T1. 解:

① $g_0(X)=a_0+b_0X$, 从 $(a_0,b_0)=\arg\min_{(a,b)} E[(Y-a-bX)^2]$

$loss = E[(Y-a-bX)^2] = E[Y^2+a^2+b^2X^2+2baX-2Y(a+bX)]$

$=a^2+2abEX+b^2EX^2+EY^2-2E[Y(a+bX)]$

$=a^2+2abEX+b^2EX^2+EY^2-2aEY-2bEXY$

$\frac{\partial loss}{\partial a}=0 \Rightarrow 2a+2bEX-2EY=0$, $a=EY-bEX$

$\frac{\partial loss}{\partial b}=0 \Rightarrow 2aEX+2bEX^2-2EXY=0$, 代入 $a=EY-bEX$ 得到

$$b_0=\frac{EXY-EXEY}{EX^2-(EX)^2}, \quad a_0=\frac{EYEX-EXEY}{EX^2-(EX)^2}\cdot EX$$

可以验证 $\frac{\partial^2 L}{\partial a^2}\Big|_{a=a_0}>0$, $\frac{\partial^2 L}{\partial b^2}\Big|_{b=b_0}>0$ ∴ $(a_0,b_0)=\arg\min_{(a,b)} L$

② $X=\alpha Y+Z$

$L[Y|X]=a_0+b_0X$, 利用①结论有 $b_0=\frac{EXY-EXEY}{EX^2-(EX)^2}$

利用 $EXY=EY(\alpha Y+Z)=\alpha EY^2+EYZ=\alpha EY^2+EYEZ$

$EXEY=E(\alpha Y+Z)EY=\alpha(EY)^2+EYZ=\alpha(EY)^2+EYEZ$

$EX=\alpha EY+EZ$, $EX^2=E(\alpha^2Y^2+Z^2+2\alpha YZ)=\alpha^2EY^2+EZ^2+2\alpha EYEZ$

代入得到 $$b_0=\frac{\alpha EY^2+EYEZ-\alpha(EY)^2-EYEZ}{-(\alpha^2(EY)^2+(EZ)^2+2\alpha EYEZ)+\alpha^2EY^2+EZ^2+2\alpha EYEZ}$$

$$=\frac{EY^2-(EY)^2}{\alpha[EY^2-(EY)^2]+\alpha[EZ^2-(EZ)^2]}$$

利用 $EY=EZ=0 \Rightarrow b_0=\frac{EY^2}{\alpha(EY^2+EZ^2)}$, $a_0=0$.

得到 $$L[Y|X]=\frac{\frac{E(\alpha^2Y^2)}{EZ^2}}{\alpha\left(\frac{E(\alpha^2Y^2)}{EZ^2}+1\right)}X=\frac{SNR}{SNR+1}\cdot\frac{1}{\alpha}X=g(X)$$

SNR↑, $L[Y|X] \to \frac{1}{\alpha}X$ 观察原式 $X=\alpha Y+Z$, $SNR=\frac{E(\alpha^2Y^2)}{E(Z^2)}$

SNR增大说明 Z相比于 Y不重要, 则 $X\simeq \alpha Y$, $Y\simeq \frac{1}{\alpha}X=g(X)$

T2. 解:

有Y的概率母函数为 $g_Y(s)=g_N(g_X(s))$

由 $N\sim P(\lambda)$, $X_i\sim B(1,p)$ 有 $g_N(s)=e^{\lambda(s-1)}$, $g_X(s)=q+ps$

$\therefore g_Y(s)=e^{\lambda((q+ps)-1)}=e^{\lambda p(s-1)}$

利用可逆性知 $Y\sim P(\lambda p)$

T3. 解:

由定义 $g_X(s)=ES^X=\sum_{j=1}^{\infty}s^jP(X=j)=\sum_{j=1}^{\infty}-\frac{q^j\cdot s^j}{\ln p\cdot j}=\sum_{j=1}^{\infty}-\frac{(qs)^j}{\ln p\cdot j}$

利用 $\ln(1-x)$ 的展开 $\ln(1-x)=-\sum_{j=1}^{\infty}\frac{x^j}{j}$, $x\in(0,1)$

$\Rightarrow g_X(s)=\frac{\ln(1-qs)}{\ln p}$

$$EX=g_X^{(1)}(0)=\frac{-q}{(1-qs)\ln p}\Big|_{s=0}=\frac{-q}{p\ln p}$$

(Ch 4.32)

T4. 解: (a).

利用性质 2.1.(3). $M(0)=1$, 而 (1.) $M(s)=e^{2(e^{e^0-1}-1)}=1$ ✓

(2.) $M(s)=e^{2(e^{e^0}-1)}=e^{2(e-1)}\neq 1$.

可知 (2.) $M(s)$ 不可作为矩母函数.

(b). 利用性质: 当X仅取非负整数值时,

$$P(X=0)=\lim_{s\to-\infty}M(s)=\lim_{s\to-\infty}e^{2(e^{e^s-1}-1)}=e^{2(e^{-1}-1)}$$

(Ch 4.33)

T5. 解: $M(s) = \frac{1}{3}\cdot\frac{2}{2-s} + \frac{2}{3}\frac{3}{3-s}$

利用可逆性. $M(s)$ 写为 $p_1\cdot\frac{\lambda_1}{\lambda_1-s} + p_2\frac{\lambda_2}{\lambda_2-s}$ 的形式

则 X 为 X_1, X_2 的混合变量. $X_1 \sim \mathcal{E}(\lambda_1) = \mathcal{E}(2)$. $X_2 \sim \mathcal{E}(\lambda_2) = \mathcal{E}(3)$

X 分别取 X_1/X_2 之一的概率分别为 $p_1 = \frac{1}{3}$, $p_2 = \frac{2}{3}$

PDF: $\begin{cases} f_X(x) = p_1 f_{X_1}(x_1) + p_2 f_{X_2}(x_2) = \frac{4}{3}e^{-2x} + e^{-3x} & (x \geq 0) \\ f_X(x) = 0 & (x < 0) \end{cases}$

(Ch 4.35)

T6. 解:

利用 $M(0)=1$ 有 $M_X(0) = C\cdot\frac{3+4+2}{3-1} = 1 \Rightarrow C = \frac{2}{9}$

对于 $M_X(s) = \frac{2}{9}\frac{3+4e^{2s}+2e^{3s}}{3-e^s}$ Taylor 展开

$\Rightarrow M_X(s) = \frac{2}{9}\times\frac{3+4(1+2s+o(s))+2(1+3s+o(s))}{3-(1+s+o(s))} = 1 + \frac{37}{18}s + o(s)$

(a) $\therefore EX = M^{(1)}(0) = \frac{37}{18}$

(b) 求解 $P_X(1)$:

利用矩母函数 (for discrete case) in Definition:

$M(s) = \sum_x e^{sx}P_X(x)$ 将 $M(s) = \frac{2}{9}\frac{3+4e^{2s}+2e^{3s}}{3-e^s}$ 按 e^s 的次数展开

$M(s) = \frac{2}{9} + \frac{2}{27}e^s + \cdots \Rightarrow$ 展开系数 $P_X(0) = \frac{2}{9}$, $P_X(1) = \frac{2}{27}$

(c). $E[X|X\neq 0]$. 令 $A = \{X \neq 0\}$

$\Rightarrow P_{X|A}(x=k) = \begin{cases} \frac{P_X(k)}{P(A)} & (k=1,2,\dots) \\ 0 & (\text{otherwise}) \end{cases}$

$E[X|X\neq 0] = E[X|A] = \sum_{k=1}^{\infty} \frac{k p_X(x)}{P(A)} = \frac{EX}{1-p_X(0)} = \frac{37/18}{1-2/9} = \frac{37}{14}$

(Ch 4.36)

T7. 解:

PMF: $X \sim B(1, 1/3)$. $Y \sim \mathcal{E}(2)$. $Z \sim P(3)$

MGF: $M_X(s) = \frac{2}{3} + \frac{1}{3}e^s$, $M_Y(s) = \frac{2}{2-s}$, $M_Z(s) = e^{3(e^s-1)}$

(a). $U = XY + (1-X)Z = \begin{cases} Y & (X=0) \\ Z & (X=1) \end{cases}$

$M_U(s) = P(X=0)M_Y(s) + P(X=1)M_Z(s) = \frac{1}{3} \cdot \frac{2}{2-s} + \frac{2}{3} e^{3(e^s-1)}$

(b). $U_2 = 2Z + 3$

$M_{U_1}(s) = e^{3s} M_Z(2s) = e^{3s} \cdot e^{3(e^{2s}-1)} = e^{3(s-1+e^{2s})}$

(c). $U_3 = Y + Z$

$M_{U_2}(s) = M_Y(s) M_Z(s) = \frac{2}{2-s} e^{3(e^s-1)}$

T8. 解:

$X \cdot Y$. i.i.d 且 $M_X(s) = M_Y(s) = e^{s+s^2}$. $s \in \mathbb{R}$.

则 (1) $Z = X - Y = M_X(s) M_Y(-s) = e^{2s^2}$

利用 $W \sim N(\mu, \sigma^2)$, $M_W(s) = e^{\mu s + (\sigma^2 s^2/2)}$ 知.

$Z \sim N(0, 4)$, 服从 $\mu=0$, $\sigma=2$ 的正态分布 (可逆性質)

(2) $EX^4 = M_X^{(4)}(0)$

$\Rightarrow EZ^4 = M_Z^{(4)}(0) = 16 + 32 = 48 = \mu_4$. $\frac{\mu_4}{\sigma^4} = 3$. ✓

$M_Z^{(1)}(s) = 4s e^{2s^2}$

$M_Z^{(2)}(s) = 4e^{2s^2} + 16s^2 e^{2s^2}$

$$M^{(3)}_{Z}(s) = 16s e^{2s^2} + 64 s^3 e^{2s^2} + 32 s e^{2s^2}$$

$$M^{(4)}_{Z}(s) = 16 e^{2s^2} + \cancel{64 s^2 e^{2s^2}} + \cancel{192 s^2 e^{2s^2}} + \cancel{256 s^4 e^{2s^2}} + 32 e^{2s^2}$$

$$+ \cancel{128 s^2 e^{2s^2}}$$