EE5138R: Solutions to Problem Set 2
Assigned: 23/01/15
Due: 30/01/15

1. Is the set
$$\{a \in \mathbb{R}^k : p(0) = 1, |p(t)| \le 1, \forall\, t \in [\alpha, \beta]\}$$
where
$$p(t) = a_1 + a_2 t + \ldots + a_k t^{k-1}$$
convex?

   Solution: Yes, it is convex. The set $S := \{a \in \mathbb{R}^k : p(0) = 1, |p(t)| \le 1, \forall\, t \in [\alpha, \beta]\}$ is the intersection of
$$T_1 := \{a \in \mathbb{R}^k : p(0) = 1\}, \quad \text{and} \quad T_2 := \{a \in \mathbb{R}^k : |p(t)| \le 1, \forall\, t \in [\alpha, \beta]\}$$
The set T_1 is the set of all vectors $a \in \mathbb{R}^k$ such that the first component $a_1 = 0$. This set is clearly convex. The set T_2 can be written as
$$T_2 = \bigcap_{t \in [\alpha,\beta]} T_2^{(t)} \quad \text{where} \quad T_2^{(t)} := \{a \in \mathbb{R}^k : -1 \le a^T [1, t, \ldots, t^{k-1}]^T \le 1\}$$
For each fixed $t \in [\alpha, \beta]$, the set $T_2^{(t)}$ is a slab, hence convex. Hence T_2 is convex. Since T_1 and T_2 are convex, so is S.

2. Prove (using the shortest argument possible) that the following set is convex:
$$\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \begin{bmatrix} x_1 + x_2 & x_1 - 2x_3 \\ x_1 - 2x_3 & x_2 + 3x_3 \end{bmatrix} \succeq 0 \right\}$$

   Solution: Consider the function $f : \mathbb{R}^3 \to \mathbf{S}^2$ satisfying
$$f(x_1, x_2, x_3) = \begin{bmatrix} x_1 + x_2 & x_1 - 2x_3 \\ x_1 - 2x_3 & x_2 + 3x_3 \end{bmatrix}$$
This function is linear, hence affine. The set of interest can be written as
$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : f(x_1, x_2, x_3) \succeq 0\} = f^{-1}(\mathbf{S}_+^3)$$
Since this $\mathbf{S}_+^3$ is convex and f is linear, $f^{-1}(\mathbf{S}_+^3)$ is convex.

3. BV Problem 2.12
Solutions:

   (a) A slab is an intersection of two halfspaces, hence it is a convex set (and a polyhedron).

   (b) As in part (a), a rectangle is a convex set and a polyhedron because it is a finite intersection of halfspaces.

(c) A wedge is an intersection of two halfspaces, so it is convex set. It is also a polyhedron. It is a cone if $b_1 = 0$ and $b_2 = 0$.

(d) This set is convex because it can be expressed as

$$\bigcap_{y \in S} \{x : \|x - x_0\|_2 \leq \|x - y\|_2\},$$

i.e., an intersection of halfspaces.

(e) In general this set is not convex, as the following example in $\mathbb{R}$ shows. With $S = \{-1, 1\}$ and $T = \{0\}$, we have

$$\{x : \mathrm{dist}(x, S) \leq \mathrm{dist}(x, T)\} = \{x \in \mathbb{R} : x \leq -1/2 \text{ or } x \geq 1/2\}$$

which is not convex.

(f) This set is convex. The condition that $x + S_2 \subset S_1$ is equivalent to $x + y \in S_1$ for all $y \in S_2$. Thus

$$A = \{x : x + S_2 \subset S_1\} = \bigcap_{y \in S_2} \{x : x + y \in S_1\} = \bigcap_{y \in S_2} (S_1 - y)$$

Since A is an intersection of convex sets $\{S_1 - y : y \in S_2\}$, A is convex.

(g) The set is convex. We have that

$$\begin{aligned} & \|x - a\|_2 \leq \theta \|x - b\|_2 \\ & \Leftrightarrow \|x - a\|_2^2 \leq \theta^2 \|x - b\|_2^2 \\ & \Leftrightarrow (1 - \theta)^2 x^T x - 2(a - \theta^2 b)^T x + (a^T a - \theta^2 b^T b) \leq 0 \end{aligned}$$

If $\theta = 1$, this is a halfspace. If $\theta < 1$ this is a ball

$$\{x : (x - x_0)^T (x - x_0) \leq R^2\}$$

where the center x_0 and radius R are

$$x_0 = \frac{a - \theta^2 b}{1 - \theta^2}, \qquad R^2 = \frac{\theta^2 \|b\|_2^2 - \|a\|_2^2}{1 - \theta^2} - \|x_0\|_2^2$$

4. BV Problem 2.16

Solutions: We need to show that if $S_1, S_2 \subset \mathbb{R}^{m+n}$ are convex sets, then so is their partial sum

$$S := \{(x, y_1 + y_2) \in \mathbb{R}^{m+n} : x \in \mathbb{R}^n, y_1, y_2 \in \mathbb{R}^m, (x, y_1) \in S_1, (x, y_2) \in S_2\}$$

Consider two points $(x, y_1 + y_2), (x', y_1' + y_2') \in S$, i.e.,

$$(x, y_1), (x', y_1') \in S_1, \qquad (x, y_2), (x', y_2') \in S_2.$$

Fix $\theta \in [0, 1]$. Then consider the point

$$\theta(x, y_1 + y_2) + (1 - \theta)(x', y_1' + y_2') = (\theta x + (1 - \theta)x', \theta y_1 + (1 - \theta)y_1' + \theta y_2 + (1 - \theta)y_2'))$$

This point is in S because by convexity of S_1 and S_2, it holds that

$$(\theta x + (1 - \theta)x', \theta y_1 + (1 - \theta)y_1') \in S_1, \qquad (\theta x + (1 - \theta)x', \theta y_2 + (1 - \theta)y_2') \in S_2$$

5. BV Problem 2.21

The conditions $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$ form a set of homogeneous linear inequalities in (a, b). Therefore this set of separating hyperplanes $\{(a, b)\}$ is the intersection of halfspaces that pass through the origin. Hence it is a convex cone.

Note that this does not require convexity of C or D.

6. BV Problem 2.24
Solutions: The set is the intersection of all supporting halfspaces at points in its boundary, which is given by $\{x \in \mathbb{R}^2_+ : x_1 x_2 = 1\}$. The supporting hyperplane at at $x = (t, 1/t)$ for $t > 0$ is given by

$$x_1/t^2 + x_2 = 2/t$$

so we can express the set as

$$\bigcap_{t>0} \{x \in \mathbb{R}^2 : x_1/t^2 + x_2 \geq 2/t\}$$

Next, let $C := \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$ be the ℓ_∞-norm unit ball in $\mathbb{R}^n$ and let $\hat{x}$ be a point on the boundary of C. We note that $s^T x \geq s^T \hat{x}$ for all $x \in C$ if and only if

$$\begin{aligned} s_i &< 0 \qquad \hat{x}_i = 1 \\ s_i &> 0 \qquad \hat{x}_i = -1 \\ s_i &= 0 \qquad -1 < \hat{x}_i < 1 \end{aligned}$$

We are going to encounter such solutions in the context of duality and in particular the KKT conditions in the sequel.

7. BV Problem 2.32
Solution: Let $K = \{Ax : x \succeq 0\}$ where $A \in \mathbb{R}^{m \times n}$. This is a cone. We prove that the dual cone is $K^* := \{y : y^T z \geq 0 \text{ for all } z \in K\} = \{y : A^T y \succeq 0\}$. Temporarily put $\tilde{K} = \{y : A^T y \succeq 0\}$ so we need to show that

$$K^* = \tilde{K}.$$

First let $y \in K^*$. Then $y^T z \geq 0$ for all $z \in K$. This means that $y^T(Ax) \geq 0$ for all $x \succeq 0$. This means that $x^T(A^T y) \geq 0$ for all $x \succeq 0$. By the same argument as the fact that the nonnegative orthant is self-dual, $A^T y \succeq 0$, i.e., $y \in \tilde{K}$.

Next, we let $y \in \tilde{K}$. This means that $A^T y \succeq 0$. This means that $x^T A^T y \geq 0$ for any $x \succeq 0$, which further implies that $y^T(Ax) \geq 0$ for any $x \succeq 0$. Since $Ax \in K$, this means that $y \in K^*$ as desired.

8. (Optional) BV Problem 2.33
Solution:

   (a) The set K_{m+} is defined by n homogeneous linear inequalities, hence it is a closed (polyhedral) cone. The interior of K_{m+} is nonempty, because there are points that satisfy the inequalities with strict inequality, for example, $x = (n, n-1, n-2, \ldots, 1)$. To show that K_{m+} is pointed, we note that if $x \in K_{m+}$, then $-x \in K_{m+}$ only if $x = 0$. This implies that the cone does not contain an entire line.

   (b) Using the hint, we see that $y^T x \geq 0$ for all $x \in K_{m+}$ if and only if

$$y_1 \geq 0, \quad y_1 + y_2 \geq 0, \quad \ldots \quad y_1 + y_2 + \ldots y_n \geq 0$$

   Therefore,

$$K^*_{m+} = \left\{ y : \sum_{i=1}^{k} y_i \geq 0, \quad \forall k = 1, \ldots, n \right\}$$