Independence

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Tossing two dice

Let an experiment consist of tossing two dice. For this experiment the sample space is

$$\mathcal{S} = \{(1,1), (1,2), \ldots, (6,6)\}$$

that is, $\mathcal{S}$ consists of the 36 ordered pairs formed from the numbers 1 to 6. Define the following events:

$$A = \{\text{doubles appear}\} = \{(1,1), \ldots, (6,6)\}$$

$$B = \{\text{the sum is between 7 and 10}\}$$

$$C = \{\text{the sum is 2 or 7 or 8}\}.$$

The probabilities can be calculated by counting among the 36 possible outcomes. We have

$$p(A) = 1/6, \quad p(B) = 1/2, \quad p(C) = 1/3.$$

Furthermore,

$$p(A \cap B \cap C) = p(\text{the sum is 8, composed of double 4s}) = 1/36 = p(A)p(B)p(C).$$

However,

$$p(B \cap C) = p(\text{sum equals 7 or 8}) = 11/36 \neq p(B)p(C).$$

So the requirement $p(A \cap B \cap C) = p(A)p(B)p(C)$ is not a strong enough condition to guarantee pairwise independence.

Letters Let the sample space $\mathcal{S}$ consists of the 3! permutations of letters a, b, and c along with the three triples of each letter. Thus,

$$\begin{bmatrix} aaa & bbb & ccc \\ abc & bca & cba \\ cab & bac & cab \end{bmatrix}$$

Furthermore, let each element of S have probability $1/9$. Define

$$A_i = \{i\text{th place in the triple is occupied by a}\}.$$

Then

$$p(A_i) = 1/3, \quad i = 1, 2, 3,$$

and

$$p(A_1 \cap A_2) = p(A_1 \cap A_3) = p(A_2 \cap A_3) = 1/9,$$

so that A_is are pairwise independent. But

$$p(A_1 \cap A_2 \cap A_3) = 1/9 \neq p(A_1)p(A_2)p(A_3),$$

so the A_is are not mutually independent.

Mutually independent

The "true" independence requires a fairly strong condition:

Definition 1. *A collection of events $A_1, \ldots, A_n$ are mutually independent if any sub collection $A_{i_1}, \ldots, A_{i_k}$, we have*

$$p\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} p(A_{i_j}).$$

[Reference]

Statistical Inference, Casella and Berge.