Homework 2

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1. EXERCISE 1.12

from the above statement, we can know the rvs. are i.i.d. then follow the instruction: $P_r(M_+ = X_1, R \le r \mid X_1 = x) = \prod_{j=2}^{n} P_r(x - r < X_j \le x)$

As for the conditional probability,

$$P_r(M_+ = X_1, R \le r) = P_r(M_+ = X_1, R \le r \mid X_1 = x) \cdot P_r(X_1 = x)$$
$$= \prod_{j=2}^{n} P_r(x - r < X_j \le x) \cdot f_x(x)$$

we can easilly get the conclusion $\left(P_r(M_+ = X_1) = \frac{1}{n}\right)$ owing to one number divide n number

$$f_{M^+ = X_1, R \le r}(R \le r) = \frac{P_r(M_+ = X_1, R \le r)}{P_r(M_+ = X_1)} = n \prod_{j=2}^{n} P_r(x - r < X_j \le x) f_x(x)$$

$$P_r(R \le r) = \int_{-\infty}^{+\infty} f_{M^+ = X_1, R \le r} \, dx = \int_{-\infty}^{+\infty} n \prod_{j=2}^{n} P_r(x - r < X_j \le x) f_x(x) \, dx$$

owing to $P_r(x - r < X_j \le x) = F(x) - F(x - r)$

so we can make another expression:

$$= \int_{-\infty}^{+\infty} n \prod_{j=2}^{n} \left[F(x) - F(x - r)\right] f_x(x) \, dx$$

$$= \int_{-\infty}^{+\infty} n f_x(x) \left[F(x) - F(x - r)\right]^{n-1} dx$$

EXERCISE 1.14

(a) From the statement, we can make sure the rvs. can have a joint density function ~~because~~, however, we do not assume that the rvs are pairwise independent. In this situation, we only can get a joint density function, not for statistically ~~over~~ independence.

(b) ~~with the~~ From the meaning of the (b) question, we can assume $f_{x_1,x_2\cdots x_n}(x_1,x_2\cdots,x_n) = f_{x_1}(x_1) f_{x_2}(x_2)\cdots f_{x_n}(x_n)$

so the expected value of the product is.

$$E[f(x_1,x_2,\cdots x_n)] = E[f_{x_1}(x_1) f_{x_2}(x_2) \cdots f_{x_n}(x_n)]$$

owing to $x_1,x_2,\cdots x_n$ are statistically independent so we can make another expression from above equatation

$$E[f_{x_1}(x_1) f_{x_2}(x_2) \cdots f_{x_n}(x_n)]$$

As for X_1, so $X_2 \cdots X_n$ seem like constant number

As for X_2, so $X_3 \cdots X_n$ seem like constant number.

and so on

(linearity of expectation.

$$= E[f_{x_1}(x_1)]\, E[f_{x_2}(x_2)\cdots f_{x_n}(x_n)]$$
$$= \bar{x}_1\, E[f_{x_2}(x_2)\cdots f_{x_n}(x_n)]$$
$$= \bar{x}_1 \cdot \bar{x}_2\, E[f_{x_3}(x_3) \cdots f_{x_n}(x_n)]$$
$$= \bar{x}_1 \cdot \bar{x}_2 \cdots \bar{x}_n$$

(c) From the statement, we can assume that

$$var[x] = E[(x-\bar{x})^2] = E[x^2] - \bar{x}^2$$

for $x_1+x_2 \cdots +x_n$ $\quad E[x_1+x_2\cdots +x_n] = \bar{x}_1+\bar{x}_2+\cdots \bar{x}_n$

$$var[x_1+x_2\cdots+x_n] = E[[(x_1+x_2-\cdots+x_n)-(\bar{x}_1+\bar{x}_2+\cdots \bar{x}_n)]^2]$$

$$= E[(x_1+x_2+\cdots+x_n)^2 - 2(x_1+x_2\cdots+x_n)(\bar{x}_1+\bar{x}_2+\cdots \bar{x}_n) + (\bar{x}_1+\bar{x}_2+\cdots \bar{x}_n)^2]$$
$$= E[(x_1+x_2+\cdots+x_n)^2] - (\bar{x}_1+\bar{x}_2+\cdots \bar{x}_n)(\bar{x}_1+\bar{x}_2+\cdots \bar{x}_n)$$

$\because x_1, x_2 \cdots x_n$ are statistically independent

$$= E[x_1^2]+E[x_2^2]+\cdots E[x_n^2] - [\bar{x}_1^2+\bar{x}_2^2+\cdots \bar{x}_n^2]$$
$$= E[x_1^2]-\bar{x}_1^2+E[x_2^2]-\bar{x}_2^2+\cdots E[x_n^2]-\bar{x}_n^2 = var(x_1)+var(x_2)+\cdots var[x_n]$$

Besform

3. EXERCISE 1.20

(a) from the statement, $F_Y(Y) = 1 - \frac{2}{(Y+1)(Y+2)}$

use 1.29 (the definition of PMF)

$$P_N(Y) = F_Y(y) - F_Y(y-1) = 1 - \frac{2}{(y+1)(y+2)} - \left[1 - \frac{2}{y\cdot(y+1)}\right]$$

$$= \frac{2}{y\cdot(y+1)} - \frac{2}{(y+1)(y+2)}$$

$Y \geq 1$

use 1.30 ($F^c_Y(Y) = 1 - F_Y(Y)$)

$$E[Y] = \int_0^\infty F^c_Y(y)\,dy = \sum_0^\infty \frac{2}{(y+1)(y+2)} = \sum_0^\infty \left(\frac{2}{y+1} - \frac{2}{y+2}\right)$$

$$= \frac{2}{0+1} - \frac{2}{0+2} + \frac{2}{1+1} - \frac{2}{1+2} + \frac{2}{2+1} - \frac{2}{2+2} + \cdots$$

when $n \to \infty$ $\quad \frac{2}{y+2} \to 0$

so the sum of this equatation is 2

as desired $E[Y] = 2$

(b). $P_N(Y) = \frac{2}{y} - \frac{2}{y+1} - \left[\frac{2}{y+1} - \frac{2}{y+2}\right]$

$$E(Y) = \sum_1^\infty P_N(Y)\,y + P_N(Y=0)\times 0 = \sum_1^\infty P_N(Y)\cdot y$$

$$= \sum_1^\infty \left[\frac{2}{y(y+1)} - \frac{2}{(y+1)(y+2)}\right] y$$

$$= \sum_1^\infty \left[\frac{2}{y+1} - \frac{2y}{(y+1)(y+2)}\right]$$

$$= \sum_1^\infty \frac{4}{(y+1)(y+2)} = \sum\left[\frac{4}{y+1} - \frac{4}{y+2}\right] = \frac{4}{1+1} - \frac{4}{1+2} + \frac{4}{2+1} + \frac{4}{2+2} + \cdots = \frac{4}{2} = 2$$

(c). $P_{X|Y}(x|Y) = \frac{1}{y}$ for $1 \le x \le y$

from the hint, we can calculate

$$E[X|Y=y] = \sum_x x \cdot P_{X|Y}(x|y) = \sum_x \frac{x}{y} = \sum_{x=1}^{y} \frac{x}{y} = \frac{[1+y]y}{y \cdot 2} = \frac{1+y}{2}$$

$$E[X] = E\left[E[X|Y=y]\right] = E\left[\frac{1+y}{2}\right]$$

$$= \frac{1}{2}\left[1 + E[Y]\right]$$

$$= \frac{3}{2}$$

(d). $P_{Z|Y}(z|Y) = \frac{1}{y^2}$ for $1 \le z \le y^2$

from the similar thought

$$E[Z|Y=y] = \sum_z z \, P_{Z|Y}(z|Y) = \sum_{z=1}^{y^2} \frac{z}{y^2} = \frac{(1+y^2)y^2}{y^2 \cdot 2} = \frac{1+y^2}{2}$$

$y \ge 1$

$$E[Z] = E\left[E[Z|Y=y]\right] = E\left[\frac{1+y^2}{2}\right] = \frac{1}{2} + \frac{1}{2}E[Y^2]$$

As for $E[Y^2] = \sum_{1}^{\infty} y^2 P_N(Y) = \sum_{1}^{\infty} \frac{2y}{(y+1)(y+2)}$

$$= \sum_{1}^{\infty}\left(\frac{2y}{y+1} - \frac{2y}{y+2}\right)$$

$$= \lim_{Y \to \infty} \sum_{y=1}^{Y}\left(\frac{2y}{y+1} - \frac{2y}{y+2}\right)$$

$\Rightarrow$ infinite

Hence the result is infinite.

4. EXERCISE 1.22

from the above statement $P_X(n) = \frac{\lambda^n \exp(-\lambda)}{n!}$

$$P_Y(n) = \frac{u^n \exp(-u)}{n!}$$

owing to the definition of Poisson

$E(X) = \lambda \quad Var[X] = \lambda$

$E(Y) = u \quad Var[Y] = u$

X and Y are independent

$Z = X+Y$

~~owing to the convolution of fx & fy~~

$$P(Z = X+Y = n) = \sum_{i=0}^{n} P(X+Y = n, X = i)$$

$$= \sum_{i=0}^{n} P(Y = n-i, X = i)$$

owing to X, Y are independent

$$= \sum_{i=0}^{n} P(Y = n-i)\, P(X = i)$$

$$= \sum_{i=0}^{n} e^{-u} \cdot \frac{u^{n-i}}{(n-i)!} \cdot \frac{e^{-\lambda} \cdot \lambda^{i}}{i!}$$

$$= e^{-(u+\lambda)} \frac{1}{n!} \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} u^{n-i} \cdot \lambda^{i}$$

$$= e^{-(u+\lambda)} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} u^{n-i} \cdot \lambda^{i}$$

$$= e^{-(u+\lambda)} \frac{1}{n!} (u+\lambda)^n$$

$$= \frac{(u+\lambda)^n \cdot e^{-(u+\lambda)}}{n!}$$

$X+Y \sim P(u+\lambda)$

and then focus on $P(Y \mid Z=n) = \dfrac{P(Y, Z=n)}{P(Z=n)}$ $\triangleq X = n-Y$

owing to the independence between Y and X

we can change the above equation

$$= \frac{P(Y, Z=n)}{P(Z=n)} = \frac{P(Y, X=n-Y)}{P(Z=n)} = \frac{P_Y(Y)\, P_X(n-Y)}{P(Z=n)}$$

as $Pr(Y) = \dfrac{u^{Y} \cdot \exp(-u)}{Y!}$, $P_X(n-Y) = \dfrac{\lambda^{n-Y} \cdot \exp(-\lambda)}{(n-Y)!}$,

$P(Z=n) = \dfrac{(u+\lambda)^{n} \cdot e^{-(u+\lambda)}}{n!}$

so the $P(Y \mid Z=n) = \dfrac{\dfrac{u^{Y} \cdot \exp(-u)}{Y!} \cdot \dfrac{\lambda^{n-Y} \cdot \exp(-\lambda)}{(n-Y)!}}{\dfrac{(u+\lambda)^{n} \cdot e^{-(u+\lambda)}}{n!}}$

$$= \frac{u^{Y} \cdot \lambda^{n-Y}}{(u+\lambda)^{n}} \cdot \frac{n!}{Y!\,(n-Y)!}$$

5. EXERCISE

(a) ~~$E[x_i] = P(x_i=0)\cdot 0 + P(x_i=1)\cdot 1 = P(x_i=1) = Y$~~

~~$E[S_n]$~~

owing to $E[x_i \mid Y] = P[x_i=0 \mid Y]\cdot 0 + P[x_i=1 \mid Y]\cdot 1 = Y$

$E[x_i] = E[E[x_i \mid Y]] = E[Y] = u$

as for $S_n = X_1 + X_2 + \cdots X_n$, and $X_1, X_2 \cdots X_n$ are conditionally independent

$E[S_n] = E[x_1 + x_2 + \cdots x_n] = \sum_{i=0}^{n} E[x_i] = n \cdot u$

(b) owing to the fact $(x_i^2 = x_i)$

$Var(x_i) = E[x_i^2] - E[x_i]^2 = E[x_i] - E[x_i]^2 = u - u^2$

(c) from above statement, ~~owing to~~ $Cov(x_i, x_j) = E[x_i x_j] - E[x_i]E[x_j]$, for $i \neq j$

$$E[x_i x_j | Y] = 1 \times P[x_i = 1, x_j = 1 | Y] + 0 \times P[x_i = 0, x_j = 1 | Y]$$
$$+ 0 \times P[x_i = 1, x_j = 0 | Y] + 0 \times P[x_i = 0, x_j = 0 | Y]$$
$$= P[x_i = 1, x_j = 1 | Y]$$
$$= P[x_i = 1 | Y] \cdot P[x_j = 1 | Y] \quad (x_i \text{ and } x_j \text{ are conditional independent}$$
$$= Y \cdot Y$$
$$= Y^2$$

so $E[x_i x_j] = E[E[x_i x_j | Y]] = E[Y^2] = Var[Y] + E^2[Y] = \sigma^2 + \mu^2$

As for $E[x_i] = \mu$ $\quad E[x_j] = \mu$

in conclusion $Cov(x_i, x_j) = E[x_i x_j] - E[x_i]E[x_j]$
$$= \sigma^2 + \mu^2 - \mu \cdot \mu = \sigma^2$$

because $\sigma^2 > 0$, so x_i, x_j are not independent

(d) As for $E[Var(S_n | Y)] = E[E(S_n^2 | Y) - E^2(S_n | Y)] = E(S_n^2) - E^2(S_n|Y)$

and for $Var[E[S_n | Y]] = E[E^2[S_n | Y]] - E^2[E[S_n | Y]] = E^2(S_n | Y) - E^2(S_n)$

so $E[Var(S_n | Y)] + Var[E[S_n | Y]]$
$$= E(S_n^2) - E^2(S_n | Y) + E^2(S_n | Y) - E^2(S_n)$$
$$= E(S_n^2) - E^2(S_n)$$

as desired

$$Var(S_n) = E(S_n^2) - E^2(S_n) = E[Var(S_n | Y)] + Var(E[S_n | Y])$$

(e). by using the formula in part (d)

$$Var(S_n) = E[Var(S_n | Y)] + Var(E[S_n | Y])$$
$$= \cancel{Var(S_n)}$$
$$= E[Var(x_1 + x_2 + \cdots x_n | Y)] + Var(E[x_1 + x_2 + \cdots + x_n | Y])$$
$$= E[n \cdot Var(x_i | Y)] + Var(n \cdot E(x_i | Y))$$
$$\cancel{= E[n \cdot Y(1-Y)] + Var(nY)}$$
$$\cancel{= n[E[Y] - E[Y^2]] + n^2 Var[Y]}$$
$$\cancel{= n[\mu - \sigma^2] + n^2 \sigma^2}$$
$$= n E[Var(x_i | Y)] + n^2 Var[E(x_i | Y)]$$
$$= n \cdot E[Y(1-Y)] + n^2 \cdot \sigma^2$$

owing to $E[Y] = u$ $E[Y^2] = 6^2 + u^2$

so the expression can be changed

$= nE[Y - Y^2] + n^2 \cdot 6^2$

$= nE[Y] - nE[Y^2] + n^2 6^2$

$= n \cdot u - n(u^2 + 6^2) + n^2 6^2$

$= n(u - u^2) + (n^2 - n) 6^2$