Gaussian MAP

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This document completes derivations for univariate Gaussian MAP (lecture 3). At the beginning of the slide, we have established that

$$(\hat{\mu}, \hat{\sigma}^2) = \underset{\mu,\sigma^2}{\operatorname{argmax}} \log \frac{1}{\sigma}\left(\frac{1}{\sigma^2}\right)^{\alpha+1} - \frac{2\beta + \gamma(\delta-\mu)^2}{2\sigma^2} - \sum_{n=1}^{N}\left[\frac{(x_n-\mu)^2}{2\sigma^2} + \log\sqrt{2\pi\sigma^2}\right]$$

- Differentiating with respect to μ, we get

$$\begin{aligned}
\frac{\gamma(\delta-\mu)}{\sigma^2} + \sum_{n=1}^{N}\frac{x_n-\mu}{\sigma^2} &= 0 \\
\gamma\delta - \gamma\mu + \sum_{n=1}^{N} x_n - N\mu &= 0 \\
\gamma\delta + \sum_{n=1}^{N} x_n &= \mu(N+\gamma) \\
\mu &= \frac{\gamma\delta + \sum_{n=1}^{N} x_n}{N+\gamma}
\end{aligned}$$

  Therefore $\hat{\mu}_{MAP} = \frac{\gamma\delta + \sum_{n=1}^{N} x_n}{N+\gamma}$

- Note that $\log\frac{1}{\sigma}\left(\frac{1}{\sigma^2}\right)^{\alpha+1} = -[2(\alpha+1)+1]\log\sigma$. Differentiating with respect to σ, we get

$$\begin{aligned}
&-\frac{2(\alpha+1)+1}{\sigma} + \frac{2\beta+\gamma(\delta-\mu)^2}{\sigma^3} + \sum_{n=1}^{N}\frac{(x_n-\mu)^2}{\sigma^3} - \frac{N}{\sigma} = 0 \\
&-\frac{1}{\sigma}(2\alpha+3+N) + \frac{1}{\sigma^3}(2\beta+\gamma(\delta-\mu)^2 + \sum_{n=1}^{N}(x_n-\mu)^2) = 0 \\
&\sigma^2 = \frac{\sum_{n=1}^{N}(x_n-\hat{\mu})^2 + 2\beta + \gamma(\delta-\hat{\mu})^2}{N+3+2\alpha}
\end{aligned}$$