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Subject： Information Theory

Assignment: Homework One

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Exercise 1.1

a). For $E[V+W] = \sum_{v,w\in\mathcal{X}} (v+w)\, P_{VW}(v,w)$

$= \sum_{v,w\in\mathcal{X}} v\, P_{VW}(v,w) + \sum_{v,w\in\mathcal{X}} w\, P_{VW}(v,w)$

$= E[V] + E[W]$

As desired, $E[V+W] = E[V] + E[W]$

b). If v and w are independent, we can get $P_{VW}(v,w) = P_V(v)\cdot P_W(w)$

$E[VW] = \sum_{v,w\in\mathcal{X}} vw\, P_{VW}(v,w)$

$= \sum_{v,w\in\mathcal{X}} vw\, P_V(v)\cdot P_W(w)$

$= \sum_{v,w\in\mathcal{X}} v\, P_V(v)\cdot w P_W(w)$

$= \sum_{w\in\mathcal{X}} v\cdot P_V(v)\cdot \sum_{v\in\mathcal{X}} w\, P_W(w)$

$= E[V]\cdot E[W]$

As desired, $E[VW] = E[V]\cdot E[W]$

c). We know, $Z = V+W$

$Var(Z) = Var(V+W) = E[(V+W)^2] - E^2[V+W]$

$= E[V^2] + E[W^2] + E[2VW] - E[V+W]\cdot E[V+W]$

$= E[V^2] + E[W^2] + 2E[V]\cdot E[W] - (E[V] + E[W])^2$ (from (a) and (b))

$= E[V^2] - E^2[V] + E[W^2] - E^2[W]$

$= Var(V) + Var(W)$

$= \sigma_V^2 + \sigma_W^2$

Exercise 1.2

a). All the possible events are listed:

the sample space

HHHH	HHHT	HTHH	HTHT
HHTH	HHTT	HTTH	HTTT
THHH	THHT	TTHH	TTHT
THTH	THTT	TTTH	TTTT

which means (X, Y)

(X=4, Y=1)	(X=3, Y=1)	(X=3, Y=1)	(X=2, Y=1)
(X=3, Y=1)	(X=2, Y=1)	(X=2, Y=1)	(X=1, Y=1)
(X=3, Y=2)	(X=2, Y=2)	(X=2, Y=3)	(X=1, Y=3)
(X=2, Y=2)	(X=1, Y=2)	(X=1, Y=4)	(X=0, Y=0)

Date

b). $P_{XY}(0,0) = \frac{1}{16}$

$P_{XY}(1,1) = \frac{1}{16}$, $P_{XY}(1,2) = \frac{1}{16}$, $P_{XY}(1,3) = \frac{1}{16}$ $P_{XY}(1,4) = \frac{1}{16}$

$P_{XY}(2,1) = \frac{3}{16}$, $P_{XY}(2,2) = \frac{1}{8}$, $P_{XY}(2,3) = \frac{1}{16}$

$P_{XY}(3,1) = \frac{3}{16}$, $P_{XY}(3,2) = \frac{1}{16}$

$P_{XY}(4,1) = \frac{1}{16}$

~~$P_X(x) = \sum_{y_j \in R_Y} P_{XY}(x, y_j)$ for any $x \in R_X$~~

so, we can get $\Pr[Y=0 \mid X=1] = \dfrac{P_{XY}(X=1, Y=0)}{P_X(X=1)} = \dfrac{0}{\frac{1}{4}} = 0$

$$\Pr[Y=1 \mid X=3] = \frac{P(X=3, Y=1)}{P_X(X=3)} = \frac{\frac{3}{16}}{\frac{3}{16}+\frac{1}{16}} = \frac{3}{4}$$

(c) $P_X(X=0) = \frac{1}{16}$ $P_Y(Y=0) = \frac{1}{16}$

$P_X(X=1) = \frac{1}{4}$ $P_Y(Y=1) = \frac{1}{2}$

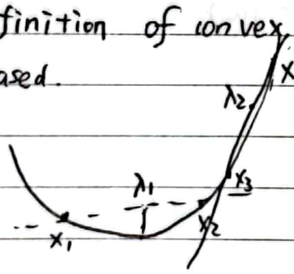
$P_X(X=2) = \frac{3}{8}$ $P_Y(Y=2) = \frac{1}{4}$

$P_X(X=3) = \frac{1}{4}$ $P_Y(Y=3) = \frac{1}{8}$

$P_X(X=4) = \frac{1}{16}$ $P_Y(Y=4) = \frac{1}{16}$

Exercise 1.3

From the definition of convex property, we know the derivative of con[vex] function is increased.



For $a \le X_1 < X_2 \le X_3 < X_4 \le b$

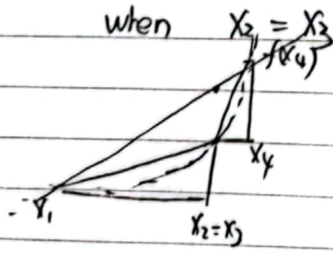
$$\frac{f(x_2)-f(x_1)}{x_2 - x_1} \le \frac{f(x_4)-f(x_3)}{x_4 - x_3}$$

But based on the simple statement of convex function

$f[\lambda_1 x_1 + (1-\lambda_1)x_2] \le \lambda_1 f(x_1) + (1-\lambda_1) f(x_2)$ $\lambda_1 \in [0,1]$

$f[\lambda_2 x_3 + (1-\lambda_2)x_4] \le \lambda_2 f(x_3) + (1-\lambda_2) f(x_4)$ $\lambda_2 \in [0,1]$

when $X_2 = X_3$



$$\frac{f(x_4)-f(x_1)}{x_4 - x_1}(x_2 - x_1) + f(x_1) \le \lambda f(x_1) + (1-\lambda) f(x_4)$$

$$\Downarrow$$

$$\frac{f(x_2)-f(x_1)}{x_2 - x_1} \le \frac{f(x_4)-f(x_3)}{x_4 - x_3}$$

Exercise 1.4

After computing the GF(8) and GF(9), I got these tables.

GF(8) addition table

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

GF(8) multiplication table

x	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	3	1	7	5
3	0	3	6	5	7	4	1	2
4	0	4	3	7	6	2	5	1
5	0	5	1	4	2	7	6	6
6	0	6	7	1	5	3	2	4
7	0	7	5	2	1	6	4	3

GF(9) addition table

+	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	0	3	2	5	4	7	6	7
2	2	3	0	1	6	7	4	5	6
3	3	2	1	0	7	6	5	4	5
4	4	5	8	7	0	1	2	3	4
5	5	4	7	6	1	0	3	2	3
6	6	8	4	8	2	8	0	1	2
7	7	6	5	4	3	2	1	0	1
8	8	7	6	5	4	3	2	1	0

GF(9) multiplication table

x	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	3
2	0	2	4	6	3	1	7	5	2
3	0	3	6	5	7	4	1	2	3
4	0	4	3	7	6	2	5	1	5
5	0	5	1	4	2	7	6	6	4
6	0	1	2	1	5	3	2	4	3
7	0	2	5	2	1	6	4	3	2
8	0	3	2	3	5	4	3	2	1

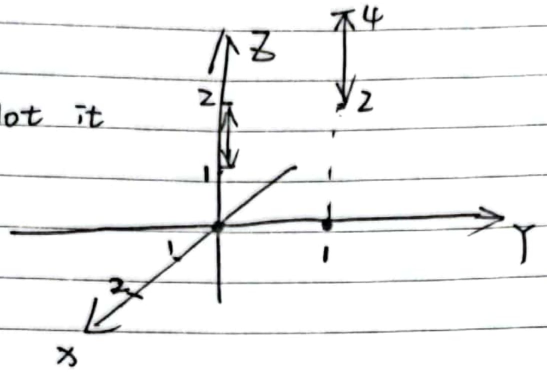
Exercise 1.5

a) From the statement, when $\begin{cases} Y=1 & Z = X + \frac{1}{2}\cdot X + \frac{1}{2}\cdot X = 2X \\ Y=0 & Z = X \end{cases}$

so we rewrite as

$$Z = 1\{Y=1\}\cdot X + X \qquad \begin{cases} Y \in \mathrm{Bern}(\frac{1}{2}) \\ X \in U[1,2] \end{cases}$$

• And I plot it



b). For the conclusion from part a, we can get

$$Pr(Y \mid Z=z) = \frac{Pr(Y, Z=z)}{Pr(z)} = \frac{Pr(Y, Z=z)}{Pr(Y=0)\,Pr(Z \mid Y=0) + Pr(Y=1)\cdot Pr(Z \mid Y=1)}$$

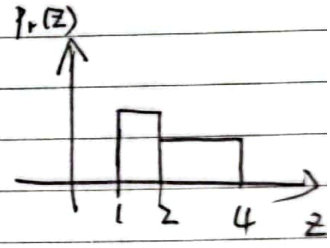
$$= \frac{2Pr(Y, Z=z)}{Pr(Z \mid Y=0) + Pr(Z \mid Y=1)} \qquad z \in [z-\varepsilon, z+\varepsilon]$$

For $1 \le z < 2$ $\quad Pr(Y=0, Z=z) = \dfrac{2\cdot\frac{1}{2\varepsilon}}{\frac{1}{2\varepsilon}\cdot 2} = 1$

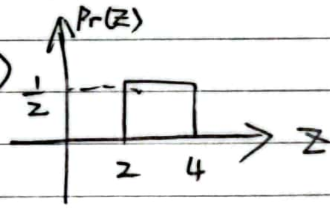
$Pr(Y=1 \mid Z=z) = 0$

For $2 < z \le 4$ $\quad Pr(Y=1 \mid Z=z) = \dfrac{2\cdot\frac{1}{2\varepsilon}}{\frac{1}{2\varepsilon}\cdot 2} = 1$

$Pr(Y=0 \mid Z=z) = 0$



Then we can get Plot $Pr(Y=0 \mid Z=z) \Rightarrow$



$=$

$= \frac{1}{2}$

Exercise 1.6

(a) $W = \begin{bmatrix} 1-\varepsilon & \varepsilon & 0 \\ 0 & 1-\varepsilon & \varepsilon \\ \varepsilon & 0 & 1-\varepsilon \end{bmatrix}$

(b) output symbols: $pW = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right] \begin{bmatrix} 1-\varepsilon & \varepsilon & 0 \\ 0 & 1-\varepsilon & \varepsilon \\ \varepsilon & 0 & 1-\varepsilon \end{bmatrix}$

$$= \left[\frac{1}{2} - \frac{\varepsilon}{4}, \frac{1}{4} + \frac{\varepsilon}{4}, \frac{1}{4}\right]$$

(c). $Pr(X=0 \mid Y=1) = \dfrac{Pr(X=0, Y=1)}{Pr(Y=1)} = \dfrac{\varepsilon}{P_0 \cdot \varepsilon + P_1 \cdot (1-\varepsilon)}$

suppose P vector

$P = [P_0, P_1, P_2]$

$Pr(X=1 \mid Y=1) = \dfrac{Pr(X=1, Y=1)}{Pr(Y=1)} = \dfrac{1-\varepsilon}{P_0 \cdot \varepsilon + P_1(1-\varepsilon)}$

$Pr(X=2 \mid Y=1) = \dfrac{Pr(X=2, Y=1)}{Pr(Y=1)} = \dfrac{0}{P_0 \cdot \varepsilon + P_1(1-\varepsilon)} = 0$