Solutions to EE5137 Exam (Semester 1 2018/9)

December 3, 2018

1 Problem 1

(a) (i) By the question, we know that

$$\mathbb{E}[X|Y=1]=3, \qquad \mathbb{E}[X|Y=2]=\mathbb{E}[X]+5, \qquad \mathbb{E}[X|Y=3]=\mathbb{E}[X]+7,$$

Hence, $a_1=0, b_1=3, a_2=1, b_2=5, a_3=1, b_3=7$.

(ii) By the law of iterated expectations,

$$\begin{aligned}\mathbb{E}[X]&=\mathbb{E}[\mathbb{E}[X|Y]]\\ &=\frac{1}{3}(3)+\frac{1}{3}(\mathbb{E}[X]+5)+\frac{1}{3}(\mathbb{E}[X]+7)\end{aligned}$$

This yields after solving the equation

$$\frac{1}{3}\mathbb{E}[X]=1+\frac{5}{3}+\frac{7}{3}=5, \quad \Longrightarrow \quad \mathbb{E}[X]=15.$$

(a) (i) We claim that $b=0$. Let's prove it. For fixed $\epsilon>0$

$$\mathbb{P}(|X_n-0|>\epsilon)=\mathbb{P}(X_n=n^2)=\frac{1}{n}\to 0$$

Hence, X_n converges to 0 in probaiblity and so $b=0$.

(ii) We claim that $c=\infty$. Let's prove it. Consider

$$\mathbb{E}[X_n]=\Pr(X_n=n^2)\cdot n^2+\Pr(X_n=0)\cdot 0=\frac{1}{n}\cdot n^2=n$$

Clearly, $\mathbb{E}[X_n]$ diverges to ∞ and so $c=\infty$.

2 Problem 2

(a) Consider the likelihood ratio test

$$\frac{f_{X|H_0}(x)}{f_{X|H_1}(x)} \overset{\text{decide } \mathsf{H}_0}{\gtrless} \frac{p_1}{p_0} =: \eta,$$

where p_1 and p_0 are the prior probabilities of H_0 and H_1 resp. Then we have

$$\frac{f_{X|H_0}(x)}{f_{X|H_1}(x)} = \frac{\frac{1}{\sqrt{2\pi\sigma_0^2}}\exp(-\frac{x^2}{2\sigma_0^2})}{\frac{1}{\sqrt{2\pi\sigma_1^2}}\exp(-\frac{x^2}{2\sigma_1^2})} = \frac{\sigma_1}{\sigma_0}\exp\left(-\frac{x^2}{2}(\sigma_0^{-2} - \sigma_1^{-2})\right).$$

Comparing this to the threshold η, we obtain

$$-\frac{x^2}{2}(\sigma_0^{-2} - \sigma_1^{-2}) \overset{\text{decide } \mathsf{H}_0}{\gtrless} \ln\left(\eta\frac{\sigma_0}{\sigma_1}\right)$$

Hence, by using the fact that $\sigma_0 < \sigma_1$

$$x^2 \overset{\text{decide } \mathsf{H}_1}{\gtrless} 2\frac{1}{\sigma_1^{-2} - \sigma_0^{-2}}\ln\left(\eta\frac{\sigma_0}{\sigma_1}\right) =: \gamma^2$$

So we decide in favor of H_1 iff

$$|x| > \gamma.$$

(b) (i) We have

$$L(X_1, X_2, X_3) = \frac{P_0(X_1)P_0(X_2)P_0(X_3)}{P_1(X_1)P_1(X_2)P_1(X_3)} = \frac{\prod_{i=1}^3(\frac{1}{2})^{X_i}(\frac{1}{2})^{1-X_i}}{\prod_{i=1}^3 p^{X_i}(1-p)^{1-X_i}} = \frac{1/8}{(2/3)^T(1/3)^{3-T}}$$

Since $L(X_1, X_2, X_3)$ depends only on T, T is a sufficient statistic.

(ii) Note that $T \in \{0, 1, 2, 3\}$. Evaluating the likelihood ratio,

$$L(X_1, X_2, X_3) = \begin{cases} 27/8 & T = 0 \\ 27/16 & T = 1 \\ 27/32 & T = 2 \\ 27/64 & T = 3 \end{cases}$$

(iii) For probability of false alarm to be $1/8$, we need to put the threshold at $(27/64, 27/32)$ and declare that if $T > 2$, then H_1 is declared. This is because $P_0(T > 2) = P_0(T = 3) = 1/8$. Hence, the best probability of detection is $P_1(T > 2) = P_1(T = 3) = (2/3)^3 = 8/27$.

(iv) For probability of false alarm to be $1/4$, we consider that $P_0(T > 1) = 1/2$ and the corresponding probability of detection is $P_1(T > 1) = (2/3)^3 + 3(2/3)^2(1/3) = 20/27$. Hence, we need to randomize between the strategy that places the threshold at $T > 2$ and $T > 1$. Now we find $\alpha \in [0, 1]$ such that

$$\alpha\frac{1}{8} + (1-\alpha)\frac{1}{2} = \frac{1}{4}, \qquad \Longrightarrow \qquad \alpha = \frac{2}{3}$$

Thus, the best probability of detection is

$$\alpha\frac{8}{27} + (1-\alpha)\frac{20}{27} = \frac{12}{27}.$$

The best test in terms of T would be to randomize between $T > 2$ and $T > 1$ where the former has probability $2/3$.

3 Problem 3

(a) States 1 and 4 are transient. States 2 and 3 constitute a class of recurrent states and state 5 constitues another class (a singleton class) of recurrent states.

(b) Let $\boldsymbol{\pi}^{(1)}$ be the be the steady-state vector for class $\{2,3\}$ and let $\boldsymbol{\pi}^{(2)}$ be the steady state vector for class $\{5\}$. For the class $\{5\}$, $\boldsymbol{\pi}^{(2)} = (0,0,0,0,1)$. For the class $\{2,3\}$, the steady-state equations (written just for the recurrent class) are

$$\pi_3^{(1)} P_{32} = \pi_2^{(1)}; \quad \pi_2^{(1)} P_{23} + \pi_3^{(1)} P_{33} = \pi_3^{(1)}; \quad \pi_2^{(1)} + \pi_3^{(1)} = 1$$

Solving these, we obtain

$$\pi_3^{(1)} = \frac{1}{P_{32}+1}, \qquad \pi_2^{(1)} = \frac{P_{32}}{P_{32}+1}.$$

(c) (i) $P_{44}^n = (1/3)^n$ since n successive self-loop transitions, each of probability 1/3, are required.

(ii) $P_{45}^n = (1/3)^1 + (1/3)^2 + \ldots + (1/3)^n = \frac{1}{2}(1 - 3^{-n})$. The reason for this is that there are n walks going from 4 to 5 in n steps; each such walk contains the 4 to 5 transition at a different time. If it occurs at time i then there are $i-1$ self-transitions from 4 to 4, so the probability of that walk is $(1/3)^i$.

(iii) $P_{41}^n = n(1/3)^n$ since there are n walks that go from 4 to 1 in n steps, one for each step in which the $4 \to 1$ transition can be made. Each walk has probability $(1/3)^n$.

(iv) $P_{43}^n + P_{42}^n = 1 - P_{44}^n - P_{45}^n - P_{41}^n = \frac{1}{2} - \frac{2n+1}{2} 3^{-n}$ since $\sum_j P_{4j}^n = 1$.

(v) $\lim_{n\to\infty} P_{43}^n$: From (iv) note that $\lim_{n\to\infty} P_{43}^n + P_{42}^n = 1/2$. Given that $X_n \in \{2,3\}$,

$$\lim_{m\to\infty} \Pr(X_m = 3 | X_n \in \{2,3\}) = \pi_3^{(1)} = \frac{1}{P_{32}+1}$$

Since $\lim_{n\to\infty} \Pr(X_n \in \{2,3\} | X_0 = 4) = 1/2$, we see that

$$\lim_{m\to\infty} P_{43}^m = \frac{1}{2(P_{32}+1)}.$$

(d) We only have the find the eigenvalue with the second largest magnitude. For this purpose, consider

$$\det(P - \lambda I) = 0 \qquad \Rightarrow \qquad \det\left(\begin{bmatrix} 0.2-\lambda & 0.8 \\ 0.5 & 0.5-\lambda \end{bmatrix} \right) = 0.$$

Multiplying out, we see that

$$(0.2-\lambda)(0.5-\lambda) - 0.4 = 0.$$

This quadratic equation has two roots, $\lambda_1 = 1$ and $\lambda_2 = -0.3$. Hence, $\phi = -0.3$.

4 Problem 4

(a) (i) Independent increments property of the Poisson process.

(ii) This distribution is exponential with rate λ.

(iii) $k = 0$.

(iv) We need to evaluate

$$\Pr(\text{no arrivals in interval } [t^* - x, t^*]) = \Pr(N(x) = 0) = e^{-\lambda x}.$$

We have shown that

$$\Pr(t^* - U > x) = e^{-\lambda x}, \qquad \Pr(t^* - U \leq x) = 1 - e^{-\lambda x}$$

so $t^* - U$ is also exponential with rate λ.

(v) Since $t^* - U$ and $V - t^*$ are independent exponentials with rate λ, their sum L is Erlang of order 2 with rate λ.

(vi) The interarrival time of a Poisson process is an exponential with rate λ. It is more likely that t^*, being observed, is in a longer interarrival interval.

(b) We have

$$\Pr(S_n \leq t) = \Pr((X_1 - \mu) + \ldots + (X_n - \mu) \leq t - n\mu) = \Pr\left(\frac{1}{n}((X_1 - \mu) + \ldots + (X_n - \mu)) \leq \frac{t}{n} - \mu\right)$$

For n large enough $\frac{t}{n} - \mu \leq -\frac{\mu}{2}$. Hence,

$$\Pr(S_n \leq t) \leq \Pr\left(\frac{1}{n}((X_1 - \mu) + \ldots + (X_n - \mu)) \leq -\frac{\mu}{2}\right)$$

Since each of the summands on the right-hand-side have zero mean, by Chebyshev's inequality or the weak law of large numbers, the right-hand-side probability converges to zero so $\Pr(S_n \leq t) \to 0$.