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Subject：　　Stochastic process

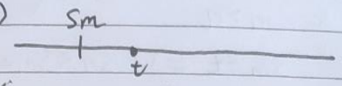
Assignment: Homework Three

1. For a Poisson process, which of the following is / are true?

(1) $\{N(t) \geq n\} = \{S_n \leq t\}$

This following is TRUE

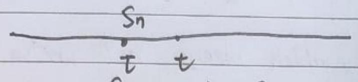
Proof: suppose $N(t) \geq n$, which means the number of arriva up to m $(m \geq n)$ in time (t)



so $S_m \leq t \Rightarrow S_n \leq S_m \leq t$

$\Rightarrow \{N(t) \geq n\} \subseteq \{S_n \leq t\}$ --- ①

~~Reversely~~ Reversely: $S_n \leq t$, which means the n^{th} arrival occured at τ, $\tau \leq t$



$\Rightarrow N(\tau) = n \Rightarrow N(t) \geq N(\tau) = n$

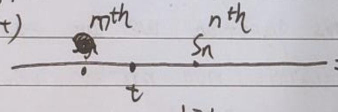
$\Rightarrow \{S_n \leq t\} \subseteq \{N(t) \geq n\}$ --- ②

combined with ① and ②: this following proves TRUE

(2) $\{N(t) < n\} = \{S_n > t\}$

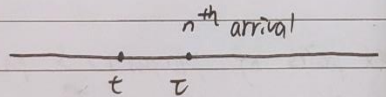
This following is TRUE

Proof: suppose $N(t) < n$, which means the number of arrival up to m $(m < n)$ in time (t)



$\Rightarrow$ $S_m < t < S_n$ $\quad S_n > t$ --- ①

Reversely: $S_n > t$, which means the n^{th} arrival occured at τ, $\tau > t$



$\Rightarrow N(\tau) = n \Rightarrow N(t) < N(\tau) = n$

$\Rightarrow \{S_n > t\} \subseteq \{N(t) < n\}$ --- ②

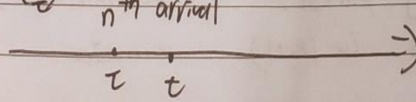
combined with ① and ②: this following proves TRUE

(3) $\{N(t) \leq n\} = \{S_n \geq t\}$

This following is FALSE

For example, when we consider this siuation (when $N(t) = n$)

Because ~~this~~ is special case, we can get, the number of arrival up to n in time (t)



$\Rightarrow S_n = \tau < t$

is self-conflict with $\{S_n \geq t\}$,

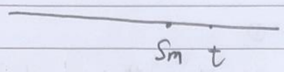
to sum up, this following is not correct,

(4) $\{N(t)>n\} = \{S_n < t\}$

This following is TRUE

Proof: suppose $N(t)>n$, which means the number of arrival is m in time (t), so $m>n$

$S_m \le t \Rightarrow S_n < S_m \le t$

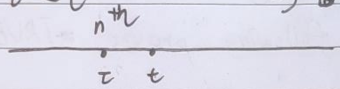


$\Rightarrow S_n < t$

so $\{N(t)>n\} \subseteq \{S_n < t\}$ --- ①

Reversely, suppose $S_n < t$, which means n^{th} arrival occured at τ

$\tau < t$ $\Rightarrow N(\tau) = n \Rightarrow N(t) > N(\tau) = n$



$\Rightarrow N(t) > n$

so $\{S_n < t\} \subseteq \{N(t)>n\}$ --- ②

combined with ① ②, this following proves TRVE

2. Through the statement of this question, we can easily find that 5 tennis courts are same siuation, not matter which pairs of players in any court. They are i.i.d. events

Now we focus on the desired pairs of players, 5 pairs of players are 5 courts, and at the same time, k pairs are waiting, which means, this pair should wait $(k+1)$ pairs of players finish their activities. In this situation. at least $(k+1)$ pairs.

Then pay attention to $(k+1)$ pairs of players, they are also same types events (exponentially distribution), and these activities are i.i.d. events. From the above statement, the mean time per pair of players is 40 minutes. Therefore, the total waiting time is $40(k+1)$ minutes.

But for 5 courts, the 5 courts are same situation,

In conclusion, the expected waiting time to get a court is $8(k+1)$ minutes

$$\frac{40(k+1)}{5} = 8(k+1)$$

4. For Geometric distribution

$$P_X(k) = (1-p)^{k-1} \cdot p, \quad k \in N = \{1, 2, \ldots\}$$

let $Pr\{X > t+x\} = 1 - Pr\{X \le t+x\} = 1 - \sum_{1}^{t+x} (1-p)^{k-1} \cdot p$

$$= 1 - p \cdot \frac{1-(1-p)^{t+x}}{1-(1-p)} = 1 - [1-(1-p)^{t+x}]$$

$$= (1-p)^{t+x}$$

Similarly $Pr\{X > t\} = 1 - Pr\{X \le t\} = 1 - \sum_{1}^{t} (1-p)^{k-1} \cdot p$

$$= 1 - [1-(1-p)^{t}] = (1-p)^{t}$$

$$Pr\{X > x\} = 1 - Pr\{X \le x\} = 1 - \sum_{1}^{x} (1-p)^{k-1} \cdot p$$

$$= 1 - [1-(1-p)^{x}] = (1-p)^{x}$$

$$Pr\{X > t+x\} = (1-p)^{t+x} = (1-p)^{t} \cdot (1-p)^{x} = Pr\{X > t\} \cdot Pr\{X > x\}$$

In conclusion, this geometric distribution has the memoryless property.

5. let X_n denote a Binomial random variable, with n trials and probability of success p_n. $E[X] = np_n = \lambda$, so $p_n = \frac{\lambda}{n}$

so $Pr(X_n = i) = \lim_{n\to\infty} \binom{n}{i} \cdot p_n^{\,i} (1-p_n)^{n-i} = \lim_{n\to\infty} \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i}$

we can rewrite this expression

$$\lim_{n\to\infty} \binom{n}{i} \left(\frac{\lambda}{n}\right) \left(1-\frac{\lambda}{n}\right)^{n-i} = \lim_{n\to\infty} \left[\frac{n\cdot(n-1)(n-2)\cdots(n-i+1)}{i!} \cdot \frac{\lambda^i}{n^i} \left(1-\frac{\lambda}{n}\right)^{n-i}\right]$$

$$= \lim_{n\to\infty} \frac{\lambda^i}{i!} \cdot \frac{n\cdot(n-1)(n-2)\cdots(n-i+1)}{n^i} \left(1-\frac{\lambda}{n}\right)^{-i} \left(1-\frac{\lambda}{n}\right)^{n}$$

owing to $\lim_{n\to\infty} \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-i+1}{n} \left(1-\frac{\lambda}{n}\right)^{-i} = 1$

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{n} = e^{-\lambda}$$

so $Pr(X_n = i) = \lim_{n\to\infty} \binom{n}{i} \cdot \left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i} = \frac{\lambda^i}{i!} e^{-\lambda} = \frac{e^{-\lambda} \cdot \lambda^i}{i!}$

as desired

6. From the statement in this question, we can get this random variable is non-negative.

Firstly, we can calculate $E[X^n] = \int_0^{\infty} x^n \cdot f_X(x)\,dx$

Based on $F_X^c(x) = 1 - F_X(x) = P(X \geq x) = \int_x^{\infty} f_X(x)\,dx$,

put this value into

so $\int_0^{\infty} n x^{n-1} F_X^c(x)\,dx = \int_0^{\infty} n \cdot x^{n-1} \cdot \int_x^{\infty} f_X(x)\,dx\,dx$

and now, we can change the order of integration

$$\int_0^{\infty} f_X(x) \int_x^{\infty} n \cdot x^{n-1}\,dx\,dx$$

$$= \int_0^{\infty} f_X(x) \int_x^{\infty} (x^n)'\,dx\,dx$$

$$= \int_0^{\infty} x^n \cdot f_X(x)\,dx$$

this result is same from the basic definition

To sum up, $E[X^n] = \int_0^{\infty} n \cdot x^{n-1} F_X^c(x)\,dx$

7. Through the information from the question, we can get $Pr(0 \leq X \leq a) = 1$, which means this random variables convergence in probability 1: $\lim_{n\to\infty} Pr\{|X - \bar{X}_n| \leq \varepsilon\} = 1$

we can find X must include $[0, a]$

for any x, and any $\bar{X}_n$ (mean expectation of x) must satisfy $-\frac{a}{2} \leq X - \bar{X}_n \leq \frac{a}{2}$

$$E[X] - \frac{a}{2} \leq X \leq E[X] + \frac{a}{2} \quad Pr\{|X - E(X)| \geq \tfrac{a}{2}\} = 1$$

Then, we use chebyshev's inequality

$$Pr(|X - E(X)| \geq \tfrac{a}{2}) \leq \frac{Var(X)}{(\frac{a}{2})^2}$$

$$Var(X) \geq (\tfrac{a}{2})^2 Pr(|X - E(X)| \geq \tfrac{a}{2})$$

$$\geq \frac{a^2}{4}$$