National University of Singapore
Department of Electrical & Computer Engineering

Examination for

EE5137 Stochastic Processes
(Semester II, 2019/20)
April/May 2020

Time Allowed: 2.5 hours

INSTRUCTIONS FOR CANDIDATES:

- Use A4 size paper and pen (blue or black ink) to write your answers.
- Write down your student number clearly on the top left of every page of the answers.
- Write the question number and page number on the top right corner of each page (e.g. Q1(a), Q1(b), . . ., Q2(a), . . .).
- This paper contains **FOUR (4)** questions, printed on **FIVE (5)** pages. Answer **ALL** questions.
- The total number of marks is **ONE HUNDRED (**100).
- This exam is **OPEN BOOK**.
- You may use any calculator. However, you should lay out systematically the various steps in the calculations.
- Join the Zoom conference and turn on the video setting at all time during the exam. Adjust your camera such that your face and upper body including your hands are captured on Zoom.
- You may go for a short toilet break (not more than 5 minutes) during the exam.
- At the end of the exam,
  - scan or take pictures of your work (make sure the images can be read clearly);
  - merge all your images into one pdf file (arrange them in the order: Q1 to Q4 in their page sequence);
  - name the pdf file by Matric No_Module Code (e.g. A123456R_EE5137);
  - upload your pdf into the LumiNUS folder "Exam Submission".
- The Exam Submission folder will close at 11.50am. After the folder is closed, exam answers that are not submitted will not be accepted, unless there is a valid reason.

Question 1 (Total 20 Marks)

Answer True or False to the following questions. No justification is required.

Two (2) marks to be awarded to each correct answer.
One (1) mark to be deducted for each wrong answer.
Zero (0) marks for each question that is not answered.
So do not guess.

(a) Let $g(y) := \mathbb{E}[X|Y = y]$. Then $\mathbb{E}[g(Y)] = \mathbb{E}[X]$.

(b) Let $f(y) := \mathrm{Var}(X|Y = y) = \mathbb{E}[X^2|Y = y] - (\mathbb{E}[X|Y = y])^2$. Then $\mathbb{E}[f(Y)] = \mathrm{Var}(X)$.

(c) Let $\gamma_X(r) = \ln \mathbb{E}[e^{rX}]$ be the cumulant generating function of the rv X with $\mathbb{E}X \neq 0$. Then

$$\gamma_X''(r)\Big|_{r=0} = \frac{\mathrm{d}^2}{\mathrm{d}r^2}\gamma_X(r)\Big|_{r=0} = \mathbb{E}[X^2].$$

(d) Let $X_1, \ldots, X_n$ be i.i.d. rvs with $\mathbb{E}[X] = 0$ and $\mathrm{Var}(X) = \sigma^2 < \infty$. Then the sequence of rvs

$$\frac{1}{n^{3/4}}\sum_{i=1}^{n} X_i, \qquad n = 1, 2, \ldots$$

converges in probability to 0.

(e) The geometric distribution $P_X(k) = (1-p)^{k-1}p$ for $k = 1, 2, \ldots$ has the memoryless property.

(f) Let $\{N_1(t) : t > 0\}$ and $\{N_2(t) : t > 0\}$ be two independent Poisson processes with rates λ_1 and λ_2 respectively. Given that there is an arrival of the combined or merged process $\{N(t) = N_1(t) + N_2(t) : t > 0\}$, the probability that it is from the first Poisson process is $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

(g) Let $\{N(t) : t > 0\}$ be a counting process. If for every $t > 0$, $N(t)$ is a Poisson rv with mean λt for some $\lambda > 0$, $\{N(t) : t > 0\}$ is a Poisson process.

(h) Consider the Markov chain with three states $\mathcal{S} = \{1, 2, 3\}$ that has the following transition matrix

$$[P] = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

State 1 is transient.

(i) Again consider the Markov chain as in Part (h) above. If $\Pr(X_1 = 1) = \Pr(X_1 = 2) = 1/4$, then $\Pr(X_1 = 3, X_2 = 2, X_3 = 1) = 1/12$.

(j) Consider a Poisson process for which the arrival rate λ is either λ_0 or λ_1 in which $\lambda_0 \neq \lambda_1$. Suppose we observe the first n interarrival times $Y_1, \ldots, Y_n$ and we would like to make a MAP decision about the arrival rate, i.e., whether it is λ_0 or λ_1. Then

$$\frac{1}{n-1}\sum_{i=1}^{n} Y_i$$

is a sufficient statistic for this binary hypothesis testing problem.

Question 2 (20 marks)

All parts can be done independently.

2(a) (10 marks) Let $\{N_1(t) : t > 0\}$ and $\{N_2(t) : t > 0\}$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively. Find the probability that the second arrival in $N_1(t)$ occurs before the third arrival in $N_2(t)$ Write your answer in the form

$$\sum_{k=m}^{n} \binom{n}{k} p^k (1-p)^{n-k},$$

by identifying the numbers m, n, and p.

2(b) (10 marks) Suppose that each arrival of a Poisson counting process $\{N(t) : t > 0\}$ with rate λ is classified as being a type-A or type-B arrival. Suppose that the probability of an arrival being classified as type-A or B depends on the time at which it occurs. If an arrival occurs at time s, then independently of all else, it is classified as being type-A with probability $Q(s) \in [0, 1]$ and being type-B with probability $1 - Q(s)$.

Show that if $N_A(t)$ and $N_B(t)$ respectively represent the number of type-A and type-B arrivals that occur by time t, then $N_A(t)$ and $N_B(t)$ are independent Poisson random variable having respective means λtp and $\lambda t(1 - p)$ where

$$p = \frac{1}{t} \int_0^t Q(s)\, \mathrm{d}s.$$

Question 3 (Total 30 Marks)

All parts can be done independently.

A Markov chain $\{X_n : n \geq 0\}$ on the state space $\mathcal{S} = \{0, 1, 2, 3, 4, 5\}$ has the transition matrix

$$[P] = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.9 & 0 \\ 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.6 & 0 & 0.3 & 0.1 \\ 0 & 0.2 & 0 & 0.2 & 0.2 & 0.4 \end{bmatrix} \end{array}$$

Drawing a state transition diagram based on $[P]$ would help in the questions below.

3(a) (5 marks) List all the classes of the Markov chain. Briefly explain your answer.

3(b) (5 marks) For each state, determine whether it is transient or recurrent.

3(c) (3 marks) Suppose the Markov chain starts from state 5. What is the long run proportion of time that the Markov chain is at state 5?

3(d) (10 marks) Suppose the Markov chain starts from any state. Find the steady-state probabilities of the Markov chain $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_5)$ as $n \to \infty$.

3(e) (7 marks) Suppose the Markov chain starts from any state. It is known that $[P^n]_{00} - \pi_0$ decays to zero exponentially fast. This means that

$$\left| [P^n]_{00} - \pi_0 \right| \leq c\, \phi^n$$

for some constant $c \in \mathbb{R}$ and $\phi \in [0, 1)$. Find the smallest possible ϕ.

Question 4 (30 marks)

All parts can be done independently.

Consider the following two state Markov chain $\{X_n : n \geq 0\}$ with state space $\mathcal{S} = \{0, 1\}$ and probability transition matrix

$$[P] = \begin{bmatrix} 1-\theta & \theta \\ \theta & 1-\theta \end{bmatrix}.$$

This means that $P_{ij} = 1 - \theta$ if $i = j$ and $P_{ij} = \theta$ if $i \neq j$. It is known that θ can take on two values

$$H_0 : \theta = \theta_0 \quad \text{and} \quad H_1 : \theta = \theta_1,$$

where $\theta_0 \neq \theta_1$. Hence,

$$\Pr(X_2 = j \mid X_1 = i, H_0) = (1 - \theta_0)^{\mathbb{1}\{i=j\}} \theta_0^{\mathbb{1}\{i \neq j\}}.$$

where $\mathbb{1}\{\text{clause}\}$ equals 1 if the clause is true and 0 otherwise. We also know that the chain (under both hypotheses) starts from state 0, i.e.,

$$\Pr(X_0 = 0) = 1.$$

4(a) (7 marks) Define the random variables

$$Y_i = \mathbb{1}\{X_i = X_{i-1}\}, \qquad \forall\, i = 1, 2, \ldots.$$

Suppose we observe X_0, X_1, X_2, and X_3. Show by considering the likelihood ratio

$$L(X_0, X_1, X_2, X_3) = \frac{\Pr(X_0, X_1, X_2, X_3 \mid H_1)}{\Pr(X_0, X_1, X_2, X_3 \mid H_0)}$$

and the likelihood ratio test that part (a) that

$$T = Y_1 + Y_2 + Y_3$$

is a sufficient statistic for deciding between hypotheses H_0 and H_1.

4(b) (5 marks) In this and the following parts, assume that

$$\theta_0 = 1/2 \quad \text{and} \quad \theta_1 = 1/4.$$

Calculate the values of the likelihood ratio $L(X_0, X_1, X_2, X_3)$ when T, defined in part (b), takes on values $\{0, 1, 2, 3\}$.

4(c) (5 marks) Suppose that the prior probability of H_0 is 1/2. What is the minimum Bayesian probability of error?

4(d) (5 marks) What is the smallest probability of missed detection Pr(declare $H_0 \mid H_1$) if we allow the probability of false alarm Pr(declare $H_1 \mid H_0$) to be at most 1/8? What is the corresponding test in terms of T?

4(e) (8 marks) What is the smallest probability of missed detection Pr(declare $H_0 \mid H_1$) if we allow the probability of false alarm Pr(declare $H_1 \mid H_0$) to be at most 1/6? What is the corresponding test in terms of T?

Hint: Here a randomization strategy is required.

END OF PAPER