# EE5137 Stochastic Processes: Problem Set 2
## Assigned: 22/01/21, Due: 29/01/21

There are five (5) non-optional problems in this problem set. You have one week to do this problem set. My advice is to get started soon.

1. Exercise 1.12 (Gallager's book)

   *Hint: For parts (a) express the CDF of $M_+$ (the maximum of the $N$ rvs) in terms of the CDFs of the individual rvs. Part (b) is analogous. Part (c) is most challenging. You may first condition on the event $\{X_1 = x\}$. Then note that $X_1 = M_+$ iff $X_j \le x$ for all $2 \le j \le n$. Also given $X_1 = M_+ = x$, we have $R = M_+ - M_- \le r$ iff $X_j > x - r$ for $2 \le j \le n$. Now since the rvs are i.i.d.,*

   $$\Pr(M_+ = X_1, R \le r \,|\, X_1 = x) = \prod_{j=2}^{n} \Pr(x - r < X_j \le x)$$

   *Continue the above argument (average over $X_1 = x$) to show that*

   $$\Pr(R \le r) = \int_{-\infty}^{\infty} n f_X(x)[F_X(x) - F_X(x-r)]^{n-1}\, \mathrm{d}x.$$

2. Exercise 1.14 (Gallager's book)

3. Exercise 1.20 (Gallager's book)

4. Exercise 1.22 (Gallager's book)

   *Note that there's a typo in the book. $p_Y(m) = \mu^n \exp(-\mu)/n!$ should be $p_Y(n) = \mu^n \exp(-\mu)/n!$*

5. We toss a biased coin $n$ times. The probability of heads, denoted by $y$, is the value of a random variable $Y$ with a given mean $\mu$ and variance $\sigma^2$. Let $X_i$ be a Bernoulli random variable that models the outcome of the $i$-th toss (i.e., $X_i = 1$ if the $i$-th toss is a head). In other words, for each $1 \le i \le n$,

   $$X_i = \begin{cases} 1 & \text{w.p. } Y \\ 0 & \text{w.p. } 1 - Y \end{cases},$$

   where $Y \in [0, 1]$ is a random variable with

   $$\mathbb{E}[Y] = \mu, \quad \text{and} \quad \mathrm{Var}(Y) = \sigma^2.$$

   We assume that $X_1, X_2, \ldots, X_n$ are conditionally independent given the event $\{Y = y\}$ for each $y \in [0, 1]$. let

   $$S_n = X_1 + X_2 + \ldots + X_n$$

   be the total number of heads in the $n$ tosses.

   (a) (5 points) Use the law of iterated expectations to find $\mathbb{E}[X_i]$ and $\mathbb{E}[S_n]$.

(b) (3 points) Using the fact that $X_i^2 = X_i$, show that $\mathrm{Var}(X_i) = \mu - \mu^2$.

(c) (5 points) Using the law of iterated expectations, find

$$\mathrm{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j], \quad \text{for} \quad i \neq j.$$

Are $X_i$ and $X_j$ independent?

(d) (7 points) By writing $\mathrm{Var}(S_n) = \mathbb{E}[S_n^2] - (\mathbb{E}[S_n])^2$, show that

$$\mathrm{Var}(S_n) = \mathbb{E}[\mathrm{Var}(S_n|Y)] + \mathrm{Var}(\mathbb{E}[S_n|Y]), \tag{1}$$

where $\mathrm{Var}(S_n|Y)$ is the random variable that takes on the value $\mathrm{Var}(S_n|Y = y)$ with probability $\Pr(Y = y)$.

(e) (5 points) Calculate the variance of $S_n$ by using the formula (1) in part (d) above.

*This was an exam question in 2017.*

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6. (Optional) Exercise 1.6 (Gallager's book)

7. (Optional) Exercise 1.16 (Gallager's book)

8. (Optional) [Reverse Markov Inequality] Derive the reverse Markov inequality: Let $X$ be a random variable such that $\Pr(X \leq a) = 1$ for some constant $a$. Then for $d < \mathbb{E}X$, we have

$$\Pr(X > d) \geq \frac{\mathbb{E}X - d}{a - d}$$

*Hint: Apply the usual Markov inequality to the new non-negative random variable $a - X$.*

9. (Optional) [Knockout Football]

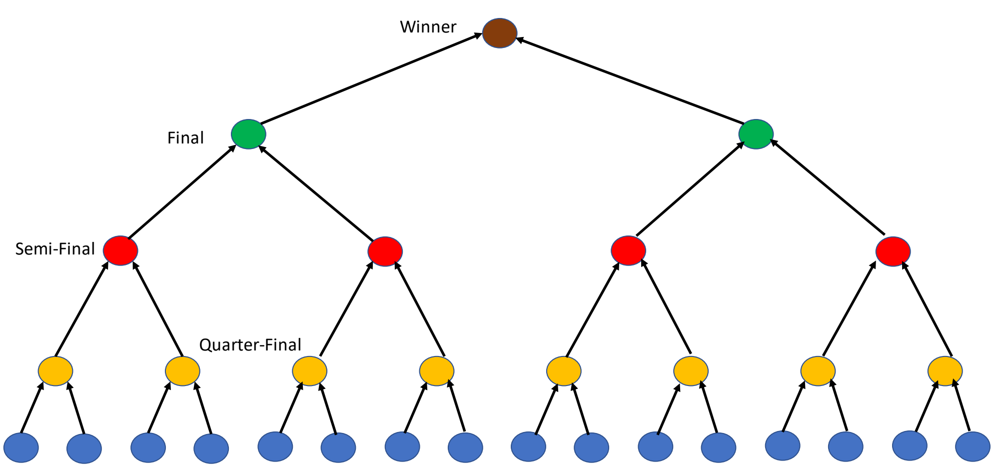


Figure 1: Figure for 16 teams

In the knockout phase of a football tournament, there are 32 teams *of equal skill* that compete in an elimination tournament. This proceeds in a number of rounds in which teams compete in pairs; any losing team retires from the tournament. See Fig. 1 for an illustration with 16 teams. What is the probability that two given teams will compete against each other? Generalize your answer to $2^k$ teams.

*The following argument is wrong but the answer is right. There has to be* 31 *games to knock out all but the ultimate winner. There are* $\binom{32}{2}$ *possible pairs, so that the probability of a given pair being selected for a particular match is* $1/\binom{32}{2} = 1/(16 \cdot 31)$. *Since the selection of the teams in the different matches is mutually exclusive, the probability of a given pair being selected is* 31 *times this, which is* $1/16$. *Why is this wrong and what's the correct way of doing it?*

This problem is taken from Problem 297 of *Five Hundred Mathematical Challenges* (Mathematical Association of America, 1996).