Solutions to Midterm Exam

EE5138 Optimization for Communication Systems
(Semester II, 19/20)

Q1. 1. (d). 2. (b). 3. (b). 4. (c). 5. (d). 6. (b). 7. (a). 8. (b). 9. (c). 10. (d).

Q2. Let $x_c + Au_1 + Bv_1$ and $x_c + Au_2 + Bv_2$ be two points in the set. Then, for any $0 \leq \theta \leq 1$, we have

$$\begin{aligned} &\theta\left(x_c + Au_1 + Bv_1\right) + (1-\theta)\left(x_c + Au_2 + Bv_2\right) \\ &= x_c + A\left(\theta u_1 + (1-\theta)u_2\right) + B\left(\theta v_1 + (1-\theta)v_2\right). \end{aligned}$$

Next, we verify

$$\begin{aligned} &||\theta u_1 + (1-\theta)u_2 + \theta v_1 + (1-\theta)v_2||_2 \\ &= ||\theta(u_1+v_1) + (1-\theta)(u_2+v_2)||_2 \\ &\overset{(a)}{\leq} ||\theta(u_1+v_1)||_2 + ||(1-\theta)(u_2+v_2)||_2 \\ &\overset{(b)}{=} \theta||u_1+v_1||_2 + (1-\theta)||u_2+v_2||_2 \\ &\overset{(c)}{\leq} \theta + (1-\theta) \\ &= 1, \end{aligned}$$

where
(a): "Triangle inequality" of norm.
(b): "Homogeneous property" of norm.
(c): Since $x_c + Au_1 + Bv_1$ and $x_c + Au_2 + Bv_2$ are assumed to be in the set, we have $||u_1 + v_1||_2 \leq 1$ and $||u_2 + v_2||_2 \leq 1$.

Thus, the point $\theta\left(x_c + Au_1 + Bv_1\right) + (1-\theta)\left(x_c + Au_2 + Bv_2\right)$ is in the set for $0 \leq \theta \leq 1$, and as a result the set is convex.

Q3. The Hessian of $f(x)$ is

$$\nabla^2 f(x) = \begin{bmatrix} 4 & -a \\ -a & 2 \end{bmatrix}.$$

Let $v = [v_1, v_2]^T \in \mathbf{R}^2$. It then follows

$$v^T \nabla^2 f(x) v = 4v_1^2 - 2av_1v_2 + 2v_2^2 = (2v_1 - (a/2)v_2)^2 + (2 - a^2/4)v_2^2.$$

If $(2 - a^2/4) \geq 0$, $v^T \nabla^2 f(x) v \geq 0$ for all v, thus $\nabla^2 f(x) \succeq 0$ and $f(x)$ is convex. In this case, we have $a^2 \leq 8$ or $-2\sqrt{2} \leq a \leq 2\sqrt{2}$.

Otherwise, if $(2 - a^2/4) < 0$, by letting $v_1 = (a/4)v_2 \neq 0$, then we have $v^T \nabla^2 f(x) v = (2 - a^2/4)v_2^2 < 0$, thus $\nabla^2 f(x) \not\succeq 0$ and $f(x)$ is not convex.

Hence, the range of values for a is $-2\sqrt{2} \leq a \leq 2\sqrt{2}$, for $f(x)$ to be a convex function.