Solutions to EE5137 Exam (Semester 1 2017/8)

November 30, 2017

1 Problem 1

(a) Using the law of iterated expectations,

$$\mathbb{E}[X_i] = \mathbb{E}[\mathbb{E}[X_i|Q]] = \mathbb{E}[Q] = \mu$$

where the second equality is because $\mathbb{E}[X_i|Q = q] = q$. Now, the expectation of the sum is

$$\mathbb{E}[S_n] = \mathbb{E}[X_1 + \ldots + X_n] = n\mu.$$

(b) The variance is

$$\mathrm{Var}(X_i) = \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 = \mathbb{E}[X_i] - (\mathbb{E}[X_i])^2 = \mu - \mu^2.$$

(c) The covariance can be computed as

$$\begin{aligned}
\mathrm{Cov}(X_i, X_j) &= \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] \\
&= \mathbb{E}[\mathbb{E}[X_i X_j|Q]] - \mu^2 \\
&= \mathbb{E}[\mathbb{E}[X_i|Q]\mathbb{E}[X_j|Q]] - \mu^2 \\
&= \mathbb{E}[Q^2] - \mu^2 \\
&= \sigma^2 + \mu^2 - \mu^2 \\
&= \sigma^2 > 0.
\end{aligned}$$

Hence, the random variables X_i and X_j are not independent.

(d) We now derive $\mathrm{Var}(S_n) = \mathbb{E}[S_n^2] - (\mathbb{E}[S_n])^2$ using the law of iterated expectations. We have

$$\begin{aligned}
\mathbb{E}[S_n^2] - \mathbb{E}[S_n]^2 &= \mathbb{E}[\mathbb{E}[S_n^2|Q]] - (\mathbb{E}[\mathbb{E}[S_n|Q]])^2 \\
&= \mathbb{E}[\mathbb{E}[S_n^2|Q] - \mathbb{E}[S_n|Q]^2] + \mathbb{E}[\mathbb{E}[S_n|Q]^2] - (\mathbb{E}[\mathbb{E}[S_n|Q]])^2 \\
&= \mathbb{E}[\mathrm{Var}(S_n|Q)] + \mathrm{Var}(\mathbb{E}[S_n|Q])
\end{aligned}$$

(e) We calculate $\mathrm{Var}(S_n|Q)$ first. By conditional independence, we have

$$\begin{aligned}
\mathrm{Var}(S_n|Q) &= \mathrm{Var}(X_1 + \ldots + X_n|Q) \\
&= nQ(1 - Q)
\end{aligned}$$

Next we recall that $\mathbb{E}[S_n|Q] = nQ$. Thus,

$$\begin{aligned}
\mathrm{Var}(S_n) &= \mathbb{E}[nQ(1 - Q)] + \mathrm{Var}(nQ) \\
&= n[\mu - \sigma^2 - \mu^2] + n^2\sigma^2 \\
&= n[\mu - \mu^2] + n(n - 1)\sigma^2.
\end{aligned}$$

2 Problem 2

(a) The Poisson process has rate $\lambda = \ln 2$. The probability that there are zero arrivals in $(3, 5]$ is

$$\Pr(N(2) = 0) = e^{-2\lambda} = e^{-2\ln 2} = \frac{1}{4}.$$

Here we used stationary increment property.

Probability that there is exactly one arrival in the three non-overlapping intervals of length 1 each is

$$\Pr(N(1) = 1)^3 = (e^{-\lambda}\lambda)^3 = (\frac{1}{2}\ln 2)^3 = \frac{1}{8}\ln^3 2.$$

(b) Assume $t_1 \geq t_2$. Consider

$$\begin{aligned}
C_N(t_1, t_2) &= \mathbb{E}[N(t_1)N(t_2)] - \mathbb{E}[N(t_1)]\mathbb{E}[N(t_2)] \\
&= \mathbb{E}[(N(t_1) - N(t_2) + N(t_2))N(t_2)] - \mathbb{E}[N(t_1)]\mathbb{E}[N(t_2)] \\
&\stackrel{(a)}{=} \mathbb{E}[N(t_1) - N(t_2)]\mathbb{E}[N(t_2)] + \mathbb{E}[N(t_2)^2] - \mathbb{E}[N(t_1)]\mathbb{E}[N(t_2)] \\
&= \lambda(t_1 - t_2) \cdot \lambda t_2 + \mathbb{E}[N(t_2)^2] - \lambda t_1 \cdot \lambda t_2 \\
&= \mathbb{E}[N(t_2)^2] - \lambda^2 t_2^2 \\
&= \mathbb{E}[N(t_2)^2] - \mathbb{E}[N(t_2)]^2 \\
&= \mathrm{Var}(N(t_2)) \\
&= \lambda t_2
\end{aligned}$$

where (a) follows from the independent increment property (i.e., that $N(t_1) - N(t_2)$ is independent of $N(t_2)$). By symmetry, if $t_1 \leq t_2$,

$$C_N(t_1, t_2) = \lambda t_1.$$

Thus,

$$C_N(t_1, t_2) = \lambda \min\{t_1, t_2\}.$$

3 Problem 3

(a) Chain 1: Recall that two states i and j in a Markov chain communicate if each is accessible from the other, i.e., if there is a walk from each to the other. Since all transitions move from left to right, each state is accessible only from those to the left, and therefore no state communicates with any other state. Thus each state is in a class by itself. States 0 to 5 (and thus the classes $\{0\}, \ldots, \{5\}$ are each transient since each is inaccessible from an accessible state (i.e., there is a path away from each from which there is no return). State 6 is recurrent. States 1 and 6 (and thus class $\{1\}$ and $\{6\}$) are each aperiodic since $P_{00}^1 \neq 0$ and $P_{66}^1 \neq 0$ The periods of classes $\{1\}$ to $\{5\}$ are undefined but no points will be taken off if some other answer was given for these periods.

Chain 2: Each state on the circle on the left communicates with all other states on the left and similarly for the circle on the right. Since there is a transition from left to right, and also from right to left, the entire set of states communicate, so there is single class containing all states. State 0 has a cycle of length 2 through state 1 and of length 7 via the left circle. The greatest common divisor of 2 and 7 is 1, so state 1 has period 1. The chain is then aperiodic since all states in a class have the same period.

(b) The transition matrix is

$$[P] = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

We will find the limiting fraction of drivers in each of these categories from the components of the stationary distribution vector π, which satisfies the following equation:

$$\pi = \pi[P].$$

This is equivalent to the following system of equations:

$$\begin{aligned} \pi_1 &= 0.6\pi_1 + 0.1\pi_2 \\ \pi_2 &= 0.4\pi_1 + 0.6\pi_2 + 0.2\pi_3 \\ \pi_3 &= 0.3\pi_2 + 0.8\pi_3 \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

This has the following solution

$$\pi = \frac{1}{11}(1, 4, 6)$$

Thus, the limiting fraction of drivers in the bad category is 1/11, in the satisfactory category 4/11 and in the preferred category 6/11.

4 Problem 4

(a) The ML decision rule decides in favor of H_1 if

$$f_{\mathbf{Y}|H}(\mathbf{y}|H_1) \geq f_{\mathbf{Y}|H}(\mathbf{y}|H_0).$$

The conditional densities are jointly Gaussian, with zero mean and the following covariance matrices

$$\Sigma_{\mathbf{Y}|H_0} = \begin{bmatrix} (\alpha^2+\sigma^2)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma^2\mathbf{I} \end{bmatrix}, \quad \Sigma_{\mathbf{Y}|H_1} = \begin{bmatrix} \sigma^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & (\alpha^2+\sigma^2)\mathbf{I} \end{bmatrix}$$

We plug these matrices into the Gaussian density functions, take logarithms and cancel constants, obtaining that we decide in favor of H_1 if

$$\frac{Y_1^2+Y_2^2}{\alpha^2+\sigma^2} + \frac{Y_3^2+Y_4^2}{\sigma^2} \geq \frac{Y_1^2+Y_2^2}{\sigma^2} + \frac{Y_3^2+Y_4^2}{\alpha^2+\sigma^2}$$

Now, we can cancel terms to get the desired result:

$$Y_3^2+Y_4^2 \geq Y_1^2+Y_2^2 \quad \Leftrightarrow \quad V_1 \geq V_0$$

to decide in favor of H_1.

(b) Simply use the hint to obtain

$$f_{V_0|H}(v_0|H_0) = \frac{1}{2(\alpha^2+\sigma^2)} \exp\left(-\frac{v_0}{2(\alpha^2+\sigma^2)}\right), \quad v_0 \geq 0$$

and

$$f_{V_1|H}(v_1|H_0) = \frac{1}{2\sigma^2} \exp\left(-\frac{v_1}{2\sigma^2}\right), \quad v_1 \geq 0$$

(c) Now given $H = H_0$, V_0 and V_1 are independent, so we can find the density of $U = V_0 - V_1$ by convolution. Let

$$a = \frac{1}{2(\alpha^2+\sigma^2)} \quad \text{and} \quad b = \frac{1}{2\sigma^2}.$$

Then

$$f_{U|H}(u|H_0) = f_{V_0|H}(u|H_0) * f_{V_1|H}(-u|H_0)$$

After convolving, we obtain

$$f_{U|H}(u|H_0) = \begin{cases} \frac{ab}{a+b}e^{bu} & u < 0 \\ \frac{ab}{a+b}e^{-au} & u \geq 0 \end{cases}.$$

(d) Given that $H = H_0$, an error occurs if $V_1 > V_0$ or equivalently $U < 0$. We compute

$$\begin{aligned} \Pr(\mathcal{E}|H=H_0) &= \int_{-\infty}^{0} f_{U|H}(u|H_0)\,\mathrm{d}u \\ &= \int_{-\infty}^{0} \frac{ab}{a+b}e^{bu}\,\mathrm{d}u \\ &= \frac{a}{a+b} = \frac{1}{2+\alpha^2/\sigma^2} \end{aligned}$$

Note that if $H = H_1$, then V_1 and V_0 switch roles in the problem. That is, they will both still be exponentially distributed, but their variances will be swapped. If we define $W = V_1 - V_0$, note that all of the above results will hold with a slight change in the subscripts.

In particular,

$$\Pr(\mathcal{E}|H=H_1) = \frac{1}{2+\alpha^2/\sigma^2} = \Pr(\mathcal{E}|H=H_0).$$

Since the two hypotheses are equally likely, this must also be the unconditional probability of error.