EE5137 Stochastic Processes: Problem Set 9
Assigned: 19/03/21, Due: 26/03/21

There are five (5) non-optional problems in this problem set.

1. Exercise 4.12 (Gallager's book)

   Hint for Part (b): Choose j to maximize $|\pi_j^{(k)}|$ for the given k.

2. Exercise 4.16 (Gallager's book)

3. Exercise 4.17 (Gallager's book)

4. Consider the Markov chain whose transition probability matrix is

$$[P] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix}.$$

   The first row/column corresponds to state 0 and the last row/column corresponds to state 5 and so on.

   (a) (4 marks) Classify the states $\{0, 1, 2, 3, 4, 5\}$ into classes.

   (b) (4 marks) Identify the recurrent and transient states.

   (c) (4 marks) Compute the period of each recurrent class.

   (d) (3 marks) Identify the ergodic states.

   (e) (5 marks) If the chain starts from state 1, find the steady state probabilities in each of the states $(\pi_0, \pi_1, \ldots, \pi_5)$.

   (f) (5 marks) Evaluate

$$\lim_{n\to\infty} -\frac{1}{n} \log\left([P^n]_{11} - \pi_1\right),$$

   where $[M]_{ij}$ is the (i, j) element of the matrix $[M]$.

   I wanted to put the above question in the exam, but then I came up with a better idea.

5. A fly moves along a straight line in unit increments. At each time period, it moves one unit to the left with probabiliy 0.3, one unit to the right with probability 0.3 and stays in place with probability 0.4, independent of past movements. Two spiders are lurking at positions 1 and M; if a fly lands in positions 1 or M, it is captured by the spider and the process terminates. Let $j \in \{1, 2, \ldots, M\}$ be the position of the fly. The Markov chain $[P]$ is thus given by

$$p_{11} = 1 \quad p_{mm} = 1, \quad p_{ij} = \begin{cases} 0.3 & \text{if } |j - i| = 1 \\ 0.4 & \text{if } j = i \end{cases} \qquad \text{for} \quad i = 2, 3, \ldots, M - 1.$$

Write down a system of linear equations to deduce the expected number of steps the fly takes given it starts from state j before being captured by one of the two spiders. This is denoted as v_j. For $M = 4$, solve your equations to find v_2 and v_3.

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6. An auto insurance company classifies its customers in three categories: bad, satisfactory and preferred. No one moves from bad to preferred or from preferred to bad in one year. 40% of the customers in the bad category become satisfactory, 30% of those in the satisfactory category moves to preferred, while 10% become bad; 20% of those in the preferred category are downgraded to satisfactory.

   - (5 points) Write the state transition matrix for the model.
   - (10 points) What is the limiting fraction of customers in each of these categories, i.e., the fraction of bad, satisfactory, and preferred customers after many years?

   This is a past year exam problem.

7. (Optional) Exercise 4.11 (Gallager's book)

8. (Optional) Exercise 4.18 (Gallager's book)