EE5137 Semester 1 2018/9: Quiz 1 (Total 24 points)

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Matriculation Number: XX

Score: 24/24

You have 1.0 hour for this quiz. There are FOUR (4) printed pages. You're allowed 1 sheet of handwritten notes. Please provide *careful explanations* for all your solutions.

1. (8 points) [Distribution Functions] Which of the following functions is a cumulative distribution function (CDF)? For those which are, compute the probability density function (PDF). For those which are not, explain what fails.

   (a)
   $$F_X(x) = \begin{cases} 1 - e^{-x^2} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

   Solution: Yes. This is a distribution function

   - $\lim_{x\to-\infty} F_X(x) = 0$;
   - $\lim_{x\to+\infty} F_X(x) = \lim_{x\to+\infty} 1 - e^{-x^2} = 1$;
   - $F_X(x)$ is continuous at every point including 0 because on the left it takes the value 0 and on the right $1 - e^{-0^2} = 0$.

   The PDF is
   $$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = 2xe^{-x^2}$$
   for $x \geq 0$ and 0 otherwise. This is known as the *Rayleigh distribution* with scale $\sigma = 1/\sqrt{2}$. See `https://en.wikipedia.org/wiki/Rayleigh_distribution`.

   (b)
   $$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{1}{3} & 0 < y \leq \frac{1}{2} \\ 1 & y > \frac{1}{2}. \end{cases}$$

   Solution: This "CDF" is not a distribution function. It is not right-continuous. Note that $F_Y(0) = 0$ while $\lim_{y\downarrow 0} F_Y(y) = 1/3$. Also $F_Y(1/2) = 1/3$ and $\lim_{y\downarrow 1/2} F_Y(y) = 1$.

2. (8 points) [Strengthened Union Bound]

Let $A_1, \ldots, A_n$ be arbitrary events. Prove that

$$\Pr\left\{\bigcup_{i=1}^n A_i\right\} \le \min_{1\le k\le n}\left(\sum_{i=1}^n \Pr\{A_i\} - \sum_{i=1\,:\,i\ne k}^n \Pr\{A_i \cap A_k\}\right).$$

Hint: For any two sets C and D,

$$C = (C \cap D) \cup (C \cap D^c)$$

Solution: Using the hint, we have

$$\bigcup_{i=1}^n A_i = \left[\left(\bigcup_{i=1}^n A_i\right) \cap A_k\right] \cup \left[\left(\bigcup_{i=1}^n A_i\right) \cap A_k^c\right].$$

for any $1 \le k \le n$. But this is equivalent to

$$\bigcup_{i=1}^n A_i = A_k \cup \left[\bigcup_{i=1}^n (A_i \cap A_k^c)\right].$$

Taking probabilities,

$$\begin{aligned}
\Pr\left\{\bigcup_{i=1}^n A_i\right\} &= \Pr\left\{A_k \cup \left[\bigcup_{i=1}^n (A_i \cap A_k^c)\right]\right\} \\
&\le \Pr\{A_k\} + \sum_{i=1}^n \Pr\{A_i \cap A_k^c\} \\
&= \Pr\{A_k\} + \sum_{i=1, i\ne k}^n \Pr\{A_i \cap A_k^c\}, \qquad \text{(because } A_k \cap A_k^c = \emptyset\text{)} \\
&= \Pr\{A_k\} + \sum_{i=1, i\ne k}^n [\Pr\{A_i\} - \Pr(A_i \cap A_k\}] \\
&= \sum_{i=1}^n \Pr\{A_i\} - \sum_{i=1, i\ne k}^n \Pr(A_i \cap A_k\}
\end{aligned}$$

Since the bound holds for all $1 \le k \le n$, we can minimize the right-hand-side to yield

$$\Pr\left\{\bigcup_{i=1}^n A_i\right\} \le \min_{1\le k\le n}\left(\sum_{i=1}^n \Pr\{A_i\} - \sum_{i=1\,:\,i\ne k}^n \Pr\{A_i \cap A_k\}\right).$$

as desired.

3. [Conditional Expectations] (8 points)

In this problem, we will calculate the expectation of a geometric random variable using the formula for iterated expectations. Let N be a geometric random variable with parameter p, i.e., N is the number of coin flips until Head appears and $\Pr(\text{Heads}) = p$. In other words $p_N(n) = (1-p)^{n-1}p$ for $n = 1, 2, \ldots$. Define the random variable

$$Y = \begin{cases} 1 & \text{first flip is Heads} \\ 0 & \text{else} \end{cases}$$

(i) Calculate $\mathbb{E}[N|Y = y]$ for $y = 1$.

Solution: If $Y = 1$, then we know that the first flip is a head. Thus the number of coin flips to a head is exactly 1, i.e.,

$$\mathbb{E}[N|Y = 1] = 1.$$

(ii) Calculate $\mathbb{E}[N|Y = y]$ for $y = 0$ in terms of $\mathbb{E}[N]$.

Solution: If $Y = 0$, then we know that the first flip is a tail. Thus the number of coin flips to a head it $\mathbb{E}[N] + 1$ (as the process is memoryless), i.e.,

$$\mathbb{E}[N|Y = 0] = \mathbb{E}[N] + 1.$$

(iii) Now use the law of iterated expectations to deduce $\mathbb{E}[N]$.

Solution: We know that

$$\mathbb{E}[N] = \Pr(Y = 0)\mathbb{E}[N|Y = 0] + \Pr(Y = 1)\mathbb{E}[N|Y = 1].$$

Substituting the above values, we have

$$\mathbb{E}[N] = (1-p) \times (\mathbb{E}[N] + 1) + p \times 1.$$

Solving this for $\mathbb{E}[N]$, we have

$$\mathbb{E}[N] = \frac{1}{p}$$

which was to be expected.