1. 解:

$$f_X(x)=\begin{cases}\frac{x}{4} & 1<x\le 3\\ 0 & \text{otherwise}\end{cases}$$ 这为什么不归一?

$A: \{X \geq 2\}$

(a).

$$EX=\int_1^3 x f_X(x)\,dx=\int_1^3 \frac{x^2}{4}\,dx=\frac{x^3}{12}\Big|_1^3=\frac{13}{6}$$

$$P(A)=\int_1^3 I_A f_X(x)\,dx=\int_2^3 \frac{x}{4}\,dx=\frac{x^2}{8}\Big|_2^3=\frac{5}{8}$$

$$f_{X|A}(x)=\begin{cases}\frac{f_X(x)}{P(A)}=\frac{2}{5}x & 2\le x\le 5\\ 0 & \text{otherwise}\end{cases}$$

$$E(X|A)=\int x f_{X|A}(x)\,dx=\int_2^3 x\cdot\frac{2}{5}x\,dx=\frac{2x^3}{15}\Big|_2^3=\frac{38}{15}$$

(b) $Y=X^2$. 求 EY 与 $\mathrm{Var}\,Y$.

$$EY=EX^2=\int_1^3 x^2 f(x)\,dx=\int_1^3 \frac{x^3}{4}\,dx=5$$

$$EY^2=EX^4=\int_1^3 \frac{x^5}{4}\,dx=91/3$$

$$\mathrm{Var}\,Y=EY^2-(EY)^2=\frac{91}{3}-5^2=\frac{16}{3}$$

2. 解: X, Y i.i.d $\sim \mathcal{E}(\lambda)$. $Z=X+Y$.

(a) 求 $f_{X|Z}(x|z)=?$

由 X, Y i.i.d. $\sim\mathcal{E}(\lambda)$ 有 $f_X(x)=\lambda e^{-\lambda x}\ (x>0)$, $f_Y(y)=\lambda e^{-\lambda y}\ (y>0)$, 且 $f_{X,Y}(x,y)=f_X(x)f_Y(y)$

$$F(x,z)=P(X\le x, Z\le z)=P(X\le x, X+Y\le z)=P(X\le x, Y\le z-X)$$

$$=\int_{-\infty}^{x}\int_{-\infty}^{z-t} f_{X,Y}(s,t)\,ds\,dt=\int_0^x\left(\int_0^{z-s} f_{X,Y}(s,t)\,dt\right)ds=\int_0^x f_X(s)\left(\int_0^{z-s} f_Y(t)\,dt\right)ds$$

$$=\int_0^x \lambda e^{-\lambda s}\left(\int_0^{z-s} \lambda e^{-\lambda t} dt\right) ds = \int_0^x \lambda e^{-\lambda s}\left(1-e^{-\lambda(z-s)}\right) ds$$

$$=\int_0^x \left(\lambda e^{-\lambda s} - \lambda e^{-\lambda z}\right) ds = 1-e^{-\lambda x} - x\cdot\lambda e^{-\lambda z}$$

$$\Rightarrow f(x,z) = \frac{\partial^2 F(x,z)}{\partial x \partial z} = \begin{cases} \lambda^2 e^{-\lambda z} & (0<x\le z) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\Rightarrow f_Z(z) = \int_0^z f(x,z)\,dx = \lambda^2 z e^{-\lambda z} \quad (z>0). \quad \text{othewise} = 0$$

$$\Rightarrow f_{X|Z}(x|z) = \frac{f_{X,Z}(x,z)}{f_Z(z)} = \frac{1}{z}$$

(b)

$$E(X^2|Z=z) = \int_0^z x^2 f_{X|Z}(x|z)\,dx = \int_0^z x^2 \cdot \frac{1}{z}\,dx = \frac{1}{z}\cdot\frac{1}{3}z^3 = \underset{m(z)}{\underset{\shortparallel}{\frac{1}{3}z^2}}$$

$$\Rightarrow E(X^2|Z) = m(Z) = \frac{1}{3}Z$$

3. 选做

4. 解(Ch4.22). 设 X_k 表示第k轮后的钱数. X_k 显然是 r.v.

根据投注策略:

$$E[X_{k+1}|X_k] = X_k\cdot\left(1+p(2p-1)-(1-p)(2p-1)\right) = X_k\cdot(1+(2p-1)^2)$$

$$E[E[X_{k+1}|X_k]] = EX_k\cdot(1+(2p-1)^2)$$

利用重期望法则 $\Rightarrow EX_{k+1} = EX_k(1+(2p-1)^2)$. 等比数列

$\therefore \quad EX_n = (1+(2p-1)^2)^n \cdot x \geq x$. — Kelly Strategy

5. 解: (2). 分支过程.

(a). $E(X_n|X_{n-1}) = \sum_{j=0}^{\infty} j P_j \cdot X_{n-1} = \mu X_{n-1}$

$$E[E(X_n|X_{n-1})] = \mu E X_{n-1}$$

由重期望法则，有 $E[E(X_n|X_{n-1})]=EX_n$

∴ $EX_n=\mu EX_{n-1} \Rightarrow EX_n=\mu^n EX_0=\mu^n X_0$

(b) 利用全方差公式

$\mathrm{var}X_n = E[\mathrm{var}(X_n|X_{n-1})] + \mathrm{var}(E[X_n|X_{n-1}])$

而 $\mathrm{Var}(X_n|X_{n-1}) = \sum_{i=1}^{X_{n-1}} \sigma_i^2 = X_{n-1}\cdot\sum_{j=0}^{\infty}(j-\mu)^2 p_j = X_{n-1}\sigma^2$

(利用细胞分裂是独立的)

$E[\mathrm{Var}(X_n|X_{n-1})]=\sigma^2 EX_{n-1}$，而 $E[X_n|X_{n-1}]=\mu X_{n-1}$

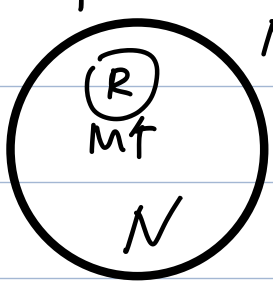
$\mathrm{var}(E[X_n|X_{n-1}])=\mu^2\mathrm{var}(X_{n-1}) \Rightarrow$

递推式： $\mathrm{var}X_n=\sigma^2\cdot\mu^{n-1}X_0+\mu^2\mathrm{Var}X_{n-1}$

边界条件： $\mathrm{var}X_0=0$, $(\mathrm{var}X_1=\sigma^2\cdot 1+0=\sigma^2)$

6. 7. 8. (Ch 3.35. Ch 4.27. Ch 4.26 选做)

9. 解：



N-M. B　取n.

$$P(X=k)=\frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}},\quad k=0,1,2,\cdots \min\{n,M\}$$

给所有的球编号1…N，记X_i为第i次取到前M个球，1~M编号为红球

则 $X=X_1+X_2+\cdots+X_n$，　$P(X_i)_{1\le i\le n}=\frac{M}{N}$

$EX=E(X_1+\cdots+X_n)=\frac{M}{N}n$

$$\mathrm{Var}X = EX^2 - (EX)^2 = E(X_1+X_2+\cdots+X_n)^2 - \left(\frac{M}{N}n\right)^2$$

利用 $E(X_i^2) = E(X_i) = \frac{M}{N}n.$

$$E(X_iX_j)_{i\neq j} = \frac{M}{N}\cdot\frac{M-1}{N-1}$$

$$\Rightarrow \mathrm{Var}X = 2\binom{n}{2}\frac{M}{N}\frac{M-1}{N-1} + \frac{nM}{N} - \frac{n^2M^2}{N^2}$$

$$\Rightarrow \mathrm{var}X = \frac{nM(N-n)(N-M)}{N^2(N-1)}$$

□

10. 解:

(a) $X_K \sim B(1,\frac{1}{6})$

$EX_K = \frac{1}{6} = EX_1 \qquad X_1^2 = X_1$

$\mathrm{var}X_1 = EX_1^2 - EX_1^2 = \frac{1}{6} - \frac{1}{36} = \frac{5}{36}$

(b). $X = X_1+\cdots+X_n$, $X \sim B(n,p)$

$EX = E(\sum_{i=1}^n X_i) = \frac{n}{6}$,

$\mathrm{var}X = \sum_{i=1}^n \mathrm{Var}X_i = \frac{5}{36}n.$ (利用 $X_1\cdots X_n$ 相互独立)

(c). 利用 $\mathrm{cov}(X_i,Y_j) = E((X_i-EX_i)(Y_j-EY_j))$

且由 $EX_i = \frac{1}{6}$, $EY_j = \frac{1}{6}$

$\Rightarrow \mathrm{cov}(X_i,Y_j) = E((X_i-\frac{1}{6})(Y_j-\frac{1}{6}))$

$= E(X_iY_j - \frac{1}{6}(X_i+Y_j) + \frac{1}{36})$

① 若 $i\neq j$, 则 $X_i \perp Y_j$. $E(X_iY_j) = E(X_i)E(Y_j) = \frac{1}{36}$

$\mathrm{cov}(X_i,Y_j) = \frac{1}{18} - \frac{1}{6}(EX_i + EY_j) = 0$

② 若 $i=j$, 则 $E(X_iY_j) = 0. \Rightarrow \mathrm{cov}(X_i,Y_j) = -\frac{1}{36}$

(d) $\mathrm{Cov}(X,Y) = \mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, Y_j) = -\frac{n}{36}$

(e). $\rho_{X,Y} = \dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{var}X}\sqrt{\mathrm{var}Y}} = \dfrac{-\frac{n}{36}}{\frac{5}{36}n} = -\frac{1}{5}$

$\rho_{X,Y} < 0$. X、Y有负相关性. 很好理解. "1"出现的机会多,

"2"出现的机会就少.