Each rv $X_n \in S = \{1, \ldots, M\}$ has finite support, a discrete rv with M possibilities.

Consider a sequence $(X_n)_{n \geq 1}$ of such rvs. that are called Markov chains.

Def: A Markov chain $(X_n)_{n \geq 0}$ is an integer-time stochastic process for which X_n depends only on the most recent rv X_{n-1}, i.e., $\forall\, n \geq 1$ & $i, j, k \in S$,

$$Pr(X_n = j \mid X_{n-1} = i, X_{n-2} = k, \ldots, X_0 = l) = Pr(X_n = j \mid X_{n-1} = i)$$

for all conditioning events $\{X_{n-2} = k\}, \{X_{n-1} = i\}, \ldots, \{X_0 = l\}$.

Furthermore $Pr(X_n = i \mid X_{n-1} = j)$ depends only on $(i, j) \in S^2$ (not n) and is denoted by

$$P_{ji} = Pr(X_n = i \mid X_{n-1} = j) \quad \text{[Homogeneous Markov chain]}$$

X_0: initial state that has an arbitrary prob. distn. on S.
S: the state space is always finite in our discussion.

Often as a shorthand, we write $Pr(X_n \mid X_{n-1}, \ldots, X_0) = Pr(X_n \mid X_{n-1})$

X_n: state of the Markov chain at time $n \geq 0$.

Ex: Consider a discrete-time stochastic process $\{Z_n; n \geq 0\}$ where Z_n is a finite integer-valued rv but Z_n depends on the past $m \geq 1$ rvs $Z_{n-1}, Z_{n-2}, \ldots, Z_{n-m}$, i.e.,

$$Pr(Z_n \mid Z_{n-1}, Z_{n-2}, \ldots, Z_0) = Pr(Z_n \mid Z_{n-1}, Z_{n-2}, \ldots, Z_{n-m}) \quad -(*)$$

Is this process a Markov chain? Not exactly. But if we define $(Z_{n-1}, Z_{n-2}, \ldots, Z_{n-m})$ as the state of the process at time $n-1$,

Subject: No: Date:

$Pr(Z_n, Z_{n-1}, \dots, Z_{n-m+1} | Z_{n-1}, \dots, Z_0) = Pr(Z_n, Z_{n-1}, \dots, Z_{n-m+1} | Z_{n-1}, \dots, Z_{n-m})$ for each n

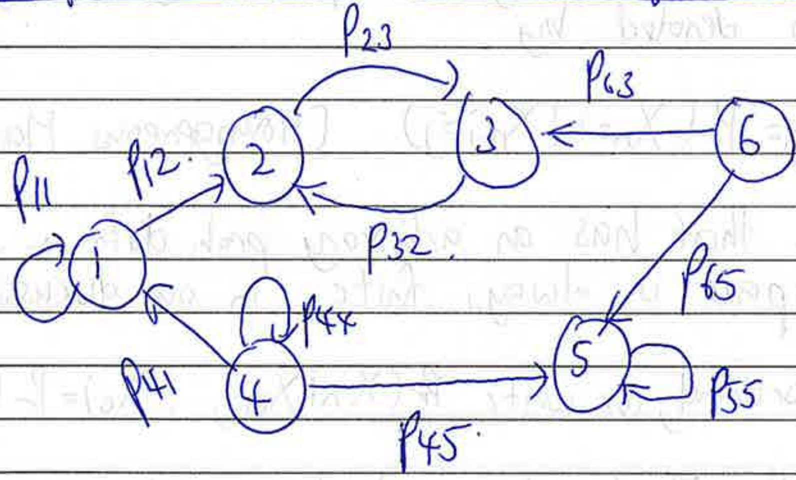
Define $X_{n-1} = (Z_{n-1}, \dots, Z_{n-m})$. The above relation reduces to

$$Pr(X_n | X_{n-1}, \dots, X_{m-1}) = Pr(X_n | X_{n-1})$$

$\Rightarrow$ By expanding the state space to include m-tuples of the rv Z_n, we have converted the m dependence over time into a unit dependence over time, a (first-order) Markov chain is defined using the expanded state space.

Initial state is $X_{m-1} = (Z_{m-1}, \dots, Z_0)$; might want to shift the time axis to start with X_0.

Description of Markov chain using directed graphs.



$S = \{1, 2, \dots, 6\}$

$P_{ij} = Pr(X_n = j | X_{n-1} = i)$

Can form a matrix of transition probabilities

$$[P] = \begin{pmatrix} P_{11} & \cdots & P_{16} \\ \vdots & \ddots & \vdots \\ P_{61} & \cdots & P_{66} \end{pmatrix}$$

$\exists$ directed arc from $i \in S$ to $j \in S$ iff $P_{ij} > 0$.

Note that $\sum_{j \in S} P_{ij} = 1, \ \forall i \in S$.

$\Rightarrow$ ~~Column~~ Row sums of $[P]$ are all one.

Subject:

No:

Date:

Classification of States.

Def: An (n-step) walk is an ordering of a string of nodes $(i_0, i_1, i_2, \ldots, i_n)$, $n \geq 1$ in which there is a directed arc from i_{m-1} to i_m for each $1 \leq m \leq n$.

Def: A path is a walk in which no nodes are repeated.

Def: A cycle is a walk in which the first and last nodes are equal and no other node is repeated.

Rmk: A walk can start and end at the same node but a path can't.

# of steps in a walk can be arbitrarily large but not for paths (# of step $\leq M-1$) & cycles (# of steps $\leq M$).

Def: A state $j \in S$ is accessible from another state $i \in S$ if $\exists$ walk from i to j. In this case we write $i \to j$.

Ex: $1 \to 3$, $5 \not\to 3$, $2 \to 2$, $6 \not\to 6$

Fact: Suppose that a walk $(i_0, i_1, \ldots, i_n)$ exists for node i_0 to node i_n. Then $P_{i_0 i_1}, P_{i_1 i_2}, P_{i_2 i_3}, \ldots, P_{i_{n-1}, i_n} > 0$.

Pf: $Pr(X_n = i_n | X_0 = i_0) = \sum_{i_1, i_2, \ldots, i_{n-1}} Pr(X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1 | X_0 = x_0)$

$= \sum_{i_1, i_2, \ldots, i_{n-1}} Pr(X_n = i_n | X_{n-1} = i_{n-1}) Pr(X_{n-1} = i_{n-1} | X_{n-2} = i_{n-2}) \cdots$
$\cdots Pr(X_1 = i_1 | X_0 = i_0)$

$\geq Pr(X_n = i_n | X_{n-1} = i_{n-1}) Pr(X_{n-1} = i_{n-1} | X_{n-2} = i_{n-2}) \cdots Pr(X_1 = i_1 | X_0 = i_0)$

$= \cancel{P_{i_{n-1}}}\ P_{i_{n-1} i_n} P_{i_{n-2} i_{n-1}} \cdots P_{i_1 i_2} P_{i_0 i_1} > 0$

$\Rightarrow$ All probabilities $P_{i_{m-1} i_m} > 0 \quad \forall\ 1 \leq m \leq n$.

Subject: No: Date:

If $\Pr(X_n = i_n | X_0 = i_0) > 0 \Rightarrow \exists$ n-step walk from i_0 to i_n.

We write $P(X_n = j | X_0 = i) = p_{ij}^n$

Eg: $p_{13}^2 = p_{12} p_{23} > 0 \qquad p_{53}^n = 0 \quad \forall n \geq 1$

Fact: If $\exists$ n-step walk from i to j & $\exists$ m-step walk from j to k, then $\exists$ (m+n)-step walk from i to k.

$\Rightarrow p_{ij}^n > 0, \; p_{jk}^m > 0 \Rightarrow p_{ik}^{n+m} > 0.$

$\Rightarrow i \to j \;\&\; j \to k \Rightarrow i \to k$

Def: Two states i & j communicate (denoted as $i \leftrightarrow j$) iff $i \to j$ & $j \to i$.

Fact: If $i \leftrightarrow j$ & $j \leftrightarrow k$, then $i \leftrightarrow k$.

Def: A class $C \subset \{1, \dots, M\}$ is a non-empty subset of states in $S = \{1, \dots, M\}$ s.t. $\forall i \in C$, each state $j \neq i$ satisfies $j \in C$ iff $i \leftrightarrow j$ & $j \notin C$ iff $j \not\leftrightarrow i$.

Eg: $\{2,3\}$ forms a class of states.
$\{1\}, \{4\}, \{5\}, \{6\}$ form other classes of states

Def: A recurrent state is a state i that is accessible from all other states that are accessible from i.

$\Rightarrow$ i is recurrent $\Leftrightarrow$ $i \to j$ implies $j \to i$.

$\Rightarrow$ There is no possibility of going to a state j from which there can be no return. If it enters a recurrent state, it returns to that state eventually w.p. 1. $\Rightarrow$ Keeps returning infinitely often. $\Rightarrow$ This explains the term recurrent.

Subject: No: Date:

Def. A transient state is one that is not recurrent

Prop 1: State j is recurrent iff

$$\sum_{n=1}^{\infty} P_{jj}^{n} = \infty$$

Pf: The number of visits to state j (if we start from state j) is infinite and thus has infinite expectation.

$\Rightarrow \mathbb{E}[\#\text{ of visits to } j \mid X_0 = j] = \infty$

Let $I_n = \begin{cases} 1 & X_n = j \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow \sum_{n=0}^{\infty} I_n$ denotes the total # of visits to state j.

$$\mathbb{E}\left[\sum_{n=0}^{\infty} I_n \,\middle|\, X_0 = j\right] = \sum_{n=0}^{\infty} \mathbb{E}[I_n \mid X_0 = j] = \sum_{n=0}^{\infty} P_{jj}^{n}$$

Thus the result follows if j is recurrent.

On the other hand, if j is transient, every time the Markov Chain visits j, there's a positive prob, say $1 - f_{jj}$ ($f_{jj} \in (0,1)$) that it will never return to j. Hence the total # of visits is geometric with finite mean $1/(1-f_{jj}) < \infty$.

Eg: 2 is a recurrent state $\because$ 3 is accessible from 2 ($2 \to 3$) But 2 is also accessible from 3. Similarly 3 is a recurrent state.

Thm: In a finite-state Markov chain, either all states in a class are transient or all are recurrent.

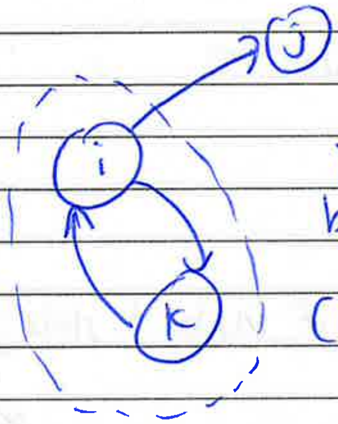
Pf: Let i be transient (i.e., $\exists j$ s.t. $i \to j$ but $j \not\to i$). Suppose i & k are in the same class, i.e., ~~[illegible]~~ $i \leftrightarrow k$. Then $k \to i$ & $i \to j$ so $k \to j$. Now if, to the contrary, if

Subject: No: Date:

$j \to k$, the walk from j to k can be extended to i ($\because$ $j \to k$ & $k \to i$ so $j \to i$). This, however, contradicts the fact that $j \not\to i$.

Thus, there is no walk from j to k ($j \not\to k$) so k is transient ($\because \exists j$ s.t. $k \to j$ but $j \not\to k$).

Hence all states in a class are transient if any of them are



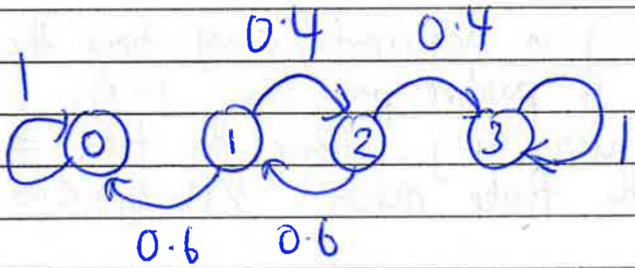
i is a transient state ($\because \exists j$ s.t. $i \to j$ but $j \not\to i$).

Class containing i & k

All states in a class are either recurrent or transient.

Ex: Gambler's Ruin Example.

$S = \{0, 1, 2, 3\}$.



State Transition Matrix
$$[P] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What are the classes here? $\{0\}$, $\{1, 2\}$, $\{3\}$

$1 \leftrightarrow 2$ / $2 \to 3$ but $3 \not\to 2$ / $1 \to 3$ but $3 \not\to 1$

Similarly for 0 $\{1, 2\}$ is a transient class

Subject:

No:

Date:

$\{0\}, \{3\}$ are recurrent classes.

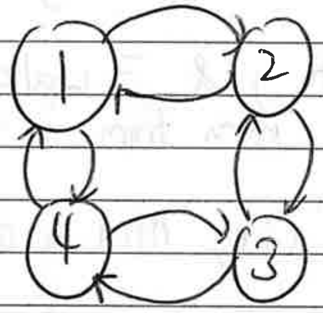
Def: The *period* of a state $i \in S$, denoted as $d(i)$, is the gcd (greatest common divisor) of those values $n \in \mathbb{N}$ for which

$$P_{ii}^{n} > 0.$$

If the period of a state i is $d(i) = 1$, the state is called aperiodic.

If $d(i) \geq 2$, the state is called periodic with period $d(i)$.

Eg:



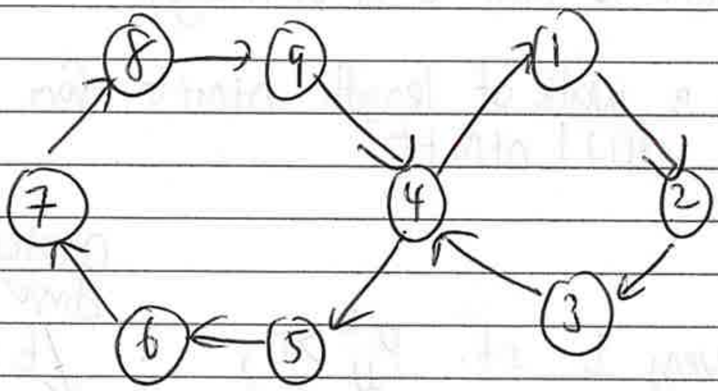
What is $d(i)$ for each $i \in \{1,2,3,4\}$?

$P_{11}^{2} > 0$, $P_{11}^{4} > 0$, $P_{11}^{6} > 0, \cdots$

$\gcd\{2,4,6,\cdots\} = 2.$

$\Rightarrow d(1) = 2$. Similarly for $d(2) = d(3) = d(4)$

Eg:



What is $d(1)$? $P_{11}^{4} > 0$, $P_{11}^{8} > 0$, $P_{11}^{10} > 0$, $P_{11}^{12} > 0, \cdots$

$\gcd\{4, 8, 10, 12, \cdots\} = 2 \quad d(1) = 2$

Check that $d(i) = 2 \;\; \forall i \in \{1, 2, \cdots, 9\}$.

Subject: No: Date:

Eg: Gambler Ruin Problem
Class $\{1,2\}$

$P_{22}^{2}, P_{22}^{4}, P_{22}^{6} > 0 \qquad P_{11}^{2}, P_{11}^{4}, P_{11}^{6}, P_{11}^{8}, \dots > 0$

$\Rightarrow d(1) = d(2) = 2$

Thm: For any Markov chain, all states in the same class have the same period.

Pf: Let $i,j \in S$ be a pair of states in the same class, say C, i.e., $i,j \in C$. Then $i \leftrightarrow j$ & $\exists\, n, m \in \mathbb{N}$ s.t. $P_{ij}^{n} > 0$, $P_{ji}^{m} > 0$.

Since $\exists$ walk of length n from i to j & $\exists$ walk of length m from j to i, $\exists$ walk of length $n+m$ from i to i (via j).

This implies that $d(i) \mid n+m$ (i.e., $n+m$ is divisible by $d(i)$)

Let t be any integer s.t. $P_{jj}^{t} > 0$. (Such a $t \in \mathbb{N}$ certainly exists, but the pt here is that t is arbitrary).

Since there is clearly a walk of length $n+m+t$ from i to i (via j), $d(i) \mid n+m+t$

$\Rightarrow \quad d(i) \mid t$

This is true for every t s.t. $P_{jj}^{t} > 0$,

greatest common divisor of those t s.t. $P_{jj}^{t} > 0$.

$\Rightarrow \quad \cancel{d(i) \mid d(j)} \quad \cancel{d(j) \mid d(i)} \quad d(i) \mid d(j)$

Reversing the roles of i & j, $\cancel{d(i) \mid d(j)}$ $d(j) \mid d(i)$.

$\Rightarrow \quad d(i) = d(j)$.

Rmk: We can say that a class has period d.
We can say that a class is transient or recurrent.

Subject:

No:

Date:

Thm: If a class C in a finite-state Markov chain has period d, the states in C can be partitioned into d subsets $S_0, S_1, \dots, S_{d-1}$ s.t. all transitions from S_ℓ go to $S_{\ell+1}$ for $\ell < d-1$ & all transitions from S_{d-1} go to S_0.

Pf: See book. Will not cover.

Def: For a Markov chain, an ergodic class of states is a class such that it is both aperiodic and recurrent

A MC containing ~~ent~~ entirely of one ergodic class is ~~et~~ called an ergodic MC.

Rmk: For an ergodic chain P_{ij}^n becomes independent of the starting state as $n \to \infty$.

Thm: For an ergodic M-state Markov chain, $P_{ij}^m > 0$ for all i, j and all $m \geq (M-1)^2 + 1$.

No:

Subject:

Date: