T1. 解：

$X \sim U(0,1)$. $\phi_X(t) = E(e^{itX}) = \int_{-\infty}^{\infty} f(x) \cdot e^{itx} dx$

$= \int_0^1 e^{itx} dx = \dfrac{e^{it} - e^0}{it} \Rightarrow \phi_X(t) = it(1 - e^{it})$

T2. 解：

$\phi_{Y_1}(t) = Ee^{itY_1}$. 由 Y_k i.i.d. 有 $\phi_{Z_n}(t) = Ee^{itZ_n} = Ee^{it\sum_{i=1}^{n} Y_i/\sqrt{n}}$

$= E(e^{itY_1/\sqrt{n}} e^{itY_2/\sqrt{n}} \cdots e^{itY_n/\sqrt{n}}) = \prod_{i=1}^{n} Ee^{it\frac{Y_1}{\sqrt{n}}} = (g_Y(t)^{\frac{1}{\sqrt{n}}})^n = (g_Y(t))^{\sqrt{n}}$

T3. 解：

proof of Chebyshev's LLN. $\{X_i, i=1,2,\cdots n\}$ independent

$var(X_i) \le C < \infty$

利用 $\frac{1}{n}\sum_{i=1}^{n} E(X_i) = E(\frac{1}{n}\sum_{i=1}^{n} X_i)$ 代入.

利用 Chebyshev's inequation 有：$P(|X - EX| \geq \varepsilon) < \dfrac{Var(X)}{\varepsilon^2}$

$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - E(\frac{1}{n}\sum_{i=1}^{n} X_i)\right| \geq \varepsilon\right) \le \dfrac{Var(\frac{1}{n}\sum_{i=1}^{n} X_i)}{\varepsilon^2} \le \dfrac{C}{n\varepsilon^2}$

事件 $\{|\frac{1}{n}\sum_{i=1}^{n} X_i - E(\frac{1}{n}\sum_{i=1}^{n} X_i)| < \varepsilon\}$ 是 $\{|\frac{1}{n}\sum_{i=1}^{n} X_i - E(\frac{1}{n}\sum_{i=1}^{n} X_i)| \geq \varepsilon\}$ 的补事件

利用 $P(|\frac{1}{n}\sum_{i=1}^{n} X_i - E(\frac{1}{n}\sum_{i=1}^{n} X_i)| < \varepsilon) = 1 - P(|\cdots| \geq \varepsilon) \geq 1 - \dfrac{C}{n\varepsilon^2}$

$\Rightarrow \lim_{n\to+\infty} P(|\frac{1}{n}\sum_{i=1}^{n} X_i - E(\frac{1}{n}\sum_{i=1}^{n} X_i)| < \varepsilon) = \lim_{n\to+\infty} (1 - \frac{C}{n\varepsilon^2}) = 1$

T4. 解： 记 $M_n = \dfrac{X_1 + \cdots + X_n}{n}$. $X_n = 1$: smoke.

利用 $P(|M_n - f| \geq \varepsilon) \le \dfrac{var(X)}{n\varepsilon^2}$. $var(x) = p(1-p) < \frac{1}{4}$

$\therefore P(|M_n - f| \geqslant \varepsilon) \leq \frac{1}{4n\varepsilon^2} = \delta$

(1) $\varepsilon \to \frac{1}{2}\varepsilon$，为了保持边界 $\frac{1}{4n\varepsilon^2}$ 移 $\Rightarrow$ $n \to 4n$

(2) $\delta \to \frac{1}{2}\delta$， $n \to 2n$

T5. 解：X_i. i.i.d. $X_1 \sim \mathcal{E}(2)$. $Y_n = \frac{\sum_{i=1}^n X_i^2}{n}$.

$EY_1 = EX_1^2 = \int_0^{+\infty} x^2 2e^{-2x} dx = \frac{1}{2}$ 且 $\text{var}\, Y_1$ 都有.

由 WLLN 可知 $\forall \varepsilon > 0$, $\lim_{n\to\infty} P(|Y_n - EY_1| > \varepsilon) = 0$.

$\therefore Y_n \xrightarrow{P} \frac{1}{2}$ 即 Y_n 依概率收敛到 $\frac{1}{2}$

T6. 解：

$X_n \sim \mathcal{E}(n\lambda)$. $Y_1 = X_1$. $Y_k = \frac{k-1}{k} Y_{k-1} + X_k$ $\quad Y_k \xrightarrow{P} \frac{1}{\lambda}$?

$EY_k = \frac{k-1}{k} EY_{k-1} + EX_k$. 由 $X_k \sim \mathcal{E}(k\lambda)$. $EX_k = \frac{1}{k\lambda}$

我们归纳. $EY_1 = EX_1 = \frac{1}{1\cdot\lambda} = \frac{1}{\lambda}$.

归纳假设 $EY_{k-1} = \frac{1}{\lambda}$. 往证 $EY_k = \frac{1}{\lambda}$. :

$EY_k = \frac{k-1}{k} EY_{k-1} + EX_k = \frac{k-1}{k} \times \frac{1}{\lambda} + \frac{1}{k\lambda} = \frac{1}{\lambda}$ 证毕!

$\text{var}\, Y_k$ 同理. $\because X_k \sim \mathcal{E}(k\lambda)$, $\text{var}\, X_k = \frac{1}{\lambda^2}$

$\text{var}\, Y_1 = \text{var}\, X_1 = \frac{1}{\lambda^2}$ 假设 $\text{var}\, Y_{k-1} = \frac{1}{(k-1)\lambda^2}$ 往证 Y_k:

$\text{Var}\, Y_k = \left(\frac{k-1}{k}\right)^2 \text{var}\, Y_{k-1} + \text{var}\, X_k + \underbrace{2\text{cov}}_{=0} = \frac{(k-1)^2}{k^2} \times \frac{1}{(k-1)\lambda^2} + \frac{1}{k^2\lambda^2} = \frac{1}{k\lambda^2}$

(由递推式, $Y_{k-1} = f(X_1, \cdots, X_{k-1}) \perp X_k$)

从而有 $\forall \varepsilon > 0$, $P(|Y_k - EY_k| \geqslant \varepsilon) \leq \frac{\text{Var}\, Y_k}{\varepsilon^2} = \frac{1}{k\lambda^2 \cdot \varepsilon^2}$

$\therefore \lim_{k\to\infty} P(|Y_k - \frac{1}{\lambda}| \geqslant \varepsilon) = 0$. $\forall \varepsilon > 0$. 即有 $Y_k \xrightarrow{P} \frac{1}{\lambda}$