Homework 1

A0224725H

LUO ZIJIAN

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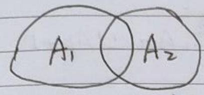
1. EXERCISE 1.1



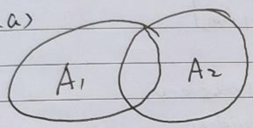
As the picture said

Sample space: $A_1 \setminus (A_1 \cap A_2)$ $A_2 \setminus (A_1 \cap A_2)$ are being counted once.

Sample space: $A_1 \cap A_2$ are being ~~counted~~ double counted on both sides

2. EXERCISE 1.2

(a)



It can be seen from the above conditions,

$A_1 \cup A_2 = A_1 \cup (A_2 - A_1)$

As the picture said

$A_1 \cap (A_2 - A_1) = \phi$

Therefore A_1 and $A_2 - A_1$ are disjoint

(b) ① Use induction: To show disjoint

First step: As the (a) said, $n=2$ $B_2 = A_2 - A_1$ $B_1 = A_1$

B_2, B_1 are disjoint

Second step: suppose $n=k$ $\underline{B_k = A_k - \cup_{m=1}^{k-1} A_m = A_k A_{k-1}^c \cdots A_1^c}$

$B_k \cdots B_2, B_1$ are disjoint

And for $n=k+1$ $\underline{B_{k+1} = A_{k+1} - \cup_{m=1}^{k} A_m = A_{k+1} \cdot A_k^c \cdot A_{k-1}^c \cdots A_1^c}$

$B_{k+1} - B_k = B_{k+1} \cdot B_k^c$ let's substitute the above expressions

$= A_{k+1} \cdot A_k^c \cdot A_{k-1}^c \cdots A_1^c [A_k A_{k-1}^c \cdots A_1^c]^c$

$= A_{k+1} \cdot A_k^c \cdot A_{k-1}^c \cdots A_1^c [\Omega - A_k A_{k-1}^c \cdots A_1^c]$

$\because A_k^c \cdot A_k = \phi$

$\therefore$ above equation becomes

$= A_{k+1} \cdot A_k^c \cdot A_{k-1}^c \cdots A_1^c - A_{k+1} \underbrace{A_k^c \cdot A_k}_{=\phi} \cdot A_{k-1}^c \cdot A_{k-1}^c \cdots A_1^c \cdot A_1^c$

$= A_{k+1} \cdot A_k^c \cdot A_{k-1}^c \cdots A_1^c$

$= B_{k+1}$

$\because B_{k+1} - B_k = B_{k+1}$ $\therefore B_{k+1}$ and B_k are disjoint

Hence this induction established

② Use induction: To show for $n \geq 2$ $\cup_{m=1}^{n} A_m = \cup_{m=1}^{n} B_m$

First step: $\cup_{m=1}^{2} A_m = A_2 \cup A_1$ $\cup_{m=1}^{2} B_m = B_2 \cup B_1$

$\cup_{m=1}^{2} A_m = \cup_{m=1}^{2} B_m$ $\because A_1 \cup A_2 = A_1 \cup (A_2 - A_1)$

Besform

Second step: suppose $n=k$ $\cup_{m=1}^{k} A_m = \cup_{m=1}^{k} B_m$

left expression:

And for $n=k+1$ $\cup_{m=1}^{k+1} A_m = \cup_{m=1}^{k} A_m \cup A_{k+1}$

right expression:

$\cup_{m=1}^{k+1} B_m = \cup_{m=1}^{k} B_m \cup B_{k+1}$

owing to $\cup_{m=1}^{k} A_m = \cup_{m=1}^{k} B_m$

$\therefore$ right expression can be replaced with

$= \cup_{m=1}^{k} A_m \cup B_{k+1}$

as for $B_{k+1} = A_{k+1} - \cup_{m=1}^{k} A_m$

$= \cup_{m=1}^{k} A_m \cup [A_{k+1} - \cup_{m=1}^{k} A_m]$

$= \cup_{m=1}^{k} A_m \cup A_{k+1}$

$\therefore$ left expression = right expression

Hence this induction established

(c) from (b) $B_1 \dots B_n$ are disjoint

$\because$ the axioms of probability $[Pr\{\cup_{n=1}^{m} A_n\} = \sum_{n=1}^{m} Pr A_n\}]$

$\therefore Pr\{\cup_{n=1}^{\infty} B_n\} = \sum_{n=1}^{\infty} Pr\{B_n\}$

$\because \cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} A_n$ $\therefore$ they are same sample space

$\therefore Pr\{\cup_{n=1}^{\infty} A_n\} = Pr\{\cup_{n=1}^{\infty} B_n\}$

Hence $Pr\{\cup_{n=1}^{\infty} A_n\} = Pr\{\cup_{n=1}^{\infty} B_n\} = \sum_{n=1}^{\infty} Pr\{B_n\}$

(d) $\because$ ~~the axioms of probability [Pr{A} ≤ Pr{B} for all A⊆B]~~

~~from (b) $\cup_{n=1}^{\infty} A_n$~~

from (c) $Pr\{\cup_{n=1}^{\infty} A_n\} = Pr\{\cup_{n=1}^{\infty} B_n\} = \sum_{n=1}^{\infty} Pr\{B_n\}$

$B_n = A_n - \cup_{m=1}^{n-1} A_m$

obviously $B_n \subseteq A_n$

from the axioms of probability $[Pr\{A\} \leq Pr\{B\}$ for all $A \subseteq B]$

$\therefore Pr\{B_n\} \leq Pr\{A_n\} \iff \sum_{n=1}^{\infty} Pr\{B_n\} \leq \sum_{n=1}^{\infty} Pr\{A_n\}$

above expression can be replaced

$Pr\{\cup_{n=1}^{\infty} A_n\} = Pr\{\cup_{n=1}^{\infty} B_n\} = \sum_{n=1}^{\infty} Pr\{B_n\} \leq \sum_{n=1}^{\infty} Pr\{A_n\}$

$\therefore Pr\{\cup_{n=1}^{\infty} A_n\} \leq \sum_{n=1}^{\infty} Pr\{A_n\}$

(e) combine (b) (c)

from (b): $\cup_{m=1}^{n} A_m = \cup_{m=1}^{n} B_m$ Hence $Pr(\bigcup_{n=1}^{\infty} A_n) = Pr(\bigcup_{n=1}^{\infty} B_n)$ ......①

$B_1 \dots B_n$ are disjoint

$\therefore Pr(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} Pr(B_n) = \lim_{m \to \infty} \sum_{n=1}^{m} Pr(B_n)$ ... ②

① ∪ ② $Pr\{\cup_{n=1}^{\infty} A_n\} = \lim_{m \to \infty} Pr\{\cup_{n=1}^{m} A_n\}$

(f) from De Morgan's equalities

$$\overline{\bigcap_{n=1}^{\infty} A_n} = \bigcup_{n=1}^{\infty} \overline{A_n} \quad \text{let } C_n = \overline{A_n} \quad D_n = C_n - \bigcup_{m=1}^{n-1} C_m$$

from (b) $D_1 \cdots D_n$ are disjoint

$$\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} D_n$$

from (c) $Pr\{\bigcup_{n=1}^{\infty} C_n\} = Pr\{\bigcup_{n=1}^{\infty} D_n\} = \sum_{n=1}^{\infty} Pr\{D_n\}$

from (e) $Pr\{\bigcup_{n=1}^{\infty} C_n\} = \lim_{m\to\infty} Pr\{\bigcup_{n=1}^{m} C_n\} = \lim_{m\to\infty} Pr\{\bigcup_{n=1}^{m} D_n\} \cdots ①$

and then

$$\overline{\bigcap_{n=1}^{\infty} A_n} = \bigcup_{n=1}^{\infty} \overline{A_n}$$

so $Pr(\overline{\bigcap_{n=1}^{\infty} A_n}) = Pr(\bigcup_{n=1}^{\infty} \overline{A_n}) \quad \cdots ②$

from the axiom of probability $[Pr(\bar{A}) = 1 - Pr(A)]$

above equatation can expressed:

$$1 - Pr(\bigcap_{n=1}^{\infty} A_n) = Pr(\bigcup_{n=1}^{\infty} \bar{A}_n) = Pr(\bigcup_{n=1}^{\infty} C_n) \cdots ③$$

from ①②③ $\Rightarrow Pr\{\bigcap_{n=1}^{\infty} A_n\} = \lim_{m\to\infty} Pr\{\bigcap_{n=1}^{m} A_m\}$

3. EXERCISE 1.3

It seems ~~the~~ that there are 52! arrangement of a deck of cards and then focus on this special situation:

all four aces are in the first five cards of a deck, which means, other (52−5) cards are randomly arranged

so 47! arrangement

then pay attention to first five cards, four ~~aces~~ cards are aces, and the other one is any cards,

so we must choose ~~one 48 different cards~~ one from 48 cards.

, which means there are 48 arrangement

Lastly, we focus on the arranged order four aces are the first five cards, so there are 4! arrangement card

∴ we only choose one position from five positions to arrange this special card

Hence, this probability of four aces are in the first five cards

$$Pr(A) = \frac{47! \times 48 \times 5 \times 4!}{52!} = \frac{48! \times 5!}{52!} = \frac{5\times4\times3\times2}{52\times51\times50\times49}$$

$= 1.847 \times 10^{-5}$

Date No.

4. Probability Review

(a) All the possible event are listed:

the sample space:

$$\begin{Bmatrix} HHHH & HHHT & HTHH & HTHT \\ HHTH & HHTT & HTTH & HTTT \\ THHH & THHT & TTHH & TTHT \\ THTH & THTT & TTTH & TTTT \end{Bmatrix}$$

which means (X, Y)

$$\begin{Bmatrix} (X=4, Y=1) & (X=3, Y=1) & (X=3, Y=1) & (X=2, Y=1) \\ (X=3, Y=1) & (X=2, Y=1) & (X=2, Y=1) & (X=1, Y=1) \\ (X=3, Y=2) & (X=2, Y=2) & (X=2, Y=3) & (X=1, Y=3) \\ (X=2, Y=2) & (X=1, Y=2) & (X=1, Y=4) & (X=0, Y=0) \end{Bmatrix}$$

$P_{XY}(x,y) \Rightarrow P_{XY}(0,0) = \frac{1}{16}$ ~~P_{XY}~~

$P_{XY}(1,1) = \frac{1}{16}$, $P_{XY}(1,2) = \frac{1}{16}$ $P_{XY}(1,3) = \frac{1}{16}$ $P_{XY}(1,4) = \frac{1}{16}$

$P_{XY}(2,1) = \frac{3}{16}$, $P_{XY}(2,2) = \frac{1}{8}$ $P_{XY}(2,3) = \frac{1}{16}$

$P_{XY}(3,1) = \frac{3}{16}$, $P_{XY}(3,2) = \frac{1}{16}$

$P_{XY}(4,1) = \frac{1}{16}$

(b) for the marginal PMF of X

$$P_X(x) = \sum_{y_j \in R_Y} P_{XY}(x, y_j), \text{ for any } x \in R_X$$

$P_X(0) = \frac{1}{16}$

$P_X(1) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4}$

$P_X(2) = \frac{3}{16} + \frac{1}{8} + \frac{1}{16} = \frac{3}{8}$

$P_X(3) = \frac{1}{4}$

$P_X(4) = \frac{1}{16}$

5. from the hint $C = (C \cap D) \cup (C \cap D^c)$

we can conclude that $\bigcup_{i=1}^{n} A_i = (\bigcup_{i=1}^{n} A_i \cap A_k) \cup (\bigcup_{i=1}^{n} A_i \cap A_k{}^c)$ $i \neq k$

let $C = \bigcup_{i=1}^{n} A_i$ $D = A_k$

$\bigcup_{i=1}^{n} A_i \cap A_k$ and $\bigcup_{i=1}^{n} A_i \cap A_k{}^c$ are disjoint

this equation ~~equistion~~ $= i \neq k$ $\because A_i \cap A_k{}^c = \phi$ when $i = k$

~~we can limit this equation~~ ~~$(\bigcup_{i=1}^{n} A_i \cap A_k) \subset A_k$~~

~~equation~~

so $Pr(\bigcup_{i=1}^{n} A_i) = Pr(\bigcup_{i=1}^{n} A_i \cap A_k) + Pr(\bigcup_{i=1}^{n} A_i \cap A_k^c)$ $i \neq k$ ---- ①

$\because \bigcup_{i=1}^{n} A_i \cap A_k \subseteq A_k$

$\therefore Pr(\bigcup_{i=1}^{n} A_i \cap A_k) \leq Pr(A_k)$ $i \neq k$ --- ②

$\because \bigcup_{i=1}^{n} A_i \cap A_k^c = \bigcup_{i=1}^{n} (A_i \cap A_k^c)$ $i \neq k$ (distributive law of unionset)

$\therefore Pr(\bigcup_{i=1}^{n} A_i \cap A_k^c) = \sum_{i=1, i\neq k}^{n} (A_i \cap A_k^c)$

from $A_i = (A_i \cap A_k) \cup (A_i \cap A_k^c)$

$\therefore Pr(A_i) = Pr(A_i \cap A_k) + Pr(A_i \cap A_k^c)$

above equation can be replaced

$Pr(\bigcup_{i=1}^{n} A_i \cap A_k^c) = \sum_{i=1, i\neq k}^{n} (A_i \cap A_k^c) = \sum_{i=1, i\neq k}^{n} Pr(A_i) - \sum_{i=1, i\neq k}^{n} Pr(A_i \cap A_k)$ ③

combine ①②③: we can conclude that

$$Pr(\bigcup_{i=1}^{n} A_i) \leq Pr(A_k) + \sum_{i=1, i\neq k}^{n} Pr(A_i) - \sum_{i=1, i\neq k}^{n} Pr(A_i \cap A_k)$$

$$= \sum_{i=1}^{n} Pr(A_i) - \sum_{i=1, i\neq k}^{n} Pr(A_i \cap A_k)$$

Hence it proves.