EE5137 Stochastic Processes: Problem Set 4
Assigned: 05/02/21, Due: 19/02/21

There are seven (7) non-optional problems in this problem set. You have two weeks to do this problem set. There are not many problems in Poisson processes as we have not covered enough, so I'm giving some practice problems on probability. You can use your spare time to read Chapter 2 of Gallager's book.

1. For a Poisson process, which of the following is/are true?
   (i) $\{N(t) \geq n\} = \{S_n \leq t\}$;
   (ii) $\{N(t) < n\} = \{S_n > t\}$;
   (iii) $\{N(t) \leq n\} = \{S_n \geq t\}$;
   (iv) $\{N(t) > n\} = \{S_n < t\}$.

2. An athletic facility has 5 tennis courts. Pairs of players arrive at the courts and use a court for an exponentially distributed time with mean 40 minutes. Suppose a pair of players arrives and finds all courts busy and k other pairs waiting in queue. What is the expected waiting time to get a court?

3. Exercise 2.3 (Gallager's book)

4. Prove that the Geometric distribution

   $$p_X(k) = (1-p)^{k-1}p, \quad k \in \mathbb{N} = \{1, 2, \ldots\}$$

   has the memoryless property.

   In fact, the Geometric distribution is the only distribution supported on $\mathbb{N}$ that is memoryless. This is analogous to the fact that the Exponential distribution is the only distribution supported on $[0, \infty)$ that is memoryless.

5. Let X_n denote a Binomial random variable with n trials and probability of success p_n. If $np_n \to \lambda$ as $n \to \infty$, show that for any fixed $i \in \mathbb{N} \cup \{0\}$,

   $$\Pr(X_n = i) \to \frac{e^{-\lambda}\lambda^i}{i!}, \quad \text{as } n \to \infty.$$

6. For a non-negative random variable X, show that for each $n \in \mathbb{N}$,

   $$\mathbb{E}[X^n] = \int_0^\infty n x^{n-1} F_X^c(x)\,\mathrm{d}x$$

   where F_X^c is the complementary CDF of X.

7. If X is a random variable with the property that $\Pr(0 \leq X \leq a) = 1$, show that

   $$\mathsf{Var}(X) \leq a^2/4.$$