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Subject: Stochastic process

Assignment: Homework TEN

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1. EXERCISE 8.1

(a) Using the sample value from 8.40

$$E[LLR(Y) | X=a] = \frac{(b-a)^T}{\sigma^2} E\left[Y - \frac{b+a}{2}\right]$$

$$= \frac{(b-a)^T}{\sigma^2}\left(a - \frac{b+a}{2}\right)$$

$$= \frac{-(b-a)^T(b-a)}{2\sigma^2} \quad \text{As desired}$$

(b) Follow the conclusion from (a), we let $\gamma = \frac{\|b-a\|}{2\sigma}$

$$E[LLR(Y) | X=a] = \frac{-\|b-a\|^2}{2\sigma^2} = -2\gamma^2$$

(c) As for the variance, we can get

$$Var[LLR(Y) | X=a] = \frac{1}{\sigma^2}(b-a)^T E[ZZ^T](b-a)\frac{1}{\sigma^2}$$

Based on the hint $LLR(Y)$ is $(\frac{1}{\sigma^2})(b-a)^T Z$, when $X=a$

$$= \frac{1}{\sigma^2}(b-a)^T \frac{E[Z \cdot Z^T]}{\sigma^2}(b-a)$$

$$= \frac{1}{\sigma^2}(b-a)^T \cdot [I] \cdot (b-a)$$

$$= \frac{1}{\sigma^2}\|b-a\|^2 = 4\gamma^2$$

(d) Conditional on $X=a$, $Y=a+Z$, $Y \sim N(a, \sigma^2)$

And then $LLR(Y)$ is also Gaussian condition on $X=a$

using the conclusion from (b)(c), mean of $LLR(Y)$ is $-2\gamma^2$

variance of $LLR(Y)$ is $4\gamma^2$, $LLR(Y) \sim N(-2\gamma^2, 4\gamma^2)$

As for scaling of $LLR(Y)$ by divided by 2γ

$$\frac{LLR(Y)}{2\gamma} \sim N\left(\frac{-2\gamma^2}{2\gamma}, \frac{4\gamma^2}{(2\gamma)^2}\right)$$

so $\frac{LLR(Y)}{2\gamma} \sim N(-\gamma, 1)$

(e) As for the definition, $\ln \gamma$ is the threshold

So the first part is proved

$$Pr\{e_n | X=a\} = Pr\{LLR(Y) \geq \ln \eta \mid X=a\}$$

As for the second equation, we know $\frac{LLR(Y)}{2\gamma} \sim N(-\gamma, 1)$, conditional on $X=a$
the mean of $\frac{LLR(Y)}{2\gamma}$ is $-\gamma$
So the probability that it exceeds $\frac{\ln\eta}{2\gamma}$ is then $Q\left(\frac{\ln\eta}{2\gamma}+\gamma\right)$

f) Follow the similar calculation from (a)(b)(c)(d)(e)
We can get $E[LLR(Y) \mid X=b] = 2\gamma^2$ (a)(b)
$Var[LLR(Y) \mid X=b] = 4\gamma^2$ (c)
condition on b, $LLR(Y) \sim N(2\gamma^2, 4\gamma^2)$, $\frac{LLR(Y)}{2\gamma} \sim N(\gamma, 1)$ (d)
In conclusion. $Pr\{e_\eta \mid X=b\} = Q\left(\frac{-\ln\eta}{2\gamma}+\gamma\right)$ (e)

. EXERCISE 8.6

a) As we all know, the requirement of ~~statistic~~ sufficient statistic is that there exist a invertible function $\Lambda(Y) = u(v(Y))$
Because the observed sample y is transformed into $v = A^{-1}y$, v is a invertible function of y, so we can replace y with $Av(y)$ in $\Lambda(y)$ to write it as a function of $v(y)$ only.

b) Using the conclusions from slides, we can get
$$LLR(v) = -\frac{1}{2}\|v - A^{-1}a\|^2 + \frac{1}{2}\|v - A^{-1}b\|^2$$
And then, we substitute $v = A^{-1}y$
$$\begin{aligned} LLR(v(y)) &= -\frac{1}{2}\|A^{-1}y - A^{-1}a\|^2 + \frac{1}{2}\|A^{-1}y - A^{-1}b\|^2 \\ &= -\frac{1}{2}(y-a)^T A^{-T}A^{-1}(y-a) + \frac{1}{2}(y-b)^T A^{-T}A^{-1}(y-b) \\ &= -\frac{1}{2}(y-a)^T(AA^T)^{-1}(y-a) + \frac{1}{2}(y-b)^T(AA^T)^{-1}(y-b) \\ &= LLR(y) \quad \text{As desired.} \end{aligned}$$

c) There are two conditions, when $X=a$, $\hat{a} = A^{-1}a$, when $X=b$, $\hat{b} = A^{-1}b$
For part b, we can let v project the direction along $\hat{b}-\hat{a}$ to get $(\hat{b}-\hat{a})v$, which satisfy sufficient statistics, put this into part b
$LLR(v(Y)) = \gamma = \frac{1}{2}\|\hat{b}-\hat{a}\|$,
Then the error probabilities are
$Pr\{Error \mid X=a\} = Q\left(\frac{\ln\eta}{2\gamma}+\gamma\right)$, $Pr\{Error \mid X=b\} = Q\left(\frac{-\ln\eta}{2\gamma}+\gamma\right)$
As for the problem in problem of part one, we can directly let y project the direction along $k_Z^{-1}(b-a)$, where $k_Z = AA^T$
$Pr\{Error \mid X=b\} = Q\left(\frac{-\ln\eta}{2\gamma}+\gamma\right)$

3. EXERCISE 8.7

(a) Through the above information, we can get

$$\begin{cases} x=0, & U=(a\cos\phi, a\sin\phi, 0, 0)^T \\ x=1, & U=(0, 0, a\cos\phi, a\sin\phi)^T \end{cases}$$

And the r.v ϕ is uniformly distributed $[0, 2\pi]$, and is independent of X and Z

As for $Z \sim N(0, \sigma^2[I])$, is independent of X.

$Y = U + Z$, and Y can be expresed as $(Y_1, Y_2, Y_3, Y_4)^T$

In order to make sense, we can let $U = ae^{i\phi}$, which means

$$\begin{cases} x=0 & U=(ae^{i\phi}, 0) \\ x=1 & U=(0, ae^{i\phi}) \end{cases}$$ ~~(owing to independe of ϕ)~~

Similarily, let $Z = (R_0 e^{i\theta_0}, R_1 e^{i\theta_1})$

$$Y = U+Z = \begin{cases} (ae^{i\phi} + R_0 e^{i\theta_0}, R_1 e^{i\theta_1}) & x=0 \\ (R_0 e^{i\theta_0}, ae^{i\phi} + R_1 e^{i\theta_1}) & x=1 \end{cases}$$

let first value became $\sqrt{V_0}\, e^{i\varphi_0}$, second value become $\sqrt{V_1}\, e^{i\varphi_1}$

Owing to the poperty of complex function, $V_0^2 = Y_1^2 + Y_2^2$, $V_1^2 = Y_3^2 + Y_4^2$

we can find, if $V_0 > V_1$, which means $\hat{X} = 0$

if $V_0 < V_1$, which means $\hat{X} = 1$

(b) $Z \sim N(0, \sigma^2[I])$, ~~when $x=0$ there are~~ $Y = U + Z$

we can get $Y \sim N(0, \sigma_Y^2]$, but there are 2 conditions

$$x=0 \quad \sigma^2 Y = \begin{bmatrix} a^2+\sigma^2 & 0 & 0 & 0 \\ 0 & a^2+\sigma^2 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & \sigma^2 \end{bmatrix} \qquad x=1 \quad \sigma^2 Y = \begin{bmatrix} \sigma^2 & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & a^2+\sigma^2 & 0 \\ 0 & 0 & 0 & a^2+\sigma^2 \end{bmatrix}$$

$$f_{Y|X}(y|0) = \frac{1}{(2\pi)^2 (a^2+\sigma^2)\sigma^2} e^{-\frac{1}{2}\left[\frac{Y_1^2+Y_2^2}{a^2+\sigma^2} + \frac{Y_3^2+Y_4^2}{\sigma^2}\right]}$$

$$f_{Y|X}(y|1) = \frac{1}{(2\pi)^2 (a^2+\sigma^2)\sigma^2} e^{-\frac{1}{2}\left[\frac{Y_1^2+Y_2^2}{\sigma^2} + \frac{Y_3^2+Y_4^2}{a^2+\sigma^2}\right]}$$

So when we use $V_1 = Y_3^2 + Y_4^2$, $V_0 = Y_1^2 + Y_2^2$

$$LLR(Y) = \frac{a^2 (V_0 - V_1)}{2\sigma^2(a^2+\sigma^2)}$$

As for $$f_{V_1|X}(v_1|x=0) = \frac{1}{2\sigma^2} e^{-\frac{v_1}{2\sigma^2}}$$

$$Pr(V_1 > v_1 | x=0) = \int_{v_1}^{+\infty} f_{V_1|X}(v_1|x=0)\, dv_1 = e^{\frac{-v_1}{2\sigma^2}}$$

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(c) As for $P_{Y_1,Y_2|X,\phi}(y_1,y_2|0,0)$ $a\cos\phi = a$ $a\sin\phi = 0$

$Y = (a,0,0,0)$ $Y_1 \sim N(a,\sigma^2)$ $Y_2 \sim N(0,\sigma^2)$

Owing to the independence between Y_1 and Y_2, we can get

$$f_{Y_1,Y_2|X,\phi}(y_1,y_2|0,0) = f_{Y_1|X,\phi}(y_1|0,0)\cdot f_{Y_2|X,\phi}(y_2|0,0)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y_1-a)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y_2^2}{2\sigma^2}}$$

$$= \frac{1}{2\pi\sigma^2} e^{\frac{-(y_1-a)^2 - y_2^2}{2\sigma^2}}$$

$$= \frac{1}{2\pi\sigma^2} \exp\left[\frac{-y_1^2 - y_2^2 + 2y_1 a - a^2}{2\sigma^2}\right]$$

(d) As for $\Pr\{V_1 > V_0 \mid X=0, \phi=0\}$

$$= \Pr\{V_1 > Y_1^2 + Y_2^2 \mid X=0, \phi=0\}$$

$$= \int f_{Y_1,Y_2|X,\phi}(y_1,y_2|0,0) \Pr\{V_1 > y_1^2+y_2^2\}\, dy_1 dy_2$$

We can use part (b) (c) conclusions

$$= \int \frac{1}{2\pi\sigma^2} e^{\frac{-y_1^2-y_2^2+2y_1a-a^2}{2\sigma^2}} \cdot e^{-\frac{v_0}{2\sigma^2}}\, dy_1 dy_2 \qquad V_0 = Y_1^2 + Y_2^2$$

$$= \int \frac{1}{2\pi\sigma^2} e^{\frac{-2y_1^2-2y_2^2+2y_1a-a^2}{2\sigma^2}}\, dy_1 dy_2$$

As for $\int \frac{1}{2\pi\sigma^2} e^{\frac{2y_1a}{2\sigma^2}}\, dy_1 dy_2 = 0$

We can get this integration of this equation

the result is $\frac{1}{2} e^{-\frac{a^2}{4\sigma^2}}$

(e) let $U = V_0 - V_1$, conditioning on V_1

$$P(U>u|X=0) = \int_0^\infty P(V_0 > u+v \mid X=0, V_1=u) f_{V_1|X}(v|0)\, dv$$

$$= \begin{cases} \frac{a^2+\sigma^2}{2\sigma^2+a^2} \exp\left[-\frac{1}{2(\sigma^2+a^2)} u\right] & u \geq 0 \\ 1 - \frac{\sigma^2}{2\sigma^2+a^2} \exp\left[\frac{1}{2\sigma^2} u\right] & u < 0 \end{cases}$$

So the PDF is $f_{U|X}(u|0) = \begin{cases} \frac{1}{2(2\sigma^2+a^2)} \exp\left(-\frac{1}{2(a^2+\sigma^2)} u\right), & u \geq 0 \\ \frac{1}{2(2\sigma^2+a^2)} \exp\left(\frac{1}{2\sigma^2} u\right) & u < 0 \end{cases}$

error probability is $P(U<0|X=0) = \int_{-\infty}^0 f_{U|X}(u|0)\, du$

$$= \int_{-\infty}^0 \frac{1}{2(2\sigma^2+a^2)} \exp\left(\frac{1}{2\sigma^2} u\right) du$$

$$= \frac{\sigma^2}{2\sigma^2+a^2}$$

Using same calculation, we can get $P\{U>0|X=1\} = \frac{\sigma^2}{2\sigma^2+a^2}$

$\Pr\{\text{error}|X=0\} = \Pr\{\text{error}|X=1\}$

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4. EXERCISE 8.9

(a) Based on the statement from equation, we know the signal a associated with $x=0$ is 5, and then the signal b associated with $x=1$ is 1.

$$\begin{cases} x=0 & Y_1 = 5+Z_1 \\ x=1 & Y_1 = 1+Z_1 \end{cases}$$

Z_1 is independent of x and Z_1 is Gaussian, $Z_1 \sim N(0, 6^2)$

so $\begin{cases} f_{Y_1|X}(y_1|0) \sim N(5, 6^2) \\ f_{Y_1|X}(y_1|0) \sim N(1, 6^2) \end{cases}$

we can get a threshold test on y

$$LLR(y) = \left[\frac{(b-a)}{6^2}\left(y - \frac{b+a}{2}\right)\right] \underset{\hat{x}(y)=a}{\overset{\hat{x}(y)=b}{\gtrless}} \ln(\eta)$$

which means $y \underset{\hat{x}(y)=a}{\overset{\hat{x}(y)=b}{\lessgtr}} \dfrac{-6^2 \ln(\eta)}{a-b} + \dfrac{b+a}{2}$ let $b=1$ $a=0$.

$$\Pr\{e|x=0\} = Q\left(6\ln\eta + \frac{1}{26}\right)$$
$$\Pr\{e|x=1\} = Q\left(6\ln\eta + \frac{1}{26}\right)$$

(b) Following the statement from question, we know $Y_2 = Y_1 + Z_2$

And Z_2 is independent of x and Y.

Thus, $Y_2 \sim N(Y_1, 6^2)$, which is independent of x.

so the map rule doesn't change.

As for $f_{Y_1,Y_2|X}(y_1,y_2|x) = f_{Y_1|X}(y_1|x)\, f_{Y_2|Y_1,X}(y_2|y_1,x)$

$$\begin{cases} f_{Y_2|Y_1,X}(y_2|y_1,0) \sim N(y_2, y_1, 6^2) \\ f_{Y_2|Y_1,X}(y_2|y_1,1) \sim N(y_2, y_1, 6^2) \end{cases}$$

So the ratio $= \dfrac{f_{Y_1,Y_2|X}(y_1,y_2|1)}{f_{Y_1,Y_2|X}(y_1,y_2|0)}$

$$= \frac{f_{Y_1|X}(y_1|1)\, f_{Y_2|Y_1,X}(y_2|y_1,1)}{f_{Y_1|X}(y_1|0)\, f_{Y_2|Y_1,X}(y_2|y_1,0)}$$

$$= \frac{f_{Y_1|X}(y_1|1)}{f_{Y_1|X}(y_1|0)}$$

So the result is similar like part (a)

(c). Through the conclusion from textbook, we know it is clear that adding an additional noise would not help us to make decision. But the existence of probability can sharpen our intuition to make something more evident.

(d) Similar like (b), we know Y_2 is independent of X, conditional on thus the decision rule and error probability does not change.

(e). When Z_1 is uniformly distributed between 0 and 1.

$$\begin{cases} Y_1 = 5 + Z_1 & \text{, when } X = 0 \\ Y_1 = 1 + Z_1 & \text{, when } X = 1 \end{cases}$$

So Y_1 is uniformly distributed between 5 and 6 (when $X=0$)

Y_1 is uniformly distributed between 1 and 2 (when $X=1$)

The decision rule is similar like (d)

However, there are no possibility of error.