ETH Zürich, FS 2019
D-MATH
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Coordinator
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Applied Stochastic Processes

Exercise sheet 10

Exercise 10.0 Frog Markov chain
Let $(X_n)_{n\geq 0}$ be the Markov chain with state space $\{1,2\}$, initial distribution $\mu = (\mu_1, \mu_2)$ and transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}, \text{ where } 0 < p, q \leq 1$$

(a) Compute $P_\mu[X_n = i]$ for every n.

(b) Deduce the value of $\lim_{n\to\infty} P_\mu[X_n = i]$.

Exercise 10.1 Let E be a a countable state space and fix $x \in E$. We consider a Markov chain $(X_n)_{n\geq 0}$ under P_x. Assume that $P_x[H_x^+ < \infty] = 1$. Define $H_x^{(1)} = H_x^+$ and

$$H_x^{(n+1)} = \min \{k > 0;\ X_{T_n+k} = x\} \text{ for } n \geq 1,$$

where $T_i = H_x^{(1)} + \cdots + H_x^{(i)}$.

(a) Show that under P_x, the random variables $(H_x^{(i)})_{i\geq 1}$ are i.i.d.

(b) Show that the process defined by $N_t = \sum_{1\leq i\leq t} 1_{\{X_i=x\}}$ is a renewal process.

Exercise 10.2 Let us consider the reflected random walk, that is, the Markov chain with state space $\mathbb{N}_0$ and transition probability given by $p_{0,1} = 1$ and $p_{x,x+1} = \alpha$, $p_{x,x-1} = 1 - \alpha$ for $x \geq 1$. Show that for $\alpha \leq 1/2$ all the states are recurrent, and for $\alpha > 1/2$ all the states are transient.

Exercise 10.3 Snakes and ladders.
A simple game of 'snakes and ladders' is played on a board of nine squares.
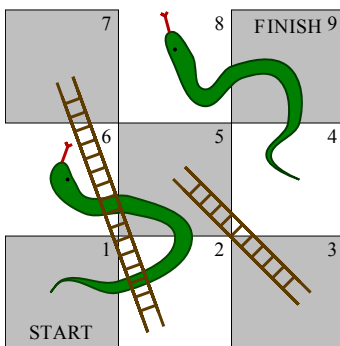
At each turn a player tosses a fair coin and advances one or two places according to whether the coin lands heads or tails. If you land at the foot of a ladder you climb to the top, but if you land at the head of a snake you slide down to the tail.

(a) How many turns on average does it take to complete the game?
   Hint: *Call $k_i = E_i[H_9]$ and find some relations between the k_i for $i \in \{1, \ldots, 9\}$.*

(b) What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

Submission deadline: 13:15, May 9.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

Students	Time & Date	Room	Assistant
A-K	Thu 09-10	HG D 7.2	Maximilian Nitzschner
L-Z	Thu 12-13	HG D 7.2	Daniel Contreras

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/