T1. Problem 3.6

解: $N=$ "the number of customers ahead"

$X=$ "time of waiting"

CDF: $F_X(x)=0 \quad (x<0)$

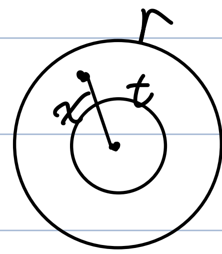
当 $x\geq 0$ 时, $F_X(x)=P(X\leq x)=\sum_{i=0}^{1}P(X\leq x|N=i)\,P(N=i)$

$\therefore F_X(x)=\frac{1}{2}P(X\leq x|N=0)+\frac{1}{2}P(X\leq x|N=1)$, $P(X\leq x|N=0)=1$

由 $X\sim \mathcal{E}(\lambda)$ 有 X 的 PDF: $f(X=x|N=1)=\lambda e^{-\lambda x} \quad (x\geq 0)$

$\Rightarrow P(X\leq x)=\int_0^x f(X=x|N=1)\,dx=1-e^{-\lambda x}$

$$\therefore F_X(x)=\begin{cases}1-\frac{1}{2}e^{-\lambda x} & (x\geq 0)\\ 0 & (x<0)\end{cases}$$



T2. Problem 3.7

解: 设 $X=$ "distance from the centre"

$F_X(x)=0 \quad (x<0)$

$F_X(x)=P(X\leq x)=\frac{\pi x^2}{\pi r^2}=\frac{x^2}{r^2} \quad (0\leq x\leq r)$

$F_X(x)=1 \quad (x>r)$

求 CDF of S: $F_S(s)=0 \quad (s<0)$

当 $0\leq s<\frac{1}{t}$ 时, 有 $x>t$, hit is outside of circle t.

$F_S(s)=P(S\leq s)=1-P(X\leq t)=1-\frac{t^2}{r^2}\in$ const

当 $s\in[\frac{1}{t},+\infty)$ 时, $0\leq X\leq t$ 有 $F_S(s)=P(S\leq s)=P(\frac{1}{s}\leq X\leq t)$

$+P(X>t)=F_X(t)-F_X(\frac{1}{s})+1-F_X(t)=1-\frac{1}{s^2r^2}$

综上所述, CDF of s: $F_S(s)=\begin{cases}0 & (s<0)\end{cases}$

$\begin{cases} 1-t^2/r^2 \ (0\le s\le t) \\ 1-\frac{1}{s^2+r^2} \ (s>t) \end{cases}$

$\lim_{s\to 0^-} F_S(s)=0 \neq \lim_{s\to 0^+} F_S(s) = 1-\frac{t^2}{r^2}$ $\therefore F_S(s)$不是连续函数.

T3. Problem 3.8

解:

(a) 由全概率公式有

$$F_X(x)=P(X\le x)= P(X=Y)P(Y\le x)+P(X=Z)P(Z\le x)$$

$$= pF_Y(x)+(1-p)F_Z(x) \quad (*)$$

对于(*)等式两边分别求导得到 $f_X(x)=pf_Y(x)+(1-p)f_Z(x)$

证毕!

(b) 由Y. Z are random variables $\sim \mathcal{E}(\lambda)$, and are 2-sided.

$$\begin{cases} f_Y(y)=\begin{cases} 0 & (y\ge 0) \\ \lambda e^{-\lambda y} & (y<0) \end{cases} \\ f_Z(z)=\begin{cases} \lambda e^{-\lambda z} & (z\ge 0) \\ 0 & (z<0) \end{cases} \end{cases}$$

由(a)问得到的等式 $\Rightarrow f_X(x)=\begin{cases} (1-p)\lambda e^{-\lambda x} & (x\ge 0) \\ p\lambda e^{-\lambda x} & (x<0) \end{cases}$

T4. problem 3.11

解: 由 $X\sim N(0,1)$ $Y\sim(1,4)$ 可写出 $f_X(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

$f_Y(y)=\frac{1}{2\sqrt{2\pi}}e^{-\frac{x^2}{8}}$

(a). X 服从标准正态分布. $P(X\le 1.5)=\Phi(1.5)=0.9332$

$P(X\le -1)=1-\Phi(1)=0.1587$

(b). 记 $Z=\frac{1}{2}(Y-1)$，则 $\bar{Z}=\frac{1}{2}\bar{Y}=1$，$\sigma_Z^2=(\frac{1}{2})^2\sigma_Y^2=1$

$Z\sim N(0,1)$

$\therefore f_{\frac{Y-1}{2}}(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$ 即 $\frac{Y-1}{2}$ 服从标准正态分布.

(c) 求 $P(-1\le Y\le 1).=P(-1\le\frac{Y-1}{2}\le 0)=P(-1\le Z\le 0)=P(0\le Z\le 1)$

$=\Phi(1)-\Phi(0)=0.8413-0.5=0.3413$

T5. $X\sim u(-1,2)$ 求 $Y=X^2$ 的 CDF.

由 $X\sim u(-1,2)$ 可知. $f_X(x)=\begin{cases}0 & (x<-1)\\ \frac{1}{3} & (-1\le x\le 2)\\ 0 & (x>2)\end{cases}$

CDF of X: $F_X(x)=\begin{cases}0 & (x<-1)\\ \frac{x+1}{3} & (-1\le x\le 2)\\ 1 & (x>2)\end{cases}$

则有 $P(Y\le x^2)=P(-x\le X\le x)=F_X(x)-F_X(-x)\ (x\ge 0)$

显然有 $P(Y\le y)=0.(y<0)$ 令 $y=x^2$. 当 $0\le y\le 1$ 时，$\pm x\in[-1,2]$

则 $P(Y\le x^2)=F_X(x)-F_X(-x)=\frac{x+1}{3}-\frac{1-x}{3}=\frac{2}{3}x$

$\therefore P(Y\le y)=\frac{2}{3}\sqrt{y}\quad(0\le y\le 1)$

当 $1<y\le 4$ 时，$x\in(1,2]\subset[-1,2]$，$-x\in[-2,-1)\subset(-\infty,-1)$

$\therefore P(Y\le x^2)=F_X(x)-F_X(-x)=\frac{x+1}{3}-0=\frac{x+1}{3}$

$\therefore P(Y\le y)=\frac{1}{3}+\frac{1}{3}\sqrt{y}\quad(1<y\le 4)$

当 $y>4$ 时，$x\in(2,+\infty)$，$-x\in(-\infty,-2)$

$\therefore P(Y\le y)=1\quad(y>4)$

$\begin{cases}0 & (y<0)\\ \frac{2}{3}\sqrt{y} & (0\le y\le 1)\end{cases}$

综上所述得到 Y 的 CDF：$F_Y(y)=\begin{cases}\frac{1}{3}+\frac{1}{3}\sqrt{y} & (1<y\leq 4)\\ 1 & (y>4)\end{cases}$

那么由 CDF ⇒ PDF：得到 $f_Y(y)=\begin{cases}0 & (y<0)\\ \frac{1}{3\sqrt{y}} & (0\leq y<1)\\ \frac{1}{6\sqrt{y}} & (1<y\leq 4)\\ 0 & (y>4)\end{cases}$

T6. $f_X(x)=\begin{cases}0 & (x\leq \nu)\\ \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1}\cdot \exp\left\{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right\} & (x>\nu)\end{cases}$

CDF and PDF of $Y=\left(\frac{x-\nu}{\alpha}\right)^{\beta}$

简：

不妨先求出 $F_X(x)=\begin{cases}0 & (x\leq \nu)\\ \int_{\nu}^{x}\frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1}\cdot \exp\left\{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right\}dx & (x>\nu)\end{cases}$

利用 $\int_{\nu}^{x}\frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1}\cdot \exp\left\{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right\}dx$

$=\int_{0}^{\left(\frac{x-\nu}{\alpha}\right)^{\beta}} e^{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}}d\left(\frac{x-\nu}{\alpha}\right)^{\beta}=\int_{0}^{y}e^{-y}dy=1-e^{-y}$

$\therefore F_X(x)=\begin{cases}0 & (x\leq \nu)\\ 1-\exp\left\{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right\} & (x>\nu)\end{cases}$

$P(Y\leq y)=\begin{cases}0 & (y\leq 0)\\ 1-e^{-y} & (y>0)\end{cases} \Rightarrow \underset{(\mathrm{CDF})}{F_Y(y)}=\begin{cases}0 & (y\leq 0)\\ 1-e^{-y} & (y>0)\end{cases}$

PDF：$f_Y(y)=\begin{cases}0 & (y\leq 0)\\ e^{-y} & (y>0)\end{cases}$

Y 服从 $\lambda=1$ 参数的指数分布. 即 $Y \sim \mathcal{E}(1)$

T7.

解: $X \sim N(0,\sigma^2)$ 则 $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

则由 $Y=\sqrt{\max(X,0)}$ 知, $Y = \begin{cases} 0 & (X \le 0) \\ \sqrt{X} & (X>0) \end{cases} \Rightarrow X>0$ 时. $\frac{dx}{dy} = 2y$

则 $P(Y=0) = P(X\le 0) = F_X(0) = \frac{1}{2}$; $f_X(\sqrt{x}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{x})^4}{2}}$

$P(Y\le y) = F_Y(y) = F_X(\sqrt{x})$

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dx} F_X(\sqrt{x}) \left.\frac{dx}{dy}\right| = f_X(\sqrt{x}) \cdot 2y = \sqrt{\frac{2}{\pi}} e^{-\frac{y^4}{2}} \cdot y$

$\therefore$ CDF of Y: $\begin{cases} P(y<0) = 0 \\ P(y=0) = \frac{1}{2} \\ f_Y(y) = \sqrt{\frac{2}{\pi}}\, y e^{-\frac{y^4}{2}} \quad (y>0) \end{cases}$