ECE 587 Midterm Review

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Concepts related to Entropy

The Asymptotic Equipartition Property (AEP)

Entropy Rates of a Stochastic Process

Data Compression

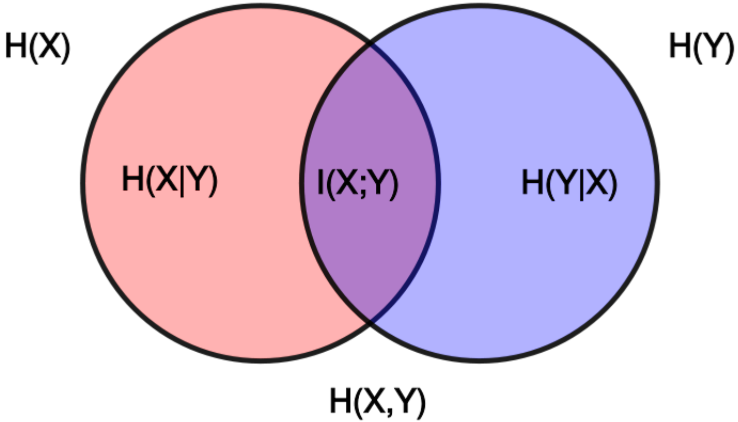

Figure: Venn diagram for entropy and mutual information

Entropy

- If the inequality is true prove it, otherwise, give a counter example
  - $H(X|Y) \leq H(X)$
  - $H(X, Y|Z) \geq H(X|Z)$
  - $H(X|Z) \leq H(Z)$
  - $H(X, Y, Z) - H(X, Y) \geq H(X, Z) - H(X)$
  - $H(X|Z) \geq H(Z)$
- Relative entropy $D(p||q)$
  - show that $D(p||q) \geq 0$
  - Is relative entropy symmetric, i.e $D(p||q) = D(p||q)$?

Law of large numbers (LLN) for product of random variables

- Since $X_i = \exp\{\ln X_i\}$
- we have $\sqrt[n]{\prod_{i=1}^{n} X_i} = \exp\{\frac{1}{n}\sum_{i=1}^{n} \ln X_i\}$.
- Hence

$$
\begin{aligned}
\sqrt[n]{\prod_{i=1}^{n} X_i} &= \exp\{\frac{1}{n}\sum_{i=1}^{n} \ln X_i\} && (1) \\
&\to \exp\{\mathbb{E} \ln X\} \quad \text{(LLN)} && (2) \\
&\leq \exp\{\ln \mathbb{E} X\} = \mathbb{E} X \quad \text{(Jensen's Inequality)} && (3)
\end{aligned}
$$

The Asymptotic Equipartition Property (AEP)

- AEP. Let X_i be *iid* $\sim p(x), x \in \{1, 2, \cdots, m\}$. Let $\mu = \mathbb{E}X$ and $H = -\sum p(x) log(x)$. Let $A^n = \{x^n \in \mathcal{X}^n : \left|-\frac{1}{n} \log p(x^n) - H\right| \leq \epsilon$. Let $B^n = \{x^n \in \mathcal{X}^n : \left|-\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| \leq \epsilon\}$
  - $Pr\{X^n \in A^n\} \rightarrow 1$?
  - Does $Pr\{X^n \in A^n \cap B^n\} \rightarrow 1$?
  - Show that $|A^n \cap B^n| \leq 2^{n(H+\epsilon)}$ for all n.
  - Show that $|A^n \cap B^n| \leq (\frac{1}{2})2^{n(H-\epsilon)}$ for n sufficiently large.

Entropy rate of random walk on a weighted graphs

- The nodes of graphs are random variables, with state distribution π_t at time t and transition probability matrix T.
- T is determined by the weights over the edges.
- Stationary distribution is a distribution *pi* over states, such that $\pi_t = \pi_{t+1}$ (shift invariant).
- The stationary distribution $\pi(i), i = 1, \cdots, |\mathcal{X}|$ satisfies

$$\pi = T\pi. \tag{4}$$

Entropy Rates of a Stochastic Process

- Monotonicity of entropy per element. For a stationary stochastic process $X_1, X_2, \cdots, X_n$, show that
  - $$\frac{H(X_1, X_2, \cdots, X_n)}{n} \leq \frac{H(X_1, X_2, \cdots, X_{n-1})}{n-1} \tag{5}$$
  - $$\frac{H(X_1, X_2, \cdots, X_n)}{n} \geq H(X_n | X_{n-1}, \cdots, X_1). \tag{6}$$
- Entropy rates of Markov Chains
  - What is the stationary distribution π
  - Find the entropy rate the two-state Markov Chain with transition matrix

$$\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

Data compression

- Huffman Coding Consider the random variable

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & X_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

  - Find the binary Huffman code for X.
  - Find the expected code length for this encoding.
  - Find a ternary Huffman code for X.

Summary

Good Luck!