EE5138R: Quiz 1 (13/02/15)
Total: 30 points

1. (10 points) For which sets of $\alpha \in \mathbf{R}$ are the following functions convex?

   (a) (2 points) $f(x) = \sin x + \alpha x$ with $\mathbf{dom}\, f = \mathbf{R}$
   Solution: $f''(x) = -\sin x$ which is not positive definite. So no α ensures that f is convex.

   (b) (3 points) $f(x) = \sin x + \alpha x^2$ with $\mathbf{dom}\, f = \mathbf{R}$
   Solution: $f''(x) = -\sin x + 2\alpha$ which is positive definite if $\alpha \geq \frac{1}{2}$.

   (c) (5 points) $f(x_1, x_2) = (5-\alpha)x_1^2 + 10x_1x_2 + x_2^2 + 4\alpha x_1$ with $\mathbf{dom}\, f = \mathbf{R}^2$
   Solution: The Hessian of f is

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 2(5-\alpha) & 10 \\ 10 & 2 \end{bmatrix}$$

   The first entry is non-negative if $\alpha \leq 5$. The determinant is non-negative if $\alpha \leq -20$. So $\nabla^2 f(x_1, x_2)$ is psd, i.e., f is convex, if $\alpha \leq -20$.

2. (10 points) Let $x_i \in (0, 1/2]$ and let $\gamma_i \in (0,1)$ for $i = 1, \ldots, n$ be real numbers satisfying $\sum_{i=1}^n \gamma_i = 1$. It is known that either

$$\text{(A)} \qquad \frac{\prod_{i=1}^n x_i^{\gamma_i}}{\prod_{i=1}^n (1-x_i)^{\gamma_i}} \leq \frac{\sum_{i=1}^n \gamma_i x_i}{\sum_{i=1}^n \gamma_i (1-x_i)}$$

   or

$$\text{(B)} \qquad \frac{\prod_{i=1}^n x_i^{\gamma_i}}{\prod_{i=1}^n (1-x_i)^{\gamma_i}} \geq \frac{\sum_{i=1}^n \gamma_i x_i}{\sum_{i=1}^n \gamma_i (1-x_i)}$$

   is true. Which is true and why?

   Hint: Consider the convexity/concavity properties of the function $f(x) = \ln x - \ln(1-x)$ with $\mathbf{dom}\, f = [0, 1/2]$ and then apply some famous inequality we talked about in class.

   Bonus question (5 points): When does equality hold?

   Solution: The hint asks us to consider the properties of $f(x) = \ln x - \ln(1-x)$. Upon differentiation, we obtain

$$f''(x) = -\frac{1}{x^2} + \frac{1}{(1-x)^2} = \frac{2x-1}{x^2(1-x)^2} \leq 0$$

   so the function f is concave. By Jensen's inequality applied to the convex combination $\sum_i \gamma_i x_i$, we obtain

$$f\left(\sum_i \gamma_i x_i\right) \geq \sum_i \gamma_i f(x_i)$$

In other words,

$$\ln\left(\frac{\sum_i \gamma_i x_i}{1-\sum_i \gamma_i x_i}\right) \geq \sum_i \gamma_i \ln\left(\frac{\sum_i x_i}{1-\sum_i x_i}\right)$$

Upon rearrangement (taking exp on both sides), we obtain

$$\frac{\sum_i \gamma_i x_i}{\sum_i \gamma_i(1-x_i)} \geq \prod_i \frac{x_i^{\gamma_i}}{(1-x_i)^{\gamma_i}}$$

so inequality (A) is true.

Equality holds if and only if all the $x_i, i = 1, \ldots, n$ are equal to some common $x \in (0, 1/2]$.

3. (10 points) The *polar* of an arbitrary set $C \subset \mathbf{R}^n$ is defined as the set

$$C^\circ := \{y \in \mathbf{R}^n : y^T x \leq 1 \text{ for all } x \in C\}$$

(a) (3 points) Let $C \subset \mathbf{R}^n$ be any set, not necessarily convex. Is C° convex? Justify your answer carefully.

Solution: Yes C° is convex. It can be written as

$$C^\circ = \bigcap_{x \in C} \{y \in \mathbf{R}^n : y^T x \leq 1\}$$

Each set $\{y \in \mathbf{R}^n : y^T x \leq 1\}$ parametrized by $x \in C$ is a halfspace hence convex. Intersection of convex sets is convex.

(b) (5 points) Recall that a *cone* K is such that if $x \in K$ and $\lambda \geq 0$, then $\lambda x \in K$. Recall that the *dual cone* is defined as

$$K^* := \{y \in \mathbf{R}^n : y^T x \geq 0 \text{ for all } x \in K\}$$

It is known that the polar of the cone K, denoted as K°, and the dual cone, denoted as K^*, are related as follows:

$$K^\circ = -cK^*$$

for some *positive* number $c > 0$. Find c.

Solution: The answer is $c = 1$. We prove this in two parts first noting that

$$-K^* := \{y : y^T x \leq 0 \text{ for all } x \in K\}$$

$K^\circ \subset -K^*$: Take $y \in K^\circ$. This means that $y^T x \leq 1$ for all $x \in K$. Since K is a cone, if $x \in K$, we have $\lambda x \in K$ for all $\lambda \geq 0$. This means that $y^T x \leq 1/\lambda$ for all $x \in K$ and all $\lambda \geq 0$. Now take $\lambda \to \infty$. Then we conclude that $y^T x \leq 0$ for all $x \in K$. This means that $y \in -K^*$.

$-K^* \subset K^\circ$: Take $y \in -K^*$. This means that $y^T x \leq 0$ for all $x \in K$. Clearly $y^T x \leq 1$ for all $x \in K$. This means that $y \in K^\circ$ as desired.

(c) (2 points) Show carefully that the polar of the unit ball $\mathcal{B}(0,1) := \{x \in \mathbf{R}^n : \|x\|_2 \leq 1\}$ is the unit ball. You may find the Cauchy-Schwarz inequality (i.e., $x^T y \leq \|x\|_2 \|y\|_2$) useful.

Solution: We need to show that $\mathcal{B}(0,1)^\circ = \mathcal{B}(0,1)$.

Take $x \in \mathcal{B}(0,1)^\circ$. This means that $x^T y \leq 1$ for all $y \in \mathcal{B}(0,1)$. Suppose $\|x\|_2 > 1$. Now let $y = x/\|x\|_2$. Then the inner product $y^T x = \|x\|_2 > 1$, which contradicts $x^T y \leq 1$. This means that $\|x\|_2 \leq 1$, i.e., $x \in \mathcal{B}(0,1)$.

Now take $x \in \mathcal{B}(0,1)$. This means that $\|x\|_2 \leq 1$. Fix $y \in \mathcal{B}(0,1)$. We have

$$x^T y \leq \|x\|_2 \|y\|_2 \leq \|y\|_2 \leq 1$$

This means that $x \in \mathcal{B}(0,1)^\circ$.