1. 解:

$X\sim P(3)$, $Y\sim P(6)$, $X\perp Y$ 设 $\lambda=3$, $\mu=6$.

则有 $Z=X+Y$. $P(Z=j)=\sum_{i=0}^{\infty}P(X=x, Y=z-x)=\sum_{i=0}^{\infty}P(X=i)P(y=j-i)$

利用 $P(X=i)=\frac{\lambda^i}{i!}e^{-\lambda}$, $P(Y=j-i)=\frac{\mu^{j-i}}{(j-i)!}e^{-\mu}$

$\Rightarrow P(Z=j)=\frac{\lambda^i\mu^{j-i}}{i!(j-i)!}e^{-(\lambda+\mu)}$

利用 $(\lambda+\mu)^j=\binom{j}{i}\lambda^i\mu^{j-i}=\frac{j!}{i!(j-i)!}\lambda^i\mu^{j-i}$ 代入得到

$P(Z=j)=\frac{(\lambda+\mu)^j}{j!}e^{-(\lambda+\mu)}$

$\therefore Z\sim P(\mu+\lambda)$，即随机变量 $Z=X+Y$ 服从参数为 $\lambda+\mu$ 的泊松分布

2. 解:

$X\sim \Gamma(\alpha,\lambda)$ 即 $f_X(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$, $x\geq 0$.

$Y\sim \Gamma(\beta,\lambda)$ 即 $f_Y(y)=\frac{\lambda^\beta}{\Gamma(\beta)}x^{\beta-1}e^{-\lambda y}$, $y\geq 0$

$Z=X+Y$. $X\perp Y$. 有 $f(z)=\int_{-\infty}^{\infty}f(x,z-x)=\int_{-\infty}^{\infty}f_X(x)f_Y(z-x)dx$

$=\int_0^z f_X(x)f_Y(z-x)dx=\int_0^z\frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(z-x)^{\beta-1}e^{-\lambda z}dx$

利用 Beta 分布: $f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $0<x<1$

$f(z)=\int_0^z\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\lambda^{\alpha+\beta}\left(\frac{x}{z}\right)^{\alpha-1}\left(1-\frac{x}{z}\right)^{\beta-1}\cdot z^{\alpha+\beta-2}e^{-\lambda z}dx\cdot\frac{1}{\Gamma(\alpha+\beta)}$

令 $z=x\cdot t\Rightarrow f(z)=\int_0^1\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\lambda^{\alpha+\beta}t^{\alpha-1}(1-t)^{\beta-1}\cdot z^{\alpha+\beta-1}dt\cdot\frac{1}{\Gamma(\alpha+\beta)}$

利用 Beta分布的归一化条件有 $\int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1}dt=1$

$$\Rightarrow f_Z(z)=\frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)} z^{\alpha+\beta-1} e^{-\lambda z}$$

$\therefore Z \sim \Gamma(\alpha+\beta, \lambda)$

3. 解:

$X \sim \mathcal{E}(\lambda)$, $Y \sim \mathcal{E}(\mu)$ $X \perp Y$. 求解 $Z=\min\{X, Y\}$的分布.

$f(x)=\lambda e^{-\lambda x}\ (x \geq 0)$, $f(y)=\mu e^{-\mu y}\ (y \geq 0)$

有 $F_X(x)=1-e^{-\lambda x}$, $F_Y(y)=1-e^{-\lambda y}$ $(x, y \geq 0)$

$\therefore P(X>x)=1-F_X(x)=e^{-\lambda x}$, $P(Y>y)=e^{-\lambda y}$

由于 $F_Z(z)=P(Z \leq z)=1-P(Z>z)=1-P(\min\{X,Y\}>z)$

则有 $F_Z(z)=1-P(X>z, Y>z) \stackrel{独立性}{=} 1-P(X>z)P(Y>z)$

$=1-e^{-(\lambda+\mu)z} \Rightarrow$ PDF of Z: $f_Z(z)=(\lambda+\mu)e^{-(\lambda+\mu)z}\ (z \geq 0)$

4. 解:

$X \sim \mathcal{E}(1)$. $Y \sim N(0,1)$. $X \perp Y$. 求解 $Z=\sqrt{2X}|Y|$ 的PDF. ?

利用PDF of X and Y: $f_X(x)=e^{-x}$, $f_Y(y)=\frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

令 $X'=\sqrt{2X}$, $Y'=|Y|$, 则有 $X=g_1(X')=\frac{X'^2}{2}$, $Y=\begin{cases} Y' & (y \geq 0) \\ -Y' & (y<0) \end{cases}$

不难得到有 $f_{X'}(x')=e^{-x'^2/2}$, $f_{Y'}(y')=\sqrt{\frac{2}{\pi}}e^{-y'^2/2}$

$(x', y' \geq 0)$ 且由 $X \perp Y$. 易证明 $g_1^{-1}(X)$ 与 $g_2^{-1}(Y)$ 独立.

$$\Rightarrow f_Z(z)=\int_{-\infty}^{\infty} f_{X',Y'}(x', \tfrac{z}{x'}) = \int_0^{\infty} f_{X'}(x') f_{Y'}(y')\,dx'$$

$\therefore f_Z(z) = \int_0^\infty e^{-x'^2/2} \cdot \sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2x'^2}} dx' = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{1}{2}(t^2+\frac{z^2}{t^2})} dt$

设 $I(z) = \int_0^\infty e^{-\frac{1}{2}(t^2+\frac{z^2}{t^2})} dt$ 有 $I'(z) = \int_0^\infty -\frac{z}{t^2} e^{-\frac{1}{2}(t^2+\frac{z^2}{t^2})} dt$

整理有 $I'(z) = \int_\infty^0 e^{-\frac{1}{2}(t^2+z^2/t^2)} d(\frac{z}{t})$ 令 $s = z/t$，得到

$I'(z) = -\int_0^\infty e^{-\frac{1}{2}(s^2+\frac{z^2}{s^2})} ds = -I(z) \Rightarrow I'(z) + I(z) = 0$

$\therefore I(z) = I(0) e^{-z}$ $(z \geq 0)$ 利用 $I(0) = \int_0^\infty e^{-\frac{1}{2}t^2} dt = \sqrt{\frac{\pi}{2}}$

$\therefore I(z) = \int_0^\infty e^{-\frac{1}{2}(t^2+\frac{z^2}{t})} dt = \sqrt{\frac{\pi}{2}} e^{-z}$ $(z \geq 0)$

$\therefore f_Z(z) = e^{-z}$ $(z \geq 0)$ 为随机变量 Z 的概率密度函数

5. 解.

$X_1 \sim U(1,2)$, X_2: $f(x_2) = \begin{cases} \frac{3}{8} x_2^2 & (x_2 \in [0,2]) \\ 0 & \text{otherwise} \end{cases}$

求 $Y = (Y_1, Y_2) = (X_1^2, X_1 + X_2)$ 的联合概率密度.(joint PDF)

利用 $f_{X_1}(x_1) = I_{\{1 \leq x_1 \leq 2\}}$，$f(x_2) = I_{\{0 \leq x_2 \leq 2\}} \frac{3}{8} x_2^2$

由 $\begin{cases} Y_1 = X_1^2 \\ Y_2 = X_1 + X_2 \end{cases}$ 有 $f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2) \left| \frac{\partial(x_1,x_2)}{\partial(y_1,y_2)} \right|$

且由 $\left| \frac{\partial(y_1,y_2)}{\partial(x_1,x_2)} \right| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2x_1 & 0 \\ 1 & 1 \end{vmatrix} = 2x_1 = 2\sqrt{y_1}$

$\therefore f_{Y_1,Y_2}(y_1,y_2) = f_{X_1}(x_1) f_{X_2}(x_2) \cdot \frac{1}{2\sqrt{y_1}}$

$= I_{\{1 \leq y_1 \leq 4\}} \cdot I_{\{0 \leq y_2 - \sqrt{y_1} \leq 2\}} \cdot \frac{3}{8} (y_2 - \sqrt{y_1})^2 \cdot \frac{1}{2\sqrt{y_1}}$

整理得到：

$$f_{Y_1,Y_2}(y_1,y_2)=\begin{cases}\frac{3}{16}(y_2-\sqrt{y_1})^2\frac{1}{\sqrt{y_1}} & (1\le y_1\le 4,\ \sqrt{y_1}\le y_2\le 2+\sqrt{y_1}) \\ 0 & (\text{otherwise})\end{cases}$$

6. 解：

$Y_2=X_2Y_1+X_2$

(a). $X_i\sim \chi^2_{r_i}$ $i=1,2,3.$ $Y_1=\frac{X_1}{X_2}.$ $Y_2=X_1+X_2.$ $\therefore X_2=\frac{Y_2}{Y_1+1}$

由 $f_{Y_1,Y_2}(y_1,y_2)=f_{X_1,X_2}(x_1,x_2)\left|\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}\right|$，且 $X_1\perp X_2$. $X_1=\frac{Y_1Y_2}{Y_1+1}$

$f_{X_1,X_2}(x_1,x_2)=f_{X_1}(x_1)f_{X_2}(x_2)$; $\left|\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}\right|=\begin{vmatrix}\frac{1}{x_2} & -\frac{x_1}{x_2^2}\\ 1 & 1\end{vmatrix}=\frac{x_1+x_2}{x_2^2}=\frac{y_2}{x_2^2}$

$=\frac{y_2}{\left(\frac{y_2}{y_1+1}\right)^2}=\frac{(y_1+1)^2}{y_2}$

$\therefore f_{Y_1,Y_2}(y_1,y_2)=\frac{1}{2^{r_1/2}\Gamma(r_1/2)}x_1^{\frac{r_1}{2}-1}e^{-\frac{x_1}{2}}\cdot\frac{1}{2^{r_2/2}\Gamma(r_2/2)}x_2^{\frac{r_2}{2}-1}e^{-\frac{x_2}{2}}\cdot\frac{y_2}{(y_1+1)^2}$

代入 $x_1=\frac{y_1y_2}{y_1+1}$，$x_2=\frac{y_2}{y_1+1}$

$\Rightarrow f_{Y_1,Y_2}(y_1,y_2)=C(r_1,r_2)\cdot y_1^{\frac{r_1}{2}-1}\cdot y_2^{\frac{r_1+r_2}{2}-1}\cdot e^{-\frac{1}{2}y_2}\cdot\left(\frac{1}{y_1+1}\right)^{\frac{r_1+r_2}{2}+2}$

$=C(r_1,r_2)\cdot y_1^{\frac{r_1}{2}-1}y_2^{\frac{r_1+r_2}{2}-1}e^{-\frac{y_2}{2}}=g_{Y_1}(y_1)\cdot g_{Y_2}(y_2)$

(可分解因子，根据 Y_1,Y_2 相互独立的充要条件)

$\therefore Y_1,Y_2$ 独立. 证毕.

且有 $f_{Y_2}(y_2)=\int_0^\infty f_{Y_1,Y_2}(y_1,y_2)dy_1$

$=\int_0^\infty\frac{1}{\Gamma(\frac{r_1}{2})}y_1^{\frac{r_1}{2}-1}\cdot\left(\frac{1}{y_1+1}\right)^{\frac{r_1+r_2}{2}+2}dy_1\frac{\Gamma(\frac{r_1+r_2}{2})}{\Gamma(r_2/2)}\frac{y_2^{\frac{r_1+r_2}{2}-1}e^{-\frac{y_2}{2}}}{\Gamma(\frac{r_1+r_2}{2})2^{r_1+r_2/2}}\cdot$

$$=\int_0^1 \frac{\Gamma(\frac{r_1}{2}+\frac{r_2}{2})}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})}\left(\frac{y_1}{1+y_1}\right)^{\frac{r_1}{2}-1}\cdot\left(\frac{1}{1+y_1}\right)^{\frac{r_2}{2}-1} d\left(\frac{y_1}{y_1+1}\right)\cdot\frac{y_2^{\frac{r_1+r_2}{2}-1}e^{-\frac{y_2}{2}}}{\Gamma(\frac{r_1+r_2}{2})2^{\frac{r_1+r_2}{2}}}$$

利用 Beta分布归一性

$$\therefore f_{Y_2}(y_2)=\frac{y_2^{\frac{r_1+r_2}{2}-1}e^{-\frac{y_2}{2}}}{\Gamma(\frac{r_1+r_2}{2})2^{\frac{r_1+r_2}{2}}} \Longrightarrow Y_2\sim\chi^2_{r_1+r_2}$$

(b) 利用F分布的定义,

$$f(x)=\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{m^{m/2}\cdot n^{n/2}\,x^{\frac{m}{2}-1}}{(mx+n)^{\frac{m+n}{2}}}$$

$$=\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\left(\frac{m}{n}\right)^{\frac{m}{2}}x^{\frac{m}{2}-1}\left(1+\frac{mx}{n}\right)^{-\frac{m+n}{2}}$$

在第(1)问中,我们得到了 $f_{Y_1,Y_2}(y_1,y_2)=g_{Y_1}(y_1)\cdot g_{Y_2}(y_2)=\cdots$

且求出了 $g_{Y_2}(y_2)=f_{Y_2}(y_2)=\dfrac{y_2^{\frac{r_1+r_2}{2}-1}e^{-\frac{y_2}{2}}}{\Gamma(\frac{r_1+r_2}{2})2^{\frac{r_1+r_2}{2}}}$

因此不难得出

$$g_{Y_1}(y_1)=g_{\frac{X_1}{X_2}}(y_1)=\frac{\Gamma(\frac{r_1}{2}+\frac{r_2}{2})}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})}\cdot y_1^{\frac{r_1}{2}-1}(1+y_1)^{-\frac{r_1+r_2}{2}}$$

利用 $\frac{r_2X_1}{r_1X_2}=\left(\frac{r_2}{r_1}\right)\cdot\left(\frac{X_1}{X_2}\right)$ 定义 $Y_3=\left(\frac{r_2}{r_1}\right)\left(\frac{X_1}{X_2}\right)=\frac{r_2}{r_1}\cdot Y_1$, $y_1=\frac{r_1}{r_2}y_3$

$$\therefore f_{Y_3}(y_3)=g_{Y_1}\left(\frac{r_1}{r_2}y_3\right)\cdot\frac{\partial(\frac{r_1}{r_2}y_3)}{\partial(y_3)}=\frac{\Gamma(\frac{r_1}{2}+\frac{r_2}{2})}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})}\cdot\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}}\cdot y_3^{\frac{r_1}{2}-1}\left(1+\frac{r_1}{r_2}y_3\right)^{-\frac{r_1+r_2}{2}}$$

$\therefore Y_3 = \frac{X_1/r_1}{X_2/r_2} \sim F(m=\frac{r_1}{2}, n=\frac{r_2}{2})$! 证毕！

同理，由 $Y_2 = X_1 + X_2 \sim \chi^2_{r_1+r_2}$，可以由一样的计算过程得到

$$Y_4 = \frac{X_3/r_3}{(X_1+X_2)/r_1+r_2} \sim F(m=\frac{r_3}{2}, n=\frac{r_1+r_2}{2}) \ !$$

证明 Y_3, Y_4 相互独立：由 $Y_2 = X_1+X_2$，且 $Y_2 \sim \chi^2_{r_1+r_2}$

即 $Y_4 = \frac{X_3/r_3}{Y_2 \ r_1+r_2}$，在第(1)问中已证明 $Y_2 \perp Y_1$. 由于 $Y_3 = \frac{r_2}{r_1} Y_1$

$\therefore Y_2 \perp Y_3$. ① 又 $\because X_3 \perp X_1$, $X_3 \perp X_2$，且 $Y_3 = \frac{r_2}{r_1}\frac{X_1}{X_2} \Rightarrow X_3 \perp Y_3$ ②

则由 $Y_4 = h_2(X_3, Y_2)$，故有 ①② $\Rightarrow Y_3 \perp Y_4$

此处引用了引理：若随机变量 $X_l \perp X_1, \cdots X_k$，则 $X_l \perp g(X_1, \cdots, X_k)$

7. 解：

$X_1 \cdots X_n$, i.i.d. $\sim f(x)$ (PDF).

则由全部次序统计量的联合分布为

$$f_{X_{(1)}, X_{(2)} \cdots X_{(n)}}(x_1, x_2, \cdots x_n) = \begin{cases} \underbrace{n! \prod_{i=1}^{n} f(x_i)}_{\text{记为} f_n} & (x_1 < x_2 \cdots < x_n) \\ 0 & (\text{otherwise}) \end{cases}$$

则前k个次序统计量的联合分布由上式积分得到：

$$f_{X_{(1)}, X_{(2)} \cdots X_{(k)}}(x_1, x_2, \cdots, x_k) = \int_{x_{k+1} < x_{k+2} < \cdots < x_n} f_n \, dx_{k+1} \, dx_{k+2} \cdots dx_n$$

利用积分结果 $\int \cdots \int f(x_1) \cdots f(x_k) \, dx_1 \cdots dx_k = \frac{1}{k!}(F(b) - F(a))^k$

那么

$$\underset{x_k<x_{k+1}<\cdots<x_n<+\infty}{\int\cdots\int}[n!f(x_1)f(x_2)\cdots f(x_k)]\cdot f(x_{k+1})\cdots f(x_n)dx_{k+1}\cdots dx_n$$

$$=n!f(x_1)\cdots f(x_k)\cdot\underset{x_k<x_{k+1}<\cdots<x_n<+\infty}{\int\cdots\int}f(x_{k+1})\cdots f(x_n)dx_1\cdots dx_n$$

$$=n!\prod_{i=1}^{k}f(x_i)\cdot\frac{1}{(n-k)!}(F(+\infty)-F(x_k))^{n-k}$$

$$=\frac{n!}{(n-k)!}\prod_{i=1}^{k}f(x_i)\cdot(1-F(x_k))^{n-k}$$

∴ 所求的前k个次序统计量的联合分布为 $f_{X_{(1)}\cdots X_{(k)}}(x_1,\cdots x_k)$

$$=n(n-1)\cdots(n-k+1)f(x_1)\cdots f(x_k)\left(1-\int_{-\infty}^{x_k}f(x)dx\right)^{n-k}$$