Name :   LUO ZIJIAN

Matric.No： A0224725H

MUSNET：   E0572844

Subject：   Information Theory

Assignment: Homework Eight

Date: October 24th

Prof: Marco Tomamichel

Exercise 8.1

a).

As for the mutual information $I(X_1, X_2; Y_1, Y_2)$, when $P_{00} = P_{01} = P_{10} = P_{11} = \frac{1}{4}$

we firstly calculate

$P_{00} = Pr(X_1=0, X_2=0) = \frac{1}{4}$ $P'_{00} = Pr(Y_1=0, Y_2=0) = \frac{1}{4}$

$P_{01} = Pr(X_1=0, X_2=1) = \frac{1}{4}$ $P'_{01} = Pr(Y_1=0, Y_2=1) = \frac{1}{4}$

$P_{10} = Pr(X_1=1, X_2=0) = \frac{1}{4}$ $P'_{10} = Pr(Y_1=1, Y_2=0) = \frac{1}{4}$

$P_{11} = Pr(X_1=1, X_2=1) = \frac{1}{4}$ $P'_{11} = Pr(Y_1=1, Y_2=1) = \frac{1}{4}$

$H(X) = 4 \times \frac{1}{4} \times \log 4 = 2$ $H(Y) = 4 \times \frac{1}{4} \times \log 4 = 2$

But for the joint PMF, we can express them like below table:

$P_{00\,00'} = 0$ $P_{00\,01'} = \frac{1}{4}$ $P_{00\,10'} = 0$ $P_{00\,11'} = 0$

$P_{01\,00'} = 0$ $P_{01\,01'} = 0$ $P_{01\,10'} = \frac{1}{4}$ $P_{01\,11'} = 0$

$P_{10\,00'} = 0$ $P_{10\,01'} = 0$ $P_{10\,10'} = 0$ $P_{10\,11'} = \frac{1}{4}$

$P_{11\,00'} = \frac{1}{4}$ $P_{11\,01'} = 0$ $P_{11\,10'} = 0$ $P_{11\,11'} = 0$

$$H(X;Y) = 4 \times \frac{1}{4} \log 4 = 2$$

$$I(X_1, X_2; Y_1, Y_2) = H(X_1, X_2) + H(Y_1, Y_2) - H(X_1, X_2; Y_1, Y_2)$$

$$= \sum_{x_1 \in X} \sum_{x_2 \in X} P(X_1=i, X_2=j) \cdot \log \frac{1}{P(X_1=i, X_2=j)} + \sum_{Y_1 \in Y} \sum_{Y_2 \in Y} P(Y_1=m, Y_2=n) \cdot \log \frac{1}{P(Y_1=m, Y_2=n)}$$

$$- \sum_{x_1 \in X} \sum_{x_2 \in X} \sum_{Y_1 \in Y} \sum_{Y_2 \in Y} P(X_1=i, X_2=j, Y_1=m, Y_2=n) \cdot \log \frac{1}{P(X_1=i, X_2=j, Y_1=m, Y_2=n)}$$

(b) Based on the above calculations, we can get

$$\text{Capacity} = \max_{P_X \in P} I(X;Y)$$

we know it is concave function, only when $P_{00} = P_{01} = P_{10} = P_{11} = \frac{1}{4}$.

(uniform distribution)

we can get Capacity $= H(X) + H(Y) - H(X;Y)$

$$= 2 + 2 - 2$$

$$= 2 \text{ bits}$$

(c). For $I(X_1; Y_1)$, $I(X_1; Y_1) = H(X_1) - H(X_1|Y_1) = H(X_1) + H(Y_1) - H(Y_1; Y_1)$

$P_0 = Pr(X_1 = 0) = \frac{1}{2}$ $P_0' = Pr(Y_1 = 0) = \frac{1}{2}$

$P_1 = Pr(X_1 = 1) = \frac{1}{2}$ $P_0' = Pr(Y_1 = 1) = \frac{1}{2}$

$H(X_1) = 2 \times \frac{1}{2} \log 2 = 1$ $H(Y_1) = 1$

$P_{00}' = Pr(X_1 = 0, Y_1 = 0) = \frac{1}{4}$ $P_{01}' = Pr(X_1 = 0, Y_1 = 1) = \frac{1}{4}$

$P_{10}' = Pr(X_1 = 1, Y_1 = 0) = \frac{1}{4}$ $P_{11}' = Pr(X_1 = 1, Y_1 = 1) = \frac{1}{4}$

$H(X_1; Y_1) = 4 \times \frac{1}{4} \cdot \log 4 = 2$

$$I(X_1; Y_1) = H(X_1) + H(Y_1) - H(X_1; Y_1) = 1 + 1 - 2 = 0$$

Exercise 8.2

Channel capacity : $Y = (X + Z) \mod m$,

where $Z = \begin{cases} Z = 1 & , \frac{3}{4} \\ Z = 0 & , \frac{1}{4} \end{cases}$, $X = 0 \cdots, m-1$

Z is independent of X

In this case, $H(Y|X) = H(Z|X) = H(Z) = \frac{1}{4} \log 4 + \frac{3}{4} \log \frac{4}{3}$

$$= 2 - \frac{3}{4} \log 3$$

Hence, the capacity of the channel is

$C = \max_{P(x)} I(X; Y)$

$= \max_{P(x)} H(Y) - H(Y|X)$

$= \max_{P(x)} H(Y) - \left(2 - \frac{3}{4} \log 3\right)$

$= \log m - 2 + \frac{3}{4} \log 3$

which is often attained when Y has a uniform distribution, which occurs (by symmetry) when X has a uniform distribution.

Exercise 8.3

Followed the statement of this additive noise channel,
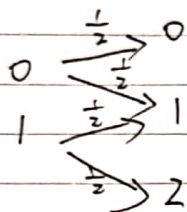
we know that Z is independent of X.
so, we write $Y = X + Z$ $X \in \{0,1\}$, $Z \in \{0, a\}$

Here are these cases:

① when $a = 0$, $Y = X$ $\Rightarrow$ $\max I(X;Y) = \max H(X) = 1$.
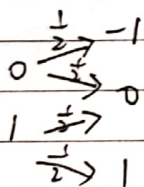
   so the capacity is 1 bit

② when ~~$a = 1$~~, Y has three possible output values, $0, 1, 2$

   Followed the conclusion from BEC (binary erasure channel)
   We can get $I(W_{BEC}) = 1 - \varepsilon = 1 - \frac{1}{2} = \frac{1}{2}$

③ when $a = -1$, Y also has three possible output values, $-1, 0, 1$

   Similarily, we can get $I(W_{BEC}) = 1 - \frac{1}{2} = \frac{1}{2}$

④ When $a \neq 0, -1, 1$, Y has four possible output values, $0, 1, a, 1+a$
Therefore, when we know Y, we certainly know the X,

$$H(X|Y) = 0$$
$$\max(I(X;Y)) = \max(H(X)) = 1$$

Exercise 8.4

(a). Following the statement of the question, we know $Y = X \cdot Z$

$$H(Y) = H(X \cdot Z) = \sum_{x_j \in X} \sum_{z_i \in Z} P(x_j, z_i) \log \frac{1}{P(x_i, z_i)}$$

$$= \sum_{x_i \in X} \sum_{z_i \in Z} P(x_i) \cdot P(z_i) \log \frac{1}{P(x_i) \cdot P(z_i)} \qquad [P(x_i, z_i) = P(x_i) \cdot P(z_i)]$$

$$= \sum_{x_i \in X} \sum_{z_i \in Z} P(x_i) \cdot P(z_i) \log \frac{1}{P(x_i)} + \sum_{x_i \in X} \sum_{z_i \in Z} P(x_i) \cdot P(z_i) \log \frac{1}{P(z_i)}$$

$$= \sum_{x_i \in X} P(x_i) \log \frac{1}{P(x_i)} + \sum_{z_i \in Z} P(z_i) \log \frac{1}{P(z_i)}$$

$$= H(X) + H(Z)$$

(b) Because X and Z are independent, $\max H(Y) = \max H(X) + \max H(Z)$

For $\max H(X)$, we take $X = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n} - \frac{1}{n})$

$$H(X) = n \cdot \frac{1}{n} \cdot \log \frac{1}{\frac{1}{n}} = \log n$$

For $\max H(Z)$, we take $p = \frac{1}{2}$

$$H(Z) = \frac{1}{2} \log \frac{1}{\frac{1}{2}} + \frac{1}{2} \log \frac{1}{\frac{1}{2}} = 1$$

so $\max(H(Y)) = \log n + 1$

(c) According to the definition of channel mutual information

$$I(P) = H(Y) - H(X) \cdot$$

$$= H(Z)$$

$$= h(P)$$

$$= p \log \frac{1}{P} + (1-P) \log \frac{1}{(1-P)}$$