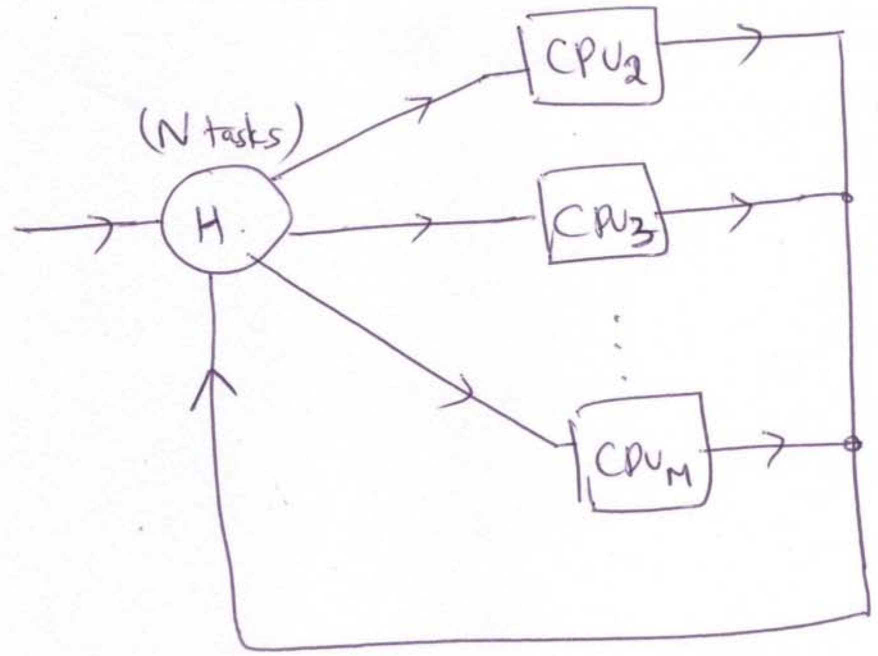
Availability / Throughput maximization for Large Scale Multiprocessor Systems



- CPUs are/need not (be) homogeneous, i.e., processing rates need not be identical
- A task can be assigned to any CPU for processing
- After processing the jobs are sent back to Host H.
- X : Throughput, ratio of jobs @ (H) to total # of tasks N in the system.

$\Rightarrow$ We need to maximize X.

N: # of tasks in the system

M: # of CPUs including the host

N_j : Expected # of tasks @ CPU_j , $j = 1, \cdots, M$

μ_j : mean service rate of CPU_j, assumed to be a poisson process

p_{1j} : probability that a task is assigned to CPU_j.

$\boxed{\text{model}}$ Closed-Queueing Model

- After processing, the jobs are sent back to (H) $(CPU_1) \Rightarrow p_{j1} = 1$ for $j = 2, \cdots, M$.
- p_{1j} are the decision variables
- State of the multiprocessor system $N = (N_1, N_2, \cdots, N_M)$

From the literature, the steady-state distribution of tasks is given by:

$$Pr\{\bar{N}\} = \frac{1}{G(N)} \prod_{j=1}^{M} (X_i)^{N_i} \quad \ldots (1)$$

$$\mu_1 X_1 = \sum_{j=2}^{M} (\mu_j \, X_j \cdot P_{1j}) \quad \ldots (2)$$

$$\mu_j X_j = \mu_1 X_1 p_{1j} \quad \ldots (3)$$

where, $(X_1, \cdots, X_M)$ is the real-positive solution to the eigenvector-like equations & $G(N)$ – normalizing constant.

Also, from the literature,

$$N_j = \sum_{k=1}^{N} \left\{ \frac{G(N-k)}{G(N)} \right\} X_j^k, \; j = 1, \cdots, M \quad (4)$$

& Utilization of CPU_j is,

$$U_j = \left\{ \frac{G(N-1)}{G(N)} \right\} X_j, \; j = 1, \cdots, M \quad (5)$$

Thus, throughput X can be obtained as,

$$X = \left(\frac{N_1}{N}\right)$$

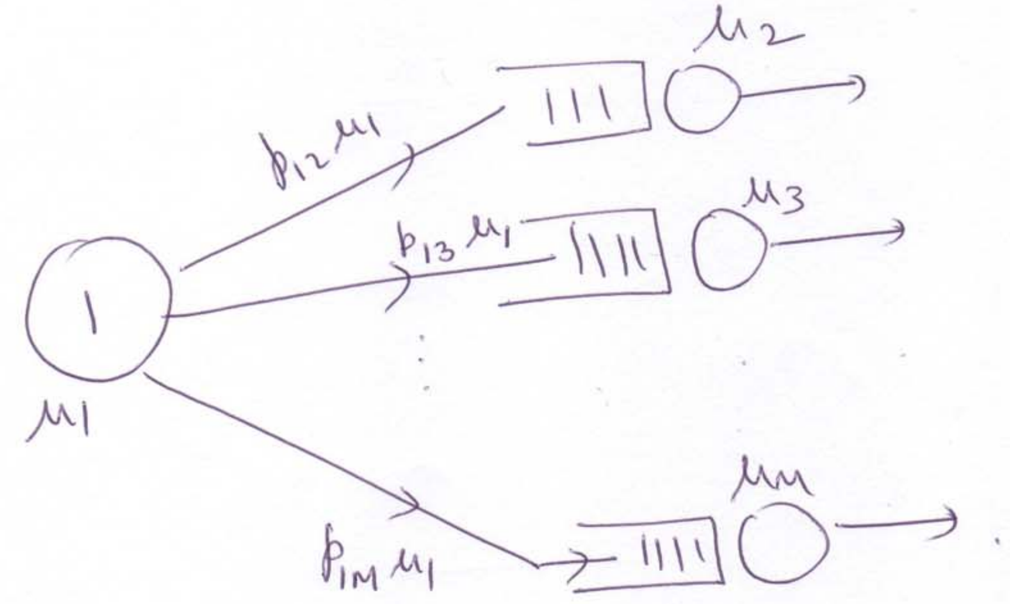
$$= \frac{1}{N} \sum_{k=1}^{N} \left\{ \frac{G(N-k)}{G(N)} \right\} X_1^k \quad (6)$$

Example: $\mu_1 = 20$, $\mu_2 = 40$, $\mu_3 = 80$, $N = 5$

(A) $p_{12} = 0.8$, $p_{13} = 0.2$, $N_1 = 4.3072$
$\Rightarrow X = 86.14\%$

(B) $p_{12} = 0.6$, $p_{13} = 0.4$, $N_1 = 4.4661 \Rightarrow X = 89.32\%$

- Computing (p_{ij}) is a difficult task.
- It is possible to model an equivalent open-queueing system, for a closed-Q model, as a limited-buffer open-Q system.
- Since N is finite, we can consider the open-queueing system to have queues of finite buffer length N



Thus, for this open-Q system,

$$N_j = \frac{p_{1j}\mu_j}{\mu_j - p_{1j}\mu_1} - \left[\frac{(N+1)\left(\frac{p_{1j}\mu_1}{\mu_j}\right)^{N+1}}{1-\left(\frac{p_{1j}\mu_1}{\mu_j}\right)^{N+1}} \right] \quad (7)$$

Thus, expected # of tasks @ CPU_1 is,

$$N_1 = N - \sum_{j=2}^{M} N_j \quad \text{——— (8)}$$

$$\Rightarrow x_1 = \left(\frac{N_1}{N}\right) = \left(\frac{N - \sum_{j=2}^{M} N_j}{N}\right) \quad \text{——— (9)}$$

From (7) we observe that for large N, we approximate N_j as,

$$(10) \leftarrow N_j = \frac{p_{1j}\,\mu_1}{\mu_j - (p_{1j}\,\mu_1)} \quad \left\{ \begin{array}{l} \text{2nd term for} \\ \text{large } N \text{ becomes} \\ \text{negligible.} \end{array} \right.$$

Example: $\mu_1 = 20$, $\mu_2 = 40$, $\mu_3 = 80$, $N = 5$
$p_{12} = 0.6$, $p_{13} = 0.4$

Using (7), we have, $x = 89.29\%$
Using (10), we have, $x_1 = 89.20\%$
} Very small error.

For this method to be valid, following condition must hold:

$$\boxed{\mu_j - p_{1j}\,\mu_1 > 0}$$

$$\Rightarrow \; p_{ij} < (\mu_j/\mu_1) \quad \text{———} \quad (11)$$

further, since $\sum_{j=2}^{M} p_{ij} = 1$ the equation

(11) also $\Rightarrow$ $\mu_1 < \sum_{j=2}^{M} \mu_j$ ..... (12)

Thus it is necessary for μ_j to satisfy (12) for the system to be valid.

Thus, for the approximate analysis, we consider each CPU_j to be an independent M/M/1 queue with arrival rate $p_{ij}\mu_1$ & service rate μ_j.

The expected # of tasks @ CPUs 2 to M is :

$$\sum_{j=2}^{M} N_j = \sum_{j=2}^{M} \left(\frac{p_{ij}\mu_1}{\mu_j - p_{ij}\mu_1} \right) \quad \text{---} (13)$$

The optimal p_{ij} are the solution to the following constrained minimization problem posed as :

$$\boxed{\begin{aligned}&\text{minimize } \sum_{j=2}^{M}\left\{\frac{p_{1j}\mu_1}{\mu_j - p_{1j}\mu_1}\right\}\\ &\text{subject to } \sum_{j=2}^{M} p_{1j} = 1\\ &\qquad p_{1j} \geq 0,\ j = 2, \ldots, M\end{aligned}} \quad (14)$$

Note that we are interested in choosing a distribution p_{1j} that satisfies (11).

The augmented cost function could be written as:

$$L = \sum_{j=2}^{M}\left\{\frac{p_{1j}\mu_1}{\mu_j - p_{1j}\mu_1}\right\} - K\left\{\sum_{j=2}^{M} p_{1j} - 1\right\} - \sum_{j=2}^{M} L_j p_{1j} \quad (15)$$

where, K & L_j $(j = 2, \ldots, M)$ are Lagrangian multipliers & L_j is such that,

$$\left.\begin{aligned} &L_j \geq 0 \text{ when } p_{1j} = 0\\ &L_j = 0, \text{ when } p_{1j} \neq 0\end{aligned}\right\} \quad (16)$$

Thus,

$\boxed{\text{Solution}}$

$$\frac{\partial L}{\partial p_{1j}} = \frac{\mu_1 \mu_j}{(\mu_j - p_{1j}\mu_1)^2} - K - L_j = 0 \quad -(17)$$

Since $L_j \neq 0$ only if $p_{1j} = 0$, we have

$$L_j = \left(\frac{\mu_1}{\mu_j}\right) - K \quad \text{for all } j \notin \{p_{1j} > 0\}$$

If $p_{1j} \neq 0$, we have $L_j = 0$. Hence from (17) for all $j \in \{p_{1j} > 0\}$,

$$\frac{\mu_1 \mu_j}{(\mu_j - p_{1j}\mu_1)^2} = K$$

or $$p_{1j} = \frac{\mu_j}{\mu_1} - \sqrt{\frac{\mu_j}{\mu_1 K}}, \quad \text{for } j \in \{p_{1j} > 0\} \quad (18)$$

Since $\sum p_{1j} = 1 \quad \forall j \in \{p_{1j} > 0\}$, we have,

$$\sum \left(\frac{\mu_j}{\mu_1} - \sqrt{\frac{\mu_j}{\mu_1 K}}\right) = 1, \quad j \in \{p_{1j} > 0\}$$

This implies,

(9)

$$k = \mu_1 \left[\frac{\sum_{\forall j} \sqrt{\mu_j}}{\sum_{\forall j} \mu_j - \mu_1} \right] , \quad j \in \{ p_{1j} > 0 \} \longrightarrow (19)$$

Using (19) in (18) we have,

$$p_{1j} = \frac{\mu_j}{\mu_1} - \frac{\sqrt{\mu_j} \left(\sum_{\forall j} \mu_j - \mu_1 \right)}{\mu_1 \sum_{\forall j} \sqrt{\mu_j}} , \quad j \in \{ p_{1j} > 0 \} \quad (20)$$

Clearly (20) satisfies (11) & (12).