T1. Problem 2.41.

解：(a) marginal PMF: $p_Y(y) = \binom{4}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{4-y}, y=0,1,2,\cdots,4$

要求的是 joint PMF: $p_{X,Y}(x,y) = p_Y(y)\, p_{X|Y}(x|y)$

而 $p_{X|Y}(x|y) = \binom{4-y}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{4-y-x}$

则有 $p_{X,Y}(x,y) = \binom{4}{y}\binom{4-y}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{4-y-x}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{4-y}$ (*)

(*)式成立在 $0\le x\le 4$, $0\le y\le 4$, 且 $0\le x+y\le 4$ 的条件下.

若不满足则 $p_{X,Y}(x,y)=0$.

(b). Conditional PMF:

$p_{X|Y}(x|y=2) = \binom{4-2}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{4-2-x} = \binom{2}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{2-x}$

T2.

解：设两次的断点坐标为 x, y, 对应随机变量 X、Y. 那由题可知

$X \sim U(0,1)$, given X, $Y \sim U(0,x)$,

要求构成三角形, 因此 $1-x<\frac{1}{2}$, $y<\frac{1}{2}$, $x-y<\frac{1}{2}$

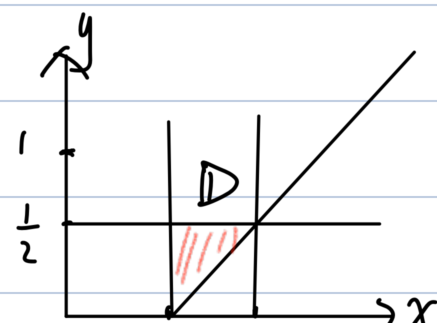
即由 $X\sim U(0,1)$ 知 PDF: $f_X(x) = I_{(0,1)}$, $Y|X \sim U(0,x)$ 知 PDF;

$f_{Y|X}(y|x) = I_{(0,x)}\frac{1}{x}$

joint PDF: $f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x) = f_{X,Y}(x,y) = I_{(0,1)\times(0,x)} \cdot \frac{1}{x}$

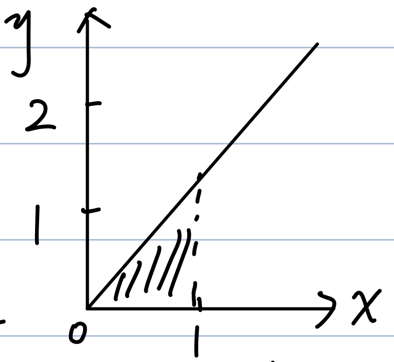
可行域: $\frac{1}{2}<x<1$, $x-\frac{1}{2}<y<\frac{1}{2}$,

$F_{X,Y}(x,y) = \iint_{\in D} I_{(0,x)}\frac{1}{x}\,dx\,dy = \iint_{\left\{\substack{\frac{1}{2}<x<1 \\ \frac{1}{2}-x<y<\frac{1}{2}}\right\}} \frac{1}{x}\,dx\,dy$



$$=\int_{\frac12<x<1} dx \int_{x-\frac12<y<\frac12} \frac1x\, dy = \int_{\frac12<x<1}\frac{1-x}{x}\,dx = \ln2-\frac12$$

0 $\frac12$ 1



13. 解: $f_X(x)=\frac67(2x^2+x)$, $0<x<1$.

$\forall\ 0<x<1$, 有 $f_{Y|X}(y|x)=\frac{2x+y}{4x+2}$, $0<y<2$

$$f_{X,Y}(x,y)=f_{Y|X}(y|x)f_X(x)=\frac{6x(2x+1)}{7}\times\frac{2x+y}{2(2x+1)}=\frac{3x(2x+y)}{7}$$

(1) $(x,y)\in(0,1)\times(0,2)$

$$P(Y<1|X=1)=\frac{\int_0^1 f_{Y|X}(y|x)\,dy}{f_X(1)}=\frac{\int_0^1\frac{2+y}{6}dy}{\frac67(2+1)}=\frac{5/12}{18/7}=\frac{35}{216}$$

(2) $$f_Y(y)=\int_0^1 f_{X,Y}(x,y)\,dx=\int_0^1\frac{3x(2x+y)}{7}dx=\frac{3y+4}{14}$$

(3) $$P(X>Y)=\int_{x>y} f_{X,Y}(x,y)\,dxdy=\int_{0<y<1}dy\int_{y<x<1}\frac{3x(2x+y)}{7}dx$$

$$=\int_0^1\frac{2+\frac32y-2y^3-\frac32y^3}{7}dy=\frac{2y+\frac34y^2-\frac12y^4-\frac38y^4}{7}\Big|_0^1=\frac{15}{56}$$

(4) $$P(Y>\tfrac12|X<\tfrac12)=\frac{P(Y>\frac12,X<\frac12)}{P(X<\frac12)}=\frac{\int_{\frac12}^2dy\int_0^{\frac12}f_{X,Y}(x,y)\,dx}{\int_0^{\frac12}f_X(x)\,dx}$$

代入 $$\int_{\frac12}^2 dy\int_0^{\frac12}\frac{3x(2x+y)}{7}dx=\int_{\frac12}^2 dy\cdot\frac{2+3y}{56}=\frac{69}{8\times56}$$

$$\int_0^{\frac12}f_X(x)\,dx=\int_0^{\frac12}\frac67(2x^2+x)\,dx=\frac67\left(\frac23x^3+\frac12x^2\right)\Big|_0^{\frac12}$$

$$=\frac67\left(\frac1{12}+\frac18\right)=\frac5{28}$$

则有 $P(Y>\frac{1}{2}|X<\frac{1}{2}) = \dfrac{69/8\times 56}{5/28} = \dfrac{69}{80}$

T4. 解:

(1) $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$，当 $f_X(x)\neq 0$ 时.

$$f_X(x) = \int_0^x f(x,y)\,dy = \begin{cases} 20(1-x)x^3 & (0<x<1) \\ 0 & (\text{otherwise}) \end{cases}$$

在 $0<x<1$ 上可定义 $f_{Y|X}(y|x) = \dfrac{6y(x-y)}{x^3}$ $(0<x<1,\ 0<y<x)$

其它区域没有定义. 或直接定义 $f_{Y|X}(y|x)=0$. otherwise

(2) 由 $T=\dfrac{Y}{\sqrt{X}}$，则有 $\dfrac{dy}{dt}=\sqrt{x}$，则 $f_{T|X}(t|x) = f_{Y|X}(y|x)\cdot\left|\dfrac{dy}{dt}\right|$

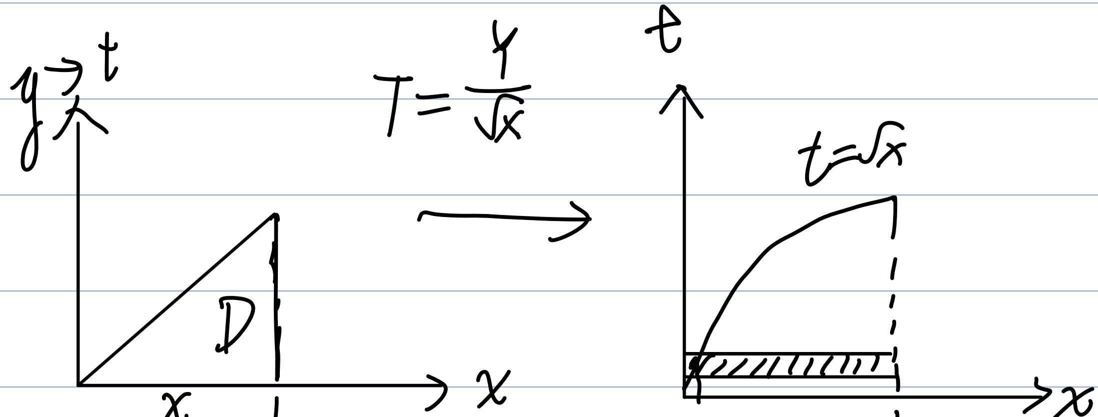
$$= \sqrt{x}\, f_{Y|X}(y|x) = \frac{6t(x-\sqrt{x}t)}{x^2} \quad (0<x<1,\ 0<t<\sqrt{x})$$

类似地也可以定义 $f_{T|X}(t|x)=0$ (otherwise)

(3) $f_T(t) = \displaystyle\int_{x \text{ in } D?} f_{X,T}(x,t)\,dx = \int_x f_{T|X}(t|x) f_X(x)\,dx$

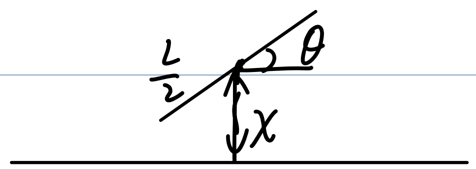
$$= \int_{t^2}^{1} 20(1-x)x^3\cdot\frac{6t(x-\sqrt{x}t)}{x^2}\,dx \quad (0<t<1)$$

积分结果即为 $f_T(t)$，T的概率密度函数



T5. Buffon 投针.

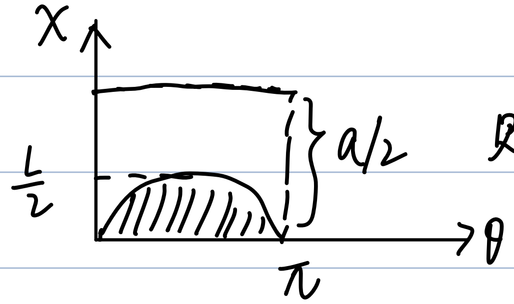
解: 取变量 x, θ 来研究问题.



x 定义为针的中心到最近的平行线的距离

θ 定义为右侧的夹角

$\therefore$ $x\in[0,\frac{a}{2}]$, $\theta\in[0,\pi]$, 与平行线相交 $\Rightarrow$ $\frac{L}{2}\sin\theta > x$



几何概型.

则 $P(\frac{L}{2}\sin\theta > x) = \frac{S(阴)}{S(全)} = \frac{\int_0^{\pi}\frac{L}{2}\sin\theta\, d\theta}{a\pi/2} = \frac{2L}{a\pi}$

$\therefore$ 针与平行线相交的概率即为 $\frac{2L}{a\pi}$.

T6. Independency. 解:

X 与 Y 独立的定义是 $\forall x,y$, $P(X\le x, Y\le y) = P(X\le x)P(Y\le y)$

对于圆盘内均匀分布的点, (X,Y) 的联合分布为 $f(x,y) = \frac{1}{\pi r^2} I_D$

D: $\{(x,y) \mid x^2+y^2\le r^2\}$. 由联合分布 $f_{X,Y}(x,y) = \frac{1}{\pi r^2} I_D$ 求 X 的分布

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{\pi r^2}\int_{-\infty}^{\infty} I_{x^2+y^2\le r^2}\,dy = \frac{1}{\pi r^2}\int_{-\infty}^{\infty} I_{|y|\le\sqrt{r^2-x^2}}\,dy$$

$$= \frac{2\sqrt{r^2-x^2}}{\pi r^2}, \quad |x|\le r$$

同理有 $f_Y(y) = \frac{2\sqrt{r^2-y^2}}{\pi r^2}$

则有由 X, Y 独立的充要条件是 (X,Y) 有联合 PDF $f_X(x)f_Y(y)$

而 $f_X(x)f_Y(y) = \frac{4\sqrt{r^2-x^2}\sqrt{r^2-y^2}}{\pi^2 r^4} \neq f_{X,Y}(x,y) = \frac{1}{\pi}$.

$\therefore$ X, Y 不独立