EE5137 Stochastic Processes: Problem Set 6
Assigned: 26/02/21, Due: 05/03/21

There are five (5) non-optional problems in this problem set.

1. Exercise 2.10 (Gallager's book)
2. Exercise 2.16 (Gallager's book)
3. Exercise 2.17 (Gallager's book)
4. (a) Let $\{N(t) : t > 0\}$ be a Poisson counting process with rate $\lambda > 0$. Let T_1 be an exponential random variable independent of $\{N(t) : t > 0\}$ with probability density function

$$f_{T_1}(t) = \begin{cases} \nu \exp(-\nu t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

for some $\nu > 0$. What is the distribution (probability mass function) of $N(T_1)$, the number of Poisson arrivals of the first process in the interval $[0, T_1]$?

   (b) Let $\{N(t) : t > 0\}$ be as in part (a). Now, let T_2 be an Erlang random variable of order 2 independent of $\{N(t) : t > 0\}$ with probability density function

$$f_{T_2}(t) = \begin{cases} \nu^2 t \exp(-\nu t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

for some $\nu > 0$. What is the distribution (probability mass function) of $N(T_2)$, the number of Poisson arrivals of the first process in the interval $[0, T_2]$?
Hint: Drawing a figure might be helpful.

5. Let $\{N(t) : t > 0\}$ be the Poisson counting process with rate λ. The *compensated Poisson process* is defined as

$$M(t) = N(t) - \lambda t$$

Let $\mathcal{F}_t := \{M(\tau) : 0 < \tau \leq t\}$ be the process up to and including time t. A *continuous-time martingale* $\{X(t) : t > 0\}$ is a stochastic process satisfying

$$\mathbb{E}[|X(t)|] < \infty \quad \text{and} \quad \mathbb{E}[X(t) \mid \mathcal{F}_s] = X(s) \quad \text{a.s.} \quad \forall\, t > s > 0.$$

   (a) Find the mean and variance of $M(t)$.
   (b) Does $\{M(\tau) : 0 < \tau \leq t\}$ have the (a) stationary increments property and (b) independent increments property?
   (c) Show that $\{M(\tau) : 0 < \tau \leq t\}$ is a continuous-time martingale.
   (d) Let $\tilde{M}(t, t+\delta) = M(t+\delta) - M(t)$. Show using the SIP, IIP, and the incremental property of the Poisson process (Eqn. (2.19) of Gallager) that

$$\mathbb{E}[\tilde{M}(t, t+\delta)^2 \mid \mathcal{F}_t] = \lambda\delta + o(\delta).$$

6. (Optional) Continuing from Problem 5(c), we have the following interesting converse result due to Shinzo Watanabe[1]. A *counting process* $\{N(t) : t > 0\}$ is a continuous-time stochastic process with $N(0) = 0$ and N is constant except for jumps of $+1$. Show that if $\{N(t) : t > 0\}$ is a counting process and $\{M(t) = N(t) - \lambda t : t > 0\}$ is a (continuous-time) martingale, then $\{N(t) : t > 0\}$ is a Poisson process of rate λ. This is yet another characterization of a Poisson process.

   Hint: It can be shown using Itô's formula and the fact that $M(t)$ is a martingale that

   $$X(t) = \exp\big(uN(t) - (e^u - 1)\lambda t\big)$$

   is a martingale. Use this and transforms (moment generating functions) that $N(t)$ is a Poisson process.

7. (Optional) Exercise 2.13 (Gallager's book)

[1] S. Watanabe, "On discontinuous additive functionals and Lévy measures of Markov processes". Japanese J. Maths. 34, 1964.