Exercise 1.1

a). For $E[V+W] = \sum_{v,w\in\mathcal{X}} (v+w)\, P_{VW}(v,w)$

$$= \sum_{v,w\in\mathcal{X}} v\, P_{VW}(v,w) + \sum_{v,w\in\mathcal{X}} w\, P_{VW}(v,w)$$

$$= E[V] + E[W]$$

As desired, $E[V+W] = E[V] + E[W]$

b). If v and w are independent, we can get $P_{VW}(v,w) = P_V(v)\cdot P_W(w)$

$$E[VW] = \sum_{v,w\in\mathcal{X}} vw\, P_{VW}(v,w)$$

$$= \sum_{v,w\in\mathcal{X}} vw\, P_V(v)\cdot P_W(w)$$

$$= \sum_{v,w\in\mathcal{X}} v\, P_V(v)\cdot w P_W(w)$$

$$= \sum_{w\in\mathcal{X}} v\cdot P_V(v)\cdot \sum_{v\in\mathcal{X}} w\, P_W(w)$$

$$= E[V]\cdot E[W]$$

As desired, $E[VW] = E[V]\cdot E[W]$

c). We know, $Z = V+W$

$$Var(Z) = Var(V+W) = E[(V+W)^2] - E^2[V+W]$$

$$= E[V^2] + E[W^2] + E[2VW] - E[V+W]\cdot E[V+W]$$

$$= E[V^2] + E[W^2] + 2E[V]\cdot E[W] - (E[V] + E[W])^2 \quad \text{(from (a) and (b))}$$

$$= E[V^2] - E^2[V] + E[W^2] - E^2[W]$$

$$= Var(V) + Var(W)$$

$$= \sigma_V^2 + \sigma_W^2$$

Exercise 1.2

a). All the possible events are listed:

the sample space

$$\left\{\begin{array}{llll} HHHH & HHHT & HTHH & HTHT \\ HHTH & HHTT & HTTH & HTTT \\ THHH & THHT & TTHH & TTHT \\ THTH & THTT & TTTH & TTTT \end{array}\right\}$$

which means (X, Y)

$$\left\{\begin{array}{llll} (X=4, Y=1) & (X=3, Y=1) & (X=3, Y=1) & (X=2, Y=1) \\ (X=3, Y=1) & (X=2, Y=1) & (X=2, Y=1) & (X=1, Y=1) \\ (X=3, Y=2) & (X=2, Y=2) & (X=2, Y=3) & (X=1, Y=3) \\ (X=2, Y=2) & (X=1, Y=2) & (X=1, Y=4) & (X=0, Y=0) \end{array}\right.$$

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b). $P_{XY}(0,0)=\frac{1}{16}$

$P_{XY}(1,1)=\frac{1}{16}$, $P_{XY}(1,2)=\frac{1}{16}$, $P_{XY}(1,3)=\frac{1}{16}$ $P_{XY}(1,4)=\frac{1}{16}$

$P_{XY}(2,1)=\frac{3}{16}$, $P_{XY}(2,2)=\frac{1}{8}$, $P_{XY}(2,3)=\frac{1}{16}$

$P_{XY}(3,1)=\frac{3}{16}$, $P_{XY}(3,2)=\frac{1}{16}$

$P_{XY}(4,1)=\frac{1}{16}$

~~$P_X(x)=\sum_{y_j\in R_Y} P_{XY}(x,y_j)$ for any $x\in R_X$~~

so, we can get $Pr[Y=0\,|\,X=1]=\frac{P_{XY}(X=1,Y=0)}{P_X(X=1)}=\frac{0}{\frac{1}{4}}=0$

$$Pr[Y=1\,|\,X=3]=\frac{P(X=3,Y=1)}{P_X(X=3)}=\frac{\frac{3}{16}}{\frac{3}{16}+\frac{1}{16}}=\frac{3}{4}$$

(c) $P_X(X=0)=\frac{1}{16}$ $P_Y(Y=0)=\frac{1}{16}$

$P_X(X=1)=\frac{1}{4}$ $P_Y(Y=1)=\frac{1}{2}$

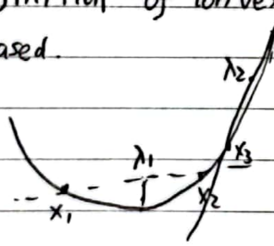
$P_X(X=2)=\frac{3}{8}$ $P_Y(Y=2)=\frac{1}{4}$

$P_X(X=3)=\frac{1}{4}$ $P_Y(Y=3)=\frac{1}{8}$

$P_X(X=4)=\frac{1}{16}$ $P_Y(Y=4)=\frac{1}{16}$

Exercise 1.3

From the definition of convex property, we know the derivative of con function is increased.



For $a\le x_1<x_2\le x_3<x_4\le b$

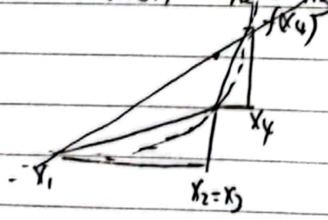
$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\le\frac{f(x_4)-f(x_3)}{x_4-x_3}$$

But based on the simple statement of convex function

$f[\lambda_1 x_1+(1-\lambda_1)x_2]\le \lambda_1 f(x_1)+(1-\lambda_1)f(x_2)$ $\lambda_1\in[0,1]$

$f[\lambda_2 x_3+(1-\lambda_2)x_4]\le \lambda_2 f(x_3)+(1-\lambda_2)f(x_4)$ $\lambda_2\in[0,1]$

when $x_2=x_3$



$$\frac{f(x_4)-f(x_1)}{x_4-x_1}(x_2-x_1)+f(x_1)\le\lambda f(x_1)+(1-\lambda)f(x_4)$$

$$\Downarrow$$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\le\frac{f(x_4)-f(x_3)}{x_4-x_3}$$