National University of Singapore

EE5139/EE6139
Homework 8

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Exercise 8.1 Deterministic Channel (EE5139)

Consider a memoryless channel that takes pairs of bits as input and produces two bits as output as follows: $00 \to 01$, $01 \to 10$, $10 \to 11$, $11 \to 00$ (to read: input $\to$ output). Let (X_1, X_2) denote the two input bits and (Y_1, Y_2) the two output bits.

a.) Calculate the mutual information $I(X_1, X_2; Y_1, Y_2)$ for a given joint PMF of the four pairs of input bits. You can express your answer in terms of

$$\begin{aligned} p_{00} &= \Pr(X_1 = 0, X_2 = 0) \\ p_{10} &= \Pr(X_1 = 1, X_2 = 0) \\ p_{01} &= \Pr(X_1 = 0, X_2 = 1) \\ p_{11} &= \Pr(X_1 = 1, X_2 = 1) \end{aligned}$$

b.) Show that the channel mutual information is 2 and indicate the units.

c.) Show that, surprisingly, $I(X_1; Y_1) = 0$ for the capacity-achieving distribution of the input you derived in part (b) (that is, information is only transferred by considering both bits). **Hint:** Find the joint pmf of X_1 and Y_1.

Exercise 8.2 Symmetric Channel (all)

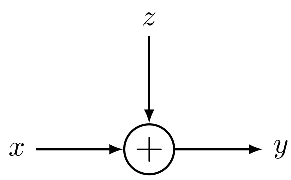
For two positive integers k and m, let $(k \bmod m)$ be the *remainder* when k is divided by m. Find the capacity of the m-input discrete memoryless channel in which

$$Y = (X + Z) \mod m,$$

where $X \in \{0, 1, \ldots, m-1\}$, $\Pr[Z = 1] = \frac{3}{4}$, and $\Pr[Z = 0] = \frac{1}{4}$.

Exercise 8.3 Additive noise channel (EE5139)

Find the channel capacity of the following discrete memoryless channel:



where $\Pr[Z = 0] = \Pr[Z = a] = \frac{1}{2}$. The alphabet for x is $\mathcal{X} = \{0, 1\}$. Assume that Z is independent of X. Observe that the channel capacity depends on the value of a.

Exercise 8.4 Channel Mutual Information (EE5139)

Let X and Z be independent random variables taking values on $\{1, \ldots, n\}$ and $\{0, 1\}$, respectively, with $p_X(i) = q_i$ (for each i) and $p_Z(1) = p$. Define the random variable $Y := X \cdot Z$.

a.) Write $H(Y)$ in terms of $H(X)$ and $H(Z)$.

b.) Find p and $\boldsymbol{q} = (q_1, \ldots, q_n)$ that maximize $H(Y)$.

c.) Suppose X and Y are input and output of a DMC channel. For a fixed $p \in [0, 1]$, what is the channel mutual information $I(p)$?

Exercise 8.5 Using two channels at once (EE6139)

Consider two discrete memoryless channels $(X_1, p(y_1|x_1), Y_1)$ and $(X_2, p(y_2|x_2), Y_2)$ with capacities C_1 and C_2, respectively. A new channel $(X_1 \times X_2, p(y_1|x_1) \times p(y_2|x_2), Y_1 \times Y_2)$ is formed in which $x_1 \in X_1$ and $x_2 \in X_2$ are sent simultaneously, resulting in y_1, y_2. Find the channel mutual information of this channel.

Exercise 8.6 Concatenation of channels (EE6139)

We concatenate n binary symmetric channels as depicted below.

$$X_0 \longrightarrow \boxed{\mathsf{BSC}_1} \longrightarrow X_1 \longrightarrow \boxed{\mathsf{BSC}_2} \longrightarrow X_2 \longrightarrow \cdots \longrightarrow \boxed{\mathsf{BSC}_n} \longrightarrow X_n$$

Let the crossover probability of all of the BECs to be p. Show that the concatenated channel is equivalent to a BSC with crossover probability

$$\frac{1}{2}\left[1 - (1 - 2p)^n\right]$$

and show that $\lim_{n\to\infty} I(X_0; X_n) = 0$ regardless of the distribution of X_0.