National University of Singapore
Department of Electrical & Computer Engineering

Examination for

EE5137 Stochastic Processes

(Semester I, 2018/19)

November/December 2018

Time Allowed: 2.5 hours

INSTRUCTIONS FOR CANDIDATES:

- This paper contains **FOUR (4)** questions, printed on **FIVE (5)** pages.
- The total number of marks is 100.
- Answer all questions.
- Programmable calculators are NOT allowed.
- Electronic communicating devices MUST be turned off and inaccessible throughout the examination. They CANNOT be used as calculators, timers or clocks.
- You are allowed to bring ONE (A4) size help sheet.
- No other material is allowed.

Question 1

1(a) A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel, The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours of travel. If we assume that the miner is at all times equally likely to choose any one of the doors, let us find the expected time for him to get to safety X using the law of iterated expectations. Let $Y \in \{1, 2, 3\}$ be the identity of the door that he initially chooses.

(i) (3 points) It is known that

$$\mathbb{E}[X|Y = 1] = a_1\mathbb{E}[X] + b_1, \quad \mathbb{E}[X|Y = 2] = a_2\mathbb{E}[X] + b_2 \quad \mathbb{E}[X|Y = 3] = a_3\mathbb{E}[X] + b_3$$

Find the constants $a_1, b_1, a_2, b_2, a_3, b_3$.

(ii) (8 points) Use the law of iterated expectations and the above part to find $\mathbb{E}[X]$.

1(b) Here we will show that convergence in probability does not imply convergence in mean, i.e., that $X_n \xrightarrow{\text{p}} b$ as $n \to \infty$ does not imply that $\mathbb{E}[X_n] \longrightarrow b$ as $n \to \infty$. Consider the sequence of random variables $\{X_n\}_{n=1}^{\infty}$, each with probability mass function

$$\Pr(X_n = n^2) = \frac{1}{n}, \qquad \Pr(X_n = 0) = 1 - \frac{1}{n}.$$

Please provide detailed justifications to each of the following problems.

(i) (7 points) It is known that $X_n \xrightarrow{\text{p}} b$ for $n \to \infty$ or some $b \in \mathbb{R} \cup \{\pm\infty\}$. Find b.

(ii) (7 points) It is known that $\mathbb{E}[X_n] \longrightarrow c$ for $n \to \infty$ for some $c \in \mathbb{R} \cup \{\pm\infty\}$. Find c.

Question 2

2(a) (6 points) Consider the hypothesis test

$$\mathsf{H}_0 : X \sim \mathcal{N}(x; 0, \sigma_0^2) \qquad \mathsf{H}_1 : X \sim \mathcal{N}(x; 0, \sigma_1^2)$$

where $\sigma_1 > \sigma_0$. Show that whatever the prior probabilities of H_0 and H_1, we will decide in favor of H_1 if and only if x belongs to

$$\mathcal{Z} := \{x : x < -\gamma\} \cup \{x : x > +\gamma\}$$

for some $\gamma > 0$.

Hint: Diagrams of the two pdfs would help. You can use the fact that the normal pdf takes the form

$$\mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

2(b) Let X_1, X_2 and X_3 be three IID Bernoulli random variables with $\Pr(X_i = 1) = p$ for $i \in \{1, 2, 3\}$. This means that $\Pr(X_i = x) = p^x(1-p)^{1-x}$ for $x \in \{0, 1\}$. It is known that p can take on two values 1/2 or 2/3. In this problem, we consider the hypothesis test

$$\mathsf{H}_0 : p = 1/2, \qquad \mathsf{H}_1 : p = 2/3$$

based on $(X_1, X_2, X_3) \in \{0, 1\}^3$.

(i) (5 points) Let $T = X_1 + X_2 + X_3$ be the number of ones in the random vector (X_1, X_2, X_3). Let P_0 and P_1 be the distributions of X_1, X_2, and X_3 under hypothesis H_0 and H_1 respectively. Write down the likelihood ratio

$$L(X_1, X_2, X_3) := \frac{P_0(X_1, X_2, X_3)}{P_1(X_1, X_2, X_3)}$$

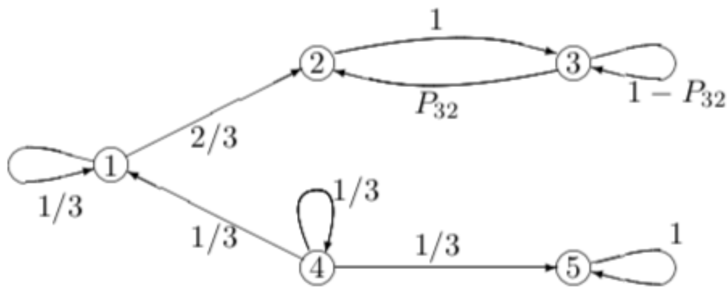
in terms of T. Hence, argue that T is a sufficient statistic for deciding between H_0 and H_1.

(ii) (4 points) Clearly $T \in \{0, 1, 2, 3\}$. Evaluate the values of the likelihood ratio in terms of T.

(iii) (3 points) What is the best probability of missed detection $P_1(\text{declare } \mathsf{H}_0)$ if we allow the probability of false alarm $P_0(\text{declare } \mathsf{H}_1)$ to be 1/8? What is the corresponding test in terms of T?

(iv) (7 points) What is the best probability of missed detection $P_1(\text{declare } \mathsf{H}_0)$ if we allow the probability of false alarm $P_0(\text{declare } \mathsf{H}_1)$ to be 1/4? What is the corresponding test in terms of T?

Hint: You need to consider randomized tests here.



Question 3

3(a) (3 points) Refer to the state transition diagram above. Identify the transient states and identify each class of recurrent states.

3(b) (5 points) For each recurrent class, find the steady-state probability vector $\boldsymbol{\pi} := (\pi_1, \pi_2, \ldots, \pi_5)$ for that class.

3(c) (12 points) Find the following n-step transition probabilities, $P_{ij}^n = \Pr\{X = j \mid X = i\}$ as a function of n. Give a brief explanation of each.

(i) P_{44}^n;

(ii) P_{45}^n;

(iii) P_{41}^n;

(iv) $P_{43}^n + P_{42}^n$;

(v) $\lim_{n\to\infty} P_{43}^n$.

3(d) (5 points) This part is not related to the above parts. We have a 2-state Markov chain whose probability transition matrix is

$$[P] = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}$$

The stationary distribution is $\boldsymbol{\pi} = (\pi_1, \pi_2)$. It is known that $[P^n]_{11} - \pi_1$ decays to zero exponentially fast. This means that

$$[P^n]_{11} - \pi_1 = c\phi^n$$

for some constant $c \in \mathbb{R}$ and $\phi \in (-1, 1)$. Find ϕ.

Question 4

4(a) Consider a Poisson process of rate $\lambda > 0$. Let t^* be a fixed time instant and consider the length of the interarrival interval $[U, V]$ that contains t^*. In this question, we would like to determine the distribution of

$$L = (t^* - U) + (V - t^*).$$

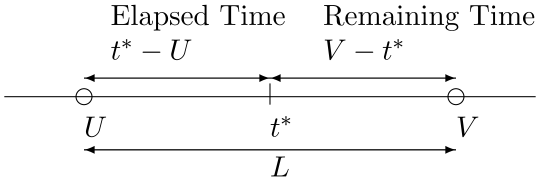


Figure 1: Here U and V are successive arrival epochs and t^* is a fixed time instance between U and V.

(i) (2 point) Give a one sentence answer as to why $V - t^*$ is independent of $t^* - U$.

(ii) (2 point) In class, we determined the distribution of $V - t^*$. What is this distribution?

(iii) (2 points) Consider the event

$$\{t^* - U > x\}.$$

This event is the same as

$$\{\text{there are } k \text{ arrivals in the interval } [t^* - x, t^*]\}.$$

Find the integer k. No explanation is needed.

(iv) (3 points) Hence, find the distribution of $t^* - U$.

(v) (3 points) By using the preceding parts, find the distribution of L.

(vi) (3 points) What is the distribution of an interarrival time of a Poisson process? Why is this the same or different from that of L in part (v)?

4(b) (10 points) Let $X_1, X_2, \ldots$ be a sequence of IID inter-renewal random variables. Let

$$S_n = X_1 + \ldots + X_n$$

be the corresponding renewal epochs for each $n \geq 1$. Assume that each X_i has a finite expectation $\mu > 0$. For any given $t > 0$, show (e.g., using Chebyshev's inequality) that

$$\lim_{n \to \infty} \Pr\{S_n \leq t\} = 0.$$

END OF PAPER