EE5137 2019/20 (Sem 2): Quiz 1 (Total 40 points)

Name:______________________________

Matriculation Number:_______________

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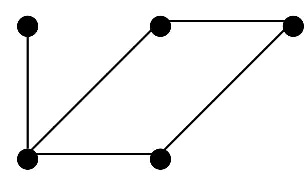
You have 1.0 hour for this quiz. There are FOUR (4) printed pages. You're allowed 1 sheet of handwritten notes. Please provide *careful explanations* for all your solutions.

1. [Random Graphs] In this problem, we consider a random (undirected) graph with n nodes. A simple model for random graphs is the *Erdös-Rényi* model $G(n,p)$. Here, every pair of nodes are connected by an edge with probability p. The occurrence of each edge in the graph is independent from other edges in the graph. The figure shows a randomly generated graph using this model. Here, $n = 5$ and p was chosen to be $1/2$.

   We say that node $i \in \{1, 2, \ldots, n\}$ is *isolated* if it is not connected to any other node. In the figure to the right, there is no isolated node.

   (a) (7 points) Let B_n be the event that a graph randomly generated according to $G(n,p)$ model has <u>at least one isolated node</u>. Use the union bound (or otherwise) to find the functions $f(p)$ and $g(n)$ such that

   $$\Pr(B_n) \le n \cdot f(p)^{g(n)}.$$

   Solution: The probability that node i is isolated is the probability that it is not connected to the other $n-1$ nodes, i.e., $(1-p)^{n-1}$. We have

   $$\Pr(B_n) = \Pr\left(\bigcup_{i=1}^{n} \{\text{node } i \text{ is isolated}\right) \le \sum_{i=1}^{n} \Pr\left(\text{node } i \text{ is isolated}\right)$$
   $$\le \sum_{i=1}^{n} (1-p)^{n-1} = n(1-p)^{n-1}$$

   Hence, $f(p) = 1 - p$ and $g(n) = n - 1$.

(b) (3 points) We may let the connection probability p be a function of n. In this case, we write p as p_n. Show that if

$$p_n = 1.01 \cdot \frac{\ln n}{n}$$

then $\Pr(B_n) \to 0$ as $n \to \infty$. That is, if p_n obeys the scaling above, then asymptotically there will be no isolated node and the graph will be connected.

Hint: You may use the fact that for any $x \in \mathbb{R}$

$$1 - x \leq e^{-x}.$$

Solution: We have

$$\begin{aligned}
\Pr(B_n) &\leq n\left(1 - \frac{1.01 \ln n}{n}\right)^{n-1} \\
&\leq n\left(1 - \frac{1.01 \ln n}{n}\right)^{n} \\
&\leq n\left[\exp\left(-\frac{1.01 \ln n}{n}\right)\right]^{n} \\
&= n \cdot \exp(-1.01 \ln n) \\
&= n \cdot \frac{1}{n^{1.01}} \\
&= \frac{1}{n^{0.01}} \to 0.
\end{aligned}$$

2. [Conditional Expectations]

   Let X and Y be independent random variables (r.v.'s), each uniformly distributed over $[0, 1]$. Define $Z = X + Y$.

   (a) (2 points) Find $\mathbb{E}[Z|X]$. Please note that this is a r.v.

   Solution: We have

   $$\mathbb{E}[Z|X] = \mathbb{E}[X + Y|X] = X + \mathbb{E}[Y|X] = X + \mathbb{E}[Y] = X + 1/2.$$

   (b) (2 points) Use your answer to part (a) and the law of iterated expectations to find $\mathbb{E}[Z]$ and verify that the value is the same as $\mathbb{E}[X] + \mathbb{E}[Y]$.

   Solution: We have

   $$\mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[Z|X]] = \mathbb{E}[X + 1/2] = \mathbb{E}[X] + 1/2 = 1/2 + 1/2 = 1,$$

   which is also the same as $\mathbb{E}[X] + \mathbb{E}[Y] = 1$.

   (c) (5 points) Find the conditional distribution (pdf) $f_{X|Z}(x|z)$. Specify the range of values of x and z.

   Hint: It would be useful to think of $z \in [0, 1]$ and $z \in [1, 2]$ separately.

   Solution: First we note that the range of values of Z is $[0, 2]$ since $X, Y \in [0, 1]$. As per the hint, we consider $z \in [0, 1]$ and $z \in [1, 2]$ separately. For the former, we know that the range of values of x is $[0, z]$, since x cannot exceed z and x is non-negative. So in this range of $[0, z]$, X must clearly be uniform and so $f_{X|Z}(x|z) = 1/z$ for $x \in [0, z]$. For $z \in [1, 2]$, we know that $x \leq 1$ and $y = z - x \leq 1$. This means that $x \geq z - 1$. As such $x \in [z - 1, 1]$. Again, X must clearly be uniform in this interval and so $f_{X|Z}(x|z) = 1/(2 - z)$ for $x \in [z - 1, 1]$.

   (d) (5 points) Find $\mathbb{E}[X|Z]$ using part (c) and the law of iterated expectations.

   Solution: If $0 \leq z \leq 1$, $\mathbb{E}[X|Z] = Z/2$ and if $1 \leq z \leq 2$, $\mathbb{E}[X|Z] = (1 + (Z - 1))/2 = Z/2$. Thus, in both cases, $\mathbb{E}[X|Z] = Z/2$.

   (e) (1 points) Use your answer to part (d) and the law of total expectation to find $\mathbb{E}[X]$ and verify that it corresponds to that of a uniform r.v. on $[0, 1]$.

   Solution: In both cases, $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Z]] = \mathbb{E}[Z/2] = \mathbb{E}[Z]/2 = 1/2$, which is the mean of a uniform random variable over $[0, 1]$.

3. [Convergence of Random Variables] In each of the following two parts, you are asked a question about the convergence of a sequence of random variables. If you say yes, provide a proof and the limiting random variable. If you say no, disprove or provide a counterexample.

   (a) (7 points) Let $A_1, A_2, \ldots$ be a sequence of *independent* events such that $\Pr(A_n) \to 1$ as $n \to \infty$. Now define a sequence of (indicator) random variables $X_n = \mathbb{1}\{A_n\}, n = 1, 2, \ldots$ Does X_n converge in probability as $n \to \infty$?

   Note: $X_n = \mathbb{1}\{A_n\}$ means that $X_n = 1$ if A_n occurs and $X_n = 0$ if A_n^c occurs.

   Solution: YES. We claim that the sequence X_n converges in probability to $X = 1$, the random variable which takes value 1 w.p. 1. For fixed $\epsilon > 0$, consider

   $$\Pr(|X_n - 1| > \epsilon) = \Pr(\mathbb{1}\{A_n^c\} > \epsilon) = \Pr(\mathbb{1}\{A_n^c\} = 1) = \Pr(A_n^c) \to 0$$

   Thus, by the definition of convergence in probability, $X_n \to X = 1$ in probability.

   (b) (8 points) Suppose X is a uniform random variable on $[-1, 1]$ and $X_n := X^n$ (this is X to the power of n). Does X_n converge almost surely as $n \to \infty$?

   Hint: For any real number a such that $|a| < 1$, it holds that $a^n \to 0$ as $n \to \infty$.

   Solution: YES. We claim that X_n converges in probability to $Z = 0$, the random variable which takes on 0 w.p. 1. Consider

   $$\begin{aligned}\Pr\left(\left\{\omega : \lim_{n\to\infty} X_n(\omega) = 0\right\}\right) &= \Pr\left(\left\{\omega : \lim_{n\to\infty} (X(\omega))^n = 0\right\}\right) \\ &\overset{(a)}{=} \Pr\left(\{\omega : X(\omega) \notin \{-1, 1\}\}\right) \\ &= 1 - \Pr\left(\{\omega : X(\omega) \in \{-1, 1\}\}\right) \\ &= 1 - \Pr(X \in \{-1, 1\}) \\ &= 1 - 0 = 1.\end{aligned}$$

   where (a) holds because for any ω such that $X(\omega) \notin \{-1, 1\}$, it holds that $X(\omega)^n \to 0$ as $n \to \infty$.