Name : LUO ZIJIAN

Matric.No： A0224725H

MUSNET： E0572844

Subject： Stochastic process

Assignment: Homework Eight

Date: Mar 19th

Prof: Vincent Tan.

1. EXERCISE 4.10

(a)

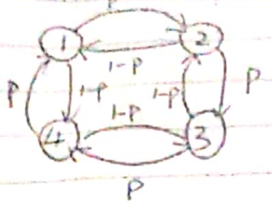


figure (a), we can get the transformation matrix

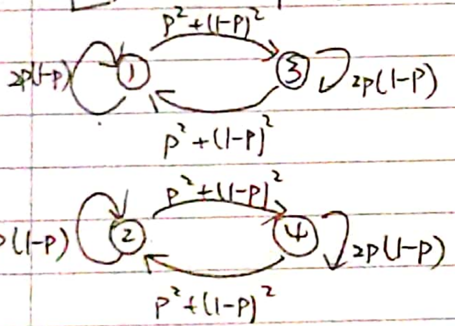
$$\begin{cases} \pi_1 = \pi_2(1-p) + \pi_4 \cdot p \\ \pi_2 = \pi_3(1-p) + \pi_1 p \\ \pi_3 = \pi_4(1-p) + \pi_2 \cdot p \\ \pi_4 = \pi_1(1-p) + \pi_3 p \end{cases} \quad \sum \pi_i = 1 \quad \Rightarrow \quad \pi_1 = \pi_2 = \pi_3 = \pi_4 = \frac{1}{4}$$

For figure (b), we know state 4 has two direction: ① to state 1 ② to state 5 $P_{41} = P_{51}$, we conclude that $p_{41} = p_{51} = \frac{1}{2}$, for others, they are all 1

$$\begin{cases} \pi_4 = \pi_9 \times 1 + \pi_3 \times 1 \\ \pi_5 = \pi_4 \times \frac{1}{2} \\ \pi_1 = \pi_4 \times \frac{1}{2} \\ \pi_9 = \pi_8 = \pi_7 = \pi_6 = \pi_5 \\ \pi_3 = \pi_2 = \pi_1 \end{cases} \Rightarrow \quad \sum \pi_i = 1 \quad \pi_4 = \frac{1}{5}, \ \pi_1 = \pi_2 = \pi_3 = \pi_5 = \pi_6 = \pi_7 = \pi_8 = \pi_9 = \frac{1}{10}$$

(b) For the first figure

$$\begin{bmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{bmatrix} \begin{bmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{bmatrix} = \begin{bmatrix} 2p(1-p) & 0 & p^2+(1-p)^2 & 0 \\ 0 & 2p(1-p) & 0 & p^2+(1-p)^2 \\ p^2+(1-p)^2 & 0 & 2p(1-p) & 0 \\ 0 & p^2+(1-p)^2 & 0 & 2p(1-p) \end{bmatrix}$$

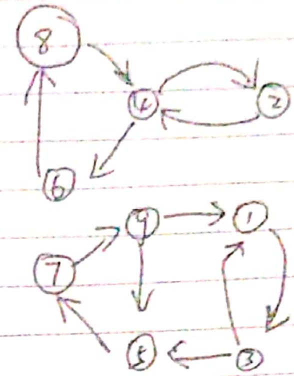


$$\Rightarrow \quad \pi_1 = \pi_3 = \frac{1}{2} \quad \text{or} \quad \pi_2 = \pi_4 = \frac{1}{2}$$

For the second figure

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}$$



$$\Downarrow$$

$$\pi_2 = \pi_6 = \pi_8 = \frac{1}{5}, \quad \pi_4 = \frac{2}{5}$$

or

$$\pi_1 = \pi_3 = \pi_5 = \pi_7 = \pi_9 = \frac{1}{5}$$

(c) According to part (b), for the first figure,

$$\lim_{n\to\infty}\left[P^{2n}\right] = \lim_{n\to\infty}\begin{bmatrix} 2p(1-p) & 0 & p^2+(1-p)^2 & 0 \\ 0 & 2p(1-p) & 0 & p^2+(1-p)^2 \\ p^2+(1-p)^2 & 0 & 2p(1-p) & 0 \\ 0 & p^2+(1-p)^2 & 0 & 2p(1-p) \end{bmatrix}^n = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{cases} \pi_1 = 2p(1-p)\pi_1 + \left[p^2+(1-p)^2\right]\cdot\pi_3 \\ \pi_2 = 2p(1-p)\pi_2 + \left[p^2+(1-p)^2\right]\cdot\pi_4 \\ \pi_3 = \left[p^2+(1-p)^2\right]\pi_1 + 2p(1-p)\pi_3 \\ \pi_4 = \left[p^2+(1-p^2\right]\pi_2 + 2p(1-p)\pi_4 \\ \sum_{i=1}^{4}\pi_i = 1 \end{cases} \Rightarrow \begin{array}{l} 2p(1-p) = p^2+(1-p)^2 \\ p = \frac{1}{2} \end{array}$$

For the second figure

$$\lim_{n\to\infty}\left[P^{2n}\right] = \begin{bmatrix} \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & \frac{1}{5} & 0 & \frac{2}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & \frac{1}{5} & 0 & \frac{2}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & \frac{1}{5} & 0 & \frac{2}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & \frac{1}{5} & 0 & \frac{2}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

similar like (b)

2.

(a) Through the above information from question, we can find this is aperiodic.

$Pr(X_{1000}=j, X_{1001}=k, X_{2000}=k \mid X_0=i)$. (using the definition of conditional)

$= Pr(X_{2000}=k, X_{1001}=k \mid X_{1000}=j, X_0=i) \cdot Pr(X_{1000}=j \mid X_0=i)$

$= Pr(X_{2000}=k \mid X_{1001}=k, X_{1000}=j, X_0=i) \cdot Pr(X_{1001}=k \mid X_{1000}=j, X_0=i) \cdot Pr(X_{1000}=j \mid X_0=i)$

Then we can simply that like this

$= Pr(X_{2000}=k \mid X_{1001}=k) \cdot Pr(X_{1001}=k \mid X_{1000}=j) \cdot Pr(X_{1000}=j \mid X_0=i)$

$= P_{kk}^{999} \cdot P_{jk} \cdot P_{ij}^{999}$

(b) For part (b), we can find it is posterior probability,

Using the bayes law, we get

$$Pr(X_{1000}=i \mid X_{1001}=j) = \frac{Pr(X_{1001}=j \mid X_{1000}=i)\, Pr(X_{1000}=i)}{Pr(X_{1001}=j)} = \frac{Pr(X_{1001}=j \mid X_{1000}=i) \cdot Pr(X_{1000}=i)}{\sum_{m=1}^{n} Pr(X_{1001}=j \mid X_{1000}=m)}$$

Because it is recurrent class and aperiodic, it must satisfy ergodic $= \frac{P_{ij} \cdot \pi_i}{\sum_{m=1}^{n} P_{kj}}$

Owing to the definition of ergodic chain, we can get $P_{kj} = P_{ij}$, for any k

So, the result is $\frac{P_{ij} \cdot \pi_i}{n \cdot P_{ij}} = \frac{\pi_i}{n}$

3. (a) For this process, we can knew. X_n can be two different directions, which can simplify like that

$$X_n = \begin{cases} X_{n-1}+1, & \text{with probability of } p. \\ 0, & \text{with probability of } (1-p). \end{cases}$$

And then, we can get the transformation equation ($\pi = \pi[P]$)

$$\begin{cases} \pi_1 = \pi_0 \cdot p \\ \pi_2 = \pi_1 \cdot p = \pi_0 \cdot p^2 \\ \vdots \\ \pi_i = \pi_{i-1} \cdot p = \pi_0 \cdot p^i \end{cases} \Rightarrow \text{using } \sum_{i=0}^{\infty} \pi_i = 1$$

$\pi_0 = \frac{1}{1-p}$ so, $\pi_i = \frac{p^i}{1-p}$

(b) Followed the statement in part (b)

T_k means the first time to get state k, so there are two possible conditions: ① it stays at state $(k-1)$, and next step is to state k with probability of p ② it stays at state $(k-1)$ but next step is to redirect to state 0, and then a new process to get state k.

$$T_k = \begin{cases} T_{k-1}+1 & \text{with probability of } p \\ T_{k-1}+1+T_k' & \text{with probability of } (1-p). \end{cases}$$

And then, we can get $E[T_k|T_{k-1}] = p\cdot(T_{k-1}+1) + (1-p)[T_{k-1}+1+E[T_k'|T_{k-1}]]$

$$= T_{k-1}+1+(1-p)E[T_k'|T_{k-1}]$$

T_k' is a renewal process like T_k, so $T_k' \overset{d}{=} T_k$

$$E[T_k'|T_{k-1}] = E[T_k|T_{k-1}]$$

Then we can get $pE[T_k|T_{k-1}] = T_{k-1}+1$

use the total expectation $E[pE[T_k|T_{k-1}]] = E[T_{k-1}+1]$

$$p\cdot E[T_k] = E[T_{k-1}]+1$$

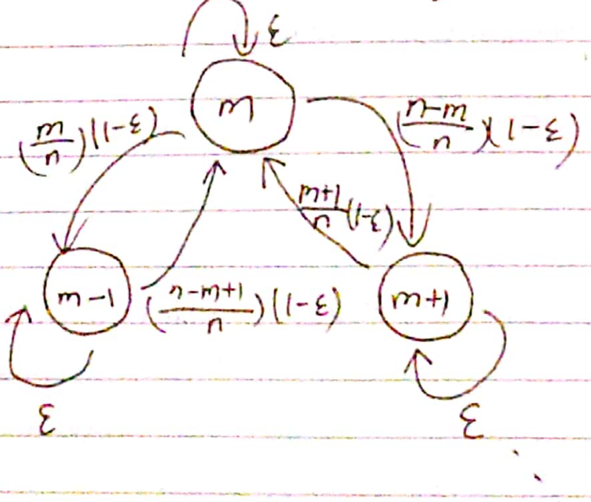
As desired. For T_1 $\quad T_1 = \begin{cases} 1, & \text{with probability of } p. \\ 1+T_1', & \text{with probability of } (1-p) \end{cases}$

$$E[T_1] = 1\cdot p + E[1+T_1']\cdot(1-p)$$
$$= p+1+E[T_1'] - p - pE[T_1']$$
$$p\cdot E[T_1] = 1 \quad (E[T_1'] = E[T_1])$$
$$E[T_1] = \frac{1}{p}$$

(c) Follow part (b), we can get $\begin{cases} p\cdot E[T_k] = E[T_{k-1}]+1 \\ E[T_1] = \frac{1}{p} \end{cases}$

$$E[T_2] = \frac{1}{p^2}+\frac{1}{p}$$
$$E[T_3] = \frac{1}{p^3}+\frac{1}{p^2}+\frac{1}{p}$$
$$\vdots$$
$$E[T_k] = \frac{1}{p^k}+\frac{1}{p^{k-1}}+\cdots\frac{1}{p} = \sum_{i=1}^{k}\left(\frac{1}{p}\right)^i$$

4. We can draw the graph of this markov chain



we suppose state m means there are m white ball

we can get this balance equati

$$\pi_i \frac{n-i}{n} = \pi_{i+1}\cdot\frac{i+1}{n}$$

$$\pi_{i+1} = \frac{n-i}{i+1}\pi_i$$

Follow this balance equation, we can get below equation

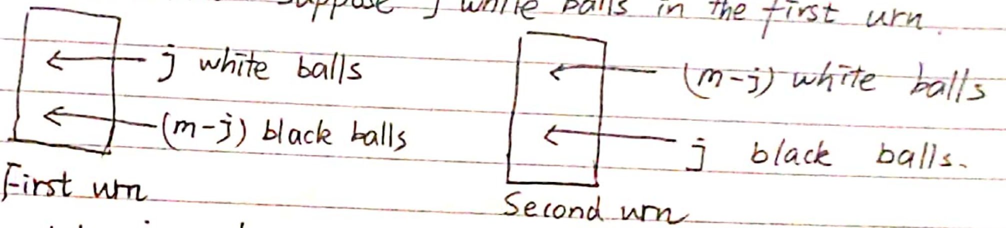
$$\begin{cases}\pi_1 = \frac{n}{1}\pi_0 \\ \pi_2 = \frac{n-1}{2}\pi_1 = \frac{(n-1)n}{2\times 1}\pi_0 \\ \vdots \\ \pi_i = \frac{n!}{(n-i)!\,i!}\pi_0 = \binom{n}{i}\pi_0\end{cases} \Rightarrow \sum_{i=0}^{n} \pi_i = 1$$

And then, we can get $\sum_{i=0}^{n} C_n^i\, 1^{n-r}\cdot 1^r = (1+1)^n \pi_0 = 1$

So $\pi_0 = \left(\frac{1}{2}\right)^n$

$\pi_i = \binom{n}{i}\left(\frac{1}{2}\right)^n$

5. With the above information in this question five, we should know the white balls in the first urn is image condition of the second urn. Suppose j white balls in the first urn.



For state j, there are three conditions.

① it can add one white ball, and then becomes state (j+1)

② it can keep same, so state j is steady

③ it can minus one white ball, and then becomes state (j-1)

$P_{j,j+1} = \left(\frac{m-j}{m}\right)\cdot\left(\frac{m-j}{m}\right) = \left(\frac{m-j}{m}\right)^2$

$P_{j,j} = \left(\frac{j}{m}\right)\left(\frac{m-j}{m}\right) + \left(\frac{m-j}{m}\right)\left(\frac{j}{m}\right) = \frac{2j(m-j)}{m^2}$

$P_{j,j-1} = \left(\frac{j}{m}\right)\cdot\left(\frac{j}{m}\right) = \frac{j^2}{m^2}$

According to the transformation, we can get balance equation

$$\pi_j\left(\frac{m-j}{m}\right)^2 = \pi_{j+1}\left(\frac{j+1}{m}\right)^2 \qquad j = 0, 1, \dots m-1$$

Follow this equation, we can get

$$\begin{cases}\pi_1 = \frac{m^2}{1^2}\pi_0 \\ \pi_2 = \left(\frac{m-1}{2}\right)^2 \pi_1 = \left(\frac{m-1}{2}\right)^2\left(\frac{m}{1}\right)^2 \pi_0 \\ \vdots \\ \pi_j = \left(\frac{m-j+1}{j}\right)^2 \pi_{j-1} = \binom{m}{j}^2 \pi_0\end{cases} \Rightarrow \text{using } \sum_{j=0}^{m} \pi_j = 1$$

$$\sum_{j=0}^{m}\binom{m}{j}^2 = \binom{2m}{m}$$

So $\pi_0 = \frac{1}{\binom{2m}{m}}$

$\pi_j = \binom{m}{j}^2 \cdot \frac{1}{\binom{2m}{m}}$