Applied Stochastic Processes

Exercise sheet 7

Exercise 7.1 Cycles of operation and repair of a machine.
Let $(U_i, V_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random vectors with $U_i \geq 0$, $V_i \geq 0$. Assume that $T_i = U_i + V_i$ is not almost surely equal to 0 and denote by F its distribution function. We interpret U_i and V_i as alternating periods when a given machine is operational or in repair. The period U_1 begins at time 0. For $t \geq 0$ we define $Y_t = 1$ if the machine is operational at time t and $Y_t = 0$ otherwise. Let $g(t) = P[Y_t = 1]$ denote the probability of the machine being operational at time $t \geq 0$, and $g(t) = 0$ for $t < 0$. We also define $h(t) = P[U_1 > t]$.

(a) Prove that for $t \geq 0$

$$g(t) = h(t) + \int_0^t g(t-s)dF(s),$$

i.e. that g is the solution of the (h, F)-renewal equation.

(b) Assume that $\mathrm{E}[U_1] < \infty$ and that F is non-arithmetic. Show that the function h is directly Riemann integrable and conclude that

$$\lim_{t \to \infty} g(t) = \frac{\mathrm{E}[U_1]}{\mathrm{E}[U_1] + \mathrm{E}[V_1]}.$$

Exercise 7.2 Let N be a renewal process with renewal times $(S_k)_{k \geq 0}$, where $S_0 = 0$, and interarrival distribution F having finite mean $\mu > 0$. Denote by A the age process of N, i.e. $A_t = t - S_{N_t}$ for $t \geq 0$. For $x \geq 0$, set $\varphi_x(t) = P[A_t \leq x]$ for $t \geq 0$, and $\varphi_x(t) = 0$ for $t < 0$.

(a) Show that φ_x satisfies the renewal equation

$$\varphi_x(t) = 1_{\{t \leq x\}}(1 - F(t)) + \int_0^t \varphi_x(t-s)dF(s) \text{ for } t \geq 0.$$

(b) Assume that F is non-arithmetic. Compute $\lim_{t \to \infty} \varphi_x(t)$. Deduce that A_t converges in distribution to some random variable A_∞ as $t \to \infty$.

Submission deadline: 13:15, Apr. 11.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

Students	Time & Date	Room	Assistant
A-K	Thu 09-10	HG D 7.2	Maximilian Nitzschner
L-Z	Thu 12-13	HG D 7.2	Daniel Contreras

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/