Lecture 21: Rate Distortion Theory

- Rate-distortion theorem, and proof ideas
- Calculation of $R(D)$
- Sphere packing for Gaussian source

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Rate-distortion theorem

- the **rate distortion region** for a source is the closure of the set of achievable rate distortion pairs (R, D)
- **rate-distortion function**: $R(D)$, is the infimum of rates R such that (R, D) is in the rate distortion region of the source for a given distortion D

Theorem. *The rate distortion function for an i.i.d. source X with distribution $p(x)$ and bounded $d(x, \hat{x})$ is equal to*

$$R(D) = \min_{p(\hat{x}|x): \sum_{(x,\hat{x})} p(x)p(\hat{x}|x)d(x,\hat{x}) \leq D} I(X; \hat{X})$$

Duality with channel capacity

- $C = \max_{p(x)} I(X;Y)$
- $R = \min_{p(\hat{x}|x): \sum_{(x,\hat{x})} p(x)p(\hat{x}|x)d(x,\hat{x}) \leq D} I(X;\hat{X})$
- $I(X;Y)$ is a "function" of $p(x)$ and $p(y|x)$
  - Concave in $p(x)$ for fixed $p(y|x)$
  - Convex in $p(y|x)$ for fixed $p(x)$

Calculation or $R(D)$

Binary Bernoulli(p) source

$$R(D) = \begin{cases} H(p) - H(D), & 0 \leq D \leq \min\{p, 1-p\} \\ 0, & D > \min\{p, 1-p\} \end{cases}$$

Gaussian source $\mathcal{N}(0, \sigma^2)$

$$R(D) = \begin{cases} \frac{1}{2}\log\frac{\sigma^2}{D}, & 0 \leq D \leq \sigma^2 \\ 0, & D > \sigma^2 \end{cases}$$

Uniform source: X uniformly distributed on (1, 2, $\ldots, m$)

$$R(D) = \begin{cases} \log m - H(D) - D\log(m-1), & 0 \leq D \leq 1 - 1/m \\ 0, & D > 1 - 1/m \end{cases}$$

Spherical packing for Gaussian source

- Gaussian source of variance σ^2
- $(2^{nR}, n)$ rate distortion code for this source with distortion D
- this sequence of code is a set of 2^{nR} sequences in $\mathbb{R}^n$ such that most source sequences of length n are within distance $\sqrt{nD}$ of some codeword
- minimum number of codewords required

$$2^{nR(D)} = \left(\frac{\sigma^2}{D}\right)^{n/2}$$

- $R(D) = \frac{1}{2}\log(\sigma^2/D)$

Proof highlights

- Converse: we cannot achieve a distortion of less than D if we describe X at rate less than $R(D)$

  Key technique: $R(\lambda D_1 + (1-\lambda)D_2) \leq \lambda R(D_1) + (1-\lambda)R(D_2)$

- Achievability: we can find a sequence of code with rate $R(D)$ such that its distortion is less than D

  Key technique: introduce another typical event:

$$|d(x^n, \hat{x}^n) - Ed(X, \hat{X})| < \epsilon$$

  Random coding, and use joint typicality for decoding