Solving this set of $m + n$ equations in the $m + n$ variables $x^\star$, $\nu^\star$ gives the optimal primal and dual variables for (5.50).

Example 5.2 *Water-filling.* We consider the convex optimization problem

$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^n \log(\alpha_i + x_i) \\ \text{subject to} & x \succeq 0, \quad \mathbf{1}^T x = 1, \end{array}$$

where $\alpha_i > 0$. This problem arises in information theory, in allocating power to a set of n communication channels. The variable x_i represents the transmitter power allocated to the ith channel, and $\log(\alpha_i + x_i)$ gives the capacity or communication rate of the channel, so the problem is to allocate a total power of one to the channels, in order to maximize the total communication rate.

Introducing Lagrange multipliers $\lambda^\star \in \mathbf{R}^n$ for the inequality constraints $x^\star \succeq 0$, and a multiplier $\nu^\star \in \mathbf{R}$ for the equality constraint $\mathbf{1}^T x = 1$, we obtain the KKT conditions

$$x^\star \succeq 0, \qquad \mathbf{1}^T x^\star = 1, \qquad \lambda^\star \succeq 0, \qquad \lambda_i^\star x_i^\star = 0, \quad i = 1, \ldots, n,$$

$$-1/(\alpha_i + x_i^\star) - \lambda_i^\star + \nu^\star = 0, \quad i = 1, \ldots, n.$$

We can directly solve these equations to find $x^\star$, $\lambda^\star$, and $\nu^\star$. We start by noting that $\lambda^\star$ acts as a slack variable in the last equation, so it can be eliminated, leaving

$$x^\star \succeq 0, \qquad \mathbf{1}^T x^\star = 1, \qquad x_i^\star \left(\nu^\star - 1/(\alpha_i + x_i^\star)\right) = 0, \quad i = 1, \ldots, n,$$

$$\nu^\star \geq 1/(\alpha_i + x_i^\star), \quad i = 1, \ldots, n.$$

If $\nu^\star < 1/\alpha_i$, this last condition can only hold if $x_i^\star > 0$, which by the third condition implies that $\nu^\star = 1/(\alpha_i + x_i^\star)$. Solving for $x_i^\star$, we conclude that $x_i^\star = 1/\nu^\star - \alpha_i$ if $\nu^\star < 1/\alpha_i$. If $\nu^\star \geq 1/\alpha_i$, then $x_i^\star > 0$ is impossible, because it would imply $\nu^\star \geq 1/\alpha_i > 1/(\alpha_i + x_i^\star)$, which violates the complementary slackness condition. Therefore, $x_i^\star = 0$ if $\nu^\star \geq 1/\alpha_i$. Thus we have

$$x_i^\star = \begin{cases} 1/\nu^\star - \alpha_i & \nu^\star < 1/\alpha_i \\ 0 & \nu^\star \geq 1/\alpha_i, \end{cases}$$

or, put more simply, $x_i^\star = \max\{0, 1/\nu^\star - \alpha_i\}$. Substituting this expression for $x_i^\star$ into the condition $\mathbf{1}^T x^\star = 1$ we obtain

$$\sum_{i=1}^n \max\{0, 1/\nu^\star - \alpha_i\} = 1.$$

The lefthand side is a piecewise-linear increasing function of $1/\nu^\star$, with breakpoints at α_i, so the equation has a unique solution which is readily determined.

This solution method is called *water-filling* for the following reason. We think of α_i as the ground level above patch i, and then flood the region with water to a depth $1/\nu$, as illustrated in figure 5.7. The total amount of water used is then $\sum_{i=1}^n \max\{0, 1/\nu^\star - \alpha_i\}$. We then increase the flood level until we have used a total amount of water equal to one. The depth of water above patch i is then the optimal value $x_i^\star$.
