

HW 4.

T1. 解:

(a). Celtic 赢 X 局 $P(X=k)$ 可视为二项分布. Celtic 赢得比赛的前提是 $X \geq \frac{n+1}{2}$

由独立性可知. $P_n(\text{win}) = \sum_{k=\frac{n+1}{2}}^n P(X=k) = \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} p^k (1-p)^{n-k}$

特别地, 对于 $k=3$, 有 $P_3(\text{win}) = P(X=2) + P(X=3)$

$$P_3(\text{win}) = \binom{3}{2} p^2 (1-p) + \binom{3}{3} p^3 = 3p^2(1-p) + p^3 = 3p^2 - 2p^3$$

$$P_5(\text{win}) = \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + \binom{5}{5} p^5 = 6p^5 - 15p^4 + 10p^3$$

$$\Rightarrow P_5(\text{win}) > P_3(\text{win}) \Rightarrow 3p^2 - 2p^3 < 6p^5 - 15p^4 + 10p^3 \Rightarrow$$

$$\text{solve: } 2p^3 - 5p^2 + 4p - 1 > 0 \Rightarrow (p - \frac{1}{2})(p-1)^2 > 0 \Rightarrow p > \frac{1}{2}$$

因此可知, 当 $p > \frac{1}{2}$ 时, 打 5 局比 3 局是更有利的.

(b). for general case: 设 i 为 Celtic 队在比赛中在前 $2k-1$ 中赢的局数

不妨作一些代数上的变形: $P_{2k+1}(\text{win}) = P_{2k-1}(i \geq k+1) + P_{2k-1}(i=k) \cdot (1-(1-p)^2)$
 $+ P_{2k-1}(i=k-1) \cdot p^2$ (动态规划的思想)

$$P_{2k-1}(\text{win}) = P_{2k-1}(i \geq k) = P_{2k-1}(i=k) + P_{2k-1}(i \geq k+1)$$

$$\therefore P_{2k+1}(\text{win}) - P_{2k-1}(\text{win}) = P_{2k-1}(i=k-1) \cdot p^2 - P_{2k-1}(i=k) \cdot (1-p)^2$$

$$= \binom{2k-1}{k-1} p^{k-1} (1-p)^k \cdot p^2 - \binom{2k-1}{k} p^k \cdot (1-p)^{k-1} (1-p)^2$$

$$= \frac{(2k-1)!}{k!(k-1)!} [p^{k+1} (1-p)^k - p^k (1-p)^{k+1}]$$

$$= \frac{(2k-1)!}{k!(k-1)!} p^k (1-p)^k (2p-1) > 0 \Rightarrow p > \frac{1}{2}$$

\therefore generally if $p > \frac{1}{2}$, namely the prob C-team wins is higher, the more games played, the more advantage!

T2. 解: key-door randomly match problem.

(a). Case (1). 具有记忆性的来试 key-door pair

PMF of X : $X = x \in \{1, 2, 3, 4, 5, 6\}$ 设事件 M_i 表示第 i 把 key 打开了门容易得到. $P(X=1) = P(M_1) = \frac{1}{5}$

$$P(X=2) = P(M_1^c)P(M_2|M_1^c) = \frac{4}{5} \times \frac{1}{4} = \frac{1}{5}$$

$$\begin{aligned} P(X=3) &= P(M_1^c \cap M_2^c)P(M_3|M_1^c \cap M_2^c) \\ &= P(M_1^c)P(M_2^c|M_1^c)P(M_3|M_1^c \cap M_2^c) \\ &= \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{5} \end{aligned}$$

$$\text{同理 } P(X=4) = P(X=5) = \frac{1}{5} \therefore P(X=i) = \frac{1}{5}, i=1, 2, 3, 4, 5$$

Case (2) 无记忆性的来试 key-door pair,

$$P(X=k) = \left(\frac{1}{5}\right) \cdot \left(\frac{4}{5}\right)^{k-1}, k=1, 2, 3, 4, \dots$$

满足几何分布.

(b) 每个 door 增加了 3 把 key:

在 Case (1) 有记忆性的.

$$P(X=1) = P(M_1) = \frac{2}{10}$$

$$P(X=2) = P(M_1^c)P(M_2|M_1^c) = \frac{8}{10} \times \frac{2}{9} = \frac{8}{45}$$

$$P(X=3) = P(M_1^c)P(M_2^c|M_1^c)P(M_3|M_1^c \cap M_2^c) = \frac{8}{10} \times \frac{7}{9} \times \frac{2}{8} = \frac{7}{45}$$

$$P(X=4) = \frac{8}{10} \times \frac{7}{9} \times \frac{6}{8} \times \frac{2}{7} = \frac{12}{90}$$

\vdots

$$P(X=k) = \frac{2 \cdot (10-k)}{90}, \quad k=1, 2, 3, \dots, 10 \text{ 为 PMF.}$$

case (2) 无记号乙生. 很明显, $P(X=k) = \left(\frac{1}{5}\right) \cdot \left(\frac{4}{5}\right)^{k-1}, (k=1, 2, 3, \dots)$

T3. 解: Form of Poisson PMF. $X \sim P(\lambda)$
由于 X satisfy Poisson distribution. $P_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$

proof of: $k \in N \subset [0, L\lambda], P_X(k)$ 单调 \uparrow

$k \in N \subset [L\lambda, +\infty), P_X(k)$ 单调 \downarrow

$$\frac{P_X(k)}{P_X(k-1)} = \frac{\lambda^k e^{-\lambda}}{k!} \times \frac{(k-1)!}{\lambda^{k-1} e^{-\lambda}} = \frac{\lambda}{k}$$

当 $k \leq \lambda$ 时, 有 $\frac{P_X(k)}{P_X(k-1)} \geq 1$. $P_X(k)$ 单调 \uparrow

当 $k > \lambda$ 时, $P_X(k) < P_X(k-1)$, $P_X(k)$ 单调 \downarrow $L\lambda \leq \lambda$.

又 $k \in N \Rightarrow$ 不对称性在 $[0, L\lambda] \uparrow, [L\lambda, +\infty) \downarrow$. 证毕!

T4. 解: $P_X = P(X=k)$ satisfy $\frac{P_n}{P_{n-1}} = \frac{\lambda}{n}, \forall n \in N, n \geq 1$.

proof of $X \sim P(\lambda)$

pro: 由递推式 $P_n/P_{n-1} = \frac{\lambda}{n}$ 有 $P_i = P_0 \frac{\lambda^i}{i!}$

利用概率归一性: $\sum_{i=0}^{\infty} P_i = \sum_{i=0}^{\infty} P_0 \frac{\lambda^i}{i!} = P_0 \cdot \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$

利用 $e^{\lambda x} = 1 + \lambda x + \frac{(\lambda x)^2}{2!} + \dots + \frac{(\lambda x)^n}{n!} = \sum_{i=0}^{\infty} \frac{(\lambda x)^i}{i!}$

取 $x=1 \Rightarrow P_0 e^{\lambda} = 1 \Rightarrow P_0 = e^{-\lambda}$

$\therefore P_i = \frac{\lambda^i}{i!} e^{-\lambda}$ 即 $P_k(\lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$ 即 $X \sim P(\lambda)$ 证毕!

T5-解: $\xrightarrow[0]{1} \rightarrow$ 单自由度的随机游荡问题.

(1) 求 $P(X=k) = ?$

由题 $P(k|k-1) = p, P(k|k+1) = 1-p$.

经过 n 步后为 X 的取值可为: $n, n-2, \dots, -n+2, -n$

$$P(X=k) = \binom{n}{\frac{n+k}{2}} p^{\frac{k+n}{2}} (1-p)^{\frac{n-k}{2}}$$

(2) 距离 $D \geq 0, D=k \in \{n, n-2, \dots\}$ 且 $k \geq 0$.

$$\begin{aligned} P(D=k) &= P(X=k) + P(X=-k) = \binom{n}{\frac{n-k}{2}} p^{\frac{k+n}{2}} (1-p)^{\frac{n-k}{2}} + \binom{n}{\frac{n+k}{2}} p^{\frac{n-k}{2}} (1-p)^{\frac{n+k}{2}} \\ &= \binom{n}{\frac{n+k}{2}} p^{\frac{n-k}{2}} (1-p)^{\frac{n-k}{2}} [p^k + (1-p)^k] \end{aligned}$$

$$= \binom{n}{\frac{n+k}{2}} (p-p^2)^{\frac{n-k}{2}} (p^k + (1-p)^k)$$

(3) 不妨设米老鼠移动 $m+n$ 步后的位置为随机变量 X .

则有 X 的分布 $P(Z=j) = \binom{m+n}{\frac{m+n-j}{2}} p^{\frac{m+n-j}{2}} (1-p)^{\frac{m+n+j}{2}}$, 取 $p = \frac{1}{2}$

$P(X=j) = \binom{m+n}{\frac{m+n-j}{2}} \cdot \frac{1}{2^{m+n}}$ 则由题意所求为 $P(Y=k|X=j)$

$$P(Y=k|X=j) = \frac{P(X=j|Y=k) P(Y=k)}{P(X=j)}, \text{ 因为容易求出 } P(Y=k) = \binom{m}{\frac{m-k}{2}} \cdot \frac{1}{2^m}$$

$$\text{且 } P(X=j|Y=k) = \binom{n}{\frac{n-(j-k)}{2}} p^{\frac{n+j-k}{2}} (1-p)^{\frac{n-j+k}{2}} = \left(\frac{1}{2}\right)^n \binom{n}{\frac{n-j+k}{2}}$$

$$\therefore P(Y=k|X=j) = \frac{\binom{n-j+k}{2} \cdot \binom{m-k}{2}}{\binom{m+n}{2}} \checkmark$$

$$\binom{m+n}{m+n-i}$$

T6. 解:

由于每个手机的传输与否是相互独立的, 故任意时刻传输的总个数 $X(n)$

满足的分布为 $P(X=k) = \binom{N}{k} p^k (1-p)^{N-k}$

当且仅当 $X=1$ 时传输可以完成, \therefore Select p . $\max P(X=1)$

$$\Rightarrow \max P(X=1) = N p (1-p)^{N-1}$$

$$\text{设 } f(p) = N p (1-p)^{N-1}, \quad \frac{\partial f}{\partial p} = N [(1-p)^{N-1} - (N-1)p(1-p)^{N-2}] = 0$$

$$\Rightarrow 1-p - (N-1)p = 0 \Rightarrow p = \frac{1}{N}. \text{ 且 } p=0 \text{ 时 } P(X=1)=0$$

\therefore 当 $p = \frac{1}{N}$ 时, $P(X=1) = N p (1-p)^{N-1}$ 取 max 值

T7. 解: $X \sim N(\mu, \sigma^2)$, $y^2 + 4y + X = 0$ 无实根的概率为 0.5 求 μ .

equation $y^2 + 4y + X = 0$, $\Delta = 16 - 4X < 0 \Rightarrow X > 4$.

$$P(X > 4) = 0.5, \text{ 由正态分布的对称性: } \mu = 4.$$

T8. 解: $T \sim E(\lambda)$. proof of "失效率" 为常数.

pro: 即求 $\lim_{\delta \downarrow 0} \frac{P(T \in (t, t+\delta] | T > t)}{\delta} = ?$

$$\text{利用 } T \sim E(\lambda), \quad f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\text{有 } \lim_{\delta \downarrow 0} P(T \in (t, t+\delta] | T > t)$$

$$= \lim_{\delta \downarrow 0} \frac{P(T \in (t, t+\delta])}{\delta}$$

$$= \lim_{\delta \downarrow 0} \frac{\int_{t=t}^{t+\delta} \lambda e^{-\lambda T} dT}{\delta \cdot \int_t^{\infty} \lambda e^{-\lambda T} dT} = \lim_{\delta \downarrow 0} \frac{-e^{-\lambda T} \Big|_t^{t+\delta}}{\delta \cdot (-e^{-\lambda T}) \Big|_t^{\infty}}$$

$$= \lim_{\delta \downarrow 0} \frac{e^{-\lambda t} (1 - e^{-\lambda \delta})}{\delta \cdot e^{-\lambda t}}, \text{ 利用 } e^{-x} \rightarrow 1 - x, x \ll 1$$

$$= \lim_{\delta \downarrow 0} \frac{\lambda \delta}{\delta} = \lambda. \quad \text{证毕!}$$

从而证明了 $T \sim \mathcal{E}(\lambda)$ 下, 其失效率为常数且恰为分布参数.

T9 解:

proof of: $X \sim \mathcal{E}(\lambda), Y = \lambda X \sim \mathcal{E}(1)$

$$\text{当 } X \sim \mathcal{E}(\lambda), \text{ 有 } f(x) = \begin{cases} \lambda e^{-\lambda x} & (x \geq 0) \\ 0 & (x < 0) \end{cases} \quad \int_0^{\infty} \lambda e^{-\lambda x} dx = \int_0^{\infty} -e^{-\lambda x} d(-\lambda x) = 1$$

$$\text{不妨计算 CDF: } F(x) = \begin{cases} 1 - e^{-\lambda x} & (x \geq 0) \\ 0 & (x < 0) \end{cases}$$

$$\Rightarrow Y = \lambda X \Rightarrow F(y) = \begin{cases} 1 - e^{-y} & (y \geq 0) \\ 0 & (y < 0) \end{cases}$$

$$\text{Y 的 PDF: } f(y) = F'(y) = 1 \cdot e^{-y} \quad (y \geq 0)$$

$$f(y) = 0, \quad (y < 0)$$

$$\therefore f(y) = \begin{cases} 1 \cdot e^{-y} & (y \geq 0) \\ 0 & (y < 0) \end{cases} \quad \therefore Y \sim \mathcal{E}(1)$$

1 0 (y < 0)

T10. 解:

$$u_1(t) = \begin{cases} \frac{1}{2} & (0 \leq t \leq 2) \\ 0 & (t < 0) \end{cases}$$

$$u_2(t) = \begin{cases} \lambda e^{-\lambda t} & (t \geq 0) \quad (\lambda = 0.1) \\ 0 & (t < 0) \end{cases}$$

$$\text{即求 } P(t \geq 4 | t \geq 1) = \frac{P(t \geq 4)}{P(t \geq 1)} = \frac{P(t \geq 4 | \text{case 2}) \cdot P(\text{case 2})}{P(t \geq 1 | \text{case 1}) P(\text{case 1}) + P(t \geq 1 | \text{case 2}) P(\text{case 2})}$$

$$= \frac{\frac{1}{2} \times \int_4^{\infty} 0.1 e^{-0.1t} dt}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \int_1^{\infty} 0.1 e^{-0.1t} dt}$$

$$= \frac{\frac{1}{2} e^{-0.4}}{\frac{1}{4} + \frac{1}{2} e^{-0.1}} = 0.477$$

∴ 待求的“已经用了一年,还能用4年的概率”为 0.477