EE5137 Stochastic Processes: Problem Set 12
Assigned: 09/04/21, Due: Never

This problem set is not due but is examinable.

1. Suppose (X, Y) is a pair of random variables. Their joint density, depicted below, is constant in the shaded area and zero elsewhere.

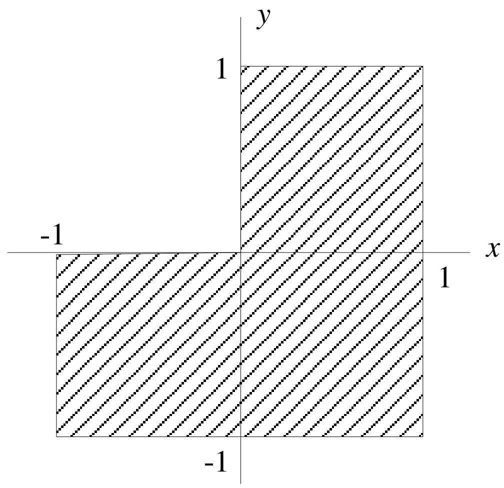

Figure 1: Joint density of (X, Y) for Question 1

   (a) Determine $\hat{x}_{\mathrm{BLS}}(Y)$, the Bayes least squares estimate of X given Y.

   (b) Determine λ_{BLS}, the error variance associated with your estimator in (a).

   (c) Consider the following "modified" cost function (corresponding to the cost of estimating x as $\hat{x}$):

$$C(x, \hat{x}) = \begin{cases} (x - \hat{x})^2 & x < 0 \\ K(x - \hat{x})^2 & x \geq 0 \end{cases},$$

   where $K > 1$ is a constant.

   Determine $\hat{x}_{\mathrm{MLS}}(Y)$, the associated Bayes estimate of X for this modified cost criterion. (MLS stands for *modified least squares.*)

   (d) Give a brief intuitive explanation for why your answers to (a) and (c) are either the same or different.

2. You are given a coin and are allowed to toss it until you see the first head. You are then asked to estimate q, the probability of heads for this particular coin. Let Y be the number of times you see tails before the first head. We note that Y is distributed according to the geometric distribution

$$P_Y(y; q) = q(1 - q)^y \quad y \in \mathbb{N} \cup \{0\}.$$

(a) Consider the maximum likelihood estimator of q. Is it unique? Is it efficient?

(b) Your Bayesian friend proposes to treat the probability of heads as random variable X with a density that is uniform in the region $[0,1]$. Find the Bayes least squares estimate for X. You can use the fact that

$$\int_0^1 x^k(1-x)^n \, \mathrm{d}x = \frac{n!k!}{(n+k+1)!}$$

(c) We define relative bias to be the expected ratio of the true parameter to the estimated value, i.e.,

$$R_{\hat{x}} = \mathbb{E}\left[\frac{X}{\hat{x}(Y)}\right].$$

Just as what we've done with the usual additive bias, relative bias can be applied to non-random parameter estimation. An estimator with a relative bias of 1 is called *relatively unbiased.*

(i) Is the ML estimator you computed in part (a) relatively unbiased?

(ii) Is the BLS estimator you computed in part (a) relatively unbiased?

3. (a) Let

$$f_Y(y;x) = \begin{cases} x & 0 \le y \le 1/x \\ 0 & \text{else} \end{cases}$$

for $x > 0$. Show that there exist no unbiased estimators $\hat{x}(Y)$ of x. (Note that because only $x > 0$ are possible values, an unbiased estimator need only be unbiased for $x > 0$ rather than all x.)

(b) Let

$$f_Y(y;x) = \begin{cases} 1/x & 0 \le y \le x \\ 0 & \text{else} \end{cases}$$

for $x > 0$. Show that there is a unique unbiased estimator.

4. Suppose that for $i = 1, 2$,

$$Y_i = x + W_i$$

where x is an unknown but non-zero constant, and where W_1 and W_2 are statistically independent, zero-mean Gaussian random variables with

$$\mathrm{Var}(W_1) = 1$$
$$\mathrm{Var}(W_2) = \begin{cases} 1 & x > 0 \\ 2 & x < 0 \end{cases}.$$

(a) Calculate the Cramér-Rao bound for unbiased estimators of x based on an observation of $\mathbf{Y} = (Y_1, Y_2)$.

(b) Show that a minimum variance unbiased estimator $\hat{x}_{\mathrm{MVU}}(Y)$ does not exist.

Hint: Consider the estimators

$$\hat{X}_1 = \frac{Y_1}{2} + \frac{Y_2}{2} \quad \textit{and} \quad \hat{X}_2 = \frac{2Y_1}{3} + \frac{Y_2}{3}.$$

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5. (Optional) Let $\mathbf{Y} = [Y_1, Y_2, \ldots, Y_n]^\top$ be an n-dimensional random vector composed of independent scalar Gaussian random variables Y_i, each with unknown, non-random mean x_i and unit variance. Our goal is to construct an estimator $\hat{x}(\mathbf{Y})$ for the vector of the mean parameters $\mathbf{x} = [x_1, x_2, \ldots, x_n]^\top$. Here, we generalize the mean-square error cost function to vector parameters in the following way:

$$\mathrm{MSE}_{\hat{\mathbf{x}}}(\mathbf{x}) = \mathbb{E}\left[\|\hat{\mathbf{x}}(\mathbf{Y}) - \mathbf{x}\|^2\right] = \mathbb{E}\left[(\hat{\mathbf{x}}(\mathbf{Y}) - \mathbf{x})^\top(\hat{\mathbf{x}}(\mathbf{Y}) - \mathbf{x})\right].$$

(a) Determine the maximum likelihood estimator $\hat{\mathbf{x}}_{\mathrm{ML}}(\mathbf{y})$.

(b) Find the bias of the maximum likelihood estimator.

(c) Determine $\mathrm{MSE}_{\hat{\mathbf{x}}_{\mathrm{ML}}}(\mathbf{x})$, the mean-square error (MSE) of the maximum likelihood estimator.

(d) Reading through a statistics book, you find a highly curious estimator

$$\hat{x}_{\mathrm{JS}}(\mathbf{y}) = \mathbf{y} - (n-2)\frac{\mathbf{y}}{\|\mathbf{y}\|_2^2}.$$

Show that

$$\mathrm{MSE}_{\hat{x}_{\mathrm{JS}}}(\mathbf{x}) = \alpha \mathrm{MSE}_{\hat{\mathbf{x}}_{\mathrm{ML}}}(\mathbf{x}) + \beta \mathbb{E}\left[\frac{1}{\|\mathbf{Y}\|_2^2}\right],$$

and find α and β.

Hint: Use this special case of Stein's lemma:

$$\mathbb{E}\left[(\mathbf{x}-\mathbf{Y})^\top \frac{\mathbf{Y}}{\|\mathbf{Y}\|_2^2}\right] = -(n-2)\mathbb{E}\left[\frac{1}{\|\mathbf{Y}\|_2^2}\right].$$

In fact, the curious estimator $\mathrm{MSE}_{\hat{x}_{\mathrm{JS}}}(\mathbf{x})$ *is known as the James-Stein estimator. If you have time, read up on it on Wikipedia.*

(e) We say that an estimator $\hat{\mathbf{x}}$ dominates another estimator $\hat{\mathbf{x}}'$ under MSE if $\mathrm{MSE}_{\hat{\mathbf{x}}}(\mathbf{x}) \leq \mathrm{MSE}_{\hat{\mathbf{x}}'}(\mathbf{x})$ for all $\mathbf{x}$ and the inequality is strict for some $\mathbf{x}$. An estimator is *admissible* if no other estimator dominates it; otherwise it is *inadmissible*.

Show that the maximum likelihood estimator $\mathrm{MSE}_{\hat{\mathbf{x}}_{\mathrm{ML}}}$ is inadmissible for $n > 2$.