EE5137 : Stochastic Processes (Spring 2021)
Some Additional Examples in Detection Theory and Hypothesis Testing

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March 13, 2021

In this document, we provide some supplementary material to supplement the lectures on detection theory. You need to know everything here.

1 Discrete Detection Example Requiring Randomization

Let K be the up time of a communications link in days. On a particular day, a link has a probability of q of failing. Under the two hypotheses, we have

$$H_0 : q = q_0 = \frac{1}{2}, \quad H_1 : q = q_1 = \frac{1}{4}.$$

We want to decide based on observing K whether H_0 or H_1 is true. The likelihood functions are as follows

$$p_{K|H}(k|H_j) = \begin{cases} q_0(1-q_0)^k & j=0 \\ q_1(1-q_1)^k & j=1 \end{cases}, k = 0, 1, \ldots.$$

That is under each hypothesis K is a geometric distribution.

1. If $p_0 = p_1$ (the priors of H_0 and H_1 are the same), find the minimum probability of error rule.

   We have that

   $$L(k) = \frac{q_1(1-q_1)^k}{q_0(1-q_0)^k} \overset{\hat{H}=H_1}{\gtrless} 1$$

   This is equivalent to

   $$k \overset{\hat{H}=H_1}{\gtrless} \frac{\log(q_0/q_1)}{\log((1-q_1)/(1-q_0))} = 1.71$$

   Thus, we decide in favor of H_1 if $k \geq 2$ and in favor of H_0 otherwise.

2. Plot the operating characteristic.

   To do so, we compute the probability of false alarm and probability of detection for all thresholds. We have

   $$P_{\mathrm{FA}} = \sum_{i=\gamma}^{\infty} q_0(1-q_0)^i = \left(\frac{1}{2}\right)^\gamma$$

   and

   $$P_{\mathrm{D}} = \sum_{i=\gamma}^{\infty} q_1(1-q_1)^i = \left(\frac{3}{4}\right)^\gamma$$

   The ROC is a set of countable points at

   $$(1,1), \left(\frac{1}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, \frac{9}{16}\right), \left(\frac{1}{8}, \frac{27}{64}\right), \ldots$$

3. Find the maximum P_{D} subject to a false alarm probability of $P_{\text{FA}} \le \alpha = 0.01$. How does the decision rule look like to attain this $(P_{\text{FA}}, P_{\text{D}})$?

   Note that $0.01 \in (\frac{1}{128}, \frac{1}{64}) = (\frac{1}{2^7}, \frac{1}{2^6})$. Thus, we need to perform a randomized test between these two operating points on the ROC. Let's figure out the proportion we need to use of each test. We have

$$P_{\text{FA}} = 0.01 = p\frac{1}{128} + (1-p)\frac{1}{64},$$

   which means after a simple calculation that $p = 0.72$. Thus the maximum P_{D} is

$$P_{\text{D}} = p\left(\frac{3}{4}\right)^7 + (1-p)\left(\frac{3}{4}\right)^6 = 0.1459.$$

   We will flip a coin C with the following outcomes:

$$C = \begin{cases} \mathsf{H} & \text{w.p. } p \\ \mathsf{T} & \text{w.p. } 1-p \end{cases}$$

   The inferred hypothesis is

$$\hat{H} = \begin{cases} H_1 & k \ge 7, C = \mathsf{H} \\ H_0 & k < 7, C = \mathsf{H} \\ H_1 & k \ge 6, C = \mathsf{T} \\ H_0 & k < 6, C = \mathsf{T} \end{cases} = \begin{cases} H_1 & k \ge 7 \\ H_0 & k < 6 \\ H_1 & k = 6 \text{ w.p. } 1-p \\ H_0 & k = 6 \text{ w.p. } p \end{cases}$$

2 Binary Communication System

We have an equally likely *message* $m \in \{0,1\}$. The *signal* $s_m = -\sqrt{\gamma}$ if $m = 0$ and $s_m = \sqrt{\gamma}$ if $m = 1$. The *receiver* $y = s_m + w$ where w is distributed as $\mathcal{N}(0, \sigma^2)$. We would like to decide based on y whether the message sent is $m = 0$ or $m = 1$. Based on the observation y, the receiver produces $\hat{m}$ obeying:

$$\hat{m} = \begin{cases} 0 & \text{if it thinks 0 was sent} \\ 1 & \text{if it thinks 1 was sent} \\ e & \text{if it thinks the received data is too noisy for reliable decoding} \end{cases}$$

The associated costs are as follows: $C_{00} = C_{11} = 0, C_{10} = C_{01} = 1, C_{e0} = C_{e1} = \frac{1}{4}$. Find the minimum average cost decision rule. First we note that

$$p_{Y|H}(y|H_0) = \mathcal{N}(y; -\sqrt{\gamma}, \sigma^2), \quad p_{Y|H}(y|H_1) = \mathcal{N}(y; \sqrt{\gamma}, \sigma^2)$$

Define $\varphi(\hat{H}_i|y)$ to be the average cost in deciding $\hat{H}_i$ ($i \in \{0, 1, e\}$) given an observation y. Then it is easy to show that

$$\varphi(\hat{H}_0|y) = \mathbb{E}[\tilde{C}(H, \hat{H}_0)|Y = y] = C_{00}p_{H|Y}(H_0|y) + C_{01}p_{H|Y}(H_1|y).$$

Please verify that you understand that the above sum is over 2 (not 3) terms (corresponding to the two original hypotheses) and that the only random quantity in the expectation is H. Thus, we have

$$\varphi(\hat{H}_0|y) = \frac{p_{Y|H}(y|H_1)\Pr(H_1)}{p_{Y|H}(y|H_1)\Pr(H_1) + p_{Y|H}(y|H_0)\Pr(H_0)} = \frac{1}{1 + \exp(-\frac{2}{\sigma^2}\sqrt{\gamma}y)}.$$

This is the *sigmoid* function. Similarly, by symmetry

$$\varphi(\hat{H}_1|y) = \frac{1}{1 + \exp(\frac{2}{\sigma^2}\sqrt{\gamma}y)}.$$

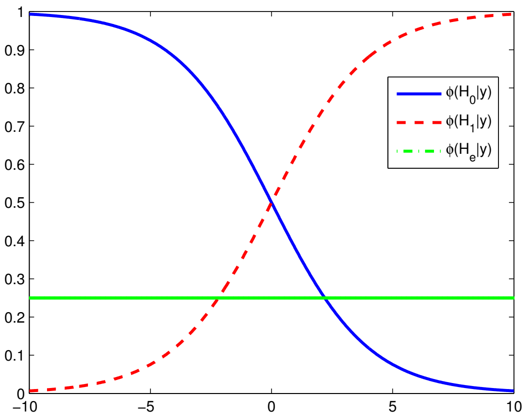

Figure 1: Decision regions for the binary communications problem for $\gamma = 1$ and $\sigma = 2$.

Furthermore, we also have (why? check!)

$$\varphi(\hat{H}_e|y) = \frac{1}{4}$$

Thus, the regions $Z_i := \{y : \hat{m}(y) = i\}$ for $i \in \{0, 1, e\}$ are given as

$$Z_0 = (-\infty, \alpha], \qquad Z_e = (-\alpha, \alpha), \qquad Z_1 = [\alpha, \infty).$$

where α is the solution of the equation

$$\frac{1}{4} = \frac{1}{1 + \exp(\frac{2}{\sigma}\sqrt{\gamma}\alpha)}.$$

See Fig. 1.

3 Resistor example

We have two resistors whose lifetimes R_i have pdfs

$$r_i(t) = \lambda \exp(-\lambda t) u(t)$$

where $u(t) = 1$ if $t \geq 0$ and 0 otherwise. Here is the hypothesis testing problem

$$H_0 : \text{resistors are connected in series}$$
$$H_1 : \text{resistors are connected in parallel}$$

We somehow know that $\Pr(H_0) = \Pr(H_1) = 1/2$. Let Y be the length of time before the light bulb goes off.

1. Find $p_{Y|H}(y|H_i)$ for $i = 0, 1$.

$$Y|H_0 \overset{d}{=} \min\{R_1, R_2\}, \qquad Y|H_1 \overset{d}{=} \max\{R_1, R_2\}$$

Now, let me just derive $p_{Y|H}(y|H_0)$. First we consider the conditional cdf.

$$\Pr(Y > y|H = H_0) = \Pr(\min\{R_1, R_2\} \geq y) \overset{(a)}{=} \Pr(R_1 \geq y)\Pr(R_2 \geq y) = \left(\int_y^\infty \lambda e^{-\lambda t}\, dt\right)^2 = e^{-2\lambda y}$$

Justify (a). Thus,

$$\Pr(Y \leq y | H = H_0) = 1 - e^{-2\lambda y}, \qquad p_{Y|H}(y|H_0) = 2\lambda e^{-2\lambda y} u(y).$$

Using a similar calculation (this is a good exercise),

$$p_{Y|H}(y|H_1) = 2\lambda(e^{-\lambda y} - e^{-2\lambda y}) u(y).$$

2. Specify a decision rule to minimize the error probability. Since the a-priori probabilities are equal, this reduces to maximum-likelihood (ML):

$$p_{Y|H}(y|H_0) \gtrless p_{Y|H}(y|H_1) \quad \Rightarrow \quad y \gtrless \frac{1}{\lambda} \log 2.$$

More precisely, we decide that $H = H_0$ if $y < \frac{1}{\lambda} \log 2$ and vice versa.

3. Find the probability of false alarm and the probability of detection

$$P_{\mathrm{FA}} = \Pr(\hat{H} = H_1 | H = H_0) = \int_{\frac{\log 2}{\lambda}}^{\infty} p_{Y|H}(y|H_0)\, dy = \frac{1}{4}, \qquad P_{\mathrm{D}} = \Pr(\hat{H} = H_1 | H = H_1) = \frac{3}{4}.$$