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EE5139/EE6139: Information Theory and its Applications

(Semester I, 2021–2022)

Disclaimer: These notes are not yet free of typos or always presented in the most clear way. Any comments that help reduce these deficiencies are very much appreciated. Parts of Chapters 0 and Chapter 7 are based on notes by Vincent Y. F. Tan. Most figures were contributed by Michael X. Cao.

Contents

$\emptyset$	empty set $\{\}$
$[M]$	the set $\{1, 2, \ldots, M\}$
$\mathcal{P}(\mathcal{X})$	the power set of $\mathcal{X}$, i.e. $\{A : A \subseteq \mathcal{X}\}$
$\mathcal{X} \times \mathcal{Y}$	the set of tuples $\{(x, y) : x \in \mathcal{X}, y \in \mathcal{Y}\}$
$\mathcal{X}^n$	the set of n-tuples with each element taking values in $\mathcal{X}$, e.g., $\mathcal{X}^2 = \mathcal{X} \times \mathcal{X}$
$\{0,1\}^n$	the set of n-bit strings
$\{0,1\}^*$	the set of bit strings of arbitrary length
$\max \mathcal{X}$	largest $x^* \in \mathcal{X}$, might not always exist
$\sup \mathcal{X}$	smallest $x^* \in \mathbb{R}$ such that $x \leq x^*$ for all $x \in \mathcal{X}$;
	equals the maximum, $\max \mathcal{X}$, if it exists
$\min \mathcal{X}$	smallest $x^* \in \mathcal{X}$, might not always exist
$\inf \mathcal{X}$	largest $x^* \in \mathbb{R}$ such that $x \geq x^*$ for all $x \in \mathcal{X}$;
	equals the minimum, $\min \mathcal{X}$, if it exists
$\mathbf{1}\{x = y\}$	indicator function, evaluates to 1 if the condition is true and 0 otherwise,
	so that, for example, $\mathbf{1}\{x = y\} + \mathbf{1}\{x \neq y\} = 1$
δ_{xy}	shorthand for $\mathbf{1}\{x = y\}$
$P_X(x)$	probability mass function (pmf), $P_X(x) = P[X = x]$
$p_X(x)$	probability density function (pdf), i.e. $P[X \in (1,2)] = \int_1^2 p_X(x) \mathrm{d}x$
$P[X \in \mathcal{A}]$	probability of a random variable X being in some set $\mathcal{A}$,
	i.e. $P[X \in \mathcal{A}] = \mathbb{P}(\{\omega : X(\omega) \in \mathcal{A}\}) = \sum_{x \in \mathcal{A}} P_X(x)$
$P[5 \leq X < 6]$	another way of writing $P[X \in [5, 6)]$
$\log$	logarithm; in these notes we take the logarithm to base 2, i.e. $\log = \log_2$

Table 1: Some basic notation used in this module.

pmf	probability mass function
pdf	probability density function
cdf	cumulative density function
rv	random variable
DMS	discrete memoryless source
DMC	discrete memoryless channel

Table 2: Some abbreviations used in this module.

Chapter 0

Review of mathematical notation and foundations

[Week 1]

Intended learning outcomes:

- You are familiar with common notation used throughout the lecture.
- You are comfortable with the main mathematical concepts needed in this module, namely basic probability theory including random variables, conditional probabilities and Markov chains.
- You can apply basic bounds on tail probabilities, and can prove the weak law of large numbers.
- You can compute vector norms and apply the Cauchy-Schwarz inequality.
- You know what convex and concave functions are and can apply Jensen's inequality.
- You know what finite fields are and how to come up with the multiplication table for simple examples.

0.1 Notation

We will use standard notation and abbreviations that you should be familiar with from other modules. Some of the less frequently encountered mathematical expressions are summarised in Tables 1 and 2 on the previous page.

0.2 Probability theory

We will not directly need the framework of probability theory in its most abstract formulation as presented in the following, but it is good to know that both discrete and continuous random variables can be seen as emanating from a shared mathematical framework.

0.2.1 Probability space

A probability space is represented by a triple $(\Omega, \Sigma, \mathbb{P})$. Here Ω is a set that is called the *sample space*. Moreover, Σ is a σ-algebra, i.e. a collection of subsets of Ω, called events, with the following properties:

- $\Omega \in \Sigma$
- If $A \in \Sigma$, then its *complement*, $A^c = \Omega \setminus A$ is also in Σ, i.e. $A^c \in \Sigma$.
- If $A_1, A_2, \ldots, A_n, \ldots \in \Sigma$, then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$

Question 0.1. *Show that the above also implies that $\emptyset \in \Sigma$ and $\bigcap_{i=1}^{\infty} A_i \in \Sigma$.*

For example, let $\Omega = [0, 1]$, and we are interested in the probability of subsets of Ω that are intervals of the form $[a, b]$ where $0 \leq a < b \leq 1$, but not individual points in Ω. Then we should also be able to say something about the probability of the union, intersections, complement and so on of such intervals. This is captured by the definition of a σ-algebra. Think of Σ as the properties of Ω that can actually be observed.

Example 0.2. *If your random variable is the location an athlete lands after a long jump then it makes sense to take Ω to be positive real numbers, $\mathbb{R}_+$ indicating the distance jumped (say, in meters). However, even with arbitrarily good equipment we cannot actually measure a real number, we can only ever say that he landed in some interval, the size of which is given by our measurement precision. Thus, Σ, comprised of the events we can actually observe, is built up by including all (arbitrarily small) intervals in $\mathbb{R}_+$ and their unions and complements. Or another way of looking at this is that the probability of the jumper landing exactly at $9m$ is always zero — it is simply the wrong question to ask. But the probability of landing within $1cm$ or some arbitrarily small interval around $9m$ might very well be nonzero.*

Question 0.3. *For the advanced reader: Note however that in the above example $\{x\} \in \Sigma$ for any $x \in \mathbb{R}^+$, that is, single points are also elements of the σ-algebra. Can you see why? Use an infinite intersection to construct it.*

Finally, the probability measure $\mathbb{P}$ is a function $\mathbb{P} : \Sigma \to [0, 1]$ defined on the measurable space (Ω, Σ), and represents your "belief" about the events in Σ. In order for $\mathbb{P}$ to be called a probability measure, it must satisfy the following two properties:

1. $\mathbb{P}(\Omega) = 1$

2. For $A_1, A_2, \ldots$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, i.e. for mutually *disjoint* sets, we have

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i) \tag{1}$$

Some basic and very useful properties that can be derived from the above definition. The union bound in particular is very often used when analysing problems in information theory.

Proposition 0.4. *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. The following holds true:*

1. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
2. *If $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$*
3. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leq \mathbb{P}(A) + \mathbb{P}(B)$, *which is called the* union bound. *Clearly, by induction, the union bound works for finitely many sets $A_i, i = 1, \ldots, k$, namely*

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i). \tag{2}$$

Proof. Property 1 follows since $A^c \cap A = \emptyset$, and thus $\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(\Omega) = 1$ by (1). For Property 2, note that $B \setminus A = B \cap A^c \in \Sigma$ and since $A \cap (B \setminus A) = \emptyset$ we again argue that $\mathbb{P}(A) + \mathbb{P}(B \setminus A) = \mathbb{P}(B)$, from which the desired inequality follows.

For Property 3 note that $A \cup B$ can be decomposed in three different ways into mutually disjoint sets:

$$A \cup B = A \cup (B \setminus A) = B \cup (A \setminus B) = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)\,. \tag{3}$$

Again using (1) for each of these decompositions we have

$$2\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \setminus B) \tag{4}$$

$$= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(A \cup B) - \mathbb{P}(A \cap B)\,, \tag{5}$$

which implies the desired equality. □

Question 0.5. *Show that $0 \leq \mathbb{P}(A) \leq 1$ for every $A \in \Sigma$.*

Sometimes we have two conflicting beliefs, or models, about the underlying probability distribution, and so we will consider two compatible probability spaces $(\Omega, \Sigma, \mathbb{P})$ and $(\Omega, \Sigma, \mathbb{Q})$. They offer different predictions about the probability with which the events in Σ occur, and one fundamental task in statistics is to find out which model is the correct one from the frequency with which certain events occur. We will cover this later in the module.

0.2.2 Random variables

We will usually not deal directly with the probability space but with random variables. A *random variable* (rv) $X : \Omega \to \mathcal{X}$ is a function from the space (Ω, Σ) to a measurable space $(\mathcal{X}, \Sigma_X)$. In order for X to make any sense, the mapping has to ensure that $\{\omega \in \Omega : X(\omega) \in \mathcal{B}\} \in \Sigma$ for all $\mathcal{B} \in \Sigma_X$, because we are restricted to observing events in Σ and our random variable can thus not be more fine-grained than what Σ allows. Functions satisfying this property are called a *measurable function.* A random variable is then more formally defined as a measurable mapping from (Ω, Σ) to $(\mathcal{X}, \Sigma_X)$.

The only two examples of interest for us in the following are discrete and continuous random variables:

discrete rv: $\mathcal{X}$ is a discrete set and Σ_X is the power set $\mathcal{P}(\mathcal{X})$ of $\mathcal{X}$, i.e. the set of all subsets of $\mathcal{X}$.

continuous rv: $\mathcal{X} = \mathbb{R}$ and $\Sigma_X = \mathcal{B}$, the Borel σ-algebra. This is the smallest σ-algebra containing all open intervals in $\mathbb{R}$.

The probability measure $\mathbb{P}$ induces a probability measure P_X on $(\mathcal{X}, \Sigma_X)$, given by

$$P_X(B) = P[X \in B] = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B\}) \tag{6}$$

for all $B \in \Sigma'$. P_X is called the *distribution* of the random variable X.

If $\mathcal{X} = \{a_1, \ldots, a_d\}$ is discrete (and Σ_X the power set of $\mathcal{X}$), then we say that X is a *discrete random variable.* The distribution of X is then also known as the *probability mass function (pmf)* of X and is fully characterised by all the events consisting of a single value, i.e. the values $P_X(a_1), P_X(a_2), \ldots, P_X(a_d)$.

Question 0.6. *Can you give a formal argument why the values at these points are sufficient?*

Some random variables are not random at all. If there is an a_i with $P_X(a_i) = 1$ (and thus $P_X(a_j) = 0$ for all $j \neq i$), then we call this random variable *deterministic.* On the other extreme we have *uniformly distributed* random variables, where $P_X(a_i) = \frac{1}{d}$ for all $i \in [d]$.

Example 0.7. *The simplest example is the Bernoulli random variable. It is defined on a binary alphabet $\mathcal{X} = \{0, 1\}$ and we write $X \sim \mathrm{Bern}(\epsilon)$ to denote the rv with $P[X = 1] = \epsilon$ and $P[X = 0] = 1 - \epsilon$.*

Let us now consider a real-valued random variable X. If there exists a function $p_X : \mathbb{R} \to [0, \infty)$ such that for all $A \in \Sigma_X$, we have

$$P[X \in A] = \int_A p_X(x)\,\mathrm{d}x \tag{7}$$

then we say that X is a *continuous random variable.* The function p_X is called the *probability density function (pdf)* of X. We also define the *cumulative distribution function* (cdf) by integrating $p_X(x)$, that is, the cdf is given by $F_X(a) = \mathbb{P}[X \leq a] = \int_{-\infty}^{a} p_X(x)\,\mathrm{d}x$.

Question 0.8. *Show that* $\int_{\mathcal{X}} p_X(x) = 1$*. Moreover, if* p_X *is continuous at some point* x*, it must satisfy* $p_X(x) \geq 0$*. Can* $p_X(x)$ *ever be larger than* 1*?*

In this class, we deal mainly with discrete rvs, although we will also encounter Gaussian random variables, which are continuous, later on.

Example 0.9. *We denote the pdf of a* Gaussian *random variable* X *as*

$$\mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \tag{8}$$

where μ *is the* mean *and* σ *the* standard deviation *of* X*. The* variance *of* X *is* σ^2*. A* normal Gaussian *random variable has* $\mu = 0$ *and* $\sigma = 1$*. The corresponding cdf is denoted as*

$$\Phi(y) = \int_{-\infty}^{y} \mathcal{N}(x; 0, 1)\,\mathrm{d}x. \tag{9}$$

Some additional notations and definitions for discrete random variables are given below. The counterparts for continuous random variables can be obtained by simply replacing pmfs with pdfs. Thus assume now that X and Y are discrete random variables taking on values in $\mathcal{X}$ and $\mathcal{Y}$ respectively. The joint pmf of X and Y is defined as

$$P_{X,Y}(x, y) = P[X = x \wedge Y = y] = \mathbb{P}\big(\{\omega \in \Omega : X(\omega) = x \wedge Y(\omega) = y\}\big)\,. \tag{10}$$

Question 0.10. *Verify that* $P_Y(y) = \sum_{x' \in \mathcal{X}} P_{X,Y}(x', y)$*.*

With this in hand we can define conditional pmf's and a notion of independence of random variables.

- The conditional pmf is given by

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x, y)}{P_Y(y)} = \frac{P_{Y|X}(y|x) P_X(x)}{P_Y(y)}, \quad \text{for} \quad P_Y(y) > 0\,, \tag{11}$$

  where the second expression is often referred to as Bayes' rule. If $P_Y(y) = 0$ then the conditional pmf is simply not defined.

- X and Y are *independent random variables*, if and only if, for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,

$$P_{X,Y}(x, y) = P_X(x) P_Y(y) \tag{12}$$

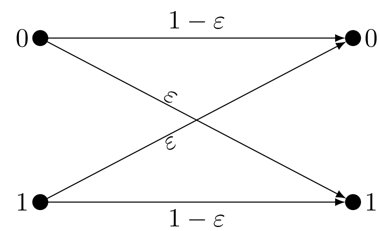
  or equivalently $P_{X|Y}(x|y) = P_X(x)$. The latter condition simply states that the conditional distribution $P_{X|Y}(x|y)$ does not depend on y.

Example 0.11 (Binary symmetric channel)**.** $X \sim \mathrm{Bern}(p)$ *is a bit that is sent over channel and is corrupted by additive noise* $Z \sim \mathrm{Bern}(\epsilon)$*, where* X *and* Z *are independent. The output of the channel is* $Y = X \oplus Z$*. The channel is fully defined by the conditional distribution* $P_{Y|X}$*, which we can compute as follows:*

$$P_{Y|X}(y|x) = P[X \oplus Z = y \mid X = x] = P[Z = y \oplus x \mid X = x] = P[Z = y \oplus x] = P_Z(y \oplus x) \tag{13}$$

Hence, the channel can be given as a matrix or pictorially as follows:

x	y	$P_{Y\|X}$
0	*0*	$1-\epsilon$
1	*0*	ϵ
0	*1*	ϵ
1	*1*	$1-\epsilon$



0.2.3 Expectation and variance

The expectation of a random variable X is defined to be

$$\mathbb{E}[X] = \int_\Omega X(\omega)\,\mathrm{d}\mathbb{P}(\omega). \tag{14}$$

This definition has a very precise mathematical meaning in measure theory, but here we are only interested in two special cases. If X is a discrete random variable this reduces to the familiar formula

$$\mathbb{E}[X] = \sum_{x\in\mathcal{X}} xP_X(x)\,. \tag{15}$$

If X is a continuous random variable with pdf $f_X(x)$, we have

$$\mathbb{E}[X] = \int_{\mathbb{R}} xp_X(x)\,\mathrm{d}x\,. \tag{16}$$

Note that the expectation is a statistical summary of the distribution of X, rather than depending on the realised value of X. If there are two different models $\mathbb{P}$ and $\mathbb{Q}$ we need to specify which probability measure we are using. We only do this when necessary (because the the model is not obvious from context) by adding a subscript $\mathbb{E}_P$ or $\mathbb{E}_Q$.

If g is a function, the expectation of $g(X)$ is given by

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x)p_X(x)\,\mathrm{d}x. \tag{17}$$

Question 0.12. *Show that the expectation is linear, i.e.* $\mathbb{E}[aX+b] = a\mathbb{E}[X]+b$.

The variance of X is the expectation of $g(X) = (X-\mathbb{E}[X])^2$. Thus,

$$\mathrm{Var}(X) = \mathbb{E}[(X-\mathbb{E}[X])^2] = \int_{\mathbb{R}} (x-\mathbb{E}[X])^2 p_X(x)\,\mathrm{d}x\,. \tag{18}$$

Question 0.13. *Check from the above definition that the variance can also be expressed as*

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2. \tag{19}$$

Question 0.14. *Verify that* $\mathcal{N}(x;\mu,\sigma^2)$ *indeed has expectation* μ *and variance* σ^2.

0.2.4 Markov chains

Markov chains describe a notion of conditional independence. Let's start with the three random variables X, Y and Z. They are said to form a *Markov chain in the order*

$$X - Y - Z$$

if their joint distribution P_{XYZ} satisfies

$$P_{XYZ}(x, y, z) = P_X(x) P_{Y|X}(y|x) P_{Z|Y}(z|y) \qquad \text{for all} \qquad (x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}. \tag{20}$$

This the same as saying that X and Z are *conditionally independent given* Y.

Question 0.15. *Assume $X - Y - Z$. Show that it is also true that $Z - Y - X$.*

Notice that if we do not assume anything about the joint distribution P_{XYZ}, then it factorizes (by repeated applications of Bayes rule) as

$$P_{XYZ}(x, y, z) = P_X(x) P_{Y|X}(y|x) P_{Z|XY}(z|x, y) \qquad \text{for all} \qquad (x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \tag{21}$$

so what Markovianty in the order $X - Y - Z$ buys us is that $P_{Z|XY}(z|x, y) = P_{Z|Y}(z|y)$ (i.e., we can drop the conditioning on X). In essence all the information that we can learn about Z is already contained in Y. No other information about Z can be gleaned from knowing X if we already know Y. Another way of saying this is that the conditional distribution of X and Z given $Y = y$ can be factorised as

$$P_{XZ|Y}(x, z|y) = P_{X|Y}(x|y) P_{Z|Y}(z|y) \qquad \text{for all} \qquad (x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}. \tag{22}$$

Notice that this is in direct analogy to the situation where X and Z are (marginally) independent. Simply set Y to be a deterministic random variable (with only one possible outcome) to recover the definition of independence.

Question 0.16. *If Z is a deterministic function of Y, show that $X - Y - Z$ is true.*

Question 0.17. *If X and Z are conditionally independent given Y, this does not imply that X and Z are marginally independent (in general). Construct a counterexample.*

0.3 Tail bounds

In this section, we summarise some bounds on probabilities that we use extensively in the sequel. More precisely, we are interested in showing that the probability of a random variable deviating too far from its expectation value is small.

0.3.1 Basic bounds

We start with the familiar Markov and Chebyshev inequalities.

Proposition 0.18 (Markov's inequality)**.** *Let X be a real-valued non-negative random variable with pdf p_X. Then for any $a > 0$, we have*

$$P[X > a] \leq \frac{\mathbb{E}[X]}{a}. \tag{23}$$

Proof. By the definition of the expectation, we have

$$\mathbb{E}[X] = \int_0^\infty x p_X(x)\, \mathrm{d}x \geq \int_a^\infty x p_X(x)\, \mathrm{d}x \geq a \int_a^\infty p_X(x)\, \mathrm{d}x = aP[X > a]. \tag{24}$$

and we are done. □

Note that this bound only becomes nontrivial if a exceeds the expectation value $\mathbb{E}[X]$.

Question 0.19. *In which step is non-negativity of X used?*

Question 0.20. *Can you do the proof also for discrete random variables?*

If we let X above be the non-negative random variable $(X - \mathbb{E}[X])^2$, we obtain Chebyshev's inequality.

Proposition 0.21 (Chebyshev's inequality)**.** *Let X be a real-valued random variable with mean μ and variance σ^2. Then for any $a > 0$, we have*

$$P\Big[|X - \mu| > a\sigma\Big] \leq \frac{1}{a^2}. \tag{25}$$

Proof. Let X in Markov's inequality be the random variable $g(X) = (X - \mathbb{E}[X])^2$. This is clearly non-negative and the expectation of $g(X)$ is $\mathrm{Var}(X) = \sigma^2$. Thus, by Markov's inequality, we have

$$P[g(X) > a^2\sigma^2] \leq \frac{\sigma^2}{a^2\sigma^2} = \frac{1}{a^2}. \tag{26}$$

Now, $g(X) > a^2\sigma^2$ if and only if $|X - \mu| > a\sigma$ so the claim is proved. □

We now consider a collection of real-valued random variables that are independent and identically distributed (i.i.d.). In particular, let $X^n = (X_1, \ldots, X_n)$ be a collection of independent random variables where each X_i has distribution P with zero mean and finite variance σ^2.

Proposition 0.22 (Weak Law of Large Numbers)**.** *For every $\epsilon > 0$, we have*

$$\lim_{n\to\infty} P\left[\left|\frac{1}{n}\sum_{i=1}^n X_i\right| > \epsilon\right] = 0. \tag{27}$$

Consequently, the average $\frac{1}{n}\sum_{i=1}^n X_i$ converges to 0 in probability.

Note that for a sequence of random variables $\{S_n\}_{n=1}^{\infty}$, we say that this sequence *converges to a number $b \in \mathbb{R}$ in probability* if for all $\epsilon > 0$,

$$\lim_{n\to\infty} P\big[|S_n - b| > \epsilon\big] = 0. \tag{28}$$

We also write this as $S_n \xrightarrow{\text{p}} b$. Contrast this to convergence of numbers: We say that a sequence of numbers $\{s_n\}_{n=1}^{\infty}$ *converges to a number $b \in \mathbb{R}$* if we have $\lim_{n\to\infty} |s_n - b| = 0$.

Proof. Let $\frac{1}{n}\sum_{i=1}^{n} X_i$ take the role of X in Chebyshev's inequality. Clearly, the mean is zero. The variance of X is

$$\text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(X_i) = \frac{\sigma^2}{n}. \tag{29}$$

Thus, we have

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| > \epsilon\right) \le \frac{\sigma^2}{n\epsilon^2} \longrightarrow 0 \tag{30}$$

as $n \to \infty$, which proves the claim. $\square$

Some further useful bounds are derived in the homework.

0.3.2 Central limit theorem

We can actually say quite a bit more than the weak law of large numbers dictates. If the scaling in front of the sum in the statement of the law of large numbers Proposition 0.22 is $1/\sqrt{n}$ instead of $1/n$, the resultant random variable $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i$ converges in distribution to a Gaussian random variable. As in Proposition 0.22, let X^n be a collection of i.i.d. random variables where each X_i is zero mean with finite variance σ^2.

Proposition 0.23 (Central limit theorem)**.** *For any $a \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} P\left(\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n} X_i < a\right) = \Phi(a). \tag{31}$$

In other words,

$$\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n} X_i \xrightarrow{\text{d}} Z \tag{32}$$

where $\xrightarrow{\text{d}}$ means convergence in distribution and Z is a standard Gaussian random variable.

For a sequence of random variables $\{S_n\}_{n=1}^{\infty}$, we say that this sequence of random variables *converges in distribution* to another random variable $\bar{S}$ if

$$\lim_{n\to\infty} P(S_n < a) = P(\bar{S} < a)$$

for all $a \in \mathbb{R}$. The proof of this statement requires tools that are outside the scope of these notes, but can be found in any textbook on probability theory.

0.4 Vector norms and Cauchy-Schwarz inequality

We can naturally interpret pmf's on an alphabet with d symbols as row vectors in a d-dimensional inner-product space. Without loss of generality we take the alphabet to be $\mathcal{X} = \{1, 2, \ldots, d\} = [d]$ and define the vector $p \in \mathbb{R}^d$ by its elements $p_x = P_X(x)$ for $x \in [d]$. The inner product is denoted by $\langle \cdot, \cdot \rangle$. For two general vectors $u, v \in \mathbb{R}^d$, it evaluates to

$$\langle u, v \rangle = uv^T = \sum_{i=1}^{d} u_i v_i \,, \tag{33}$$

where v^T denotes the transpose of the vector v, and is a column vector. The Cauchy-Schwarz inequality then states that for any two vectors we have

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle \,. \tag{34}$$

On these vector spaces we can also define the p-norms for $p \geq 1$ as

$$\|u\|_p = \left(\sum_{x=1}^{d} |u_x|^p \right)^{\frac{1}{p}} \tag{35}$$

We will mostly encounter the 1-norm and the 2-norm, the latter being the usual Euclidian norm of the vector. The following special case of the Cauchy-Schwarz inequality will be encountered later.

Lemma 0.24. *Let $u, v \in \mathbb{R}^d$. Then,*

$$\|u \cdot v\|_1 \leq \|u\|_2\| \, \|v\|_2 \,, \tag{36}$$

where $\cdot$ denotes the element-wise product of the vectors, i.e. $(u \cdot v)_i = u_i v_i$.

Proof. Define $k \in$ using $k_i = \mathrm{sgn}^*(u_i v_i)$, where sgn^* is the modified sign function, i.e.

$$\mathrm{sgn}^*(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} . \tag{37}$$

Then since $\mathrm{sgn}^*(x)^2 = 1$ for all $x \in \mathbb{R}$, the Cauchy-Schwarz inequality yields

$$\big|\langle k \cdot u, v \rangle\big| \leq \langle u, u \rangle \langle v, v \rangle = \|k \cdot u\|_2 \|v\|_2 = \|u\|_2 \|v\|_2 \,. \tag{38}$$

Moreover, we have

$$\langle k \cdot u, v \rangle = \sum_{x=1}^{d} k_i u_i v_i = \sum_{x=1}^{d} |u_i v_i| = \|u \cdot v\|_1 \,. \tag{39}$$

□

Question 0.25. *Using the above, can you show that $\|u\|_1 \leq \sqrt{d} \|u\|_2$?*

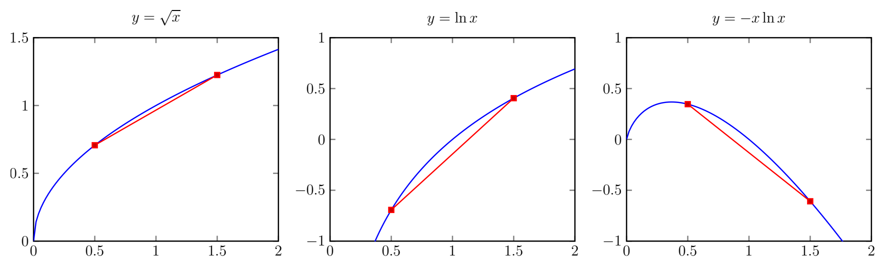


Figure 0.1: Examples of concave (left and centre) and convex (right) functions. The straight line between two points of the curve is either below or above the plot of the function, which is exactly what the definition requires.

0.5 Convexity and Jensen's inequality

A function $f(x)$ is said to be *convex* on [a, b] if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y). \tag{40}$$

If we do not mention any interval then we mean that the function is convex on its full domain, i.e. the statement $\log(x)$ is concave should be understood as $\log(x)$ is concave on $(0, \infty)$.

The function f is *strictly convex* if equality in (40) holds only if $\lambda = 0$ or 1, or $x = y$. The function f is *concave* if $-f$ is convex, and *strictly concave* if $-f$ is strictly convex.

In the homework you will show the following lemma:

Lemma 0.26. *If f is convex on $[a, b]$, then for any $a \leq x_1 < x_2 \leq x_3 < x_4 \leq b$, we have*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_4) - f(x_3)}{x_4 - x_3} \tag{41}$$

Proposition 0.27 (Jensen's inequality)**.** *If $f(x)$ is convex and X is a random variable on $\mathbb{R}$, then*

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)] \tag{42}$$

We only give a proof for discrete distributions here.

Proof. We give a proof by induction. Due to convexity, we have

$$p_1 f(x_1) + p_2 f(x_2) \geq f(p_1 x_1 + p_2 x_2), \tag{43}$$

which proves the statement if $|\mathcal{X}| = 2$.

Suppose the statement $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$ is true when $|\mathcal{X}| = k - 1$. Then consider a pmf with k mass points $\{p_1, p_2, \ldots, p_k\}$. Define another pmf on $k-1$ points given by the probabilities

$$p'_i = \frac{p_i}{1 - p_k}, \quad i = 1, \ldots, k-1\,. \tag{44}$$

We then have

$$\sum_{i=1}^{k} p_i f(x_i) = p_k f(x_k) + (1-p_k) \sum_{i=1}^{k-1} p_i' f(x_i) \tag{45}$$

$$\geq p_k f(x_k) + (1-p_k) f\left(\sum_{i=1}^{k-1} p_i' x_i\right) \tag{46}$$

$$\geq f\left(p_k x_k + (1-p_k) \sum_{i=1}^{k-1} p_i' x_i\right) \tag{47}$$

$$= f\left(\sum_{i=1}^{k} p_i x_i\right) \tag{48}$$

where the first inequality is from the induction hypothesis and the second by convexity (of two points). By the definition of expectation we have $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$. $\square$

Often it is hard to check convexity directly. But for twice differentiable functions, this is easy.

Proposition 0.28. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable. The function f is convex if and only if $f''(x) \geq 0$ for all $x \in (a,b)$, and strictly convex if $f''(x) > 0$ for all $x \in (a,b)$.*

Proof. Assume $f''(x) > 0$ for all $x \in [a,b]$. By Taylor expansion of f around $x_0 \in (a,b)$, we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x^*)}{2}(x - x_0)^2 \tag{49}$$

where $x^* \in [x_0, x]$. By assumption $f''(x^*) > 0$ so the quadratic term is strictly positive unless $x = x_0$, in which case it is still non-negative. Now let $x_0 = \lambda x_1 + (1-\lambda)x_2$. Further let $x = x_1$. Then we have

$$f(x_1) \geq f(x_0) + f'(x_0)((1-\lambda)(x_1 - x_2)). \tag{50}$$

Now let $x = x_2$. Then we have

$$f(x_2) \geq f(x_0) + f'(x_0)(\lambda(x_2 - x_1)). \tag{51}$$

Both of these inequalities are strict unless $\lambda \in \{0,\}]$ or $x_1 = x_2$. Multiplying the first inequality by λ and the second by $1-\lambda$ and adding them up, we recover the definition of strict convexity. If we instead had assumed only $f''(x) \geq 0$ the same argument would ensure convexity (but no longer strict convexity).

For the other direction, choose $a < x_1 < x_2 < x_3 < x_4 < b$. By the property shown in Lemma 0.26,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_4) - f(x_3)}{x_4 - x_3} \tag{52}$$

Now let $x_2 \searrow x_1$ and $x_3 \nearrow x_4$. We see that $f'(x_1) \leq f'(x_4)$, and since these were arbitrary points, f' is increasing on (a,b). So $f''(x) \geq 0$ for all $x \in (a,b)$. $\square$

0.6 Finite field arithmetic

This is a rather informal discussion, but it is sufficient for our purposes.

A finite field is a field (on which addition, subtraction, multiplication and division are defined) with a finite number of elements. Such fields are denoted by F_q where q is the number of elements in the field, or its *dimension*. The idea is that such fields behave like $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, with the usual rules for addition and multiplication.

A bit more formally, we have two binary operations on F_q denoted by $+$ and $\cdot$ and the following properties (here a, b and c are any elements of F_q):

Associativity: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Commutativity: $a + b = b + a$ and $a \cdot b = b \cdot a$.

Identities: There exist two different elements $0, 1$ such that $a + 0 = a$ and $a \cdot 1 = a$.

Additive inverse: Every a has an additive inverse, denoted $-a$, such that $a + (-a) = 0$.

Multiplicative inverse: Every $a \neq 0$ has a multiplicative inverse, denoted a^{-1}, such that $a \cdot a^{-1} = 1$.

Distributivity: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Such fields exist only for particular numbers of elements, for example when $q = p^\ell$ for some prime p and $\ell \in \mathbb{N}$. For F_q where q is prime we can always simply denote the elements of F_q by the integers $\{0, 1, \ldots, q - 1\}$ and use integer addition and multiplication modulo q as our operations.

Question 0.29. *Verify the above properties for F_2 and F_3. Can you find the inverse of 2 for a general F_q with prime q? Use that $q + 1$ is even...*

When q is not a prime but a prime power we can construct the arithmetic by constructing a polynomial ring. To see how this works with an example, let us look at $q = 2^2 = 4$. Let us first try something that does not work. As for primes, we may denote the elements by $\{0, 1, 2, 3\}$ and use addition modulo 4, which works fine, but the problem is that using multiplication modulo 4 we would, for example, get

$$
\begin{aligned}
&2 \times 0 = 0 && (53)\\
&2 \times 1 = 2 && (54)\\
&2 \times 2 = 4 \mod 4 = 0 && (55)\\
&2 \times 3 = 6 \mod 4 = 2\,, && (56)
\end{aligned}
$$

and hence 2 does not have a multiplicative inverse.

The arithmetic is thus defined differently, by interpreting $\{0, 1, 2, 3\}$ instead as the polynomials, 0, 1, x and $x + 1$, respectively. We can add these polynomials modulo 2 for each coefficient individually, so in particular for the binary case the negation of each number is just the number itself. For multiplication, we simply do this modulo an irreducible polynomial, for example $x^2 + x + 1$ in this case. So for the above labelings $\{0, 1, 2, 3\}$ of elements (which, admittedly, is more confusing than helpful), we get

$$2 \times 0 \to x \times 0 = 0 \to 0 \qquad \Longrightarrow\ 2 \times 0 = 0 \tag{57}$$
$$2 \times 1 \to x \times 1 = x \to 2 \qquad \Longrightarrow\ 2 \times 1 = 2 \tag{58}$$
$$2 \times 2 \to x \times x = x^2 \mod x^2 + x + 1 = -x - 1 \to 3 \qquad \Longrightarrow\ 2 \times 2 = 3 \tag{59}$$
$$2 \times 3 \to x \times (x + 1) = x^2 + x \mod x^2 + x + 1 = -1 \to 1 \qquad \Longrightarrow\ 2 \times 3 = 1\,. \tag{60}$$

Hence, 2 and 3 are multiplicative inverses of each other. The full addition and multiplication tables can then be written down as follows:

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Similar constructions can be done for every prime power, and, quite importantly for practical applications, all of this arithmetic can be implemented highly efficiently in computer programs.

Chapter 1

Information measures

[Week 2]

Intended learning outcomes:

- You can compute the entropy and conditional entropy for any discrete random variable and understand the basic properties of these two quantities, e.g. you can apply the chain rule or sub-additivity.
- You can compute mutual information and now how it relates to entropy and conditional entropy. You can apply the data-processing inequality for mutual information.
- You can compute the relative entropy and understand how entropy and mutual information can be expressed in terms of the relative entropy.

Book reference: Chapter 2 in Cover & Thomas [1], but we are not following it too closely.

1.1 Surprisal and entropy

It is not immediately clear how to model our intuitive notion of "information" in a mathematical language. In this chapter we take a somewhat axiomatic approach to information measures, i.e. we try to build them up from our intuitive understanding of what entropy and information "should" be. But we will only really be able to justify the choices we make here once we start analysing practical problems in information theory, and see that the quantities we derive here pop up again and again.

1.1.1 Surprisal

It turns out to be fruitful to start not by finding an expression for the information contained in a random variable, but rather the lack of information, or uncertainty inherent in a random experiment. Let us consider a discrete random variable X taking values in $\mathcal{X}$ following the pmf $P_X(x) = p_x$. How surprised are we to see a particular outcome $x \in \mathcal{X}$ of this random experiment? Clearly this depends on the probability p_x and not the value of x itself. In fact, we do not even need to know what $\mathcal{X}$ really is. On the one hand, if $p_x = 1$ we are not surprised at all since we already knew that we would see x. On the other hand, the smaller p_x is the more surprised we are to see this particular outcome. If $p_x = 0$ we will never see x, so our surprise when seeing it anyway would be literally off the scale. Furthermore—and this turns out to be a very convenient choice—if we do a random experiment twice independently and both times observe x, we say that we will be twice as surprised as if we had seen x once in a single random experiment.

The above notions can be formalised, and that is essentially what Shannon did when he introduce the notion of *surprisal.* Let us denote the surprisal of x as $s(p_x)$. We want this function to satisfy the following three conditions:

1. **Monotonicity:** $s(p_x) = 0$ if $p_x = 1$ and $s(p_x)$ increases monotonically as p_x decreases.

2. **Additivity:** The surprisal of seeing a pair of outcomes of independent random experiments is simply the sum of the individual surprisals, i.e. $s(p_x p_y) = s(p_x) + s(p_y)$.

3. **Normalisation:** $s(\frac{1}{2}) = 1$

Question 1.1. *We do not really need the condition $s(p_x) = 0$ if $p_x = 1$ under Point 1 as it follows directly from additivity. Can you see how?*

It turns out that the only function that satisfies these three properties is the logarithm. To show this one uses a result by Erdös that characterises additive functions, but that is beyond the scope here. We therefore pick

$$s(p_x) = \log \frac{1}{p_x} . \tag{1.1}$$

where the logarithm is taken to base 2 (as everywhere in these notes) so that the normalisation requirement is satisfied.

We can see the surprisal as another random variable, say S, that takes the value $s(p_x) = \log \frac{1}{p_x}$ with probability p_x. Since $S = S(X)$ is a function of X we usually simply write this new random variable as

$$S(X) = \log \frac{1}{P_X(X)} . \tag{1.2}$$

1.1.2 Entropy

Entropy measures how much we can learn by looking at the outcome of a random experiments, or, in other words, how much uncertainty there is about the outcome. It is simply the expected surprisal of X.

Definition 1.2. *Given a discrete random variable X, the* entropy *of X is defined as*

$$H(X) := \mathbb{E}[S(X)] = \mathbb{E}\left[\log \frac{1}{P_X(X)}\right] = \sum_{x \in \mathcal{X}} p_x \log \frac{1}{p_x} \tag{1.3}$$

Here and throughout we use the convention that $0 \log 0 = 0$. This is reasonable since $\lim_{\epsilon \to 0} \epsilon \log \epsilon = 0$, and thus we simply continuously extend the function to the point 0.

Question 1.3. *Can you verify this limit?*

Note again that the entropy X is really only a function of the pmf of X, and in particular independent of the alphabet $\mathcal{X}$, in contrast to potential alternative uncertainty measures like the variance of X.

Sometimes we are interested in more than just the expected surprisal. The minimum surprisal, or min-entropy, for example, has applications in cryptography and the variance of $S(X)$ has itself operational meaning in many information-theoretic problems when we go beyond first order asymptotics.

Question 1.4. *Can you find an expression for* $\mathrm{Var}[S(X)]$ *in terms of the probabilities* p_x*?*

Now let us explore this formula a bit. First we want to show the following basic property.

Proposition 1.5. *Let X be a discrete random variable taking values in $\mathcal{X}$. We have*

$$H(X) \geq 0, \tag{1.4}$$

with equality if and only if X is deterministic.

Proof. Since $p_x \leq 1$, we have $\log \frac{1}{p_x} \geq 0$ for every $x \in \mathcal{X}$, so the expectation of this quantity over x must be positive too. In fact, $\log \frac{1}{p_x}$ equals 0 if and only if $p_x = 1$ and hence $H(X) = 0$ only if there exists an $x \in \mathcal{X}$ for which $p_x = 1$, which is the hallmark of a deterministic rv. □

The entropy is a strictly concave function of the probability mass function P_X. To see this, we first verify that $f(t) = t \log \frac{1}{t} = -t \log t$ is concave on $(0, 1)$ by taking its second derivative:

$$f'(t) = -\log t - \log e, \qquad f''(t) = -\frac{\log e}{t}\,. \tag{1.5}$$

Since the latter is always negative for $t \in (0, 1)$, the function is indeed strictly concave according to Lemma 0.28. Now since the entropy is simply the sum $\sum_{x \in \mathcal{X}} f(p_x)$ it is indeed a concave function of the pmf. This simple property, together with Jensen's inequality, has profound implications. The first one is that the entropy has a unique maximum. Intuitively we would want that entropy is maximal when uncertainty about the outcome is greatest, namely when the rv is uniformly distributed. And this is indeed the case.

Proposition 1.6. *Let X be a discrete random variable taking values in $\mathcal{X}$. We have*

$$H(X) \leq \log |\mathcal{X}|, \tag{1.6}$$

with equality if and only if X is uniformly distributed.

The general case will be covered in the homework but here we give a proof for the case when the set $\mathcal{X}$ is a bit, i.e. when the random variable is binary.

Proof for $\mathcal{X} = \{0, 1\}$. It is easy to verify by a simple computation that $H(X) = \log |\mathcal{X}|$ for a uniformly distributed random variable. Let $\{p, 1-p\}$ be the pmf for X. We use $f(t) = -t \log t$ Then we can write

$$
\begin{aligned}
H(X) &= f(p) + f(1-p) && (1.7)\\
&= \frac{1}{2}\left(f(p) + f(1-p)\right) + \frac{1}{2}\left(f(p) + f(1-p)\right) && (1.8)\\
&\leq f\left(\frac{1}{2}p + \frac{1}{2}(1-p)\right) + f\left(\frac{1}{2}p + \frac{1}{2}(1-p)\right) && (1.9)\\
&= f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) && (1.10)\\
&= 1\,. && (1.11)
\end{aligned}
$$

Moreover, due to the strict concavity of f, equality only holds if $p = 1 - p = \frac{1}{2}$, i.e. when the random variable X follows the uniform distribution. $\square$

Concavity in fact has even stronger consequences, and we will show a few additional properties of entropy later on.

1.2 Joint entropy, conditional entropy, and mutual information

1.2.1 Joint entropy

For two discrete random variables X and Y with joint pmf $P_{XY}(x, y) = p_{xy}$ we can simply consider (X, Y) as one single random variable and use the same construction to define the surprisal of a tuple (X, Y) as $S(X, Y) = -\log P_{XY}(X, Y)$. Its expectation is the *joint entropy*, $H(XY)$, given by

$$
H(XY) := \mathbb{E}[S(X, Y)] = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p_{xy} \log \frac{1}{p_{xy}} \tag{1.12}
$$

The first thing to note is that — if X and Y are independent — then $p_{xy} = p_x \cdot p_y$ and thus the expression simplifies to

$$H(XY) = \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p_{xy} \log \frac{1}{p_x p_y} \tag{1.13}$$

$$= \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p_{xy} \log \frac{1}{p_x} + \log \frac{1}{p_y} \tag{1.14}$$

$$= \sum_{x\in\mathcal{X}} p_x \log \frac{1}{p_x} + \sum_{y\in\mathcal{Y}} p_y \log \frac{1}{p_y} \tag{1.15}$$

$$= H(X) + H(Y)\,. \tag{1.16}$$

This is not true in general though if the two random variables are correlated.

Question 1.7. *Find an example for which $H(XY) = H(X) = H(Y) = 1$.*

1.2.2 Conditional entropy

So why do these entropies not just add up? Fundamentally, this is because once we learn X we might not be so surprised seeing some particular outcomes of the random variable Y anymore. In fact, in the most extreme case, we have $Y = f(X)$ for some function f; hence, once we know that X takes on the value x, we can immediately deduce that Y will take on the value $f(x)$ with probability one, and thus there is no surprisal anymore! We model this "conditional suprisal" using the conditional pmfs, $P_{Y|X}(y|x) = p_{y|x}$, which leads us to conditional entropy.

Definition 1.8. *The* conditional entropy *of Y given X is defined as*

$$H(Y|X) = \mathbb{E}\left[\log \frac{1}{P_{Y|X}(Y|X)}\right] = \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p_{xy} \log \frac{1}{p_{y|x}}\,. \tag{1.17}$$

This can be interpreted as the expectation of the entropy of Y over all outcomes X. We sometimes use the notation $H(Y|X = x)$ to denote the entropy of the random variable Y_x that follows the pmf $\{p_{y|x}\}_{y\in\mathcal{Y}}$, i.e., the pmf of Y when we already know that $X = x$. Using this and the expression in (1.17) we can simply write the conditional entropy as

$$H(Y|X) = \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p_{xy} \log \frac{1}{p_{y|x}} \tag{1.18}$$

$$= \sum_{x\in\mathcal{X}} p_x \sum_{y\in\mathcal{Y}} p_{y|x} \log \frac{1}{p_{y|x}} \tag{1.19}$$

$$= \sum_{x} p_x\, H(Y|X = x)\,. \tag{1.20}$$

The last line which expresses the conditional entropy in terms of an average of (unconditional) entropies is particularly useful since it allows us to immediately conclude that the conditional entropy is also bounded from below and above, like the entropy. We thus have

$$0 \leq H(Y|X) \leq \log |\mathcal{Y}| \,. \tag{1.21}$$

Moreover, our definition of conditional entropy also allows us to establish a *chain rule* for the conditional entropy, which sometimes is in fact used as the definition of conditional entropy itself. This rule is very useful because it allows us to write the joint entropy as a sum of its parts, even if the two random variables are not independent.

Proposition 1.9. *We have $H(XY) = H(X) + H(Y|X)$.*

Proof. We simply write

$$H(XY) = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{xy} \log \frac{1}{p_{xy}} \tag{1.22}$$

$$= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{xy} \log \frac{1}{p_{x}} + \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{xy} \log \frac{1}{p_{y|x}} \tag{1.23}$$

$$= \sum_{x\in\mathcal{X}} p_{x} \log \frac{1}{p_{x}} + \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{xy} \log \frac{1}{p_{y|x}} \tag{1.24}$$

$$= H(X) + H(Y|X) \,. \tag{1.25}$$

□

Question 1.10. *Using the chain rule, find a different proof that $H(XY) = H(X) + H(Y)$ for independent random variables.*

Now we have put everything in place to show our first entropic inequality, which relates the entropy of two random variables with their joint entropy. This result shows the *sub-additivity* of entropy.

Proposition 1.11. *Let X and Y be two discrete random variables. Then*

$$H(XY) \leq H(X) + H(Y) \qquad \text{or, equivalently} \qquad H(X|Y) \leq H(X) \,. \tag{1.26}$$

Proof. The equivalence of the two relations follows directly from the chain rule, we thus only need to show the second statement.

We start with Eq. (1.20), which states that

$$H(Y|X) = \sum_{x} p_x H(Y|X=x) \tag{1.27}$$

$$= \sum_{x} p_x \sum_{y} p_{y|x} \log \frac{1}{p_{y|x}} \tag{1.28}$$

$$= \mathbb{E}\left[\sum_{y} p_{y|X} \log \frac{1}{p_{y|X}} \right] \tag{1.29}$$

Note that sum inside the expectation is simply another expectation, as in the definition of entropy — but since we only want to apply Jensen's inequality on the outer expectation we spell this one out explicitly. Moreover, by definition of the conditional pmf we have $\mathbb{E}[p_{y|X}] = \sum_x p_x p_{y|x} = \sum_x p_{xy} = p_y$. Hence, using concavity of the entropy as a function of the pmf and Jensen's inequality for the outer expectation, we find

$$
\begin{aligned}
H(Y|X) &= \mathbb{E}\left[\sum_y p_{y|X} \log \frac{1}{p_{y|X}}\right] && (1.30)\\
&\leq \sum_y \left(\mathbb{E}[p_{y|X}]\right) \log \frac{1}{\mathbb{E}[p_{y|X}]} && (1.31)\\
&= \sum_y p_y \log \frac{1}{p_y} && (1.32)\\
&= H(Y)\,. && (1.33)
\end{aligned}
$$

□

The second relation in Eq. (1.26) can be strengthened by considering three random variables X, Y and Z. In that case, we have

$$H(X|YZ) \leq H(X|Z)\,. \tag{1.34}$$

This is sometimes referred to as *strong sub-additivity*. The proof follows from (regular) sub-additivity, applied to the entropies $H(X|Y, Z = z)$ and $H(X|Z = z)$, and averaging the resulting inequalities.

Question 1.12. *Can you construct a formal proof out of the above sketch?*

1.2.3 Mutual information

We have already established that $H(XY) \neq H(X) + H(Y)$ in general, and hence also $H(Y|X) \neq H(Y)$ by the chain rule. The difference between these two quantities clearly tells us something about how much the uncertainty about Y changes when we learn X, or in other words, about how much information X contains about Y. This leads us to the definition of mutual information,

Definition 1.13. *The* mutual information *between X and Y is defined as*

$$I(X:Y) := H(Y) - H(Y|X) \tag{1.35}$$

It is not evident immediately from the way we defined it here but this expression is symmetric between X and Y. However, using the chain rule for conditional entropy (recall Proposition 1.9) twice, we can write

$$I(X:Y) = H(Y) - H(Y|X) = H(Y) + H(X) - H(XY) = H(X) - H(X|Y)\,. \tag{1.36}$$

The mutual information is a measure of the correlation between the two random variables.

Using these various equivalent expressions it is then easy to derive some bounds on the mutual information. First, sub-additivity of the entropy directly implies that $I(X:Y) \geq 0$, so the mutual information is non-negative, and it vanishes only if the two random variables are independent. This is consistent with our intuitive notion of information — we cannot know less than nothing after all! We also cannot know more than everything, i.e. the mutual information can never exceed the minimal entropy of its constituent parts.

Question 1.14. *Using the bounds on entropies established in the previous sections, show that* $I(X:Y) \leq \log \min\{|\mathcal{X}|, |\mathcal{Y}|\}$. *Give an example that saturates the bound.*

If we have three random variables X, Y and Z we can ask for the mutual information between X and Y conditioned on knowing Z, the *conditional mutual information*. It is defined as

$$I(X:Y|Z) := \sum_z P_Z(z) I(X:Y|Z=z)\,. \tag{1.37}$$

Various equivalent expressions can then be readily derived, e.g.,

$$I(X:Y|Z) = H(Y|Z) - H(Y|XZ) = H(X|Z) - H(X|YZ)\,. \tag{1.38}$$

Moreover, the *chain rule* for the mutual information states that

$$I(X:YZ) = I(X:Y) + I(X:Z|Y)\,, \tag{1.39}$$

which can be verified by a close inspection of the definition of both conditional and unconditional mutual information.

Consider now the special case where $X - Y - Z$ form a Markov chain. In this case $P_{X|YZ} = P_{X|Y}$ and thus $H(X|YZ) = H(X|Y)$. As a consequence, the conditional mutual information $I(X:Z|Y)$ vanishes.

One of the most intriguing properties of the mutual information is the *data-processing inequality* for mutual information. It states that the mutual information can never increase when we apply an operation that only acts on one of the parts. Intuitively this tells us that by manipulating one of the random variables without looking at the other we cannot increase the correlations between the pair.

We can formalise this using the notion of Markov chains.

Proposition 1.15. *Let $X - Y - Z$ form a Markov chain. Then, $I(X:Y) \geq I(X:Z)$.*

Proof. Since $I(X:Z|Y) = 0$, the chain rule for mutual information implies that $I(X:Y) = I(X:YZ)$. It thus remains to show that

$$I(X:Z) \leq I(X:YZ)\,. \tag{1.40}$$

But, since $I(X:Z) = H(X) - H(X|Z)$ and $I(X:YZ) = H(X) - H(X|YZ)$, the relation in Eq. (1.40) is equivalent to the condtion $H(X|Z) \geq H(X|YZ)$, which is ensured by the strong sub-additivity of entropy. □

Question 1.16. *Can you also show that $I(Y:Z) \geq I(X:Z)$ under the same assumption?*

1.3 Relative entropy

The relative entropy appears when we want to compare two different probability distributions. We define it here only for discrete random variables (or rather the respective pmfs), but this can be readily generalised to other probability measures.

Definition 1.17. *Let P and Q be two pmfs on an alphabet $\mathcal{X}$. The* relative entropy *of P with regards to Q is defined as*

$$D(P\|Q) := \sum_{x\in\mathcal{X}} P(x) \log \frac{P(x)}{Q(x)} \,. \tag{1.41}$$

if $P(x) > 0 \implies Q(x) > 0$ for all $x \in \mathcal{X}$, and $+\infty$ otherwise.

We can alternatively see the relative entropy as the expectation value of the *log-likelihood ratio*, namely we can write

$$D(P\|Q) = \mathbb{E}\left[Z(X)\right], \qquad \text{where} \qquad Z(X) = \log \frac{P(X)}{Q(X)} \tag{1.42}$$

and X is distributed according to P. The random variable $Z(X)$ is called the log-likelihood ratio. It takes on the role of the surprisal in the definition of entropy. We will explore this random variable and its distribution much more when we discuss hypothesis testing later on.

Just by manipulating the definition, we are able to show the following equivalences.

Proposition 1.18. *Let X and Y be random variables on alphabets $\mathcal{X}$ and $\mathcal{Y}$. Moreover, let U be a uniform random variable on $\mathcal{X}$. Then the following relations are true:*

$$H(X) = \log|\mathcal{X}| - D(P_X\|U_X) \tag{1.43}$$
$$H(X|Y) = \log|\mathcal{X}| - D(P_{XY}\|U_X \times P_Y) \tag{1.44}$$
$$I(X:Y) = D(P_{XY}\|P_X \times P_Y)\,. \tag{1.45}$$

You will prove these equivalences in the homework. We will need two important properties of the relative entropy. The first proposition establishes that the relative entropy is always positive.

Proposition 1.19. *For any two pmfs P and Q, we have $D(P\|Q) \geq 0$ with equality if and only if $P = Q$.*

Proof. We can assume without loss of generality that $Q(x) > 0$ for all x. We first note that

$x \mapsto -\log x$ is strictly convex. Hence,

$$D(P\|Q) = \sum_x P(x) \log \frac{P(x)}{Q(x)} \tag{1.46}$$

$$= \sum_x P(x) \left(-\log \frac{Q(x)}{P(x)} \right) \tag{1.47}$$

$$\geq -\log \left(\sum_x P(x) \frac{Q(x)}{P(x)} \right) \tag{1.48}$$

$$= -\log \left(\sum_x Q(x) \right) = 0\,. \tag{1.49}$$

Moreover, equality only holds if $\frac{Q(x)}{P(x)}$ is independent of x, which is only true if $P(x) = Q(x)$ for all $x \in \mathcal{X}$, or if $P(x) \in \{0, 1\}$ for all $x \in \mathcal{X}$. In the latter case we must have

$$D(P\|Q) = -\log Q(x^*) \tag{1.50}$$

for the $x^* \in \mathcal{X}$ for which $P(x^*) = 1$. Hence, $D(P\|Q)$ vanishes only if $Q(x^*) = 1$ as well. □

An immediate corollary of Propositions 1.18 and 1.19 is that $I(X : Y)$ is positive and zero only if X and Y are independent.

Question 1.20. *Can you see why?*

Finally, there is one property of the relative entropy that implies all other properties of both entropy and mutual information. It states that applying a noisy operation, i.e. a stochastic map or channel, on both arguments of the relative entropy will never increase the relative entropy. Together with the positivity of relative entropy this justifies that we think of it as a measure of similarity or distinguishability. If the relative entropy is small the two pmfs are similar and hard to distinguish by observing the outcomes of a random experiment. Observing the outcomes after further noise has been applied should make distinguishing them even harder, and that is exactly what the *data-processing inequality* for relative entropy tells us.

Proposition 1.21. *Let P_X and Q_X be two pmfs on an alphabet $\mathcal{X}$, and let $P_{Y|X}$ be a conditional pmf. Define the marginals*

$$P_Y(y) = \sum_{x \in \mathcal{X}} P_{Y|X}(y|x) P_X(x) \qquad \textit{and} \qquad Q_Y(y) = \sum_{x \in \mathcal{X}} P_{Y|X}(y|x) Q_X(x)\,. \tag{1.51}$$

Then, the data-processing inequality *(DPI) states that*

$$D(P_X\|Q_X) \geq D(P_Y\|Q_Y)\,. \tag{1.52}$$

Proof. Consider now the joint distributions $P_{XY}(x,y) = P_{Y|X}(y|x)P_X(x)$ and $Q_{XY}(x,y) = P_{Y|X}(y|x)Q_X(x)$, using the usual shorthand notation for conditional and marginal distributions. We first show that

$$D(P_{XY}\|Q_{XY}) - D(P_Y\|Q_Y) = \left(\sum_{x,y} p_{xy}\log\frac{p_{xy}}{q_{xy}}\right) - \left(\sum_{y} p_y\log\frac{p_y}{q_y}\right) \tag{1.53}$$

$$= \sum_{x,y} p_{xy}\left(\log\frac{p_{xy}}{q_{xy}} - \log\frac{p_y}{q_y}\right) \tag{1.54}$$

$$= \sum_y p_y \sum_x p_{x|y}\log\frac{p_{x|y}}{q_{x|y}} \tag{1.55}$$

$$= \sum_y p_y\, D(P_{X|Y=y}\|Q_{X|Y=y}) \geq 0\,, \tag{1.56}$$

where we have used the positivity of relative entropy in the last step. Similarly, we have

$$D(P_{XY}\|Q_{XY}) - D(P_X\|Q_X) = \sum_x p_x\, D(P_{Y|X=x}\|Q_{Y|X=x}) = 0 \tag{1.57}$$

since $Q_{Y|X} = P_{Y|X}$ by construction of the joint distribution. Combining Eqs. (1.53)–(1.56) and (1.57) yields the desired inequality. □

It turns out that all the properties of entropy, conditional entropy and mutual information we discussed previously can be derived form the DPI. As an example we give here a strengthening of the strong sub-additivity, which we call the data-processing inequality for conditional entropy. It intuitively states that any processing of the side information can at most increase the conditional entropy.

Corollary 1.22. *Let P_{XY} be a joint pmf and $P_{Z|Y}$ a conditinal pmf. Define now the pmf*

$$P_{XZ}(x,z) = \sum_y P_{XY}(x,y)P_{Z|Y}(z|y)\,. \tag{1.58}$$

Then, we have $H(X|Y) \leq H(X|Z)$.

Proof. Let us use the inequality in terms of relative entropies using Proposition 1.18. This reads

$$\log|\mathcal{X}| - D(P_{XY}\|\, U_X \times P_Y) \leq \log|\mathcal{X}| - D(P_{XZ}\|U_X \times P_Z)\,. \tag{1.59}$$

or simply $D(P_{XY}\|\, U_X \times P_Y) \geq D(P_{XZ}\|U_X \times P_Z)$. But this is imply the DPI applied to the channel $P_{Z|Y}$ that happens to leave X untouched. □