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1. EXERCISE 4.12

(a) According to above information, we can get

$\pi^{(k)} P = \lambda_k \pi^{(k)} \Leftrightarrow \lambda_k \pi^{(k)} = \pi^{(k)} \cdot P$, then

$\lambda_k^n \pi^k = \lambda_k^{n-1} \pi^{(k)} P = \cdots \pi^k [P^n]$.

It means, $\lambda_k^n \pi_j^{(k)} = \sum_i \pi_i^{(k)} P_{ij}^{(n)}$ As desired.

(b) with the proof of (a), we can get that

$$\left|\lambda_k^n \pi_j^{(k)}\right| = \left|\sum_i \pi_i^{(k)} P_{ij}^{(n)}\right|$$

$$|\lambda_k^n| = \frac{|\sum_i \pi_i^{(k)} P_{ij}^{(n)}|}{|\pi_j^k|}$$

As for $\pi_j^{(k)}$, we can let $|\pi_j^k| = \max\{\pi_i^k\}$

Then we take this value into last equation, we can get

$$|\lambda_k^n| \le \frac{\sum_{i=1}^{M} \pi_j \cdot P_{ij}^n}{\pi_j^k} = \sum_{i=1}^{M} P_{ij}^n \le M$$

(c) With similar method in (b), we can prove like this

$$\lambda_k \pi_j^k = \sum_i \pi_i^k P_{ij}$$

$$|\lambda_k| = \frac{|\sum_i \pi_i^k P_{ij}|}{|\pi_j^k|} \le \sum_i P_{ij} = 1$$

so $|\lambda_k| \le 1$

2. EXERCISE 4.16

(a) According to the statement, $[[A] - \lambda v \pi]^2 = [A^2] - \lambda v\pi[A] - \lambda v \pi[A] + \lambda^2 v \pi v \pi$

$$= [A^2] - \lambda[A] \cdot v\pi - \lambda v \pi[A] + \lambda^2 v\pi$$

based on $\begin{cases} \pi v = 1 \\ \lambda[A] = \lambda \end{cases}$ $\Longrightarrow$

$$= [A^2] - \lambda^2 v\pi - \lambda^2 v\pi + \lambda^2 v\pi$$

$$= [A^2] - \lambda^2 v\pi$$

(b) We can use similar method to analyse in part (b)

$$[[A^n] - \lambda^n v\pi][[A] - \lambda v\pi]$$

$$= [A]^{n+1} - \lambda^n v\pi[A] - \lambda[A^n] v\pi + \lambda^{n+1} v\pi v\pi \quad (\pi v = 1,\ \lambda[A] = \lambda)$$

$$= [A^{n+1}] - \lambda^{n+1} v\pi$$

(c) Using the induction:

Firstly, when we find $n=1$, it satisfy

$$[[A] - \lambda v \pi] = [A] - \lambda v \pi$$

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Secondly, we suppose when $n=k$, it satisfy

$$[[A]-\lambda v\pi]^{k} = [A^{k}] - \lambda^{k} v\pi$$

Finally, we let $n=k+1$

$$[[A]-\lambda v\pi]^{k+1} = [[A]-\lambda v\pi]^{k}[[A]-\lambda v\pi]$$

$$= [A^{n+1}] - \lambda^{n+1} v\pi \quad \text{(using the idea from b)}$$

it satisfy the ~~suppos~~ assumption

In conclusion, it proves.

3. EXERCISE 4.17

(a) Through the proof of slides, we know $P = \begin{bmatrix} P_T & P_{TR} \\ 0 & P_R \end{bmatrix}$

As for the ~~upe~~ upper right brock $P_{TR} = \begin{bmatrix} P_{1,t+1} \cdots P_{1,t+r} \\ P_{t,t+1} \cdots P_{t,t+r} \end{bmatrix}$

$$[P^{2}] = \begin{bmatrix} P_T & P_{TR} \\ 0 & P_R \end{bmatrix}^{2} = \begin{bmatrix} P_T^2 & P_T P_{TR} + P_{TR} P_R \\ 0 & P_R^2 \end{bmatrix} = \begin{bmatrix} P_T^2 & P_R + P_T \\ 0 & P_R^2 \end{bmatrix}$$

$$\Downarrow$$

$$[P]^{n} = \begin{bmatrix} P_T & P_{TR} \\ 0 & P_R \end{bmatrix}^{n} = \begin{bmatrix} [P_T^{n}] & [P_R^{n} + P_T^{n}] \\ 0 & [P_R^{n}] \end{bmatrix}$$

we can set $P_x^{n} = P_R + R_T$, which means whatever it turns out to be.

As desired $[P^{n}] = \begin{bmatrix} [P_T^{n}] & [P_x^{n}] \\ 0 & [P_R^{n}] \end{bmatrix}$

(b) As for this part, $q_i = \sum_{t \le j \le t+r} P_{ij}^{t}$

Firstly, we know $P_{ij} > 0$ in the transition matrix for an aperiodic markov unichain

, which means $P_{ij}^{t} > 0$ for any t in $(t \le j \le t+r)$

In conclusion $q_i = \sum_{t \le j \le t+r} P_{ij}^{t} > 0$

(c) Based on part (b) ~~$q = \min(q_i)$~~, $q_i = \sum_{t \le j \le t+r} P_{ij}^{t}$

we can get $1-q_i$ is the probability of the ~~t~~ chain will still be transient state after t transistions.

let $q = \min(q_i)$, we can get $(1-q) = \max(q_i)$

$P_{ij}^{nt} = (1-q)^{n}$

As desired.

(d) According to the statement of (d), we know $\pi = (\pi_T, \pi_R)$, it is also the left ~~eor~~ eigenvector of P, with eigenvalue 1.

$$\pi P = \lambda \pi$$

$$\Downarrow$$

$$(\pi_T, \pi_R) P^n = (\pi_T, \pi_R)$$

$$[\pi_T, \pi_R]\begin{bmatrix} [P_T^n] & [P_x^n] \\ 0 & [P_R^n] \end{bmatrix} = \begin{bmatrix} P_T^n \pi_T, & P_x^n \pi_T + P_R^n \cdot \pi_R \end{bmatrix}$$

In order to satisfy, $\begin{cases} P_T^n \pi_T = \pi_T \\ P_x^n \pi_T + P_R^n \pi_R = \pi_R \end{cases}$

we can not let $P_T^n = 1$ (based on aperiodic union chain)

so, only when $\pi_T = 0$, it does satisfy,

$\pi_T P_x^n + P_R^n \pi_R = \pi_R \Rightarrow P_R^n \pi_R = \pi_R$

it can get (π_R is a left eigen vector of P_R)

And it must be positive

(e) As for this part (d), we know π is unique, through the conclusion from slides, we know there must exist $\lambda = 1$ value

for other lambda, we $|\lambda_2| < 1$, $[P^n]$ approaches the steady state matrix, then we get $\lim_{n\to\infty}[P^n] = e\pi$, $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

4. (a) For this matrix, we know

~~transient 3~~

~~recurrent 1, 2, 1, 4, 5, 6~~

~~recurre~~

class = {3}, {2, 4, 6} {1, 5}

(b) $\begin{cases} \text{transient} = 3 \\ \text{recurrent} = 1, 2, 4, 5, 6 \end{cases}$

(c) For recurrent class {2, 4, 6}

the period of this is 3

For recurrent class {1, 5}

the period of this is 2

(f). From the state 1 stating, we can calculate

$$Q = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

As for $\det(Q - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ \frac{1}{2} & \frac{1}{2}-\lambda \end{vmatrix} = \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$

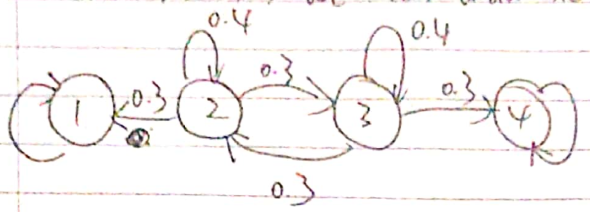
we can get $\lambda_1 = 1$ $\lambda_2 = -\frac{1}{2}$

As for $\lim_{n\to\infty}([P^n]_{11} - \pi_1)$, we should use the conclusion from slides, to choose second largest lambda

so $\lim_{n\to\infty} -\frac{1}{n}\log([P^n]_{11} - \pi_1) = -\log\frac{1}{2} = \log 2$

5. EXERCISE

For $M=4$, we can draw the graph



$$P=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since state 1 and state 4 are recurrent states,

we can get below equations

$$\begin{cases} V_1 = 0 \\ V_2 = 1 + 0.3V_1 + 0.4V_2 + 0.3V_3 \\ V_3 = 1 + 0.3V_2 + 0.4V_3 + 0.3V_4 \\ V_4 = 0 \end{cases}$$

$$V_2 = V_3 = \frac{10}{3}$$

So the expected number (V_2, V_3) is $\frac{10}{3}$