# EE5137 Semester 1 2018/9: Quiz 2 (Total 30 points)

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Matriculation Number: XX

Score: 30/30

You have 75 mins for this quiz. There are FOUR (4) printed pages. You're allowed 1 sheet of handwritten notes. Please provide *careful explanations* for all your solutions.

1. [Covariances in the Poisson Process] (10 points)

   Given two random variables $X$ and $Y$, their *covariance* is defined as

   $$\mathsf{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

   Answer the following questions.

   (i) (2 points) Show that if $X$ and $Y$ are independent, then $\mathsf{Cov}(X,Y) = 0$.

   **Solution:** We have

   $$\begin{aligned}\mathbb{E}[XY] &= \int\int f_{XY}(x,y)xy\,\mathrm{d}x\,\mathrm{d}y = \int\int f_X(x)f_Y(y)xy\,\mathrm{d}x\,\mathrm{d}y \\ &= \left(\int f_X(x)\,\mathrm{d}x\right)\left(\int f_Y(y)y\,\mathrm{d}y\right) = \mathbb{E}[X]\mathbb{E}[Y]\end{aligned}$$

   Hence,

   $$\mathsf{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0.$$

   (ii) (8 points) $N(t)$ is a Poisson counting process with rate $\lambda$. Find the covariance between $N(3) - N(1)$ and $N(4) - N(2)$.

   **Solution:** We first establish that

   $$\mathsf{Cov}(X+Y,Z) = \mathsf{Cov}(X,Z) + \mathsf{Cov}(Y,Z).$$

This can be shown by appealing to the definition of covariance above. Next, we consider

$$
\begin{aligned}
&\mathsf{Cov}(N(3) - N(1), N(4) - N(2)) \\
&= \mathsf{Cov}((N(3) - N(2)) + (N(2) - N(1)), (N(4) - N(3)) + (N(3) - N(2))) \\
&= \mathsf{Cov}(N(3) - N(2), N(4) - N(3)) + \mathsf{Cov}(N(3) - N(2), N(3) - N(2)) \\
&\qquad + \mathsf{Cov}(N(2) - N(1), N(4) - N(3)) + \mathsf{Cov}(N(2) - N(1), N(3) - N(2)) \\
&= \mathsf{Cov}(N(3) - N(2), N(3) - N(2)) = \mathsf{Var}(N(3) - N(2)) = \lambda
\end{aligned}
$$

Those terms in covariances that have no overlap have covariance zero by the independent increments property.

2. [Poisson Arrivals] (10 points)

   Suppose customers arrive to a shop according to a Poisson process with intensity/rate of $\lambda = 8$ per hour.

   (i) (2 points) If $X$ is an Exp($\lambda$) random variable, what is the variance of $X$?

   **Solution:** By straightforward integration,

$$\mathbb{E}[X] = 1/\lambda, \quad \mathbb{E}[X^2] = 2/\lambda^2$$

   Hence,

$$\mathsf{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1/\lambda^2.$$

   (ii) (3 points) What is the variance of the time when the *fourth* ($4^{\text{th}}$) customer arrives?

   **Solution:** The arrival time of the fourth customer is $S_4 = X_1 + X_2 + X_3 + X_4$ where $X_i$'s are independent Exp(8) random variables. Hence,

$$\mathsf{Var}(S_4) = 4/8^2 = 1/16$$

   (iii) (5 points) Assume that 25% of the customers are men and 75% percent are women (and that whether each customer is a man/woman is independent of every other customer). What is the expected time when the *fifth* ($5^{\text{th}}$) woman arrives?

   **Solution:** Let the arrival epochs of women be $W_i$. Then $W_i$ is a Poisson process with rate $\lambda_w = (3/4) \times 8 = 6$. The expected arrival time of $5^{\text{th}}$ woman is

$$\mathbb{E}\left[\sum_{i=1}^{5} W_i\right] = 5\mathbb{E}[W_1] = 5/6.$$

3. [Travelers and Poisson Processes] (10 points)

Suppose that travelers arrive at a train depot according to a Poisson process with rate $\lambda$. The train departs at time $t$. We would like to compute the expected sum of the waiting times of travelers arriving in the interval $(0, t)$, i.e., we want to find the number

$$\alpha := \mathbb{E}\left[\sum_{i=1}^{N(t)} (t - S_i)\right]$$

where $S_i$ are the arrival epochs of the travelers.

(i) (6 points) Find a simple expression in terms of $n$ and $t$ for

$$\mathbb{E}\left[\sum_{i=1}^{N(t)} (t - S_i) \,\middle|\, N(t) = n\right].$$

That is, we condition on the event $\{N(t) = n\}$.

**Solution:** We have

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^{N(t)} (t - S_i) \,\middle|\, N(t) = n\right] &= \mathbb{E}\left[\sum_{i=1}^{n} (t - S_i) \,\middle|\, N(t) = n\right] \\ &= nt - \mathbb{E}\left[\sum_{i=1}^{n} S_i \,\middle|\, N(t) = n\right]\end{aligned}$$

There are many ways to calculate the final expectation. Here, we note that

$$\mathbb{E}\left[\sum_{i=1}^{n} S_i \,\middle|\, N(t) = n\right] = \mathbb{E}\left[\sum_{i=1}^{n} U_{(i)}\right] = \mathbb{E}\left[\sum_{i=1}^{n} U_i\right]$$

where $U_i$ are independent uniform random variables on $(0, t)$ and $U_{(i)}$ is the $i$-th smallest one. However, the final expectation is $nt/2$ so

$$\mathbb{E}\left[\sum_{i=1}^{N(t)} (t - S_i) \,\middle|\, N(t) = n\right] = \frac{nt}{2}.$$

(ii) (4 points) Hence, find

$$\alpha := \mathbb{E}\left[\sum_{i=1}^{N(t)} (t - S_i)\right].$$

**Solution:** By iterated expectations, we have

$$\begin{aligned}\alpha = \mathbb{E}\left[\sum_{i=1}^{N(t)}(t - S_i)\right] &= \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N(t)}(t - S_i) \;\middle|\; N(t)\right]\right] \\ &= \mathbb{E}\left[\frac{N(t)t}{2}\right] = \frac{\lambda t^2}{2}.\end{aligned}$$