EE5137 2019/20 (Sem 2): Quiz 2 (Total 30 points)

Name:______________________________

Matriculation Number:_______________

Score:______________________________

You have 1.0 hour for this quiz. There are SEVEN (7) printed pages. You're allowed 1 sheet of handwritten notes. Please provide *careful explanations* for all your solutions.

1. (a) (5 points) Let $\{N(t) : t > 0\}$ be a Poisson counting process with rate $\lambda > 0$. Let T_1 be an exponential random variable with probability density function

$$f_{T_1}(t) = \begin{cases} \nu \exp(-\nu t) & t \ge 0 \\ 0 & t < 0 \end{cases}$$

for some $\nu > 0$. What is the distribution (probability mass function) of $N(T_1)$, the number of Poisson arrivals of the first process in the interval $[0, T_1]$?

Solution: Let the Poisson processes with rates λ and ν be called PP of the first and second types respectively. Given an arrival, it is of the first type with probability $p = \frac{\lambda}{\lambda+\nu}$. Thus $N(T_1)$ represents the number of arrivals of the first type before the first arrival of the second type. This is a geometric random variable starting from 0, i.e.,

$$\Pr(N(T_1) = n) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^n, \quad n = 0, 1, 2, \ldots$$

(b) (5 points) Let $\{N(t) : t > 0\}$ be as in part (a). Now, let T_2 be an Erlang random variable of order 2 with probability density function

$$f_{T_2}(t) = \begin{cases} \nu^2 t \exp(-\nu t) & t \ge 0 \\ 0 & t < 0 \end{cases}$$

for some $\nu > 0$. What is the distribution (probability mass function) of $N(T_2)$, the number of Poisson arrivals of the first process in the interval $[0, T_2]$?

Hint: Drawing a figure might be helpful.

Solution: Let the Poisson processes with rates λ and ν be called PP of the first and second types respectively. Given an arrival, it is of the first type with probability $p = \frac{\lambda}{\lambda+\nu}$. Thus $N(T_2)$ represents the number of arrivals of the first type before the second arrival of the second type. This is a negative binomial random variable starting from 0, i.e.,

$$\begin{aligned}
\Pr(N(T_2) = n) &= \binom{n+2-1}{1} \left(\frac{\nu}{\lambda+\nu}\right)^2 \left(\frac{\lambda}{\lambda+\nu}\right)^n \\
&= (n+1) \left(\frac{\nu}{\lambda+\nu}\right)^2 \left(\frac{\lambda}{\lambda+\nu}\right)^n, \quad n = 0, 1, 2, \ldots
\end{aligned}$$

2. Buses arrive at a certain bus stop according to a Poisson process with rate 2 per hour. Passengers arrive according to an independent Poisson process with rate 10 per hour. The instant a bus arrives, all passengers at the stop at that instant board the bus and the bus departs.

   You may use the following fact without proof: An exponential random variable with rate λ, i.e., density

   $$f_X(x) = \lambda \exp(-\lambda x) \mathbb{1}_{x \geq 0}$$

   has mean $1/\lambda$ and variance $1/\lambda^2$.

   (a) (2 points) Assume that there are currently no passengers at the bus stop. What is the probability that the next bus will pick up no passengers? Explain.

   Solution: The sum of the two Poisson processes has rate $10 + 2 = 12$, and each event in the combined process is a bus with probability $2/12$. So the probability that the next event is a bus (which is equivalent to the next bus picking up no passengers) is $2/12$.

   (b) (2 points) If you arrive at the stop at noon, what is the expected amount of time you will have to wait until the next arrival of any type (bus or passenger)? Explain.

   Solution: The rate of the combined process is 12, so we have to wait on average $1/12$ hour, or 5 minutes. By the memoryless property, we do not have to know how long before noon the last event happened.

   (c) (2 points) At 2:00pm, there are 2 passengers waiting for the bus. Given this information, what is the expected arrival time of the next bus after 2:00pm? Explain.

   Solution: The next bus will arrive in $1/2$ hour on average, or at 2:30. By the memoryless property, the information about the number of passengers present at 2:00 is irrelevant.

   (d) (2 points) Show that

   $$\mathrm{Var}(Y) = \mathbb{E}[\mathrm{Var}(Y|T)] + \mathrm{Var}(\mathbb{E}[Y|T])$$

   where $\mathrm{Var}(Y|T) := \mathbb{E}[Y^2|T] - (\mathbb{E}[Y|T])^2$.
   Solution: We have

   $$\begin{aligned}
   \mathrm{Var}(Y) &= \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 \\
   &= \mathbb{E}[\mathbb{E}[Y^2|T]] - (\mathbb{E}[\mathbb{E}[Y|T]])^2 \\
   &= \mathbb{E}[\mathrm{Var}(Y|T) + (\mathbb{E}[Y|T])^2] - (\mathbb{E}[\mathbb{E}[Y|T]])^2 \\
   &= \mathbb{E}[\mathrm{Var}(Y|T)] + \mathbb{E}[(\mathbb{E}[Y|T])^2 - (\mathbb{E}[\mathbb{E}[Y|T]])^2] \\
   &= \mathbb{E}[\mathrm{Var}(Y|T)] + \mathrm{Var}(\mathbb{E}[Y|T]).
   \end{aligned}$$

(e) (4 points) Assume that there are currently no passengers at the bus stop. Let Y be the number of people at the stop when the next bus arrives. Find $\mathbb{E}[Y]$ and $\text{Var}(Y)$, showing all your work. For $\text{Var}(Y)$, use part (d).

Solution: Let T be the time (in hours from now) when the next bus will arrive. Then T is exponential with mean $1/2$ and variance $1/4$. Also, given T, Y is Poisson with mean $10T$, which means that $\mathbb{E}[Y|T] = \text{Var}(Y|T) = 10T$. Therefore, averaging over T (using the law of iterated expectation and the law of total variance as in (d)) gives

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y|T]] = \mathbb{E}[10T] = 10\mathbb{E}[T] = 5 \\
\text{Var}(Y) &= \mathbb{E}[\text{Var}(Y|T)] + \text{Var}(\mathbb{E}[Y|T]) \\
&= \mathbb{E}[10T] + \text{Var}(10T) = 10\mathbb{E}[T] + 10^2\text{Var}(T) = 5 + 25 = 30.
\end{aligned}$$

3. There are three types of MRT trains—Red, Blue and Green. MRT trains arrive at Kent Ridge NUS station according to a Poisson process with an arrival rate of λ trains per minute. Trains arriving after 5.00pm always arrive in the following order: Red always comes first; followed by Blue; then by Green. An EE5137 student Alice *must* take the Green train. In the following problems, δ is a very small constant. You may use the fact that an Erlang distribution of order k with rate λ is

$$f_{S_k}(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}, \qquad t \geq 0$$

and that for any continuous function g,

$$\int_t^{t+\delta} g(s)\,\mathrm{d}s \approx g(t)\delta.$$

(a) (2 points) On week 1, Alice arrives at Kent Ridge at exactly 5:00pm. What is the probability that she will wait between t and $t+\delta$ minutes for her green MRT train?

Solution: The probability in question is that of the event $\{t \leq S_3 < t+\delta\}$ where S_3 is an Erlang of order 3, i.e,

$$\Pr\{t \leq S_3 < t+\delta\} = \int_t^{t+\delta} f_{S_3}(s)\,\mathrm{d}s \approx \frac{\lambda^3 t^2 e^{-\lambda t}}{2!} \cdot \delta,$$

where $t \geq 0$.

(b) (3 points) However, Alice's lecturer has a poor grasp of time, ending his lecture late on week 2. Therefore, Alice reaches the Kent Ridge station at 5.05pm. What is the probability that her green MRT train arrives between τ and $\tau+\delta$ minutes <u>after</u> Alice? Note that τ may be negative.

Solution: The waiting time in this part is five minutes less than the waiting time in the previous part. The random variable in question is $Z = S_3 - 5$ minutes. Hence,

$$\Pr\{\tau \leq Z < \tau+\delta\} = \Pr\{(\tau+5) \leq S_3 < (\tau+5)+\delta\} \approx \frac{\lambda^3 (\tau+5)^2 e^{-\lambda(\tau+5)}}{2!} \cdot \delta$$

where $\tau \geq -5$.

(c) (3 points) Fortunately, Alice's friend Bob has been Kent Ridge since 5:00pm. He told Alice that she hasn't missed her Green bus; the Red train came at 5:03pm, but the Blue and Green trains have not arrived yet. Assuming that Bob is reliable, what is the probability that Alice will wait between τ and $\tau+\delta$ minutes for her bus?

Solution: Since the Red train has come and gone, Alice only has to wait for two trains to arrive. The waiting time for the second arrival in a Poisson process is a second-order Erlang and so,

$$\Pr\{\tau \leq S_2 < \tau + \delta\} \approx \frac{\lambda^2 \tau^1 e^{-\lambda\tau}}{1!} \cdot \delta,$$

where $\tau \geq 0$. The fact that the Red train arrived at 5:03pm is irrelevant; the only thing that matters is that it has already arrived.