T1. Ch 5.9

解：(a) $S = X_1 + \cdots + X_n$，$n=50$，$X_i \sim B(1,p)$，$p=0.95$

则有 近似为 normal distribution.

则 $\mu = np = 47.5$，$\sigma_S = \sqrt{n}\,\sigma_X = \sqrt{np(1-p)} \simeq 1.54$

$P(S \geq 45) = P\left(\frac{S-\mu}{\sigma_S} \geq \frac{45-\mu}{\sigma_S}\right) = P(x \geq -1.62) = 1 - \Phi(-1.62)$

$= \Phi(1.62)$

查表可知 $\Phi(1.62) \simeq 0.947$

若引入 0.5 整数修正，得到 $P(S \geq 45) \simeq P(S > 44.5)$

$= P\left(\frac{S-\mu}{\sigma_S} \geq \frac{44.5-47.5}{1.54}\right) = \Phi(1.95) = 0.9744$

(b) $S = X_1 + \cdots + X_n$. 若用 Poisson 分布来估计事件发生的概率，$n \geq 43$ $np \ll 1$

故采用 $(50-S)$ 来估计比较合适

$P(S \geq 45) = P(50 - S \leq 5) = \sum_{k=0}^{5} P(n-S=k) = \sum_{k=0}^{5} e^{-\lambda}\frac{\lambda^k}{k!} = 0.958$

T2. Ch 5.10

解：$S_n = X_1 + \cdots + X_n$，$\mu_X = 5$，$\sigma_X^2 = 9$，$\sigma_X = 3$，$\sigma_S = \sqrt{n}\,\sigma_X = 30$

(a) $P(S_{100} < 440) = P(S_{100} \leq 439.5) = P\left(\frac{S_{100} - \mu_S}{\sigma_S} < \frac{439.5 - 500}{30}\right)$

$= 1 - \Phi(2.02) = 0.022$

(b) $P(S_n \geq 200 + 5n) \leq 0.05 \Rightarrow P\left(\frac{S_n - 5n}{3\sqrt{n}} \geq \frac{200}{3\sqrt{n}}\right) \leq 0.05$

$\Rightarrow 1 - \Phi\left(\frac{300}{3\sqrt{n}}\right) \leq 0.05$，利用 $\Phi(1.65) \simeq 0.95$

解出有 $\frac{200}{3\sqrt{n}} \geq 1.65 \Rightarrow n \leq 1632$

(c) $P(N \geq 220) = P(S_{219} \leq 1000) = P(S_{219} \leq 999.5)$

$= P\left(\frac{S_{219} - 219\times 5}{3\times\sqrt{219}} \leq \frac{999.5 - 5\times 219}{3\times\sqrt{219}}\right) = 1 - \Phi(2.14)$

$\simeq 0.016$

T3. Ch 5.11

解：$X_1 \cdots X_{16}, Y_1 \cdots Y_{16} \sim$ uniform distribution.

$$W = \frac{X_1 + \cdots + X_{16} - (Y_1 + \cdots + Y_{16})}{16} \quad . \quad EX^2 - (EX)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

求 $P(|W - E[W]| < 0.001)$

令 $Z_i = X_i - Y_i$，则由 $EZ_i = EX_i - EY_i = \frac{1}{2} - \frac{1}{2} = 0$，

$var(Z_i) = var X_i + var(-Y_i) = 2 var X_i = 2 \times \frac{1}{12} = \frac{1}{6}$

∴ 可用 normal distribution 对于 W 进行近似 ⇒

$$P(|W| < 0.001) = \Phi(0.001 \times \sqrt{96}) - \Phi(-0.001 \times \sqrt{96})$$
$$= 2\Phi(0.001 \times \sqrt{96}) - 1 = 0.008$$

T4. 证明：

$$\lim_{n \to +\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}$$

考虑 poisson 分布：$P_\lambda(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$，$P_n(X=k) = \frac{n^k}{k!} e^{-n}$

利用 CLT 定理，当 $n \to +\infty$ 时，有 Poisson distribution → Normal distribution

则用 $EX = n$，$var X = n$，$f(x) = \frac{1}{\sqrt{2\pi n}} e^{-\frac{x^2}{2n}}$

$$P_n(X \geq 0) = \sum_{k=0}^{n} \frac{n^k}{k!} e^{-n} = e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}$$

$$\lim_{n \to \infty} P_n(X \geq 0) = \lim_{n \to \infty} \int_0^{\infty} f(x) dx = \frac{1}{2} \quad \therefore \lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}$$

T5. 解：

$S_N = X_1 + \cdots + X_N$, $X_i \sim B(1, p)$. $N = 10000 \gg 1$

求 $P(S_N > m) \leq 5\%$ $S_N \sim N(Np, Np(1-p))$

$$P\left(\frac{S_N - Np}{\sqrt{Np(1-p)}} > \frac{m-2000}{100\times 0.4}\right) \leq 0.05$$

$$1 - \Phi\left(\frac{m-2000}{40}\right) \leq 0.05 \Rightarrow \frac{m-2000}{40} \geq \Phi^{-1}(0.95)$$

利用 $\Phi^{-1}(0.95) = 1.65$ $\frac{m-2000}{40} \geq 1.65$ $m \geq 2066$

$\Rightarrow m \simeq 2066$ 个人