EE5137: Quiz 1

Name: ___________

Matriculation Number: ___________

Total Score: ___________

September 7, 2017

You have 1.0 hours for this exam. You're allowed 1 sheet of handwritten notes (both sides). Please show provide *careful explanations* for all your solutions.

1. [Conditional Expectations] (10 points) Consider the joint probability density function (pdf)

$$f_{X,Y}(x,y) = \frac{1}{y^2} e^{-x/y^2} e^{-y}, \quad x \ge 0, y > 0.$$

You may use the following fact without proof in this problem

$$\int_0^\infty t^z e^{-t\lambda} \, \mathrm{d}t = \frac{z!}{\lambda^{z+1}}, \quad \forall\, z \in \mathbb{N}, \lambda > 0.$$

(a) (2 points) Find the marginal pdf $f_Y(y)$. Please specify the range of y.

(b) (2 points) Hence find the conditional pdf $f_{X|Y}(x|y)$. Please specify the ranges of x and y.

(c) (2 points) Find $\mathbb{E}[X|Y = y]$ for each $y > 0$.

(d) (1 points) Write down $\mathbb{E}[X|Y]$.

(e) (3 points) Find $\mathbb{E}[X]$ using parts (a) and (d).

2. [Convergence of Random Variables] (5 points) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with zero mean and finite variance σ^2. Consider the following sequence of random variables

$$T_n = \frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i, \quad n = 1, 2, 3, \ldots$$

Does T_n converge in probability to a constant? If so, to what?

Consider the standard proof for convergence of $\frac{1}{n} S_n$.

3. [Gaussian Rate Function] (5 points) Let $X_1, X_2, \ldots, X_n$ be i.i.d. Poisson random variables with mean (expectation) $\lambda = \mathbb{E}[X_1] > 0$, i.e.,

$$P_{X_1}(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \forall\, k = 0, 1, 2, \ldots$$

It is known that the moment generating function of X_1 is

$$g_{X_1}(r) = \exp(\lambda(e^r - 1)), \quad \forall\, r \in \mathbb{R}.$$

Suppose $\lambda = 2$. Using the Chernoff bound, find the exponent $E < 0$ in

$$\Pr\left(\frac{1}{n}\sum_{i=1}^{n} X_i > 2e^2\right) \le \exp(nE).$$