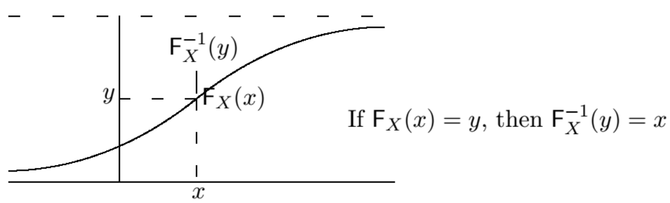
# EE5137 Stochastic Processes: Problem Set 3
## Assigned: 29/01/21, Due: Never

All problems here are optional. Please use these practice problems to prepare for the quiz.

1. Exercise 1.28 (Gallager's book) Suppose the rv $X$ is continuous and has the CDF $F_X(x)$. Consider another rv $Y = F_X(X)$. That is, for each sample $\omega$ such that $X(\omega) = x$, we have $Y(\omega) = F_X(x)$. Show that $Y$ is uniformly distributed in the interval 0 to 1.
   **Solution** For simplicity, first assume that $F_X(x)$ is strictly increasing in $x$, thus having the following appearance:

   Since $F_X(x)$ is continuous in $x$ and strictly increasing from 0 to 1, there must be an inverse function $F_X^{-1}(x)$ such that for each $y \in (0,1), F_X^{-1}(y) = x$ for that x such that $F_X(x) = y$. For this $y$, then, the event $\{F_X(X) \le y\}$ is the same as the event $\{X \le F_X^{-1}(y)\}$. This is illustrated in the above figure. Using this equality for the given $y$,

   $$\Pr\{Y \le y\} = \Pr\{F_X(X) \le y\} = \Pr\{X \le F_X^{-1}(y)\} \tag{1}$$
   $$= F_X(F_X^{-1}(y)) = y, \tag{2}$$

   where in the final equation, we have used the fact that $F_X^{-1}$ is the inverse function of $F_X$. This relation, for all $y \in (0,1)$, shows that $Y$ is uniformly distributed between 0 and 1.

   If $F_X$ is not strictly increasing, i.e., if there is any interval over which $F_X(x)$ has a constant value $y$, then we can define $F_X^{-1}(y)$ to have any given value within that interval. The above argument then still holds, although $F_X^{-1}$ is no longer the inverse of $F_X$.

   If there is any discrete point, say $z$ at which $\Pr\{X = z\} > 0$, then $F_X(x)$ cannot take on values in the open interval between $F_X(z) - a$ and $F_X(z)$ where $a = \Pr\{X = z\}$. Thus $F_X$ is uniformly distributed only for continuous rv's.

2. Exercise 1.32 (Gallager's book) (The one-sided Chebyshev inequality) This inequality states that if a zero-mean rv $X$ has a variance $\sigma^2$, then it satisfies the inequality

   $$\Pr\{X \ge b\} \le \frac{\sigma^2}{\sigma^2 + b^2} \quad \text{for every} \quad b > 0, \tag{3}$$

   which equality for some $b$ only if $X$ is binary and $\Pr\{X = b\} = \sigma^2/(\sigma^2 + b^2)$. We prove this here using the same approach as in Excercise 1.31. Let $X$ be a non-zero rv that satisfies $\Pr\{X \ge b\} = \beta$ for some given $b > 0$ and $0 < \beta < 1$. The variance $\sigma^2$ of $X$ can be expressed as

$$\sigma^2 = \int_{-\infty}^{b-} x^2 f_X(x)dx + \int_b^{\infty} x^2 f_X(x)dx. \tag{4}$$

We will first minimize $\sigma^2$ over all zero-mean $X$ satisfying $\Pr\{X \geq b\} = \beta$.

(a) Show that the second integral in (4) satisfies $\int_b^\infty x^2 f_X(x)dx \geq b^2\beta$.

(b) Show that the first integral in (4) is constrained by

$$\int_{-\infty}^{b-} f_X(x)dx = 1 - \beta, \quad \text{and} \quad \int_{-\infty}^{b-} x f_X(x)dx \leq -b\beta. \tag{5}$$

(c) Minimize the first integral in (4) subject to the constraint in (b). Hint: If you scale $f_X(x)$ up by $1/(1-\beta)$, it integrates to 1 over $(-\infty, b)$ and the second constraint becomes an expectation. You can then minimize the first integral in (4) by inspection.

(d) Combine the results in a) and c) to show that $\sigma^2 \geq b^2\beta/(1-\beta)$. Find the minimizing distribution. Hint: It is binary.

(e) Use (d) to establish (3). Also show (trivially) that if $Y$ has a mean $\bar{Y}$ and variance $\sigma^2$, then $\Pr\{Y - \bar{Y} \geq b\} \leq \sigma^2/(\sigma^2 + b^2)$.

**Solution:**

(a) We first explain why we started trying to establish an upper bound to $\Pr\{X \geq b\}$ and then switched to minimizing the variance such that $\bar{X} = 0$ and $\Pr\{X \geq b\} = \beta$ for some given $\beta$. We will find that $\sigma^2_{\min} = b^2\beta/(1-\beta)$, or, equivalently, $\beta = \sigma^2_{\min}/(b^2 + \sigma^2_{\min})$. Thus for each zero-mean $X$ satisfying $\Pr\{X \geq b\} = \beta$, we have (4), i.e.,

$$\Pr\{X \geq b\} = \beta = \frac{\sigma^2_{\min}}{\sigma^2_{\min} + b^2} \leq \frac{\sigma^2}{\sigma^2 + b^2}. \tag{6}$$

Solving part (a),

$$\int_b^\infty x^2 f_X(x) \geq \int_b^\infty b^2 f_X(x)dx = b^2 F^c_X(b) = b^2\beta. \tag{7}$$

(b) $\int_{-\infty}^{b-} f_X(x)dx = 1 - \int_b^\infty f_X(x)dx = 1 - \beta$. Similarly, since $\bar{X} = 0$,

$$\int_{-\infty}^{b-} x f_X(x)dx = 0 - \int_b^\infty x f_X(x)dx \leq -\int_b^\infty b f_X(x)dx = -b\beta. \tag{8}$$

(c) Let $h(x) = f_X(x)/(1-\beta)$ over $x \leq b$ and $h(x) = 0$ elsewhere. Then $h(x)$ is a PDF and the corresponding mean is $-b\beta/(1-\beta)$. The corresponding second moment is lower bounded by the square of that mean, so

$$\int_{-\infty}^{b-} x^2 h(x)dx \geq \frac{b^2\beta^2}{(1-\beta)^2}. \tag{9}$$

Since $h(x) = f_X(x)/(1-\beta)$, we have $\int_{-\infty}^{b-} x^2 f_X(x)dx \geq b^2\beta^2/(1-\beta)$.

(d) Substituting the results of (a) and (c) into (4)

$$\sigma^2 \geq \frac{b^2\beta^2}{1-\beta} + b^2\beta = \frac{b^2\beta}{1-\beta}. \tag{10}$$

This is met with equality when the inequalities in (a) and (c) are met with equality. Thus $X = b$ whenever $X \geq b$ and $X = -b\beta/(1-\beta)$ when $X < b$. Since $\beta$ is the probability that $X \geq b$, we see that $p_X(b) = \beta$ and $p_X(-b\beta/(1-\beta)) = 1 - \beta$.

(e) From (10), $\sigma^2(1-\beta) \geq b^2\beta$, from which $\beta \leq \sigma^2/(b^2+\sigma^2)$, i.e., $\Pr\{X \geq b\} \leq \sigma^2/(b^2+\sigma^2)$. The conditions for equality are clearly the same as in (d). The final result follows by letting $X$ be the fluctuation of $Y$ and applying (3) to $X$.

3. Exercise 1.43 (Gallager's book) (**MS convergence $\rightarrow$ convergence in probability**) Assume that $\{Z_n : n \geq 1\}$ is a sequence of rv's and $\alpha$ is a number with the property that $\lim_{n\to\infty} \mathbb{E}[(Z_n - \alpha)^2] = 0$.

   (a) Let $\varepsilon > 0$ be arbitrary and show that for each $n \geq 0$,

   $$\Pr\{|Z_n - \alpha| \geq \varepsilon\} \leq \frac{\mathbb{E}[(Z_n-\alpha)^2]}{\varepsilon^2}. \tag{11}$$

   (b) For the $\varepsilon$ above, let $\delta > 0$ be arbitrary. Show that there is an integer $m$ such that $\mathbb{E}[(Z_n-\alpha)^2] \leq \varepsilon^2\delta$ for all $n \geq m$.

   (c) Show that this implies convergence in probability.

   **Solution:**

   (a) It is a direct result of the Chebysev's inequality.

   (b) By the definition of a limit, $\lim_{n\to\infty} \mathbb{E}[(Z_n-\alpha)^2] = 0$ means that for all $\varepsilon_1 > 0$, there is an $m$ large enough that $\mathbb{E}[(Z_n-\alpha)^2] < \varepsilon_1$ for all $n \geq m$. Choosing $\varepsilon_1 = \varepsilon^2\delta$ established the desired result.

   (c) Substituting $\mathbb{E}[(Z_n-\alpha)^2] \leq \varepsilon^2\delta$ into (11), we see that for all $\varepsilon, \delta > 0$, there is an $m$ such that $\Pr\{|Z_n-\alpha|^2 \geq \varepsilon\} \leq \delta$ for all $n \geq m$. This is convergence in probability.

4. Exercise 1.48 (Gallager's book) Let $\{Y_n : n \geq 1\}$ be a sequence of rv's and assume that $\lim_{n\to\infty} \mathbb{E}[|Y_n|] = 0$. Show that $\{Y_n : n \geq 1\}$ converges to 0 in probability.

   **Solution**: Let $\epsilon > 0$ be fixed. Consider

   $$\Pr(|Y_n - 0| > \epsilon) \leq \frac{\mathbb{E}[|Y_n|]}{\epsilon}$$

   Since $\lim_{n\to\infty} \mathbb{E}[|Y_n|] = 0$, we have that

   $$\lim_{n\to\infty} \Pr(|Y_n - 0| > \epsilon) = 0$$

   which means that $Y_n \rightarrow 0$ in probability.

5. [Convergence of RVs] Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(1/2). Define $Y_n = 2^n \prod_{i=1}^n X_i$. Does $Y_n$ converge to 0 almost surely (with probability 1)? Does $Y_n$ converge to 0 in mean square?

   **Solution:** $Y_n$ converges to 0 a.s. Fix $m \in \mathbb{N}$ and then fix $0 < \epsilon < 2^m$. Then for this fixed $m$, and consider

   $$\Pr(|Y_n - 0| < \epsilon, \forall\, n \geq m) = \Pr(X_n = 0 \text{ for some } n \leq m) = 1 - \Pr(X_n = 1, \forall\, n \leq m) = 1 - (1/2)^m$$

   which converges to 1 as $m \rightarrow \infty$. Hence $Y_n \rightarrow 0$ a.s.

   $Y_n$ does not converge to 0 in m.s. Since

   $$\mathbb{E}[(Y_n - 0)^2] = (1/2)^n 2^{2n} = 2^n \rightarrow \infty,$$

   the sequence doesn't converge in m.s.

6. [Two Independent Random Variables]

   Let $X$ and $Y$ be independent random variables, uniformly distributed on $[0, 2]$.

(a) Find the mean and variance of $XY$.

(b) Calculate the probability $\Pr(XY \le 1)$.

You may use, without proof, the fact that for a uniform random variable on $[a, b]$, the variance is $(b-a)^2/12$.

**Solution:**

(a) The mean is

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 1 \times 1 = 1$$

Consider the second moment:

$$\mathbb{E}[(XY)^2] = \mathbb{E}[X^2]\mathbb{E}[Y^2] = [\mathsf{Var}(X) + (\mathbb{E}[X])^2][\mathsf{Var}(Y) + (\mathbb{E}[Y])^2] = (4/3)^2 = 16/9.$$

Hence the variance is

$$\mathsf{Var}(XY) = \mathbb{E}[(XY)^2] - (\mathbb{E}[XY])^2 = 16/9 - 1^2 = 7/9.$$

(b) We have

$$\Pr(XY \le 1) = \Pr(Y \le 1/X) \overset{(a)}{=} \int_0^2 f_X(x) \Pr(Y \le 1/x)\,\mathrm{d}x = \int_0^2 F_Y(1/x) f_X(x)\,\mathrm{d}x,$$

where (a) holds because $X$ and $Y$ are independent. Now observe that when $X \le 1/2$, then $Y \le 1/X$ with probability 1; and when $X \in (1/2, 2)$, then $Y \le 1/X$ with probability $(1/X)/2$. Thus,

$$\Pr(XY \le 1) = \int_0^{1/2} 1 \cdot \frac{1}{2}\,\mathrm{d}x + \int_{1/2}^2 \frac{1}{2x} \cdot \frac{1}{2}\,\mathrm{d}x = \frac{1}{4} + \frac{\log 2}{2}.$$

7. [Laws of Large Numbers]

(a) State Chebychev's Inequality.

(b) Suppose $\{X_i\}_{i=1}^{\infty}$ is a sequence of uncorrelated random variables (i.e., $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j]$ for $i \ne j$), each of which has finite mean, and assume that for all $n$, $\mathrm{Var}(X_n) \le M < \infty$. Define

$$\mu_n = \frac{1}{n}\sum_{i=1}^n \mathbb{E}[X_i].$$

and suppose that $\mu_n \to \mu$ as $n \to \infty$ and $|\mu| < \infty$. Let $S_n = \sum_{i=1}^n X_i$. Show that $S_n/n \to \mu$ in probability.

**Solution:**

(a) Suppose that the random variable $X$ has finite mean $\mu$. Then for any $c > 0$,

$$\Pr(|X - \mu| > c) \le \frac{\mathsf{Var}(X)}{c^2}.$$

(b) Choose any $\epsilon > 0$. There exists $N$ such that $|\mu_n - \mu| < \epsilon/2$ for all $n > N$. Thus for such $n > N$,

$$\begin{aligned}
\Pr\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) &\le \Pr\left(\left|\frac{S_n}{n} - \mu_n\right| + |\mu_n - \mu| > \epsilon\right) \\
&\le \Pr\left(\left|\frac{S_n}{n} - \mu_n\right| > \epsilon/2\right) \\
&\le \frac{4\,\mathsf{Var}(S_n/n)}{\epsilon^2} \le \frac{4M}{n\epsilon^2}
\end{aligned}$$

Thus for any $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) = 0$$

which implies $S_n/n \xrightarrow{\text{p}} \mu$.

8. [Convergence in Probability Implies Convergence in Distribution]

   Show if $\{X_n\}$ convergence in distribution to $X$, then $\{X_n\}$ converges in distribution to the same rv $X$.

   *Hint: First convince yourself that*

$$F_{X_n}(x) \le F_X(x+\epsilon) + \Pr(|X_n - X| > \epsilon)$$

   *Consider the "flipped" inequality and use the definition of convergence in probability.*

   **Solution**: First we note that

$$\{X_n \le x\} \subset \{X \le x + \epsilon\} \cup \{|X_n - X| > \epsilon\}.$$

   Why? Otherwise, if $X > x + \epsilon$ and $|X_n - X| \le \epsilon$ (equivalently $X - \epsilon \le X_n \le X + \epsilon$), then $X_n > x$. By the union bound, we have

$$F_{X_n}(x) \le F_X(x+\epsilon) + \Pr(|X_n - X| > \epsilon).$$

   In the same way, one can show that

$$F_X(x-\epsilon) - \Pr(|X_n - X| > \epsilon) \le F_{X_n}(x).$$

   From convergence in probability, we know that for any $\epsilon > 0$, $\Pr(|X_n - X| > \epsilon) \to 0$. Thus, we have

$$F_X(x-\epsilon) \le \liminf_{n\to\infty} F_{X_n}(x) \le \limsup_{n\to\infty} F_{X_n}(x) \le F_X(x+\epsilon),$$

   and this statement holds for all $\epsilon > 0$. If $x$ is a point of continuity of $F_X$, then $\lim_{\epsilon\to 0} F_X(x+\epsilon) = F_X(x)$. Thus, taking $\epsilon \to 0$ in the above display, and using the squeeze theorem (to be really pedantic), we get

$$F_X(x) = \lim_{n\to\infty} F_{X_n}(x)$$

   for all points of continuity $x$ of $F_X$, showing convergence in distribution.

9. [Probability Generating Function ]

   If $X$ is a non-negative integer-valued rv then the function $Q(z)$ defined for $|z| \le 1$ by

$$Q(z) = \mathbb{E}[z^X] = \sum_{j=0}^{\infty} z^j \Pr(X = j)$$

   is called the probability generating function of $X$.

   In this problem you may interchange the differentiation operation and the infinite sum operation. The reason we can do this (for the more mathematically inclined students) is due to the so-called differentiable limit theorem and Abel's theorem.

   (a) Show that

$$\left.\frac{d^k}{dz^k} Q(z)\right|_{z=0} = k! \Pr(X = k).$$

(b) With 0 considered even, show that

$$\Pr(X \text{ is even}) = \frac{Q(-1) + Q(1)}{2}.$$

(c) If $X$ is binomial with parameters $n$ and $p$, show that

$$\Pr(X \text{ is even}) = \frac{1 + (1-2p)^n}{2}.$$

(d) If $X$ is Poisson with mean $\lambda$, show that

$$\Pr(X \text{ is even}) = \frac{1 + e^{-2\lambda}}{2}.$$

**Solutions**:

(a) Consider the first derivative

$$\frac{d}{dz}\sum_{j=0}^{\infty} z^j p_X(j) = \sum_{j=0}^{\infty} \frac{d}{dz}(z^j p_X(j)) = \sum_{j=1}^{\infty} j z^{j-1} p_X(j)$$

Considering the $k^{\text{th}}$ derivative, we get

$$\frac{d^k}{dz^k}\sum_{j=0}^{\infty} z^j p_X(j) = \sum_{j=k}^{\infty} j(j-1)\dots(j-k+1) z^{j-k} p_X(j)$$

Plugging $z = 0$ into the above display yields

$$\frac{d^k}{dz^k}\sum_{j=0}^{\infty} z^j p_X(j)\bigg|_{z=0} = k(k-1)\dots 1 \cdot p_X(k) = k! \Pr(X = k).$$

(b) We consider

$$Q(-1) + Q(1) = \sum_{j=0}^{\infty}((-1)^j p_X(j) + p_X(j)) = 2\sum_{k=0}^{\infty} p_X(2k) = 2\Pr(X \text{ is even})$$

as desired.

(c) For a binomial distribution,

$$Q(z) = \sum_{j=0}^{n} z^j \binom{n}{j} p^j (1-p)^{n-j} = \sum_{j=0}^{n} \binom{n}{j} (pz)^j (1-p)^{n-j} = [pz + (1-p)]^n$$

Thus, $Q(1) = 1$ and $Q(-1) = (1-2p)^n$, demonstrating the desired result upon using part (b).

(d) For a Poisson distribution,

$$Q(z) = \sum_{j=0}^{\infty} z^j \frac{e^{-\lambda}\lambda^j}{j!} = \sum_{j=0}^{\infty} \frac{e^{-\lambda}(z\lambda)^j}{j!} = e^{-\lambda+\lambda z}\sum_{j=0}^{\infty} \frac{e^{-\lambda z}(z\lambda)^j}{j!} = e^{\lambda(z-1)}.$$

Thus, $Q(1) = 1$ and $Q(-1) = e^{-2\lambda}$, demonstrating the desired result upon using part (b).