

EE5137 Stochastic Processes: Problem Set 4

Assigned: 05/02/21, Due: 19/02/21

There are seven (7) non-optional problems in this problem set. You have two weeks to do this problem set. There are not many problems in Poisson processes as we have not covered enough, so I'm giving some practice problems on probability. You can use your spare time to read Chapter 2 of Gallager's book.

1. For a Poisson process, which of the following is/are true?

- (i) $\{N(t) \geq n\} = \{S_n \leq t\}$;
- (ii) $\{N(t) < n\} = \{S_n > t\}$;
- (iii) $\{N(t) \leq n\} = \{S_n \geq t\}$;
- (iv) $\{N(t) > n\} = \{S_n < t\}$.

Solution: It is easy to see the only parts (i) and (ii) are true by drawing a timeline. More rigorously, first we show that (i) is true as follows. First, realize that if $S_n \leq t$ then we also have $S_1 \leq S_2 \leq \dots \leq S_n \leq t$. This means that there are at least n arrivals (at time S_1, S_2, \dots, S_n) from time 0 to t , which leads to $N(t) \geq n$. Conversely, if $N(t) \geq n$, we have $S_n \leq t$. Combining these facts, we have $\{N(t) \geq n\} = \{S_n \leq t\}$. Since (i) is true, (ii) is true by taking complement.

Now, it is also easy to see that a realization that $s_1 < s_2 < \dots < s_{n-1} < s_n < t < s_{n+1}$ belongs to $\{N(t) \leq n\}$, but not belongs to $\{S_n \geq t\}$. It follows that (iii) is not true. Since (iii) is not true, (iv) is not true by taking complement.

2. An athletic facility has 5 tennis courts. Pairs of players arrive at the courts and use a court for an exponentially distributed time with mean 40 minutes. Suppose a pair of players arrives and finds all courts busy and k other pairs waiting in queue. What is the expected waiting time to get a court?

Solution: As long as the pair of players is waiting, all five courts are occupied by other players. When all five courts are occupied, the time until a court is freed up is exponentially distributed with mean $40/5 = 8$. For our pair of players to get a court, a court must be freed up $k + 1$ times. Thus, the expected waiting time is $8(k + 1)$.

3. (Optional) Exercise 2.3 (Gallager's book) The purpose of this exercise is to give an alternate derivation of the Poisson distribution for $N(t)$, the number of arrivals in Poisson process up to time t . Let λ be the rate of the process.

- (a) Find the conditional probability $\Pr\{N(t) = n | S_n = \tau\}$ for all $\tau \leq t$.
- (b) Using the Erlang density for S_n , use (a) to find $\Pr\{N(t) = n\}$.

Solution:

- (a) The condition $S_n = \tau$ means that the epoch of the n th arrival is τ . Conditional on this, the event $\{N(t) = n\}$ for some $t > \tau$ means there have been no subsequent arrivals from τ to t . In other words, it means that the $(n + 1)$ -st interarrival time, X_{n+1} exceeds $t - \tau$. This interarrival time is independent of S_n and thus,

$$\Pr\{N(t) = n | S_n = \tau\} = \Pr\{X_{n+1} > t - \tau\} = e^{-\lambda(t-\tau)}, \quad \text{for } t > \tau. \quad (1)$$

(b) We find $\Pr\{N(t) = n\}$ simply by averaging (1) over S_n .

$$\Pr\{N(t) = n\} = \int_0^\infty \Pr\{N(t) = n | S_n = \tau\} f_{S_n}(\tau) d\tau \quad (2)$$

$$= \int_0^t e^{-\lambda(t-\tau)} \frac{\lambda^n \tau^{n-1} e^{-\lambda\tau}}{(n-1)!} d\tau \quad (3)$$

$$= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \int_0^t \tau^{n-1} d\tau \quad (4)$$

$$= \frac{(\lambda t)^n e^{-\lambda t}}{n!}. \quad (5)$$

4. Prove that the Geometric distribution

$$p_X(k) = (1-p)^{k-1}p, \quad k \in \mathbb{N} = \{1, 2, \dots\}$$

has the memoryless property.

In fact, the Geometric distribution is the only distribution supported on \mathbb{N} that is memoryless. This is analogous to the fact that the Exponential distribution is the only distribution supported on $[0, \infty)$ that is memoryless.

Solution: To show that any distribution has the memoryless property, we need to show that

$$\Pr(X > x+t) = \Pr(X > x) \Pr(X > t). \quad (6)$$

For the Geometric distribution $p_X(k)$, we have

$$\Pr(X > x) = \sum_{k=x+1}^{\infty} (1-p)^{k-1}p = p \cdot \frac{(1-p)^x}{1-(1-p)} = (1-p)^x$$

Now clearly, the memoryless property in (6) holds true.

5. Let X_n denote a Binomial random variable with n trials and probability of success p_n . If $np_n \rightarrow \lambda$ as $n \rightarrow \infty$, show that for any fixed $i \in \mathbb{N} \cup \{0\}$,

$$\Pr(X_n = i) \rightarrow \frac{e^{-\lambda} \lambda^i}{i!}, \quad \text{as } n \rightarrow \infty.$$

Solution: Fix $i \in \{0, 1, \dots, n\}$. We have

$$\begin{aligned} \Pr(X_n = i) &= \binom{n}{i} p_n^i (1-p_n)^{n-i} \\ &= \frac{n!}{i!(n-i)!} p_n^i (1-p_n)^{n-i} \\ &= \frac{n(n-1)(n-2)\dots(n-i+1)}{i!} p_n^i (1-p_n)^{n-i} \\ &= \frac{(np_n)((n-1)p_n)((n-2)p_n)\dots((n-i+1)p_n)}{i!} (1-p_n)^n (1-p_n)^{-i} \end{aligned}$$

We note that $np_n \rightarrow \lambda$ and so $(n-j)p_n \rightarrow \lambda$ for each $j = 0, 1, \dots, i-1$. Furthermore, $(1-p_n)^n \rightarrow e^{-\lambda}$ because $np_n \rightarrow \lambda$. More precisely here, for fixed $\epsilon > 0$ and for n large enough, $|np_n - \lambda| < \epsilon$. Hence, $(1-p_n)^n \leq (1 - (\lambda - \epsilon)/n)^n \rightarrow e^{-\lambda + \epsilon}$. Similarly, $(1-p_n)^n \geq e^{-\lambda - \epsilon}$. By the arbitrariness of $\epsilon > 0$, we have $(1-p_n)^n \rightarrow e^{-\lambda}$. Finally, since i is fixed $(1-p_n)^i \rightarrow 1$. Putting these facts together, we get

$$\Pr(X_n = i) \rightarrow \frac{\lambda^i e^{-\lambda}}{i!} \quad \text{as } n \rightarrow \infty.$$

6. For a non-negative random variable X , show that for each $n \in \mathbb{N}$,

$$\mathbb{E}[X^n] = \int_0^\infty nx^{n-1}F_X^c(x) \, dx$$

where F_X^c is the complementary CDF of X .

Solution: Since X is a non-negative random variable, so is X^n . Hence, by the layer cake representation,

$$\mathbb{E}[X^n] = \int_0^\infty F_{X^n}^c(x) \, dx$$

However, we have

$$F_{X^n}^c(x) = \Pr(X^n \geq x) = \Pr(X \geq x^{1/n}) = F_X^c(x^{1/n})$$

where the second equality holds because X is a nonnegative random variable. Hence,

$$\mathbb{E}[X^n] = \int_0^\infty F_X^c(x^{1/n}) \, dx$$

By letting $y = x^{1/n}$, we get $\frac{dy}{dx} = \frac{1}{n}x^{1/n-1}$ so $dx = nx^{-1/n+1}dy$ which means that $dx = ny^{-1}y^n dy = ny^{n-1}dy$. Thus,

$$\mathbb{E}[X^n] = \int_0^\infty ny^{n-1}F_X^c(y) \, dy$$

as desired.

7. If X is a random variable with the property that $\Pr(0 \leq X \leq a) = 1$, show that

$$\text{Var}(X) \leq a^2/4.$$

Solution: Define the function $g(t) = \mathbb{E}[(X - t)^2]$. Then clearly, $g'(t^*) = 0$ implies that $t^* = \mathbb{E}X$. Furthermore, $g''(t^*) = 2 > 0$ so the function g achieves its unique minimum at $t^* = \mathbb{E}X$. This means that

$$\text{Var}(X) = g(\mathbb{E}X) \leq g\left(\frac{a}{2}\right) = \mathbb{E}\left[\left(X - \frac{a}{2}\right)^2\right] = \frac{1}{4}\mathbb{E}[(2X - a)^2]$$

Since $X \in [0, a]$ almost surely, we know that $2X - a \in [-a, a]$ almost surely. This means that $\mathbb{E}[(2X - a)^2] \leq a^2$. This proves the claim. It is also easy to see that the random variable that ensures that equality is met is X such that $X = 0$ and $X = a$ with probability $1/2$.