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Subject: Information Theory

Assignment: Homework Five

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Exercise 5.1

a). For $\alpha \to 0^+$, $H_\alpha(X) = \frac{1}{1-\alpha} \log \sum_x P(x)^\alpha$

$$\Rightarrow = \log |X|$$

For $\alpha \to 1$, $H_\alpha(X) = \cancel{\frac{1}{1-\alpha}\Big|_{\alpha\to 1} \log \sum_x P(x)} \frac{1}{1-\alpha} \log \sum_x P(x)$

$$\cancel{\Rightarrow = \frac{1}{1-\alpha}\Big|_{\alpha\to 1} \log} \Rightarrow \sum_x P_x \log \frac{1}{P(x)}$$

~~we should rewrite this equatition as continuous form~~

As desired, this is shannon entropy

for $\alpha \to +\infty$, $H_\alpha(X) = \frac{1}{1-\alpha} \log(\sum_x P(x)^\alpha) = \frac{\alpha}{1-\alpha} \log \sum_x P(x)$

As $\frac{\alpha}{1-\alpha}\Big|_{\alpha\to+\infty} = -1$

Therefore, $H_\alpha(X) = -\log \max P(x)$ [norm equation]

As desired, this is min-entropy

b). I would upload the figure which is generated by Matlab

c). For $H_\alpha(X) = \frac{1}{1-\alpha} \log \sum_x P(x)^\alpha$

① in the range of $(0,1)$, we consider $\frac{dH_\alpha(X)}{d\alpha} = \frac{1}{(1-\alpha)}' \log \sum_x P_x(x)^\alpha$

$$+ \frac{1}{1-\alpha} \log \sum_x P_x(X)$$

$$= \frac{-2\alpha}{(1-\alpha)^2} \log \sum_x P_x(X) \le 0.$$

② Similarily, in the range of $(1, +\infty)$

$$\frac{d H_\alpha(X)}{d\alpha} \le 0.$$

In a word, this $H_\alpha(X)$ is non-increasing in the parameter.

$$H_{min}(X) \le H(X) \le \log |X|$$

d). For the min-entropy $H_{min}(X|Y)$

$H_{min}(X|Y) = -\log \max P(X|Y)$

And for $P(X|Y) \Rightarrow$

$P(X=0|Y=0) = \frac{1}{2}$ $P(X=0|Y=1) = \frac{1}{4}$ $P(X=0|Y=2) = \frac{1}{4}$

$P(X=1|Y=0) = \frac{1}{4}$ $P(X=1|Y=1) = \frac{1}{2}$ $P(X=1|Y=2) = \frac{1}{4}$

$P(X=2|Y=0) = \frac{1}{4}$ $P(X=2|Y=1) = \frac{1}{4}$ $P(X=2|Y=2) = \frac{1}{2}$

$\max P(X|Y) = \frac{1}{2}$

Therefore $H_{min}(X|Y) = -\log\frac{1}{2} = 1$

Exercise 5.2

a). $H(X_n) \geq (1-\varepsilon)\log n$

$H_{min}(X_n) = C$

In order to satisfy above requirement, we set this sequence

Any sequence can satisfy this requirement, obviously.

b) $H(X_n) = H_{min}(X_n) = \log n$

$H(X_n|Y_n) \geq (1-\varepsilon)\log n$

$H_{min}(X_n|Y_n) = C$

In order to satisfy above requirement, we set this sequence

① X_n and Y_n are i.i.d. variable.

② And then it satisfy a) part requirement is used.

Exercise 5.3

a). Using the definition of typical set

$$2^{nH(x)-\varepsilon} \le Pr\,[A_\varepsilon^{(n)}] \le 2^{nH(x)+\varepsilon}$$

So, $W(x_j) = \sum_{j \in x} W(x_j = x) = \underset{(j=a)}{W(x_j = a)} + \underset{(j=b)}{W(x_j = b)}$

$$= -\log\frac{2}{3} - \log\frac{1}{3}$$

$$= -\log\frac{2}{9}$$

$$= \log\frac{9}{2}$$

For the bound of this typical set, we can get

$$2^{n\cdot\log\frac{9}{2} - 0.01} \le Pr[A_\varepsilon^{(n)}] \le 2^{n\log\frac{9}{2} + 0.01}$$

b). For this statement, we know N_a be the number of a' in the string $x^n = (X_1, \cdots, X_n)$.

In this part, we know DMS (two symbol alphabet $\{a, b\}$)

Therefore, these variables are x_j who belongs to $\{a\}$

c) For the definition, $W(X^n)$ ~~as a function of the~~

$$= \sum_{N=1}^{n} W(X_j)$$

$$= \sum^{N_a} W(x_j = a) + \sum^{N_b} W(X_j = b)$$

$$= N_a\, W(x_j) + (n - N_a)\, W(x_j)$$

$$\cancel{= n \cdot W(x_j)}$$

In a word, $W(x^n)$ as a function of N_a

And at the same time, it depends on n. $(n = 100000)$

d).

Using Chebyshev's inequality, and we assume that $\alpha < N_\alpha < \beta$

so, we can get these findings.

$$2^{nW(x_i)-\varepsilon} \leq Pr(A_\varepsilon^{(n)}) \leq 2^{nW(x_i)+\varepsilon}$$

Therefore, this typical set is in terms of bounds on N_x

e). $Pr(Na=0) = 2^{-1}$

$Pr(Na=1) = 2^{-2}$

$Pr(Na=2) = 2^{-3}$

Therefore, the particular string x^n that has maximum probability over all sample values of x^n.

Next most probable n-strings = aaa

Exercise 5.4.

a). For the statement of this question, following the illustration,

i). X and Y are independent

$$R^*(X|Y) = H(X|Y) = H(X)$$

ii) $X = Y$

$$R^*(X|Y) = H(X|X) = 0.$$

b).

By explicitly constructing a code for the source (X, Y) using codes for the sources Y and X (with side information Y).

$$R^*(X, Y) = \lim_{n\to\infty} \frac{H(X, Y)}{n} \quad (\text{using the definition})$$

$$\le \lim_{n\to\infty} \frac{H(X|Y) + H(Y)}{n}$$

$$\le \lim_{n\to\infty} \frac{H(X|Y)}{n} + \lim_{n\to\infty} \frac{H(Y)}{n}$$

$$\le R^*(X|Y) + R^*(Y)$$

c) For the converse part,

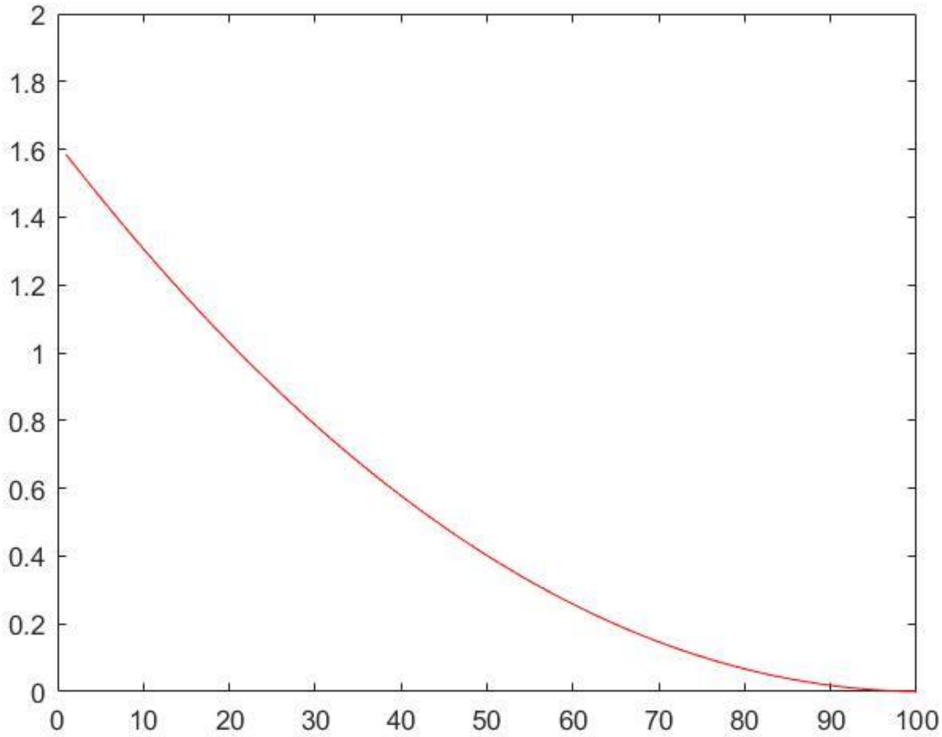
$$R^*(X|Y)$$
$$\ge R^*(X^n | \hat{X}^n)$$
$$\ge H(X^n | \hat{X}^n)$$
$$= H(X^n | Y^n M)$$
$$= H(X^n M | Y^n) - H(M | Y^n)$$
$$\ge H(X^n M | Y^n) - L$$
$$\ge H(X^n | Y^n) - L$$

It is desired, $R^*(X|Y) \ge H(X|Y)$

d). Using the typical set, $A_\varepsilon^{(n)}(X|Y) := \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \left| \frac{1}{n} \log \frac{1}{P_{X^n|Y^n}(x^n|y^n)} - H(X|Y) \right| \le \varepsilon \right\}$

Therefore, we can prove that $R^*(X|Y) \le H(X|Y)$.

Here is the Exercise5.1 (b) figure



Here is the Matlab code to plot the figure

```
q=[1/2,1/4,1/4];
renyi=[];
for alpha=[0:0.01:1,1.01:1:101.01]
    renyi=[renyi,(1/1-alpha).* log2(sum(q .^alpha))]
end
SHANNON=-(sum (q .* log2 (q)))
figure(1)
plot(renyi,'r')
xlim([0 100])
ylim([0 2])
```