EE5138R: Problem Set 5
Assigned: 16/02/15
Due: 06/03/15

1. BV Problem 4.1
   Solutions: The feasible set is the convex hull of $(0, \infty)$, $(0, 1)$, $(2/5, 1/5)$, $(1, 0)$ and $(-\infty, 0)$.
   (a) $x^* = (2/5, 1/5)$
   (b) Unbounded below
   (c) $X_{\text{opt}} = \{(0, x_2) : x_2 \geq 1\}$
   (d) $x^* = (1/3, 1/3)$
   (e) $x^* = (1/2, 1/6)$. This is optimal because it satisfies $2x_1 + x_2 = 7/6 > 1$, $x_1 + 3x_2 = 1$, and
   $$\nabla f_0(x^*) = (1, 3)$$
   is perpendicular to the line $x_1 + 3x_2 = 1$.

2. (Optional) "Hello World" in CVX. Use CVX to verify the optimal values you obtained (analytically) for exercise 4.1 in Convex Optimization.
   I strongly encourage you to install CVX on Matlab and try this out so that you're familiar in using CVX in your own research.

3. BV Problem 4.7
   Solutions:
   (a) The domain of the objective is convex because $f_0(x)$ is convex. The sublevel sets are convex because $f_0(x)/(c^T x + d) \leq \alpha$ iff $c^T x + d > 0$ and $f_0(x) \leq \alpha(c^T x + d)$.
   (b) Suppose x is feasible in the original problem. Define $t = 1/(c^T x + d)$ (a positive number), $y = x/(c^T x + d)$. Then $t > 0$ and it is easily verified that t, y are feasible in the transformed problem, with the objective value $g_0(y, t) = f_0(x)/(c^T x + d)$.
   Conversely, suppose y, t are feasible for the transformed problem. We must have $t > 0$, by definition of the domain of the perspective function. Define $x = y/t$. We have $x \in \mathbf{dom}\, f_i$ for $i = 0, 1, \ldots, m$ (again, by definition of perspective). x is feasible in the original problem, because
   $$f_i(x) = g_i(y, t)/t \leq 0, \quad i = 1, \ldots, m, \qquad Ax = A(y/t) = b.$$
   From the last equality, $c^T x + d = (c^T y + dt)/t = 1/t$, and hence,
   $$t = \frac{1}{c^T x + d}, \qquad \frac{f_0(x)}{c^T x + d} = t f_0(x) = g_0(y, t).$$
   Therefore x is feasible in the original problem, with the objective value $g_0(y, t)$. In conclusion, from any feasible point of one problem we can derive a feasible point of the other problem, with the same objective value.

(c) The problem is clearly quasiconvex. The convex formulation as above is

$$\min g_0(y,t)$$

subject to

$$\begin{aligned} g_i(y,t) &\le 0, \quad i=1,\dots,m \\ Ay &= bt \\ \tilde{h}(t,t) &\le -1 \end{aligned}$$

where g_i is the perspective of f_i and $\tilde{h}$ is the perspective of $-h$. For the example, we have the equivalent problem

$$\min \frac{1}{m}\mathbf{tr}(tF_0 + y_1F_1 + \dots, +y_nF_n) \quad \text{s.t.} \quad \det(tF_0 + y_1F_1 + \dots + y_nF_n)^{1/m} \ge 1$$

with domain

$$\{(y,t) : t > 0, tF_0 + y_1F_1 + \dots + y_nF_n \succ 0\}$$

4. BV Problem 4.8 (a)–(c)

To help you, the optimal value for (a) is

$$p^* = \begin{cases} +\infty & b \notin \mathcal{R}(A) \\ \lambda^T b & c = A^T\lambda \text{ for some } \lambda \\ -\infty & \lambda \text{ otherwise} \end{cases}.$$

For part (b),

$$p^* = \begin{cases} \lambda b & c = a\lambda \text{ for some } \lambda \le 0 \\ -\infty & \lambda \text{ otherwise} \end{cases}$$

For part (c),

$$p^* = l^T c^+ + u^T c^-$$

where $c_i^+ = \max\{c_i, 0\}$ *and similarly for* c_i^-.

Solutions:

(a) There are three possibilities

- The problem is infeasible $b \notin \mathcal{R}(A)$. The optimal value is ∞.
- The problem is feasible, and c is orthogonal to the nullspace of A. We can decompose c as

  $$c = A^T\lambda + \hat{c}, \quad A\hat{c} = 0$$

  Here $\hat{c}$ is the component in the nullspace of A and $A^T\lambda$ is the component orthogonal to the nullspace. If $\hat{c} = 0$, then on the feasible set the objective function reduces to a constant:

  $$c^Tx = \lambda^T Ax + \hat{c}^T x = \lambda^T b.$$

  The optimal value is $\lambda^T b$. All feasible solutions are optimal.
- The problem is feasible, and c is not in the range of A^T and $\hat{c} \ne 0$. The problem is unbounded $p^* = -\infty$. To verify this, note that $x = x_0 - t\hat{c}$ feasible for all t; as t goes to infinity, the objective value decreases unboundedly.

So we obtain the first part.

(b) This problem is always feasible. The vector c can be decomposed into a component parallel to a and a component orthogonal to a:

$$c = a\lambda + \hat{c}.$$

with $a^T\hat{c} = 0$.

- If $\lambda > 0$, the problem is unbounded below. Choose $x = -ta$ and let t go to infinity:

$$c^T x = -tc^T a = -t\lambda\|a\|_2^2 \to -\infty.$$

  and

$$a^T x - b = -ta^T a - b \leq 0$$

  for large t, so x is feasible for large t. Intuitively, by going very far in the direction $-a$, we find feasible points with arbitrarily negative objective values.
- If $\hat{c} \neq 0$, the problem is unbounded below. Choose $x = ba - t\hat{c}$ and let $t \to \infty$.
- If $c = a\lambda$ for some $\lambda \leq 0$, the optimal value is $c^T ab = \lambda b$.

So we obtain the second part.

(c) The objective and the constraints are separable: The objective is a sum of terms $c_i x_i$, each dependent on one variable only; each constraint depends on only one variable. We can therefore solve the problem by minimizing over each component of x independently. The optimal x_i^* minimizes $c_i x_i$ subject to the constraint $l_i \leq x_i \leq u_i$. If $c_i > 0$, then $x_i^* = l_i$; if $c_i < 0$, then $x_i^* = u_i$; if $c_i = 0$, then any x_i in the interval $[l_i, u_i]$ is optimal. Therefore, the optimal value of the problem is

$$p^* = l^T c^+ + u^T c^-$$

where $c_i^+ = \max\{c_i, 0\}$ and $c_i^- = \max\{-c_i, 0\}$.

5. BV Problem 4.9
Solution: Make a change of variables $y = Ax$. The problem is equivalent to

$$\min_y \; c^T A^{-1} y \quad \text{s.t.} \quad y \preceq b$$

If $A^{-T}c \preceq 0$, the optimal solution is $y = b$, with $p^* = c^T A^{-1} b$. Otherwise, the LP is unbounded below.

6. BV Problem 4.11
Solutions:

(a) Equivalent to LP

$$\min_{t,x} \; t \quad \text{s.t.} \quad Ax - b \preceq t\mathbf{1}, Ax - b \succeq -t\mathbf{1}$$

(b) Equivalent to LP

$$\min_{s,x} \; \mathbf{1}^T s \quad \text{s.t.} \quad Ax - b \preceq s, Ax - b \succeq -s$$

Assume x is fixed in this problem, and we optimize only over s. The constraints say that

$$-s_k \leq a_k^T x - b_k \leq s_k.$$

The objective function of the LP is separable, so we achieve the optimum over s by choosing

$$s_k = |a_k^T x - b_k|$$

and obtain the optimal value $p^*(x) = \|Axb\|_1$. Therefore optimizing over t and s simultaneously is equivalent to the original problem.

(c) Equivalent to the LP

$$\min_{x,y} \; \mathbf{1}^T y \quad \text{s.t.} \quad -y \preceq Ax - b \preceq y, -\mathbf{1} \preceq x \preceq \mathbf{1}$$

(d) Equivalent to the LP

$$\min_{x,y} \ \mathbf{1}^T y \quad \text{s.t.} \quad -y \preceq x \preceq y, -\mathbf{1} \preceq Ax - b \preceq \mathbf{1}$$

Another good solution is to write $x = x^+ - x^-$ and to express the problem as

$$\min_{x^+,x^-} \ \mathbf{1}^T x^+ + \mathbf{1}^T x^- \quad \text{s.t.} \quad -\mathbf{1} \preceq A(x^+ - x^-) - b \preceq \mathbf{1}, x^+, x^- \succeq 0.$$

(e) Equivalent to the LP

$$\min_{x,y,t} \ \mathbf{1}^T y + t \quad \text{s.t.} \quad -y \preceq Ax - b \preceq y, -\mathbf{1}t \preceq x \preceq t\mathbf{1}$$

7. (Optional) BV Problem 4.12
Solution: This can be formulated as the LP

$$\min \ C = \sum_{i,j=1}^{n} c_{ij} x_{ij}$$

subject to

$$b + \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = 0, \qquad , i = 1, \ldots, n$$
$$l_{ij} \leq x_{ij} \leq u_{ij}.$$

8. BV Problem 4.21(a)
Solutions: If $A \succ 0$, the solution is

$$x^* = -\frac{1}{\sqrt{c^T A^{-1} c}} A^{-1} c, \qquad p^* = -\|A^{-1/2} c\|_2$$

This can be shown as follows. We make a change of variables $y = A^{1/2}x$, and write $\tilde{c} = A^{-1/2}c$. With this new variable the optimization problem becomes

$$\min_{y} \ \tilde{c}^T y, \quad \text{s.t.} \quad y^T y \leq 1$$

The answer is $y^* = -\tilde{c}/\|\tilde{c}\|_2$.
In the general case, we can make a change of variables based on the eigenvalue decomposition

$$A = Q\text{diag}(\lambda)Q^T = \sum_{i=1}^{n} \lambda_i q_i q_i^T.$$

We define $y = Qx$, $b = Qc$, and express the problem as

$$\min \ \sum_{i=1}^{n} b_i y_i, \quad \text{s.t.} \quad \sum_{i=1}^{n} \lambda_i y_i^2 \leq 1$$

If $\lambda_i > 0$ for all i, the problem reduces to the case we already discussed. Otherwise, we can distinguish several cases.

- $\lambda_n < 0$: The problem is unbounded below. By letting $y_n \to \pm\infty$, we can make any point feasible.
- $\lambda_n = 0$: If for some i, $b_i \neq 0$ and $\lambda_i = 0$, the problem is unbounded below.

- $\lambda_n = 0$, and $b_i = 0$ for all i with $\lambda_i = 0$. In this case we can reduce the problem to a smaller one with all $\lambda_i > 0$.

9. (Optional) BV Problem 4.23
Solution: We can rewrite the the l_4 norm approximation problem as

$$\min_{y,z} \sum_{i=1}^{m} z_i^2$$

subject to

$$a_i^T x - b_i = y_i, \quad y_i^2 \le z_i, \quad i = 1, \ldots, m.$$

This is exactly a QCQP.

10. BV Problem 4.28
Solutions:

(a) The objective function is a maximum of convex function, hence convex. We can write the problem as

$$\min\ t \quad \text{s.t.} \quad \frac{1}{2} x^T P_i x + q^T x + r \le t, i = 1, \ldots, K, \quad Ax \preceq b$$

which is a QCQP in the variables x and t.

(b) For given x, the supremum of $x^T \Delta P x$ over $-\gamma I \preceq \Delta P \preceq \gamma I$ is given by

$$\sup_{-\gamma I \preceq \Delta P \preceq \gamma} x^T \Delta P x = \gamma x^T x$$

Therefore we can express the robust QP as

$$\min\ \frac{1}{2} x^T (P_0 + \gamma I) x + q^T x + r, \quad \text{s.t.} \quad Ax \preceq b$$

which is a QP.

(c) For given x, the quadratic objective function is

$$\frac{1}{2} \left(x^T P_0 x + \sup_{\|u\|_2 \le 1} \sum_{i=1}^{K} u_i (x^T P_i x) \right) + q^T x + r = \frac{1}{2} x^T P_0 x + \frac{1}{2} \left(\sum_{i=1}^{K} (x^T P_i x)^2 \right)^{1/2} + q^T x + r.$$

This is a convex function of x: each of the functions $x^T P_i x$ is convex since $P_i \succeq 0$. The second term is a composition $h(g_1(x), \ldots, g_K(x))$ of $h(y) = \|y\|_2$ with $g_i(x) = x^T P_i x$. The functions g_i are convex and nonnegative. The function h is convex and, for $y \in \mathbf{R}_+^K$, nondecreasing in each of its arguments. Therefore the composition is convex.
The resulting problem can be expressed as

$$\min\ \frac{1}{2} x^T P_0 x + \|y\|_2 + q^T x + r$$

subject to

$$\frac{1}{2} x^T P_i x \le y_i, \quad i = 1, \ldots, K, \qquad Ax \preceq b$$

which can be further reduced to an SOCP

$$\min\ u + t$$

subject to

$$\left\|\begin{bmatrix} P_0^{1/2}x \\ 2u - 1/4 \end{bmatrix}\right\|_2 \leq 2u + 1/4, \quad \left\|\begin{bmatrix} P_i^{1/2}x \\ 2y_i - 1/4 \end{bmatrix}\right\|_2 \leq 2y_i + 1/4, \quad i = 1, \ldots, K, \quad \|y\|_2 \leq t, \quad Ax \preceq b.$$

The variables are x, u, t, and $y \in \mathbf{R}^K$.

Note that if we square both sides of the first inequality, we obtain

$$x^T P_0 x + (2u - 1/4)^2 \leq (2u + 1/4)^2$$

i.e., $x^T P_0 x \leq 2u$. Similar, the other constraints are equivalent to $\frac{1}{2} x^T P_i x \leq y_i$.

11. BV Problem 4.40(a)-(b)
Solution:

(a) The LP can be expressed as

$$\min_x \; c^T x + d \quad \text{s.t.} \quad \mathbf{diag}(Gx - h) \preceq 0, Ax = b$$

(b) With $P = WW^T$ and $W \in \mathbf{R}^{n \times r}$, the QP can be expressed as

$$\min_{x \in \mathbf{R}^n, t \in \mathbf{R}} \; t + 2q^T x + r \quad \text{s.t.} \quad \begin{bmatrix} I & W^T x \\ x^T W & tI \end{bmatrix} \succeq 0, \mathbf{diag}(Gx - h) \preceq 0, Ax = b$$

(c) With $P_i = W_i W_i^T$ and $W_i \in \mathbf{R}^{n \times r_i}$, the QCQP can be expressed as

$$\min_{x \in \mathbf{R}^n, t_i \in \mathbf{R}, i \in [m]} \; t_0 + 2q_0^T x + r_0$$

subject to

$$t_i + 2q_i^T x + r_i \leq 0, i \in [m], \quad \begin{bmatrix} I & W_i^T x \\ x^T W_i & t_i I \end{bmatrix} \succeq 0, i \in [m], \quad Ax = b.$$

(d) The SOCP can be expressed as

$$\min_x \; c^T x \quad \text{s.t.} \quad \begin{bmatrix} (c_i^T x + d_i)I & A_i x + b_i \\ (Ax_i + b_i)^T & (c_i^T x + d_i)I \end{bmatrix} \succeq 0, i \in [N], Fx = g$$

By the result in the hint, the constraint is equivalent with $\|A_i x + b_i\|_2 < c_i^T x + d_i$ when $c_i^T x + d_i > 0$. We have to check the case $c_i^T x + d_i = 0$ separately. In this case, the LMI constraint means $A_i x + b_i = 0$, so we can conclude that the LMI constraint and the SOC constraint are equivalent.

12. (Optional) BV Problem 4.43(a)-(c)
Solution:

(a) We use the property that $\lambda_1(x) \leq t$ if and only if $A(x) \preceq tI$. We minimize the maximum eigenvalue by solving the SDP

$$\min_{t,x} \; t \quad \text{s.t.} \quad A(x) \preceq tI$$

(b) $\lambda_1(x) \leq t_1$ if and only if $A(x) \preceq t_1 I$ and $\lambda_m(A(x)) \geq t_2$ if and only if $A(x) \succeq t_2 I$ so we can minimize $\lambda_1 - \lambda_m$ by solving

$$\min_{t_1, t_2, x} \; t_1 - t_2 \quad \text{s.t.} \quad t_2 I \preceq A(x) \preceq t_1 I$$

(c) We first note that the problem is equivalent to

$$\min\ \lambda/\gamma \quad \text{s.t.} \quad \gamma I \preceq A(x) \preceq \lambda I \tag{1}$$

if we take as domain of the objective $\{(\lambda, \gamma) : \gamma > 0\}$. This problem is quasiconvex, and can be solved by bisection: The optimal value is less than or equal to α if and only if the inequalities

$$\lambda \leq \gamma\alpha, \quad \gamma I \preceq A(x) \preceq \lambda I, \quad \gamma > 0$$

(with variables γ, λ, x) are feasible.
Following the hint we can also pose the problem as the SDP

$$\min\ t \quad \text{s.t.} \quad I \preceq sA_0 + y_1 A_1 + \ldots + y_n A_n \preceq tI, s \geq 0 \tag{2}$$

We now verify more carefully that the two problems are equivalent. Let p^* be the optimal value of (1), and p^*_{sdp} is the optimal value of the SDP (2).
Let λ/γ be the objective value of (1), evaluated at a feasible point (γ, λ, x). Define $s = 1/\gamma, y = x/\gamma, t = \lambda/\gamma$. This yields a feasible point in (2), with objective value $t = \lambda/\gamma$. This proves that $p^* \geq p^*_{\text{sdp}}$.
Now suppose that s, y, t are feasible in (2). If $s > 0$, then $\gamma = 1/s, x = y/s, \lambda = t/s$ are feasible in (1) with objective value t. If $s = 0$, we have

$$I \preceq y_1 A_1 + \ldots + y_n A_n \preceq tI$$

Choose $x = \tau y$ with τ sufficiently large so that $A(\tau y) \succeq A_0 + \tau I \succ 0$. We have

$$\lambda_1(\tau y) \leq \lambda_1(0) + \tau t, \quad \lambda_m(\tau y) \geq \lambda_m(0) + \tau.$$

The first inequality is justified as follows:

$$\begin{aligned} A(\tau y) &= A_0 + \tau(y_1 A_1 + \ldots + y_n A_n) \\ &\preceq A_0 + \tau t I \\ &\preceq \lambda_1(0) I + \tau t I \\ &= (\lambda_1(0) + \tau t) I \end{aligned}$$

By using the fact that $\lambda_1(x) \leq t_1$ if and only if $A(x) \leq t_1 I$, we recover the first inequality.
Hence, for τ sufficiently large

$$\kappa(x_0 + \tau y) \leq \frac{\lambda_1(0) + \tau t}{\lambda_m(0) + \tau}$$

Letting τ go to infinity, we can construct feasible points in (1), with objective value arbitrarily close to t. We conclude that $t \geq p^*$ if (s, y, t) are feasible in (2). Minimizing over t yields $p^*_{\text{sdp}} \geq p^*$.