

HW4.

T1. 前言:

(a). Celtics 队赢得比赛的条件是  $X \geq \frac{n+1}{2}$   
 由对称性可知.  $P_n(\text{win}) = \sum_{k=\frac{n+1}{2}}^n P(X=k) = \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} p^k (1-p)^{n-k}$

特别地, 对于  $k=3$ , 有  $P_3(\text{win}) = P(X=2) + P(X=3)$

$$P_3(\text{win}) = \binom{3}{2} p^2 (1-p) + \binom{3}{3} p^3 = 3p^2(1-p) + p^3 = 3p^2 - 2p^3$$

$$P_5(\text{win}) = \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + \binom{5}{5} p^5 = 6p^5 - 15p^4 + 10p^3$$

$$\Rightarrow P_5(\text{win}) > P_3(\text{win}) \Rightarrow 3p^2 - 2p^3 < 6p^5 - 15p^4 + 10p^3 \Rightarrow$$

$$\text{solve: } 2p^3 - 5p^2 + 4p - 1 > 0 \Rightarrow (p-\frac{1}{2})(p-1)^2 > 0 \Rightarrow p > \frac{1}{2}$$

因此可知, 当  $p > \frac{1}{2}$  时, 5 局比 3 局更有利。

(b). for general case: 设  $i$  为 Celtics 队在比赛中赢得  $2k-1$  局的局数

不妨作一些代数上的变形:  $P_{2k-1}(\text{win}) = \sum_{i=k}^{2k-1} P(i \geq k) + P(i=k) \cdot (1-(1-p)^2) + P(i=k-1) \cdot p^2$   
(因为这样找 i 会更整)

$$P_{2k-1}(\text{win}) = \sum_{i=k}^{2k-1} P(i \geq k) = \sum_{i=k}^{2k-1} P(i=k) + P(i \geq k+1)$$

$$\therefore P_{2k-1}(\text{win}) - P_{2k-1}(\text{win}) = P_{2k-1}(i=k-1) \cdot p^2 - P_{2k-1}(i=k) \cdot (1-p)^2$$

$$= \binom{2k-1}{k-1} p^{k-1} (1-p)^k \cdot p^2 - \binom{2k-1}{k} p^k (1-p)^{k-1} (1-p)^2$$

$$= \frac{(2k-1)!}{k! (k-1)!} [ p^{k+1} (1-p)^k - p^k (1-p)^{k+1} ]$$

$$= \frac{(2k-1)!}{k! (k-1)!} p^k (1-p)^k (2p-1) > 0 \Rightarrow p > \frac{1}{2}$$

∴ generally if  $p > \frac{1}{2}$ , namely the prob C-team wins is higher,  
the more games played, the more advantage!

T2. 题目: Key-door randomly match problem.

(a). Case(1) 有 24 种不同的 key-door pair

PMF of  $X$ ;  $X = x \in \{1, 2, 3, 4, 5, 6\}$  事件  $M_i$  表示第  $i$  把 key 对应  $j$

容易得到.  $P(X=1) = P(M_1) = \frac{1}{5}$

$$P(X=2) = P(M_1^c)P(M_2|M_1^c) = \frac{4}{5} \times \frac{1}{4} = \frac{1}{5}$$

$$\begin{aligned} P(X=3) &= P(M_1^c \cap M_2^c) P(M_3|M_1^c \cap M_2^c) \\ &= P(M_1^c) P(M_2^c|M_1^c) P(M_3|M_1^c \cap M_2^c) \\ &= \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{5} \end{aligned}$$

同理  $P(X=4) = P(X=5) = \frac{1}{5} \therefore P(X=i) = \frac{1}{5}, i=1, 2, 3, 4, 5$

Case(2) 有 24 种不同的 key-door pair,

$$P(X=k) = \left(\frac{1}{5}\right) \cdot \left(\frac{4}{5}\right)^{k-1}, k=1, 2, 3, 4, \dots$$

满足  $\text{N}(1)$  分布.

(b) 每个 door 带 3 把 key.

在 Case(1) 有 24 种.

$$P(X=1) = P(M_1) = \frac{2}{10}$$

$$P(X=2) = P(M_1^c)P(M_2|M_1^c) = \frac{8}{10} \times \frac{2}{9} = \frac{8}{45}$$

$$P(X=3) = P(M_1^c)P(M_2^c|M_1^c)P(M_3|M_1^c \cap M_2^c) = \frac{8}{10} \times \frac{7}{9} \times \frac{2}{8} = \frac{7}{45}$$

$$P(X=4) = \frac{8}{10} \times \frac{7}{9} \times \frac{6}{8} \times \frac{2}{7} = \frac{12}{90}$$

⋮

$$P(X=k) = \frac{2 \cdot (10-k)}{90}, k=1, 2, 3, \dots, 10 \text{ 为 PMF.}$$

case (2). 无规律产生. 很明显,  $P(X=k) = \left(\frac{1}{5}\right) \cdot \left(\frac{4}{5}\right)^{k-1}, (k=1, 2, 3, \dots)$

T3. 解: Form of Poisson PMF.  $X \sim P(\lambda)$   
由于  $X$  satisfy Poisson distribution.  $P_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$

proof of:  $k \in N \subset [0, \lfloor \lambda \rfloor]$ ,  $P_X(k)$  单调↑

$k \in N \subset [\lfloor \lambda \rfloor + \infty)$ ,  $P_X(k)$  单调↓

$$\frac{P_X(k)}{P_X(k-1)} = \frac{\lambda^k e^{-\lambda}}{k!} \times \frac{(k-1)!}{\lambda^{(k-1)} e^{-\lambda}} = \frac{\lambda}{k}$$

当  $k \leq \lambda$  时, 有  $\frac{P_X(k)}{P_X(k+1)} \geq 1$ .  $P_X(k)$  单调↑

当  $k > \lambda$  时,  $P_X(k) < P_X(k-1)$ ,  $P_X(k)$  单调↓  $\lfloor \lambda \rfloor \leq \lambda$ .

又  $k \in N \Rightarrow$  不对称分布在  $[0, \lfloor \lambda \rfloor]$  上↑,  $[\lfloor \lambda \rfloor, +\infty)$  上↓. 没错!

T4. 解:  $P_X = P(X=k)$  satisfy  $\frac{P_n}{P_{n-1}} = \frac{\lambda}{n}$ ,  $\forall n \in N, n \geq 1$ .

proof of  $X \sim P(\lambda)$

pro: 由连乘式  $P_n / P_{n-1} = \frac{\lambda}{n}$  有  $P_i = P_0 \frac{\lambda^i}{i!}$

利用概率原理:  $\sum_{i=0}^{\infty} P_i = \sum_{i=0}^{\infty} P_0 \frac{\lambda^i}{i!} = P_0 \cdot \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$

利用  $e^{\lambda x} = 1 + \lambda x + \frac{(\lambda x)^2}{2!} + \dots + \frac{(\lambda x)^n}{n!} = \sum_{i=0}^{\infty} \frac{(\lambda x)^i}{i!}$

取  $x=1$ .  $\Rightarrow P_0 e^{\lambda} = 1 \Rightarrow P_0 = e^{-\lambda}$

$\therefore P_i = \frac{\lambda^i}{i!} e^{-\lambda}$  且  $P_k(\lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$  且  $X \sim P(\lambda)$  证毕!

T5. 题目:  $\xrightarrow{0}$  单身狗的随机游走问题.

(1) 求  $P(X=k) = ?$

由题  $P(k|k-1) = p, P(k|k+1) = 1-p$ .

经过  $n$  步运动  $X$  的位置为:  $n, n-2, \dots, -n+2, -n$

$$P(X=k) = \binom{n}{\frac{n+k}{2}} p^{\frac{k+n}{2}} (1-p)^{\frac{n-k}{2}}$$

(2) 距离  $D \geq 0, D = k \in \{n, n-2, \dots\}$  且  $k > 0$ .

$$\begin{aligned} P(D=k) &= P(X=k) + P(X=-k) = \binom{n}{\frac{n-k}{2}} p^{\frac{k+n}{2}} (1-p)^{\frac{n-k}{2}} + \binom{n}{\frac{n+k}{2}} p^{\frac{n-k}{2}} (1-p)^{\frac{n+k}{2}} \\ &= \binom{n}{\frac{n+k}{2}} p^{\frac{n-k}{2}} (1-p)^{\frac{n-k}{2}} [p^k + (1-p)^k] \\ &= \binom{n}{\frac{n+k}{2}} (p-p^2)^{\frac{n-k}{2}} (p^k + (1-p)^k) \end{aligned}$$

(3) 不妨设粒子移动  $m+n$  步后的位置为 原点  $X$ .

则有  $X$  的分布  $P(Z=j) = \binom{m+n}{\frac{m+n-j}{2}} p^{\frac{m+n-j}{2}} (1-p)^{\frac{m+n+j}{2}}$ , 取  $p=\frac{1}{2}$

$$P(X=j) = \binom{m+n}{\frac{m+n-j}{2}} \cdot \frac{1}{2^{m+n}}$$
 (由题目所求为  $P(Y=k|X=j)$ )

$$P(Y=k|X=j) = \frac{P(X=j|Y=k) P(Y=k)}{P(X=j)}, \text{ 因为容易看出 } P(Y=k) = \binom{m}{\frac{m-k}{2}} \cdot \frac{1}{2^m}$$

$$\text{且 } P(X=j|Y=k) = \binom{n}{\frac{n-(j+k)}{2}} p^{\frac{n+j-k}{2}} (1-p)^{\frac{n-j+k}{2}} = \left(\frac{1}{2}\right)^n \binom{n}{\frac{n-j+k}{2}}$$

$$\therefore P(Y=k|X=j) = \frac{\binom{n-j+k}{2} \cdot \binom{m-k}{2}}{\binom{m+n}{2}}, \quad v.$$

$$\binom{m+n-j}{2}$$

T6. 解：

由于每个手机的行驶与否是相互独立的，故任意时刻行驶的总个数  $X(n)$

满足分布为  $P(X=k) = \binom{N}{k} p^k (1-p)^{N-k}$

当且仅当  $X=1$  时行驶可以完成， $\therefore$  Select  $p$ . max  $P(X=1)$

$$\Rightarrow \max P(X=1) = N p (1-p)^{N-1}$$

$$\text{设 } f(p) = N p (1-p)^{N-1}, \frac{\partial f}{\partial p} = N [(1-p)^{N-1} - (N-1)p(1-p)^{N-2}] = 0$$

$$\Rightarrow 1-p - (N-1)p = 0 \Rightarrow p = \frac{1}{N}. \text{ 且 } p=0, \text{ inf } P(X=1)=0$$

$$\therefore p = \frac{1}{N} \text{ 时, } P(X=1) = N p (1-p)^{N-1} \text{ 取 max 值}$$

T7. 解： $X \sim N(\mu, \sigma^2)$  .  $y^2 + 4y + X = 0$  无实根的概率为 0.5 求  $\mu$ .

equation  $y^2 + 4y + X = 0$ ,  $\Delta = 16 - 4X < 0 \Rightarrow X > 4$ .

$P(X > 4) = 0.5$  , 由正态分布的对称性： $\mu = 4$ .

T8. 解： $T \sim \mathcal{E}(\lambda)$ . proof of “失效率”为常数.

PRO:

$$\text{即求 } \lim_{\delta \downarrow 0} \frac{P(T \in (t, t+\delta] | T > t)}{\delta} = ?$$

利用  $T \sim \mathcal{E}(\lambda)$ ,  $f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$$\text{有 } \lim_{\delta \downarrow 0} \frac{P(T \in (t, t+\delta] | T > t)}{\delta}$$

$$\begin{aligned}
 &= \lim_{\delta \downarrow 0} \frac{P(T \in (t, t+\delta])}{\int_t^{t+\delta} \lambda e^{-\lambda T} dT} \\
 &= \lim_{\delta \downarrow 0} \frac{\delta P(T \in (t, t+\delta])}{\delta \cdot \int_t^{\infty} \lambda e^{-\lambda T} dT} = \lim_{\delta \downarrow 0} \frac{-e^{-\lambda t} / t}{\delta \cdot (-e^{-\lambda t}) \Big|_t^{\infty}} \\
 &= \lim_{\delta \downarrow 0} \frac{e^{-\lambda t} (1 - e^{-\lambda \delta})}{\delta \cdot e^{-\lambda t}}, \text{利用 } e^{-x} \rightarrow 1-x, x \ll 1 \\
 &= \lim_{\delta \downarrow 0} \frac{\lambda \delta}{\delta} = \lambda. \quad \text{证毕!}
 \end{aligned}$$

从而证明了  $T \sim \mathcal{E}(\lambda)$  下，其失效率为常数且恰为分布参数。

T9. 解：

proof of:  $X \sim \mathcal{E}(\lambda)$ ,  $Y = \lambda X \sim \mathcal{E}(1)$

$$\begin{aligned}
 \text{由 } X \sim \mathcal{E}(\lambda), \text{ 有 } f(x) = \begin{cases} \lambda e^{-\lambda x} & (x \geq 0) \\ 0 & (x < 0) \end{cases} \quad \int_0^{\infty} \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_0^{\infty} = 1
 \end{aligned}$$

$$\text{不妨计算 CDF: } F(x) = \begin{cases} 1 - e^{-\lambda x} & (x \geq 0) \\ 0 & (x < 0) \end{cases}$$

$$\Rightarrow Y = \lambda X \Rightarrow F(y) = \begin{cases} 1 - e^{-y} & (y \geq 0) \\ 0 & (y < 0) \end{cases}$$

$$\text{由 PDF: } f(y) = F'(y) = 1 \cdot e^{-y} \quad (y \geq 0)$$

$$f(y) = 0, \quad (y < 0)$$

$$\therefore f(y) = \begin{cases} 1 \cdot e^{-y} & (y \geq 0) \\ 0 & (y < 0) \end{cases} \quad \therefore Y \sim \mathcal{E}(1)$$

18 (y<0)

T10. 解:

$$u_1(t) = \begin{cases} \frac{1}{2} & (0 \leq t \leq 2) \\ 0 & (t < 0) \end{cases}$$

$$u_2(t) = \begin{cases} xe^{-\lambda t} & (t \geq 0) \\ 0 & (t < 0) \end{cases} \quad (\lambda = 0.1)$$

$$\text{即求 } P(t \geq 4 | t \geq 1) = \frac{P(t \geq 4)}{P(t \geq 1)} = \frac{P(t \geq 4 | \text{case 2}) \cdot P(\text{case 2})}{P(t \geq 1 | \text{case 1}) P(\text{case 1}) + P(t \geq 1 | \text{case 2}) P(\text{case 2})}$$

$$= \frac{\frac{1}{2} \times \int_4^\infty 0.1 e^{-0.1t} dt}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \int_1^\infty 0.1 e^{-0.1t} dt}$$
$$= \frac{\frac{1}{2} e^{-0.4}}{\frac{1}{4} + \frac{1}{2} e^{-0.1}} = 0.477$$

∴ 得求的“已经用了一年，还能用4年”的概率为 0.477