



# NUS

National University  
of Singapore

Name : LUO ZIJIAN

Matric.No: A0224725H

MUSNET: E0572844

Subject: Stochastic process

Assignment: Homework Four

Date: Feb 19<sup>th</sup>

Prof: Vincent Tan.

1. For a Poisson process, which of the following is/are true?

(1)  $\{N(t) \geq n\} = \{S_n \leq t\}$

This following is TRUE

Proof: suppose  $N(t) \geq n$ , which means the number of arrivals up to  $m$  ( $m \geq n$ ) in time  $t$ )  $S_m$  so  $S_m \leq t \Rightarrow S_n \leq S_m \leq t$

$\Rightarrow \{N(t) \geq n\} \subseteq \{S_n \leq t\} \dots \textcircled{1}$

Reversely:  $S_n \leq t$ , which means the  $n^{\text{th}}$  arrival occurred at  $\tau$ ,  $\tau \leq t$

$\Rightarrow N(\tau) = n \Rightarrow N(t) \geq N(\tau) = n$

$\Rightarrow \{S_n \leq t\} \subseteq \{N(t) \geq n\} \dots \textcircled{2}$

combined with  $\textcircled{1}$  and  $\textcircled{2}$ : this following proves TRUE

(2)  $\{N(t) < n\} = \{S_n > t\}$

This following is TRUE

Proof: suppose  $N(t) < n$ , which means the number of arrivals up to  $m$  ( $m < n$ )

in time  $t$ )  $S_m < t < S_n \Rightarrow S_n > t \dots \textcircled{1}$

Reversely:  $S_n > t$ , which means the  $n^{\text{th}}$  arrival occurred at  $\tau$ ,  $\tau > t$

$\Rightarrow N(\tau) = n \Rightarrow N(t) < N(\tau) = n$

$\Rightarrow \{S_n > t\} \subseteq \{N(t) < n\} \dots \textcircled{2}$

combined with  $\textcircled{1}$  and  $\textcircled{2}$ : this following proves TRUE

(3)  $\{N(t) \leq n\} = \{S_n \geq t\}$

This following is FALSE

For example, when we consider this situation (when  $N(t) = n$ )

Because this special case, we can get, the number of arrivals up to  $n$  in time  $t$ )

$\Rightarrow S_n = t < t$

is self-conflict with  $\{S_n \geq t\}$ ,

to sum up, this following is not correct,

$$(4) \{N(t) > n\} = \{S_n < t\}$$

This following is TRUE

Proof: suppose  $N(t) > n$ , which means the number of arrival is  $m$  in time  $t$ , so  $m > n$

$$\begin{array}{c} \text{---} \\ \text{S}_m \quad t \\ \Rightarrow S_n < t \end{array} \quad \begin{array}{l} S_m \leq t \Rightarrow S_n < S_m \leq t \\ \Rightarrow S_n < t \end{array}$$

$$\text{so } \{N(t) > n\} \subseteq \{S_n < t\} \quad \text{--- ①}$$

Reversely, suppose  $S_n < t$ , which means  $n^{\text{th}}$  arrival occurred at  $\tau$

$$\begin{array}{c} \tau < t \\ \text{---} \\ \tau \quad t \end{array} \quad \begin{array}{l} \Rightarrow N(\tau) = n \Rightarrow N(t) > N(\tau) = n \\ \Rightarrow N(t) > n \end{array}$$

$$\text{so } \{S_n < t\} \subseteq \{N(t) > n\} \quad \text{--- ②}$$

combined with ① ②, this following proves TRUE

2. Through the statement of this question, we can easily find that 5 tennis courts are same situation, not matter which pairs of players in any court. They are i.i.d. events

Now we focus on ~~the~~ desired pairs of players, 5 pairs of players are 5 courts, and at the same time,  $k$  pairs are waiting, which means, this pair should wait  $(k+1)$  pairs of players finish their activities. In this situation, at least  $(k+1)$  pairs.

Then pay attention to  $(k+1)$  pairs of players, they are also same types events (exponentially distribution), and these activities are i.i.d. events. From the above statement, the mean time per pair of players is 40 minutes. Therefore, the total waiting time is  $40(k+1)$  minutes.

But for 5 courts, the 5 courts are same situation,

In conclusion, the expected waiting time to get a court

$$\frac{40(k+1)}{5} = 8(k+1) \quad \text{is } 8(k+1) \text{ minutes}$$



4. For Geometric distribution

$$P_X(k) = (1-p)^{k-1} \cdot p, \quad k \in \mathbb{N} = \{1, 2, \dots\}$$

$$\begin{aligned} \text{let } \Pr\{X > t+x\} &= 1 - \Pr\{X \leq t+x\} = 1 - \sum_{k=1}^{t+x} (1-p)^{k-1} \cdot p \\ &= 1 - p \cdot \frac{1 - (1-p)^{t+x}}{1 - (1-p)} = 1 - [1 - (1-p)^{t+x}] \\ &= (1-p)^{t+x} \end{aligned}$$

$$\begin{aligned} \text{Similarly } \Pr\{X > t\} &= 1 - \Pr\{X \leq t\} = 1 - \sum_{k=1}^t (1-p)^{k-1} \cdot p \\ &= 1 - [1 - (1-p)^t] = (1-p)^t \end{aligned}$$

$$\begin{aligned} \Pr\{X > x\} &= 1 - \Pr\{X \leq x\} = 1 - \sum_{k=1}^x (1-p)^{k-1} \cdot p \\ &= 1 - [1 - (1-p)^x] = (1-p)^x \end{aligned}$$

$$\Pr\{X > t+x\} = (1-p)^{t+x} = (1-p)^t \cdot (1-p)^x = \Pr\{X > t\} \cdot \Pr\{X > x\}$$

In conclusion, this geometric distribution has the memoryless property.

5. let  $X_n$  denote a Binomial random variable, with  $n$  trials and probability of success  $p_n$ .  $E[X] = np_n = \lambda$ , so  $p_n = \frac{\lambda}{n}$

$$\text{so } \Pr(X_n = i) = \lim_{n \rightarrow \infty} \binom{n}{i} \cdot p_n^i \cdot (1-p_n)^{n-i} = \lim_{n \rightarrow \infty} \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

we can rewrite this expression

$$\begin{aligned} \lim_{n \rightarrow \infty} \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i} &= \lim_{n \rightarrow \infty} \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-i+1)}{i!} \cdot \frac{\lambda^i}{n^i} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-i} \\ &= \lim_{n \rightarrow \infty} \frac{\lambda^i}{i!} \cdot \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-i+1)}{n^i} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-i} \end{aligned}$$

$$\text{owing to } \lim_{n \rightarrow \infty} \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-i+1}{n} \left(1 - \frac{\lambda}{n}\right)^{n-i} = 1$$

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$\text{so } \Pr(X_n = i) = \lim_{n \rightarrow \infty} \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i} = \frac{\lambda^i}{i!} e^{-\lambda} = \frac{e^{-\lambda} \cdot \lambda^i}{i!}$$

as desired

6. From the statement in this question, we can get the random variable is non-negative.

Firstly, we can calculate  $E[X^n] = \int_0^{\infty} x^n \cdot f_X(x) dx$

Based on,  $F_X(x) = 1 - F_X^c(x) = P(X \geq x) = \int_x^{\infty} f_X(x) dx$

Put this value into

$$\text{so } \int_0^{\infty} n x^{n-1} F_X^c(x) dx = \int_0^{\infty} n x^{n-1} \cdot \int_x^{\infty} f_X(x) dx dx$$

and now, we can change the order of integration

$$\int_0^{\infty} f_X(x) \int_x^{\infty} n x^{n-1} dx dx$$

$$= \int_0^{\infty} f_X(x) \left( \int_x^{\infty} x^n \right)' dx dx$$

$$= \int_0^{\infty} x^n \cdot f_X(x) dx$$

this result is same from the basic definition

$$\text{To sum up, } E[X^n] = \int_0^{\infty} n x^{n-1} F_X^c(x) dx$$

7. Through the information from the question,

we can get  $Pr(0 \leq X \leq a) = 1$ , which means this random variables convergence in probability 1:  $\lim_{n \rightarrow \infty} Pr\left\{ \left| \bar{X}_n - \bar{X} \right| \leq \frac{a}{2} \right\} = 1$

we can find  $X$  must include  $[0, a]$

for any  $x$ , and any  $\bar{X}_n$  (mean expectation of  $x$ ) must satisfy  $-\frac{a}{2} \leq x - \bar{X}_n \leq \frac{a}{2}$

$$E[X] - \frac{a}{2} \leq x \leq E[X] + \frac{a}{2} \quad Pr\left\{ \left| x - E(X) \right| \leq \frac{a}{2} \right\} = 1$$

Then, we use chebyshev's inequality

$$Pr\left( \left| x - E(X) \right| \geq \frac{a}{2} \right) \leq \frac{\text{Var}(X)}{\left( \frac{a}{2} \right)^2}$$

$$\text{Var}(X) \geq \left( \frac{a}{2} \right)^2 Pr\left( \left| x - E(X) \right| \geq \frac{a}{2} \right)$$

$$\geq \frac{a^2}{4}$$