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Subject：　　Stochastic process

Assignment: Homework Six

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1. EXERCISE

① For $N(t+s) = N(t) + \tilde{N}(t, t+s)$ (using S.I.P.)

let $N(t) = k$ $\quad N(t+s) = m$, so $\tilde{N}(t, t+s) = m-k$, $m \geq k$

$$P_{N(t)\cdot N(t+s)}(k,m) = Pr\{N(t) = k\}\, Pr\{\tilde{N}(t, t+s) = m-k\}$$

$$= \frac{(\lambda t)^k \cdot \exp(-\lambda t)}{k!} \cdot \frac{(\lambda s)^{m-k} \cdot \exp(-\lambda s)}{(m-k)!}$$

② $E[N(t)\cdot N(t+s)] = E[N(t)\cdot[N(t) + \tilde{N}(t, t+s)]]$

$$= E[N^2(t) + N(t)\tilde{N}(t, t+s)]$$

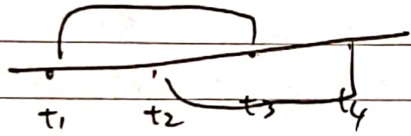
Using the Independent increment property of counting process, we can get that $N(t)$ is independent with $\tilde{N}(t, t+s)$ and at the same time $\tilde{N}(t, t+s) = N(s)$

Therefore, the result can be rewrite as

$$= E[N^2(t)] + E[N(t)]\cdot E[N(s)]$$

$$= (\lambda t)^2 + \lambda t + \lambda t \cdot \lambda s$$

$$= (\lambda t)^2 + \lambda t + \lambda^2 \cdot t \cdot s$$

③ 

We can $\tilde{N}(t_1, t_3) = \tilde{N}(t_1, t_2) + \tilde{N}(t_2, t_3)$

$\tilde{N}(t_2, t_4) = \tilde{N}(t_2, t_3) + \tilde{N}(t_3, t_4)$

As for $E[\tilde{N}(t_1, t_3)\cdot \tilde{N}(t_2, t_4)] = E[[\tilde{N}(t_1, t_2) + \tilde{N}(t_2, t_3)][\tilde{N}(t_2, t_3) + \tilde{N}(t_3, t_4)]]$

And $\tilde{N}(t_1, t_2)$, $\tilde{N}(t_2, t_3)$, $\tilde{N}(t_3, t_4)$ are independent with each other

We can get $= E[\tilde{N}(t_1, t_2)]E[\tilde{N}(t_2, t_3)] + E[\tilde{N}(t_1, t_2)]\cdot E[\tilde{N}(t_3, t_4)]$

$+ E[\tilde{N}^2(t_2, t_3)] + E[\tilde{N}(t_2, t_3)]E[\tilde{N}(t_3, t_4)]$

$$= \lambda^2(t_2 - t_1)(t_3 - t_2) + \lambda^2(t_2 - t_1)(t_4 - t_3) + \lambda^2(t_3 - t_2)^2 + \lambda(t_3 - t_2)$$

$$+ \lambda^2(t_3 - t_2)(t_4 - t_3)$$

$$= \lambda^2(t_2 - t_1)(t_4 - t_2) + \lambda^2(t_3 - t_2)(t_4 - t_2) + \lambda(t_3 - t_2)$$

$$= \lambda^2(t_3 - t_1)(t_4 - t_2) + \lambda(t_3 - t_2)$$

2.(a) For a Poisson counting process, this event $\{S_1, S_2 \cdots S_{n-1} | S_n = t\}$ using the definition of conditional probability

$$f_{S_1, S_2 \cdots S_{n-1} | S_n = t} = \frac{f_{S_1\cdot S_2 \cdots S_{n-1}\cdot S_n}}{f(S_n = t)} = \frac{\lambda^n \exp(-\lambda S_n)}{\lambda^n \cdot t^{n-1}\cdot \exp(-\lambda t)/(n-1)!}$$

$$= \frac{\exp(-\lambda S_n + \lambda t)\cdot(n-1)!}{t^{n-1}}$$

$$= \frac{(n-1)!}{t^{n-1}}$$

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ⓑ Similarily, we ~~also~~ apply bayesian law ~~conditional probability~~ into part (b)

$$f_{x_1|S_n}(\tau|t) = \frac{f_{x_1}(\tau)\cdot f_{S_n|x_1}(t|\tau)}{f_{S_n}(t)}$$

As for $f_{S_n|x_1}(t|\tau)$, we can regard as during $(t-\tau)$ this period of time, $(n-1)$ th erlang distribution function $f_{S_{n-1}}(t-\tau)$

So, we can rewrite

$$= \frac{\lambda\cdot\exp(-\lambda\tau)\cdot\lambda^{n-1}(t-\tau)^{n-2}\cdot\exp(-\lambda t+\lambda\tau)/(n-2)!}{\lambda^n\cdot t^{n-1}\cdot\exp(-\lambda t)/(n-1)!}$$

$$= \frac{(t-\tau)^{n-2}}{t^{n-1}}\frac{(n-1)!}{(n-2)!} = \frac{(t-\tau)^{n-2}}{t^{n-1}}(n-1)$$

$$P_r\{x_1>\tau|S_n=t\} = \int_\tau^t \frac{(t-\tau)^{n-2}}{t^{n-1}}(n-1)\,d\tau = \left[\frac{t-\tau}{t}\right]^{n-1}$$

ⓒ As for x_i, we know this counting process with independent increment property, ~~$x_1 \ldots S_i \ldots S_1$~~ $X_1, X_2, X_3 \cdots X_i \cdots X_n$ are independent, and at the same time, $x_i = S_i - S_{i-1}$,

In a word, $x_i \overset{d}{=} X_1$, so the analyse to x_i is similar to X_1, with same result in part (b).

$$P_r\{x_i>\tau|S_n=t\} = \left[\frac{t-\tau}{t}\right]^{n-1}$$

ⓓ · Apply bayesian law into part (d)

$$f_{S_i|S_n}(\tau|t) = \frac{f_{S_i}(\tau)\cdot f_{S_n|S_i}(t|\tau)}{f_{S_n}(t)}$$

As for $f_{S_n|S_i}(t|\tau)$, we can regard as a new erlang distribution function $f_{S_{n-i}}(t-\tau)$

So, we can rewrite

$$= \frac{\lambda^i\tau^{i-1}\exp(-\lambda\tau)/(i-1)!\cdot\lambda^{n-i}\cdot(t-\tau)^{n-i-1}\exp(-\lambda t+\lambda\tau)/(n-i-1)!}{\lambda^n\cdot t^{n-1}\exp(-\lambda t)/(n-1)!}$$

$$= \frac{\tau^{i-1}\cdot(t-\tau)^{n-i-1}\cdot(n-1)!}{t^{n-1}\,(i-1)!\,(n-i-1)!}$$

let $\tau = S_i$

$$= \frac{S_i^{\,i-1}(t-S_i)^{n-i-1}\cdot(n-1)!}{t^{n-1}(i-1)!(n-i-1)!}$$

e.

For $N(t)=n-1$, means the first arrival after $(0,t]$ is strictly after t. , the number of arrivals in $(0,t]$ is $n-1$

And For $S_n=t$, means the first arrival after $(0,t)$ is exactly at t. , the number of arrivals in $(0,t)$ is $n-1$

These two conditions are similar, but different.

3. EXERCISE 2.17

a. For $\Pr\{N(t)=n \mid S_1=\tau\}$, we can regard as a new ~~order~~ counting distribution function. during (τ, t), there are $(n-1)$ arrivals

So $$\Pr\{N(t)=n \mid S_1=\tau\} = \Pr\{\tilde{N}(\tau,t)=n-1\} = \frac{\lambda^{n-1}\cdot(t-\tau)^{n-1}\cdot \exp(-\lambda t+\lambda\tau)}{(n-1)!}$$

b. Using bayesian law, we can get

$$f_{S_1}(\tau \mid N(t)=n) = \frac{f(N(t)=n \mid S_1=\tau)\cdot f(S_1=\tau)}{f[N(t)=n]}$$

$$= \frac{\lambda^{n-1}(t-\tau)^{n-1}\exp(-\lambda t+\lambda\tau)\cdot\lambda\exp(-\lambda\tau)}{(n-1)!\cdot\lambda^n t^n\cdot\exp(-\lambda t)/n!}$$

$$= \frac{n!\cdot\lambda^n\,(t-\tau)^{n-1}}{(n-1)!\,\lambda^n\, t^n}$$

$$= n\cdot\frac{(t-\tau)^{n-1}}{t^n}$$

c. For Equation 2.41 is $\Pr\{S_1>\tau \mid N(t)=n\} = \left[\frac{t-\tau}{t}\right]^n$

We can use part (b)

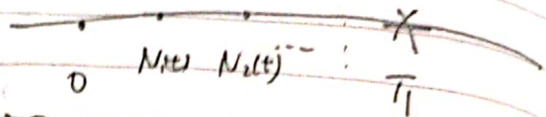
$$\Pr\{S_1>\tau \mid N(t)=n\} = \int_{\tau}^{+\infty} \frac{n\cdot(t-\tau)^{n-1}}{t^n}\,d\tau$$

$$= \left[\frac{t-\tau}{t}\right]^n$$

Therefore, it is correct

4. a.

$\cancel{P_{N(T_1)|T_1} = \frac{(\lambda T_1)^n \cdot \exp(-\lambda T_1)}{n!}}$



$$f_{T_1} = \begin{cases} v \cdot \exp(-vt) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

so the rate is v.

We can regard as there are two independent poisson process,

the total process is a new poisson process. with the rate $(\lambda + v)$

the rate of original poisson process is λ, with $\frac{\lambda}{\lambda+v}$ probability.

the rate of new poisson process is v, with $\frac{v}{\lambda+v}$ probability.

let ~~the numb~~ n denotes the number of Poisson arrivals of the first process in the interval $[0, T_1]$, $(n+1)^{th}$ must be second process.

$$Pr[N(T_1) = n] = \left(\frac{\lambda}{\lambda+v}\right)^n \cdot \left(\frac{v}{\lambda+v}\right) = \left(\frac{\lambda}{\lambda+v}\right)^n \cdot \frac{v}{(\lambda+v)}$$

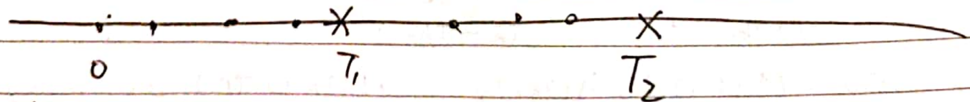
$$= \left(\frac{\lambda}{\lambda+v}\right)^n \cdot \frac{v}{(\lambda+v)}$$

b. Using the same analysis, we can regard part (b) as.

the first process's rate is λ, with $\frac{\lambda}{\lambda+v}$ probability,

the second process's rate is v, with $\frac{\lambda}{\lambda+v}$ probability.

In part (b)'s condition, we can conclude that $(n+2)^{th}$ process must be second process,



let n denotes the number of Poisson arrivals of the first process

$$Pr[N(T_2) = n] = \binom{n+1}{n} \left(\frac{\lambda}{\lambda+v}\right)^n \left(\frac{v}{\lambda+v}\right)^2 = \frac{(n+1)!}{n!\,1!} \left(\frac{\lambda}{\lambda+v}\right)^n \left(\frac{v}{\lambda+v}\right)^2$$

$$= (n+1) \cdot \left(\frac{\lambda}{\lambda+v}\right)^n \cdot \left(\frac{v}{\lambda+v}\right)^2$$

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5.

a. For $M(t) = N(t) - \lambda t$

$$E[M(t)] = E[N(t)] - E[\lambda t] = \lambda t - \sum_{n=0}^{\infty} \frac{(\lambda t)^{n+1} \cdot e^{-\lambda t}}{n!} = 0$$

$$Var[M(t)] = E\{[N(t) - \lambda t]^2\}$$

$$= E[N^2(t) - 2N(t) \cdot \lambda t + \lambda^2 t^2]$$

$$= (\lambda t)^2 + \lambda t - 2(\lambda t)^2 + \lambda^2 t^2 = \lambda t$$

b. ⓐ Stationary increments property.

$$M(\tau') - M(\tau) \overset{d}{=} M(\tau' - \tau).$$

$0 < \tau < \tau' \le t$

For $M(\tau') = N(\tau') - \lambda\tau'$

$M(\tau) = N(\tau) - \lambda\tau$

So $M(\tau') - M(\tau) = N(\tau') - N(\tau) - \lambda(\tau' - \tau)$

As for $M(\tau' - \tau) = N(\tau' - \tau) - \lambda(\tau' - \tau)$

$[N(\tau' - \tau) \overset{d}{=} N(\tau') - N(\tau)]$

$= N(\tau') - N(\tau) - \lambda(\tau' - \tau)$

As desired, $\{M(\tau): 0 < \tau \le t\}$ have the stationary increments property.

ⓑ independent increament property.

We all know $N(t)$ has the independent increment property,
$N(t_1), \hat{N}(t_1, t_2), \hat{N}(t_2, t_3) \cdots \hat{N}(t_{k-1}, t_k)$ are independent.

As for $M(t) = N(t) - \lambda t$ (using the stationary increment property)

$M(t_1)$, $\hat{M}(t_1, t_2) = M(t_2) - M(t_1)$, $\hat{M}(t_2, t_3) = M(t_3) - M(t_2)$

$\hat{M}(t_{k-1}, t_k) = M(t_k) - M(t_{k-1})$

$= N(t_k) - \lambda t_k - N(t_{k-1}) + \lambda t_{k-1}$

$= N(t_k) - N(t_{k-1}) + \lambda(t_k - t_{k-1})$

$= \hat{N}(t_{k-1}, t_k) + \lambda(t_k - t_{k-1})$

Therefore, $M(t_1)$, $\hat{M}(t_1, t_2)$, $\hat{M}(t_2, t_3) \cdots \hat{M}(t_{k-1}, t_k)$ are independent.

In a word, $\{M(\tau): 0 < \tau \le t\}$ have the ~~increment~~ independent increment property.

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c. For $\{M(\tau): 0<\tau\le t\}$

① $E[|M(t)|] = E[|N(t)-\lambda t|] \le E[N(t)+\lambda t]$

(using the triangle inequality)

$|N t-\lambda t| \le N(t)+\lambda t$

And then, we know $N(t)$ and λt are independent.

so we can get $E[N(t)+\lambda t] = E[N(t)] + E[\lambda t]$

$= \lambda t + \lambda t$

$= 2\lambda t < +\infty$

As desired, it satisfy the first condition

② Using the ~~stationary increment property~~ statement from this question, then we can analyse s is the stopping time for $M(\tau)$, $0<\tau\le s$

Firstly, use the independent increment property, we can get $M(\tau)$ is independent of event $M(s-\tau)$, and this $M(\tau)$ only cares about the length of time $(s-0)$

$F_s \Rightarrow 0<\tau\le s \qquad f_{M(\tau)|F_s} = u(0<\tau\le s)$

Therefore $E[M(\tau)|F_s] = \int_0^s M(\tau)\, u(0<\tau\le s)\, d\tau$

$N(s)-\lambda s - N(0)-\lambda\cdot 0$

$= \int_0^s M(\tau)\, d\tau =$

$= M(s) - M(0) =$

As for $M(0) = N(0) - \lambda\cdot 0 = 0$.

Therefore, we can get $E[M(\tau)|F_s] = M(s)$

In a word, $\{M(\tau): 0<\tau\le t\}$ is a continuous-time martingale

d. For $\hat{M}(t,t+\delta) = M(t+\delta) - M(t)$

$E[\hat{M}(t,t+\delta)^2 / F_t] = E[(M(t+\delta)-M(t))^2] / F_t]$

$= E[M^2(t+\delta) - 2M(t+\delta)M(t) + M^2(t) / F_t]$

$= E[[N(t+\delta)-\lambda(t+\delta)]^2 - 2[N(t+\delta)-\lambda(t+\delta)][N(t)-\lambda t] + [N(t)-\lambda t]^2 / F_t]$

We only focus on $[N^2(t+\delta) - N(t)]^2$, other elements can be canceled

$= E[[N(t+\delta) - N(t)]^2 | F_t]$

$= E[N^2(\delta) | F_t]$

$= Var[N(\delta)] + E^2[N(\delta)]$

$= \lambda\cdot\delta + (\lambda\delta)^2$

we all know $\delta\to 0$, so $\delta^2 \to o(\delta)$

In conclusion, we can get $E[\hat{M}(t,t+\delta)^2 / F_t] = \lambda\delta + o(\delta)$