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Subject： Stochastic process

Assignment: Homework Five

Date: Feb 26th

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1. EXERCISE 2.1

(b) $E[e^{rx}] = \int_{-\infty}^{+\infty} e^{rx} \cdot \lambda \cdot e^{-\lambda x} dx$

$= \int_{-\infty}^{+\infty} \lambda \cdot e^{(r-\lambda)x} dx$

$= \frac{\lambda}{\lambda-r}[e^{(r-\lambda)x}]\Big|_0^{+\infty}$

$= \frac{\lambda}{\lambda-r}, \quad r<\lambda$

Then we use $S_n = x_1+x_2+\cdots x_n$ to get S_n's moment generating function

$E[e^{r \cdot S_n}] = E[e^{\sum_{i=1}^n r\cdot x_i}] = E[\prod_{i=1}^n e^{rx_i}] = \prod_{i=1}^n g_{x_i}(r) = \left(\frac{\lambda}{\lambda-r}\right)^n$, $r<\lambda$.

So $\left(\frac{\lambda}{\lambda-r}\right)^n = \int_{-\infty}^{+\infty} e^{rx} \cdot f_{S_n}(x)\,dx = \int_0^{+\infty} e^{rt} f_{S_n}(t)\,dt$

Using the laplace in verse ~~transmission~~ transform

$\frac{1}{(n-1)!} t^{n-1} \cdot e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{(s+a)^n}, \quad Re\{s\} > Re\{-a\}$

We cat get $\frac{1}{(\lambda-r)^n} \longleftrightarrow \frac{1}{(n-1)!} t^{n-1} \cdot e^{\lambda t} u(t)$

so $\left(\frac{\lambda}{\lambda-r}\right)^n \Longleftrightarrow \frac{1}{(n-1)!} t^{n-1} e^{-\lambda t} (\lambda)^n$

, which means $f_{S_n}(t) = \frac{\lambda^n \cdot t^{n-1} \cdot e^{-\lambda t}}{(n-1)!}$

2. EXERCISE 2.2

(a). For the mean of counting process $N(t)$

$E[N(t)] = \sum_{n=0}^{\infty} n \cdot \frac{(\lambda t)^n e^{-\lambda t}}{n!} = (\lambda t) \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!}$

We can change this form $\left(\sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1} \cdot e^{-\lambda t}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \right)$

so the equation can be rewrite as $\lambda t \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!}$

And then this can be concluded by (the sum of the Poisson PMF is

, which means $\sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} = 1$

so the result is λt

For the variance of counting process $(N(t))$

$E[N^2(t)] = \sum_{n=0}^{\infty} \cdot n^2 \cdot \frac{(\lambda t)^n e^{-\lambda t}}{n!} = (\lambda t) \sum_{n=0}^{\infty} n \cdot \frac{(\lambda t)^{n-1} \cdot e^{-\lambda t}}{(n-1)!}$

then we let $n = (n-1)+1$

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we can change that to be

$$(\lambda t)\sum_{n=0}^{\infty}[(n-1)+1]\frac{(\lambda t)^{n-1}\cdot e^{-\lambda t}}{(n-1)!}$$

$$=\lambda t\sum_{n=0}^{\infty}\frac{(\lambda t)^{n-1}\cdot e^{-\lambda t}}{(n-2)!}+\lambda t\cdot\sum_{n=0}^{\infty}\frac{(\lambda t)^{n-1}\cdot e^{-\lambda t}}{(n-1)!}$$

$$=(\lambda t)^2\sum_{n=2}^{\infty}\frac{(\lambda t)^{n-2}\cdot e^{-\lambda t}}{(n-2)!}+\lambda t\cdot\sum_{n=0}^{\infty}\frac{(\lambda t)^{n-1}\cdot e^{-\lambda t}}{(n-1)!}$$

$$=(\lambda t)^2\sum_{n=0}^{\infty}\frac{(\lambda t)^{n}\cdot e^{-\lambda t}}{n!}+(\lambda t)\cdot\sum_{n=0}^{\infty}\frac{(\lambda t)^{n}e^{-\lambda t}}{n!}$$

With same epoch like above proof ($\sum_{n=0}^{\infty}\frac{(\lambda t)^n\cdot e^{-\lambda t}}{n!}=1$)

$$=(\lambda t)^2+\lambda t$$

so the variance of counting process

$$Var[N(t)]=E[N^2(t)]-E^2[N(t)]$$
$$=(\lambda t)^2+\lambda t-(\lambda t)^2=\lambda t$$

For the MGF of counting process ($N(t)$)

$$g_x(r)=E[e^{rN(t)}]=\cancel{\int_{-\infty}^{+\infty}e^{rt}\cdot\frac{(\lambda t)^n e^{-\lambda t}}{n!}dt}$$

$$\cancel{=\int_0^{+\infty}e^{(r-\lambda)t}\cdot\frac{(\lambda t)^n}{n!}dt}$$

$$=\sum_0^{+\infty}\frac{e^{r\cdot n}\cdot(\lambda t)^n\cdot e^{-\lambda t}}{n!}$$

$$=\sum_0^{+\infty}\frac{(e^r\cdot\lambda t)^n\cdot e^{-\lambda t}}{n!}$$

$$=\sum_0^{+\infty}\frac{(e^r\cdot\lambda t)^n\cdot e^{-\lambda t\cdot e^r}}{n!}\cdot e^{-\lambda t}\cdot e^{\lambda t\cdot e^r}$$

Using the same method ($\sum_0^{+\infty}\frac{(e^r\cdot\lambda t)^n\cdot e^{-\lambda t\cdot e^r}}{n!}=1$)

so the MGF of $N(t)$ is $e^{-\lambda t}\cdot e^{\lambda t\cdot e^r}$

$$=e^{\lambda t(e^r-1)}$$

(b). For two independent Poisson rvs, X, Y

they have seperate poisson rate λ_A, λ_B

PDF: A $= f_X(x) = \frac{(\lambda_A)^x \cdot e^{-\lambda_A}}{x!}$. PDF: B : $f_Y(Y) = \frac{(\lambda_B)^Y \cdot e^{-\lambda_B}}{Y!}$

For $Z = X + Y$ (owing to X, Y are independent)

so $f_Z(Z) = \sum_0^{+\infty} f_X(x)\, f_Y(z-x)$

$$= \sum_0^{+\infty} \frac{(\lambda_A)^x \cdot e^{-\lambda_A}}{x!} \cdot \frac{(\lambda_B)^{z-x} \cdot e^{-\lambda_B}}{(z-x)!}$$

$$= \sum_0^{+\infty} e^{-(\lambda_A+\lambda_B)} \cdot \frac{(\lambda_A)^x \cdot (\lambda_B)^{z-x}}{x!\,(z-x)!}$$

$$= e^{-(\lambda_A+\lambda_B)} \sum_0^{+\infty} \frac{z!}{x!\,(z-x)!} \cdot \frac{(\lambda_A)^x \cdot (\lambda_B)^{z-x}}{z!}$$

Using the Binomial theorem, we can rewrite above equation

$$= e^{-(\lambda_A+\lambda_B)} \sum_0^{+\infty} C_z^x (\lambda_A)^x \cdot (\lambda_B)^{z-x} \cdot \frac{1}{z!}$$

$$= e^{-(\lambda_A+\lambda_B)} \cdot (\lambda_A+\lambda_B)^z \frac{1}{z!}$$

$$= \frac{(\lambda_A+\lambda_B)^z \cdot e^{-(\lambda_A+\lambda_B)}}{z!}$$

so the sum of two independent Poisson rvs must be Poisson.

3. EXERCISE 2.4

(a) For $Pr\{X_1 > x\}$, which means $(0, x]$, $N(t) = 0$

(using 2.17) $P_{N(t)}(n) = \frac{(\lambda t)^n \cdot \exp(-\lambda t)}{n!}$

let $n=0$ $\quad P_{N(t)}(0) = \frac{(\lambda t)^0 \cdot \exp(-\lambda t)}{0!} = e^{-\lambda t}$

In conclusion $Pr\{X_1 > x\} = e^{-\lambda x}$

(b) According to above information from this part

S_{n-1} be the epoch of the $(n-1)$ th arrival

so $Pr\{X_n > x \mid S_{n-1} = \tau\} = Pr\{[\tau, \tau+x]$ there is no arrival $\mid S_{n-1} = \tau$

$= Pr\{N(t) = 0 \quad [0, x]\}$ (renewal process)

$= \frac{(\lambda t)^{n=0} \cdot \exp(-\lambda t)}{n!}$ $t = x$ $(n=0)$

$= \exp(-\lambda x)$

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(c) For the above information in (c) statement, when $n \geq 1$, $Pr\{X_n > x\}$

From the above provement from (b).
We can get $Pr\{X_n > x \mid S_{n-1} = \tau\} = e^{-\lambda x}$

Because the probability $\{S_{n-1} = \tau\}$ is independent on x
we can conclude that, X_n is independent of S_{n-1}.
In other words, we can get $Pr\{X_n > x\} = e^{-\lambda\tau}$
Based on the definition of conditional probability
($Pr\{A|B\} = Pr\{A\}$, A, B are not dependent

As desired, $Pr\{X_n > x\} = e^{-\lambda x}$ and X_n is independent of S_{n-1}.

(d). For $X_1, X_2 \cdots X_{n-1}$, using the idea from (c) part.
X_n is independent S_{n-1}.
When $n=2$, X_2 and $S_1 = X_1$ are independent
$n=3$, X_3 and $S_2 = X_1 + X_2$ are independent
⋮
n, X_n and $S_{n-1} = X_1 + X_2 + \cdots X_{n-1}$ are independent

From the definition of lectures, we use the I.I.P.
the above event is then independent of the rv $\tilde{N}(S_{n-1}, S_{n-1}+x)$
It is obvious to get the conclusion that
X_n is independent of $X_1, X_2 \cdots X_{n-1}$

4. EXERCISE 2.7

(a) For $F_0(\tau) = \Pr\{N(\tau) = 0\}$

Using the hint $\left(\left.\frac{df(t)}{dt}\right|_{t=\tau} = \lim_{\delta \downarrow 0} \frac{f(\tau+\delta) - f(\tau)}{\delta} \right)$

We can $F_0(\tau) = \Pr\{N(\tau)=0\}$ $\cancel{= 1-\lambda\delta+o(\delta)}$

$F_0(\tau+\delta) = \Pr\{\tilde{N}(\tau+\delta)=0\}$ $\cancel{= 1-\lambda(\tau+\delta)+o(\delta)}$

$$\frac{dF_0(\tau)}{d\tau} = \lim_{\delta\to 0} \frac{F_0(\tau+\delta) - F_0(\tau)}{d\delta} = \lim_{\delta\to 0} \frac{\Pr\{\tilde{N}(\tau+\delta)=0\} - \Pr\{N(\tau)=0\}}{\delta}$$

$$= \lim_{\delta\to 0} \Pr\{N(\tau)=0\} \frac{\left[\Pr\{\tilde{N}(\tau,\tau+\delta)=0\} - 1\right]}{\delta}$$

This is based on the counting process is memoless.

And then we change this form, we can get

$\Pr\{\tilde{N}(\tau,\tau+\delta)=0\} = 1-\lambda\delta+o(\delta)$

Let put this equation into above expression

$$= \lim_{\delta\to 0} \Pr\{N(\tau)=0\} \frac{1-\lambda\delta+o(\delta)-1}{\delta}$$

$$= -\lambda \Pr\{N(\tau)=0\}$$

In conclusion, we can get $\frac{dF_0(\tau)}{d\tau} = -\lambda F_0(\tau)$

(b). For X_1 (the first arrival), which means ~~$\Pr\{N(\ldots)\}$~~

when $\tau \le$ time of first arrival, it still satisfy the condition in (a)

$$\frac{dF_0(\tau)}{d\tau} = -\lambda F_0(\tau)$$

$$\frac{dF_0(\tau)}{F_0(\tau)} = -\lambda d\tau$$

And then use the definition of differental equation

$$\ln F_0(\tau) = -\lambda\tau + C$$

$$F_0(\tau) = e^{-\lambda\tau+C}$$

When we focus on $\tau=0$, $F_0(\tau)=1$, put this restriction into this equation, we can find $C=0$.

So $F_0(\tau) = e^{-\lambda\tau}$

To sum up, X_1, is exponential with pareameter λ.

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(c) Let $F_n^c(\tau) = P_r\{\check{N}(t, t+\tau)=0 \mid S_{n-1}=t\}$

~~F_n~~ Using the similar method like (a)

But for counting process $N(t)$, which is independent $\{S_{n-1}=t\}$, so we can rewrite this $F_n^c(\tau)$

$$F_n^c(\tau) = P_r\{\check{N}(t, t+\tau)=0 \mid S_{n-1}=t\}$$
$$= P_r\{\check{N}(t, t+\tau)=0\}$$
$$F_n^c(\tau+\delta) = P_r\{\check{N}(t, t+\tau+\delta)=0\}$$

And then get deriavation (memoryless property)

$$\frac{dF_n^c(\tau)}{d\tau} = \lim_{\delta\to 0}\frac{F_n^c(\tau+\delta)-F_n^c(\tau)}{\delta} = \lim_{\delta\to 0}\frac{F_n^c(\tau)[P_r\{\check{N}(t,t+\delta)=0\}-}{\delta}$$

Let $P_r\{\check{N}(t, t+\delta)=0\} = 1-\lambda\delta+o(\delta)$

Obviously, we can get $\frac{dF_n^c(\tau)}{d\tau} = \lim_{\delta\to 0}\frac{F_n^c(\tau)[1-\lambda\delta+o(\delta)-1]}{\delta}$

$$= -\lambda F_n^c(\tau)$$

As desired, $\frac{dF_n^c(\tau)}{d\tau} = -\lambda F_n^c(\tau)$

(d). For X_n, it means there is no (n^{th}) arrival in $[t, t+\tau]$

We can get it from (c). X_n satisfy the condition in (c),

So it doesn't about the first $(n-1)$ arrival time (t)

In other words, it is dependent of earlier arrival time

Using similar method like (b) (differental equation)

$$\frac{dF_n^c(\tau)}{d\tau} = -\lambda F_n^c(\tau)$$

$$\frac{dF_n^c(\tau)}{F_n^c(\tau)} = -\lambda d\tau$$

$$F_n^c(\tau) = e^{-\lambda\tau+C}$$

When we focus on $\tau=0$, $F_n^c(\tau)=1$, use this restriction into this equation, we also can get $C=0$.

So $F_n^c(\tau) = e^{-\lambda\tau}$

In conclusion, X_n is exponential with parameter λ, and at the same time, it is independent of earlier arrival time $\{S_{n-1}=t\}$

5. EXERCISE 2.9

(a). From the illustration is section 2.2.5.

For any series of Bernoulli process is an IID sequence, $Y_1+Y_2+\cdots Y_m$ at any time slot, ~~we can prefer~~ $(\delta = 2^{-j})$, j^{th} time slot, we can set the probability of arrival is $\lambda\delta$, and the probability of no arrival is $(1-\lambda\delta)$, use the definition of Binominal process,

$$Pr(n=k) = \binom{n}{k} p^k \cdot (1-p)^{n-k}$$

and take this equation to this expression, we can get that

$$Pr(N_\delta(m\delta) = n) = \binom{m}{n} (\lambda\delta)^n (1-\lambda\delta)^{m-n}$$

(b). Let $t = m\cdot\delta$, we can get $\delta = \frac{t}{m}$, which means this is the smallest time slot. In this slot, we can regard this Binominal process as binary process in some aspects.

There are two possibilities, the first one is that arrival process with $(\lambda\delta)$ probability, and the second one is that no arrival process with $(1-\lambda\delta)$ probability.

Through the analyse of above information, we can rewrite.

$$\lim_{\delta\to 0} Pr\{N_\delta(t) = n\} = \lim_{m\to\infty} \binom{m}{n} \left(\frac{\lambda t}{m}\right)^n \left(1-\frac{\lambda t}{m}\right)^{m-n}$$

Especially for $m\to\infty$, we can let $\frac{t}{m} = \delta$.

(c). From the above information from (c)

$$\lim_{m\to\infty} \binom{m}{n} \frac{1}{m^n} = \lim_{m\to\infty} \frac{m!}{n!(m-n)!} \cdot \frac{1}{m^n} = \lim_{m\to\infty} \frac{m(m-1)\cdots(m-n+1)}{n!\, m\cdot m\cdots m}$$

When the m satisfy the condition ($m\to\infty$)

we can cancel some items on fraction.

$$\lim_{m\to\infty} \frac{m\cdot(m-1)\cdots(m-n+1)}{m\cdot m\cdots m\,(n!)} = \frac{1}{n!}$$

For $\lim_{m\to\infty}\left(1-\frac{\lambda t}{m}\right)^{m-n} = \lim_{m\to\infty}\left(1-\frac{\lambda t}{m}\right)^{m-n} = \dfrac{\lim_{m\to\infty}\left(1-\frac{\lambda t}{m}\right)^m}{\left(1-\frac{\lambda t}{m}\right)^n} = e^{-\lambda t}$

As desired, we can prove two equation

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(d). We can use above equation from (c)

$$\lim_{\delta\to 0} Pr\{N_\delta(m\delta)=n\} = \binom{m}{n}(\lambda\delta)^n(1-\lambda\delta)^{m-n}$$

$$= \frac{(\lambda t)^n}{n!}\cdot e^{-\lambda t}$$

As we all know $Pr\{N(t)=n\} = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$

These two definition has same value, so we can conclude that

$$\lim_{\delta\to 0} Pr\{N_\delta(t)=n\} = Pr\{N(t)=n\}$$

When it comes to $\delta\to 0$, it satisfy the same value, the Poisson PMF is the limit of shrinking Bernoulli PMF

6. (a). From the statement from above information

Because the $X+Y$ is still poisson process,

The receiver can receive 9 message, is still poisson process.

The rate is $(\lambda_A+\lambda_B)$

$$Pr(n=k) = \frac{[(\lambda_A+\lambda_B)t]^k}{k!}e^{-(\lambda_A+\lambda_B)t}$$

When it comes to 9 messages, we can set $k=9$, which means 9 messages has been received

$$Pr(n=9) = \frac{[(\lambda_A+\lambda_B)t]^9\cdot e^{-(\lambda_A+\lambda_B)t}}{9!}$$

(b). Let N to be the total number of words received during an interval of duration t., Let k display the k^{th} message,

$$E[N] = E\left[\sum_0^{+\infty} k\cdot\frac{[(\lambda_A+\lambda_B)t]^k}{k!}e^{-(\lambda_A+\lambda_B)t}\right]\Big|_{W=1} * \frac{2}{6}$$

$$+ E\left[\sum_0^{+\infty} k\cdot\frac{[(\lambda_A+\lambda_B)t]^k}{k!}e^{-(\lambda_A+\lambda_B)t}\right]\Big|_{W=2} *2* \frac{3}{6}$$

$$+ E\left[\sum_0^{+\infty} k\cdot\frac{[(\lambda_A+\lambda_B)t]^k}{k!}e^{-(\lambda_A+\lambda_B)t}\right]\Big|_{W=3} *3* \frac{1}{6}$$

$$+0$$

As for the expected number of counting process, we can use the idea from Q2

So $E\left[\sum_{0}^{+\infty} k \cdot \frac{[(\lambda_A+\lambda_B)t]^k}{k!} e^{-(\lambda_A+\lambda_B)t}\right] = (\lambda_A+\lambda_B)t$

To sum up, the total result is $\frac{11}{6}(\lambda_A+\lambda_B)t$

(c). As for the above information, we only need pay attention to three-word message, ~~other~~ not other types.

We can ~~consider~~ regard ~~their~~ these two transmitter A and B only send 3-word message, under the condition ($\frac{1}{6}$ propabability).

When it comes to the receiver, the rate became $\frac{\lambda_A+\lambda_B}{6}$

$$Pr\{S_n(t) = k\} = \frac{\left[\frac{(\lambda_A+\lambda_B)}{6}\right]^k t^{k-1} e^{-\frac{(\lambda_A+\lambda_B)t}{6}}}{(k-1)!}$$ (Erlang PMF)

let $k=8$, we can get

$$Pr\{S_n(t) = 8\} = \frac{\left(\frac{\lambda_A+\lambda_B}{6}\right)^8 t^7 e^{-\frac{(\lambda_A+\lambda_B)}{6}t}}{7!}$$

(d) When it comes to exactly 8 out of the 12 messages.

we can conclude this is Binominal process in some aspects.

j^{th} message from transmitter A, with $p = \frac{\lambda_A}{\lambda_A+\lambda_B}$,

may from transmitter B, with $1-p = \frac{\lambda_B}{\lambda_A+\lambda_B}$.

Obviously, $Pr\{$ 8 messages from A out of 12 messages $\}$

$$= \binom{12}{8}\left(\frac{\lambda_A}{\lambda_A+\lambda_B}\right)^8 \left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^{12-8}$$

$$= \binom{12}{8}\left(\frac{\lambda_A^8 \cdot \lambda_B^4}{(\lambda_A+\lambda_B)^{12}}\right)$$