EE5137: Quiz 1

Name: ___________

Matriculation Number: ___________

Total Score: ___________

September 7, 2017

You have 1.0 hours for this exam. You're allowed 1 sheet of handwritten notes (both sides). Please show provide *careful explanations* for all your solutions.

1. [Conditional Expectations] (10 points) Consider the joint probability density function (pdf)

$$f_{X,Y}(x,y) = \frac{1}{y^2} e^{-x/y^2} e^{-y}, \quad x \ge 0, y > 0.$$

You may use the following fact without proof in this problem

$$\int_0^\infty x^z e^{-t\lambda} \,\mathrm{d}t = \frac{z!}{\lambda^{z+1}} \quad \forall\, z \in \mathbb{N}, \lambda > 0.$$

(a) (2 points) Find the marginal pdf $f_Y(y)$. Please specify the range of y.
Solution: Consider,

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)\,\mathrm{d}x = \int_0^\infty \frac{1}{y^2} e^{-x/y^2} e^{-y}\,\mathrm{d}x = -e^{-x/y^2} e^{-y}\Big|_{x=0}^{\infty} = e^{-y}$$

for $y > 0$. Thus f_Y is indeed an exponential distribution with mean 1.

(b) (2 points) Hence find the conditional pdf $f_{X|Y}(x|y)$. Please specify the ranges of x and y.
Solution: Consider,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{y^2} e^{-x/y^2} e^{-y}}{e^{-y}} = \frac{1}{y^2} e^{-x/y^2}, \quad x \ge 0, y > 0.$$

Thus $f_{X|Y}(\cdot|y)$ is indeed an exponential distribution with mean y^2 (as we will show in the next part).

(c) (2 points) Find $\mathbb{E}[X|Y = y]$ for each $y > 0$.

Solution: Consider

$$\begin{aligned}
\mathbb{E}[X|Y=y] &= \int_0^\infty x f_{X|Y}(x|y)\,\mathrm{d}x \\
&= \int_0^\infty x \frac{1}{y^2} e^{-x/y^2}\,\mathrm{d}x \\
&= \frac{1}{y^2}\left[x(-y^2 e^{-x/y^2})\Big|_{x=0}^{\infty} - \int_0^\infty -y^2 e^{-x/y^2}\,\mathrm{d}x \right] \\
&= \int_0^\infty e^{-x/y^2}\,\mathrm{d}x \\
&= -y^2 e^{-x/y^2}\Big|_0^\infty \\
&= y^2
\end{aligned}$$

We could also have used the formula with $z = 1$ and $\lambda = 1/y^2$.

(d) (1 points) Write down $\mathbb{E}[X|Y]$.

Solution: This is merely $\mathbb{E}[X|Y] = Y^2$.

(e) (3 points) Find $\mathbb{E}[X]$ using parts (a) and (d).

Solution: We use iterated expectations as follows:

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[Y^2] = \int_0^\infty y^2 e^{-y}\,\mathrm{d}y = 2! = 2$$

where the integral is evaluated using the given formula.

2. [Convergence of Random Variables] (5 points) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with zero mean and finite variance σ^2. Consider the following sequence of random variables

$$T_n = \frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i, \quad n = 1, 2, 3, \ldots$$

Does T_n converge in probability to a constant? If so, to what?

Consider the standard proof for convergence of $\frac{1}{n} S_n$.

Solutions: Fix $\epsilon > 0$. Then from Markov's inequality

$$\begin{aligned} \Pr\left(\left|\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i\right| > \epsilon\right) &= \Pr\left(\left(\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i\right)^2 > \epsilon^2\right) \\ &\le \frac{\mathbb{E}\left[\left(\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i\right)^2\right]}{\epsilon^2}. \end{aligned}$$

Evaluating the numerator,

$$\begin{aligned} \mathbb{E}\left[\left(\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i\right)^2\right] &= \frac{1}{n^{3/2}} \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] \\ &\overset{(a)}{=} \frac{1}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E}[X_i^2] \\ &= \frac{1}{n^{3/2}} \cdot n\sigma^2 \\ &= \frac{\sigma^2}{n^{1/2}}, \end{aligned}$$

where (a) holds because the X_i's are independent and zero mean so $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j] = 0$. Now,

$$\Pr\left(\left|\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i\right| > \epsilon\right) \le \frac{\sigma^2}{n^{1/2}\epsilon^2} \to 0, \quad \text{as} \quad n \to \infty.$$

Hence by the definition of convergence in probability, $T_n \to 0$ in probability.

3. [Gaussian Rate Function] (5 points) Let $X_1, X_2, \ldots, X_n$ be i.i.d. Poisson random variables with mean (expectation) $\lambda = \mathbb{E}[X_1] > 0$, i.e.,

$$P_{X_1}(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \forall\, k = 0, 1, 2, \ldots$$

It is known that the moment generating function of X_1 is

$$g_{X_1}(r) = \exp(\lambda(e^r - 1)), \quad \forall\, r \in \mathbb{R}.$$

Suppose $\lambda = 2$. Using the Chernoff bound, find the exponent $E < 0$ in

$$\Pr\left(\frac{1}{n}\sum_{i=1}^{n} X_i > 2e^2\right) \le \exp(nE).$$

Solutions: From the lectures

$$E = \inf_{r \ge 0} \log g_{X_1}(r) - 2e^2 \cdot r.$$

Plugging in the form of the MGF, we obtain

$$E = \inf_{r \ge 0} 2(e^r - 1) - 2e^2 \cdot r.$$

Differentiating w.r.t. r and setting to zero yields

$$2e^r - 2e^2 = 0$$

so $r^* = 2$ which is positive. Hence, the exponent is

$$E = 2(e^2 - 1) - 4e^2 = -2e^2 - 2.$$