EE5138R: Quiz 2 (20/03/15)
Total: 30 points

1. (10 points) Let $f : \mathbf{R}^n \to \mathbf{R}$ be a differentiable convex function. Consider the following problem:

$$\text{(P)} \qquad \min_x \; f(x) \quad \text{s.t.} \quad x \succeq 0.$$

Let $v(x), w(x) : \mathbf{R}^n \to \mathbf{R}^n$ be two vector-valued functions of x. Show using KKT conditions that $\bar{x} \in \mathbf{R}^n$ is an optimal solution to problem (P) *if and only if* $\bar{x}$ is a solution to the following system:

$$\begin{aligned} \nabla f(x) &\succeq 0 \\ x &\succeq 0 \\ w(x)^T v(x) &= 0 \end{aligned}$$

In the process, find the functions $v(x)$ and $w(x)$.

Advice: Be sure to justify the "if and only if".

Solution: Since the problem is convex and a Slater vector obviously exists, the KKT conditions are necessary and sufficient. They state that $(\bar{x}, \bar{\lambda})$ is primal-dual feasible if

$$\begin{aligned} \bar{x} &\succeq 0 \\ \bar{\lambda} &\succeq 0 \\ -\bar{x}^T \bar{\lambda} &= 0 \\ \nabla f(\bar{x}) - \bar{\lambda} &= 0 \end{aligned}$$

Combining the second and fourth condition, we obtain

$$\nabla f(\bar{x}) \succeq 0.$$

Combining the third and fourth, we obtain

$$\nabla f(\bar{x})^T \bar{x} = 0.$$

So $w(x) = \nabla f(x)$ and $v(x) = x$.

2. (10 points) Consider the two-dimensional problem

$$\begin{array}{ll} \min & e^{x_2} \\ \text{s.t.} & \|x\|_2 - x_1 \leq 0 \end{array}$$

where the domain of the optimization variable $x = (x_1, x_2)$ is $\mathbf{R}^2$.

(a) (1 points) Is this a convex program?
Solution: Yes. e^{x_2} is convex and so is $\|x\| - x_1$.

(b) (1 points) Calculate the primal optimal value p^*.
Solution: By definition

$$p^* = \inf\left\{e^{x_2} : \sqrt{x_1^2 + x_2^2} - x_1 \leq 0\right\}$$

The constraint says that $x_2^* = 0$. Thus $p^* = 1$.

(c) (7 points) Calculate the dual optimal value d^*. What is the duality gap?
Solution: The Lagrange dual function is

$$g(\lambda) = \inf_x \left\{e^{x_2} + \lambda\left(\sqrt{x_1^2 + x_2^2} - x_1\right)\right\}$$

The dual optimal value is

$$d^* = \sup\{g(\lambda) : \lambda \geq 0\}$$

Consider any $\lambda \geq 0$. We claim that $g(\lambda) = 0$.
Let $u = \|x\| - x_1$. Suppose $u < 0$. Then there is no x such that $\|x\| - x_1 = u$. For $u = 0$, the vectors x such that $\|x\| - x_1 = u$ are precisely those of the form $(x_1, 0)$. The objective in this case is 1. For $u > 0$, for each x_2, the equation $\|x\| - x_1 = u$ has a solution in x_1 given by

$$x_1 = \frac{x_2^2 - u^2}{2u}.$$

Now we set $x_2 = -L$ and $x_1 = (L^2 - u^2)/(2u)$ giving $e^{x_2} + \lambda u = e^{-L} + \lambda u$. But u can be made arbitrarily small and L arbitrarily large so in the case $u > 0$, the objective can be made arbitrarily small. Thus, $g(\lambda) = 0$ for all $\lambda \geq 0$. Thus $d^* = 0$ and the duality gap is $p^* - d^* = 1$.

(d) (1 point) Can you find a Slater vector? Explain intuitively why there is or isn't a duality gap.
Solution: There is no Slater vector $\bar{x}$ because the constraint $\|x\| - x_1 \leq 0$ implies that $x_2 = 0$ and so $\|x\| - x_1 = 0$. Hence, there is no vector $\bar{x}$ such that $\|\bar{x}\| - x_1 < 0$. Since Slater's condition is not satisfied, we cannot guarantee that the problem has no duality gap.

3. (10 points) Consider the problem of finding an approximate solution to an overdetermined system of linear equations

$$a_i^T x \approx y_i, \qquad y_i > 0, i = 1, \ldots, m.$$

In certain applications, it makes sense to use an error criterion based on the maximum of the ratios $(a_i^T x)/y_i$ instead of the usual least squares criterion. The log-Chebyshev approximation problem thus amounts to solving

$$\text{(LC0)} \qquad \min_{x \in \mathbf{R}^n} \max_{i=1,\ldots,m} \left|\log(a_i^T x) - \log(y_i)\right|, \quad \text{s.t.} \quad a_i^T x > 0, i = 1, \ldots, m.$$

(a) (1 points) Show that Problem (LC0) is equivalent to

$$\text{(LC1)} \qquad \min_{x \in \mathbf{R}^n} \max_{i=1,\ldots,m} \max \left\{ \frac{a_i^T x}{y_i}, \frac{y_i}{a_i^T x} \right\} \quad \text{s.t.} \quad a_i^T x > 0, i = 1, \ldots, m.$$

Solution: Maximizing $|\log(a_i^T x) - \log(y_i)|$ is obviously equivalent to maximizing the maximum of $a_i^T x / y_i$ and $y/a_i^T x$.

(b) (3 points) By introducing an auxiliary scalar variable t and expressing the problem in epigraphic form, show that Problem (LC1) is equivalent to

$$\begin{aligned}
\text{(LC2)} \qquad & \min_{x \in \mathbf{R}^n, t \in \mathbf{R}_{++}} \quad t \\
\text{s.t.} \qquad & \frac{a_i^T x}{y_i} \le f(t), \qquad i = 1, \ldots, m \\
& \frac{a_i^T x}{y_i} \ge g(t), \qquad i = 1, \ldots, m \\
& a_i^T x > 0, \qquad i = 1, \ldots, m
\end{aligned}$$

Find the functions $f(t)$ and $g(t)$.

Solution: Let t be an upper bound for the maximum of $a_i^T x / y_i$ and $y / a_i^T x$ over all i's. Then we minimize t and the constraints can be written as

$$\frac{a_i^T x}{y_i} \le t \quad \text{and} \quad \frac{y_i}{a_i^T x} \le t$$

Because $a_i^T x > 0$, the second constraint can be written as

$$\frac{a_i^T x}{y_i} \ge \frac{1}{t}$$

Thus, $f(t) = t$ and $g(t) = 1/t$.

(c) (1 points) Show that for any $(\alpha, \beta, \gamma) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}$, we have

$$\|\alpha\|_2^2 \le 2\beta\gamma, \quad \beta \ge 0, \quad \gamma \ge 0 \quad \Longleftrightarrow \quad \left\| \begin{bmatrix} \alpha \\ \frac{1}{\sqrt{2}}(\beta - \gamma) \end{bmatrix} \right\|_2 \le \frac{1}{\sqrt{2}}(\beta + \gamma)$$

Solution: This is algebra. Expanding the norm on the right-hand-side, we obtain

$$\left\| \begin{bmatrix} \alpha \\ \frac{1}{\sqrt{2}}(\beta - \gamma) \end{bmatrix} \right\|_2^2 = \|\alpha\|^2 + \frac{1}{2}\beta^2 - \beta\gamma + \frac{1}{2}\gamma^2$$

If this is no greater than $\frac{1}{2}(\beta + \gamma)^2$, the square of the right-hand-side, we recover

$$\|\alpha\|^2 \le 2\beta\gamma.$$

(d) (5 points) Using part (c), express (LC2) as a second-order cone program. In particular, show that (LC2) is equivalent to

$$\begin{array}{lll}(\text{LC3}) & \displaystyle\min_{x\in\mathbf{R}^n, t\in\mathbf{R}_{++}} & t \\ \text{s.t.} & \dfrac{a_i^T x}{y_i} \le f(t), & i = 1,\dots,m \\ & \left\| \begin{bmatrix} 2 \\ h_1(t, a_i, x, y_i) \end{bmatrix} \right\|_2 \le h_2(t, a_i, x, y_i) & i = 1,\dots,m \end{array}$$

for some functions h_1 and h_2 that you should write down explicitly. Explain briefly why this is a second-order cone program.

Solution: The constraint $\frac{a_i^T x}{y_i} \ge \frac{1}{t}$ can be expressed as

$$t\frac{a_i^T x}{y_i} \ge 1$$

Now we take $\alpha = \sqrt{2}$, $\beta = t$ and $\gamma = \frac{a_i^T x}{y_i}$. Using part (c), we obtain

$$\left\| \begin{bmatrix} \sqrt{2} \\ \frac{1}{\sqrt{2}}(t - \frac{a_i^T x}{y_i}) \end{bmatrix} \right\|_2 \le \frac{1}{\sqrt{2}}\Big(t + \frac{a_i^T x}{y_i}\Big)$$

Multiply by $\sqrt{2}$ everywhere we obtain

$$h_1(t, a_i, x, y_i) = t - \frac{a_i^T x}{y_i}, \qquad h_2(t, a_i, x, y_i) = t + \frac{a_i^T x}{y_i}$$

This is an SOCP because if we define $\tilde{x} = (x, t)$, then the objective is linear in $\tilde{x}$ and the constraints are of the form

$$\|A_i\tilde{x} + b_i\|_2 \le c_i^T \tilde{x} + d_i, \quad i = 1,\dots,m.$$