离散(2) hw9

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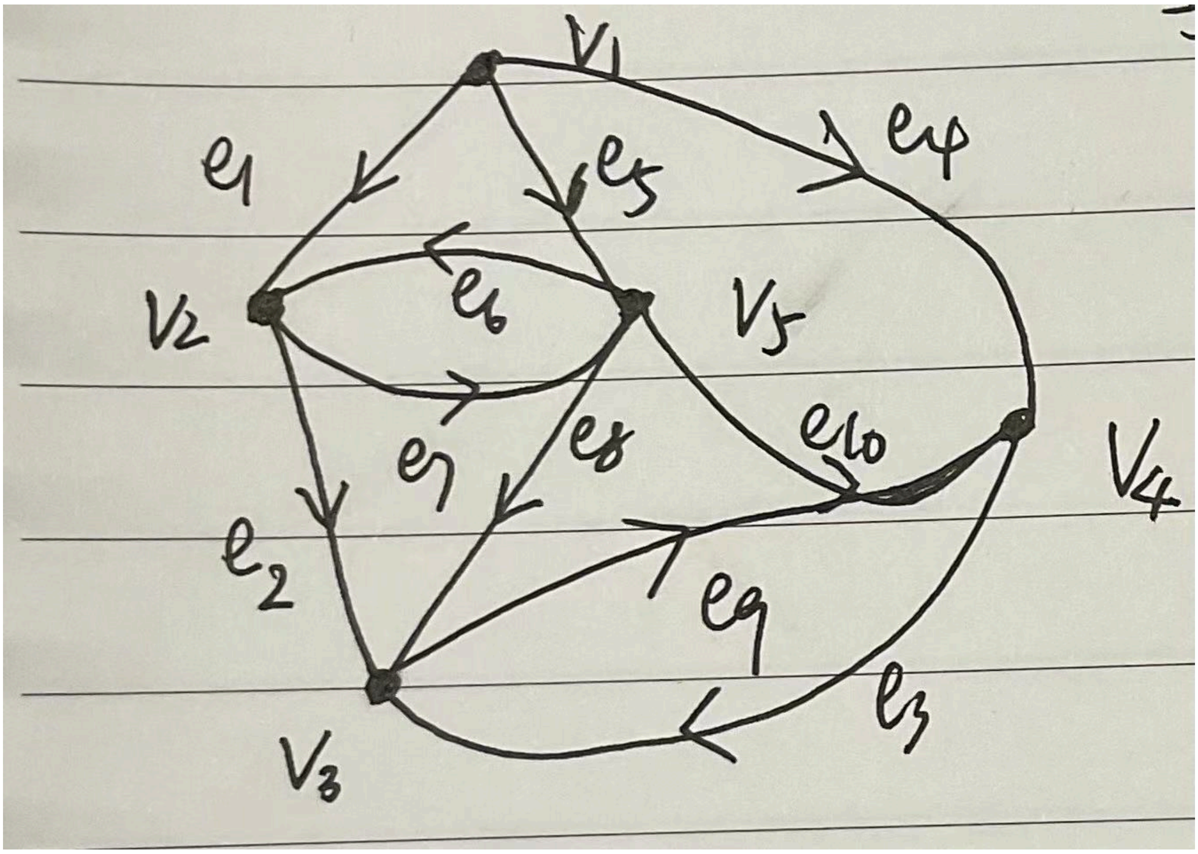
P83 T2

> 求图`3.28`中，以(1)v_1为根的根树的数目 (2)以v_1为根，不含有(v_1, v_5)的根树的数目 (3)以v_1为根，不含有(v_2, v_3)的数目

解：



（1）

构造有向图的关联矩阵B：

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 1 \end{bmatrix}$$

造基本关联矩阵B_1（去除第一行）

$$B_1 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 1 \end{bmatrix}$$

构造变换矩阵$\vec{B}_1$（将B_1中所有的1替换为0）

$$\vec{B}_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

计算$\vec{B}_1 \cdot B_1^T$

$$\vec{B}_1 \cdot B_1^T = \begin{bmatrix} 2 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$

$$det(\vec{B}_1 \cdot B_1^T) = 24$$

因此，以v_1为根的根树的数目为24

（2）

从关联矩阵中删除边(v_1, v_5)对应的列。

构造基本关联矩阵B_1

$$B_1 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 1 \end{bmatrix}$$

删除边(v_1, v_5)对应的列（第5列）：

$$B_1' = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{bmatrix}$$

构造变换矩阵$\vec{B}_1'$（将B_1'中所有的1替换为0）：

$$\vec{B}_1{}' = \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

1. 计算$\vec{B}_1{}' \cdot (B_1')^T$:

   $$\vec{B}_1{}' \cdot (B_1')^T = \begin{bmatrix} 2 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

2. 计算行列式:

   $\vec{B}_1' \cdot B_1'^T = 8$

因此，以v_1为根，不含有(v_1, v_5)的根树的数目为8。

（3）

```
import numpy as np
# 有向图的关联矩阵
B = np.array([
[ 1, 0, 0, 1, 1, 0, 0, 0, 0, 0],
```

```
[-1, 1, 0, 0, 0,-1, 1, 0, 0, 0],
[ 0,-1,-1, 0, 0, 0, 0,-1, 1, 0],
[ 0, 0, 1,-1, 0, 0, 0, 0,-1,-1],
[ 0, 0, 0, 0,-1, 1,-1, 1, 0, 1]
])
B_1 = B[1:5]
B_1_v = np.where(B_1 == 1, 0, B_1)
det_all = np.linalg.det(B_1_v.dot(B_1.T))
print("以v_1为根的所有根树数目 =", round(det_all))
B_1_no_v2v3 = np.delete(B_1, 2, axis=1)
B_1_v_no_v2v3 = np.where(B_1_no_v2v3 == 1, 0, B_1_no_v2v3)
det_no_v2v3 = np.linalg.det(B_1_v_no_v2v3.dot(B_1_no_v2v3.T))
print("以v_1为根不含(v_2,v_3)边的根树数目 =", round(det_no_v2v3))
det_with_v2v3 = det_all - det_no_v2v3
print("以v_1为根必含(v_2,v_3)边的根树数目 =", round(det_with_v2v3))
###
"""
以v_1为根的所有根树数目 = 24
以v_1为根不含(v_2,v_3)边的根树数目 = 18
以v_1为根必含(v_2,v_3)边的根树数目 = 6
"""
```

P85 T27

> 已知图G的基本关联矩阵，求解以$\{e_3, e_4, e_6, e_7\}$为树的基本关联矩阵

解：

$$B = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

给定树边为e_3, e_4, e_6, e_7，对应矩阵B_5的第3、4、6、7列。
弦边为e_1, e_2, e_5, e_8，对应矩阵B_5的第1、2、5、8列。

树矩阵B_{11}：

$$B_{11} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

余树边矩阵B_{12}：

$$B_{12} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$B_{12}^{-T} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

$$C_{f12} = B_{11}^T B_{12}^{-T} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

P86 T38

> 求解赋权图中的最小通信代价

解：

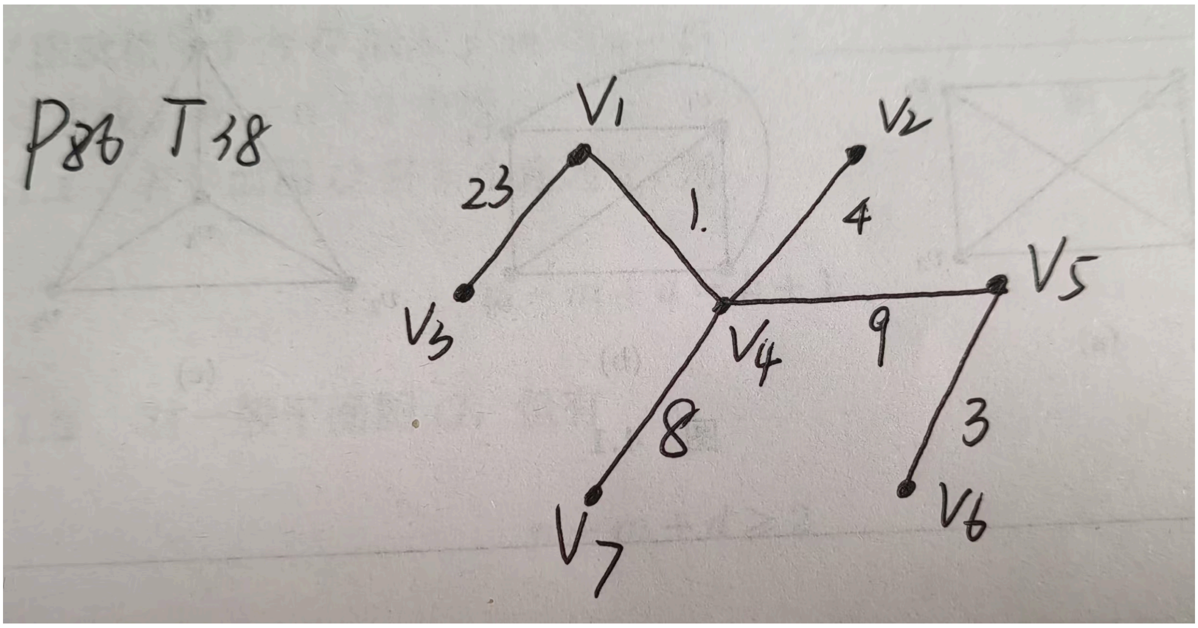
使用Prim算法，计算城市节点图的最小生成树，实现的算法代码如下：

```
import numpy as np
import matplotlib.pyplot as plt
def prim(graph):
    n = len(graph)
    visited = [False] * n  # 记录顶点是否已加入MST
    key = [float('inf')] * n  # 记录每个顶点到MST的最小权重
    parent = [-1] * n  # 记录每个顶点在MST中的父节点
    key[0] = 0
    for _ in range(n):
        min_key = float('inf')
        min_index = -1
        for i in range(n):
            if not visited[i] and key[i] < min_key:
                min_key = key[i]
                min_index = i
        visited[min_index] = True
        for i in range(n):
            if (graph[min_index][i] > 0 and  # 存在边
                not visited[i] and  # 顶点未访问
                graph[min_index][i] < key[i]):  # 权重更小
                key[i] = graph[min_index][i]
                parent[i] = min_index
    mst = []
    for i in range(1, n):
        mst.append((parent[i], i, graph[parent[i]][i]))

    return mst
```

给出方案：

在$v_3 - v_1, v_1 - v_4, v_4 - v_2, v_4 - v_5, v_5 - v_6, v_4 - v_7$之间假设通信线路

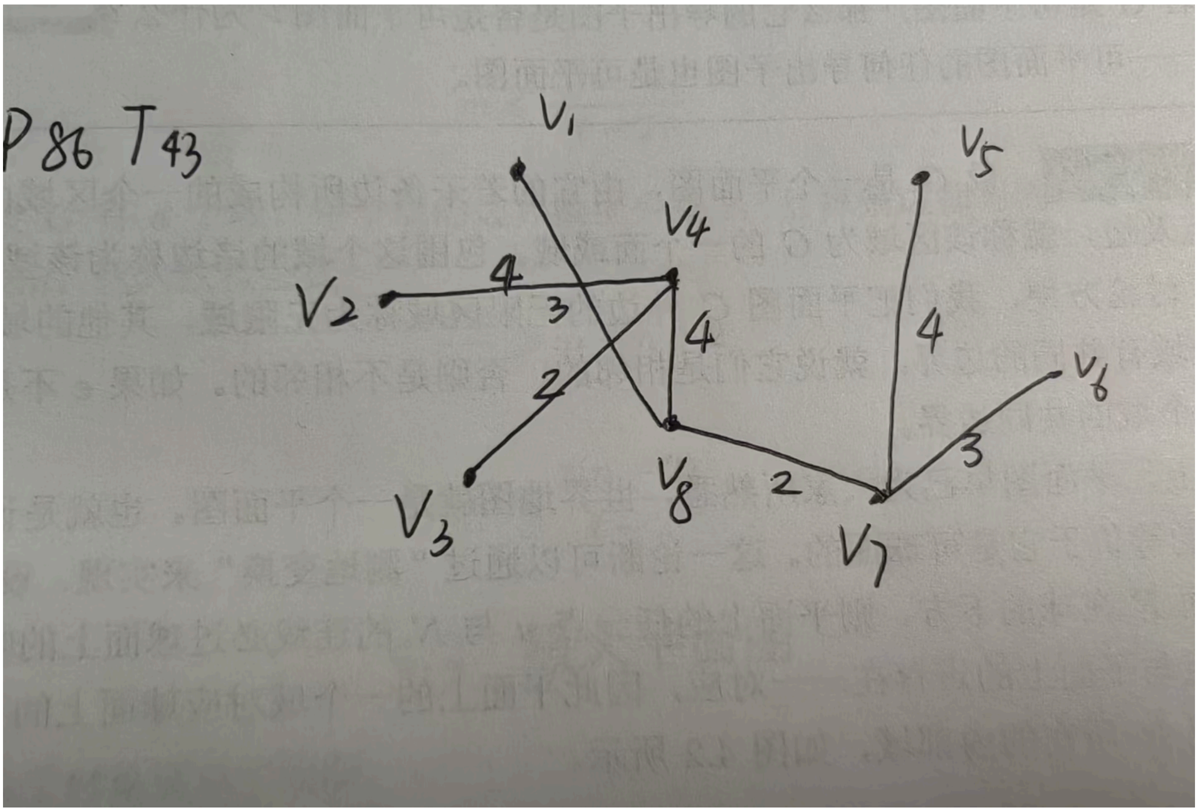


总最小代价为48

P86 T43

> 求解图上的最短树

解：算法同上，使用Prim算法



附录

```
### P86 T38
def create_example_graph():
    n = 8
    graph = np.zeros((n, n))
    edges = [
        (4, 3, 36),
```

```python
        (4, 1, 1),
        (4, 2, 4),
        (4, 5, 9),
        (4, 6, 16),
        (4, 7, 8),
        (1, 2, 20),
        (2, 5, 15),
        (5, 6, 3),
        (6, 7, 17),
        (7, 3, 28),
        (3, 1, 23),
    ]

    for u, v, w in edges:
        graph[u][v] = w
        graph[v][u] = w

    return graph

def main():
    graph = create_example_graph()
    mst = prim(graph)
    print("\n最小生成树的边:")
    total_weight = 0
    for u, v, weight in mst:
        print(f"边 {u}-{v}: 权重 = {weight}")
        total_weight += weight
    print(f"最小生成树总权重: {total_weight}")

main()
### output
"""
最小生成树的边:
边 4-1: 权重 = 1.0
边 4-2: 权重 = 4.0
边 1-3: 权重 = 23.0
边 7-4: 权重 = 8.0
边 4-5: 权重 = 9.0
边 5-6: 权重 = 3.0
边 7-7: 权重 = 0.0
最小生成树总权重: 48.0
"""
### P43 T43

def create_example_graph():
    n = 9
    graph = np.zeros((n, n))
    edges = [
        (1, 2, 5),
        (1, 8, 3),
        (2, 4, 4),
        (2, 3, 7),
        (3, 1, 6),
        (4, 1, 4),
        (4, 3, 2),
        (4, 6, 5),
```

```python
        (4, 8, 4),
        (5, 4, 7),
        (5, 7, 4),
        (6, 5, 5),
        (7, 6, 3),
        (8, 3, 5),
        (8, 7, 2),
    ]

    for u, v, w in edges:
        graph[u][v] = w
        graph[v][u] = w

    return graph

def main():
    graph = create_example_graph()
    mst = prim(graph)
    print("\n最小生成树的边:")
    total_weight = 0
    for u, v, weight in mst:
        print(f"边 {u}-{v}: 权重 = {weight}")
        total_weight += weight
    print(f"最小生成树总权重: {total_weight}")

# 运行主函数
main()
### output
"""
最小生成树的边:
边 8-1: 权重 = 3.0
边 4-2: 权重 = 4.0
边 4-3: 权重 = 2.0
边 8-4: 权重 = 4.0
边 7-5: 权重 = 4.0
边 7-6: 权重 = 3.0
边 8-7: 权重 = 2.0
边 8-8: 权重 = 0.0
最小生成树总权重: 22.0
"""
```

最短树如下图所示: