HW 1

王子轩
2023011307
wang-zx23@mails.tsinghua.edu.cn

1

> Calculate and comparee on both sides of De'Moron'theorem:
> $(A \cup B)^c = A^c \cap B^c$
> $(A \cap B)^c = A^c \cup B^c$

$$\begin{gathered}\Omega = 1,2,3,4,5,6 \\ A = \{x|x = 2k, k = 1,2,3\} = \{2,4,6\} \\ B = \{x|x \geq 3\} = \{4,5,6\} \\ A^c = \{1,3,5\} \\ B^c = \{1,2,3\}\end{gathered}$$

(1)

$A \cup B = \{2,4,5,6\}$, $(A \cup B)^c = \{1,3\}$ and $A^c \cap B^c = \{1,3\}$

$$(A \cup B)^c = A^c \cap B^c$$

(2)

$A \cap B = \{4,6\}$, $(A \cap B)^c = \{1,2,3,5\}$ and $A^c \cup B^c = \{1,2,3,5\}$

$$(A \cap B)^c = A^c \cup B^c$$

2

(a)

(a.1)

> Show the equation:
> $A^c = (A^c \cap B) \cup (A^c \cap B^c)$

$$\begin{aligned}\mathbf{RHS} &= (A^c \cap B) \cup (A^c \cap B^c) \\ &= [(A^c \cap B) \cup A^c] \cap [(A^c \cap B) \cup B^c] \\ &= [A^c \cap (B \cup A^c)] \cap [(A^c \cup B^c) \cap (B \cup B^c)] \\ &= A^c \cap [(A^c \cup B^c) \cap \Omega] \\ &= A^c \cap (A^c \cup B^c) \\ &= A^c \\ &= \mathbf{LHS}\end{aligned}$$

□

(a.2)

> show the equation:
> $B^c = (A \cap B^c) \cup (A^c \cap B^c)$

$$\begin{aligned}
\mathbf{RHS} &= (A \cap B^c) \cup (A^c \cap B^c) \\
&= [(A \cap B^c) \cup A^c] \cap [(A \cap B^c) \cup B^c] \\
&= [\Omega \cap (B^c \cup A^c)] \cap [(A \cup B^c) \cap (B^c \cup B^c)] \\
&= (B^c \cup A^c) \cap [(A \cup B^c) \cap B^c] \\
&= (B^c \cup A^c) \cap B^c \\
&= B^c \\
&= \mathbf{LHS}
\end{aligned}$$

□

(b)

> show the equation:
> $(A \cap B)^c = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c)$

$$\begin{aligned}
\mathbf{LHS} &= A^c \cup B^c \\
&= (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c) \cup (A^c \cap B^c) \\
&= (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c) \\
&= \mathbf{RHS}
\end{aligned}$$

□

(c)

$$\begin{gathered}
A = \{1, 3, 5\} \\
B = \{1, 2, 3, 4\} \\
A^c = \{2, 4, 6\} \\
B^c = \{5, 6\} \\
\Omega = \{1, 2, 3, 4, 5, 6\} \\
A \cap B = \{1, 3\} \\
A^c \cap B = \{2, 4\} \\
A \cap B^c = \{5\} \\
A^c \cap B^c = \{6\} \\
\mathbf{LHS} = (A \cap B)^c = \{2, 4, 5, 6\} \\
\mathbf{RHS} = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c) = \{2, 4, 5, 6\} \\
thus \quad \mathbf{LHS} = \mathbf{RHS}
\end{gathered}$$

□

3

> Six cups and saucers come in pairs: there are two cups and saucers which are red, two white, and two with stars on. If the cups are placed randomly onto the saucers (one each), find the probability that no cup is upon a saucer of the same pattern.

我们不妨记花色为W、R、S,并且给每一个cup和saucer都编号，样本空间$|\Omega| = A_6^6 = 720$.
事件A = “没有一对花色相同”,事件A的可以分割为两类，第一类如下:

Cup	W	W	R	R	S	S
Saucer	R	R	S	S	W	W

$$n(A_1) = 4 \times 1 \times 2 \times 2 = 16$$

第二类如下:

Cup	W	W	R	R	S	S
Saucer	S	R	W	S	W	R

$$n(A_2) = 4 \times 2 \times (1 \times 2 \times 2 + 2 \times 1 \times 2) = 64$$

$$P(A) = \frac{n(A)}{n(\Omega)} = \frac{n(A_1) + n(A_2)}{|\Omega|} = \frac{16 + 64}{720} = \frac{1}{9}$$

4

> 为不均匀的色子分配概率，已知偶点数出现的概率为奇点数出现概率的2倍.

由题$\Omega = \{1,2,3,4,5,6\}$, $P(1) = P(3) = P(5) = P(A_1)$,$P(2) = P(4) = P(6) = P(A_2)$,$A_1 \cap A_2 = 0$. 且由概率公理$P(\Omega) = P(A_1 \cup A_2) = P(A_1) + P(A_2) = 1$, $P_1 = \frac{1}{2}P_2$,得到:

$$\begin{aligned} P(1) = P(3) = P(5) = P(A_1) = \frac{1}{9} \\ P(2) = P(4) = P(6) = P(A_2) = \frac{2}{9} \\ P(\{1,2,3\}) = P(1) + P(2) + P(3) = \frac{4}{9} \end{aligned}$$

5

> 小清、小华、小紫三人轮流抛一枚硬币，第一次抛出正面朝上的人获胜。试验的样本空间可以定义为：S = {1,01,001,0001, ⋯ }. 请根据S，写出下列事件的具体表达：(i) 小清胜，记作事件A；(ii)小华胜，记作事件B；(iii)$A \cup B^c$.

(i) $A = \{1, 0001, 0000001, 0000000001, \cdots\}$
(ii) $B = \{01, 00001, 00000001, 00000000001, \cdots\}$
(iii) 注意到$\Omega\ = A \cup B \cup C$, C=“小紫胜利”, $C = \{001，\ 000001，\ 000000001，\ 000000000001，\ \cdots\}$.很明显$A, B, C$互不相交 故$B^c = \Omega - B = A \cup C$. $A \cup B^c = A \cup C = \{1, 001, 0001, 000001, 0000001, 000000001, 0000000001, \cdots\}$.

6

> 某商店售卖一种盲盒，每盒内有一个印有24节气之一的书签。试求购买n盒这种盲盒而集齐24节气书签的概率。(Hint:古典概型、定义合适的事件来表达目标事件、可用课上已学的概率性质；写出最终含n的计算式即可，无需化简)

定义事件A_i = “购买的n个盲盒中不包括第i个节气的书签”,那么购买n盒这种盲盒而集齐24节气书签的概率可以表示为$P = P((\bigcup_{i=1}^{24} A_i)^c)$.

$$P = P((\bigcup_{i=1}^{24} A_i)^c) = 1 - P(\bigcup_{i=1}^{i=24} A_i)$$
$$P = 1 - \Sigma_{k=1}^{n=24}(-1)^{k-1}P_k$$
$$P_k = \Sigma_{1\le j1<j2<\cdots<jk\le n}P(A_{j1}\cap A_{j2}\cap\cdots\cap A_{jk})$$
$$P(A_i) = (1-\frac{1}{24})^n$$
$$P(A_i\cap A_j) = (1-\frac{2}{24})^n \quad (i\neq j)$$
$$P(A_{j1}\cap A_{j2}\cap\cdots\cap A_{jk}) = (1-\frac{k}{24})^k$$
$$P = 1 - \Sigma_{k=1}^{n=24}(-1)^{k-1}P_k = 1 + \Sigma_{k=1}^{n=24}(-1)^k C_n^k((1-\frac{k}{24})^k)$$

7

Problem 9. A partition of the sample space Ω is a collection of disjoint events $S_1, \ldots, S_n$ such that $\Omega = \cup_{i=1}^n S_i$.

(a) Show that for any event A, we have

$$\mathbf{P}(A) = \sum_{i=1}^n \mathbf{P}(A \cap S_i).$$

(b) Use part (a) to show that for any events A, B, and C, we have

$$\mathbf{P}(A) = \mathbf{P}(A\cap B) + \mathbf{P}(A\cap C) + \mathbf{P}(A\cap B^c \cap C^c) - \mathbf{P}(A\cap B\cap C).$$

Problem 10. Show the formula

$$\mathbf{P}\big((A\cap B^c)\cup(A^c\cap B)\big) = \mathbf{P}(A) + \mathbf{P}(B) - 2\mathbf{P}(A\cap B),$$

(a)

$\{S_1, S_2, \ldots Sn\}$是Ω的一个分割，满足$\Omega = \bigcup_{i=1}^n S_i, \quad S_i\cap S_j = \emptyset, \quad i\neq j$.即全集是$S_i$的不交并，不难得到$A = \bigcup_{i=1}^n (A\cap S_i)$. 而且有$(A\cap S_i)\cap(A\cap S_j) = \emptyset, \quad i\neq j$. 利用概率的性质，有限可加性我们可以得到$P(A) = P(\bigcup_{i=1}^n(A\cap S_i)) = \Sigma_{i=1}^n P(A\cap S_i)$.

□

(b)

利用韦恩图不难看出，$\{B^c\cap C^c, B^c\cap C$，$B\cap C^c, B\cap C\}$构成Ω的一个分割，那么利用(a)导出的公式$P(A)=\Sigma_{i=1}^n P(A\cap S_i)$. 我们有$P(A)=P(A\cap B^c\cap C^c)+P(A\cap B^c\cap C)+P(A\cap B\cap C^c)+P(A\cap B\cap C)$. 同时我们有$\{B,B^c\}$也是$\Omega$的一个分割，因此$P(A\cap C)=P(A\cap B\cap C)+P(A\cap B^c\cap C)$. 同理$P(A\cap B)=P(A\cap B\cap C)+P(A\cap B\cap C^c)$.
将$P(A\cap C)$,$P(A\cap B)$代入$P(A)=P(A\cap B^c\cap C^c)+P(A\cap B^c\cap C)+P(A\cap B\cap C^c)+P(A\cap B\cap C)$有$P(A)=P(A\cap C)+P(A\cap B)+P(A\cap B^c\cap C^c)-P(A\cap B\cap C)$

□

8

which gives the probability that exactly one of the events A and B will occur. [Compare with the formula $\mathbf{P}(A\cup B)=\mathbf{P}(A)+\mathbf{P}(B)-\mathbf{P}(A\cap B)$, which gives the probability that at least one of the events A and B will occur.]

因为$A\cap A^c=\emptyset$. 而且$A\cap B^c\subseteq A, A^c\cap B\subseteq A^c$所以$(A\cap B^c)\cap(A^c\cap B)=\emptyset$. 由概率的可加性有待证式$LHS=P(A^c\cap B)+P(A\cap B^c)$
再利用$A\cup B$的不交并(一个分割)是$\{A^c\cap B, A\cap B^c, A\cap B\}$(一个分割)依然由概率的可加性有$P(A\cup B)=P(A^c\cap B)+P(A\cap B^c)+P(A\cap B)$,又由二集合的容斥原理有$P(A\cup B)=P(A)+P(B)-P(A\cap B)$联系两式得到$P(A^c\cap B)+P(A\cap B^c)=P(A)+P(B)-2P(A\cap B)$

□

9

7. (Bonferroni's inequality) 如果 $A_i\in\mathscr{F}$, $i=1,\dots,n$, 则

$$\mathbb{P}\Big(\bigcup_{i=1}^n A_i\Big)\ge\sum_{1\le i\le n}\mathbb{P}(A_i)-\sum_{1\le i<j\le n}\mathbb{P}(A_i\cap A_j).$$

(Kounias's inequality)

$$\mathbb{P}\Big(\bigcup_{i=1}^n A_i\Big)\le\min_k\left\{\sum_{1\le i\le n}\mathbb{P}(A_i)-\sum_{i:i\ne k}\mathbb{P}(A_i\cap A_k)\right\}.$$

Proof of Bonferroni's inequality

proof: $P(\bigcup_{i=1}^{n} A_i) \geq \sum_{1\le i\le n} P(A_i) - \sum_{1\le i<j\le n} P(A_i\cap A_j)$

① $P(A_1)\geq P(A_1)$

$P(A_1\cup A_2)\geq P(A_1)+P(A_2)-P(A_1\cap A_2)$

② 假设对(n-1)成立(*)式，即 $P(\bigcup_{i=1}^{n-1}A_i)\geq \sum_{1\le i\le n-1}P(A_i)-\sum_{1\le i<j\le n-1}P(A_i\cap A_j)$

③ 则有 $P(\bigcup_{i=1}^{n-1}A_i\cup A_n)=P(\bigcup_{i=1}^{n}A_i) = P(\bigcup_{i=1}^{n-1}A_i)+P(A_n)-P(\bigcup_{i=1}^{n-1}A_i\cap A_n)$

由于 $P(\bigcup_{i=1}^{n-1}A_i\cap A_n)=P(\bigcup_{i=1}^{n-1}(A_i\cap A_n))\le\sum_{i=1}^{n-1}P(A_i\cap A_n)$

$\therefore P(\bigcup_{i=1}^{n}A_i)\geq P(A_n)+P(\bigcup_{i=1}^{n-1}A_i)-\sum_{i=1}^{n-1}P(A_i\cap A_n)$

利用 $P(\bigcup_{i=1}^{n-1}A_i)\geq\sum_{1\le i\le n-1}P(A_i)-\sum_{1\le i<j\le n-1}P(A_i\cap A_j)$

$\therefore P(\bigcup_{i=1}^{n}A_i)\geq\sum_{1\le i\le n}P(A_i)-\sum_{1\le i<j\le n-1}P(A_i\cap A_j)-\sum_{1\le i\le n-1,\ j=n}P(A_i\cap A_j)$

$\text{RHS}=\sum_{i=1}^{n}P(A_i)-\sum_{1\le i<j\le n}P(A_i\cap A_j)$

证毕！

① 归纳奠基. ② 归纳假设 ③ 递推证明

□.

10

> You are given that at least one of the events Ar , 1 ≤ r ≤ n, is certain to occur, but certainly no more than two occur. If $\mathbb{P}$ Ar = p, ∀ 1 ≤r ≤ n and $\mathbb{P}$ Ar ∩ As = q, ∀ 1 ≤ r ≠ s ≤ n, show that p ≥ 1/n and q ≤ 2/n

在上题中我们证明$P(\bigcup_{i=1}^n A_i) \geq \Sigma_{1\leq i\leq n}P(A_i) - \Sigma_{1\leq i<j\leq n}(A_i \cap A_j)$. 实际上这只是Bonferroni Inequalities的一个不等式，我们还可以写出

$$P(\bigcup_{i=1}^n A_i) \leq \Sigma_{1\leq i\leq n}P(A_i) - \Sigma_{1\leq i<j\leq n}(A_i \cap A_j) + \Sigma_{1\leq i<j<k\leq n}P(A_i \cap A_j \cap A_k)$$

(1) proof of $p \geq 1/n$

利用题目给出的条件$P(A_i \cap A_j \cap A_k) = 0 \quad$. (i,j,k互不相等)以及$P(\bigcup_{i=1}^n A_i) = 1$我们可以得到$\Sigma_{1\leq i\leq n}P(A_i) \geq 1 + \Sigma_{1\leq i<j\leq n}(A_i \cap A_j) \geq 1$而$P(A_1) = ... = P(A_i)$ 所以我们可以得到 $P(A_r) \geq \frac{1}{n}, \quad \forall r \in 1, 2, \ldots n$

(2) proof of $q \leq 2/n$

利用$\Sigma_{1\leq i\leq n}P(A_i) \geq 1 + \Sigma_{1\leq i<j\leq n}(A_i \cap A_j)$则$\Sigma_{1\leq i<j\leq n}(A_i \cap A_j) \leq \Sigma_{1\leq i\leq n}P(A_i) - 1 \leq n - 1$那么$\frac{n(n-1)}{2}P(A_i \cap A_j) \leq n - 1$于是$P(A_i \cap A_j) \leq \frac{2}{n}$