Exercise 9.1

For z-channel, we can get $p(Y|X)$ with this following matrix

$$Q = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad x, y \in \{0,1\}$$

And we suppose $Pr(X=1) = P$,

$$H(Y|X) = Pr(X=0)\cdot 0 + Pr(X=1)\cdot 1 = P$$

$$H(Y) = H[Pr(X=1)] = H[\tfrac{P}{2}]$$

So, we can $I(X;Y) = H(Y) - H(Y|X) = H(\frac{P}{2}) - P$

In conclusion: this channel mutual information $H(\frac{P}{2}) - P$

Exercise 9.2

For empirical distribution, we know $f_X(x) := n^{-1}\cdot \sum_{i=1}^{n} \delta_{x_i, x}$

And then x^n are n i.i.d. copies of X, $P_{X^n} = P_X^{\otimes n}$

$$\Downarrow$$

$$P_{X^n}(x) = \prod_{i=1}^{n} P(x_i)$$

$$= 2^{\sum_{i=1}^{n} \log P(x_i)}$$

$$= 2^{\sum_{i=1}^{n} P(x_i|x^n) \log P(x_i)}$$

$$= 2^{-n\left[\frac{\sum \delta x_i}{n} \cdot \log \frac{1}{P(X_i)}\right]} \quad (\text{we use the } f_X(x) \text{ definition})$$

$$= 2^{-n\left[\frac{\sum \delta x_i}{n} \log \cdot \frac{1}{P(x_i)} \frac{f_X(x)}{f_X(x)}\right]}$$

$$= 2^{-n\left[H[f_X(x)] + D(f_X(x) \| P(X))\right]}$$

$$= 2^{-n(H(f_X) + D(f_X \| P_X))}$$

It is desired, the probability that x^n being any sequence $x \in \mathcal{X}^n$ depends only on its type and P_X

Exercise 9.5

Following the instructions from lecture notes and the paper of vincent Poor, for this $(2^{nR}, n)$ code, we let encoder and decoder to use random mappings, which means $P[M \neq \hat{M}] \leq \varepsilon$

And then use the Proposition 6.9

$$|M| \leq \max_{P_X \in P(X)} \min_{Q_Y \in P(Y)} \frac{1}{\beta_\varepsilon^*(P_{XY} || P_X \times Q_Y)}$$

where $P_{XY}(x,y) = P_X(x) \cdot W_{Y|X}(y|x)$ is the joint distribution of channel input and output and $\beta_\varepsilon^*(P_{XY} || P_X \times Q_Y)$ is the minimal error of the second kind.

And we know the encoder generate a codeword $x^n(m, W)$, and the decoder generate an estimate $\hat{m}(y^n, W)$

Therefore, we construct these sets $A_x = \{y \in Y : (x, y) \notin A\}$ for all $x \in X$

Given a fixed channel output y, its probability of (assigns $\hat{M} = m$)

$$P(\hat{M} = m \mid Y = y) = \frac{I\{y \in A_{e(m)}\}}{\sum_{m'} I\{y \in A_{e(m')}\}} = \frac{I\{y \in A_{e(m)}\}}{I\{y \in A_{e(m)}\} + \sum_{m' \neq m} I\{y \in A_{e(m')}\}}$$

Then to analysis the error for this code

$$P(M \neq \hat{M} \mid M = m, E = e) = 1 - \sum_{y \in Y} W(y \mid e(m)) \cdot P[\hat{M} = m \mid Y = y]$$

$$= \sum_{y \in Y} W(y \mid e(m)) \left(1 - \frac{I\{y \in A_{e(m)}\}}{I\{y \in A_{e(m)}\} + \sum_{m' \neq m} I\{y \in A_{e(m')}\}} \right)$$

$$\leq \sum_{x \in X, y \in Y} P_X(x) W(y \mid x) \left(I\{y \notin A_x\} + (|M| - 1) \sum_{x' \in X} P_X(x') I\{y \in A_{x'}\} \right)$$

And summarising this, we can get

$$P(M \neq \hat{M}) \leq \varepsilon + |M| \beta_\varepsilon^*(P_{XY} || P_X \times P_Y)$$

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And then use the part of (Theorem 6.8 in lecture notes)

$$2^{\lceil nR \rceil} \leq \varepsilon \cdot \frac{1}{\beta_\varepsilon^{*}(P_{X^nY^n} \| P_{X^n} \times P_{Y^n})}$$

And take logrithm, and choose P_X as the maximiser in the definition of mutual information $I(X;Y) = I(W)$

$$R \leq \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\beta_\varepsilon^{*}(P_{XY}^{n} \| P_X^n \times P_Y^n)} = D(P_{XY} \| P_X \times P_Y) = I(X;Y) = C(W)$$

As desired, we prove that this randomization does not increase the capacity of the DMC