On Minumum Spanning Trees and Steiner Trees

Mehul Motani

TL;DR

- Minimum spanning trees (MST) and Steiner trees are different problems.
- The MST of an edge-weighted undirected graph is a subgraph which is a tree and has minimum total edge weight.
- The Steiner tree is an MST for a subset S of the vertices but can include more than S.
- The MST can be solved in polynomial time by Prim's or Kruskal's algoritm.
- The Steiner tree problem is NP-complete.

Let's start with some definitions:

- Let G be a connected, edge-weighted undirected with vertices V and edges E.
- A graph F is a subgraph of G if every edge in F belongs to G.
- A tree T is a connected undirected graph with no cycles (or loops).
- A spanning tree T of a connected graph G is a subgraph that is a tree which spans G (that is, it includes every vertex of G).
- A spanning tree T of a connected graph G can also be defined as a subgraph with maximal set of edges of G that contains no cycle.
- A spanning tree T of a connected graph G can also be defined as a subgraph with minimal set of edges of G that connect all vertices.

Let's talk about Minimum Spanning Trees (MST)

- Let $n = |V|$ and $m = |E|$, i.e., n = number of vertices in G and m = number of edges in G.
- A minimum spanning tree T is a subgraph that connects all the vertices together, without any cycles and with the minimum possible total edge weight.
- If the all the edge weights in G are identical, then every spanning tree is a MST.
- The MST is also called the minimum weight spanning tree or minimum cost spanning tree.
- Prim's algorithm computes the MST in $O(m\, log\, n)$ or $O(m + n\, log\, n)$ time, depending on the data-structures used.
- Kruskal's algorithm computes the MST in $O(m\, log\, n)$ time.

Let's talk about Steiner Trees.

- A MST of a graph must include exactly all vertices of that graph (and no other vertices).
- Steiner trees can be viewed as a generalization of MST's.
- Suppose we have a graph with vertices V. A Steiner tree of a subgraph with vertices $S \subseteq V$ needs to include all the vertices of $S \subseteq V$, but may include other vertices from V. Note that the Steiner tree can include more than S.
- If $S = V$, then a MST for S is a Steiner tree for S.
- If $S \subset V$, then a MST for S could be a Steiner tree for S but not necessarily.
- Similarly, if $S \subset V$, a Steiner tree for S may not be an MST for S, since it can include nodes not in S.
- The general Steiner tree problem is NP-complete.

Note: This information is sourced from WikiPedia (https://en.wikipedia.org) and other online sources.