National University of Singapore
Department of Electrical & Computer Engineering

Examination for

EE5137 Stochastic Processes

(Semester I, 2017/18)
November/December 2017

Time Allowed: 2.5 hours

INSTRUCTIONS FOR CANDIDATES:

- This paper contains **FOUR (4)** questions, printed on **FIVE (5)** pages.
- The total number of marks is 100.
- Answer all questions.
- Programmable calculators are NOT allowed.
- Electronic communicating devices MUST be turned off and inaccessible throughout the examination. They CANNOT be used as calculators, timers or clocks.
- You are allowed to bring ONE (A4) size help sheet.
- No other material is allowed.

Question 1

We toss a biased coin n times. The probability of heads, denoted by q, is the value of a random variable Q with a given mean μ and variance σ^2. Let X_i be a Bernoulli random variable that models the outcome of the i-th toss (i.e., $X_i = 1$ if the i-th toss is a head). In other words, for each $1 \le i \le n$,

$$X_i = \begin{cases} 1 & \text{w.p. } Q \\ 0 & \text{w.p. } 1-Q \end{cases},$$

where $Q \in [0,1]$ is a random variable with

$$\mathbb{E}[Q] = \mu, \quad \text{and} \quad \mathrm{Var}(Q) = \sigma^2.$$

We assume that $X_1, X_2, \dots, X_n$ are conditionally independent given $\{Q = q\}$. let

$$S_n = X_1 + X_2 + \ldots + X_n$$

be the total number of heads in the n tosses.

1(a) (5 points) Use the law of iterated expectations to find $\mathbb{E}[X_i]$ and $\mathbb{E}[S_n]$.

1(b) (3 points) Using the fact that $X_i^2 = X_i$, show that $\mathrm{Var}(X_i) = \mu - \mu^2$.

1(c) (5 points) Using the law of iterated expectations, find

$$\mathrm{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j], \quad \text{for} \quad i \neq j.$$

Are X_i and X_j independent?

1(d) (7 points) By writing $\mathrm{Var}(S_n) = \mathbb{E}[S_n^2] - (\mathbb{E}[S_n])^2$, show that

$$\mathrm{Var}(S_n) = \mathbb{E}[\mathrm{Var}(S_n|Q)] + \mathrm{Var}(\mathbb{E}[S_n|Q]), \tag{1}$$

where $\mathrm{Var}(S_n|Q)$ is the random variable that takes on the value $\mathrm{Var}(S_n|Q = q)$ with probability $\Pr(Q = q)$.

1(e) (5 points) Calculate the variance of S_n by using the formula (1) in part 1(d).

Question 2

2(a) Let $\{N(t) : t > 0\}$ be a Poisson counting process with rate $\lambda = \ln 2 > 0$.

- (6 points) Find the probability that there are *no arrivals* in $(3, 5]$.
  Express your answer as a rational number (fraction).
- (7 points) Find the probability that there is *exactly one arrival* in each of the intervals $(0, 1]$, $(2, 3]$, and $(99, 100]$.
  Express your answer in terms of $\ln 2$.

2(b) (12 points) Let $\{N(t) : t > 0\}$ be a Poisson counting process with rate $\lambda > 0$. Find the covariance function of this process, i.e.,

$$C_N(t_1, t_2) := \mathrm{Cov}\big(N(t_1), N(t_2)\big), \quad \text{for } t_1, t_2 \in [0, \infty)$$

Hints: (i) First assume that $t_1 > t_2$. (ii) Write

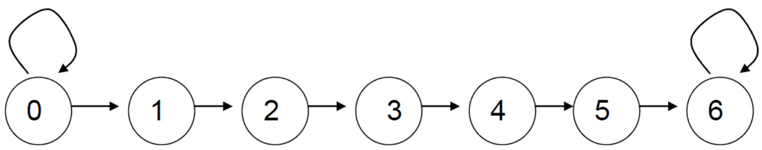
$$N(t_1) = \big[N(t_1) - N(t_2)\big] + \big[N(t_2)\big]$$

in some formula. (iii) You may use the fact that the variance of a Poisson random variable is the same as its mean.

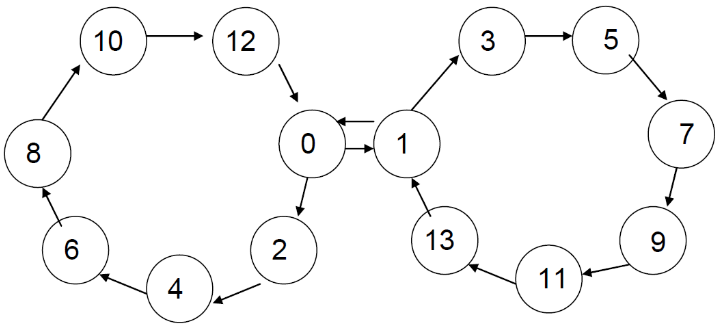
Question 3

3(a) For the following finite-state Markov chains, each transition is marked with $\longleftarrow$ or $\longrightarrow$, the transition probability is nonzero. For each chain, identify all classes, determine the period of each class, and specify whether each class is recurrent or transient. Explain your answer carefully.

- (5 points) Chain 1:



- (5 points) Chain 2:



3(b) An auto insurance company classifies its customers in three categories: bad, satisfactory and preferred. No one moves from bad to preferred or from preferred to bad in one year. 40% of the customers in the bad category become satisfactory, 30% of those in the satisfactory category moves to preferred, while 10% become bad; 20% of those in the preferred category are downgraded to satisfactory.

- (5 points) Write the state transition matrix for the model.
- (10 points) What is the limiting fraction of customers in each of these categories, i.e., the fraction of bad, satisfactory, and preferred customers after many years?

Question 4

Binary frequency shift keying (FSK) on a Rayleigh fading channel can be modeled in terms of a 4-dimensional observation vector

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}$$

which is given by $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$ where $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ and $\mathbf{Z}$ is independent of $\mathbf{X}$. Under the two hypotheses, $\mathbf{X}$ takes the following two values

$$\mathbf{X} = \begin{cases} \begin{bmatrix} X_1 \\ X_2 \\ 0 \\ 0 \end{bmatrix} & \text{if } H = H_0 \\ \begin{bmatrix} 0 \\ 0 \\ X_3 \\ X_4 \end{bmatrix} & \text{if } H = H_1 \end{cases}$$

The X_i's are i.i.d. $\mathcal{N}(0, \alpha^2)$ random variables. Furthermore, the two hypotheses are equally likely. Assume that $\alpha, \sigma \neq 0$.

4(a) (10 points) Show that the maximum likelihood receiver calculates $V_0 = Y_1^2 + Y_2^2$ and $V_1 = Y_3^3 + Y_4^2$ and chooses $\hat{H} = H_0$ if $V_0 \geq V_1$ and chooses $\hat{H} = H_1$ otherwise.

4(b) (2 points) It is known that if $A \sim \mathcal{N}(0, \nu)$ and $B \sim \mathcal{N}(0, \nu)$ are independent Gaussians, then the distribution (pdf) of $R = A^2 + B^2$ is exponential

$$f_R(t) = \frac{1}{2\nu} e^{-t/(2\nu)}, \quad \forall\, t \geq 0.$$

Using this fact, write down the distributions (pdfs) $f_{V_0|H}(v_0|H_0)$ and $f_{V_1|H}(v_1|H_0)$.

4(c) (6 points) Let $U = V_0 - V_1$. Using convolutions, show that $f_{U|H}(u|H_0)$ is

$$f_{U|H}(u|H_0) = \begin{cases} \frac{ab}{a+b} e^{bu} & u < 0 \\ \frac{ab}{a+b} e^{-au} & u \geq 0 \end{cases}$$

and identify the constants a and b in terms of σ^2 and α^2.

4(d) (7 points) Define the error event

$$\mathcal{E} := \{\hat{H} \neq H\}.$$

Find an expression in terms of a and b (and hence σ^2 and α^2 if you manage to get part 4(c)) for the conditional probability of error $\Pr(\mathcal{E}|H = H_0)$. Also find $\Pr(\mathcal{E}|H = H_1)$ and hence the unconditional probability of error $\Pr(\mathcal{E})$.

END OF PAPER