EXERCISE 4.1

(a) For the entropy of X, $H(x) = \sum_{x \in \mathcal{X}} P(x=i) \log \frac{1}{P(x=i)}$

$$= \sum_{x \in \mathcal{X}} 2^{-i} \log 2^{i}$$
$$= \sum_{x \in \mathcal{X}} i \cdot 2^{-i}$$

$$H(x) = 1 \times 2^{-1} + 2 \times 2^{-2} + 3 \times 2^{-3} + \cdots i \cdot 2^{-i} \quad \cdots \text{①}$$
$$\tfrac{1}{2} H(x) = \quad + 1 \times 2^{-2} + 2 \times 2^{-3} + \cdots (i-1) \cdot 2^{-i} + i \cdot 2^{-i-1} \cdots \text{②}$$

① − ②, we get $\frac{1}{2} H(x) = 1 - (\frac{i}{2}+1)(\frac{1}{2})^{i}$

$$H(x) = 2 - (i+2)(\tfrac{1}{2})^{i}$$
$$= 2 - \frac{i+2}{2^{i}}$$
$$= 2$$

(b). For Huffman coding, we set $P(x=1) = 2^{-1} \Rightarrow 0$

$P(x=2) = 2^{-2} \Rightarrow 10$

$\vdots$

$P(x=i-1) = 2^{-(i-1)} \Rightarrow 111\cdots0$

$P(x=i) = 2^{-i} \Rightarrow 111\cdots1$

Expected length $=$ $1 \times 2^{-1} + 2 \times 2^{-2} + \cdots + (i-1) \times 2^{-(i-1)} + (i-1) \times 2^{-i}$

We know the entropy of x [from part (a)]

$$H(x) = 1 \times 2^{-1} + 2 \times 2^{-2} + \cdots + (i-1) \times 2^{-(i-1)} + i \times 2^{-i}$$

Expected length − H(X) = -2^{-i}

so Expected length $\le$ H(x) $\cdots$ ①

And at the same time Expected length $\ge$ H(x) $\cdots$ ②

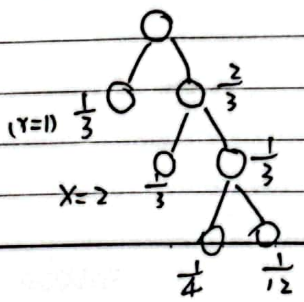
From ①, ②, so Expected length = H(X)

In a word, it is indeed optimal code.

Exercise 4.2

a). Through the order from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12})$

$Pr(X=1) = \frac{1}{3}$, $Pr(X=2) = \frac{1}{3}$, $Pr(X=3) = \frac{1}{4}$, $Pr(X=4) = \frac{1}{12}$



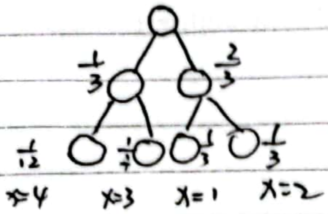
$x=1 \Rightarrow 0$

$x=2 \Rightarrow 10$

$x=3 \Rightarrow 110$

$x=4 \Rightarrow 111$

(b) However, it exists a different optimal set



$x=1 \Rightarrow 10$

$x=2 \Rightarrow 11$ $\quad H(x) = 1.855$ bit

$x=3 \Rightarrow 01$

$x=4 \Rightarrow 00$

Therefore $(1,2,3,3)$ and $(2,2,2,2)$ all exists.

Next, we prove why these two codeword length assignments are both optimal.

For $(1,2,3,3)$, we get expected length $1\times\frac{1}{3} + 2\times\frac{1}{3} + 3\times\frac{1}{4} + 3\times\frac{1}{12} = 2$ bits

$1.855 \text{ bits} < 2 \text{ bits} < 1.855+1 \text{ bits}$

Therefore, it is optimal

For $(2,2,2,2)$, we get expected length $2\times\frac{1}{3} + 2\times\frac{1}{3} + 2\times\frac{1}{4} + 2\times\frac{1}{12} = 2$ bits.

with same calculation, we also can adust this assignment lengths set is also optimal.

c) No, there are no any optimal codes with codeword lengths can exceed the shannon code length $\lceil \log \frac{1}{p(x)} \rceil$

For shannon code: $\frac{1}{3}\times\lceil\log_2 3\rceil + \frac{1}{3}\times\lceil\log_2 3\rceil + \frac{1}{4}\times\lceil\log_2 4\rceil + \frac{1}{12}\times\lceil\log 12\rceil$

$= \frac{2}{3} + \frac{2}{3} + \frac{1}{2} + \frac{1}{3}$

$= 2.166$ bit

Obviously, $2.166 \text{ bits} > 2 \text{ bits}$

Therefore, there are no any optimal codes with codeword lengths can exceed the shannon code length $\lceil \log \frac{1}{p(x)} \rceil$