ECN-614 ASSIGNMENT:

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1. Given the AR process with the following statistics

a) $x[n] = a_1 x[n-1] + a_2 x[n-2] + w[n]$

$a_1 = 1.2 \quad a_2 = -0.8 \quad \sigma_w^2 = 0.3$

The AR process is hence defined by the following difference equation

$x[n] = 1.2\,x[n-1] + w[n] - 0.8\,x[n-2]$ — (1)

To solve for the autocorrelation function values:
Multiply on both sides with $x[n-2]$ and taking expectations

$r(2) = 1.2\,r(1) + 0 - 0.8\,r(0)$ — (2)

[$r(\ell-k) = E[x[n-k]\,x^*[n-\ell]]$
real valued signals hence
$x^*[n-\ell] = x[n-\ell]$]

Multiply on both sides with $x[n-1]$ and take expectations:

$r(1) = 1.2\,r(0) + 0 - 0.8\,r(1)$

$1.8\,r(1) = 1.2\,r(0) \Rightarrow r(0) = 1.5\,r(1)$ — (3)

Multiply on both sides with $x[n]$ and take expectations

$r(0) = 1.2\,r(1) + 0.3 - 0.8\,r(2)$ — (4)

[$E[w[n] * x[n]] = \sigma_w^2$
$E[w[n] * x[n-k]] = 0$
$\forall k > 0$]

Substitute (3) into (4)

$1.5\,r(1) = 1.2\,r(1) - 0.8\,r(2) + 0.3$

$\Rightarrow 0.3\,r(1) + 0.8\,r(2) = 0.3$ — (5)

Substitute (3) into (2)

$r(2) = 1.2\,r(1) - 0.8 \times 1.5\,r(1)$

$\Rightarrow \boxed{r(2) = 0}$ — (6)

Substitute (6) into (5)

$\Rightarrow 0.3\,r(1) + 0.8 \times 0 = 0.3$

$\Rightarrow \boxed{r(1) = 1}$ — (7)

Substitute (7) into (3)

$r(0) = 1.5 \times 1$

$\boxed{r(0) = 1.5}$

The final autocorrelation values are

$$\boxed{\begin{aligned} r_{xx}(0) &= 1.5 \\ r_{xx}(1) &= 1.0 \\ r_{xx}(2) &= 0 \end{aligned}}$$

b) For a forward prediction filter; the weiner filter equation is characterized by:

$$R_{xx} W_{f,opt} = r \quad - \; (1)$$

For a 2 tap weiner filter ~~characterized by a real signal~~

$$R_{xx} = \begin{bmatrix} r_{xx}(0) & r_{xx}(1) \\ r^*_{xx}(1) & r_{xx}(0) \end{bmatrix} \qquad r = [r^*(1) \;\; r^*(2)]$$

From the values found out in previous question

$$R_{xx} = \begin{bmatrix} 1.5 & 1 \\ 1 & 1.5 \end{bmatrix} \qquad r = [1 \;\; 0]$$

Put these values into (1)

$$\begin{bmatrix} 1.5 & 1 \\ 1 & 1.5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \left[\text{Assume } W_{f,opt} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right]$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \\ 1 & 1.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{(1.5)^2 - (1)^2} \begin{bmatrix} 1.5 & -1 \\ -1 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right]$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{1.25} \begin{bmatrix} 1.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 1.2 \\ -0.8 \end{bmatrix}$$

$$\Rightarrow W_{f,opt} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1.2 \\ -0.8 \end{bmatrix}$$

$\Rightarrow$ Optimum tap weight vector of the forward predictor is

$$\boxed{W_{opt} = \begin{bmatrix} 1.2 \\ -0.8 \end{bmatrix}}$$

c) J_{min} for a forward prediction filter is given by

$$J_{min} = r(0) - r^H w_{f,opt}$$

$r(0) = 1.5$

$r^H = [1 \quad 0]$

$w_{f,opt} = \begin{bmatrix} 1.2 \\ -0.8 \end{bmatrix}$

$$J_{min} = 1.5 - [1 \quad 0]\begin{bmatrix} 1.2 \\ -0.8 \end{bmatrix}$$

$$\boxed{J_{min} = 1.5 - 1.2 = 0.3}$$

⇒ The minimum mean square prediction error J_{min} produced ~~predicted~~ ~~given~~ by the predictor is given by 0.3.

d) Let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$J(w)$ for a weiner filter is given by the following equation

$$J(w) = \sigma_d^2 - w^H p - p^H w + w^H R w$$

for a forward prediction filter:

$\sigma_d^2 = r(0)$

$p = r$

$$\Rightarrow J(w) = r(0) - w^H p - p^H w + w^H R w \quad -(1)$$

The mean square prediction error when the filter tap-weight vector $w = 0$ is given by

$$J(0) = r(0) - 0 - 0 + 0 \Rightarrow \boxed{J(0) = 1.5} \quad [\text{from } 1a)]$$

Mean square prediction error when the filter top weight vector $w=0$ is $\boxed{J(0) = 1.5}$

To plot the contour for $J(0)$; substitute $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and $J(w) = 1.5$ into equation (1)

$$J(0) = 1.5 - [w_1 \quad w_2]\begin{bmatrix} 1 \\ 0 \end{bmatrix} - [1 \quad 0]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + [w_1 \quad w_2]\begin{bmatrix} 1.5 & 1 \\ 1 & 1.5 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$J(0) = 1.5 - w_1 - w_1 + [1.5w_1 + w_2 \quad w_1 + 1.5w_2]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$J(0) = 1.5 - 2w_1 + 1.5w_1^2 + w_1w_2 + w_1w_2 + 1.5w_2^2$$

$\cancel{J(\omega)} = 1.5\omega_1^2 + 1.5\omega_2^2 + 2\omega_1\omega_2 - 2\omega_1 \quad [J(0) = 1.5]$

$$\cancel{J(\omega)} = \frac{3\omega_1^2 + 3\omega_2^2 + 4\omega_1\omega_2 - 4\omega_1}{4}$$

$$\Rightarrow \boxed{3\omega_1^2 + 3\omega_2^2 + 4\omega_1\omega_2 - 4\omega_1 = 0}$$ Contour equation for $J(\omega)$

A standard 2 dimensional conic section is given by:

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

compare this 2 equations $[\omega_1 = x ; \omega_2 = y]$

$a = 3$ $\quad 2g = -4 \Rightarrow g = -2$

$2h = 4 \Rightarrow h = 2$ $\quad 2f = 0 \Rightarrow f = 0$

$b = 3$ $\quad c = 0$

Calculate the discriminant $\Rightarrow \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 3 & 2 & -2 \\ 2 & 3 & 0 \\ -2 & 0 & 0 \end{vmatrix} \neq 0$

$[\cancel{h^2 - ab \Rightarrow (2)^2 - 3\times 3} < 0]$

Since $(h^2 - ab) < 0$ and discriminant $\neq 0 \Rightarrow$ the equation represents an ellipse.

Standard equation of an ellipse is given by: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

centre $= (h, k)$

major axis $= a$

minor axis $= b$

The equation of the ellipse is

$$3\omega_1^2 + 3\omega_2^2 + 4\omega_1\omega_2 - 4\omega_1 = 0$$

Comparing this equation with the standard equation for a conic section

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad [x = \omega_1, \ y = \omega_2]$$

we get:

$$A = 3 \quad B = 4 \quad C = 3 \quad D = -4 \quad E = 0 \quad F = 0$$

Centre of this ellipse is given by:

$$x_0 = \frac{2 \times C \times D - B \times E}{(B^2 - 4AC)} = \frac{(2 \times -4 \times 3) - 0}{((4)^2 - 4 \times 3 \times 3)} = \frac{-24}{-20} = \frac{-6}{-5} = +1.2$$

$$y_0 = \frac{2 \times A \times E - B \times D}{(B^2 - 4AC)} = \frac{0 - 4 \times (-4)}{(4)^2 - 4 \times 3 \times 3} = \frac{+16}{-20} = -\frac{4}{5} = -0.8$$

$$\text{Centre} = \begin{bmatrix} 1.2 \\ -0.8 \end{bmatrix}$$ which is also the value of the optimum weights for this Weiner filter.

$$a, b = \frac{-\sqrt{2(AE^2 + CD^2 - BDE + (B^2 - 4AC)F)((A+C) \pm \sqrt{(A-C)^2 + B^2})}}{(B^2 - 4AC)}$$

Taking the positive sign we get a (length of semi major axis) and taking the negative sign we get b (length of semi minor axis)

$$a = \frac{\sqrt{2(0 + 48 - 0 + 0)(6 + 4)}}{-(16 - 4 \times 3 \times 3)} = \frac{\sqrt{2 \times 48 \times 10}}{20} = \frac{8\sqrt{15}}{20} = \frac{2}{5}\sqrt{15}$$

$$b = \frac{\sqrt{2(0 + 48 + 0 - 0)(6 - 4)}}{-(16 - 4 \times 3 \times 3)} = \frac{\sqrt{2 \times 48 \times 2}}{+20} = \frac{8\sqrt{3}}{20} = \frac{2}{5}\sqrt{3}$$

$$e = \text{eccentricity} = \sqrt{1 - \left(\frac{b}{a}\right)^2} = \frac{2}{\sqrt{5}}$$

$$\theta = \arctan\left[\frac{1}{B}\left(C - A - \left(\sqrt{(A-C)^2+B^2}\right)\right)\right] = \arctan\left[\frac{-B}{B}\right] = -45^\circ$$

⇒ major axis is inclined at an angle of -45° with the ω_1-axes

⇒ minor axis is inclined at an angle of $+45^\circ$ with the ω_1 axis

Since both major and minor axis pass through the centre

⇒ Major axis:

$$(\omega_2 + 0.8) = -1(\omega_1 - 1.2)$$

$$\omega_2 + 0.8 = -\omega_1 + 1.2$$

$$\boxed{\omega_2 + \omega_1 = 0.4} \quad - (1)$$

Minor axis:

$$(\omega_2 + 0.8) = 1(\omega_1 - 1.2)$$

$$\omega_2 + 0.8 = \omega_1 - 1.2$$

$$\boxed{\omega_2 = \omega_1 - 2} \quad - (2)$$

To find the vertices of major axis put (1) into the ellipse equation:

$$3\omega_1^2 + 3\omega_2^2 + 4\omega_1\omega_2 - 4\omega_1 = 0$$

$$3\omega_1^2 + 3(0.4-\omega_1)^2 + 4\omega_1(0.4-\omega_1) - 4\omega_1 = 0$$

$$3\omega_1^2 + 0.48 + 3\omega_1^2 - 2.4\omega_1 + 1.6\omega_1 - 4\omega_1^2 - 4\omega_1 = 0$$

$$2\omega_1^2 - 4.8\omega_1 + 0.48 = 0 \Rightarrow \boxed{\begin{array}{ll} \omega_1 = 2.2954 & \Rightarrow \omega_2 = -1.8954 \\ \omega_1 = 0.1046 & \Rightarrow \omega_2 = 0.2954 \end{array}}$$

⇒ $(2.2954, -1.8954)$ and $(0.1046, 0.2954)$ are the vertices of the major axis

To find vertices of the minor axis put (2) into the ellipse equation:

$$3\omega_1^2 + 3(\omega_1 - 2)^2 + 4\omega_1(\omega_1 - 2) - 4\omega_1 = 0$$

$$3\omega_1^2 + 3\omega_1^2 + 12 - 12\omega_1 + 4\omega_1^2 - 8\omega_1 - 4\omega_1 = 0$$

$$10\omega_1^2 - 24\omega_1 + 12 = 0$$

$$5\omega_1^2 - 12\omega_1 + 6 = 0 \Rightarrow \boxed{\begin{array}{ll} \omega_1 = 1.6899 & \Rightarrow \omega_2 = -0.3101 \\ \omega_1 = 0.7101 & \Rightarrow \omega_2 = -1.2899 \end{array}}$$

$(1.6899, -0.3101)$ and $(0.7101, -1.2899)$ are the vertices of the minor axis

These are plotted in the plot attached with this assignment