Solutions to EE5137 Exam (Semester 2 2019/20)

April 28, 2020

1 Problem 1

(a) True. Law of iterated expectations.

(b) False. $\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y])$. This was proved in Quiz 2.

(c) False. $\gamma_X''(r)\Big|_{r=0} = \mathrm{Var}(X) \neq \mathbb{E}[X^2]$ if $\mathbb{E}X \neq 0$.

(d) True. By Chebyshev's inequality.

(e) True. Can directly verify that $\Pr(X > k + l \mid X > l) = \Pr(X > k)$ for any $k, l \in \mathbb{N}$.

(f) True. By splitting of Poisson process.

(g) False. We additionally need the independent increments property and the stationary increments property.

(h) False. It is recurrent.

(i) True. $\Pr(X_1 = 3, X_2 = 2, X_3 = 1) = \Pr(X_3 = 1 \mid X_2 = 2)\Pr(X_2 = 2 \mid X_1 = 3)\Pr(X_1 = 3) = 1/3 \times 1/2 \times (1 - 1/4 - 1/4) = 1/12$.

(j) True. Example 8.2.7 in the book says that $S_n = \sum_{i=1}^n Y_i$ is a sufficient statistic. Any one-to-one function of a sufficient statistic such as $\frac{1}{n-1}\sum_{i=1}^n Y_i$ is a sufficient statistic.

2 Problem 2

(a) Let $N(t)$ be a Poisson process with rate $\lambda = \lambda_1 + \lambda_2 = 3$. We split $N(t)$ into two processes $N_1(t)$ and $N_2(t)$ in the following way. For each arrival, a coin with $\Pr(H) = p = 1/3$ is tossed. If the coin lands heads up, the arrival is sent to the first process $N_1(t)$, otherwise it is sent to the second process. The coin tosses are independent of each other and are independent of $N(t)$. Then

- $N_1(t)$ is a Poisson process with rate $\lambda p = 1$.
- $N_2(t)$ is a Poisson process with rate $\lambda(1 - p) = 2$.
- $N_1(t)$ and $N_2(t)$ are independent.

Thus, $N_1(t)$ and $N_2(t)$ have the same probabilistic properties as the ones stated in the problem. We can now restate the probability that the second arrival in $N_1(t)$ occurs before the third arrival in $N_2(t)$ as the probability of observing at least two heads in four coin tosses, which is

$$\sum_{k=2}^{4} \binom{4}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{4-k}.$$

Hence, $m = 2, n = 4$ and $p = 1/3$.

(b) We compute the joint distribution of $N_A(t)$ and $N_B(t)$ by conditioning on $N(t)$:

$$\begin{aligned}
&\Pr(N_A(t) = n, N_B(t) = m) \\
&= \sum_{k=0}^{\infty} \Pr(N_A(t) = n, N_B(t) = m \mid N(t) = k) \Pr(N(t) = k) \\
&= \Pr(N_A(t) = n, N_B(t) = m \mid N(t) = n+m) \Pr(N(t) = n+m).
\end{aligned}$$

Now consider an arbitrary event that occurred during the interval $[0, t]$. If it had occurred at time s, the probability that it is of type-A is $Q(s)$. Hence, by the fact that the event occurred uniformly at random in the interval $[0, t]$, it follows that it is a type-A event with probability

$$p = \frac{1}{t} \int_0^t Q(s) \, \mathrm{d}s$$

independently of all the other events. Hence, $\Pr(N_A(t) = n, N_B(t) = m \mid N(t) = n+m)$ will just equal the probability of n successes and m failures in a total of $n+m$ independent trials when p is the probability of success of each trial. That is,

$$\Pr(N_A(t) = n, N_B(t) = m \mid N(t) = n+m) = \binom{n+m}{n} p^n (1-p)^m.$$

Using the fact that $N(t)$ is a Poisson with rate λt, we have

$$\begin{aligned}
\Pr(N_A(t) = n, N_B(t) = m) &= \frac{(n+m)!}{n!m!} p^n (1-p)^m e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!} \\
&= e^{-\lambda t p} \frac{(\lambda t p)^n}{n!} e^{-\lambda t(1-p)} \frac{(\lambda t(1-p))^m}{m!}
\end{aligned}$$

which completes the proof.

3 Problem 3

(a) The transition graph is

$$\begin{array}{ccccc}
0 & \leftarrow & 3 & & 2 \\
\updownarrow & \swarrow & \updownarrow & \searrow & \updownarrow \\
1 & \leftarrow & 5 & \leftrightarrow & 4
\end{array}$$

Clearly, $\{0, 1\}$ is a class and $\{2, 3, 4, 5\}$ is also a class.

(b) The class $\{0, 1\}$ is recurrent. The class $\{2, 3, 4, 5\}$ is transient.

(c) Since state 5 is transient, it will only be visited finite number of times. So the long run proportion of time that state 5 is visited is 0.

(d) Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_5)$ be the stationary distribution when we start anywhere for this ergodic unichain. Clearly, $\pi_2 = \pi_3 = \pi_4 = \pi_5 = 0$. To determine $\bar{\boldsymbol{\pi}} = (\pi_0, \pi_1)$, we solve the equation

$$\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}[P],$$

or

$$\pi_0 = 0.5\pi_0 + 0.3\pi_1 \qquad \pi_1 = 0.5\pi_0 + 0.7\pi_1,$$

which is the same as $0.5\pi_0 = 0.3\pi_1$. Since $\pi_0 + \pi_1 = 1$, we have

$$\pi_0 + \frac{0.5}{0.3}\pi_0 = 1$$

or

$$\pi_0 = \frac{3}{8} \quad \text{and} \quad \pi_1 = \frac{5}{8}.$$

Thus, the stationary distribution is

$$\boldsymbol{\pi} = \left(\frac{3}{8}, \frac{5}{8}, 0, 0, 0, 0\right).$$

(e) We have to find the eigenvalue with the second largest magnitude for the transition matrix

$$[Q] = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}$$

For this purpose

$$\det([Q] - \lambda[I]) = 0, \quad \Longrightarrow \quad (0.5 - \lambda)(0.7 - \lambda) - 0.15 = 0.$$

Solving this yields $\lambda_1 = 1$ and $\lambda_2 = 0.2$. Thus $\phi = 0.2$.

4 Problem 4

(a) Let $\mathbf{X} = (X_0, X_1, X_2, X_3)$. The likelihood under H_0 is

$$\begin{aligned} P_{\mathbf{X}|H_0}(\mathbf{x}) &= \Pr(X_0 = x_0) \prod_{i=1}^{3} \Pr(X_i = x_i \mid X_{i-1} = x_{i-1}) \\ &= 1 \cdot \prod_{i=1}^{3} (1 - \theta_0)^{\mathbb{1}\{x_{i-1} = x_i\}} \theta_0^{\mathbb{1}\{x_{i-1} \neq x_i\}} \\ &= \prod_{i=1}^{3} (1 - \theta_0)^{y_i} \theta_0^{1 - y_i} \\ &= \theta_0^3 \left(\frac{1 - \theta_0}{\theta_0}\right)^{\sum_{i=1}^{3} y_i} \end{aligned}$$

Similarly,

$$P_{\mathbf{X}|H_1}(\mathbf{x}) = \theta_1^3 \left(\frac{1 - \theta_1}{\theta_1}\right)^{\sum_{i=1}^{3} y_i}$$

The LRT states that we decide in favor of H_1 if

$$\log L(x_0, x_1, x_2, x_3) = \log \frac{P_{\mathbf{X}|H_1}(\mathbf{x})}{P_{\mathbf{X}|H_0}(\mathbf{x})} \geq \eta$$

for some $\eta \in \mathbb{R}$. This is equivalent to

$$3 \log \frac{\theta_1}{\theta_0} + \left(\sum_{i=1}^{3} y_i\right) \left(\log \frac{1 - \theta_1}{\theta_1} - \log \frac{1 - \theta_0}{\theta_0}\right) \geq \eta.$$

Since the test is in terms of $T = \sum_{i=1}^{3} y_i$ and nothing else, T is a sufficient statistic for discriminating between the two hypotheses.

(b) Since $\theta_0 = 1/2$ and $\theta_1 = 1/4$, the log-likelihood ratio is

$$\log L(X_0, X_1, X_2, X_3) = 3 \log \frac{1}{2} + T \log 3.$$

Hence, the likelihood ratio is

$$L(X_0, X_1, X_2, X_3) = \frac{1}{8} \cdot 3^T$$

This means that

$$L(X_0, X_1, X_2, X_3) = \begin{cases} 1/8 & \text{if } T = 0 \\ 3/8 & \text{if } T = 1 \\ 9/8 & \text{if } T = 2 \\ 27/8 & \text{if } T = 3 \end{cases}.$$

(c) If $P(H_0) = 1/2$, this means we employ ML decoding, i.e., the threshold on the likelihood ratio is 1. Thus, we declare H_0 is true if $T \le 1$ and H_1 is true if $T > 1$. We have the probability of false alarm being

$$P_0(\text{declare } H_1) = P_0(T > 1) = P_0(T = 2) + P_0(T = 3) = 3(\frac{1}{2})^3 + (\frac{1}{2})^3 = \frac{1}{2}.$$

We also have the probability of mis-detection being

$$P_1(\text{declare } H_1) = P_1(T \le 1) = P_1(T = 0) + P_1(T = 1) = (\frac{1}{4})^3 + 3(\frac{3}{4})(\frac{1}{4})^2 = \frac{10}{64}.$$

Thus, the Bayesian error probability is

$$P_{\text{err}} = \frac{1}{2} P_0(T > 1) + \frac{1}{2} P_1(T \le 1) = \frac{21}{64}.$$

(d) Note that the likelihood ratio test simplifies to deciding in favor of H_1 if

$$L(X_0, X_1, X_2, X_3) = \frac{1}{8} \cdot 3^T \ge \eta'.$$

We want $P_0(\text{declare } H_1) \le 1/8$. To do so, we need to put the threshold $\eta' \in (9/8, 27/8)$ and declare that if $T > 2$ (i.e., $T = 3$), then H_1 is declared. This results in $P_0(T > 2) = P_0(T = 3) = (1/2)^3 = 1/8$.

The corresponding largest probability of detection is $P_1(\text{declare } H_1) = P_1(T > 2) = P_1(T = 3) = (3/4)^3 = 27/64$, hence the smallest probability of missed detection is $1 - 27/64 = 37/64$.

(e) We now want $P_0(\text{declare } H_1) \le 1/6$. We consider that $P_0(T > 1) = P_0(T = 2 \text{ or } T = 3) = 1/2$ and the corresponding probability of detection is $P_1(T > 1) = (3/4)^3 + 3(3/4)^2(1/4) = 27/32$. Hence, we need to randomize between the strategy that places the threshold at $T > 2$ and $T > 1$. Now we find an $\alpha \in [0, 1]$ such that

$$\alpha \frac{1}{8} + (1 - \alpha) \frac{1}{2} = \frac{1}{6} \quad \Longrightarrow \quad \alpha = \frac{8}{9}$$

Thus, we use the decision rule corresponding to declaring H_1 is true if $T > 2$ for a fraction of $8/9$. Thus, the best probability of detection is

$$\alpha \frac{27}{64} + (1 - \alpha) \frac{27}{32} = \frac{15}{32}$$

The best probability of mis-detection is $17/32$ and the optimal test in terms of T would be to randomize between $T > 2$ and $T > 1$ where the former has probability $8/9$.