EE5137 Lecture 4:
Motivating the Poisson Process using Goals in World Cup Games

Vincent Y. F. Tan

Department of Electrical and Computer Engineering,
National University of Singapore

Feb 2021

The Poisson Process

- The inter-arrival times of a Poisson process is the i.i.d. process $\{X_i\}_{i=1}^{\infty}$, where X_i is an exponential rv with rate λ, i.e., $X_i \sim \mathrm{Exp}(\lambda)$ or

$$f_X(x) = \lambda \exp(-\lambda x), \qquad x \geq 0$$

The Poisson Process

- The inter-arrival times of a Poisson process is the i.i.d. process $\{X_i\}_{i=1}^{\infty}$, where X_i is an exponential rv with rate λ, i.e., $X_i \sim \text{Exp}(\lambda)$ or

$$f_X(x) = \lambda \exp(-\lambda x), \qquad x \geq 0$$

- The arrivals of a Poisson process is the sequence of increasing random variables $0 < S_1 < S_2 < \ldots$. In particular,

$$S_n = \sum_{j=1}^{n} X_j, \qquad n \in \mathbb{N}.$$

It is known that S_n has an Erlang pdf with shape parameter n.

- The Poisson counting process $\{N(t) : t > 0\}$ is the process in which $N(t)$ represents the number of arrivals up to and including time t.
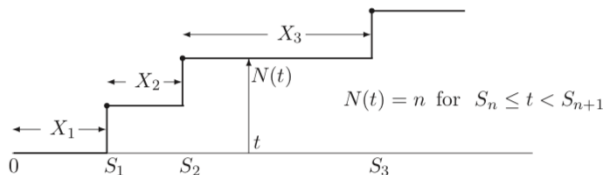
- The Poisson counting process $\{N(t) : t > 0\}$ is the process in which $N(t)$ represents the number of arrivals up to and including time t.

- $N(t)$ has the Poisson pmf

$$p_{N(t)}(n) = \frac{(\lambda t)^n \exp(-\lambda t)}{n!}, \qquad n \in \mathbb{N}$$

The Poisson Counting Process

- The Poisson counting process $\{N(t) : t > 0\}$ is the process in which $N(t)$ represents the number of arrivals up to and including time t.

- $N(t)$ has the Poisson pmf

$$p_{N(t)}(n) = \frac{(\lambda t)^n \exp(-\lambda t)}{n!}, \qquad n \in \mathbb{N}$$

Sample path of a Poisson process

- Focus on 232 World cup (soccer) games from 1990 to 2002

- Focus on 232 World cup (soccer) games from 1990 to 2002

- In the first game of 1990, Cameroon scored a single goal against Argentina at the 67th minute.

Is the Poisson process a good model? An Application

- Focus on 232 World cup (soccer) games from 1990 to 2002

- In the first game of 1990, Cameroon scored a single goal against Argentina at the 67th minute.

- In the second game, Romania scored 2 goals against the Soviet Union at the 42nd and 57th minutes.

- Focus on 232 World cup (soccer) games from 1990 to 2002

- In the first game of 1990, Cameroon scored a single goal against Argentina at the 67th minute.
- In the second game, Romania scored 2 goals against the Soviet Union at the 42nd and 57th minutes.
- Times to these goals are $65 = (90 - 67) + 42$ and $15 = 57 - 42$ min.

Is the Poisson process a good model? An Application

- Focus on 232 World cup (soccer) games from 1990 to 2002

- In the first game of 1990, Cameroon scored a single goal against Argentina at the 67th minute.

- In the second game, Romania scored 2 goals against the Soviet Union at the 42nd and 57th minutes.

- Times to these goals are $65 = (90 - 67) + 42$ and $15 = 57 - 42$ min.

	Game 1	Game 2	....	Game 232
Goals	*	* *	....	* *
Time between Goals				

Process of the arrival of goals

Property 1: Mean and Standard Deviation Coincide

- For $X \sim \text{Exp}(\lambda)$, we have

$$\mathbb{E}[X] = \frac{1}{\lambda}, \qquad \text{var}(X) = \frac{1}{\lambda^2}$$

  so the mean and standard deviation coincide.

Property 1: Mean and Standard Deviation Coincide

- For $X \sim \mathrm{Exp}(\lambda)$, we have

$$\mathbb{E}[X] = \frac{1}{\lambda}, \qquad \mathrm{var}(X) = \frac{1}{\lambda^2}$$

so the mean and standard deviation coincide.

- Empirically,

$$\frac{1}{n} \sum_{i=1}^{n} X_i = 36.25 \text{ min}, \qquad \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(X_i - \frac{1}{n} \sum_{j=1}^{n} X_j \right)^2} = 36.68 \text{ min}$$

Property 1: Mean and Standard Deviation Coincide

- For $X \sim \text{Exp}(\lambda)$, we have

$$\mathbb{E}[X] = \frac{1}{\lambda}, \qquad \text{var}(X) = \frac{1}{\lambda^2}$$

  so the mean and standard deviation coincide.

- Empirically,

$$\frac{1}{n} \sum_{i=1}^{n} X_i = 36.25 \text{ min}, \qquad \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(X_i - \frac{1}{n} \sum_{j=1}^{n} X_j \right)^2} = 36.68 \text{ min}$$

- There were 574 goals in the 232 games. Thus, $\lambda = 574/232$ goals per game.

Property 1: Mean and Standard Deviation Coincide

- For $X \sim \mathrm{Exp}(\lambda)$, we have

$$\mathbb{E}[X] = \frac{1}{\lambda}, \qquad \mathrm{var}(X) = \frac{1}{\lambda^2}$$

  so the mean and standard deviation coincide.

- Empirically,

$$\frac{1}{n} \sum_{i=1}^{n} X_i = 36.25 \text{ min}, \qquad \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(X_i - \frac{1}{n} \sum_{j=1}^{n} X_j \right)^2} = 36.68 \text{ min}$$

- There were 574 goals in the 232 games. Thus, $\lambda = 574/232$ goals per game.

- These compare well to the theoretical expectation

$$\frac{90}{\lambda} \approx 36.31 \text{ min}$$

Inter-goal Duration (minutes)	Actual	Empirical Probability	Theoretical Probability	Expected
0-10	144	0.2504	0.2407	138
10-20	106	0.1843	0.1828	105
20-30	86	0.1496	0.1388	80
30-40	52	0.0904	0.1054	60
40-50	46	0.0800	0.0800	46
50-60	27	0.0470	0.0607	35
60-70	35	0.0626	0.0461	26
70-80	16	0.0278	0.0350	20
80-90	22	0.0383	0.0266	15
90-100	12	0.0209	0.0202	12
100-110	3	0.0052	0.0153	9
110-120	3	0.0052	0.0116	7
120-130	6	0.0104	0.0088	5
130 or more	16	0.0278	0.0279	16
Total	574	1	1	574

Times Between Goals

Property 2: Memorylessness

- The interarrival times are also memoryless, i.e., that

$$\Pr(X > t + x) = \Pr(X > x)\Pr(X > t), \qquad \forall\, t, x \geq 0.$$

or

$$\Pr(X > t + x | X > t) = \Pr(X > x)$$

Property 2: Memorylessness

- The interarrival times are also memoryless, i.e., that

$$\Pr(X > t + x) = \Pr(X > x)\Pr(X > t), \qquad \forall\, t, x \geq 0.$$

or

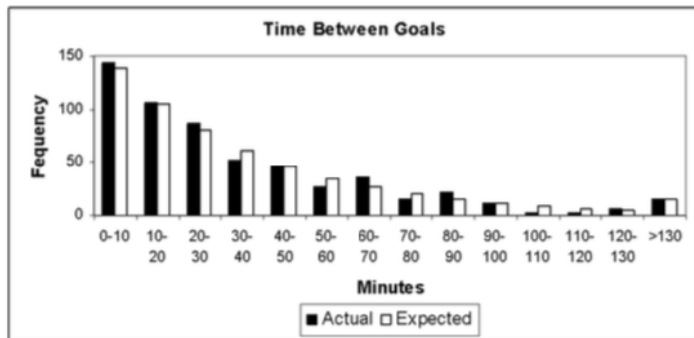$$\Pr(X > t + x | X > t) = \Pr(X > x)$$

- We calculate empirically

$$\Pr(X > 10) = 1 - \frac{144}{574} \approx 0.7491,$$

$$\Pr(X > 20 | X > 10) = \frac{574 - 144 - 106}{574 - 144} \approx 0.7534,$$

$$\Pr(X > 30 | X > 20) \approx 0.7346$$

Property 2: Memorylessness

Times Between Goals

- The X_i's (Inter-Goal or Inter-Arrival Times) are independent.

Property 3: Independence of Inter-arrival Times

- The X_i's (Inter-Goal or Inter-Arrival Times) are independent.

- Autocorrelations of times between goals

Lag	1	2	3	4	5	6	7	8	9
Autocorrelation	-0.0122	-0.0086	0.0021	-0.0033	0.0128	-0.0051	0.0003	0.0092	0.0033

Autocorrelation Between Goals

- All these times lie inside a band of plus or minus twice the standard error i.e., $2/\sqrt{574}$ or 0.0835.

Property 3: Independence of Inter-arrival Times

- The X_i's (Inter-Goal or Inter-Arrival Times) are independent.

- Autocorrelations of times between goals

| Lag	1	2	3	4	5	6	7	8	9
Autocorrelation	-0.0122	-0.0086	0.0021	-0.0033	0.0128	-0.0051	0.0003	0.0092	0.0033

Autocorrelation Between Goals

- All these times lie inside a band of plus or minus twice the standard error i.e., $2/\sqrt{574}$ or 0.0835.

- So empirically inter-arrival times also appear approximately independent.

- We will prove in class that $N(t)$ is Poisson.

- We will prove in class that $N(t)$ is Poisson.
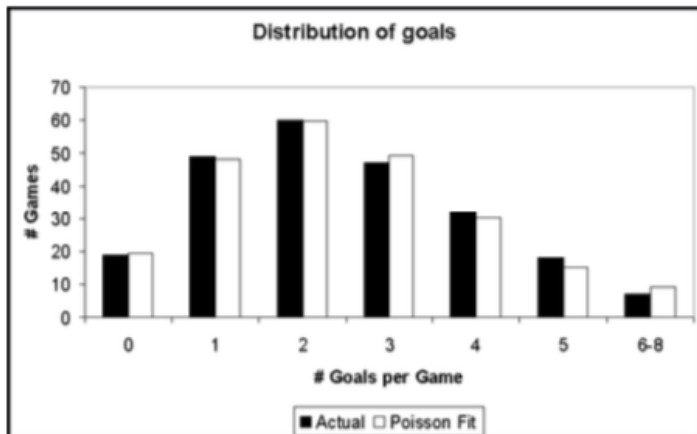
Distribution of Goals per Game

- Singfat Chu, (2003) Using Soccer Goals to Motivate the Poisson Process. INFORMS Transactions on Education 3(2):64-70. https://doi.org/10.1287/ited.3.2.64