主观题 HW6

HW 6.1

在雨课堂的填空题中，选择一个普遍有效的公式给出证明，并选择一个不是普遍有效的公式举出反例。（10 分）

解：

普遍有效的公式$((\exists x)P(x) \to ((\forall x)Q(x)) \to (\forall x)(P(x) \to Q(x))$

证明：

$$
\begin{aligned}
&((\exists x)P(x) \to ((\forall x)Q(x)) \to (\forall x)(P(x) \to Q(x))\\
&= (\neg(\exists x)P(x) \lor (\forall x)Q(x)) \to (\forall x)(P(x) \to Q(x))\\
&= ((\forall x)\neg P(x) \lor ((\forall x)Q(x)) \to (\forall x)(P(x) \to Q(x))\\
&= (\forall x)(\neg P(x) \to Q(x)) \to (\forall x)(P(x) \to Q(x))\\
&= (\forall x)(P(x) \to Q(x)) \to (\forall x)(P(x) \to Q(x))\\
&= T
\end{aligned}
$$

不是普遍有效的公式$(\forall x)(\exists y)P(x,y) \to (\exists y)(\forall x)P(x,y)$

反例：取$D=\{1,2\}, P(1,1)=P(2,2)=T, P(1,2)=P(2,1)=F$

$$
\begin{aligned}
(\forall x)(\exists y)P(x,y) = (P(1,1) \lor P(1,2)) \land (P(2,1) \lor P(2,2)) = T\\
(\exists x)(\forall y)P(x,y) = (P(1,1) \land P(1,2)) \lor (P(2,1) \land P(2,2)) = F
\end{aligned}
$$

有

$$(\forall x)(\exists y)P(x,y) \to (\exists y)(\forall x)P(x,y) = F$$

HW 6.2

证明下列等值式和蕴含式：

（1）$(\forall x)(P(x) \lor q) \to (\exists x)(P(x) \land q) = ((\exists x)\neg P(x) \land \neg q) \lor ((\exists x)P(x) \land q)$ (5分)

proof:

$$
\begin{aligned}
&(\forall x)(P(x) \lor q) \to (\exists x)(P(x) \land q)\\
&= \neg(\forall x)(P(x) \lor q) \lor (\exists x)(P(x) \land q)\\
&= (\exists x)(\neg P(x) \land \neg q) \lor (\exists x)(P(x) \land q)
\end{aligned}
$$

（2）$(\forall y)(\exists x)((P(x) \to q) \lor S(y)) = ((\forall x)P(x) \to q) \lor (\forall y)S(y)$ (5 分)

proof:

$$
\begin{aligned}
&(\forall y)(\exists x)((P(x) \to q) \lor S(y))\\
&= (\forall y)((\exists x)(P(x) \to q) \lor S(y))\\
&= (\forall y)((\exists x)\neg P(x) \lor q) \lor S(y))\\
&= (\forall y)(\neg(\forall x)P(x) \lor q) \lor S(y))\\
&= (\forall y)(((\forall x)P(x) \to q) \lor S(y)\\
&= ((\forall x)P(x) \to q) \lor (\forall y)P(y)
\end{aligned}
$$

（3）$(\forall x)P(x) \to q = (\exists x)(P(x) \to q)$ (5分)

proof:

$$
\begin{aligned}
&(\forall x)P(x) \to q\\
&= \neg(\forall x)P(x) \lor q\\
&= (\exists x)\neg P(x) \lor q\\
&= (\exists x)(\neg P(x) \lor q)\\
&= (\exists x)(P(x) \to q)
\end{aligned}
$$

（4）$(\exists x)(P(x) \to Q(x)) = (\forall x)P(x) \to (\exists x)Q(x)$ (5分)

proof:

$$
\begin{aligned}
&(\exists x)(P(x)\to Q(x))\\
&=(\exists x)(\neg P(x)\vee Q(x))\\
&=(\exists x)\neg P(x)\vee(\exists x)Q(x)\\
&=\neg(\forall x)P(x)\vee(\exists Q(x)\\
&=\forall P(x)\to(\exists x)Q(x)
\end{aligned}
$$

（5）$(\exists x)P(x)\to(\forall x)Q(x)\Rightarrow(\forall x)(P(x)\to Q(x))$ (5分)

proof:

$$
\begin{aligned}
&(\exists x)P(x)\to(\forall x)Q(x)\\
&=\neg(\exists x)P(x)\vee(\forall x)Q(x)\\
&=(\forall x)\neg P(x)\vee(\forall x)Q(x)\\
&\Rightarrow(\forall x)(\neg P(x)\vee Q(x))\\
&=(\forall x)(P(x)\to Q(x))
\end{aligned}
$$

（6）$(\exists x)P(x)\wedge(\forall x)Q(x)\Rightarrow(\exists x)(P(x)\wedge Q(x))$(5分)

proof：

$$
\begin{aligned}
&(\exists x)P(x)\wedge(\forall x)Q(x)\\
&=(\exists x)P(x)\wedge(\forall y)Q(y)\\
&=(\exists x)(P(x)\wedge(\forall y)Q(y))\\
&=(\exists x)(\forall y)(P(x)\wedge Q(y))\\
&\Rightarrow(\exists x)(P(x)\wedge Q(x))
\end{aligned}
$$

（7）

$$
\begin{aligned}
&((\forall x)P(x)\wedge(\forall x)Q(x)\wedge(\exists x)R(x))\\
&\vee((\forall x)P(x)\wedge(\forall x)Q(x)\wedge(\exists x)S(x)) \quad \text{(5分)}\\
&=(\forall x)(P(x)\wedge Q(x))\wedge(\exists x)(R(x)\vee S(x))
\end{aligned}
$$

proof:

$$
\begin{aligned}
&((\forall x)P(x)\wedge(\forall x)Q(x)\wedge(\exists x)R(x))\vee((\forall x)P(x)\wedge(\forall x)Q(x)\wedge(\exists x)S(x))\\
&=(\forall x)(P(x)\wedge Q(x))\wedge(\exists x)R(x)\vee((\forall x)(P(x)\wedge Q(x))\wedge(\exists x)S(x))\\
&=(((\forall x)(P(x)\wedge Q(x)))\vee((\forall x)(P(x)\wedge Q(x))\wedge(\exists x)R(x)))\wedge(((\forall x)(P(x)\wedge Q(x)))\vee((\forall x)(P(x)\wedge Q(x))\wedge(\exists x)S(x)))
\end{aligned}
$$

形式地记$A=(\forall x)(P(x)\wedge Q(x)),B=(\exists x)R(x),C=(\exists x)S(x)$

则有

$$
\begin{aligned}
&(A\wedge B)\vee(A\wedge C)\\
&=(A\vee(A\wedge C))\wedge(B\vee(A\wedge C)\\
&=A\wedge(B\vee A)\wedge(B\vee C)=A\wedge(B\vee C)
\end{aligned}
$$

$$
\begin{aligned}
\text{从而}&(((\forall x)(P(x)\wedge Q(x)))\vee((\forall x)(P(x)\wedge Q(x))\wedge(\exists x)R(x)))\wedge(((\forall x)(P(x)\wedge Q(x)))\vee((\forall x)(P(x)\wedge Q(x))\wedge(\exists x)S(x)))\\
&=((\forall x)(P(x)\wedge Q(x)))\wedge((\exists x)R(x)\vee(\exists x)S(x))\\
&=((\forall x)(P(x)\wedge Q(x)))\wedge(\exists x)(R(x)\vee S(x))
\end{aligned}
$$

HW 6.3

求下列（1）到（5）的前束范式，以及（6）和（7）的 Skolem 标准型，即仅保留全称量词的前束范式。

（1）$(\forall x)(P(x)\to(\exists y)Q(x,y))$(5分)

解：

$$
\begin{aligned}
&(\forall x)(\neg P(x)\vee(\exists y)Q(x,y))\\
&=(\forall x)(\exists y)Q(\neg P(x)\vee Q(x,y))
\end{aligned}
$$

（2）$(\forall x)(\forall y)(\forall z)(P(x,y,z)\wedge((\exists u)Q(x,u)\to(\exists w)Q(y,w)))$(5分)

解：

$$
\begin{aligned}
&(\forall x)(\forall y)(\forall z)(P(x,y,z) \wedge ((\exists u)Q(x,u) \rightarrow (\exists w)Q(y,w))) \\
&= (\forall x)(\forall y)(\forall z)(P(x,y,z) \wedge (\neg(\exists u)Q(x,u) \vee (\exists w)Q(y,w)) \\
&= (\forall x)(\forall y)(\forall z)(P(x,y,z) \wedge ((\forall u)\neg Q(x,u) \vee (\exists w)Q(y,w)) \\
&= (\forall x)(\forall y)(\forall z)(P(x,y,z) \wedge (\forall u)(\exists w)(\neg Q(x,u) \vee Q(y,w)) \\
&= (\forall x)(\forall y)(\forall z)(\forall u)(\exists w)(P(x,y,z) \wedge (\neg Q(x,u) \vee Q(y,w))
\end{aligned}
$$

(3) $(\exists x)P(x,y) \leftrightarrow (\forall z)Q(z)$(5分)

解:

$$
\begin{aligned}
&(\exists x)P(x,y) \leftrightarrow (\forall z)Q(z) \\
&= ((\exists x)P(x,y) \wedge (\forall z)Q(z)) \vee (\neg(\exists x)P(x,y) \wedge \neg(\forall z)Q(z)) \\
&= (\neg(\exists x)P(x,y) \vee (\forall z)Q(z)) \wedge ((\exists x)P(x,y) \vee \neg(\forall z)Q(z))(\text{这里利用了}(A \wedge B) \vee (\neg A \wedge \neg B = (A \vee \neg B) \wedge (\neg A \vee B)) \\
&= ((\forall x)\neg P(x,y) \vee (\forall z)Q(z)) \wedge ((\exists v)P(v,y) \vee (\exists u)\neg Q(u)) \\
&= (\forall x)(\forall z)(\exists v)(\exists u)((\neg P(x,y) \vee Q(z)) \wedge (P(v,y) \vee \neg Q(u)))
\end{aligned}
$$

(4) $(\neg(\exists x)P(x) \vee (\forall y)Q(y)) \rightarrow (\forall z)R(z)$ (5分)

解:

$$
\begin{aligned}
&(\neg(\exists x)P(x) \vee (\forall y)Q(y)) \rightarrow (\forall z)R(z) \\
&= \neg((\forall x)\neg P(x) \vee (\forall y)Q(y) \vee (\forall z)R(z) \\
&= ((\exists x)P(x) \wedge (\exists y)\neg Q(y)) \vee (\forall z)R(z) \\
&= (\exists x)(\exists y)(\forall z)(P(x) \wedge \neg Q(y) \vee R(z)
\end{aligned}
$$

(5) $(\forall x)(P(x) \rightarrow (\forall y)((P(y) \rightarrow (Q(x) \rightarrow Q(y))) \vee (\forall z)P(z)))$(5分)

解:

$$
\begin{aligned}
&(\forall x)(P(x) \rightarrow (\forall y)((P(y) \rightarrow (Q(x) \rightarrow Q(y))) \vee (\forall z)P(z))) \\
&= (\forall x)(\neg P(x) \vee (\forall y)(\neg(P(y) \vee (\neg Q(x) \vee Q(y))) \vee (\forall z)P(z))) \\
&= (\forall x)(\forall y)(\forall z)(\neg P(x) \vee \neg P(y) \vee \neg Q(x) \vee Q(y) \vee P(z))
\end{aligned}
$$

(6) $(\forall x)(P(x) \rightarrow (\exists y)Q(x,y)) \vee (\forall z)R(z)$(5分)

解:

$$
\begin{aligned}
&(\forall x)(P(x) \rightarrow (\exists y)Q(x,y)) \vee (\forall z)R(z) \\
&= (\forall x)(\neg P(x) \vee (\exists y)Q(x,y)) \vee (\forall z)R(z) \\
&= (\forall x)(\exists y)(\forall z)(\neg P(x) \vee Q(x,y)) \vee R(z)) \\
&(Skolem\text{范式}) \\
&= (\forall x)(\forall z)(\neg P(x) \vee Q(x,f(x)) \vee R(z))
\end{aligned}
$$

(7) $(\exists y)(\forall x)(\forall z)(\exists u)(\forall v)P(x,y,z,u,v)$(5分)

解:

$$
\begin{aligned}
&Skolem\text{范式} \\
&= (\forall x)(\forall z)(\forall v)P(x,a,z,f(x,z),v)
\end{aligned}
$$