EE5137 Lecture 9: Motivating Markov Chains: The PageRank Algorithm

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- So if you profess to be an expert in Markov chains, you don't have to write any papers on Markov chains.
- Just put 10^6 occurrences of "Markov chains" on your website!

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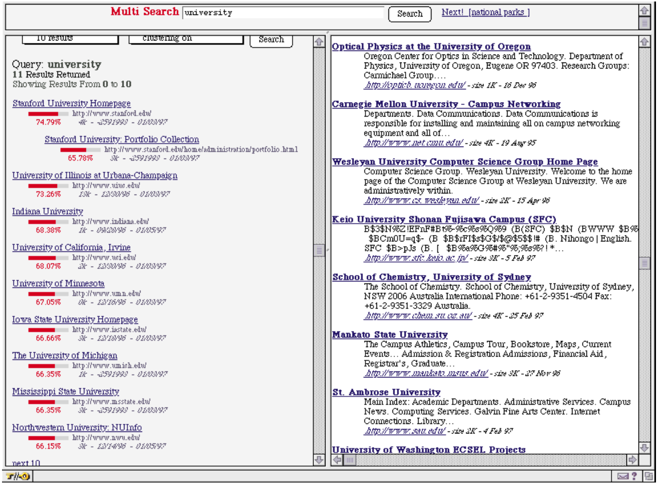

Taken from Page et al. (1999), “The PageRank Citation Ranking: Bringing Order to the Web”

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- Each web page i has an associated importance, or score r_i. This is a positive number.

Intuition Behind Pagerank

- The importance rule: If a page P links to m other pages $Q_1, Q_2, \ldots, Q_m$ then each page Q_i inherits $1/m$ of P's importance.

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- Random surfer interpretation:
  - A "random surfer" just sits at his computer all day, randomly clicking on links.
  - The pages he spends the most time on should be the most important.
  - Important pages are those where a random surfer will end up most often. This measure turns out to be equivalent to the score.

Importance Matrix

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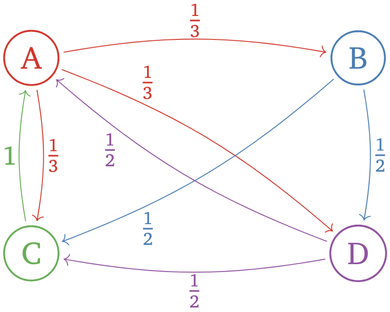
- Observe that the importance matrix is a row stochastic matrix, assuming every page contains a link.
- If page i has m outgoing links, then the i-th row contains the number $1/m$ a total of m times, and the number zero in the other entries.

Example of Importance Matrix

Internet with only 4 pages.



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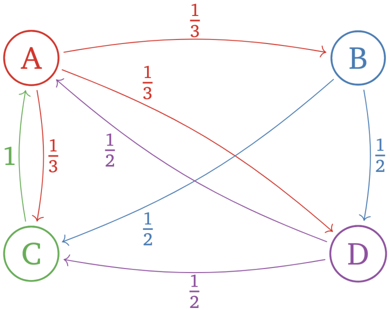

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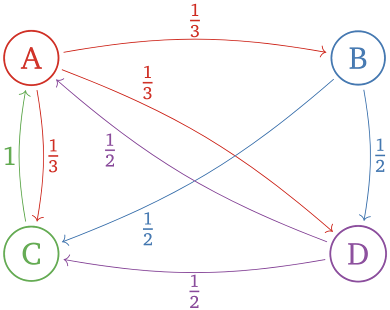

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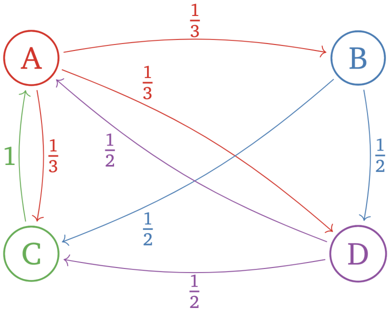

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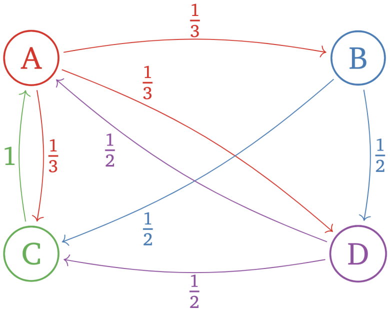

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- Page B has 2 links, so it passes $1/2$ of its impt to pages C, D;
- Page C has one link, so it passes all of its impt to page A;
- Page D has 2 links, so it passes $1/2$ of its impt to pages A, C.

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- By construction,

$$\begin{bmatrix} r_1 & r_2 & r_3 & r_4 \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 & r_4 \end{bmatrix}$$

because

$$\begin{aligned} r_1 &= r_3 + r_4/2, & r_2 &= r_1/3, \\ r_3 &= r_1/3 + r_2/2 + r_4/2, & r_4 &= r_1/3 + r_2/2. \end{aligned}$$

Equality expresses the importance rule.

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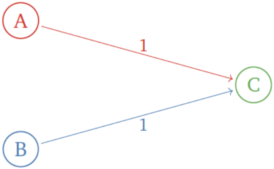
- r is a steady-state vector of $[Q]$
- So Google can construct $[Q]$ to rank webpages by learning r.
- Unfortunately, in real-life there are problems with this approach, e.g., $[Q]$ need not be ergodic!

Problem 1: Page with No Links



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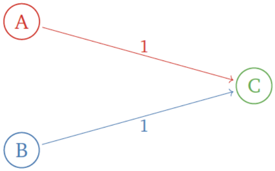

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- The characteristic polynomial is

$$\det([Q] - \lambda[I]) = \det\left(\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{bmatrix}\right) = -\lambda^3$$

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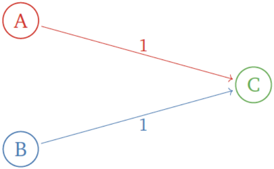

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- 1 is not an eigenvalue because $[Q]$ is not row stochastic!

Problem 2: Disconnected Internet

- The importance matrix is

$$[Q] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

- Two steady-state distributions

$$r^{(1)} = [1/2 \quad 1/2 \quad 0 \quad 0 \quad 0] \qquad r^{(2)} = [0 \quad 0 \quad 1/3 \quad 1/3 \quad 1/3]$$

Not ergodic! Two recurrent classes.

Page and Brin's Solution: Fix of First Problem

- Replace zero rows by adding a row of $1/n$'s, where n is the total number of pages

$$[Q'] = [Q] + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \quad \text{so} \quad [Q'] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

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- Now $[Q']$ is indeed row stochastic.

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- In the random surfer interpretation, this matrix says:
  - With probability p, our surfer will surf to a completely random page;
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  - Otherwise, he'll click a random link on the current page;
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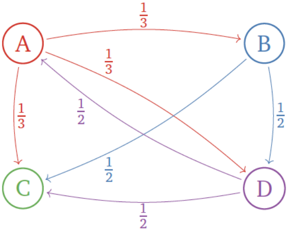
- $[P]$ is ergodic $\implies$ Has a unique steady-state vector r or π!

Page and Brin's Solution: Example



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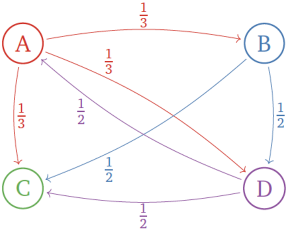

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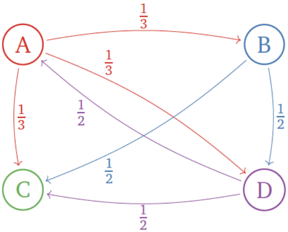

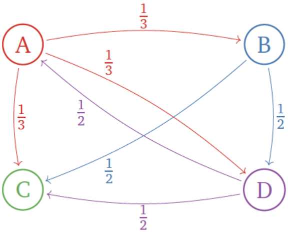

- Fix the second problem

$$[P] = 0.85 \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} + 0.15 \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.0375 & 0.3208 & 0.3208 & 0.3208 \\ 0.0375 & 0.0375 & 0.4625 & 0.4625 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.4625 & 0.0375 & 0.4625 & 0.0375 \end{bmatrix}$$

Learned Score Vector

- The Google Matrix is

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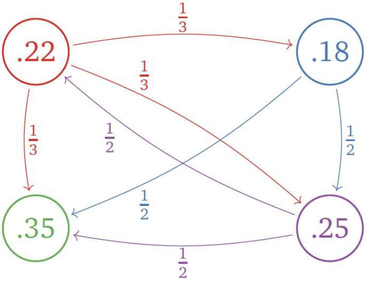
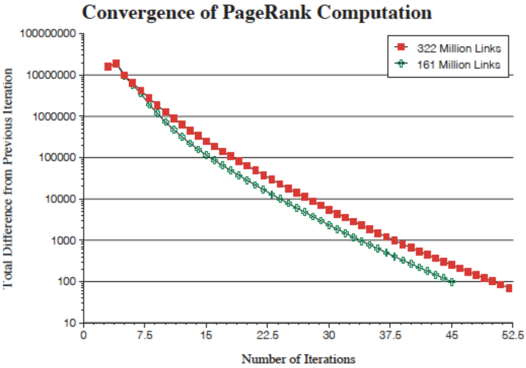
- The left-eigenvector with eigenvalue 1 is

Pagerank solution $\quad \pi = r = \begin{bmatrix} 0.2192 & 0.1752 & 0.3558 & 0.2498 \end{bmatrix}.$

- and the ranking of the webpages is C, D, A, B.



Rate of Convergence



Taken from “The PageRank Citation Ranking: Bringing Order to the Web”

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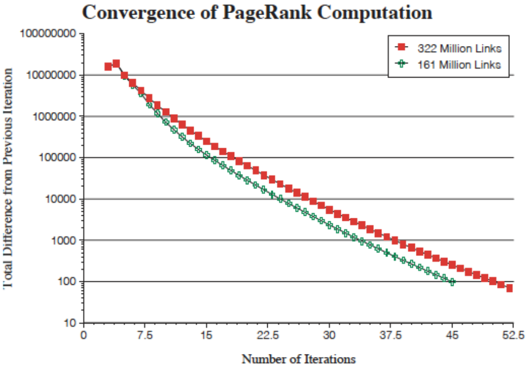

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- Fast convergence due to large spectral gap – difference between largest eigenvalue 1 and $1 - p = 0.85$.

Moral of the Story

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"Beautiful math tends to be useful; useful things tend to have beautiful math." ... Statistics is often where it comes together.

References

- https://textbooks.math.gatech.edu/ila/stochastic-matrices.html
- http://statweb.stanford.edu/ tibs/sta306bfiles/pagerank/ryan/01-24-pr.pdf