Exercise 8.1

a).

As for the mutual information $I(X_1, X_2; Y_1, Y_2)$, when $P_{00} = P_{01} = P_{10} = P_{11} = \frac{1}{4}$

we firstly calculate

$P_{00} = P_r(X_1=0, X_2=0) = \frac{1}{4}$ $\quad$ $P'_{00} = P_r(Y_1=0, Y_2=0) = \frac{1}{4}$

$P_{01} = P_r(X_1=0, X_2=1) = \frac{1}{4}$ $\quad$ $P'_{01} = P_r(Y_1=0, Y_2=1) = \frac{1}{4}$

$P_{10} = P_r(X_1=1, X_2=0) = \frac{1}{4}$ $\quad$ $P'_{10} = P_r(Y_1=1, Y_2=0) = \frac{1}{4}$

$P_{11} = P_r(X_1=1, X_2=1) = \frac{1}{4}$ $\quad$ $P'_{11} = P_r(Y_1=1, Y_2=1) = \frac{1}{4}$

$H(X) = 4 \times \frac{1}{4} \times \log 4 = 2$ $\quad$ $H(Y) = 4 \times \frac{1}{4} \times \log 4 = 2$

But for the joint PMF, we can express them like below table:

$P_{00\,00'} = 0$	$P_{00\,01'} = \frac{1}{4}$	$P_{00\,10'} = 0$	$P_{00\,11'} = 0$
$P_{01\,00'} = 0$	$P_{01\,01'} = 0$	$P_{01\,10'} = \frac{1}{4}$	$P_{01\,11'} = 0$
$P_{10\,00'} = 0$	$P_{10\,01'} = 0$	$P_{10\,10'} = 0$	$P_{10\,11'} = \frac{1}{4}$
$P_{11\,00'} = \frac{1}{4}$	$P_{11\,01'} = 0$	$P_{11\,10'} = 0$	$P_{11\,11'} = 0$

$$H(X;Y) = 4 \times \frac{1}{4} \cdot \log 4 = 2$$

$$I(X_1, X_2; Y_1, Y_2) = H(X_1, X_2) + H(Y_1, Y_2) - H(X_1, X_2; Y_1, Y_2)$$

$$= \cancel{\sum_{X_1 \in X} \sum_{X_2 \in X} P(X_1=i, X_2=j) + \sum_{Y_1 \in Y} \sum_{Y_2 \in Y} P(Y_1=m, Y_2=n)}$$

$$\cancel{= \sum_{X_1 \in X} \sum_{X_2 \in X} \sum_{Y_1 \in Y} \sum_{Y_2 \in Y}}$$

$$= \sum_{X_1 \in X} \sum_{X_2 \in X} P(X_1=i, X_2=j) \cdot \log \frac{1}{P(X_1=i, X_2=j)} + \sum_{Y_1 \in Y} \sum_{Y_2 \in Y} P(Y_1=m, Y_2=n) \cdot \log \frac{1}{P(Y_1=m, Y_2=n)}$$

$$- \sum_{X_1 \in X} \sum_{X_2 \in X} \sum_{Y_1 \in Y} \sum_{Y_2 \in Y} P(X_1=i, X_2=j, Y_1=m, Y_2=n) \cdot \log \frac{1}{P(X_1=i, X_2=j, Y_1=m, Y_2=n)}$$

(b) Based on the above calculations, we can get

$\cancel{H(X;Y)}$

$$\text{Capacity} = \max_{P_X \in P} I(X;Y)$$

we know it is concave function, only when $P_{00} = P_{01} = P_{10} = P_{11} = \frac{1}{4}$.

(uniform distribution)

we can get Capacity $= H(X) + H(Y) - H(X;Y)$

$= 2 + 2 - 2$

$= 2$ bits

(c). For $I(X_1;Y_1)$, $I(X_1;Y_1) = H(X_1) - H(X_1|Y_1) = H(X_1) + H(Y_1) - H(X_1;Y_1)$

$P_0 = Pr(X_1=0) = \frac{1}{2}$ $\quad$ $P_0' = Pr(Y_1=0) = \frac{1}{2}$

$P_1 = Pr(X_1=1) = \frac{1}{2}$ $\quad$ $P_0' = Pr(Y_1=1) = \frac{1}{2}$

$H(X_1) = 2\times\frac{1}{2}\log 2 = 1$ $\quad$ $H(Y_1) = 1$

$P_{00}' = Pr(X_1=0, Y_1=0) = \frac{1}{4}$ $\quad$ $P_{01}' = Pr(X_1=0, Y_1=1) = \frac{1}{4}$

$P_{10}' = Pr(X_1=1, Y_1=0) = \frac{1}{4}$ $\quad$ $P_{11}' = Pr(X_1=1, Y_1=1) = \frac{1}{4}$

$H(X_1;Y_1) = 4\times\frac{1}{4}\cdot\log 4 = 2$

$$I(X_1;Y_1) = H(X_1) + H(Y_1) - H(X_1;Y_1) = 1+1-2 = 0$$

Exercise 8.2

Channel capacity : $Y = (X+Z) \bmod m$,

where $Z = \begin{cases} Z=1 & , \frac{3}{4} \\ Z=0 & , \frac{1}{4} \end{cases}$, $X = 0 \cdots, m-1$

Z is independent of X

In this case, $H(Y|X) = H(Z|X) = H(Z) = \frac{1}{4}\log 4 + \frac{3}{4}\log\frac{4}{3}$

$= 2 - \frac{3}{4}\log 3$

Hence, the capacity of the channel is

$$C = \max_{P(X)} I(X;Y)$$
$$= \max_{P(X)} H(Y) - H(Y|X)$$
$$= \max_{P(X)} H(Y) - (2 - \frac{3}{4}\log 3)$$
$$= \log m - 2 + \frac{3}{4}\log 3$$

which is attained when Y has a uniform distribution, which occurs (by symmetry) when X has a uniform distribution.

Exercise 8.3

Followed the statement of this additive noise channel,

we know that Z is independent of X.

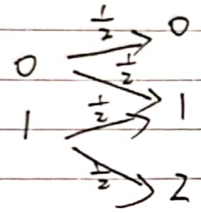
so, we write $Y = X + Z$ $\quad X \in \{0,1\}$, $Z \in \{0, a\}$

Here are these cases:

① when $a=0$, $Y = X \Rightarrow \max I(X;Y) = \max H(X) = 1$.

so the capacity is 1 bit

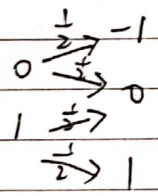
② when $a=1$, Y has three possible output values, 0, 1, 2



Followed the conclusion from BEC (binary erasure channel)

We can get $I(W_{BEC}) = 1 - \varepsilon = 1 - \frac{1}{2} = \frac{1}{2}$

③ when $a=-1$, Y also has three possible output values, $-1, 0, 1$



Similarily, we can get $I(W_{BEC}) = 1 - \frac{1}{2} = \frac{1}{2}$

④ When $a \neq 0, -1, 1$, Y has four possible output values, $0, 1, a, 1+a$

Therefore, when we know Y, we certainly know the X,

$\Downarrow$

$H(X|Y) = 0$

$\max(I(X;Y)) = \max(H(X)) = 1$

Exercise 8.4

(a). Following the statement of the question, we know $Y = X \cdot Z$

$$H(Y) = H(X \cdot Z) = \sum_{x_i \in X} \sum_{z_i \in Z} P(x_i, z_i) \log \frac{1}{P(x_i, z_i)}$$

$$= \sum_{x_i \in X} \sum_{z_i \in Z} P(x_i) \cdot P(z_i) \log \frac{1}{P(x_i) \cdot P(z_i)} \quad [P(x_i, z_i) = P(x_i) \cdot P(z_i)]$$

$$= \sum_{x_i \in X} \sum_{z_i \in Z} P(x_i) \cdot P(z_i) \log \frac{1}{P(x_i)} + \sum_{x_i \in X} \sum_{z_i \in Z} P(x_i) \cdot P(z_i) \log \frac{1}{P(z_i)}$$

$$= \sum_{x_i \in X} P(x_i) \log \frac{1}{P(x_i)} + \sum_{z_i \in Z} P(z_i) \log \frac{1}{P(z_i)}$$

$$= H(X) + H(Z)$$

(b) Because X and Z are independent, $\max H(Y) = \max H(X) + \max H(Z)$

For $\max H(X)$, we take $q = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n} \cdots \frac{1}{n})$

$$H(X) = n \cdot \frac{1}{n} \cdot \log \frac{1}{\frac{1}{n}} = \log n$$

For $\max H(Z)$, we take $p = \frac{1}{2}$

$$H(Z) = \frac{1}{2} \log \frac{1}{\frac{1}{2}} + \frac{1}{2} \log \frac{1}{\frac{1}{2}} = 1$$

so $\max(H(Y)) = \log n + 1$

(c) According to the definition of channel mutual information

$$I(P) = H(Y) - H(X)$$

$$= H(Z)$$

$$= h(P)$$

$$= p \log \frac{1}{p} + (1-p) \log \frac{1}{(1-p)}$$

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