EE5139
Homework 1

Semester I, AY2021/2022
Marco Tomamichel

Exercise 1.1 Expectation value and variance [EE5139]

Let V and W be discrete random variables defined on some probability space with a joint pmf $P_{VW}(v, w)$. We do not assume independence.

a.) Prove that $\mathbb{E}[V + W] = \mathbb{E}[V] + \mathbb{E}[W]$.

b.) Prove that if V and W are independent, then $\mathbb{E}[VW] = \mathbb{E}[V]\mathbb{E}[W]$.

c.) Let V and W be independent and let σ_V^2 and σ_W^2 be their respective variances. Find the variance of $Z = V + W$.

Exercise 1.2 Coin flips [EE5139]

Flip a fair coin four times. Let X be the number of Heads obtained, and let Y be the position of the first Heads i.e. if the sequence of coin flips is TTHT, then $Y = 3$, if it is THHH, then $Y = 2$. If there are no heads in the four tosses, then we define $Y = 0$.

a.) Model the experiment completely, i.e. define the sample space and the random variables X and Y as functions from that sample space.

b.) Find the joint pmf of X and Y. What is $\Pr[Y = 0|X = 1]$ and $\Pr[Y = 1|X = 3]$?

c.) Using the joint pmf, find the marginal pmf of X and Y.

Exercise 1.3 Property of convex functions [EE5139]

Let f be convex on $[a, b]$. Using only the defining property of convex functions, show that for any $a \le x_1 < x_2 \le x_3 < x_4 \le b$, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_4) - f(x_3)}{x_4 - x_3}.$$

Hint: First show the simpler statement for the case when $x_2 = x_3$, and then apply that results twice to get the general statement.

Exercise 1.4 Finite fields [EE5139]

Derive the addition and multiplication tables for F_8 and F_9. You should use the construction described in the lecture notes and the irreducible polynomials $x^3 + x + 1$ for F_8 and $x^2 + 1$ for F_9. **Hint:** You may want to use Matlab to solve this problem. However, you will need to compute some elements by hand to verify the computer-generated output.

Exercise 1.5 Continuous and discrete random variables [all]

Consider the following random experiment. A ball is thrown and lands after X meters, where X is distributed uniformly in the interval $[1, 2]$. It either stays there or bounces off and jumps again an additional distance of $\frac{1}{2}X$. The binary random variable $Y \in \{0, 1\}$ takes the value 0 (with probability 50%), indicating that the ball stays put, and 1 (with probability 50%), indicating that the ball jumps again. After this additional bounce the ball rests.

a.) Express the total distance Z that the ball travels in terms of X and Y. Compute and plot the pdf for Z.

b.) Find the pmf for Y given $Z = z$. Plot $\Pr[Y = 0|Z = z]$ as a function of z.

Hint: We would be inclined to use Bayes' rule here, but the problem is that the $\Pr[Z = z] = 0$ for each z. To avoid this, consider an interval $z \pm \epsilon$ and compute the pmf for Y given $Z \in [z - \epsilon, z + \epsilon]$ and then let $\epsilon \to 0$.

Exercise 1.6 Matrix representation of a communication channel [all]

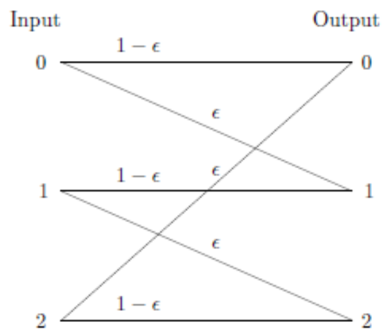

Figure 1: Ternary communication channel

A ternary communication channel is shown in Figure 1.

a.) Represent the channel as a matrix W such that the output distribution of the channel can be written as the matrix product pW, where p is a row vector containing the three input probabilities.

Hint: The matrix W should be a 3×3 matrix.

b.) Suppose that the input probabilities are given by the vector $p = [\frac{1}{2}, \frac{1}{4}, \frac{1}{4}]$. Find the probabilities of the output symbols.

c.) Suppose that 1 was observed at the output. What's the probability that the input was 0? 1? 2?

Exercise 1.7 Further tail bounds [EE6139]

a.) For a nonnegative integer-valued random variable N, show that $\mathbb{E}[N] = \sum_{n>0} \Pr(N \geq n)$.

b.) Derive the Cauchy-Schwarz inequality, which says that $\mathbb{E}[AB] \leq \sqrt{\mathbb{E}[A^2]\mathbb{E}[B^2]}$.

Hint: Consider the non-negative random variable $(X - \alpha Y)^2$ and compute its expectation, then choose α appropriately.

c.) Derive the one-sided Cheybyshev inequality, which says that $\Pr(Y \geq a) \leq \sigma_Y^2/(\sigma_Y^2 + a^2)$ if $\mathbb{E}[Y] = 0$ and $a > 0$.

d.) Derive the reverse Markov inequality: Let X be a random variable such that $\Pr(X \leq a) = 1$ for some constant a. Then for $d < \mathbb{E}[X]$, we have

$$\Pr(X > d) \geq \frac{\mathbb{E}[X] - d}{a - d}.$$

e.) Chernoff Bound: Let $X_1, \ldots, X_n$ be a sequence of i.i.d. rvs with zero-mean and moment generating function $M_X(s) := \mathbb{E}[e^{sX}]$. Show that for any $\epsilon > 0$,

$$\Pr\left(\frac{1}{n}(X_1 + \ldots + X_n) > \epsilon\right) \leq \exp\left[-n \max_{s \geq 0}(\epsilon s - \log M_X(s))\right].$$

Hint: Note that the event $\{\frac{1}{n}(X_1 + \ldots + X_n) > \epsilon\}$ occurs if and only if $\{\exp(s(X_1 + \ldots + X_n)) > \exp(ns\epsilon)\}$ occurs for any fixed $s \geq 0$. Now apply Markov's inequality.