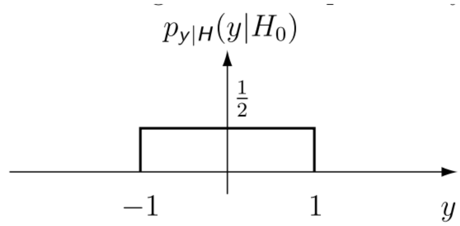
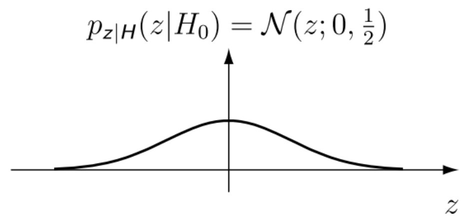
EE5137 Stochastic Processes: Problem Set 11
Assigned: 02/04/21, Due: 09/04/21

There are four (4) non-optional problems in this problem set. This is the last problem set.

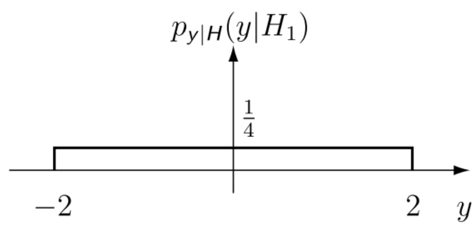
1. Exercise 8.15 (Gallager's book)

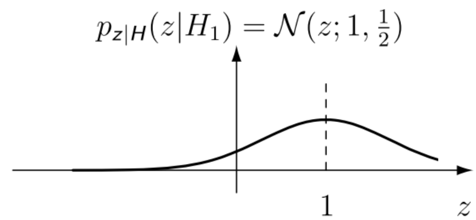
2. Consider the problem of deciding between two equally likely hypotheses based on two random variables, Y and Z. Specifically, under the null hypothesis H_0, Y and Z are independent and have the following conditional probability densities:





Under the alternative hypothesis H_1, Y and Z are independent and have the following conditional probability densities:





   (a) Specify a decision rule for deciding between H_0 and H_1, based on Y and Z, in order to minimize the probability of error.

   (b) Compute $P_D = \Pr(\text{decide } H_1|H_1)$ and $P_F = \Pr(\text{decide } H_1|H_0)$ for the decision rule in part (a), expressing your answer in terms of

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,\mathrm{d}t.$$

3. Let Y_1, Y_2 and Y_3 be three IID Bernoulli random variables with $\Pr(Y_i = 1) = p$ for $i \in \{1, 2, 3\}$. This means that $\Pr(Y_i = y) = p^y(1-p)^{1-y}$ for $y \in \{0, 1\}$. It is known that p can take on two values 1/2 or 2/3. In this problem, we consider the hypothesis test

$$\mathsf{H}_0 : p = 1/2, \qquad \mathsf{H}_1 : p = 2/3$$

based on $(Y_1, Y_2, Y_3) \in \{0, 1\}^3$.

(i) (5 points) Let $T = Y_1 + Y_2 + Y_3$ be the number of ones in the random vector (Y_1, Y_2, Y_3). Let P_0 and P_1 be the distributions of Y_1, Y_2, and Y_3 under hypothesis H_0 and H_1 respectively. Write down the likelihood ratio

$$L(Y_1, Y_2, Y_3) := \frac{P_0(Y_1, Y_2, Y_3)}{P_1(Y_1, Y_2, Y_3)}$$

in terms of T. Hence, argue that T is a sufficient statistic for deciding between H_0 and H_1.

(ii) (4 points) Clearly $T \in \{0, 1, 2, 3\}$. Evaluate the values of the likelihood ratio in terms of T.

(iii) (3 points) What is the best probability of missed detection $P_1(\text{declare } \mathsf{H}_0)$ if we allow the probability of false alarm $P_0(\text{declare } \mathsf{H}_1)$ to be $1/8$? What is the corresponding test in terms of T?

(iv) (7 points) What is the best probability of missed detection $P_1(\text{declare } \mathsf{H}_0)$ if we allow the probability of false alarm $P_0(\text{declare } \mathsf{H}_1)$ to be $1/4$? What is the corresponding test in terms of T?
Hint: You need to consider randomized tests here.

4. A binary random variable X with prior $p_X(\cdot)$ takes values in $\{-1, 1\}$. It is observed via n separate sensors; Y_i denotes the observation at sensor i. The $Y_1, \ldots, Y_n$ are conditionally independent given X, i.e.,

$$p_{Y_1,\ldots,Y_n|X}(y_1, \ldots, y_n|x) = \prod_{i=1}^{n} p_{Y_i|X}(y_i|x).$$

A *local* decision $\hat{x}_i(y_i) \in \{-1, 1\}$ about the value of X is made at each sensor.

(a) In this part of the problem, each sensor sends its local decision to a fusion center. The fusion center combines the local decisions from all sensors to produce a global decision $\hat{x}(\hat{x}_1, \ldots, \hat{x}_n)$. Consider the special case in which: i) $p_X(1) = p_X(-1) = 1/2$; ii) $Y_i = X + W_i$, where $W_1, \ldots, W_n$ are independent and each uniformly distributed over the interval $[-2, 2]$; and iii) the local decision rule is a simple thresholding of the observation, i.e.,

$$y_i \underset{\hat{x}_i(y_i)=-1}{\overset{\hat{x}_i(y_i)=1}{\gtrless}} 0.$$

Determine the minimum probability of error decision rule, $\hat{x}(\cdot, \ldots, \cdot)$, at the fusion center.

In the remainder of the problem, there is no fusion center. The prior $p_X(\cdot)$, observation model $p_{Y_i|X}(\cdot|x)$, $i = 1, 2$, and local decision rules $\hat{x}_i(\cdot)$, are no longer restricted as in part (a). However, we restrict our attention to the two-sensor case ($n = 2$).

Consider local decisions $\hat{x}_i(y_i)$, $i = 1, 2$, that minimize the expected cost, where the cost is defined for the two local rules jointly. Specifically, $C(\hat{x}_1, \hat{x}_2, x)$ is the cost of deciding $\hat{x}_1$ at sensor 1 and deciding $\hat{x}_2$ at sensor 2 when the true value of X is x. The cost C strictly increases with the number of errors made by the two sensors, but is not necessarily symmetric. Assuming conditional independence, the expected cost is

$$\begin{aligned}\mathbb{E}\Big[C(\hat{X}_1, \hat{X}_2, X)\Big] &= \mathbb{E}_{Y_1,X}\Big[\mathbb{E}_{Y_2|Y_1,X}\big[C(\hat{X}_1(Y_1), \hat{X}_2(Y_2), X) \mid Y_1, X\big]\Big] \\ &= \mathbb{E}_{Y_1,X}\Big[\mathbb{E}_{Y_2|X}\big[C(\hat{X}_1(Y_1), \hat{X}_2(Y_2), X) \mid X\big]\Big]\end{aligned}$$

You can define another cost function

$$\tilde{C}(x, \hat{x}_1(y_1)) = \mathbb{E}_{Y_2|X}[C(\hat{x}_1(y_1), \hat{X}_2(Y_2), X)|X = x]$$

(b) First, assume $\hat{x}_2(\cdot)$ is given. Show that the choice $\hat{x}_1^*(\cdot)$ for $\hat{x}_1(\cdot)$ that minimizes the expected (joint) cost is a likelihood ratio test of the form

$$\frac{p_{Y_1|X}(y_1|1)}{p_{Y_1|X}(y_1|-1)} \underset{\hat{x}_1^*(y_1)=-1}{\overset{\hat{x}_1^*(y_1)=1}{\gtrless}} \gamma_1,$$

where γ_1 is a threshold that depends on the rule $\hat{x}_2(\cdot)$. Determine the threshold γ_1.

(c) Assuming, instead, that $\hat{x}_1(\cdot)$ is given, determine the choice $\hat{x}_2^*(\cdot)$ for $\hat{x}_2(\cdot)$ that minimizes the expected joint cost.

(d) Consider a joint cost function $C(\hat{x}_1, \hat{x}_2, x)$ such that the cost is: 0 if both sensors making correct decisions; 1 if exactly one sensor makes an error; and L if both sensors make an error. Determine the value of L such that the optimal local decision rules at the two sensors are decoupled, i.e., the optimal threshold γ_1 does not depend on $\hat{x}_2^*(\cdot)$, and *vice versa*.

This was a question I designed for a quiz while I was a Ph.D. student at MIT.

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5. (Optional) Attempt all the hypothesis testing problems in the past year exam papers.