1. 解：(Ch2.2.16)

$$P_X(x)=\begin{cases} x^2/a, & x=-3,-2,-1,0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

(a) $\sum P_X(x)=1 \Rightarrow \frac{1}{a}(9+4+1)\times 2=1 \Rightarrow a=28$

$E(X)=\sum x P_X(x)=0$ (由对称性)

(b) $Z=(X-E(X))^2$. PMF.

由 $EX=0$, $Z=X^2$. $x=\sqrt{z} \Rightarrow P_Z(z)=P_X(\sqrt{z})+P_X(-\sqrt{z})$

当 $(z=1,4,9) \Rightarrow P_Z(z)=\begin{cases} z/14, & z=1,4,9 \\ 0 & \text{otherwise} \end{cases}$

(c) $\text{var}(X)=E(X-EX)^2=EZ=\sum P_Z(z)\cdot z=\frac{1}{14}+\frac{16}{14}+\frac{81}{14}=7$

(d) $\text{var}(X)=\sum_x (x-EX)^2 P_X(x)$

$=\left(9\times\frac{9}{28}+4\times\frac{4}{28}+1\times\frac{1}{28}\right)\times 2=7$

2. 解 (Ch2.22)

(a). 同时抛，结果可为 H_1H_2, H_1T_2, T_1H_2, T_1T_2 由独立性

记事件 $A=\{H_1T_2, H_2T_1\}$. 则在一次抛掷中，

$P(A)=p(1-q)+q(1-p)$

记事件 A_k 为在第 k 次抛掷中 A 发生，记 X 为题中所求的次数.

所求 $P_X(k)=P(\overline{A_1}\,\overline{A_2}\cdots\overline{A_{k-1}}A_k)=(1-p(1-q)-q(1-p))^{k-1}(p(1-q)+q(1-p))$

$(k=1,2,\cdots)$, $X\sim G(p(1-q)+q(1-p))$, 记 $m=$ 参数.

因此有（上课结论），$EX=\frac{1}{m}=\frac{1}{\quad}$

$p(1-q)+q(1-p)$

$$EX^2 = E(X(X-1)) + EX = \sum_{j=1}^{\infty} j(j-1)(1-m)^{j-1} \cdot m + \frac{1}{m}$$

$$= \sum_{j=0}^{\infty} (1-m)\frac{\partial^2}{\partial m^2}(1-m)^j \cdot m + \frac{1}{m} = (1-m)\cdot\frac{2}{m^3}\cdot m + \frac{1}{m}$$

$$\Rightarrow \mathrm{Var}X = EX^2 - (EX)^2 = \frac{2(1-m)}{m^2} + \frac{1}{m} - \frac{1}{m^2} = \frac{2-2m+m-1}{m^2} = \frac{1-m}{m^2}$$

$\Rightarrow$ 本题 $\mathrm{var}(X) = \dfrac{pq + (1-p)(1-q)}{[p(1-q)+q(1-p)]^2}$

(b) 所求概率 $P(H_1T_2 \mid \{H_1T_2, H_2T_1\}) = \dfrac{P(H_1T_2)}{P(H_1T_2 \cup H_2T_1)} = \dfrac{P(H_1T_2)}{P(H_1T_2)+P(H_2T_1)}$

$= \dfrac{p(1-q)}{p(1-q)+q(1-p)}$

3. 解 (ch 3.13)

$X \sim N(\mu, \sigma^2) = N(10, 10^2)$ 令 $Z = \dfrac{X-\mu}{\sigma}$. (Standardization)

X: in Celsius. Y: in Fahrenheit.

relation between X and Y: $X = \frac{5}{9}(Y-32)$

$\Rightarrow P(Y \le 59) = P(X \le 15) = P(Z \le \frac{1}{2}) = \Phi(0.5) = 0.692$

$\therefore P(Y \le 59) = 0.692$

4. 解:

① $X \sim \chi_n^2$. proof of $EX = n$, $\mathrm{Var}(X) = 2n$.

由 χ_n^2 分布的定义 $\Rightarrow X = Z_1^2 + Z_2^2 + \cdots + Z_n^2$. $Z_1 \cdots Z_n$ 为 i.i.d, $\sim N(0,1)$

$\Rightarrow$ 则 $EX = E(Z_1^2 + Z_2^2 + \cdots + Z_n^2) = nEZ_1^2$

$Z_1 \sim N(0,1)$ 即有 $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$，$\int_{-\infty}^{\infty} z^2 f_Z(z) dz = 1$

$\Rightarrow EX = n \times 1 = n$.

$Var(X) = Var(Z_1^2 + Z_2^2 + \cdots + Z_n^2)$，由 $Z_1 \cdots Z_n$ 独立，$Z_1^2 \cdots Z_n^2$ 也 independent

$\therefore Var(X) = \sum_{i=1}^{n} Var(Z_1^2) = n \cdot var(Z_1^2)$

$var(Z_1^2) = E(Z_1^4) - (EZ_1^2)^2 = \int_{-\infty}^{\infty} z^4 f_Z(z) dz - 1 = 2$，$Var(X) = 2n$

（利用了 $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z^4 e^{-z^2/2} dz = 3$，由 Gaussian 积分得到）

thus, $EX = n$, $Var(X) = 2n$.

② $T \sim t_n$.

$Z \sim N(0,1)$, $X \sim \chi_n^2$, $X \perp Z$. $T = \frac{Z}{\sqrt{\frac{X}{n}}}$, $T \sim t_n$.

由 $X \perp Z \Rightarrow g(X) \perp Z$ ∴ 有如下的式子成立

$$ET = E\left(\frac{Z}{\sqrt{\frac{X}{n}}}\right) = EZ \cdot E\left(\frac{1}{\sqrt{\frac{X}{n}}}\right)$$

由 $EZ = 0$，且 $n \geq 2$ 时，$E\left(\frac{1}{\sqrt{\frac{X}{n}}}\right)$ 存在. ∴ $ET = 0$. $(n \geq 2)$.

$var(T) = ET^2 - (ET)^2$ $\qquad (EZ^2 = 1)$

$$= E\left(\frac{Z^2}{\frac{X}{n}}\right) = nE\left(\frac{Z^2}{X}\right) = nEZ^2 E\left(\frac{1}{X}\right) = nE\left(\frac{1}{X}\right)$$

只要证明 $E\left(\frac{1}{X}\right) = \frac{1}{n-2}$ 即可，由 $X \sim \chi_n^2$，有 $f_X = \frac{(\frac{1}{2})^{\frac{n}{2}} \cdot x^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})} e^{-\frac{x}{2}}$

$$\Rightarrow E\left(\frac{1}{X}\right) = \int_0^{\infty} x f(x) dx$$

$$= \int_0^{\infty} \frac{1}{x} \frac{(\frac{1}{2})^{\frac{n}{2}} \cdot x^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})} e^{-\frac{x}{2}} dx$$

$$= \frac{(\frac{1}{2})^{n}}{\Gamma(\frac{n}{2})} \int_0^{\infty} x^{\frac{n}{2}-2} e^{-\frac{x}{2}} dx$$

$(n>2)$

$$= \frac{(\frac{1}{2})^{n}}{\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{n}{2}-1)}{(\frac{1}{2})^{\frac{n}{2}-1}} = \frac{1}{n-2} \quad (n \geq 3)$$

$\therefore var(T) = \frac{n}{n-2}$ 证毕！

$(n \geq 3)$

③ $F = \frac{X_1/m}{X_2/n} \sim F(m,n)$. $X_1 \sim \chi^2_m$, $X_2 \sim \chi^2_n$

$$EF = E\left(\frac{X_1/m}{X_2/n}\right) = \frac{n}{m} E\left(\frac{X_1}{X_2}\right) = \frac{n}{m} E(X_1) E\left(\frac{1}{X_2}\right)$$

利用 ①问结论有 $E(X_1) = m$，②问结论有 $E\left(\frac{1}{X_2}\right) = \frac{1}{n-2}$

$\therefore EF = \frac{n}{n-2}$ 证毕！ $(n \geq 3)$

补充：求 F 分布的方差.

$$\mathrm{Var} F = EF^2 - (EF)^2 = EF^2 - \frac{n^2}{(n-2)^2}.$$

而由 $EF^2 = \frac{n^2}{m^2} EX_1^2 E\left(\frac{1}{X_2^2}\right)$

EX_1^2： $X_1^2 \sim \chi^2_m$，

$$EX_1^2 = \int_0^\infty x^2 \cdot \frac{(\frac{1}{2})^{m/2}}{\Gamma(\frac{m}{2})} x^{\frac{m}{2}-1} e^{-\frac{x}{2}} dx = \frac{(\frac{1}{2})^{m/2}}{\Gamma(\frac{m}{2})} \int_0^\infty x^{\frac{m}{2}+1} e^{-\frac{x}{2}} dx$$

$$= \frac{(\frac{1}{2})^{m/2}}{\Gamma(m/2)} \times \frac{\Gamma(\frac{m}{2}+1)}{(\frac{1}{2})^{\frac{m}{2}+2}} = m(m+1)$$

$E\frac{1}{X_2^2}$： $X_2^2 \sim \chi^2_n$

$$E\frac{1}{X_2^2} = \int_0^\infty \frac{1}{x^2} \cdot \frac{(\frac{1}{2})^{n/2}}{\Gamma(n/2)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} dx = \frac{(\frac{1}{2})^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \times \frac{\Gamma(\frac{n}{2}-2)}{(\frac{1}{2})^{\frac{n}{2}-2}} = \frac{1}{(n-2)(n-4)}$$

$$\therefore EF^2 = EF^2 - (EF)^2 = \frac{2n^2(m+n-2)}{m(n-2)(n-4)}$$

5. 解:

Chebyshev's inequation: $P(|X-EX|\geqslant a)\leq \dfrac{Var(X)}{a^2}$

设 X 表示正面朝上的次数. $X\sim B(N,p)$. 有 $\mu=EX=Np=\frac{1}{2}N$

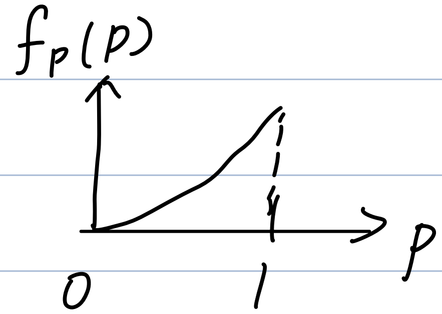
$\sigma^2=EX^2=Np(1-p)=\frac{1}{4}N$

$$\therefore\ P(|X-\tfrac{1}{2}N|\geqslant \tfrac{1}{4}N|\leq \frac{\frac{1}{4}N}{(\frac{1}{16}N)^2}=\frac{4}{N}$$

我得到了这个事件发生概率的一个上界, $\frac{4}{N}$.

6. 解: (选做)

7. 解: (Ch 3.34)



(a). $f_p(p)=\begin{cases} pe^p & p\in[0,1] \\ 0 & \text{otherwise}\end{cases}$

$$E(p)=\int_0^1 p f_p(p)\,dp=\int_0^1 p^2e^p\,dp=\int_0^1 p^2\,de^p=e-2\int_0^1 pe^p\,dp$$

而 $\int_0^1 pe^p\,dp=1$ $\therefore$ $E(p)=e-2$

$\therefore$ probability of $\{H\}=e-2$

(b). 求 $f_{P|H}(p|H)=\dfrac{f(p,H)}{P(H)}=\dfrac{P(H|p)\cdot f_p(p)}{E(p)}=\dfrac{p\cdot f_p(p)}{e-2}$

$\therefore$ $f_{p|H}=\begin{cases} \dfrac{p^2e^p}{e-2} & p\in[0,1] \\ 0 & \text{otherwise}\end{cases}$

(c) 求 $P(H_2|H_1)$

$$P(H_2|H_1) = \frac{P(H_1\cap H_2)}{P(H_1)} = \frac{E_p(P(H_1\cap H_2))}{E(p)} = \frac{\int_0^1 p^2 f_p(p)dp}{e-2}$$

$$\int_0^1 p^2 f_p(p)dp = \int_0^1 p^3 e^p dp = 6-2e$$

$$\therefore P(H_2|H_1) = \frac{6-2e}{e-2} \simeq 0.784 > P(H_1)$$

大于号说明我们在得知H_1的信息后，更新了对于p的认知。

8. 解:

$$(a-1) \xleftarrow{\frac{1}{2}} (a) \xrightarrow{\frac{1}{2}} (a+1)$$ 状态转移

停止时一共只有了X_a局，本金为a，求EX_a.

由全期望定理:

$$EX_a = \frac{1}{2}\times(EX_{a+1}+1)+\frac{1}{2}(EX_{a-1}+1)$$

$\therefore$ 得到递推关系 $EX_{a+1}-2EX_a+EX_{a-1}+2=0$ (*)

边界条件: $EX_0=0$, $EX_{a+b}=0$.

由(*)式有 $\underbrace{EX_{a+1}-EX_a}_{S_{a+1}} = \underbrace{EX_a-EX_{a-1}}_{S_a}-2$

$$\Rightarrow S_{a+1}-S_a=-2 \Rightarrow S_a=-2\cdot a+m$$

待定系数

$$\therefore \begin{cases} EX_a-EX_{a-1}=-2a+n & ① \\ EX_{a-1}-EX_{a-2}=-2(a-1)+n & ② \\ \vdots & \vdots \\ EX_1-EX_0=-2\cdot 1+n & ⓐ \end{cases}$$

Σ :　$EX_a = -2 \cdot \frac{a+1}{2} a + an \Rightarrow$

而由 $EX_{a+b} = -2 \cdot \frac{a+b+1}{2}(a+b) + n(a+b) = 0$

$\therefore n = a+b+1$

$\therefore EX_a = -(a+1)a + a(a+b+1) = ab$!