HW 2

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1

> 设样本空间 Ω 是由 n 个互不相交的事件 A_i 的并集，即：$\Omega=\bigcup_{i=1}^{n} A_i$其中，事件 A_i（$i=1,2,\ldots,n$）互不相交，并且每个事件 A_i 发生的概率为正数。设 $\mathcal{F}$ 是包含所有事件 A_i 的最小事件域。问$\mathcal{F}$ 中有多少个元素?

SOLUTION
由事件域的定义和性质由$\Omega\in\mathcal{F}$，因此$\mathcal{F}$ 中至少包含 Ω,因此最小的$\mathcal{F}$由Ω的一切子集构成，包含不可能事件$\emptyset$,$\binom{n}{1}$个单点集，...,$\binom{n}{n-1}$个$n-1$点集，还有必然事件Ω因此$\mathcal{F}$共有$\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{n-1}=2^n$个元素.

1*

> 设 $\mathcal{F}_1$ 和 $\mathcal{F}_2$ 都是 Ω 上的事件域，验证 $\mathcal{F}_1\cap\mathcal{F}_2$ 也是 Ω 上的事件域

SOLUTION
首先，定义 $\mathcal{F}_1\cap\mathcal{F}_2$：

$$\mathcal{F}_1\cap\mathcal{F}_2=\{A\subset\Omega\mid A\in\mathcal{F}_1\text{ 且 }A\in\mathcal{F}_2\}$$

由于 $\mathcal{F}_1$ 和 $\mathcal{F}_2$ 都是 σ-代数，根据定义，$\Omega\in\mathcal{F}_1$ 且 $\Omega\in\mathcal{F}_2$。因此，$\Omega\in\mathcal{F}_1\cap\mathcal{F}_2$。
设 $A\in\mathcal{F}_1\cap\mathcal{F}_2$，则 $A\in\mathcal{F}_1$ 且 $A\in\mathcal{F}_2$。由于 $\mathcal{F}_1$ 和 $\mathcal{F}_2$ 都是 σ-代数，所以 $A^c\in\mathcal{F}_1$ 且 $A^c\in\mathcal{F}_2$。因此，$A^c\in\mathcal{F}_1\cap\mathcal{F}_2$。
设 $A_1,A_2,\ldots\in\mathcal{F}_1\cap\mathcal{F}_2$，则对于每个 i，$A_i\in\mathcal{F}_1$ 且 $A_i\in\mathcal{F}_2$。由于 $\mathcal{F}_1$ 和 $\mathcal{F}_2$ 都是 σ-代数，所以 $\bigcup_{i=1}^{\infty}A_i\in\mathcal{F}_1$ 且 $\bigcup_{i=1}^{\infty}A_i\in\mathcal{F}_2$。因此，$\bigcup_{i=1}^{\infty}A_i\in\mathcal{F}_1\cap\mathcal{F}_2$。
综上所述，$\mathcal{F}_1\cap\mathcal{F}_2$ 满足 σ-代数的所有条件，因此 $\mathcal{F}_1\cap\mathcal{F}_2$ 是一个 σ-代数，即 $\mathcal{F}_1\cap\mathcal{F}_2$ 也是 Ω 上的事件域。

□

2

> 设 $\{A_n,n=1,2,\ldots\}$ 是 $\mathcal{F}$ 中的事件列，证明：如果 $\lim_{n\to\infty}A_n$ 存在，那么 $P(\lim_{n\to\infty}A_n)=\lim_{n\to\infty}P\left(A_n\right)$.

PRPOOF
由题目可知，事件列A_i的极限$\lim_{n\to\infty}A_n$存在，$\limsup_{n\to\infty}A_n=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}A_k$与$\liminf_{n\to\infty}A_n=\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}A_k$均存在且相等. 即$\lim_{n\to\infty}A_n=\limsup_{n\to\infty}A_n=\liminf_{n\to\infty}A_n$ 我们不妨令$B_n=\bigcap_{k=n}^{\infty}A_k$，则有$B_1\subset B_2\subset\cdots\subset\cdots\subset B_\infty$，是不减的事件列，因此我们不妨构造$T_n=B_n\setminus B_{n-1}$. 则有$\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}A_k=\bigcup_{n=1}^{\infty}B_n=\bigcup_{n=1}^{\infty}T_n$ 利用T_n的互斥性，以及可列可加性，我们可以得到$P(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}A_k)=P(\bigcup_{n=1}^{\infty}T_n)=P(\bigcup_{n=1}^{\infty}B_n)=\Sigma_{n=1}^{\infty}P(T_n)=\Sigma_{n=1}^{\infty}P(B_n\setminus B_{n-1})=\Sigma_{n=2}^{\infty}(P(B_n)-P(B_{n-1}))+P(B_1)=\lim_{n\to\infty}P(B_n)=\lim_{n\to\infty}P(\bigcap_{n=1}^{\infty}A_n)$. 那么由$\bigcap_{n=1}^{k}A_k$是单调递减事件列，我们不妨继续构造$U_k=\bigcap_{n=1}^{k}A_k$,则$U_k^c=\Omega-U_k$为递增列，由前述的性质可以得到$P(\bigcup_{k=1}^{\infty}(\Omega-U_k))=P(\lim_{k=1}^{\infty}(\Omega-U_k))=P(\Omega-\lim_{k=1}^{\infty}U_k)=1-P(\lim_{k=1}^{\infty}U_k)=\lim_{k\to\infty}P(\Omega-U_k)=1-\lim_{k\to\infty}P(U_k)$. 所以有$P(\lim_{k\to\infty}U_k)=\lim_{k\to\infty}P(U_k)$即$P(\lim_{n\to\infty}\bigcap_1^n A_n)=\lim_{n\to\infty}P(\bigcap_1^n A_n)=\lim_{n\to\infty}P(A_n)$. 综合上述两步证明的过程，我们最终得到了$P(\lim_{n\to\infty}A_n)=\liminf_{n\to\infty}A_n=\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}A_k=\lim_{n\to\infty}P(A_n)$

□

2*

> 证明Borel-Cantelli引理.
> (1)如果一串事件 $\{A_n\}$ 的概率之和是有限的，即 $\sum_{n=1}^{\infty}P(A_n)<\infty$，那么这些事件几乎不会无限多次发生。$P(\limsup_{n\to\infty}A_n)=0$
> (2)如果一串事件 $\{A_n\}$ 的概率之和是无限的，即 $\sum_{n=1}^{\infty}P(A_n)=\infty$，并且这些事件是相互独立的，那么这些事件几乎必然无限多次发生事件$P(\limsup_{n\to\infty}A_n)=1$ （Hint:证明（2）时，可利用不等式$1-|x|\le e^{-|x|}$）

PROOF

(1)

根据概率的性质不难得到$P(\bigcup_{n=k}^{\infty} A_n) \leq \Sigma_{n=k}^{\infty} P(A_n) < \infty$. 利用事件列上极限的定义$\limsup_{n\to\infty} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$，我们可以得到$P(\limsup_{n\to\infty} A_n) = \lim_{k\to\infty} P(\bigcup_{n=k}^{\infty} A_n)$那么$\lim_{k\to\infty} P(\bigcup_{n=k}^{\infty} A_n) \leq \lim_{k\to\infty} \Sigma_{n=k}^{\infty} P(A_n)$ 利用当$\leq \Sigma_{n=k}^{\infty} P(A_n) < \infty$时，$\lim_{k\to\infty} \Sigma_{n=k}^{\infty} P(A_n) = 0$ 从而我们得到了$P(\limsup_{n\to\infty} A_n) = 0$

□

(2)

设A_n是相互独立的时间列，则A_n^c也是相互独立的事件列，由独立事件的概率乘法公式. 有$P(\bigcap_{n=k}^{\infty} A_n^c) = \prod_{n=k}^{\infty} P(A_n^c) = \prod_{n=k}^{\infty} P(1 - A_n)$利用Hint中给出的提示$1 - |x| \leq e^{-|x|}$，我们可以得到$P(\bigcap_{n=k}^{\infty} A_n^c) = \prod_{n=k}^{\infty} P(1 - A_n) \leq \exp(-\Sigma_{n=k}^{\infty} P(A_n))$ 由于$\sum_{n=1}^{\infty} P(A_n) = \infty$可以推出$\lim_{k\to\infty} \sum_{n=k}^{\infty} P(A_n) = \infty$我们可以得到$\lim_{k\to\infty} P(\bigcap_{n=k}^{\infty} A_n^c) = 0$,所以$\lim_{k\to\infty} P(\limsup_{n\to\infty} A_n) = 1 - \lim_{k\to\infty} P(\bigcap_{n=k}^{\infty} A_n^c) = 1$

□

3

> A coin is tossed twice. Alice claims that the event of two heads is at least as likely if we know that the first toss is a head than if we know that at least one of the tosses is a head. Is she right? Does it make a difference if the coin is fair or unfair? How can we generalize Alice's reasoning?

SOLUTION

我们不妨记事件A = “第一次投掷HEAD朝上”，记录事件B = “第二次投掷HEAD朝上”；那么题目中所述“ the event of two heads” = $A \cap B$,根据Alice所说的两种条件概率分别可以表示为

$$P_1 = P(A \cap B|A) = \frac{P(A \cap B)}{P(A)}$$
$$P_2 = P(A \cap B|A \cup B) = \frac{P(A \cap B)}{P(A \cup B)}$$

显然由于$A \subset A \cup B$,因此无论正反两面的概率是否相同（即硬币是否公平）有$P(A \cap B) \geq P(A)$，所以$P_1 \geq P_2$，因此Alice的观点是正确的。
如果硬币是公平的，那么$P(A) = \frac{1}{2}$ $\quad P(A \cup B) = \frac{3}{4}$，且$P(A \cap B) = \frac{1}{4}$因此$P_1 = \frac{1}{2}$,同时$P_2 = \frac{1}{3}$。
对于Alice的论断，我们可以提出一个通用正确的表述，即对于事件S, T, R如果满足$T \subset R$而且$S \cap T = S \cap R$我们可以得到$P(S|T) \geq P(S|R)$,在本例中$S = A \cap B, T = A, R = A \cup B$，因此$P(A \cap B|A) \geq P(A \cap B|A \cup B)$。

4

> （Galton’s Paradox）投掷三枚质地均匀的硬币，至少两个硬币朝上的面是一样的，同时第三个硬币正与反是等概率的。因此三个硬币都是同一个面朝上的概率是1/2，你同意这个论断吗？

SOLUTION

不同意。我们先来在"至少两个面朝向一样"条件下，计算三个面都是一样的条件概率$P(\text{三个面都是一样的}|\text{至少两个面朝向一样}) = \frac{P(\text{三个面都是一样的})}{P(\text{至少两个面朝向一样})} = \frac{n}{|\Omega|}/1 = \frac{2}{2^3} = \frac{1}{4}$
题目中概率计算错误的原因：混淆了“前两个面朝向一样”和“三个硬币中至少有两个面朝向一样”这两个事件

5

> 某年级一共三个班级，一班3个男生、7个女生；二班7个男生、13个女生；三班12个男生、8个女生。新年时要抽取两位给幸运奖。先按人数比例，以1:2:2的概率选择其中一个班级，再从中先后选出两位同学。求：
> ① 先抽到男生的概率；
> ② 已知第二个抽到的是女生，求先抽到的是男生的概率。

SOLUTION

① 先抽到男生的概率,使用全概率公式,记录A = “先抽到男生”
C_i= "选中了第i个班级"

$$P(A) = \Sigma_{i=1}^{3} P(C_i)P(A|C_i) = \frac{1}{5} \times \frac{3}{10} + \frac{2}{5} \times \frac{7}{20} + \frac{2}{5} \times \frac{12}{20} = \frac{11}{25}$$

② 已知第二个抽到的是女生，求先抽到的是男生的概率，用条件概率公式
记B为“第二个抽到的是女生”，A为“第一个抽到的是男生”，则有

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\Sigma_i P(C_i)P(B\cap A|C_i)}{\Sigma_i P(C_i)(P(A^c\cap C_i)P(B|A^C\cap C_i) + P(A\cap C_i)P(B|A\cap C_i))}$$
$$= \frac{(1/5\times 3/10\times 7/9 + 2/5\times 7/20\times 13/19 + 2/5\times 12/20\times 8/19)}{(1/5\times 7/10\times 6/9 + 2/5\times 13/20\times 12/19 + 2/5\times 8/20\times 7/19) + (1/5\times 3/10\times 7/9 + 2/5\times 7/20\times 13/19 + 2/5\times 12/20\times 8/19)}$$
$$= \frac{347}{798}$$

6

> Problem 22. Each of k jars contains m white and n black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $m/(m+n)$。

PROOF

我们不妨采用数学归纳法，假设从第i个袋子中取出的球是白球的概率为$P_i(W) = \frac{m}{m+n}$.
归纳奠基：$P_1(W) = \frac{m}{m+n}$
归纳假设：$P_{i-1}(W) = \frac{m}{m+n}$
归纳递推：$P_i(W) = P_{i-1}(W)P_i(W|W) + P_{i-1}(B)P_i(W|B) = \frac{m}{m+n}\frac{m+1}{m+n+1} + \frac{n}{m+n}\frac{m}{m+n+1} = \frac{m}{m+n}$
从而，the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $m/(m+n)$.

□

7

> **Problem 25. The Two Envelopes Paradox.** You are handed two envelopes, and you know that each contains a positive integer amount of dollars, and that the amounts are different. The values of these amounts are modeled as unknown constants. Without knowing the amounts, you randomly choose one of the two envelopes, and after seeing the amount inside, you can switch envelopes if you wish. A friend claims that the following strategy will give you a probability greater than 1/2 of ending up with the envelope containing the larger amount: Toss a coin repeatedly, let X equal 1/2 plus the number of tosses required to obtain heads. If the amount in the chosen envelope is less than the value of X, then switch envelopes. Is your friend correct?

PROOF

是对的。我们不妨设较大的数额为M，较小的数额为m；定义事件$A = \{X < m\}, B = \{m < X < M\}, C = \{X > M\}, S =$"最终选择的红包是较大的那个",$S_1$表示最初选择的红包是较大的那个。那么在题目中所给的策略之下，$P(S) = P(A)P(S|A) + P(B)P(S|B) + P(C)P(C|B) = P(S) = P(A)P(S|(A\cap S)\cup(A\cap S^c)) + P(B)P(S|(B\cap S)\cup(B\cap S^c)) + P(S)P(S|(C\cap S)\cup(C\cap S^c)) = P(A)(P(S_1)P(S|A\cap S_1) + P(S_1^c)P(S|A\cap S_1^c)) + e.t.c.$. 利用$P(S_1) = P(S_1^c) = \frac{1}{2}$，我们有

$$P(S|A) = \frac{1}{2}(1+0) = \frac{1}{2}$$
$$P(S|C) = \frac{1}{2}(0+1) = \frac{1}{2}$$
$$P(S|B) = \frac{1}{2}(1+1) = 1$$

则$P(S) = \frac{1}{2}P(A) + \frac{1}{2}P(C) + P(B) = \frac{1}{2} + \frac{1}{2}P(B)$，因此，题目中给出的策略是有价值的。实际上我们会发现，无论采取什么样的判断标准，比如说我定X为随机一个正数，以此作为是否交换的标准，都可以提高拿到更大红包的概率。

□

> **Problem 51.** (选答题)A jar contains m red and n white balls.
> (a) We randomly select two balls simultaneously. Describe the sample space, and calculate the probability that the selected balls are of different colors using two methods: one based on counting with the discrete uniform law, and the other based on the sequential method using the multiplication rule.
> (b) We roll a fair three-sided die with faces labeled 1, 2, 3. If k appears, we randomly remove k balls from the jar and set them aside. Describe the sample space, and use the divide-and-conquer approach and the law of total probability to calculate the probability that all the removed balls are red.

略

8

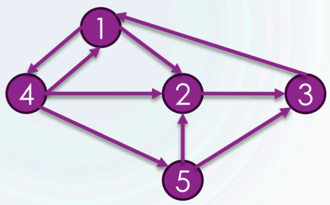
> (PageRank) The World Wide Web is a collection of linked web pages. We can use a graph to represent these pages and their links: the nodes of the graph are pages $\mathcal{A}$ and there is a directed edge from i to j if page i has a link to j. We can define the rank of page i as π i = σ$j\in\mathcal{A}$ π j $P(j, i)$, where $P(j, i)$ is the fraction of links in j that point to i and is zero if there is no such link. (The basic idea of the algorithm is due to Larry Page, hence the

> name PageRank. Since it ranks pages, the name is doubly appropriate). For convenience, we further restrict σj∈$\mathcal{A}$ π j = 1. Please solve π i , i = 1, … , 5 in the following graph.

作业（共3页，第三页）

- 全概率：

8. (**PageRank**) The World Wide Web is a collection of linked web pages. We can use a graph to represent these pages and their links: the nodes of the graph are pages $\mathcal{A}$ and there is a directed edge from i to j if page i has a link to j. We can define the rank of page i as $\pi(i) = \sum_{j\in\mathcal{A}} \pi(j)P(j,i)$, where $P(j,i)$ is the fraction of links in j that point to i and is zero if there is no such link. (*The basic idea of the algorithm is due to Larry Page, hence the name PageRank. Since it ranks pages, the name is doubly appropriate*). For convenience, we further restrict $\sum_{j\in\mathcal{A}} \pi(j) = 1$. Please solve $\pi(i), i = 1, \dots, 5$ in the following graph.



For example, we have $P(3,2) = 1, P(4,2) = \frac{1}{3}, P(5,2) = 0$, and $\pi(1) = \pi(2)\cdot 0 + \pi(3)\cdot 1 + \pi(4)\cdot\frac{1}{3} + \pi(5)\cdot 0$ in this graph.

SOLUTION

$$
\begin{aligned}
\pi(1) &= \pi(3) + \frac{1}{3}\pi(4) \\
\pi(2) &= \frac{1}{2}\pi(1) + \frac{1}{3}\pi(4) + \frac{1}{2}\pi(5) \\
\pi(3) &= \pi(2) + \frac{1}{2}\pi(5) \\
\pi(4) &= \frac{1}{2}\pi(1) \\
\pi(5) &= \frac{1}{3}\pi(4) \\
\Sigma_{i=1}^{5}\pi(i) &= 1
\end{aligned}
$$

可以解出$\pi(1) = \frac{4}{13}, \pi(2) = \frac{3}{13}, \pi(3) = \frac{10}{39}, \pi(4) = \frac{2}{13}, \pi(5) = \frac{2}{39}$。