \beta v_{P_{n+1}})^{-1} Bv_{P_n}. \tag{31}$$
Note this equation can be interpreted as Newton's method if we regard $(\beta P_{n+1} - I)$ as the gradient of Bv_{P_n}.

2.5 Numerical implementation

We shall illustrate Howard's policy improvement algorithm by applying it to our savings example. Consider a policy function $a' = g(a,s)$. For each j, define the $n \times n$ matrices J_j by
$$J_j(a,a') = \begin{cases} 1 & \text{if } g(a,s_j) = a' \\ 0 & \text{otherwise.} \end{cases}$$
Here $j = 1, 2, \cdots, m$ where m is the number of possible values for s_t, and $J_j(a,a')$ is the element of J_j with rows corresponding to initial assets a and columns to terminal assets a'. For a given policy function $a' = g(a,s)$ define the $n \times 1$ vectors r_j with rows corresponding to
$$r_j(a) = u[(r+1)a + ws_j - g(a,s_j)], \tag{32}$$
for $j = 1, \cdots, m$.

Suppose the policy function $a' = g(a,s)$ is used forever. Let the value be the m $(n \times 1)$ vectors $[v_1, \cdots, v_m]$, where $v_j(a_i)$ is the value starting from state (a_i, s_j). Suppose that $m = 2$. The vector $[v_1, v_2]$ obey
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + \begin{bmatrix} \beta\mathcal{P}_{11}J_1 & \beta\mathcal{P}_{12}J_1 \\ \beta\mathcal{P}_{21}J_2 & \beta\mathcal{P}_{22}J_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$
Then
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left[I - \beta \begin{pmatrix} \beta\mathcal{P}_{11}J_1 & \beta\mathcal{P}_{12}J_1 \\ \beta\mathcal{P}_{21}J_2 & \beta\mathcal{P}_{22}J_2 \end{pmatrix} \right]^{-1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{33}$$
Here is how to implement the Howard policy improvement algorithm.

1. For an initial feasible policy function $g_\tau(a,s)$ for $\tau = 1$, form the r_j matrices using equation 32, then use equation 33 to evaluate the vectors of value $[v_1^\tau, v_2^\tau]$ implied by using that policy forever.

2. Use $[v_1^\tau, v_2^\tau]$ as the terminal value vectors in 25, and perform one step on the Bellman equation to find a new policy function $g_{\tau+1}(a,s)$ for $\tau + 1 = 2$. Use this policy function, increment τ by 1, and repeat step 1.

3. Iterate to convergence on steps 1 and 2.

[3]Newton's method for finding the solution of $G(z) = 0$ is to iterate on $z_{n+1} = z_n - G'(z_n)^{-1}G(z_n)$.

2.5.1 Modified policy iteration

For $k \geq 2$, iterate k times on the Bellman equation. Take the resulting policy function and use equation 33 to produce a new candidate value function. Then starting from this terminal value function, perform another k iterations on the Bellman equation. Continue in this fashion until the decision rule converges.

2.6 Sample Bellman equations

2.6.1 Example 1: calculating expected utility

Suppose $u(c) = c^{1-\gamma}/(1-\gamma)$. Suppose that $c_{t+1} = \lambda_{t+1} c_t$ and that $\{\lambda_t\}$ is an n-state Markov process with transition matrix $P_{ij} = \text{Prob}(\lambda_{t+1} = \bar{\lambda}_j | \lambda_t = \bar{\lambda}_i)$. Suppose that we want to evaluate discounted expected utility

$$V(c_0, \lambda_0) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t). \tag{34}$$

We can express this equation recursively:

$$V(c_t, \lambda_t) = u(c_t) + \beta E_t V(c_{t+1}, \lambda_{t+1}) \tag{35}$$

We use a guess-and-verify technique to solve equation 35 for $V(c_t, \lambda_t)$. Guess that $V(c_t, \lambda_t) = u(c_t) w(\lambda_t)$. Substitute the guess into equation 35, divide both sides by $u)c_t$ to get

$$w(\lambda_t) = 1 + \beta E_t \left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma} w(\lambda_{t+1})$$

or

$$w_i = 1 + \beta \sum_j P_{ij} (\lambda_j)^{1-\gamma} w_j. \tag{36}$$

Equation 36 is a system of linear equations in $w_i, i = 1, \cdots, n$ whose soulution can be expressed as

$$w = \left[1 - \beta P \text{diag}(\lambda_1^{1-\gamma}, \cdots, \lambda_n^{1-\gamma}\right]^{-1} \mathbf{1} \tag{37}$$

where $\mathbf{1}$ is an $n \times 1$ vector of ones.

2.6.2 Example 2: risk-sensitive preferences

Modify the preferences to be of the recursive form

$$V(c_t, \lambda_t) = u(c_t) + \beta \mathcal{R}_t V(c_{t+1}, \lambda_{t+1}), \tag{38}$$

where

$$\mathcal{R}_t(V) = \left(\frac{2}{\sigma}\right) \log E_t \left[\exp\left(\frac{\sigma V_{t+1}}{2}\right)\right] \tag{39}$$

is an operator used by Jacobson (1973), Whittle (1990), and Hansen and Sargent (1995) to induce a preference for robustness to model misspecification. The method used in example 1 will not apply directly because the homogeneity property exploited there fails to prevail now.

2.6.3 Example 3: costs of business cycles

2.7 Polynonial approximations

3 Search and Unemployment

3.1 Preliminaries

3.1.1 Nonnegative random variables

We begin with some properties of nonnegative random variables that possess finite first moment. Consider a random variable p with a cumulative probability distribution function $F(P)$. We assume that $F(0) = 0$, that is, that p is nonnegative. We assume that F, a non-decreasing function, is continuous from the right. We also assume that there is an upper bound $B < \infty$ such that $F(B) = 1$, so that p is bounded with probability 1.

The mean of p is defined by

$$Ep = \int_0^B p \, dF(p). \tag{40}$$

Let $u = 1 - F(p)$ and $v = p$ and use the integration-by-parts formula to verify that

$$\int_0^B p \, dF(p) = \int_0^B [1 - F(p)] dp = B - \int_0^B F(p) \, dp. \tag{41}$$

Now consider two independent random variables p_1 and p_2 drawn from the distribution F. Consider the event $\{(p_1 < p) \cap (p_2 < p)\}$, which has probability $F(p)^2$. The event is equivalent to the event $\{\max(p_1, p_2) < p\}$. Therefore, if we use formula 41, the random variable $\max(p_1, p_2)$ has mean

$$E \max(p_1, p_2) = B - \int_0^B F(p)^2 \, dp. \tag{42}$$

Similarly, if $p_1, p_2, \cdots, p_n$ are n independent random variables drawn from F, we have

$$M_n \equiv E \max(p_1, p_2, \cdots, p_n) = B - \int_0^B F(p)^n \, dp. \tag{43}$$

3.1.2 Mean-preserving spreads

Consider a class of distributions with the same mean. We index this class by a parameter r belonging to some set R. For the rth distribution we denote $\text{Prob}\{p < P\} = F(P, r)$ and assume that $F(P, r)$ is differentiable with respect to r for all $P \in [0, B]$. We assume that there is a single finite B such that $F(B, r) = 1$ for all r in R and that $F(0, r) = 0$ for all r in R, so that we are considering a class of distributions R for nonnegative, bounded random variables.

From 41, we have

$$Ep = B - \int_0^B F(p, r) \, dp. \tag{44}$$

Therefore, two distributions with the same value of $\int_0^B F(\theta, r) \, \theta$ have identical means. We write this as the **identical means condition**:

(i) $$\int_0^B [F(\theta, r_1) - F(\theta, r_2)] \, d\theta = 0$$

Two distributions r_1, r_2 are said to satisfy the **single-crossing property** if there exists a $\hat{\theta}$ with $0 < \hat{\theta} < B$ such that

(ii) $$F(\theta, r_2) - F(\theta, r_1) \leq 0 (\geq 0) \qquad \text{when} \quad \theta \geq (\leq) \, \hat{\theta}.$$

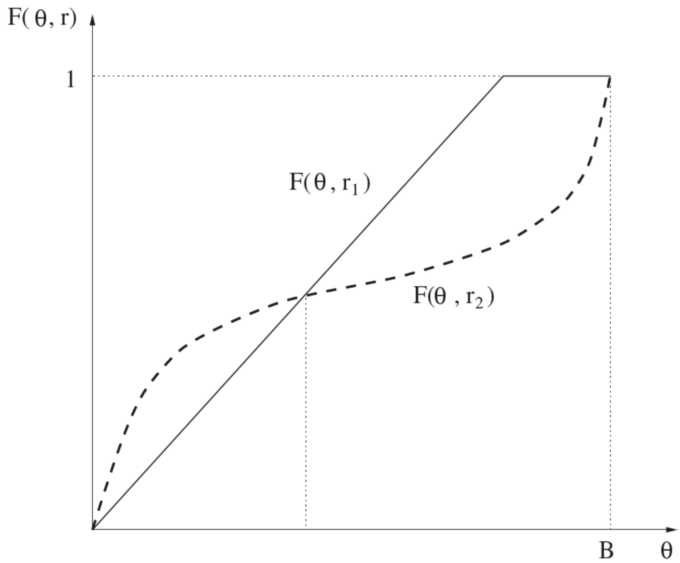

Figure 1: Two distributions, r_1 and r_2, that satisfy the single-crossing property.

Figure 1 illustrates the single-crossing property. If two distributions, r_1 and r_2, that satisfy properties (i) and (ii), we can regard distribution r_2 as having been obtained from r_1 by a process that shifts probability toward the tails of the distribution while keeping the mean constant.

Properties (i) and (ii) implies the following property:

(iii) $$\int_0^y [F(\theta, r_2) - F(\theta, r_1)]\, d\theta \geq 0, \qquad 0 \leq y \leq B.$$

A distribution indexed by r_2 is said to have been obtained from a distribution indexed by r_1 by a mean-preserving spread if the two distributions satisfy (i) and (iii).

For infinitesimal changes in r, Diamond and Stiglitz use the differntial versions of properties (i) and (iii) to rank distributions with the same mean in order of riskiness. An increment in r is said to represent a mean-preserving increase in risk if

(iv) $$\int_0^B F_r(\theta, r)\, d\theta = 0$$

(v) $$\int_0^y F_r(\theta, r)\, d\theta \geq 0, \qquad 0 \leq y \leq B,$$

where $F_r(\theta, r) = \partial F(\theta, r)/\partial r$.

3.2 McCall's model of intertemporal job search

Consider an unemployed worker who is searching for a job under the following circumstances: Each period the worker draws one offer w from the same wage distribution $F(W) = \text{Prob}\{w \leq W\}$, with $F(0) = 0, F(B) = 1$ for $B < \infty$. The worker has the option of rejecting the offer, receiving c in unemployment compensition, and waiting until next period to draw another offer from F; alternatively, the worker can accept the offer and receives w per period forever.

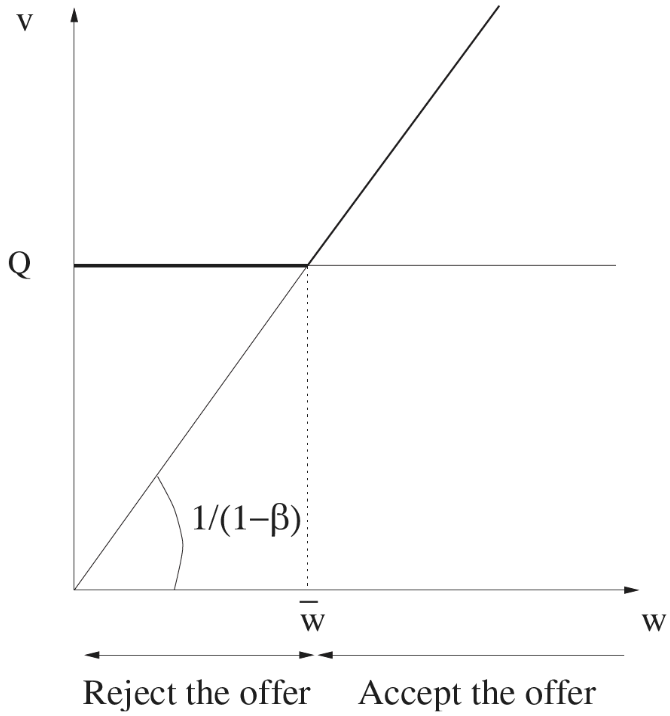

Figure 2: The function $v(w) = max\{w/(1-\beta), c + \beta \int_0^B v(w')\,dF(w')$. The reservation wage $\bar{w} = (1-\beta)[c + \int_0^B v(w')\,dF(w')]$.

Let y_t be the worker's income in period t. Then, $y_t = c$ if the worker is unemployed and $y_t = w$ if the worker has accepted an offer. The unemployed worker devises a stragtgy to maximize $\sum_{t=0}^{\infty} \beta^t y_t$.

Let $v(w)$ be the expected value of $\sum_{t=0}^{\infty} \beta^t y_t$ for a previousy unemployed worker who has offer w in hand, who is deciding whether to accept or to reject it, and who behaves optimally. The Bellman equation is

$$v(w) = \max_{\text{accept,reject}} \left\{ \frac{w}{1-\beta}, c + \beta \int_0^B v(w')\,dF(w') \right\} \tag{45}$$

Figure 2 graphs the functional equation 45 and reveals that its solution is of the form

$$v(w) = \begin{cases} \frac{\bar{w}}{1-\beta} = c + \beta \int_0^B v(w')\,dF(w') & \text{if} \quad w \le \bar{w} \\ \frac{w}{1-\beta} & \text{if} \quad w \ge \bar{w}. \end{cases} \tag{46}$$

Using 46, we can convert the functional equation 45 in the value function $v(w)$ into an ordinary equation in the reservation wage $\bar{w}$:

$$\frac{\bar{w}}{1-\beta} = c + \beta \int_0^{\bar{w}} \frac{\bar{w}}{1-\beta}\,dF(w') + \beta \int_{\bar{w}}^B \frac{w'}{1-\beta}\,dF(w')$$

or

$$\frac{\bar{w}}{1-\beta} \int_0^{\bar{w}} dF(w') + \frac{\bar{w}}{1-\beta} \int_{\bar{w}}^B dF(w') = c + \beta \int_0^{\bar{w}} \frac{\bar{w}}{1-\beta}\,dF(w') + \beta \int_{\bar{w}}^B \frac{w'}{1-\beta}\,dF(w')$$

or

$$\bar{w} \int_0^{\bar{w}} dF(w') - c = \frac{1}{1-\beta} \int_{\bar{w}}^B (\beta w' - \bar{w})\,dF(w').$$

Adding $\bar{w} \int_{\bar{w}}^{B} dF(w')$ to both sides gives

$$(\bar{w} - c) = \frac{\beta}{1-\beta} \int_{\bar{w}}^{B} (w' - \bar{w})\, dF(w'). \tag{47}$$

47 is often used to characterize the reservation wage $\bar{w}$. The left side is the cost of searching one more time when an offer $\bar{w}$ is in hand. The right side is the expected benefit of searching one more time interms of the expeted present value associated with drawing $w' > w$. 47 instructs the agent to set $\bar{w}$ so that the cost of searchign one more time equals the benefit.

3.2.1 Characterizing reservation wage

Define the right sife of equation 47 as

$$h(w) = \frac{\beta}{1-\beta} \int_{w}^{B} (w' - w)\, dF(w'). \tag{48}$$

Notice that $h(0) = Ew\beta/(1-\beta)$, that $h(B) = 0$, and

$$h'(w) = -\frac{\beta}{1-\beta}[1 - F(w)] < 0.$$

We also have

$$h''(w) = \frac{\beta}{1-\beta} F'(w) > 0,$$

so that $h(w)$ is convex to the origin. Figure 3 graphs $h(w)$ against $(w - c)$ and indicates how $\bar{w}$ is determined. To get another useful characterization of $\bar{w}$, we express 47 as

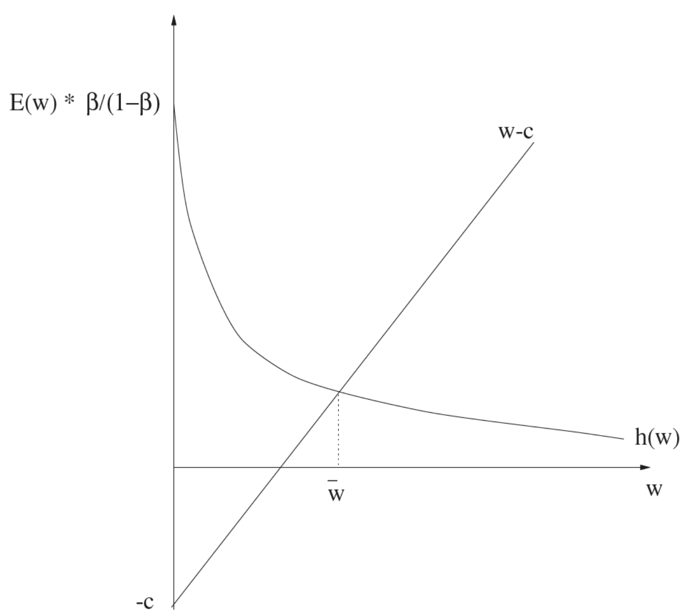

Figure 3: The reservation wage $\bar{w}$ that satisfies $\bar{w} - c = [\beta/1-\beta] \int_{\bar{w}}^{B} (w' - \bar{w})\, dF(w') \equiv h(\bar{w})$

$$\begin{aligned}\bar{w} - c &= \frac{\beta}{1-\beta} \int_{\bar{w}}^{B} (w' - \bar{w})\, dF(w') + \frac{\beta}{1-\beta} \int_{0}^{\bar{w}} (w' - \bar{w})\, dF(w') - \frac{\beta}{1-\beta} \int_{0}^{\bar{w}} (w' - \bar{w})\, dF(w') \\ &= \frac{\beta}{1-\beta} Ew - \frac{\beta}{1-\beta} \bar{w} - \frac{\beta}{1-\beta} \int_{0}^{\bar{w}} (w' - \bar{w})\, dF(w')\end{aligned}$$

or

$$\bar{w} - (1-\beta)c = \beta E w - \beta \int_0^{\bar{w}} (w' - \bar{w})\, dF(w').$$

Applying integration by parts to the last integral on the right side and rearranging, we have

$$\bar{w} - c = \beta(Ew - c) + \beta \int_0^{\beta} F(w')\, dw' \tag{49}$$

At this point it is useful to define

$$g(s) = \int_0^{s} F(p)\, dp \tag{50}$$

This function has the characteristics that $g(0) = 0, g(s) > 0, g'(s) = F(s) > 0$, and $g''(s) = F'(S) > 0$ for $s > 0$. Then equation 49 can be represented as $\bar{w} - c = \beta(Ew - c) + \beta g(\bar{w}.)$ Figure 4 uses equation 49 to determine $\bar{w}$.

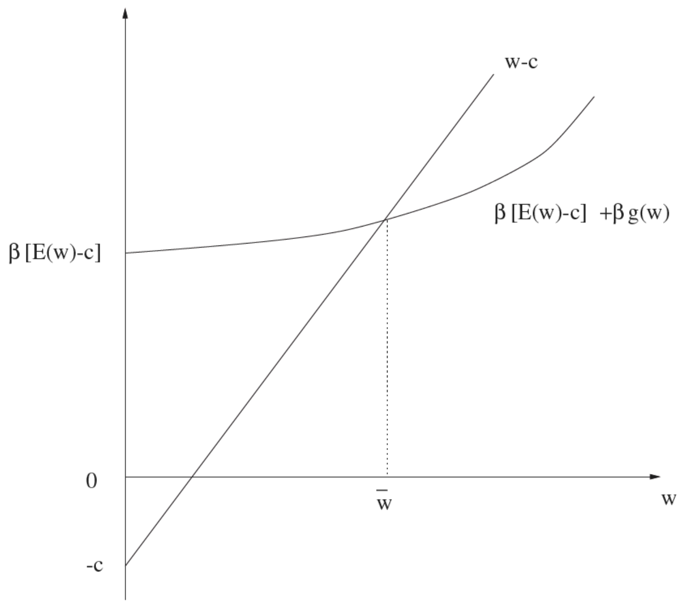

Figure 4: The reservation wage, w, that satisfies $\bar{w} - c = \beta(Ew - c) + \beta \int_0^{\bar{w}} F(w')\, dw' \equiv \beta(Ew - c) + \beta g(\bar{w})$.

3.2.2 Effects of mean-preserving spreads

Figure 4 can be used to establish two properties about $\bar{w}$. First, given F, $\bar{w}$ increases when c increases. Second, given c, a mean-preserving increase in risk causes $\bar{w}$ to increase. The latter causes an increase in $g(w)$ and an upward shift in $\beta(Ew - c) + \beta g(w)$.

Since an increase in unemployment compensation and a mean-preserving increase in risk both raise the reservation wage, it follows 46 that unemployed workers are also better off with both such increases. Intuition for this latter finding can be gleaned from the result in option pricing theory that the value of an option is an increasing function of the variance in the price of the underlying asset. In our context, the unemployed worker has the option to accept a job. Under a mean-preserving increase in risk, the higher incidence of very good wage offers increases the value of searching for a job while the higher incidence of very bad wage offers is not detrimental because the option to work will not be exercised at such low wages.

3.2.3 Allowing quits

In this model, had we allowed the worker to quit and search again, after being unemployed one period, he would never exercise that option. To see this point, recall that the reservation wage $\bar{w}$ satisfies

$$v(\bar{w}) = \frac{\bar{w}}{1-\beta} = c + \beta \int v(w')\, dF(w'). \tag{51}$$

We compute the lifetime utility associated with three mutually exclusive alternative ways of responding to that offer:

A1. Accept the wage and keep the job forever:

$$\frac{w}{1-\beta}.$$

A2. Accept the wage but quit after t periods:

$$\frac{w-\beta^t w}{1-\beta} + \beta^t (c + \beta \int w(w')\, dF(w')) = \frac{w}{1-\beta} - \beta^t \frac{w-\bar{w}}{1-\beta}.$$

A3. Reject the wage:

$$\frac{\bar{w}}{1-\beta}.$$

We conclude that if $w < \bar{w}$,

$$A1 \succ A2 \succ A3,$$

and if $w > \bar{w}$,

$$A1 \prec A2 \prec A3.$$

3.2.4 Waiting times

Let N be the random variable "length of time until a successful offer is encountered," with the understanding that $N = 1$ if the first job offer is accepted. Let $\lambda = \int_0^{\bar{w}} dF(w')$ be the probability that a job offer is accepted. Then $\text{Prob}\{N = j\} = (1-\lambda)\lambda^{j-1}$, so the waiting time is geometrically distributed. The mean waiting time $\bar{N} = \frac{1}{1-\lambda}$. It can be proved that the mean waiting time increases with increases in the rate of unemployment compensation, c.

3.2.5 Firing

We now consider a modification of the job search model in which each period after the first period on the job the worker faces probability α of being fired. A worker who is fired becomes unemployed for one period before drawing a new wage. Only previously employed workers are fired. A previously employed worker who is fired at the beginning of a period cannot draw a new wage offer that period but must be unemployed for one period.

Let $\hat{v}(w)$ be the expected present value of income of a previously unemployed worker who has offer w in hand and who behaves optimally. If she accepts the offer

$$\hat{v}(w) = w + \beta(1-\alpha)\hat{v}(w) + \beta\alpha \left[c + \beta \int \hat{v}(w')\, dF(w') \right].$$

Thus, the Bellman equation becomes

$$\hat{v}(w) = \max_{\text{accept,reject}} \{w + \beta(1-\alpha)\hat{v}(w) + \beta\alpha[c + \beta E\hat{v}],\, c + \beta E\hat{v}\},$$

where $E\hat{v} = \int \hat{v}(w')\, dF(w')$. This equation has a solution of the form

$$\hat{v}(w) = \begin{cases} \frac{w+\beta\alpha[c+\beta E\hat{v}]}{1-\beta(1-\alpha)}, & \text{if} \quad w \geq \bar{w} \\ c + \beta E\hat{v} & \text{if} \quad w \leq \bar{w}. \end{cases}$$

where $\bar{w}$ solves

$$\frac{w + \beta\alpha[c + \beta E\hat{v}]}{1 - \beta(1-\alpha)} = c + \beta E\hat{v},$$

which can be rearranged as

$$\frac{\hat{w}}{1-\beta} = c + \beta \int \hat{v}(w')\, dF(w'). \tag{52}$$

51 and 52 look identical but the reservation wages differ because the value functions differ. In particular, $\hat{v}(w)$ is strictly less than $v(w)$. This is an immediate implication of our argument that it cannot be optimal to quit if you have accepted a wage strictly greater than the reservation wage in the situation without possible firings. Since the employed workers in the situation where they face possible firings are worse off than employed workers in the situation without possible firings, it follows that $\hat{v}(w)$ lies strictly below $v(w)$ over the whole domain because, even at wages that are rejected, the value function partly reflects a stream of future outcomes whose expectation is less favorable in the situation in which workers face a chance of being fired.

Since $\hat{v}(w)$ lies strictly below $v(w)$, it follows from 51 and 52 that the reservation wage w is strictly lower with firings. There is less of a reason to hold out for high-paying jobs when a job is expected to last for a shorter period of time. That is, unemployed workers optimally invest less in search when the payoffs associated with wage offers have gone down because of the probability of being fired.

3.2.6 A lake model

3.2.7 A model of career choice

A worker chooses career-job (θ, ϵ) pairs subject to the following conditions: There is no unemployment. The worker's earnings at time t equal $\theta_t + \epsilon_t$, where θ_t is a career and ϵ_t is a job. The worker maximizes $E\sum_{t=0}^{\infty}\beta^t(\theta_t + \epsilon_t)$. A career θ is drawn from c.d.f. F ; a job ϵ is drawn from c.d.f. G. Successive draws are independent, and $G(0) = F(0) = 0$, $G(B_\epsilon) = F(B_\theta) = 1$. The worker can draw a new career only if he also draws a new job. The worker is free to retain his existing career θ, and to draw a new job ϵ'. The worker decides at the beginning of a period.

Let $v(\theta, \epsilon)$ be the optimal value of the problem at the beginning of a period for a worker currently having inherited career-job pair (θ, ϵ) and who is about to decide whether to draw a new career and or job. The value function $v(\theta, \epsilon)$ satisfies the Bellman equation

$$\begin{aligned} v(\theta,\epsilon) = \max\Big\{&\theta + \epsilon + \beta v(\theta,\epsilon),\, \theta + \int [\epsilon' + \beta v(\theta,\epsilon')]\, dG(\epsilon'), \\ &\int\int [\theta' + \epsilon' + \beta v(\theta',\epsilon')]\, dF(\theta') dG(\epsilon')\Big\}. \end{aligned} \tag{53}$$

When the career-job pair (θ, ϵ) is such that the worker chooses to stay put, the value function in 53 attains the value $\theta + \epsilon)/(1\beta)$. This happens when the decision to stay put weakly dominates the other two actions, which occurs when

$$\frac{\theta + \epsilon}{1 - \beta} \geq \max C(\theta, Q), \tag{54}$$

where Q is the value of drawing both a new job and a new career,

$$Q \equiv \int \int [\theta' + \epsilon' + \beta v(\theta', \epsilon')] \, dF(\theta') dG(\epsilon'),$$

and $C(\theta)$ is the value of keeping but drawing a new job,

$$C(\theta) = \theta + \int [\epsilon' + \beta v(\theta, \epsilon')] \, dG(\epsilon').$$

For a given career θ, a job $\bar{\epsilon}(\theta)$ makes 54 hold with equality. Evidently, $\bar{\epsilon}(\theta)$ solves

$$\bar{\epsilon}(\theta) = \max[(1 - \beta)C(\theta) - \theta, (1 - \beta)Q - \theta].$$