EE5137: Quiz 2

Name: __________

Matriculation Number: __________

Total Score: __________

October 12, 2017

You have 1.0 hour for this quiz. You're allowed 1 sheet of handwritten notes (both sides). Please show provide *careful explanations* for all your solutions.

1. [Merged Processes] Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates λ_1 and λ_2 respectively. Let $N(t) = N_1(t) + N_2(t)$ be the merged process.

   (a) (5 points) Find the probability that $N(1) = 2$ and $N(2) = 5$. Express your answer in terms of λ_1 and λ_2.

   Solution: $N(t)$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$. Then

$$\Pr(N(1) = 2, N(2) = 5) = \Pr(N(1) = 2) \Pr(N(2) = 5 | N(1) = 2) = \frac{e^{-\lambda}\lambda^2}{2!} \cdot \frac{e^{-\lambda}\lambda^3}{3!}$$

   (b) (5 points) Given that $N(1) = 2$, find the probability that $N_1(1) = 1$. Express your answer as a fraction and in terms of λ_1 and λ_2.

   Solution: We have

$$\begin{aligned}
\Pr(N_1(1) = 1 | N(1) = 2) &= \frac{\Pr(N_1(1) = 1, N(1) = 2)}{\Pr(N(1) = 2)} \\
&= \frac{\Pr(N_1(1) = 1, N_2(1) = 1)}{\Pr(N(1) = 2)} \\
&= \frac{\Pr(N_1(1) = 1) \Pr(N_2(1) = 1)}{\Pr(N(1) = 2)} \\
&= \frac{e^{-\lambda_1}\lambda_1 e^{-\lambda_2}\lambda_2}{e^{-\lambda}\lambda^2/2!} \\
&= \frac{2\lambda_1\lambda_2}{(\lambda_1 + \lambda_2)^2}
\end{aligned}$$

2. [Waiting Time]

Patients arrive at the doctor's office according to a Poisson process with rate $\lambda = 1/10$ minutes. The doctor will not see a patient until <u>at least three patients</u> are in the waiting room.

(a) (5 points) Find the expected waiting time in minutes until the first patient is admitted to see the doctor.

Solution: Let X_n be the interarrival time, i.e., the time between the arrival of the $(n-1)^{\text{th}}$ and n^{th} patients. Then the X_i's are iid exponential random variables with mean λ. Let S_n be the arrival time of the n^{th} patient. Then $S_n = \sum_{i=1}^{n} X_i$. Clearly,

$$\mathbb{E}S_n = \frac{n}{\lambda}.$$

Thus, the expected waiting time until the first patient is admitted to see the doctor is

$$\mathbb{E}S_3 = \frac{3}{1/10} = 30 \text{ mins.}$$

(b) (5 points) What is the probability that nobody is admitted to see the doctor in the first hour (1 hour = 60 minutes)?

Write your answer as $c \times e^{-6}$ and find the constant c.

Solution: Let $N(t)$ be the Poisson counting process with mean $\lambda t = \frac{t}{10}$. Then we have

$$\begin{aligned}
&\Pr(\text{Nobody admitted to see the doc in the first hour}) \\
&= \Pr(N(60) \leq 2) \\
&= \Pr(N(60) = 0) + \Pr(N(60) = 1) + \Pr(N(60) = 2) \\
&= e^{-60/10} + e^{-60/10}\frac{60}{10} + e^{-60/10}(\frac{60}{10})^2/2! \\
&= 25 \cdot e^{-6}
\end{aligned}$$

3. [Arrival Times] (5 points)

Let $S_1, \ldots, S_n$ be the arrival times of a Poisson process with rate λ. Find

$$\mathbb{E}[S_1 + S_2 + \ldots + S_n | N(1) = n].$$

Hint: You may use the fact that $\sum_{i=1}^n i = n(n+1)/2$.

Solution: From class we know that

$$\mathbb{E}[S_i | N(t) = n] = \frac{it}{n+1}.$$

Hence,

$$\mathbb{E}[S_1 + S_2 + \ldots + S_n | N(1) = n] = \sum_{i=1}^{n} \mathbb{E}[S_i | N(1) = n] = \sum_{i=1}^{n} \frac{i}{n+1} = \frac{n}{2}$$

where the last equality is from the hint.