EE5137 Stochastic Processes: Problem Set 3
Assigned: 29/01/21, Due: Never

All problems here are optional. Please use these practice problems to prepare for the quiz.

1. Exercise 1.28 (Gallager's book)
2. Exercise 1.32 (Gallager's book)
3. Exercise 1.43 (Gallager's book)
4. Exercise 1.44 (Gallager's book)
5. Exercise 1.48 (Gallager's book)
6. [Convergence of RVs] Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(1/2). Define $Y_n = 2^n \prod_{i=1}^n X_i$. Does Y_n converge to 0 almost surely (with probability 1)? Does Y_n converge to 0 in mean square?
7. [Two Independent Random Variables]

   Let X and Y be independent random variables, uniformly distributed on $[0, 2]$.

   (a) Find the mean and variance of XY.

   (b) Calculate the probability $\Pr(XY \le 1)$.

   You may use, without proof, the fact that for a uniform random variable on $[a, b]$, the variance is $(b-a)^2/12$.
8. [Laws of Large Numbers]

   (a) State Chebychev's Inequality.

   (b) Suppose $\{X_i\}_{i=1}^{\infty}$ is a sequence of uncorrelated random variables (i.e., $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j]$ for $i \neq j$), each of which has finite mean, and assume that for all n, $\mathrm{Var}(X_n) \le M < \infty$. Define

   $$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[X_i].$$

   and suppose that $\mu_n \to \mu$ as $n \to \infty$ and $|\mu| < \infty$. Let $S_n = \sum_{i=1}^n X_i$. Show that $S_n/n \to \mu$ in probability.
9. [Convergence in Probability Implies Convergence in Distribution]

   Show if $\{X_n\}$ convergence in distribution to X, then $\{X_n\}$ converges in distribution to the same rv X.

   Hint: First convince yourself that

   $$F_{X_n}(x) \le F_X(x+\epsilon) + \Pr(|X_n - X| > \epsilon)$$

   Consider the "flipped" inequality and use the definition of convergence in probability.

10. [Probability Generating Function]

If X is a non-negative integer-valued rv then the function $Q(z)$ defined for $|z| \leq 1$ by

$$Q(z) = \mathbb{E}[z^X] = \sum_{j=0}^{\infty} z^j \Pr(X = j)$$

is called the probability generating function of X.

In this problem you may interchange the differentiation operation and the infinite sum operation. The reason we can do this (for the more mathematically inclined students) is due to the so-called differentiable limit theorem and Abel's theorem.

(a) Show that

$$\left.\frac{d^k}{dz^k} Q(z)\right|_{z=0} = k! \Pr(X = k).$$

(b) With 0 considered even, show that

$$\Pr(X \text{ is even}) = \frac{Q(-1) + Q(1)}{2}.$$

(c) If X is binomial with parameters n and p, show that

$$\Pr(X \text{ is even}) = \frac{1 + (1 - 2p)^n}{2}.$$

(d) If X is Poisson with mean λ, show that

$$\Pr(X \text{ is even}) = \frac{1 + e^{-2\lambda}}{2}.$$