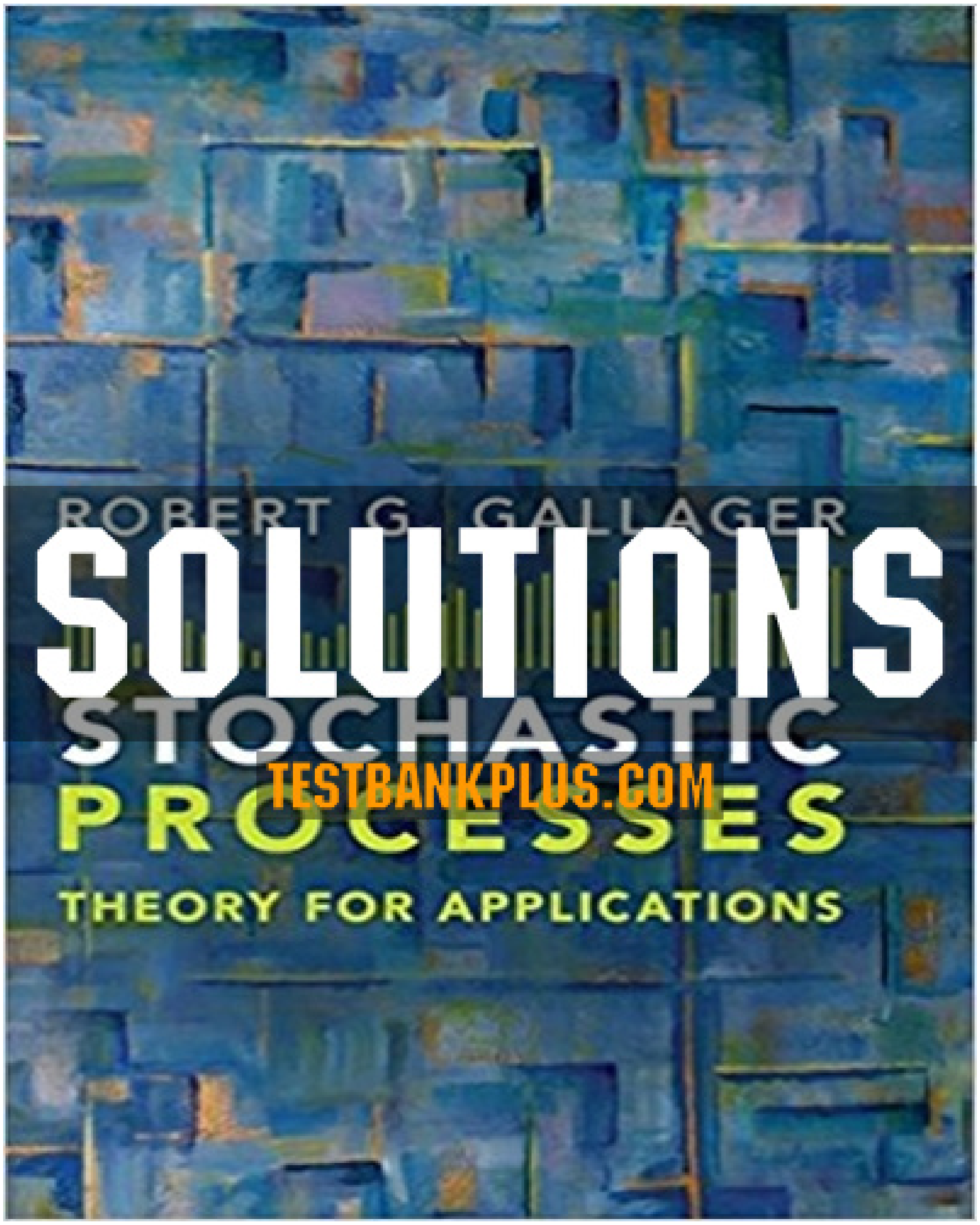
SOLUTIONS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A.2 Solutions for Chapter 2

Exercise 2.1: **a)** Find the Erlang density $\mathsf{f}_{S_n}(t)$ by convolving $\mathsf{f}_X(x) = \lambda\exp(-\lambda x)$, $x \geq 0$ with itself n times.

Solution: For $n = 2$, we convolve $\mathsf{f}_X(x)$ with itself.

$$\mathsf{f}_{S_2}(t) \;=\; \int_0^t \mathsf{f}_{X_1}(x)\mathsf{f}_{X_2}(t-x)\,dx \;=\; \int_0^t \lambda e^{-\lambda x}\lambda e^{-\lambda(t-x)}\,dx \;=\; \lambda^2 t e^{-\lambda t}.$$

For larger n, convolving $\mathsf{f}_X(x)$ with itself n times is found by taking the convolution $n-1$ times, *i.e.*, $\mathsf{f}_{S_{n-1}}(t)$, and convolving this with $\mathsf{f}_X(x)$. Starting with $n = 3$,

$$\begin{aligned}
\mathsf{f}_{S_3}(t) \;&=\; \int_0^t \mathsf{f}_{S_2}(x)\mathsf{f}_{X_3}(t-x)\,dx \;=\; \int_0^t \lambda^2 x e^{-\lambda x}\lambda e^{-\lambda(t-x)}\,dx \;=\; \frac{\lambda^3 t^2}{2}e^{-\lambda t}\\
\mathsf{f}_{S_4}(t) \;&=\; \int_0^t \frac{\lambda^3 x^2}{2}e^{-\lambda x}\cdot \lambda e^{\lambda(t-x)}\,dx \;=\; \frac{\lambda^4 t^3}{3!}e^{-\lambda t}.
\end{aligned}$$

We now see the pattern; each additional integration increases the power of λ and t by 1 and multiplies the denominator by $n-1$. Thus we hypothesize that $\mathsf{f}_{S_n}(t) = \frac{\lambda^n t^{n-1}}{n!}e^{-\lambda t}$. If one merely wants to verify the well-known Erlang density, one can simply use induction from the beginning, but it is more satisfying, and not that much more difficult, to actually derive the Erlang density, as done above.

b) Find the moment generating function of X (or find the Laplace transform of $\mathsf{f}_X(x)$), and use this to find the moment generating function (or Laplace transform) of $S_n = X_1 + X_2 + \cdots + X_n$.

Solution: The formula for the MGF is almost trivial here,

$$\mathsf{g}_X(r) = \int_0^\infty \lambda e^{-\lambda x}e^{rx}\,dx \;=\; \frac{\lambda}{\lambda - r} \qquad \text{for } r < \lambda.$$

Since S_n is the sum of n IID rv's,

$$\mathsf{g}_{S_n}(r) \;=\; \big[\mathsf{g}_X(r)\big]^n \;=\; \left(\frac{\lambda}{\lambda - r}\right)^n.$$

c) Find the Erlang density by starting with (2.15) and then calculating the marginal density for S_n.

Solution: To find the marginal density, $\mathsf{f}_{S_n}(s_n)$, we start with the joint density in (2.15) and integrate over the region of space where $s_1 \leq s_2 \leq \cdots \leq s_n$. It is a peculiar integral, since the integrand is constant and we are just finding the volume of the $n-1$ dimensional space in $s_1, \dots, s_{n-1}$ with the inequality constraints above. For $n = 2$ and $n = 3$, we have

$$\begin{aligned}
\mathsf{f}_{S_2}(s_2) \;&=\; \lambda^2 e^{-\lambda s_2}\int_o^{s_2} ds_1 \;=\; \left(\lambda^2 e^{-\lambda s_2}\right)s_2\\
\mathsf{f}_{S_3}(s_3) \;&=\; \lambda^3 e^{-\lambda s_3}\int_0^{s_3}\left[\int_0^{s_2} ds_1\right]ds_2 \;=\; \lambda^3 e^{-\lambda s_3}\int_0^{s_3} s_2\,ds_2 \;=\; \left(\lambda^3 e^{-\lambda s_3}\right)\frac{s_3^2}{2}.
\end{aligned}$$

The critical part of these calculations is the calculation of the volume, and we can do this inductively by guessing from the previous equation that the volume, given s_n, of the $n-1$ dimensional space where $0 < s_1 < \cdots < s_{n-1} < s_n$ is $s_n^{n-1}/(n-1)!$. We can check that by

$$\int_0^{s_n}\left[\int_0^{s_{n-1}}\cdots\int_0^{s_2} ds_1\ldots ds_{n-2}\right] ds_{n-1} \;=\; \int_0^{s_n}\frac{s_{n-1}^{n-2}}{(n-2)!}ds_{n-1} \;=\; \frac{s_n^{n-1}}{(n{-}1)!}.$$

This volume integral, multiplied by $\lambda^n e^{-\lambda s_n}$, is then the desired marginal density.

A more elegant and instructive way to calculate this volume is by first observing that the volume of the $n-1$ dimensional cube, s_n on a side, is s_n^{n-1}. Each point in this cube can be visualized as a vector $(s_1, s_2, \ldots, s_{n-1})$. Each component lies in $(0, s_n)$, but the cube doesn't have the ordering constraint $s_1 < s_2 < \cdots < s_{n-1}$. By symmetry, the volume of points in the cube satisfying this ordering constraint is the same as the volume in which the components $s_1, \ldots s_{n-1}$ are ordered in any other particular way. There are $(n-1)!$ different ways to order these $n-1$ components (*i.e.*, there are $(n-1)!$ permutations of the components), and thus the volume with the ordering constraints, is $s_n^{n-1}/(n-1)!$.

Exercise 2.2: **a)** Find the mean, variance, and moment generating function of $N(t)$, as given by (2.17).

Solution:

$$\mathsf{E}[N(t)] \;=\; \sum_{n=0}^{\infty}\frac{n(\lambda t)^n e^{-\lambda t}}{n!} \;=\; \lambda t\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}e^{-\lambda t}}{(n-1)!} \;=\; \lambda t\sum_{m=0}^{\infty}\frac{(\lambda t)^m e^{-\lambda t}}{m!} \;=\; \lambda t,$$

where in the last step, we recognized the terms in the sum as the Poisson PMF of parameter λt; since the PMF must sum to 1 the mean is λt as shown. Using the same approach, the second moment is $(\lambda t)^2 + \lambda t$, so the variance is λt. For the MGF,

$$\mathsf{E}\left[e^{rN(t)}\right] \;=\; \sum_{n=0}^{\infty}\frac{(\lambda t e^r)^n e^{-\lambda t}}{n!} \;=\; e^{-\lambda t}\sum_{n=0}^{\infty}\left[\frac{(\lambda t e^r)^n e^{-\lambda t e^r}}{n!}\right]e^{\lambda t e^r}.$$

Recognizing the term in brackets as the PMF of a Poisson rv of parameter λe^r in this expression, we get

$$\mathsf{E}\left[e^{rN(t)}\right] \;=\; e^{-\lambda t}e^{\lambda t e^r} \;=\; \exp[\lambda t(e^r - 1)].$$

b) Show by discrete convolution that the sum of two independent Poisson rv's is again Poisson.

Solution: Let X and Y be independent Poisson with parameters λ and μ respectively. Then

$$\begin{aligned}\mathsf{p}_{X+Y}(m) \;&=\; \sum_{n=0}^{m}\mathsf{p}_X(n)\mathsf{p}_Y(m-n) \;=\; e^{-\lambda-\mu}\sum_{n=0}^{m}\frac{\lambda^n\mu^{m-n}}{n!(m-n)!}\\ &=\; \frac{e^{-\lambda-\mu}}{m!}\sum_{n=0}^{m}\binom{m}{n}\lambda^n\mu^{m-n} \;=\; \frac{e^{-\lambda-\mu}(\lambda+\mu)^m}{m!},\end{aligned}$$

where we recognized the final sum as a binomial sum.

c) Show by using the properties of the Poisson process that the sum of two independent Poisson rv's must be Poisson.

Solution: For any $t, \tau > 0$ in a Poisson process we know that $N(t+\tau) = N(t) + \widetilde{N}(t, t+\tau)$ is Poisson and $\widetilde{N}(t, t+\tau)$ is Poisson in τ. Since $N(t)$ and $\widetilde{N}(t, \tau)$ are independent and t and τ are arbitrary positive numbers, the sum of 2 independent Poisson rv's is Poisson.

Exercise 2.3: The purpose of this exercise is to give an alternate derivation of the Poisson distribution for $N(t)$, the number of arrivals in a Poisson process up to time t. Let λ be the rate of the process.

a) Find the conditional probability $\Pr\{N(t) = n \mid S_n = \tau\}$ for all $\tau \le t$.

Solution: The condition $S_n = \tau$ means that the epoch of the nth arrival is τ. Conditional on this, the event $\{N(t) = n\}$ for some $t > \tau$ means there have been no subsequent arrivals from τ to t. In other words, it means that the $(n+1)$th interarrival time, X_{n+1} exceeds $t - \tau$. This interarrival time is independent of S_n and thus

$$\Pr\{N(t) = n \mid S_n = \tau\} = \Pr\{X_{n+1} > t - \tau\} = e^{-\lambda(t-\tau)} \qquad \text{for } t > \tau. \tag{A.24}$$

b) Using the Erlang density for S_n, use (a) to find $\Pr\{N(t) = n\}$.

Solution: We find $\Pr\{N(t) = n\}$ simply by averaging (A.24) over S_n.

$$\begin{aligned}
\Pr\{N(t){=}n\} &= \int_0^\infty \Pr\{N(t){=}n \mid S_n{=}\tau\}\, \mathsf{f}_{S_n}(\tau)\, d\tau \\
&= \int_0^t e^{-\lambda(t-\tau)} \frac{\lambda^n \tau^{n-1} e^{-\lambda\tau}}{(n-1)!}\, d\tau \\
&= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \int_0^t \tau^{n-1}\, d\tau \;=\; \frac{(\lambda t)^n e^{-\lambda t}}{n!}.
\end{aligned}$$

Exercise 2.4: Assume that a counting process $\{N(t); t{>}0\}$ has the independent and stationary increment properties and satisfies (2.17) (for all $t > 0$). Let X_1 be the epoch of the first arrival and X_n be the interarrival time between the $(n{-}1)$st and the nth arrival. Use only these assumptions in doing the following parts of this exercise.

a) Show that $\Pr\{X_1 > x\} = e^{-\lambda x}$.

Solution: The event $\{X_1 > x\}$ is the same as the event $\{N(x) = 0\}$. Thus, from (2.17), $\Pr\{X_1 > x\} = \Pr\{N(x) = 0\} = e^{-\lambda x}$.

b) Let S_{n-1} be the epoch of the $(n{-}1)$st arrival. Show that $\Pr\{X_n > x \mid S_{n-1} = \tau\} = e^{-\lambda x}$.

Solution: The conditioning event $\{S_{n-1} = \tau\}$ is somewhat messy to deal with in terms of the parameters of the counting process, so we start by solving an approximation of the desired result and then go to a limit as the approximation becomes increasingly close. Let $\delta > 0$ be a small positive number which we later allow to approach 0. We replace the event $\{S_{n-1} = \tau\}$ with $\{\tau{-}\delta < S_{n-1} \le \tau\}$. Since the occurrence of two arrivals in a small interval of size δ is very unlikely (of order $o(\delta)$), we also include the condition that $S_{n-2} \le \tau{-}\delta$ and $S_n \ge \tau$. With this, the approximate conditioning event becomes

$$\{S_{n-2} \le \tau{-}\delta < S_{n-1} \le \tau < S_n\} \;=\; \{N(\tau{-}\delta) = n{-}2,\ \widetilde{N}(\tau{-}\delta, \tau) = 1\}.$$

Since we are irretrievably deep in approximations, we also replace the event $\{X_n > x\}$ (conditional on this approximating condition) with $\{\widetilde{N}(\tau, \tau{+}x) = 0\}$. Note that this approximation is exact for $\delta = 0$, since in that case $S_{n-1} = \tau$, so $X_n > x$ means that no arrivals occur in $(\tau, \tau{+}x)$.

We can now solve this approximate problem precisely,

$$\begin{aligned}\Pr\Big\{\widetilde{N}(\tau, \tau{+}x) = 0 \mid N(\tau{-}\delta) = n-2, \widetilde{N}(\tau{-}\delta, \tau) = 1\Big\} &= \Pr\Big\{\widetilde{N}(\tau, \tau{+}x) = 0\Big\} \\ &= e^{-\lambda x}.\end{aligned}$$

In the first step, we used the independent increment property and in the second, the stationary increment property along with (a).

In the limit $\delta \to 0$, the conditioning event becomes $S_{n-1} = \tau$ and the conditioned event becomes $X_n > x$. The argument is very convincing, and becomes more convincing the more one thinks about it. At the same time, it is somewhat unsatisfactory since both the conditioned and conditioning event are being approximated. One can easily upper and lower bound the probability that $X_n > x$ for each δ but the 'proof' then requires many uninsightful and tedious details.

c) For each $n > 1$, show that $\Pr\{X_n > x\} = e^{-\lambda x}$ and that X_n is independent of S_{n-1}.

Solution: We have seen that $\Pr\{X_n > x \mid S_{n-1}{=}\tau\} = e^{-\lambda x}$. Since the value of this probability conditioned on $\{S_{n-1} = \tau\}$ does not depend on τ, X_n must be independent of S_{n-1}.

d) Argue that X_n is independent of $X_1, X_2, \dots X_{n-1}$.

Solution: Equivalently, we show that X_n is independent of $\{S_1{=}s_1,\ S_2{=}s_2,\ \dots,\ S_{n-1}{=}s_{n-1}\}$ for all choices of $0 < s_1 < s_2 < \cdots < s_{n-1}$. Using the same artifice as in (b), this latter event is the same as the limit as $\delta \to 0$ of the event

$$\{N(s_1{-}\delta){=}0,\ \ \widetilde{N}(s_1{-}\delta, s_1){=}1,\ \ \widetilde{N}(s_1, s_2{-}\delta){=}0,\ \ \widetilde{N}(s_2{-}\delta, s_2){=}1,\ \ \dots,\ \ \widetilde{N}(s_{n-1}{-}\delta, s_{n-1}){=}1\}.$$

From the independent increment property, the above event is then independent of the rv $\widetilde{N}(s_{n-1},\ s_{n-1}{+}x)$ for each $x > 0$. As in (b), this shows that X_n is independent of $S_1, \dots, S_{n-1}$ and thus of $X_1, \dots, X_{n-1}$.

The most interesting part of this entire exercise is that the Poisson CDF was used only to derive the fact that X_1 has an exponential CDF. In other words, we have shown quite a bit more than Definition 2 of a Poisson process. We have shown that if X_1 is exponential and the stationary and independent increment properties hold, then the process is Poisson. On the other hand, we have shown that a careful derivation of the properties of the Poisson process from this definition requires a great deal of intricate, uninsightful, and tedious analysis.

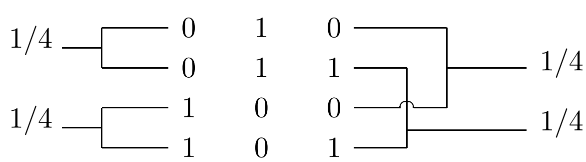
Exercise 2.5: The point of this exercise is to show that the sequence of PMF's for the counting process of a Bernoulli process does not specify the process. In other words, knowing that $N(t)$ satisfies the binomial distribution for all t does not mean that the process is Bernoulli. This helps us understand why the second definition of a Poisson process requires stationary and independent increments along with the Poisson distribution for $N(t)$.

a) For a sequence of binary rv's $Y_1, Y_2, Y_3, \dots,$ in which each rv is 0 or 1 with equal probability, find a joint distribution for Y_1, Y_2, Y_3 that satisfies the binomial distribution, $p_{N(t)}(k) = \binom{t}{k}2^{-t}$ for $t = 1, 2, 3$ and $0 \le k \le t$, but for which Y_1, Y_2, Y_3 are not independent.

Your solution should contain four 3-tuples with probability 1/8 each, two 3-tuples with probability 1/4 each, and two 3-tuples with probability 0. Note that by making the subsequent arrivals IID and equiprobable, you have an example where $N(t)$ is binomial for all t but the process is not Bernoulli. Hint: Use the binomial for $t = 3$ to find two 3-tuples that must have probability 1/8. Combine this with the binomial for $t = 2$ to find two other 3-tuples with probability 1/8. Finally look at the constraints imposed by the binomial distribution on the remaining four 3-tuples.

Solution: The 3-tuples 000 and 111 each have probability 1/8, and are the unique tuples for which $N(3) = 0$ and $N(3) = 3$ respectively. In the same way, $N(2) = 0$ only for $(Y_1, Y_2) = (0, 0)$, so (0,0) has probability 1/4. Since $(0, 0, 0)$ has probability 1/8, it follows that $(0, 0, 1)$ has probability 1/8. In the same way, looking at $N(2) = 2$, we see that $(1, 1, 0)$ has probability 1/8.

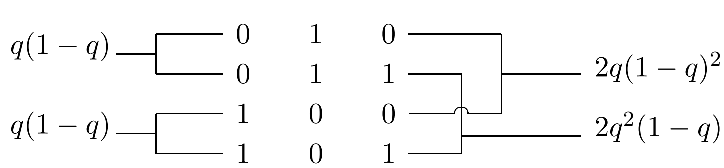
The four remaining 3-tuples are illustrated below, with the constraints imposed by $N(1)$ and $N(2)$ on the left and those imposed by $N(3)$ on the right.



It can be seen by inspection from the figure that if $(0, 1, 0)$ and $(1, 0, 1)$ each have probability 1/4, then the constraints are satisfied. There is one other solution satisfying the constraints: choose $(0, 1, 1)$ and $(1, 0, 0)$ to each have probability 1/4.

b) Generalize (a) to the case where Y_1, Y_2, Y_3 satisfy $\Pr\{Y_i = 1\} = q$ and $\Pr\{Y_i = 0\} = 1 - q$. Assume $q < 1/2$ and find a joint distribution on Y_1, Y_2, Y_3 that satisfies the binomial distribution, but for which the 3-tuple $(0, 1, 1)$ has zero probability.

Solution: Arguing as in (a), we see that $\Pr\{(0, 0, 0)\} = (1 - q)^3$, $\Pr\{(0, 0, 1)\} = (1 - q)^2 p$, $\Pr\{(1, 1, 1)\} = q^3$, and $\Pr\{(1, 1, 0)\} = q^2(1-q)$. The remaining four 3-tuples are constrained as shown below.



If we set $\Pr\{(0, 1, 1)\} = 0$, then $\Pr\{0, 1, 0)\} = q(1 - q)$, $\Pr\{(1, 0, 1)\} = 2q^2(1 - q)$, and $\Pr\{(1, 0, 0)\} = q(1 - q) - 2q^2(1 - q) = q(1 - q)(1 - 2q)$. This satisfies all the binomial constraints.

c) More generally yet, view a joint PMF on binary t-tuples as a nonnegative vector in a 2^t dimensional vector space. Each binomial probability $p_{N(\tau)}(k) = \binom{\tau}{k}q^k(1 - q)^{\tau-k}$ constitutes a linear constraint on this vector.

For each τ, show that one of these constraints may be replaced by the constraint that the components of the vector sum to 1.

Solution: There are 2^t binary n-tuples and each has a probability, so the joint PMF can be viewed as a vector of 2^t numbers. The binomial probability $p_{N(\tau)}(k) = \binom{\tau}{k} q^k(1-q)^{\tau-k}$ specifies the sum of the probabilities of the n-tuples in the event $\{N(\tau) = k\}$, and thus is a linear constraint on the joint PMF. Note: Mathematically, a linear constraint specifies that a given weighted sum of components is 0. The type of constraint here, where the weighted sum is a nonzero constant, is more properly called a first-order constraint. Engineers often refer to first order constraints as linear, and we follow that practice here.

Since $\sum_{k=0}^{\tau} \binom{\tau}{k} p^k q^{\tau-k} = 1$, one of these $\tau + 1$ constraints can be replaced by the constraint that the sum of all 2^t components of the PMF is 1.

d) Using (c), show that at most $(t + 1)t/2 + 1$ of the binomial constraints are linearly independent. Note that this means that the linear space of vectors satisfying these binomial constraints has dimension at least $2^t - (t + 1)t/2 - 1$. This linear space has dimension 1 for $t = 3$, explaining the results in parts a) and b). It has a rapidly increasing dimension for $t > 3$, suggesting that the binomial constraints are relatively ineffectual for constraining the joint PMF of a joint distribution. More work is required for the case of $t > 3$ because of all the inequality constraints, but it turns out that this large dimensionality remains.

Solution: We know that the sum of all the 2^t components of the PMF is 1, and we saw in (c) that for each integer τ, $1 \le \tau \le t$, there are τ additional linear constraints on the PMF established by the binomial terms $N(\tau = k)$ for $0 \le k \le \tau$. Since $\sum_{\tau=1}^{t} \tau = (t + 1)t/2$, we see that there are $t(t + 1)/2$ independent linear constraints on the joint PMF imposed by the binomial terms, in addition to the overall constraint that the components sum to 1. Thus the dimensionality of the 2^t vectors satisfying these linear constraints is at least $2^t - 1 - (t + 1)t/2$.

Exercise 2.6: Let $h(x)$ be a positive function of a real variable that satisfies $h(x + t) = h(x) + h(t)$ and let $h(1) = c$.

a) Show that for integer $k > 0$, $h(k) = kc$.

Solution: We use induction. We know $h(1) = c$ and the inductive hypothesis is that $h(n) = nc$, which is satisfied for $n = 1$. We then have $h(n + 1) = h(n) + h(1) = nc + c = (n + 1)c$. Thus if the hypothesis is satisfied for n it is also satisfied for $n + 1$, which verifies that it is satisfied for all positive integer n.

b) Show that for integer $j > 0$, $h(1/j) = c/j$.

Solution: Repeatedly adding $h(1/j)$ to itself, we get $h(2/j) = h(1/j) + h(1/j) = 2h(1/j)$, $h(3/j) = h(2/j) + h(1/j) = 3h(1/j)$ and so forth to $h(1) = h(j/j) = jh(1/j)$. Thus $h(1/j) = c/j$.

c) Show that for all positive integers k, j, $h(k/j) = ck/j$.

Solution: Since $h(1/j) = c/j$, for each positive integer j, we can use induction on positive integers k for any given $j > 0$ to get $h(k/j) = ck/j$

d) The above parts show that $h(x)$ is linear in positive *rational* numbers. For very picky mathematicians, this does not guarantee that $h(x)$ is linear in positive *real* numbers. Show that if $h(x)$ is also monotonic in

x, then $h(x)$ is linear in $x > 0$.

Solution: Let $x > 0$ be a real number and let $x_1, x_2, \dots$ be a sequence of increasing rational numbers approaching x. Then $\lim_{i\to\infty} h(x_i) = c \lim_{i\to\infty} x_i = cx$. Thus $h(x) \geq cx$. If we look at a similarly decreasing sequence, we see that $h(x) \leq cx$, so $h(x) = cx$.

Exercise 2.7: Assume that a counting process $\{N(t);\, t{>}0\}$ has the independent and stationary increment properties and, for all $t > 0$, satisfies

$$\begin{aligned} \Pr\left\{\widetilde{N}(t, t+\delta) = 0\right\} &= 1 - \lambda\delta + o(\delta) \\ \Pr\left\{\widetilde{N}(t, t+\delta) = 1\right\} &= \lambda\delta + o(\delta) \\ \Pr\left\{\widetilde{N}(t, t+\delta) > 1\right\} &= o(\delta). \end{aligned} \tag{A.25}$$

a) Let $\mathsf{F}^{\mathsf{c}}_1(\tau) = \Pr\{N(\tau) = 0\}$ and show that $d\mathsf{F}^{\mathsf{c}}_1(\tau)/d\tau = -\lambda\mathsf{F}^{\mathsf{c}}_1(\tau)$.

Solution: Note that $\mathsf{F}^{\mathsf{c}}_1$ is the complementary CDF of X_1. Using the fundamental definition of a derivative,

$$\begin{aligned} \frac{d\mathsf{F}^{\mathsf{c}}_1(\tau)}{d\tau} &= \lim_{\delta\to 0} \frac{\mathsf{F}^{\mathsf{c}}_1(\tau{+}\delta) - \mathsf{F}^{\mathsf{c}}_1(\tau)}{\delta} = \lim_{\delta\to 0} \frac{\Pr\{N(\tau{+}\delta) = 0\} - \Pr\{N(\tau) = 0\}}{\delta} \\ &= \lim_{\delta\to 0} \frac{\Pr\{N(\tau) = 0\}\left(\Pr\left\{\widetilde{N}(\tau,\, \tau{+}\delta) = 0\right\} - 1\right)}{\delta} &\text{(A.26)} \\ &= \lim_{\delta\to 0} \frac{\Pr\{N(\tau) = 0\}\left(1 - \lambda\delta + o(\delta) - 1\right)}{\delta} &\text{(A.27)} \\ &= \Pr\{N(\tau) = 0\}(-\lambda) \;=\; -\lambda\mathsf{F}^{\mathsf{c}}_1(\tau), \end{aligned}$$

where (A.26) resulted from the independent increment property and (A.27) resulted from (A.25).

b) Show that X_1, the time of the first arrival, is exponential with parameter λ.

Solution: The complementary CDF of X_1 is $\mathsf{F}^{\mathsf{c}}_1(\tau)$, which satisfies the first order linear differential equation in (a). The solution to that equation, with the boundary point $\mathsf{F}^{\mathsf{c}}_1(0) = 1$ is $e^{-\lambda\tau}$ for $\tau > 0$, showing that X_1 is exponential.

c) Let $\mathsf{F}^{\mathsf{c}}_n(\tau) = \Pr\left\{\widetilde{N}(t, t+\tau) = 0 \mid S_{n-1} = t\right\}$ and show that $d\mathsf{F}^{\mathsf{c}}_n(\tau)/d\tau = -\lambda\mathsf{F}^{\mathsf{c}}_n(\tau)$.

Solution: By the independent increment property, $\{\widetilde{N}(t,\, t{+}\tau) = 0\}$ is independent of $\{S_{n-1} = t\}$, and by the stationary increment property, it has the same probability as $N(\tau) = 0$. Thus $d\mathsf{F}^{\mathsf{c}}_n(\tau)/d\tau = -\lambda\mathsf{F}^{\mathsf{c}}_n(\tau)$ follows by the argument in (a).

d) Argue that X_n is exponential with parameter λ and independent of earlier arrival times.

Solution: Note that $\mathsf{F}^{\mathsf{c}}_n(\tau)$ is the complementary CDF of X_n conditional on $\{S_{n-1} = t\}$, and as shown in (c), it's distribution does not depend on S_{n-1}. In other words, $\mathsf{F}^{\mathsf{c}}_n(\tau)$ as found in (c) is the complementary CDF of X_n. It is exponential by the argument given in (b) for X_1, and it is independent of earlier arrivals since $\{\widetilde{N}(t,\, t{+}\tau) = 0\}$ is independent of arrivals before t.

This was also shown in the solution to Exercise 2.4 (c) and (d). In other words, definitions 2 and 3 of a Poisson process both follow from the assumptions that X_1 is exponential and that the stationary and independent increment properties hold.

Exercise 2.8: For a Poisson process, let $t > 0$ be arbitrary and let Z_1 be the duration of the interval from t until the next arrival after t. Let Z_m, for each $m > 1$, be the interarrival time from the epoch of the $(m-1)$st arrival after t until the mth arrival after t.

a) Given that $N(t) = n$, explain why $Z_1 = X_{n+1} - t + S_n$ and, for each $m > 1$, $Z_m = X_{m+n}$.

Solution: Given $N(t) = n$, the mth arrival after t for $m \geq 1$ must be the $(n+m)$th arrival overall. Its arrival epoch is then

$$S_{n+m} = S_{n+m-1} + X_{n+m}. \tag{A.28}$$

By definition for $m > 1$, Z_m is the interval from S_{n+m-1} (the time of the $(m-1)$st arrival after t) to S_{n+m}. Thus, from (A.28), $Z_m = X_{n+m}$ for $m > 1$. For $m = 1$, Z_1 is the interval from t until the next arrival, *i.e.*, $Z_1 = S_{n+1} - t$. Using $m = 1$ in (A.28), $Z_1 = S_n + X_{n+1} - t$.

b) Conditional on $N(t) = n$ and $S_n = \tau$, show that $Z_1, Z_2, \dots$ are IID.

Solution: The condition $N(t) = n$ and $S_n = \tau$ implies that $X_{n+1} > t - \tau$. Given this condition, $X_{n+1} - (t-\tau)$ is exponential, so $Z_1 = X_{n+1} - (t-\tau)$ is exponential. For $m > 1$, and for the given condition, $Z_m = X_{n+m}$. Since X_{n+m} is exponential and independent of $X_1, \dots, X_{n+m-1}$, we see that Z_m is also exponential and independent of $Z_1, \dots, Z_{m-1}$. Since these exponential distributions are the same, $Z_1, Z_2, \dots,$ are IID conditional on $N(t) = n$ and $S_n = \tau$.

c) Show that $Z_1, Z_2, \dots$ are IID.

Solution: We have shown that$\{Z_m;\, m \geq 1\}$ are IID and exponential conditional on $N(t) = n$ and $S_n = \tau$. The joint distribution of $Z_1, \dots, Z_m$ is thus specified as a function of $N(t) = n$ and $S_n = \tau$. Since this function is constant in n and τ, the joint conditional distribution must be the same as the joint unconditional distribution, and therefore $Z_1, \dots, Z_m$ are IID for all $m > 0$.

Exercise 2.9: Consider a "shrinking Bernoulli" approximation $N_\delta(m\delta) = Y_1 + \cdots + Y_m$ to a Poisson process as described in Subsection 2.2.5.

a) Show that

$$\Pr\{N_\delta(m\delta) = n\} = \binom{m}{n} (\lambda\delta)^n (1 - \lambda\delta)^{m-n}.$$

Solution: This is just the binomial PMF in (1.23)

b) Let $t = m\delta$, and let t be fixed for the remainder of the exercise. Explain why

$$\lim_{\delta \to 0} \Pr\{N_\delta(t) = n\} = \lim_{m \to \infty} \binom{m}{n} \left(\frac{\lambda t}{m}\right)^n \left(1 - \frac{\lambda t}{m}\right)^{m-n},$$

where the limit on the left is taken over values of δ that divide t.

Solution: This is the binomial PMF in (a) with $\delta = t/m$.

c) Derive the following two equalities:

$$\lim_{m \to \infty} \binom{m}{n} \frac{1}{m^n} = \frac{1}{n!}; \qquad \text{and} \qquad \lim_{m \to \infty} \left(1 - \frac{\lambda t}{m}\right)^{m-n} = e^{-\lambda t}.$$

Solution: Note that

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} = \frac{1}{n!}\prod_{i=0}^{n-1}(m-i).$$

When this is divided by m^n, each term in the product above is divided by m, so

$$\binom{m}{n}\frac{1}{m^n} = \frac{1}{n!}\prod_{i=0}^{n-1}\frac{(m-i)}{m} = \frac{1}{n!}\prod_{i=0}^{n-1}\Big(1-\frac{i}{m}\Big). \tag{A.29}$$

Taking the limit as $m \to \infty$, each of the n terms in the product approaches 1, so the limit is $1/n!$, verifying the first equality in (c). For the second,

$$\begin{aligned}\Big(1-\frac{\lambda t}{m}\Big)^{m-n} &= \exp\Big[(m-n)\ln\Big(1-\frac{\lambda t}{m}\Big)\Big] = \exp\Big[(m-n)\Big(\frac{-\lambda t}{m} + \mathrm{o}(1/m)\Big)\Big] \\ &= \exp\Big[-\lambda t + \frac{n\lambda t}{m} + (m-n)\mathrm{o}(1/m)\Big].\end{aligned}$$

In the second equality, we expanded $\ln(1-x) = -x + x^2/2\cdots$. In the limit $m \to \infty$, the final expression is $\exp(-\lambda t)$, as was to be shown.

If one wishes to see how the limit in (A.29) is approached, we have

$$\frac{1}{n!}\prod_{i=0}^{n-1}\Big(1-\frac{i}{m}\Big) = \frac{1}{n!}\exp\Big(\sum_{i=1}^{n-1}\ln\Big(1-\frac{i}{m}\Big)\Big) = \frac{1}{n!}\exp\Big(\frac{-n(n-1)}{2m} + \mathrm{o}(1/m)\Big).$$

d) Conclude from this that for every t and every n, $\lim_{\delta\to 0}\Pr\{N_\delta(t){=}n\} = \Pr\{N(t){=}n\}$ where $\{N(t);\, t > 0\}$ is a Poisson process of rate λ.

Solution: We simply substitute the results of (c) into the expression in (b), getting

$$\lim_{\delta\to 0}\Pr\{N_\delta(t) = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}.$$

This shows that the Poisson PMF is the limit of shrinking Bernoulli PMF's, but recall from Exercise 2.5 that this is not quite enough to show that a Poisson process is the limit of shrinking Bernoulli processes. It is also necessary to show that the stationary and independent increment properties hold in the limit $\delta \to 0$. It can be seen that the Bernoulli process has these properties at each increment δ, and it is intuitively clear that these properties should hold in the limit, but it seems that carrying out all the analytical details to show this precisely is neither warranted or interesting.

Exercise 2.10: Let $\{N(t);\, t > 0\}$ be a Poisson process of rate λ.

a) Find the joint probability mass function (PMF) of $N(t)$, $N(t+s)$ for $s > 0$.

Solution: Note that $N(t+s)$ is the number of arrivals in $(0, t]$ plus the number in $\big(t,\, t{+}s\big)$. In order to find the joint distribution of $N(t)$ and $N(t{+}s)$, it makes sense to express $N(t{+}s)$

as $N(t) + \widetilde{N}(t, t{+}s)$ and to use the independent increment property to see that $\widetilde{N}\big(t, t{+}s)\big)$ is independent of $N(t)$. Thus for $m > n$,

$$\begin{aligned}
\mathsf{p}_{N(t)N(t+s)}(n,m) &= \Pr\{N(t){=}n\} \Pr\Big\{\widetilde{N}(t,t{+}s){=}m{-}n)\Big\} \\
&= \frac{(\lambda t)^n e^{-\lambda t}}{n!} \times \frac{\big(\lambda s\big)^{m-n} e^{-\lambda s}}{(m-n)!},
\end{aligned}$$

where we have used the stationary increment property to see that $\widetilde{N}(t, t{+}s)$ has the same distribution as $N(s)$. This solution can be rearranged in various ways, of which the most interesting is

$$\mathsf{p}_{N(t)N(t+s)}(n,m) = \frac{\big(\lambda\big(t{+}s)\big)^m e^{-\lambda(s+t)}}{m!} \times \binom{m}{n} \Big(\frac{t}{t{+}s}\Big)^n \Big(\frac{s}{t{+}s}\Big)^{m-n},$$

where the first term is $\mathsf{p}_{N(t+s)}(m)$ (the probability of m arrivals in $(0, t{+}s]$) and the second, conditional on the first, is the binomial probability that n of those m arrivals occur in $(0, t)$.

b) Find $\mathsf{E}\left[N(t) \cdot N(t+s)\right]$ for $s > 0$.

Solution: Again expressing $N(t{+}s) = N(t) + \widetilde{N}(t, t{+}s)$,

$$\begin{aligned}
\mathsf{E}\left[N(t) \cdot N(t{+}s)\right] &= \mathsf{E}\left[N^2(t)\right] + \mathsf{E}\left[N(t)\widetilde{N}(t, t{+}s)\right] \\
&= \mathsf{E}\left[N^2(t)\right] + \mathsf{E}\left[N(t)\right] \mathsf{E}\left[N(s)\right] \\
&= \lambda t + \lambda^2 t^2 + \lambda t \lambda s.
\end{aligned}$$

In the final step, we have used the fact (from Table 1.2 or a simple calculation) that the mean of a Poisson rv with PMF $(\lambda t)^n \exp(-\lambda t)/n!$ is λt and the variance is also λt (thus the second moment is $\lambda t + (\lambda t)^2$). This mean and variance was also derived in Exercise 2.2 and can also be calculated by looking at the limit of shrinking Bernoulli processes.

c) Find $\mathsf{E}\left[\widetilde{N}(t_1, t_3) \cdot \widetilde{N}(t_2, t_4)\right]$ where $\widetilde{N}(t, \tau)$ is the number of arrivals in $(t, \tau]$ and $t_1 < t_2 < t_3 < t_4$.

Solution: This is a straightforward generalization of what was done in (b). We break up $\widetilde{N}(t_1, t_3)$ as $\widetilde{N}(t_1, t_2) + \widetilde{N}(t_2, t_3)$ and break up $\widetilde{N}(t_2, t_4)$ as $\widetilde{N}(t_2, t_3) + \widetilde{N}(t_3, t_4)$. The interval $(t_2, t_3]$ is shared. Thus

$$\begin{aligned}
\mathsf{E}\left[\widetilde{N}(t_1,t_3)\widetilde{N}(t_2,t_4)\right] &= \mathsf{E}\left[\widetilde{N}(t_1,t_2)\widetilde{N}(t_2,t_4)\right] + \mathsf{E}\left[\widetilde{N}^2(t_2,t_3)\right] + \mathsf{E}\left[\widetilde{N}(t_2,t_3)\widetilde{N}(t_3,t_4)\right] \\
&= \lambda^2(t_2{-}t_1)(t_4{-}t_2) + \lambda^2(t_3{-}t_2)^2 + \lambda(t_3{-}t_2) + \lambda^2(t_3{-}t_2)(t_4{-}t_3) \\
&= \lambda^2(t_3{-}t_1)(t_4{-}t_2) + \lambda(t_3{-}t_2).
\end{aligned}$$

Exercise 2.11: An elementary experiment is independently performed N times where N is a Poisson rv of mean λ. Let $\{a_1, a_2, \ldots, a_K\}$ be the set of sample points of the elementary experiment and let p_k, $1 \le k \le K$, denote the probability of a_k.

a) Let N_k denote the number of elementary experiments performed for which the output is a_k. Find the PMF for N_k $(1 \le k \le K)$. (Hint: no calculation is necessary.)

Solution: View the experiment as a combination of K Poisson processes where the kth has rate $\mathsf{p}_k\lambda$ and the combined process has rate λ. At $t = 1$, the total number of experiments is then Poisson with mean λ and the kth process is Poisson with mean $\mathsf{p}_k\lambda$. Thus $\mathsf{p}_{N_k}(n) = (\lambda\mathsf{p}_k)^n e^{-\lambda\mathsf{p}_k}/n!$.

b) Find the PMF for $N_1 + N_2$.

Solution: By the same argument,

$$\mathsf{p}_{N_1+N_2}(n) = \frac{[\lambda(\mathsf{p}_1+\mathsf{p}_2)]^n e^{-\lambda(\mathsf{p}_1+\mathsf{p}_2)}}{n!}.$$

c) Find the conditional PMF for N_1 given that $N = n$.

Solution: Each of the n combined arrivals over $(0, 1]$ is then a_1 with probability p_1. Thus N_1 is binomial given that $N = n$,

$$\mathsf{p}_{N_1|N}(n_1|n) = \binom{n}{n_1}(\mathsf{p}_1)^{n_1}(1-\mathsf{p}_1)^{n-n_1}.$$

d) Find the conditional PMF for $N_1 + N_2$ given that $N = n$.

Solution: Let the sample value of $N_1 + N_2$ be n_{12}. By the same argument in (c),

$$\mathsf{p}_{N_1+N_2|N}(n_{12}|n) = \binom{n}{n_{12}}(\mathsf{p}_1+\mathsf{p}_2)^{n_{12}}(1-\mathsf{p}_1-\mathsf{p}_2)^{n-n_{12}}.$$

e) Find the conditional PMF for N given that $N_1 = n_1$.

Solution: Since N is then n_1 plus the number of arrivals from the other processes, and those additional arrivals are Poisson with mean $\lambda(1-\mathsf{p}_1)$,

$$\mathsf{p}_{N|N_1}(n|n_1) = \frac{[\lambda(1-\mathsf{p}_1)]^{n-n_1} e^{-\lambda(1-\mathsf{p}_1)}}{(n-n_1)!}.$$

Exercise 2.12: Starting from time 0, northbound buses arrive at 77 Mass. Avenue according to a Poisson process of rate λ. Customers arrive according to an independent Poisson process of rate μ. When a bus arrives, all waiting customers instantly enter the bus and subsequent customers wait for the next bus.

a) Find the PMF for the number of customers entering a bus (more specifically, for any given m, find the PMF for the number of customers entering the mth bus).

Solution: Since the customer arrival process and the bus arrival process are independent Poisson processes, the sum of the two counting processes is a Poisson counting process of rate $\lambda + \mu$. Each arrival for the combined process is a bus with probability $\lambda/(\lambda + \mu)$ and a customer with probability $\mu/(\lambda + \mu)$. The sequence of choices between bus or customer arrivals is an IID sequence. Thus, starting immediately after bus $m - 1$ (or at time 0 for $m = 1$), the probability of n customers in a row followed by a bus, for any $n \geq 0$, is $\big[\mu/(\lambda+\mu)\big]^n\lambda/(\lambda+\mu)$. This is the probability that n customers enter the mth bus, *i.e.*, defining N_m as the number of customers entering the mth bus, the PMF of N_m is

$$\mathsf{p}_{N_m}(n) = \left(\frac{\mu}{\lambda+\mu}\right)^n \frac{\lambda}{\lambda+\mu}. \qquad \text{(A.30)}$$

b) Find the PMF for the number of customers entering the mth bus given that the interarrival interval between bus $m-1$ and bus m is x.

Solution: For any given interval of size x (*i.e.*, for the interval $(s, s+x]$ for any given s), the number of customer arrivals in that interval has a Poisson distribution of rate μ. Since the customer arrival process is independent of the bus arrivals, this is also the distribution of customer arrivals between the arrival of bus $m-1$ and that of bus m given that the interval X_m between these bus arrivals is x. Thus letting X_m be the interval between the arrivals of bus $m-1$ and m,

$$\mathsf{p}_{N_m|X_m}(n|x) = (\mu x)^n e^{-\mu x}/n!.$$

c) Given that a bus arrives at time 10:30 PM, find the PMF for the number of customers entering the next bus.

Solution: First assume that for some given m, bus $m-1$ arrives at 10:30. The number of customers entering bus m is still determined by the argument in (a) and has the PMF in (A.30). In other words, N_m is independent of the arrival time of bus $m-1$. From the formula in (A.30), the PMF of the number entering a bus is also independent of m. Thus the desired PMF is that on the right side of (A.30).

d) Given that a bus arrives at 10:30 PM and no bus arrives between 10:30 and 11, find the PMF for the number of customers on the next bus.

Solution: Using the same reasoning as in (b), the number of customer arrivals from 10:30 to 11 is a Poisson rv, say N' with PMF $\mathsf{p}_{N'}(n) = (\mu/2)^n e^{-\mu/2}/n!$ (we are measuring time in hours so that μ is the customer arrival rate in arrivals per hour.) Since this is independent of bus arrivals, it is also the PMF of customer arrivals in (10:30 to 11] given no bus arrival in that interval.

The number of customers to enter the next bus is N' plus the number of customers N'' arriving between 11 and the next bus arrival. By the argument in (a), N'' has the PMF in (A.30). Since N' and N'' are independent, the PMF of $N' + N''$ (the number entering the next bus given this conditioning) is the convolution of the PMF's of N' and N'', *i.e.*,

$$\mathsf{p}_{N'+N''}(n) = \sum_{k=0}^{n} \left(\frac{\mu}{\lambda+\mu}\right)^k \frac{\lambda}{\lambda+\mu} \frac{(\mu/2)^{n-k} e^{-\mu/2}}{(n-k)!}.$$

This does not simplify in any nice way.

e) Find the PMF for the number of customers waiting at some given time, say 2:30 PM (assume that the processes started infinitely far in the past). Hint: think of what happens moving backward in time from 2:30 PM.

Solution: Let $\{Z_i; -\infty < i < \infty\}$ be the (doubly infinite) IID sequence of bus/customer choices where $Z_i = 0$ if the ith combined arrival is a bus and $Z_i = 1$ if it is a customer. Indexing this sequence so that -1 is the index of the most recent combined arrival before 2:30, we see that if $Z_{-1} = 0$, then no customers are waiting at 2:30. If $Z_{-1} = 1$ and $Z_{-2} = 0$, then one customer is waiting. In general, if $Z_{-n} = 0$ and $Z_{-m} = 1$ for $1 \le m < n$, then n

customers are waiting. Since the Z_i are IID, the PMF of the number N_{past} waiting at 2:30 is

$$\mathsf{p}_{N\text{past}}(n) = \left(\frac{\mu}{\lambda+\mu}\right)^n \frac{\lambda}{\lambda+\mu}.$$

This is intuitive in one way, *i.e.*, the number of customers looking back toward the previous bus should be the same as the number of customers looking forward to the next bus since the bus/customer choices are IID. It is paradoxical in another way since if we visualize a sample path of the process, we see waiting customers gradually increasing until a bus arrival, then going to 0 and gradually increasing again, etc. It is then surprising that the number of customers at an arbitrary time is statistically the same as the number immediately before a bus arrival. This paradox is partly explained at the end of (f) and fully explained in Chapter 5.

Mathematically inclined readers may also be concerned about the notion of 'starting infinitely far in the past.' A more precise way of looking at this is to start the Poisson process at time 0 (in accordance with the definition of a Poisson process). We can then find the PMF of the number waiting at time t and take the limit of this PMF as $t \to \infty$. For very large t, the number M of combined arrivals before t is large with high probability. Given $M = m$, the geometric distribution above is truncated at m, which is a neglibible correction for t large. This type of issue is handled more cleanly in Chapter 5.

f) Find the PMF for the number of customers getting on the next bus to arrive after 2:30. Hint: this is different from (a); look carefully at (e).

Solution: The number getting on the next bus after 2:30 is the sum of the number N_{p} waiting at 2:30 and the number of future customer arrivals N_{f} (found in (c)) until the next bus after 2:30. Note that N_{p} and N_{f} are IID. Convolving these PMF's, we get

$$\begin{aligned}
\mathsf{p}_{N_{\text{p}}+N_{\text{f}}}(n) &= \sum_{m=0}^{n} \left(\frac{\mu}{\lambda+\mu}\right)^m \frac{\lambda}{\lambda+\mu} \left(\frac{\mu}{\lambda+\mu}\right)^{n-m} \frac{\lambda}{\lambda+\mu} \\
&= (n+1) \left(\frac{\mu}{\lambda+\mu}\right)^n \left(\frac{\lambda}{\lambda+\mu}\right)^2.
\end{aligned}$$

This is very surprising. It says that the number of people getting on the first bus after 2:30 is the sum of two IID rv's, each with the same distribution as the number to get on the mth bus. This is an example of the 'paradox of residual life,' which we discuss very informally here and then discuss carefully in Chapter 5.

Consider a very large interval of time $(0, t_o]$ over which a large number of bus arrivals occur. Then choose a random time instant T, uniformly distributed in $(0, t_o]$. Note that T is more likely to occur within one of the larger bus interarrival intervals than within one of the smaller intervals, and thus, given the randomly chosen time instant T, the bus interarrival interval around that instant will tend to be larger than that from a given bus arrival, $m-1$ say, to the next bus arrival m. Since 2:30 is arbitrary, it is plausible that the interval around 2:30 behaves like that around T, making the result here also plausible.

g) Given that I arrive to wait for a bus at 2:30 PM, find the PMF for the number of customers getting on the next bus.

Solution: My arrival at 2:30 is in addition to the Poisson process of customers, and thus the number entering the next bus is $1 + N_\text{p} + N_\text{f}$. This has the sample value n if $N_\text{p} + N_\text{f}$ has the sample value $n - 1$, so from (f),

$$\mathsf{p}_{1+N_\text{p}+N_\text{f}}(n) = n\left(\frac{\mu}{\lambda+\mu}\right)^{n-1}\left(\frac{\lambda}{\lambda+\mu}\right)^2.$$

Do not be discouraged if you made a number of errors in this exercise and if it still looks very strange. This is a first exposure to a difficult set of issues which will become clear in Chapter 5.

Exercise 2.13: **a)** Show that the arrival epochs of a Poisson process satisfy

$$\mathsf{f}_{\boldsymbol{S}^{(n)}|S_{n+1}}(\boldsymbol{s}^{(n)}|s_{n+1}) = n!/s_{n+1}^n.$$

Hint: This is easy if you use only the results of Section 2.2.2.

Solution: Note that $\boldsymbol{S}^{(n)}$ is shorthand for $S_1, \dots, S_n$. Using Bayes' law,

$$\mathsf{f}_{\boldsymbol{S}^{(n)}|S_{n+1}}(\boldsymbol{s}^{(n)}|s_{n+1}) \quad = \quad \frac{\mathsf{f}_{\boldsymbol{S}^{(n)}}(\boldsymbol{s}^{(n)})\mathsf{f}_{S_{n+1}|\boldsymbol{S}^{(n)}}(s_{n+1}|\boldsymbol{s}^{(n)})}{\mathsf{f}_{S_{n+1}}(s_{n+1})}$$

From (2.15), $\mathsf{f}_{\boldsymbol{S}^{(n)}}(\boldsymbol{s}^{(n)}) = \lambda^n e^{-\lambda s_n}$. Also, $\mathsf{f}_{S_{n+1}|\boldsymbol{S}^{(n)}}(s_{n+1}|\boldsymbol{s}^{(n)}) = \lambda e^{-\lambda(s_{n+1}-s_n)}$. Finally $f_{S_{n+1}}(s_{n+1})$ is Erlang. Combining these terms,

$$\mathsf{f}_{\boldsymbol{S}^{(n)}|S_{n+1}}(\boldsymbol{s}^{(n)}|s_{n+1}) \quad = \quad n!/s_{n+1}^n.$$

b) Contrast this with the result of Theorem 2.5.1

Solution: (a) says that $S_1, \dots, S_n$ are uniformly distributed (subject to the ordering constraint) between 0 and t for any sample value t for S_{n+1}. Theorem 2.5.1 says that they are uniformly distributed between 0 and t given that $N(t) = n$. The conditions $\{S_{n+1} = t\}$ and $\{N(t) = n\}$ each imply that there are n arrivals in $(0, t)$. The condition $\{S_{n+1} = t\}$ in addition specifies that arrival $n + 1$ is at epoch t, whereas $\{N(t) = n\}$ specifies that arrival n+1 is in (t, ∞). From the independent increment property, the arrival epochs in $(0, t)$ are independent of those in $[t, \infty)$ and thus the conditional joint distribution of $\boldsymbol{S}^{(n)}$ is the same for each conditioning event.

One might ask whether this equivalence of conditional distributions provides a rigorous way of answering (a). The answer is yes if the above argument is spelled out in more detail. It is simpler, however, to use the approach in (a). The equivalence approach is more insightful, on the other hand, so it is worthwhile to understand both approaches.

Exercise 2.14: Equation (2.42) gives $\mathsf{f}_{S_i|N(t)}(s_i \mid n)$, which is the density of random variable S_i conditional on $N(t) = n$ for $n \geq i$. Multiply this expression by $\Pr\{N(t) = n\}$ and sum over n to find $\mathsf{f}_{S_i}(s_i)$; verify that your answer is indeed the Erlang density.

Solution: It is almost magical, but of course it has to work out.

$$\mathsf{f}_{S_i|N(t)}(s_i|n) = \frac{(s_i)^{i-1}}{(i-1)!}\frac{(t-s_i)^{n-i}}{(n-i)!}\frac{n!}{t^n}; \qquad \mathsf{p}_{N(t)}(n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}.$$

$$
\begin{aligned}
\sum_{n=i}^{\infty} \mathsf{f}(s_i|n)\mathsf{p}(n) &= \frac{s_i^{i-1}}{(i-i)!}\sum_{n=i}^{\infty}\frac{(t-s_i)^{n-i}}{(n-i)!}\lambda^n e^{-\lambda t} \\
&= \frac{\lambda^i s_i^{i-1}e^{-\lambda s_i}}{(i-i)!}\sum_{n=i}^{\infty}\frac{\lambda^{n-i}(t-s_i)^{n-i}e^{-\lambda(t-s_i)}}{(n-i)!} \\
&= \frac{\lambda^i s_i^{i-1}e^{-\lambda s_i}}{(i-i)!}.
\end{aligned}
$$

This is the Erlang distribution, and it follows because the preceding sum is the sum of terms in the PMF for the Poisson rv of rate $\lambda(t-s_i)$

Exercise 2.15: Consider generalizing the bulk arrival process in Figure 2.5. Assume that the epochs at which arrivals occur form a Poisson process $\{N(t); t > 0\}$ of rate λ. At each arrival epoch S_n, the number of arrivals Z_n, satisfies $\Pr\{Z_n{=}1\} = p$, $\Pr\{Z_n{=}2\} = 1-p$. The variables Z_n are IID.

a) Let $\{N_1(t); t > 0\}$ be the counting process of the epochs at which single arrivals occur. Find the PMF of $N_1(t)$ as a function of t. Similarly, let $\{N_2(t); t \geq 0\}$ be the counting process of the epochs at which double arrivals occur. Find the PMF of $N_2(t)$ as a function of t.

Solution: Since the process of arrival epochs is Poisson, and these epochs are split into single and double-arrival epochs by an IID splitting, the process of single-arrival epochs is Poisson with rate λp and the process of double-arrival epochs is Poisson with rate $\lambda(1-p)$. Thus, letting $q = 1-p$,

$$
\mathsf{p}_{N_1(t)}(n) = \frac{(\lambda p)^n e^{-\lambda p}}{n!}; \qquad \mathsf{p}_{N_2(t)}(m) = \frac{(\lambda q)^m e^{-\lambda q}}{m!}.
$$

b) Let $\{N_B(t); t \geq 0\}$ be the counting process of the total number of arrivals. Give an expression for the PMF of $N_B(t)$ as a function of t.

Solution: Since there are two arrivals at each double-arrival epoch, we have $N_B(t) = N_1(t) + 2N_2(t)$, and as seen above $N_1(t)$ and $N_2(t)$ are independent. This can be done as a digital convolution of the PMF's for $N_1(t)$ and $2N_2(t)$, but this can be slightly confusing since $2N_2(t)$ is nonzero only for even integers. Thus we revert to the general approach, which also reminds you where convolution comes from and thus how it can be done in general (PDF's, PMF's, etc.)

$$
\Pr\{N_B(t) = n, N_2(t) = m\} = \mathsf{p}_{N_1(t)}(n{-}2m)\mathsf{p}_{N_2(t)}(m).
$$

The marginal PMF for $N_B(t)$ is then given by

$$
\begin{aligned}
\mathsf{p}_{N_B(t)}(n) &= \sum_{m=0}^{\lfloor n/2 \rfloor} \mathsf{p}_{N_1(t)}(n{-}2m)\mathsf{p}_{N_2(t)}(m) \\
&= \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(\lambda pt)^{n-2m}e^{-\lambda pt}}{(n-2m)!}\frac{(\lambda qt)^m e^{-\lambda qt}}{m!} \\
&= e^{-\lambda t}\sum_{m=0}^{\lfloor n/2 \rfloor}\frac{(\lambda pt)^{n-2m}(\lambda qt)^m}{(n-2m)!m!}.
\end{aligned}
$$

Exercise 2.16: a) For a Poisson counting process of rate λ, find the joint probability density of $S_1, S_2, \ldots, S_{n-1}$ conditional on $S_n = t$.

Solution: The joint density of $S_1, \ldots, S_n$ is given in (2.15) as $\mathsf{f}_{S_1,\ldots,S_n}(s_1, \ldots, s_n) = \lambda^n \exp(-\lambda s_n)$. The marginal density of S_n is the Erlang density. The conditional density is the ratio of these, *i.e.*,

$$\mathsf{f}_{S_1,\ldots,S_{n-1}|S_n}(s_1, \ldots, s_{n-1}|s_n) = \frac{\lambda^n \exp(-\lambda s_n)}{\lambda^n s_n^{n-1} e^{-\lambda s_n}/(n-1)!} = \frac{(n-1)!}{s_n^{n-1}}.$$

b) Find $\Pr\{X_1 > \tau \mid S_n{=}t\}$.

Solution: We first use Bayes' law to find the density, $\mathsf{f}_{X_1|S_n}(\tau|t)$.

$$\begin{aligned}
\mathsf{f}_{X_1|S_n}(\tau|t) &= \frac{\mathsf{f}_{X_1}(\tau)\mathsf{f}_{S_n|X_1}(t|\tau)}{\mathsf{f}_{S_n}(t)} = \frac{\lambda e^{-\lambda\tau}\ \lambda^{n-1}(t-\tau)^{n-2}e^{-\lambda(t-\tau)}/(n-2)!}{\lambda^n t^{n-1}e^{-\lambda t}/(n-1)!} \\
&= \frac{(t-\tau)^{n-2}(n{-}1)}{t^{n-1}} \qquad \text{for } \tau < t
\end{aligned}$$

Integrating this with respect to τ, we get

$$\Pr\{X_1 > \tau \mid S_n{=}t\} = \left[\frac{t-\tau}{t}\right]^{n-1}.$$

c) Find $\Pr\{X_i > \tau \mid S_n{=}t\}$ for $1 \le i \le n$.

Solution: The condition here is $X_1 + \cdots + X_n = t$. Since $X_1, \ldots, X_n$ are IID without the condition, and the condition is symmetric in $X_1, \ldots, X_n$, we see that $X_1, \ldots, X_n$ are identically distributed conditional on $S_n = t$. Thus, from (b),

$$\Pr\{X_i > \tau \mid S_n{=}t\} = \left[\frac{t-\tau}{t}\right]^{n-1}; \qquad \text{for } 1 \le i \le n.$$

d) Find the density $\mathsf{f}_{S_i|S_n}(s_i|t)$ for $1 \le i \le n-1$.

Solution: We can use Bayes' law in the same way as (b), getting

$$\mathsf{f}_{S_i|S_n}(s_i|t) = \frac{s_i^{i-1}(t-s_i)^{n-i-1}(n-1)!}{t^{n-1}(i-1)!(n-i-1)!}.$$

e) Give an explanation for the striking similarity between the condition $N(t) = n-1$ and the condition $S_n = t$.

Solution: The solutions to (c) and (d) are the same as (2.45) and (2.46) respectively for $N(t) = n-1$. The condition $N(t) = n-1$ and the condition $S_n = t$ both imply that the number of arrivals in $(0,t)$ is $n-1$. In addition, $N(t) = n-1$ implies that the first arrival after $(0,t)$ is strictly after t, whereas $S_n = t$ implies that the first arrival after $(0,t)$ is at t. Because of the independent increment property, this additional implication does not affect the distribution of $S_1, \ldots, S_{n-1}$. See the solution to Exercise 2.13(b) for a further discussion of this point.

The important fact here is that the equivalence of the arrival distributions in $(0, t)$, given these slightly different conditions, is a valuable aid in problem solving, since either approach can be used.

Exercise 2.17: **a)** For a Poisson process of rate λ, find $\Pr\{N(t){=}n \mid S_1{=}\tau\}$ for $t > \tau$ and $n \geq 1$.

Solution: Given that $S_1 = \tau$, the number, $N(t)$, of arrivals in $(0, t]$ is 1 plus the number in $(\tau, t]$. This latter number, $\widetilde{N}(\tau, t)$ is Poisson with mean $\lambda(t - \tau)$. Thus,

$$\Pr\{N(t){=}n \mid S_1{=}\tau\} \;=\; \Pr\left\{\widetilde{N}(\tau, t) = n{-}1\right\} \;=\; \frac{[\lambda(t-\tau)]^{n-1} e^{-\lambda(t-\tau)}}{(n-1)!}.$$

b) Using this, find $\mathsf{f}_{S_1}(\tau) \mid N(t){=}n)$.

Solution: Using Bayes' law,

$$\mathsf{f}_{S_1|N(t)}(\tau|n) = \frac{n(t-\tau)^{n-1}}{t^n}.$$

c) Check your answer against (2.41).

Solution: Eq. (2.41) is $\Pr\{S_1 > \tau \mid N(t) = n\} = [(t-\tau)/t]^n$. The derivative of this with respect to τ is $-\mathsf{f}_{S_1|N(t)}(\tau|t)$, which clearly checks with (b).

Exercise 2.18: Consider a counting process in which the rate is a rv Λ with probability density $\mathsf{f}_\Lambda(\lambda) = \alpha e^{-\alpha\lambda}$ for $\lambda > 0$. Conditional on a given sample value λ for the rate, the counting process is a Poisson process of rate λ (i.e., nature first chooses a sample value λ and then generates a sample path of a Poisson process of that rate λ).

a) What is $\Pr\{N(t){=}n \mid \Lambda{=}\lambda\}$, where $N(t)$ is the number of arrivals in the interval $(0, t]$ for some given $t > 0$?

Solution: Conditional on $\Lambda = \lambda$, $\{N(t); t > 0$ is a Poisson process, so

$$\Pr\{N(t){=}n \mid \Lambda{=}\lambda\} = (\lambda t)^n e^{-\lambda t}/n!.$$

b) Show that $\Pr\{N(t){=}n\}$, the unconditional PMF for $N(t)$, is given by

$$\Pr\{N(t){=}n\} = \frac{\alpha t^n}{(t+\alpha)^{n+1}}.$$

Solution: The straightforward approach is to average the conditional distribution over λ,

$$\begin{aligned}
\Pr\{N(t) = n\} &= \int_0^\infty \frac{(\lambda t)^n e^{-\lambda t}}{n!} \alpha e^{-\lambda\alpha}\, d\lambda \\
&= \frac{\alpha t^n}{(t+\alpha)^n} \int_0^\infty \frac{[\lambda(t{+}\alpha)]^n e^{-\lambda(t+\alpha)}}{n!}\, d\lambda \\
&= \frac{\alpha t^n}{(t+\alpha)^{n+1}} \int_0^\infty \frac{x^n e^{-x}}{n!}\, dx \;=\; \frac{\alpha t^n}{(t+\alpha)^{n+1}},
\end{aligned}$$

where we changed the variable of integration from λ to $x = \lambda(t{+}\alpha)$ and then recognized the integral as the integral of an Erlang density of order $n{+}1$ with unit rate.

The solution can also be written as pq^n where $p = \alpha/(t+\alpha)$ and $q = t/(t+\alpha)$. This suggests a different interpretation for this result. $\Pr\{N(t) = n\}$ for a Poisson process (PP) of rate λ is a function only of λt and n. The rv $N(t)$ for a PP of rate λ thus has the same distribution as $N(\lambda t)$ for a PP of unit rate and thus $N(t)$ for a PP of variable rate Λ has the same distribution as $N(\Lambda t)$ for a PP of rate 1.

Since Λt is an exponential rv of parameter α/t, we see that $N(\Lambda t)$ is the number of arrivals of a PP of unit rate before the first arrival from an independent PP of rate α/t. This unit rate PP and rate α/t PP are independent and the combined process has rate $1 + \alpha/t$. The event $\{N(\Lambda t) = n\}$ then has the probability of n arrivals from the unit rate PP followed by one arrival from the α/t rate process, thus yielding the probability $q^n p$.

c) Find $\mathsf{f}_\Lambda(\lambda \mid N(t){=}n)$, the density of Λ conditional on $N(t){=}n$.

Solution: Using Bayes' law with the answers in (a) and (b), we get

$$\mathsf{f}_{\Lambda|N(t)}(\lambda \mid n) = \frac{\lambda^n e^{-\lambda(\alpha+t)}\,(\alpha+t)^{n+1}}{n!}.$$

This is an Erlang PDF of order $n{+}1$ and can be interpreted (after a little work) in the same way as (b).

d) Find $\mathsf{E}[\Lambda \mid N(t){=}n]$ and interpret your result for very small t with $n = 0$ and for very large t with n large.

Solution: Since Λ conditional on $N(t) = n$ is Erlang, it is the sum of $n+1$ IID rv's, each of mean $1/(t+\alpha)$. Thus

$$\mathsf{E}[\Lambda \mid N(t){=}n] \;=\; \frac{n+1}{\alpha+t}.$$

For $N(t) = 0$ and $t << \alpha$, this is close to $1/\alpha$, which is $\mathsf{E}[\Lambda]$. This is not surprising since it has little effect on the distribution of Λ. For n large and $t >> \alpha$, $\mathsf{E}[\Lambda \mid N(t){=}n] \approx n/t$.

e) Find $\mathsf{E}[\Lambda \mid N(t){=}n, S_1, S_2, \dots, S_n]$. (Hint: consider the distribution of $S_1, \dots, S_n$ conditional on $N(t)$ and Λ). Find $\mathsf{E}[\Lambda \mid N(t){=}n, N(\tau){=}m]$ for some $\tau < t$.

Solution: From Theorem 2.5.1, $S_1, \dots, S_n$ are uniformly distributed, subject to $0 < S_1 < \cdots < S_n < t$, given $N(t) = n$ and $\Lambda = \lambda$. Thus, conditional on $N(t) = n$, Λ is statistically independent of $S_1, \dots, S_n$.

$$\mathsf{E}[\Lambda \mid N(t){=}n, S_1, S_2, \dots, S_n] \;=\; \mathsf{E}[\Lambda \mid N(t){=}n] = \frac{n+1}{\alpha+t}.$$

Conditional on $N(t) = n$, $N(\tau)$ is determined by $S_1, \dots, S_n$ for $\tau < t$, and thus

$$\mathsf{E}[\Lambda \mid N(t){=}n, N(\tau) = m] \;=\; \mathsf{E}[\Lambda \mid N(t){=}n] = \frac{n+1}{\alpha+t}.$$

This corresponds to one's intuition; given the number of arrivals in $(0, t]$, it makes no difference where the individual arrivals occur.

Exercise 2.19: a) Use Equation (2.42) to find $\mathsf{E}[S_i \mid N(t)=n]$. Hint: When you integrate $s_i \mathsf{f}_{S_i}(s_i \mid N(t)=n)$, compare this integral with $\mathsf{f}_{S_{i+1}}(s_i \mid N(t)=n+1)$ and use the fact that the latter expression is a probability density.

Solution: We can find $\mathsf{E}[S_i \mid N(t)=n]$ from $\mathsf{f}_{S_i|N(t)}(s_i|n)$ (as given in (2.42)) by integration,

$$\mathsf{E}[S_i \mid N(t)=n] \;=\; \int_0^\infty x \mathsf{f}_{S_i|N(t)}(x|n)\,dx \;=\; \int_0^\infty \frac{x^i(t-x)^{n-i} n!}{t^n(n-i)!(i-1)!}\,dx. \tag{A.31}$$

Using the hint,

$$f_{S_{i+1}|N(t)}(x|n+1) = \frac{x^i(t-x)^{n-i}(n+1)!}{t^{n+1}(n-i)!\,i!}. \tag{A.32}$$

The factors involving x are the same as in (A.31), and substituting (A.32) into (A.31)

$$\mathsf{E}[S_i \mid N(t)=n] \;=\; \int_0^\infty \frac{it}{n+1} f_{S_{i+1}|N(t)}(x|n+1)\,dx = \frac{it}{n+1}.$$

As a check, we see that this agrees with the more elegant derivation in (2.43). The derivation in (2.43), however, does not generalize as easily to the second moment.

b) Find the second moment and the variance of S_i conditional on $N(t)=n$. Hint: Extend the previous hint.

Solution: Using the same approach as in (AP19)

$$\mathsf{E}\left[S_i^2 \mid N(t)=n\right] \;=\; \int_0^\infty x^2 \mathsf{f}_{S_i|N(t)}(x|n)\,dx \;=\; \int_0^\infty \frac{x^{i+1}(t-x)^{n-i} n!}{t^n(n-i)!(i-1)!}\,dx. \tag{A.33}$$

This suggests comparing with the density of S_{i+2} conditional on $N(t) = n+2$,

$$f_{S_{i+2}|N(t)}(x|n+2) = \frac{x^{i+1}(t-x)^{n-i}(n+2)!}{t^{n+2}(n-i)!(i+1)!}. \tag{A.34}$$

$$\mathsf{E}\left[S_i^2 \mid N(t)=n\right] \;=\; \frac{(i+1)\,i\,t^2}{(n+2)(n+1)}.$$

Finally, calculating the variance as the second moment minus the mean squared,

$$\mathsf{VAR}\left[S_i|N(t)=n\right] = \frac{(i+1)it^2}{(n+2)(n+1)} - \left[\frac{it}{n+1}\right]^2 \;=\; \frac{it^2(n+1-i)}{(n+1)^2(n+2)}.$$

c) Assume that n is odd, and consider $i = (n+1)/2$. What is the relationship between S_i, conditional on $N(t)=n$, and the sample median of n IID uniform random variables.

Solution: We have seen that the first n arrival epochs of a Poisson process, conditional on $N(t) = n$ have the same joint probability distribution as the order statistics of n IID rv's that are uniform over $(0,t]$. Thus the sample value of the rv $S_{(n+1)/2}$ is the sample median of those rv's . From (a) and (b),

$$\mathsf{E}\left[S_{(n+1)/2}\right] = t/2; \qquad \mathsf{VAR}\left[S_{(n+1)/2}\right] = \frac{t^2}{4(n+2)}.$$

d) Give a weak law of large numbers for the above median.

Solution: The median $S_{(n+1)/2}$ is a rv, just like the sample average of n rv's is a rv . By a WWLN for the median, we mean a result that says that $S_{(n+1)/2}$ converges in probability to a limit, in this case the mean of $S_{(n+1)/2}$ as $n \to \infty$. Using the Chebyshev inequality,

$$\Pr\left\{\left|S_{(n+1)/2} - \frac{t}{2}\right| \geq \epsilon\right\} \leq \frac{\mathsf{VAR}\left[S_{(n+1)/2}\right]}{\epsilon^2} = \frac{t^2}{4(n+2)\epsilon^2}.$$

Thus, for any fixed ϵ, t,

$$\lim_{n\to\infty} \Pr\left\{\left|S_{(n+1)/2} - \frac{t}{2}\right| \geq \epsilon\right\} = 0.$$

The limit here does not correspond to any limit in a given Poisson process, but it does correspond to a limit of the median of n uniformly distributed IID rv's. By combining this result with Exercise (1.28), it is possible to extend this result to the median of IID rv's with an arbitrary probability density.

Exercise 2.20: Suppose cars enter a one-way infinite length, infinite lane highway at a Poisson rate λ. The ith car to enter chooses a velocity V_i and travels at this velocity. Assume that the V_i's are independent positive rv's having a common CDF F. Derive the distribution of the number of cars that are located in an interval $(0, a)$ at time t.

Solution: This is a thinly disguised variation of an M/G/∞ queue. The arrival process is the Poisson process of cars entering the highway. We then view the service time of a car as the time interval until the car reaches or passes point a. All cars then have IID service times, and service always starts at the time of arrival (*i.e.*, this can be viewed as infinitely many independent and identical servers). To avoid distractions, assume initially that V is a continuous rv. The CDF $G(\tau)$ of the service time X is then given by the equation

$$G(\tau) = \Pr\{X \leq \tau\} = \Pr\{a/V \leq \tau\} = \Pr\{V \geq a/\tau\} = \mathsf{F}^{\mathsf{c}}_V(a/\tau).$$

The PMF of the number $N_1(t)$ of cars in service at time t is then given by (2.36) and (2.37) as

$$\mathsf{p}_{N_1(t)}(n) = \frac{m^n(t)\exp[-m(t)]}{n!},$$

where

$$m(t) = \lambda\int_0^t [1 - G(\tau)]\, d\tau = \lambda\int_0^t \mathsf{F}_V(a/\tau)\, d\tau.$$

Since this depends only on the CDF, it can be seen that the answer is the same if V is discrete or mixed.

Exercise 2.21: Consider an M/G/∞ queue, i.e., a queue with Poisson arrivals of rate λ in which each arrival i, independent of other arrivals, remains in the system for a time X_i, where $\{X_i;\, i \geq 1\}$ is a set of IID rv's with some given CDF $\mathsf{F}(x)$.

You may assume that the number of arrivals in any interval $(t, t+\epsilon)$ that are still in the system at some later time $\tau \geq t+\epsilon$ is statistically independent of the number of arrivals in that same interval $(t, t+\epsilon)$ that have departed from the system by time τ.

a) Let $N(\tau)$ be the number of customers in the system at time τ. Find the mean, $m(\tau)$, of $N(\tau)$ and find $\Pr\{N(\tau) = n\}$.

Solution: The answer is worked out and explained in (2.36) and (2.37). We have

$$m(\tau) = \lambda \int_0^\tau [1 - \mathsf{F}(t)]\, dt; \qquad \Pr\{N(\tau) = n\} = \frac{m(\tau)^n \exp(-m(\tau))}{n!}.$$

Note that $m(\tau)$ is non-decreasing in τ and has the limit $\lim_{\tau\to\infty} m(\tau) = \lambda\overline{X}$.

b) Let $D(\tau)$ be the number of customers that have departed from the system by time τ. Find the mean, $\mathsf{E}[D(\tau)]$, and find $\Pr\{D(\tau) = d\}$.

Solution: As illustrated in Figure 2.8, The number of arrivals that have departed and those that remain can be treated as a non-homogeneous splitting of the Poisson arrival process. Thus the departures can be handled in the same way as those still in the system. Let $\mu(\tau) = \mathsf{E}[D(\tau)]$ be the mean of the number of arrivals that have departed by time t. Then $\mu(\tau)$ and $\Pr\{D(\tau) = d\}$ are given by

$$\mu(\tau) = \lambda \int_0^\tau \mathsf{F}(t)\, dt; \qquad \Pr\{D(\tau) = d\} = \frac{\mu(\tau)^d \exp(-\mu(\tau))}{d!}.$$

Note that $m(\tau) + \mu(\tau) = \lambda \int_0^\tau dt = \lambda\tau$ so that $\mu(\tau)$ tends to $\lambda\tau - \lambda\overline{X}$ as τ increases.

c) Find $\Pr\{N(\tau) = n, D(\tau) = d\}$.

Solution: By the same argument as used in Section 2.3 on the splitting of homogeneous Poisson processes, the processes $\{N(\tau); \tau > 0\}$ and the process $\{D(\tau); \tau > 0\}$ are statistically independent. The assumption in the exercise is one step in that argument. Combining (a) and (b), we then have

$$\Pr\{N(\tau) = n, D(\tau) = d\} = \frac{m(\tau)^n \mu(\tau)^d \exp\big(-m(\tau) - \mu(\tau)\big)}{n!\, d!}. \tag{A.35}$$

d) Let $A(\tau)$ be the total number of arrivals up to time τ. Find $\Pr\{N(\tau) = n \mid A(\tau) = a\}$.

Solution: Since $A(\tau) = N(\tau) + D(\tau)$, we can let $a = n + d$ for $a \geq n$ and rewrite (A.35) as

$$\Pr\{N(\tau) = n, A(\tau) = a\} = \frac{m(\tau)^n \mu(\tau)^{a-n} \exp\big(-m(\tau) - \mu(\tau)\big)}{n!\,(a-n)!}. \tag{A.36}$$

Since $A(\tau)$ is Poisson with rate $\lambda = [m(\tau) + \mu(\tau)]/\tau$, the conditional PMF is

$$\begin{aligned}\Pr\{N(\tau) = n \mid A(\tau) = a\} &= \frac{m(\tau)^n \mu(\tau)^{a-n} \exp\big(-\lambda\tau\big)}{n!\,(a-n)!} \times \frac{a!}{(\lambda t)^a \exp(-\lambda t)} \\ &= \binom{a}{n} \left[\frac{m(\tau)}{\lambda\tau}\right]^n \left[\frac{\mu(\tau)}{\lambda\tau}\right]^{a-n}.\end{aligned}$$

e) Find $\Pr\{D(\tau+\epsilon) - D(\tau) = d\}$.

Solution: Since $\{D(\tau); \tau > 0\}$ is a non-homogeneous Poisson process, $Y = D(\tau+\epsilon) - D(\tau)$ is a Poisson random variable of mean $\overline{Y} = \lambda \int_{\tau}^{\tau+\epsilon} \mathsf{F}(t)\, dt$. Thus

$$\Pr\{Y = d\} = \frac{\overline{Y}^d \exp(-\overline{Y})}{d!}.$$

Note that Y is the number of departures in $(\tau,\, \tau+\epsilon]$. This is a Poisson rv which has a mean approaching $\lambda\epsilon$ for $\tau \to \infty$, The output process is a non-homogeneous Poisson process which becomes stationary (homogeneous) as $\tau \to \infty$.

Exercise 2.22: The voters in a given town arrive at the place of voting according to a Poisson process of rate $\lambda = 100$ voters per hour. The voters independently vote for candidate A and candidate B each with probability $1/2$. Assume that the voting starts at time 0 and continues indefinitely.

a) Conditional on 1000 voters arriving during the first 10 hours of voting, find the probability that candidate A receives n of those votes.

Solution: Intuitively, each of the 1000 voters votes independently for A with probability $1/2$. Thus, from the binomial formula,

$$\Pr\{n \text{ of } 1000 \text{ votes for } A\} = \binom{1000}{n} (1/2)^{1000}.$$

Being more careful, the number of votes for A in 10 hours is Poisson with mean 500 and the number for B is independent and Poisson with mean 500. The total number of votes in 10 hours is the sum of these and also Poisson. Using Bayes' law to find the votes for A conditional on the overall number, we get the same answer.

b) Again conditional on 1000 voters during the first 10 hours, find the probability that candidate A receives n votes in the first 4 hours of voting.

Solution: Intuitively, each of the 1000 voters comes in the first 4 hours independently with probability .4, and out of those, each independently votes for A with probability $1/2$. Thus each voter independently both comes in the first 4 hours and votes for A with probability 0.2. Thus

$$\Pr\{n \text{ votes for } A \text{ in first 4 hours} \mid 1000 \text{ in 10 hours}\} = \binom{1000}{n} (0.2)^n (0.8)^{1000-n}.$$

Being more careful (or first stating a general theorem about these conditional Poisson probabilities), the number in the first 4 hours is Poisson, the number of those who vote for A is Poisson and the overall number to vote in 10 hours is Poisson and the sum of those in the first 4 hours and those in the last 6 hours, further broken into A and B are independent Poisson. We can then find the conditional probability as in (a).

c) Let T be the epoch of the arrival of the first voter voting for candidate A. Find the density of T.

Solution: View the voters for A as a splitting of the overall arrival process. Thus the voters for A form a Poisson process of rate 50 and $\mathsf{f}_T(t) = 50\exp(-50t)$.

d) Find the PMF of the number of voters for candidate B who arrive before the first voter for A.

Solution: n B voters arrive before the first A if the first n voters are B and the $(n+1)$st is A; this is an event of probability $(1/2)^{n+1}$ for $n \geq 0$. Thus this number is a geometric rv.

e) Define the nth voter as a *reversal* if the nth voter votes for a different candidate than the $(n-1)^{st}$. For example, in the sequence of votes $AABAABB$, the third, fourth, and sixth voters are reversals; the third and sixth are A to B reversals and the fourth is a B to A reversal. Let $N(t)$ be the number of reversals up to time t (t in hours). Is $\{N(t);\, t > 0\}$ a Poisson process? Explain.

Solution: The first voter can not be a reversal. Every subsequent voter is a reversal with probability 1/2. Thus, after the first arrival, each inter-reversal interval is exponential with mean 1/50 hours. The process is not Poisson, however, because the interval until the first reversal is not exponential with mean 1/50 hours.

When we study renewal processes, we denote renewal processes with a non-standard first renewal interval as 'delayed renewal processes' and find that most of the renewal results also apply to delayed renewal processes. Thus we might call this a delayed Poisson process.

f) Find the expected time (in hours) between reversals.

Solution: Starting from one reversal we can view subsequent reversals as an equi-probable splitting of the arrivals. Thus the expected time between reversals is 1/50.

g) Find the probability density of the time between reversals.

Solution: As explained in (f), the time X between reversals is exponential with mean 1/50, so $\mathsf{f}_X(x) = 50\exp(-50t)$.

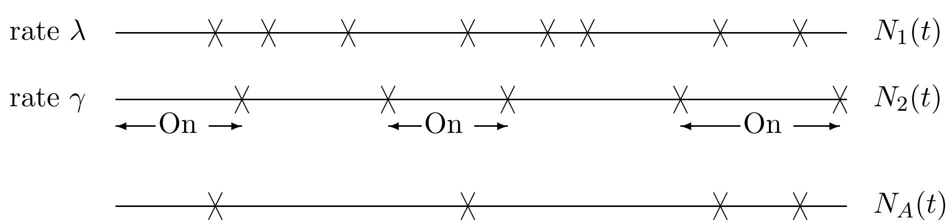
h) Find the density of the time from one A to B reversal to the next A to B reversal.

Solution: Since A to B reversals alternate with B to A reversals, the time Y from one A to B reversal to the next is the sum of 2 independent exponential rv's. One can either calculate this density by convolution or recognize it as an Erlang rv of order 2.

$$\mathsf{f}_Y(y) = \lambda^2 y \exp(-\lambda y) \qquad \text{where } \lambda = 50 \text{ and } y \geq 0.$$

An alternate approach here is to see that after an A to B reversal, the time to the next A to B reversal is the sum of the time to the next A followed by the time to the next B.

Exercise 2.23: Let $\{N_1(t);\, t > 0\}$ be a Poisson counting process of rate λ. Assume that the arrivals from this process are switched on and off by arrivals from a second independent Poisson process $\{N_2(t); t > 0\}$ of rate γ.



Let $\{N_A(t);\, t{>}0\}$ be the switched process; that is $N_A(t)$ includes the arrivals from $\{N_1(t);\, t > 0\}$ during periods when $N_2(t)$ is even and excludes the arrivals from $\{N_1(t);\, t > 0\}$ while $N_2(t)$ is odd.

a) Find the PMF for the number of arrivals of the first process, $\{N_1(t);\, t > 0\}$, during the nth period when the switch is on.

Solution: We have seen that the combined process $\{N_1(t) + N_2(t)\}$ is a Poisson process of rate $\lambda+\gamma$. For any even numbered arrival to process 2, subsequent arrivals to the combined process independently come from process 1 or 2, and come from process 1 with probability $\lambda/(\lambda+\gamma)$. The number N_s of such arrivals before the next arrival to process 2 is geometric with PMF $\mathsf{p}_{N_s}(n) = \big[\lambda/(\lambda+\gamma)\big]^n\big[\gamma/(\lambda+\gamma)\big]$ for integer $n \geq 0$.

b) Given that the first arrival for the second process occurs at epoch τ, find the conditional PMF for the number of arrivals N_a of the first process up to τ.

Solution: Since processes 1 and 2 are independent, this is equal to the PMF for the number of arrivals of the first process up to τ. This number has a Poisson PMF, $(\lambda\tau)^n e^{-\lambda\tau}/n!$.

c) Given that the number of arrivals of the first process, up to the first arrival for the second process, is n, find the density for the epoch of the first arrival from the second process.

Solution: Let N_a be the number of process 1 arrivals before the first process 2 arrival and let X_2 be the time of the first process 2 arrival. In (a), we showed that $\mathsf{p}_{N_a}(n) = \big[\lambda/(\lambda+\gamma)\big]^n\big[\gamma/(\lambda+\gamma)\big]$ and in (b) we showed that $\mathsf{p}_{N_a|X_2}(n|\tau) = (\lambda\tau)^n e^{-\lambda\tau}/n!$. We can then use Bayes' law to find $\mathsf{f}_{X_2|N_a}(\tau \mid n)$, which is the desired solution. We have

$$\mathsf{f}_{X_2|N_a}(\tau \mid n) \;=\; \mathsf{f}_{X_2}(\tau)\frac{\mathsf{p}_{N_a|X_2}(n|\tau)}{\mathsf{p}_{N_a}(n)} \;=\; \frac{(\lambda+\gamma)^{n+1}\tau^n e^{-(\lambda+\gamma)\tau}}{n!},$$

where we have used the fact that X_2 is exponential with PDF $\gamma\exp(-\gamma\tau)$ for $\tau \geq 0$. It can be seen that the solution is an Erlang rv of order $n+1$. To interpret this (and to solve the exercise in a perhaps more elegant way), note that this is the same as the Erlang density for the epoch of the $(n{+}1)$th arrival in the combined process. This arrival epoch is independent of the process 1/process 2 choices for these $n{+}1$ arrivals, and thus is the arrival epoch for the particular choice of n successive arrivals to process 1 followed by 1 arrival to process 2.

d) Find the density of the interarrival time for $\{N_A(t); t \geq 0\}$. Note: This part is quite messy and is done most easily via Laplace transforms.

Solution: The process $\{N_A(t);\, t > 0$ is not a Poisson process, but, perhaps surprisingly, it is a renewal process; that is, the interarrival times are independent and identically distributed. One might prefer to postpone trying to understand this until starting to study renewal processes, but we have the necessary machinery already.

Starting at a given arrival to $\{N_A(t);\, t > 0\}$, let X_A be the interval until the next arrival to $\{N_A(t);\, t > 0\}$ and let X be the interval until the next arrival to the combined process. Given that the next arrival in the combined process is from process 1, it will be an arrival to $\{N_A(t);\, t > 0\}$, so that under this condition, $X_A = X$. Alternatively, given that this next arrival is from process 2, X_A will be the sum of three independent rv's, first X, next, the interval X_2 to the following arrival for process 2, and next the interval from that point to the following arrival to $\{N_A(t);\, t > 0\}$. This final interarrival time will have the same distribution as X_A. Thus the unconditional PDF for X_A is given by

$$\begin{aligned} \mathsf{f}_{X_A}(x) &= \frac{\lambda}{\lambda+\gamma}\mathsf{f}_X(x) + \frac{\gamma}{\lambda+\gamma}\mathsf{f}_X(x) \otimes \mathsf{f}_{X_2}(x) \otimes \mathsf{f}_{X_A}(x) \\ &= \lambda \exp(-(\lambda+\gamma)x) + \gamma \exp(-(\lambda+\gamma)x) \otimes \gamma \exp(-\gamma x) \otimes \mathsf{f}_{X_A}(x). \end{aligned}$$

where $\otimes$ is the convolution operator and all functions are 0 for $x < 0$.

Solving this by Laplace transforms is a mechanical operation of no real interest here. The solution is

$$\mathsf{f}_{X_A}(x) = B\,\exp\left[-\frac{x}{2}\left(2\gamma+\lambda + \sqrt{4\gamma^2+\lambda^2}\right)\right] + C\,\exp\left[-\frac{x}{2}\left(2\gamma+\lambda - \sqrt{4\gamma^2+\lambda^2}\right)\right],$$

where

$$B = \frac{\lambda}{2}\left(1 + \frac{\lambda}{\sqrt{4\gamma^2+\lambda^2}}\right); \qquad C = \frac{\lambda}{2}\left(1 - \frac{\lambda}{\sqrt{4\gamma^2+\lambda^2}}\right).$$

Exercise 2.24 : Let us model the chess tournament between Fisher and Spassky as a stochastic process. Let X_i, for $i \geq 1$, be the duration of the ith game and assume that $\{X_i;\ i{\geq}1\}$ is a set of IID exponentially distributed rv's each with density $\mathsf{f}_X(x) = \lambda e^{-\lambda x}$. Suppose that each game (independently of all other games, and independently of the length of the games) is won by Fisher with probability p, by Spassky with probability q, and is a draw with probability $1-p-q$. The first player to win n games is defined to be the winner, but we consider the match up to the point of winning as being embedded in an unending sequence of games.

a) Find the distribution of time, from the beginning of the match, until the completion of the first game that is won (i.e., that is not a draw). Characterize the process of the number $\{N(t);\ t > 0\}$ of games won up to and including time t. Characterize the process of the number $\{N_F(t); t \geq 0\}$ of games won by Fisher and the number $\{N_S(t); t \geq 0\}$ won by Spassky.

Solution: The Poisson game process is split into 3 independent Poisson processes, namely the draw process of rate $(1-p-q)\lambda$, the Fisher win process of rate $p\lambda$ and the Spassky win process of rate $q\lambda$. The process of wins is the sum of the Fisher and Spassky win processes, and is independent of the draw process. Thus the time to the first win is an exponential rv X_W of rate $(p+q)\lambda$. Thus the density and CDF are

$$\mathsf{f}_{X_W}(x) = (p+q)\lambda \exp(-(p+q)\lambda x), \qquad \mathsf{F}_{X_W}(x) = 1 - \exp(-(p+q)\lambda x).$$

b) For the remainder of the problem, assume that the probability of a draw is zero; i.e., that $p+q = 1$. How many of the first $2n-1$ games must be won by Fisher in order to win the match?

Solution: Note that (a) shows that the process of wins is Poisson within the process of games including draws, and thus the assumption that there are no draws (*i.e.*, $p+q=1$) only simplifies the notation slightly. Note also that it makes no difference to the probability of winning the match whether they continue playing beyond the game in which one of them first wins n games.

If Fisher wins n or more of the first $2n{-}1$ games, then Spassky wins at most $n{-}1$, so Fisher wins the match. Conversely if Fisher wins the match, he must win n or more of the first $2n{-}1$ games. Thus Fisher wins if and only if he wins n or more of the first $2n{-}1$ games.

c) What is the probability that Fisher wins the match? Your answer should not involve any integrals. Hint: consider the unending sequence of games and use (b).

Solution: The sequence of games is Bernoulli with probabiity p of a Fisher game win. Thus, using the binomial PMF for $2n-1$ plays, the probability that Fisher wins the match is

$$\Pr\{\text{Fisher wins match}\} = \sum_{k=n}^{2n-1} \binom{2n-1}{k} p^k q^{2n-1-k}.$$

Without the hint, the problem is more tricky, but no harder computationally. The probability that Fisher wins the match at the end of game k, for $n \le k \le 2n-1$, is the probabiity that he wins $n-1$ games out the first $k-1$ and then wins the kth. This is p times $\binom{k-1}{n-1} p^{n-1} q^{k-1}$. Thus

$$\Pr\{\text{Fisher wins match}\} = \sum_{k=n}^{2n-1} \binom{k-1}{n-1} p^n q^{k-n}.$$

It is surprising that these very different appearing expressions are the same.

d) Let T be the epoch at which the match is completed (i.e., either Fisher or Spassky wins). Find the CDF of T.

Solution: Let T_f be the time at which Fisher wins his nth game and T_s be the time at which Spassky wins his nth game (again assuming that playing continues beyond the winning of the match). The Poisson process of Fisher wins is independent of that of Spassky wins. Also, the time T at which the match ends is the minimum of T_f and T_s, so, for any $t > 0$, $\Pr\{T > t\} = \Pr\{T_f > t, T_s > t\}$. Thus

$$\Pr\{T > t\} = \Pr\{T_f > t\} \Pr\{T_s > t\}. \tag{A.37}$$

Now T_f has an Erlang distribution so its complementry CDF is equal to the probability that fewer than n Fisher wins have occured by time t. The number of Fisher wins is a Poisson rv, and Spassky wins are handled the same way. Thus,

$$\Pr\{T > t\} = \sum_{k=0}^{n-1} \frac{(\lambda pt)^k e^{-\lambda pt}}{k!} \sum_{j=0}^{n-1} \frac{(\lambda qt)^j e^{-\lambda qt}}{j!} \; = \; e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda pt)^k}{k!} \sum_{j=0}^{n-1} \frac{(\lambda qt)^j}{j!}.$$

Finally $\mathsf{F}_T(t) = 1 - \Pr\{T > t\}$.

e) Find the probability that Fisher wins and that T lies in the interval $(t, t+\delta)$ for arbitrarily small δ.

Solution: From (A.37), the PDF $\mathsf{f}_T(t)$ is given by

$$\mathsf{f}_T(t) = \mathsf{f}_{T_f}(t)\Pr\{T_s > t\} + \mathsf{f}_{T_s}(t)\Pr\{T_f > t\}.$$

The first term on the right is associated with a Fisher win, and the second with a Spassky win. Thus

$$\lim_{\delta \to 0} \frac{\Pr\{\text{Fisher wins}, T \in [t, t+\delta)\}}{\delta} = \frac{(\lambda p)^n t^{n-1} e^{-\lambda pt}}{(n-1)!} \sum_{j=0}^{n-1} \frac{(\lambda qt)^j e^{-\lambda qt}}{j!}. \tag{A.38}$$

The main reason for calculating this is to point out that T and the event that Fisher wins the match are not independent events. The number of games that Fisher wins up to time t is independent of the number that Spassky wins up to t. Also, given that k games have been played by time t, the distribution of the number of those games won by Fisher does not vary with t. The problem here, given a Fisher win with $T = t$, the most likely number of Spassky wins is 0 when t is very small and $n-1$ when t is very large. This can be seen by comparing the terms in the sum of (A.38). This is not something you could reasonably be expected to hypothesize intuitively; it is simply something that indicates that one must sometimes be very careful.

Exercise 2.25: **a)** For $1 \le i < n$, find the conditional density of S_{i+1}, conditional on $N(t) = n$ and $S_i = s_i$.

Solution: Recall from (2,41) that

$$\Pr\{S_1 > \tau \mid N(t) = n\} \;=\; \Pr\{X_1 > \tau \mid N(t) = n\} \;=\; \left(\frac{t-\tau}{t}\right)^n.$$

Given that $S_i = s_i$, we can apply this same formula to $\widetilde{N}(s_i, t)$ for the first arrival after s_i.

$$\Pr\{X_{i+1}{>}\tau \mid N(t){=}n, S_i{=}s_i\} \;=\; \Pr\left\{X_{i+1}{>}\tau \mid \widetilde{N}(s_i,t){=}n{-}i, S_i{=}s_i\right\} \;=\; \left(\frac{t{-}s_i{-}\tau}{t{-}s_i}\right)^{n-i}.$$

Since $S_{i+1} = S_i + X_{i+1}$, we get

$$\begin{aligned}
\Pr\{S_{i+1}{>}s_{i+1} \mid N(t){=}n, S_i{=}s_i\} &= \left(\frac{t{-}s_{i+1}}{t{-}s_i}\right)^{n-i} \\
\mathsf{f}_{S_{i+1}|N(t)S_i}\big(s_{i+1} \mid n, s_i\big) &= \frac{(n-i)(t-s_{i+1})^{n-i-1}}{(t-s_i)^{n-i}}.
\end{aligned} \tag{A.39}$$

b) Use (a) to find the joint density of $S_1, \dots, S_n$ conditional on $N(t) = n$. Verify that your answer agrees with (2.38).

Solution: For each i, the conditional probability in (a) is clearly independent of $S_{i-2}, \dots, S_1$. Thus we can use the chain rule to multiply (A35) by itself for each value of i. We must also include $\mathsf{f}_{S_1|N(t)}\big(s_1 \mid n\big) = n(t-s_1)^{n-1}/t^n$. Thus

$$\begin{aligned}
\mathsf{f}_{\boldsymbol{S}^{(n)}|N(t)}\big(\boldsymbol{s}^{(n)}|n\big) &= \frac{n(t{-}s_1)^{n-1}}{t^n} \cdot \frac{(n{-}1)(t{-}s_2)^{n-2}}{(t-s_1)^{n-1}} \cdot \frac{(n{-}2)(t{-}s_3)^{n-3}}{(t-s_2)^{n-2}} \cdots \frac{(t{-}s_n)^0}{t{-}s_{n-1}} \\
&= \frac{n!}{t^n}.
\end{aligned}$$

Note: There is no great insight to be claimed from this exercise. it is useful, however, in providing some additional techniques for working with such problems.

Exercise 2.26: A two-dimensional Poisson process is a process of randomly occurring special points in the plane such that (i) for any region of area A the number of special points in that region has a Poisson distribution with mean λA, and (ii) the number of special points in nonoverlapping regions is independent.

For such a process consider an arbitrary location in the plane and let X denote its distance from its nearest special point (where distance is measured in the usual Euclidean manner). Show that

a) $\Pr\{X > t\} = \exp(-\lambda\pi t^2)$.

Solution: Given an arbitrary location, $X > t$ if and only if there are no special points in the circle of radius t around the given point. The expected number in that circle is $\lambda\pi t^2$, and since the number in that circle is Poisson with expected value $\lambda\pi t^2$, the probability that number is 0 is $e^{-\lambda\pi t^2}$. Thus $\Pr\{X > t\} = e^{-\lambda\pi t^2}$.

b) $\mathsf{E}[X] = 1/(2\sqrt{\lambda})$.

Solution: Since $X \geq 0$, we have

$$\mathsf{E}[X] \;=\; \int_0^\infty \Pr\{X > t\}\, dt \;=\; \int_0^\infty \exp(-\lambda\pi t^2)\, dt.$$

We can look this up in a table of integrals, or recognize its resemblance to the Gaussian PDF. If we define $\sigma^2 = 1/(2\pi\lambda)$, the above integral is

$$\mathsf{E}[X] \;=\; \sigma\sqrt{2\pi}\int_0^\infty \frac{1}{\sigma\sqrt{2\pi}}\exp\left[\frac{-t^2}{2\sigma^2}\right] dt \;=\; \frac{\sigma\sqrt{2\pi}}{2} \;=\; \frac{1}{2\sqrt{\lambda}}.$$

Exercise 2.27: This problem is intended to show that one can analyze the long term behavior of queueing problems by using just notions of means and variances, but that such analysis is awkward, justifying understanding the strong law of large numbers. Consider an M/G/1 queue. The arrival process is Poisson with $\lambda = 1$. The expected service time, $\mathsf{E}[Y]$, is 1/2 and the variance of the service time is given to be 1.

a) Consider S_n, the time of the nth arrival, for $n = 10^{12}$. With high probability, S_n will lie within 3 standard derivations of its mean. Find and compare this mean and the 3σ range.

Solution: Let X_i be the ith interarrival time. Then $S_n = \sum_{i=1}^n X_i$, so $\mathsf{E}[S_n] = n$ and $\mathsf{VAR}[S_n] = n$. Thus $\mathsf{E}[S_{10^{12}}] = 10^{12}$ with 3×10^6 as the 3 sigma point.

b) Let V_n be the total amount of time during which the server is busy with these n arrivals (i.e., the sum of 10^{12} service times). Find the mean and 3σ range of V_n.

Solution: In the same way, $\mathsf{E}[V_{10^{12}}] = 5 \times 10^{11}$ with 3×10^6 as the 3 sigma point.

c) Find the mean and 3σ range of I_n, the total amount of time the server is idle up until S_n (take I_n as $S_n - V_n$, thus ignoring any service time after S_n).

Solution: As before $\mathsf{E}[I_{10^{12}}] = 5 \times 10^{11}$. Since interarrival times and service times are independent and $-V$ and V have the same variance, the 3 sigma point of I_n is 6×10^6.

d) An idle period starts when the server completes a service and there are no waiting arrivals; it ends on the next arrival. Find the mean and variance of an idle period. Are successive idle periods IID?

Solution: The time from the beginning of an idle period to the next arrival is exponential with mean 1 and variance 1. Since the time at which the nth idle period starts is a function only of the first n interarrival times and service times, and since for each sample value of the beginning of the nth idle period, the interval until the next arrival is independent of those arrivals (by the independent increment property) and independent of those service times, it follows that the interval until the next arrival is independent of those earlier interarrival intervals and service times. This interval (*i.e.*, this idle period) is thus independent of all earlier idle periods.

This is one of those peculiar situations where the independence of idle durations is virtually obvious but a careful demonstration is quite tricky. This type of argument will become clearer when we study renewal processes.

e) Combine (c) and (d) to estimate the total number of idle periods up to time S_n. Use this to estimate the total number of busy periods.

Solution: The aggregate idle time up to S_n can be expressed as $n/2 \pm 6\sqrt{n}$ and the aggregate duration of m idle periods can be expressed as $m \pm 3\sqrt{m}$. Letting $m(n)$ be the number of idle periods up to S_n, we then have $n/2 \pm 6\sqrt{n} \approx m \pm 3\sqrt{m}$. For n very large (10^{12}), the square roots are small relative to the linear terms, so we can argue that $m(n) \approx n/2 \pm [6\sqrt{n} + 3\sqrt{n/2}]$. One can do this more carefully by looking at the maximum with 3 sigma limits of one expression and the minimum of the other, but however this is expressed, the number or idle periods per arrival is increasingly close to 1/2 with high probability as the number of arrivals is increased.

The period from 0 to S_n starts with an idle period and ends with a busy period, so that $m(n)$ is also the number of busy periods.

f) Combine (e) and (b) to estimate the expected length of a busy period.

Solution: Let B be the expected duration of a busy period. Then the expected aggregate duration of m busy periods is mB. Since the expected aggregate busy time for large m is equal to the expected aggregate idle time, we have $\mathsf{E}[B] \approx 1$.

It is curious to note that this does not depend on the variance of the service time (beyond ensuring that the standard deviation of 10^{12} service times is small compared to the mean).

Exercise 2.28: The purpose of this problem is to illustrate that for an arrival process with independent but not identically distributed interarrival intervals, $X_1, X_2, \dots$, the number of arrivals $N(t)$ in the interval $(0, t]$ can be a defective rv. In other words, the 'counting process' is not a stochastic process according to our definitions. This illustrates that it is necessary to prove that the counting rv's for a renewal process are actually rv's .

a) Let the CDF of the ith interarrival interval for an arrival process be $F_{X_i}(x_i) = 1 - \exp(-\alpha^{-i} x_i)$ for some fixed $\alpha \in (0, 1)$. Let $S_n = X_1 + \cdots + X_n$ and show that

$$\mathsf{E}[S_n] = \frac{\alpha(1 - \alpha^n)}{1 - \alpha}.$$

Solution: Each X_i is an exponential rv, but the rate, α^{-i}, is rapidly increasing with i and the expected interarrival time, $\mathsf{E}[X_i] = \alpha^i$, is rapidly decreasing with i. Thus

$$\mathsf{E}[S_n] = \alpha + \alpha^2 + \cdots \alpha^n.$$

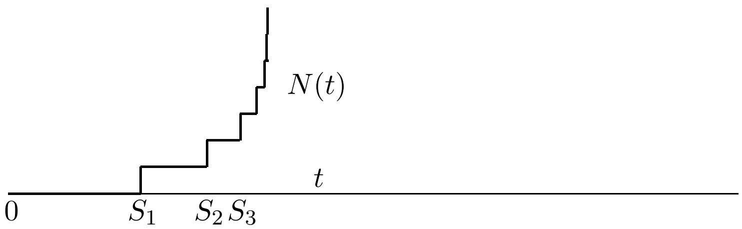
Recalling that $1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} = (1 - \alpha^n)/(1 - \alpha)$,

$$\begin{aligned} \mathsf{E}[S_n] &= \alpha(1 + \alpha + \cdots \alpha^{n-1}) \\ &= \frac{\alpha(1 - \alpha^n)}{1 - \alpha} < \frac{\alpha}{1 - \alpha}. \end{aligned}$$

In other words, not only is $\mathsf{E}[X_i]$ decaying to 0 geometrically with increasing i, but $\mathsf{E}[S_n]$ is upper bounded, for all n, by $\alpha/(1-\alpha)$.

b) Sketch a 'reasonable' sample function for $N(t)$.

Solution: Since the expected interarrival times are decaying geometrically and the expected arrival epochs are bounded for all n, it is reasonable for a sample path to have the following shape:



Note that the question here is not precise (there are obviously many sample paths, and which are 'reasonable' is a matter of interpretation). The reason for drawing such sketches is to acquire understanding to guide the solution to the following parts of the problem.

c) Find $\sigma^2_{S_n}$.

Solution: Since X_i is exponential, $\sigma^2_{X_i} = \alpha^{2i}$. Since the X_i are independent,

$$\begin{aligned}
\sigma^2_{S_n} &= \sigma^2_{X_1} + \sigma^2_{X_2} \cdots + \sigma^2_{X_n} \\
&= \alpha^2 + \alpha^4 + \cdots + \alpha^{2n} \\
&= \alpha^2(1 + \alpha^2 + \cdots \alpha^{2(n-1)}) \\
&= \frac{\alpha^2(1-\alpha^{2n})}{1-\alpha^2} < \frac{\alpha^2}{1-\alpha^2}.
\end{aligned}$$

d) Use the Markov inequality on $\Pr\{S_n \geq t\}$ to find an upper bound on $\Pr\{N(t) \leq n\}$ that is smaller than 1 for all n and for large enough t. Use this to show that $N(t)$ is defective for large enough t.

Solution: The figure suggests (but does not prove) that for typical sample functions (and in particular for a set of sample functions of non-zero probability), $N(t)$ goes to infinity for finite values of t. If the probability that $N(t) \leq n$ (for a given t) is bounded, independent of n, by a number strictly less than 1, then that $N(t)$ is a defective rv rather than a true rv.

By the Markov inequality,

$$\begin{aligned}
\Pr\{S_n \geq t\} &\leq \frac{\overline{S}_n}{t} \leq \frac{\alpha}{t(1-\alpha)} \\
\Pr\{N(t) < n\} &= \Pr\{S_n > t\} \leq \Pr\{S_n \geq t\} \leq \frac{\alpha}{t(1-\alpha)}.
\end{aligned}$$

where we have used (2.3). Since this bound is independent of n, it also applies in the limit, *i.e.*,

$$\lim_{n\to\infty} \Pr\{N(t) \leq n\} \leq \frac{\alpha}{t(1-\alpha)}.$$

For any $t > \alpha/(1-\alpha)$, we see that $\frac{\alpha}{t(1-\alpha)} < 1$. Thus $N(t)$ is defective for any such t, *i.e.*, for any t greater than $\lim_{n\to\infty} \mathsf{E}[S_n]$.

Actually, by working harder, it can be shown that $N(t)$ is defective for all $t > 0$. The outline of the argument is as follows: for any given t, we choose an m such that $\Pr\{S_m \le t/2\} > 0$ and such that $\Pr\{S_\infty - S_m \le t/2\} > 0$ where $S_\infty - S_m = \sum_{i=m+1}^{\infty} X_i$. The second inequality can be satisfied for m large enough by the Markov inequality. The first inequality is then satisfied since S_m has a density that is positive for $t > 0$.