Exercise 4.4 $H(X,Y) = \lim_{n\to\infty} \frac{1}{n} \cdot H(X^1, X^2 \cdots X^n, Y^1, Y^2 \cdots Y^n)$

$H(X) = \lim_{n\to\infty} \frac{1}{n} H(X^1, X^2, \cdots X^n)$

$H(Y) = \lim_{n\to\infty} \frac{1}{n} H(Y^1, Y^2, \cdots Y^n)$

$H(X|Y) = H(X,Y) - H(Y)$ $\quad R(Y|X) = H(X,Y) - H(X)$

And, we also know $\lim_{n\to\infty} P\{(\hat{X}^n, \hat{Y}^n) \neq (X^n, Y^n)\} = 0$

From the conclusion $[R X_1 \geq H(X)]$ is achievable. ... ①

For (1)

$H(X) - H(X|Y) = \sum_{x\in X} P_X \log \frac{1}{P_X} - \sum_{x\in X, Y\in Y} P_{X|Y} \log \frac{1}{P_{X|Y}}$

$= \sum_{X\in X, Y\in Y} P_Y \cdot P_{X|Y} \log \frac{1}{P_{X|Y}} \frac{1}{P_Y} - \sum_{X\in X, Y\in Y} P_{X|Y} \log \frac{1}{P_{X|Y}}$

$= \sum_{X\in X, Y\in Y} P_{X|Y} \log \frac{P_{X,Y}}{P_{X|Y}}$.

Therefore $H(X) - H(X|Y) \geq 0$ ... ②

Combined with ① and ②, we got $R_1 \geq H(X|Y)$

For (2)

This proof is similar, so, we also can get $R_2 \geq H(Y|X)$

For (3) $R_1 + R_2 \geq H(X) + H(Y)$ ..... ③

$H(X) + H(Y) - H(X,Y) = \sum_{X\in X} P_X \log \frac{1}{P_X} + \sum_{Y\in Y} P_Y \log \frac{1}{P_Y} - \sum_{X\in X}\sum_{Y\in Y} P_{X,Y} \log \frac{1}{P_{X,Y}}$

$= \sum_{X\in X}\sum_{Y\in Y} \log \frac{P_{X,Y}}{P_X \cdot P_Y}$.

≥ 0

So, $H(X) + H(Y) \geq H(X,Y)$ ... ④

Combined with ③ and ④, we can get $R_1 + R_2 \geq H(X,Y)$