Lecture 6: Reading 2.3.2, 2.5.

Eg: Let S_{1k} be the epoch of the k^{th} arrival of Poisson process 1.
Let S_{2j} be the epoch of the j^{th} arrival of Poisson process 2.

PP1 has rate λ_1; PP2 has rate λ_2.

Qn: $Pr(S_{1k} < S_{2j})$: Prob. that the kth epoch of the 1st process comes before the jth epoch of the second process.

Conside a combined/merged process with rate $\lambda_1 + \lambda_2$.

Say $k=3$ (arrivals indicated by x) & $j=4$ (arrivals indicated by o).

$0 \quad S_{21} \quad S_{11} \quad S_{12} \quad S_{22} \quad S_{23} \quad S_{13} \quad \Rightarrow S_{13} < S_{24}$

What do we need for $S_{1k} < S_{2j}$? (or $S_{13} < S_{24}$?)

We need, out of the first $k+j-1$ arrivals of the combined process k or more of them to be switched to the first process.

Define $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$.

$Pr(\text{exactly } k \text{ are switched to the first process}) = \binom{k+j-1}{k} p^k (1-p)^{j-1}$.

$Pr(\text{exactly } i \geq k \text{ are switched to the first process}) = \binom{k+j-1}{i} p^i (1-p)^{k+j-1-i}$.

$\Rightarrow Pr(S_{1k} < S_{2j}) = Pr(\text{more than or equal to } k \text{ are switched to the first process})$

$$= \sum_{i=k}^{k+j-1} \binom{k+j-1}{i} p^i (1-p)^{k+j-1-i}$$

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Ex: M/M/1 Queue

First character (M): M stands for memoryless & mean Poisson arrival process.

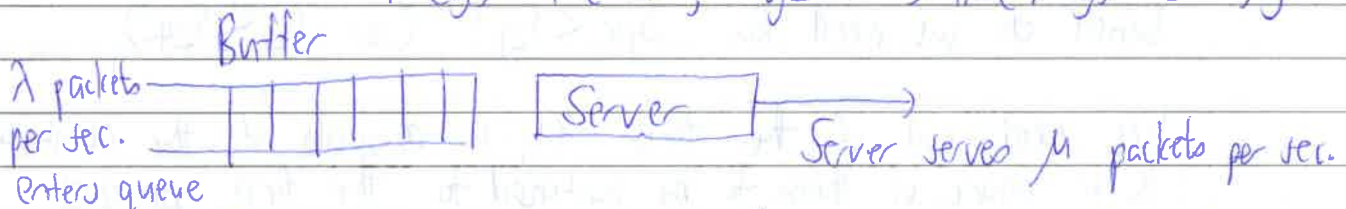
Second character (M): Service process is also memoryless.

Third character (1): Number of servers.

Consider a queuing system with arrival of customers following $PP(\lambda)$, a Poisson process with rate λ.

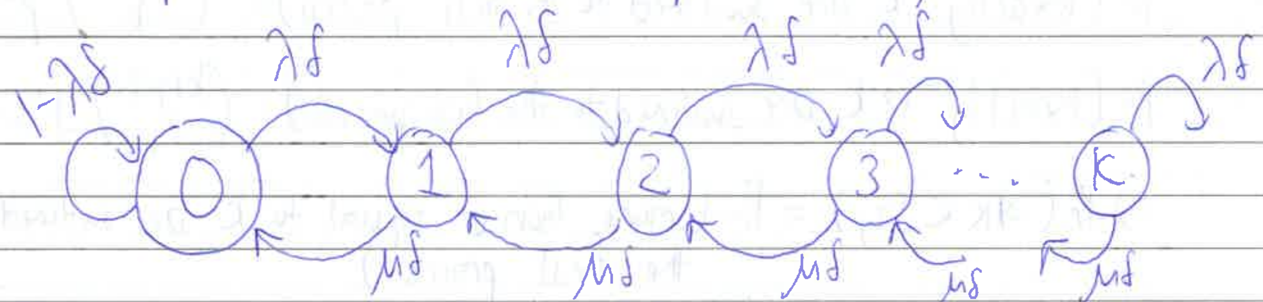
Also consider a single server queue that serves customers with a service time distribution that is ~~also~~ $Exp(\mu)$.

$$F(y) = 1-e^{-\mu y}, \quad y \geq 0. \Rightarrow Pr(Y>y) = e^{-\mu y}, y \geq 0.$$



Qn: $Pr(S_{1k} < S_{2j})$ = prob. that k^{th} packet arrives before j^{th} departure

Arrival epochs of a $PP(\lambda)$ Arrival epochs of a $PP(\mu)$.



State K: K customers (or packets) in the system.

$P(i,j)$ = Prob. transition ~~at~~ from state i to state j $(i,j \in \mathbb{N} \cup \{0\})$

Also, let δ be an infinitesimally small time interval. $\delta \to 0^+$

$P(0,0) = 1-\lambda\delta$, $P(j,j+1) = \lambda\delta$, $P(j,j-1) = \mu\delta$, $P(j,j) = 1-(\lambda+\mu)\delta$.

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This is an example of a "birth-death" process that we'll study in detail when we talk about Markov chains.

$\lambda\delta$: flow rate from j to $j+1$.
$\mu\delta$: flow rate from j to $j-1$. $\rho = \frac{\lambda}{\mu}$.

Let $P(k)$ be the probability of being in state k.

At "equilibrium", $\lambda P(k) = \mu P(k+1) \Rightarrow P(k) = \left(\frac{\lambda}{\mu}\right)^k P(0)$.

Since $\sum_{k=0}^{\infty} P(k) = 1 \Rightarrow \sum_{k=0}^{\infty} P(0)\rho^k = 1 \Rightarrow P(0)\frac{1}{1-\rho} = 1 \Rightarrow P(0) = 1-\rho$.

$\Rightarrow P(k) = \rho^k(1-\rho)$. (requires that $\rho < 1$, i.e., stability).

Average queue size $X \sim P$ $\mathbb{E}X = \sum_{k=0}^{\infty} k P(k) = \sum_{k=0}^{\infty} k\rho^k(1-\rho) = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$

Note that if $\lambda < \mu$, the arrival rate of packets to the buffer is smaller than the service rate.

Intuitively, all the packets will be served before they "accumulate" in the buffer, causing an overflow.

Example: # of Poisson Arrivals during an Exponentially distributed Int.

Consider a $PP(\lambda)$ & an independent rv $T \sim Exp(\nu)$. We would like to find the pmf of the # of Poisson arrivals in $[0,T]$, i.e., the pmf of the rv $N(T)$, where $\{N(t): t>0\}$ is the Poisson counting process with rate $\lambda > 0$.

Answer: View T as the first arrival of an indep. Poisson process with rate ν and merge this process with the original one.

Each arrival in this merged process comes from the original one with prob. $\lambda/(\lambda+\nu)$ independent of all other arrivals.

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K = # of arrivals until the first success (where success = arrival of the form of ~~a~~ the new process)

$$P_K(k) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^{k-1}, \quad k=1,2,3,\cdots$$

L = # of arrivals of the original Poisson process until the first success.

$$L = N(T) = K-1.$$

$$\Rightarrow \quad P_{N(T)}(\ell) = P_L(\ell) = P_K(\ell+1) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^{\ell}, \quad \ell=0,1,2,\cdots$$

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Conditional Arrival Densities & Order Statistics.

Question: Conditioned on the event $\{N(t)=n\}$, there are n arrivals in $(0,t]$. What is the distribution of the n arrival epochs $S^n = S^{(n)} = (S_1, S_2, \ldots, S_n)$?

Thm. Let $f_{S^n|N(t)}(s^n|n)$ be the joint density of S^n given $N(t)=n$. The density is constant over the region $0 < s_1 < s_2 < \cdots < s_n < t$ and has value

$$f_{S^n|N(t)}(s^n|n) = n!/t^n$$

Pf: Recall the joint density of $S^{n+1} = (S_1, S_2, \ldots, S_n, S_{n+1})$

$$f_{S_1 \ldots S_{n+1}}(s_1, s_2, \ldots, s_{n+1}) = \lambda^{n+1} \exp(-\lambda s_{n+1}), \; 0 \le s_1 \le s_2 \le \cdots \le s_n \le s_{n+1}.$$

By Bayes rule,

$$f_{S^{n+1}|N(t)}(s^{n+1}|n)\, p_{N(t)}(n) = p_{N(t)|S^{n+1}}(n|s^{n+1})\, f_{S^{n+1}}(s^{n+1})$$

Note that $N(t)=n$ iff $t \le S_n$ & $S_{n+1} > t$.
Thus $p_{N(t)|S^{n+1}}(n|s^{n+1}) = 1$ if $s_n \le t$ & $s_{n+1} > t$ and 0 otherwise.

Restricting our attention to the case $N(t)=n$, $S_n \le t$ & $S_{n+1} > t$, we have

$$f_{S^{n+1}|N(t)}(s^{n+1}|n) = \frac{f_{S^{n+1}}(s^n)}{p_{N(t)}(n)} = \frac{\lambda^{n+1}\exp(-\lambda s_{n+1})}{(\lambda t)^n \exp(-\lambda t)/n!}$$

$$= \frac{n!\,\lambda \exp(\lambda(s_{n+1}-t))}{t^n}.$$

Now $f_{S^{n+1}|N(t)}(s^{n+1}|n) = f_{S^n|N(t)}(s^n|n)\, \underbrace{f_{S^{n+1}|S^n,N(t)}(s_{n+1}|s^n,n)}_{\lambda \exp(-\lambda(s_{n+1}-t))}$

We thus obtain $f_{S^n|N(t)}(s^n|n) = n!/t^n$, $0 \le s_1 \le s_2 \le \cdots \le s_n$.

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How about the joint density of the interarrival times $X^n = (X_1, \dots, X_n)$ given $N(t) = n$?

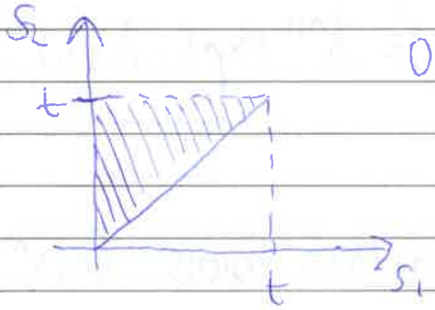
Recall that there is a one-to-one transformation between S^n & X^n

$$S_1 = X_1, \qquad X_i = S_i - S_{i-1}, \quad i \in \{2, \dots, n\}.$$

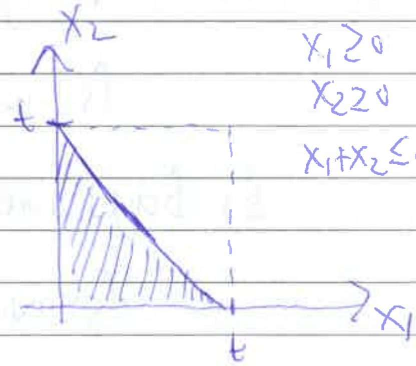
Hence, the value of the density is unchanged but the region in which it is non-zero is changed

$$f_{X^n|N(t)}(x^n|n) = n!/t^n \quad \text{for} \quad x^n > 0 \ \& \ \sum_{i=1}^{n} x_i < t.$$

For example, for $n=2$.



$0 \le S_1 \le S_2 \le t$



$X_1 \ge 0$
$X_2 \ge 0$
$X_1 + X_2 \le t$

Connection to Order Statistics.

Let $U^n = (U_1, \dots, U_n)$ be iid rvs in $(0, t]$. $U_i \sim \text{Unif}(0,t]$.

For any point $u^n \in (0,t]^n$, $f_{U^n}(u^n) = \frac{1}{t^n}$, $0 < u_i \le t$, $1 \le i \le n$.

Fact: Both $f_{S^n|N(t)}(\cdot|n)$ and $f_{U^n}(\cdot)$ [as well as $f_{X^n|N(t)}(\cdot|n)$ and $f_{U^n}(\cdot)$] are uniform over the space they are defined on and are non-zero but the value of the latter is $n!$ smaller than that of the former.

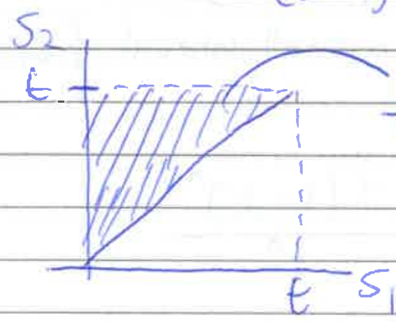
Indeed $f_{S^n|N(t)}(s^n|n) = n!/t^n$, $\quad f_{U^n}(u^n) = 1/t^n$.
$0 \le S_1 \le S_2 \le \dots \le S_{n-1} \le S_n \qquad 0 < u_i \le t, \ 1 \le i \le n.$

To see this more explicitly, let $S^n = (S_1, S_2, \ldots, S_n)$ be defined, not as the ~~arrivals of~~ epochs of a Poisson process, but as the order statistics of the U^n iid random process, i.e.,

$$S_1 = \min\{U_1, \ldots, U_n\}, \quad S_2 = \min\{\{U_1, \ldots, U_n\} \setminus S_1\}$$

$\Rightarrow$ S_2 is the 2nd min of the U^n process

$$S_k = \min\{\{U_1, \ldots, U_n\} \setminus \{S_1, \ldots, S_{k-1}\}\}$$



$f_{S^2|N(t)}(s^2|n) = \frac{2}{t^2}$

$f_{U^2}(u^2) = \frac{1}{t^2}$

Note that the n-cube (cube of side length t in dim n) is partitioned into $n!$ regions, one in which $0 < u_1 < u_2 < \cdots < u_n$.

For each permutation $\pi: \{1, \ldots, n\} \to \{1, \ldots, n\}$, there is a region $0 < u_{\pi(1)} < u_{\pi(2)} < \cdots < u_{\pi(n-1)} < u_{\pi(n)}$.

By symmetry, each of these regions has the same volume, which ~~means that~~ then must be $1/n!$ of the volume of the cube which is t^n

Hence, what is the density $f_{S_i|N(t)}(s_i|n)$, the prob. density of the i^{th} arrival epoch given that there are n in $(0,t]$?

Consider $S_1 = \min\{U_1, \ldots, U_n\}$.

$$\Pr(S_1 > \tau \mid N(t) = n) = \left(\frac{t-\tau}{t}\right)^n \quad \text{for } 0 < \tau \leq t$$

Prob. of all U_i exceeds τ.

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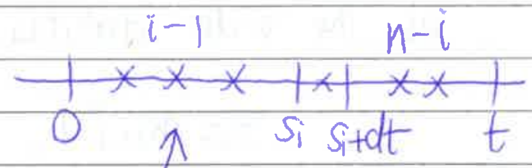
To derive the density of S_i, $i \in \{1, \dots, n\}$, given $\{N(t)=n\}$, we conside

$f_{S_i|N(t)=n}(s_i)\, dt$



$$= \binom{n-1}{i-1} \left(\frac{s_i}{t}\right)^{i-1} \left(1-\frac{s_i}{t}\right)^{n-i} \underbrace{\frac{n\, dt}{t}}_{\text{One of the } n \text{ arrivals lands in a small interval } [s_i, s_i+dt]}$$

prob. of success $\frac{s_i}{t}$

$\Rightarrow$ Density of the i^{th} arrival epoch is

$$f_{S_i|N(t)=n}(s_i) = \frac{n!}{(n-i)!\,(i-1)!} \frac{s_i^{\,i-1}(t-s_i)^{n-i}}{t^n}$$

We can also find the expectations of S_i given $N(t)=n$.

$$E[S_1|N(t)=n] = \int_0^t Pr(S_1 > \tau | N(t)=n)\, d\tau$$
$$= \int_0^t \left(\frac{t-\tau}{t}\right)^n d\tau = \frac{1}{t^n} \left[\frac{-(t-\tau)^{n+1}}{n+1}\right]_0^t$$
$$= \frac{t}{n+1}$$

For the other $E[S_i|N(t)=n]$, recall that $f_{X^n|N(t)}(x^n|n) = \frac{1}{t^n}$ for all $x^n > 0$ & $\sum_{i=1}^n X_i < t$.

This density is symmetric in the x_i's, thus all the densities $f_{X_i|N(t)}(x_i|n)$ must be the same by symmetry.

Thus $Pr(X_i > \tau | N(t)=n) = Pr(X_1 > \tau | N(t)=n) = Pr(S_1 > \tau | N(t)=n)$
$$= \left(\frac{t-\tau}{t}\right)^n, \quad 1 \le i \le n, \quad 0 < \tau \le t.$$

$\Rightarrow E[X_i|N(t)=n] = E[X_1|N(t)=n] = \frac{t}{n+1}.$

$\Rightarrow E[S_i|N(t)=n] = E\left[\sum_{j=1}^{i} X_j \,\middle|\, N(t)=n\right] = \frac{it}{n+1}$ ※

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Counting Process with a Random Rate (Gallager Ex 2.18).

Conside a Poisson process with a random rate Λ

$$f_\Lambda(\lambda) = \alpha \exp(-\alpha\lambda)\, 1\{\lambda \geq 0\}$$

That is, Λ is an exponential random variable with mean $1/\alpha$.

a) What is $\Pr(N(t)=n \mid \Lambda=\lambda)$?

$$\Pr(N(t)=n \mid \Lambda=\lambda) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n=0,1,2,3,\ldots$$

b) What is $\Pr(N(t)=n)$?

Pf 1: $\Pr(N(t)=n) = \int_0^\infty \Pr(N(t)=n \mid \Lambda=\lambda) f_\Lambda(\lambda)\, d\lambda$

$$= \int_0^\infty \frac{(\lambda t)^n e^{-\lambda t}}{n!} \alpha \exp(-\alpha\lambda)\, d\lambda$$

$$= \frac{\alpha t^n}{(t+\alpha)^n} \int_0^\infty \frac{[\lambda(t+\alpha)]^n \exp(-\lambda(t+\alpha))}{n!}\, d\lambda$$

$$\underset{x=\lambda(t+\alpha)}{=} \frac{\alpha t^n}{(t+\alpha)^{n+1}} \int_0^\infty \underbrace{\frac{x^n e^{-x}}{n!}}_{\text{Erlang of order } n+1}\, dx = \frac{\alpha t^n}{(t+\alpha)^{n+1}}.$$

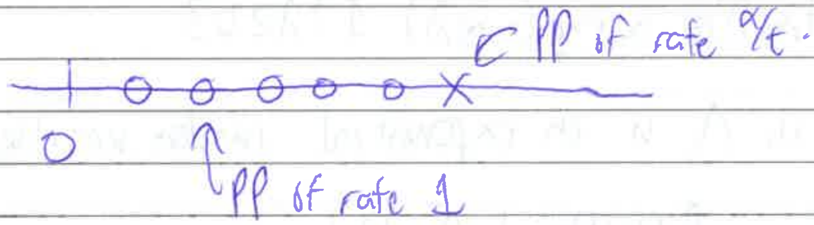
Pf 2: Smarter / Less mechanical proof.

Observe that the solution can be written as pq^n where $p = \frac{\alpha}{\alpha+t}$ and $q = \frac{t}{\alpha+t}$.

Note that ~~#~~ $\Pr(N(t)=n)$ for a PP of fixed deterministic rate λ is a function of λt and n only

$N(t)$ for a Poisson Process of rate $\lambda \overset{d}{=} N(\lambda t)$ for a PP with rate 1.

Since Λt is an exponential rv with parameter α/t, $N(\Lambda t)$ represents the # of arrivals of a PP of unit rate before the first arrival of an indep PP of rate α/t.



The two PP are independent & the combined process has rate $1+\alpha/t$.

Event $\{N(t)=n\}$ is the prob. of n arrivals from the unit rate PP followed by 1 arrival of the PP with rate α/t.

This yields the prob $q^n p$ (a geometric dist^n), $n=0,1,2,\ldots$

c) $f_\Lambda(\lambda \mid N(t)=n) = f_{\Lambda|N(t)}(\lambda|n)$

By Bayes rule $\quad f_{\Lambda|N(t)}(\lambda|n) = \dfrac{Pr(N(t)=n \mid \Lambda=\lambda)\, f_\Lambda(\lambda)}{Pr(N(t)=n)}$

$$= \frac{(\lambda t)^n e^{-\lambda t}}{n!}\, \alpha e^{-\alpha\lambda} \Big/ \frac{\alpha t^n}{(t+\alpha)^{n+1}}$$

$$= \frac{\lambda^n e^{-\lambda(\alpha+t)} (\alpha+t)^{n+1}}{n!} \qquad \lambda \geq 0.$$

This is an Erlang pdf of order $n+1$ with rate $\alpha+t$.

d) $E[\Lambda \mid N(t)=n]$

Since conditioned on $N(t)=n$, Λ is Erlang of order $n+1$, it is the sum of $n+1$ independent Exp$(\alpha+t)$ rvs. Each of these rvs has mean $(\alpha+t)^{-1}$.

$$E[\Lambda \mid N(t)=n] = \frac{n+1}{t+\alpha}.$$

For $N(t)=0$ & $t \ll \alpha$, this is close to $1/\alpha = E\Lambda$. Intuitive since now the data has little effect on Λ, so we revert to prior knowledge.