Exercise 5.1

a). For $\alpha \to 0^+$, $H_\alpha(X) = \frac{1}{1-\alpha} \log \sum_x P(x)^\alpha$

$$\Rightarrow = \log |X|$$

For $\alpha \to 1$, $H_\alpha(X) = \cancel{\frac{1}{1-\alpha}\Big|_{\alpha\to 1} \log \sum_x P(x)} \ \frac{1}{1-\alpha} \log \sum_x P(x)$

$$\cancel{\Rightarrow = \frac{1}{\alpha}\Big|_{\alpha\to 1} \log} \Rightarrow \sum_x P_x \log \frac{1}{P(x)}$$

~~we should rewrite this equatition as continuus form~~

As desired, this is shannon entropy

for $\alpha \to +\infty$, $H_\alpha(X) = \frac{1}{1-\alpha} \log(\sum_x P(x)^\alpha) = \frac{\alpha}{1-\alpha} \log \sum_x P(x)$

As $\frac{\alpha}{1-\alpha}\Big|_{\alpha \to +\infty} = -1$

Therefore, $H_\alpha(X) = -\log \max P(x)$ [norm equation]

As desired, this is min-entropy

b). I would upload the figure which is generated by Matlab

c). For $H_\alpha(X) = \frac{1}{1-\alpha} \log \sum_x P(x)^\alpha$

① in the range of $(0,1)$, we consider $\frac{dH_\alpha(X)}{d\alpha} = \frac{1}{(1-\alpha)}' \log \sum_x P_x(x)^\alpha$

$$+ \frac{1}{1-\alpha} \log \sum_x P_x(x)$$

$$= \frac{-2\alpha}{(1-\alpha)^2} \log \sum_x P_x(x) \leq 0.$$

② Similarily, in the range of $(1, +\infty)$

$$\frac{d H_\alpha(X)}{d\alpha} \leq 0.$$

In a word, this $H_\alpha(X)$ is non-increasing in the parameter.

$$H_{min}(X) \leq H(X) \leq \log |X|$$

d). For the min-entropy $H_{min}(X|Y)$

$H_{min}(X|Y) = -\log \max P(X|Y)$

And for $P(X|Y) \Rightarrow$

$P(X=0|Y=0) = \frac{1}{2}$ $P(X=0|Y=1) = \frac{1}{4}$ $P(X=0|Y=2) = \frac{1}{4}$

$P(X=1|Y=0) = \frac{1}{4}$ $P(X=1|Y=1) = \frac{1}{2}$ $P(X=1|Y=2) = \frac{1}{4}$

$P(X=2|Y=0) = \frac{1}{4}$ $P(X=2|Y=1) = \frac{1}{4}$ $P(X=2|Y=2) = \frac{1}{2}$

$\max P(X|Y) = \frac{1}{2}$

Therefore $H_{min}(X|Y) = -\log\frac{1}{2} = 1$

Exercise 5.2

a). $H(X_n) \geq (1-\varepsilon)\log n$

$H_{min}(X_n) = C$

In order to satisfy above requirement, we set this sequence

Any sequence can satisfy this requirement, obviously.

b) $H(X_n) = H_{min}(X_n) = \log n$

$H(X_n|Y_n) \geq (1-\varepsilon)\log n$

$H_{min}(X_n|Y_n) = C$

In order to satisfy above requirement, we set this sequence

① X_n and Y_n are i.i.d. variable.

② And then it satisfy a) part requirement is used.

Exercise 5.3

a). Using the definition of typical set

$$2^{nH(x)-\varepsilon} \le Pr\left[A_\varepsilon^{(n)}\right] \le 2^{nH(x)+\varepsilon}$$

So, $W(x_j) = \sum_{j\in *} W(x_j = *) = \sum_{(j=a)} W(x_j=a) + \sum_{(j=b)} W(x_j=b)$

$$= -\log\frac{2}{3} - \log\frac{1}{3}$$

$$= -\log\frac{2}{9}$$

$$= \log\frac{9}{2}$$

For the bound of this typical set, we can get

$$2^{n\cdot\log\frac{9}{2} - 0.01} \le Pr\left[A_\varepsilon^{(n)}\right] \le 2^{n\log\frac{9}{2}+0.01}$$

b). For this statement, we know N_a be the number of a' in the string $x^n = (X_1, \cdots, X_n)$.

In this part, we know DMS (two symbol alphabet $\{a, b\}$)

Therefore, these variables are x_j who belongs to $\{a\}$

c) For the definition, $W(X^n)$ ~~as a function of the~~

$$= \sum_{N=1}^{n} W(X_j)$$

$$= \sum^{N_a} W(x_j=a) + \sum^{N_b} W(X_j=b)$$

$$= N_a\, W(x_j) + (n-N_a)\, W(x_j)$$

~~$= n\cdot W(x_j)$~~

In a word, $W(x^n)$ as a function of N_a

And at the same time, it depends on n. $(n = 100000)$

d).

Using Chebyshev's inequality, and we assume that $\alpha < N_\alpha < \beta$

so, we can get these findings.

$$2^{nW(x_i)-\varepsilon} \leq Pr(A_\varepsilon^{(n)}) \leq 2^{nW(x_i)+\varepsilon}$$

Therefore, this typical set is in terms of bounds on N_x

e). $Pr(Na = 0) = 2^{-1}$

$Pr(Na = 1) = 2^{-2}$

$Pr(Na = 2) = 2^{-3}$

Therefore, the particular string x^n that has maximum probability over all sample values of x^n.

Next most probable n-strings = aaa

Exercise 5.4.

a). For the statement of this question, following the illustration,

i). X and Y are independent

$$R^*(X|Y) = H(X|Y) = H(X)$$

ii) $X = Y$

$$R^*(X|Y) = H(X|X) = 0.$$

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b).

By explicitly constructing a code for the source (X, Y) using codes for the sources Y and X (with side information Y).

$$R^*(X,Y) = \lim_{n\to\infty} \frac{H(X,Y)}{n} \quad \text{(using the definition)}$$

$$\leq \lim_{n\to\infty} \frac{H(X|Y) + H(Y)}{n}$$

$$\leq \lim_{n\to\infty} \frac{H(X|Y)}{n} + \lim_{n\to\infty} \frac{H(Y)}{n}$$

$$\leq R^*(X|Y) + R^*(Y)$$

c) For the converse part,

$$
\begin{aligned}
R^*(X|Y) &\geq R^*(X^n | \hat{X}^n) \\
&\geq H(X^n | \hat{X}^n) \\
&= H(X^n | Y^n M) \\
&= H(X^n M | Y^n) - H(M | Y^n) \\
&\geq H(X^n M | Y^n) - L \\
&\geq H(X^n | Y^n) - L
\end{aligned}
$$

It is desired, $R^*(X|Y) \geq H(X|Y)$

d). Using the typical set, $A_\varepsilon^{(n)}(X|Y) := \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \left| \frac{1}{n} \log \frac{1}{P_{X^n|Y^n}(x^n|y^n)} - H(X|Y) \right| \leq \varepsilon \right\}$

Therefore, we can prove that $R^*(X|Y) \leq H(X|Y)$.