Motivation

Quantum theory is a mathematical model of the physical world.

physics of the atomic and subatomic world

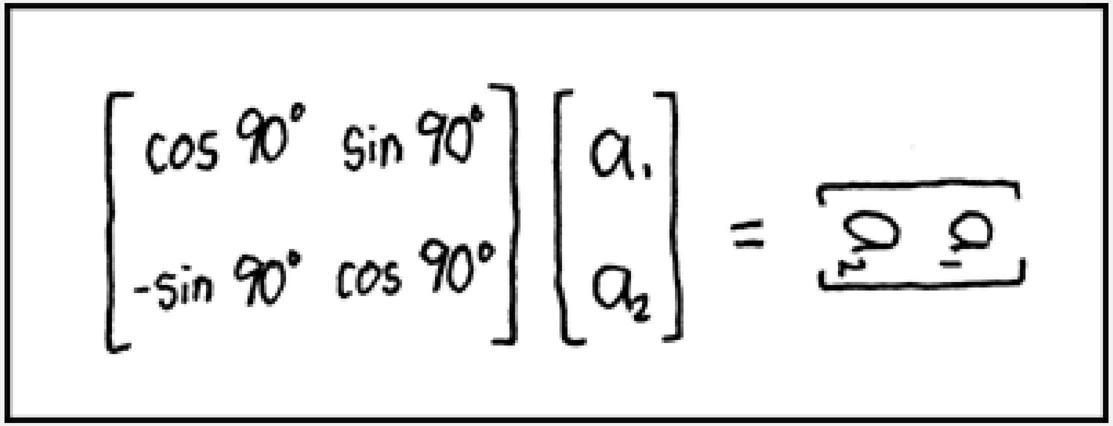
many behaviors have no analog in the classical world

GOAL:

Comfort level with the mathematics and concepts that determine behavior and interactions of qubits.

Understanding of the constraints and opportunities of quantum computation.

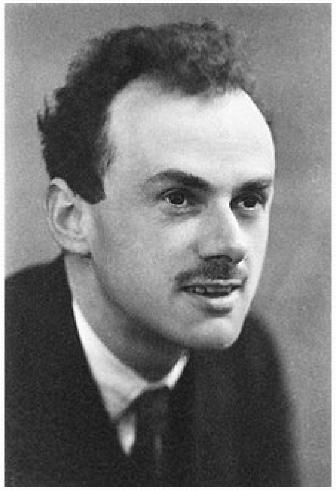
Linear Algebra



https://xkcd.com/184/

Vectors and Dirac (bra-ket) Notation

We are going to be working with n-dimensional vectors of **complex** numbers.


Paul Dirac

A ***ket*** is a column vector: $|a\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

A ***bra*** is a row vector, with the complex conjugates of the corresponding ket: $\langle a| = (\alpha_0^* \quad \alpha_1^* \quad \alpha_2^* \quad \alpha_3^*)$

A bra is the **adjoint** (complex conjugate **transpose**) of a ket.

Vector Space

A vector space is a collection of vectors
that is closed under addition and scalar multiplication.

Can add vectors together, or multiply any vector by a scalar,
and the result is still in the set.

Vector Space Properties

A vector space with vectors A, B, and C has the following properties:

- Commutativity: $A+B=B+A$
- Associativity of vector addition: $(A+B)+C=A+(B+C)$
- Additive identity: $0+A=A+0=A$, for all A
- Existence of additive inverse:
  For any A, there exists a $(-A)$ such that $A+(-A)=0$
- Scalar multiplication identity: $1A=A$
- Given scalars r and s

  Associativity of scalar multiplication: $r(sA)=(rs)A$

  Distributivity of scalar sums: $(r+s)A=rA+sA$

  Distributivity of vector sums: $r(A+B)=rA+rB$

Basis Vectors

A set of elements (vectors) in a vector space (V) is called a basis, or a set of basis vectors, if:

a) The vectors are linearly independent, and
b) Every vector in V is a linear combination of these vectors.

Linearly independent = no vector in the set can be written as a linear combination of other vectors in the set.

Example: Standard Basis

For a 4-dimensional vector space over complex numbers, these vectors are known as the standard basis:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Any vector can be written as: $|a\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle + c_3|3\rangle$

Hilbert Space: Inner Product

A Hilbert space is a vector space over complex numbers with an inner product $\langle b|a\rangle$.

David Hilbert

$$|a\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

$$\langle b|a\rangle = (\beta_0^* \quad \beta_1^*) \cdot \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \alpha_0\beta_0^* + \alpha_1\beta_1^*$$

$$|b\rangle = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

https://en.wikipedia.org/wiki/Hilbert_space

Properties of the Inner Product

Conjugate symmetric: $\langle b|a\rangle = \langle a|b\rangle^*$

Linear: $\langle b|(\alpha|x\rangle) = \alpha\langle b|x\rangle$

Positive definite: $\langle a|a\rangle > 0$, for $|a\rangle \neq 0$

Norm: $||a|| = \sqrt{\langle a|a\rangle}$

Inner Product and Basis Vectors

For orthonormal basis vectors, inner product is either 0 or 1.

$$\langle e_i | e_j \rangle = \delta_{ij}$$

Any vector can be expressed as a linear combination of basis vectors:

$$|\psi\rangle = \sum_i \alpha_i |e_i\rangle$$

Coefficients are derived from the inner products:

$$|\psi\rangle = \sum_i \langle \psi | e_i \rangle \, |e_i\rangle$$

Outer Product

$$|a\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

$$|a\rangle\langle b| = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \begin{pmatrix} \beta_0^* & \beta_1^* \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0^* & \alpha_0\beta_1^* \\ \alpha_1\beta_0^* & \alpha_1\beta_1^* \end{pmatrix}$$

$$|b\rangle = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

Can be viewed as an operator that transforms $|b\rangle$ to $|a\rangle$.

Unitary Matrix

A unitary matrix is a square complex matrix whose adjoint equals its inverse.

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad U^{\dagger} = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$$

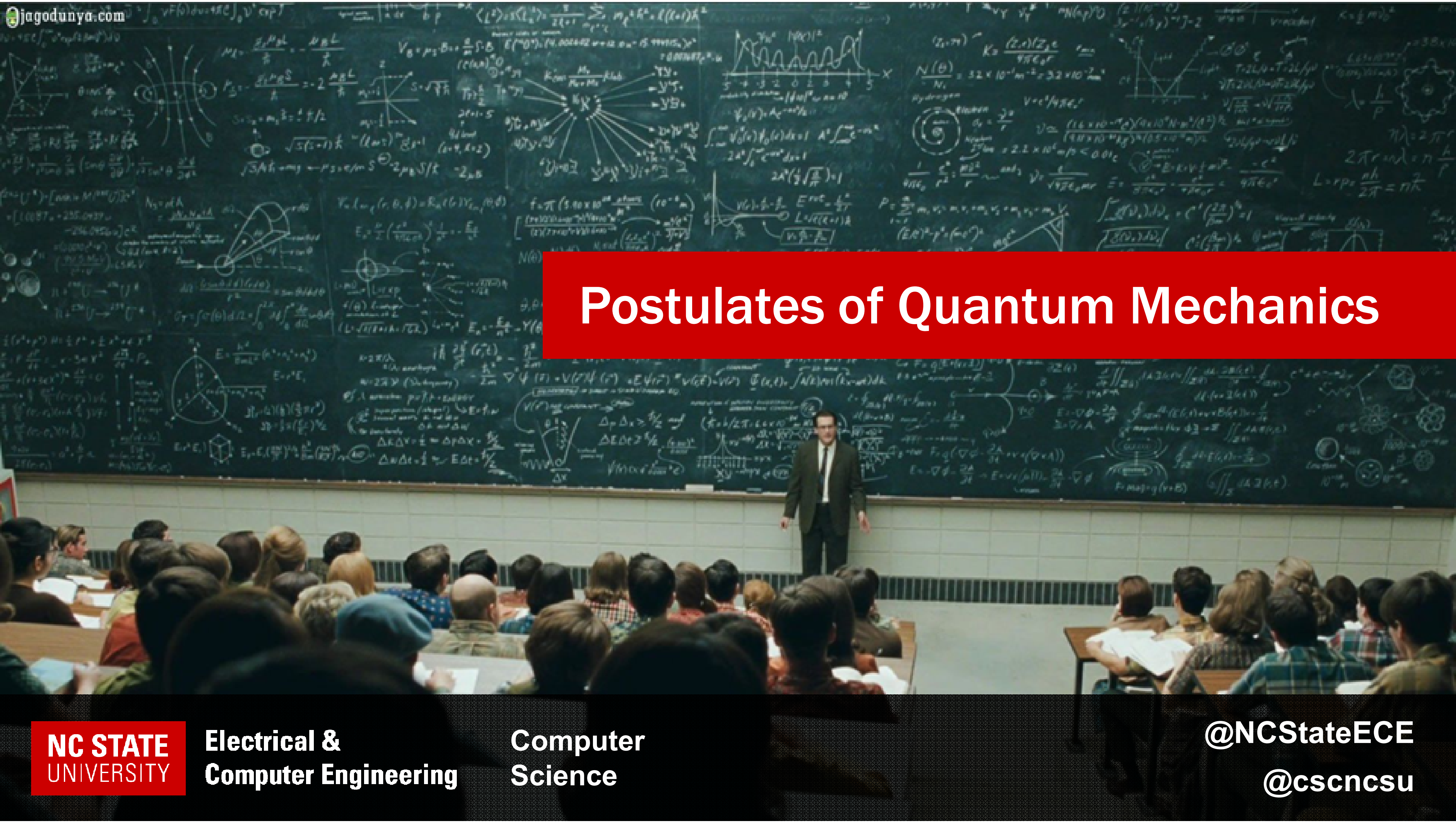
Also called the *conjugate transpose* or the *Hermitian transpose*.

Unitary if $UU^{\dagger} = U^{\dagger}U = I$

Unitary transform preserves the norm of a vector.

Charles Hermite

https://en.wikipedia.org/wiki/Charles_Hermite



Postulates of Quantum Mechanics

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Postulate 1

Associated to any *isolated* physical system is a Hilbert space known as the system's state space.

Also known as its *wave function*.

The system is completely specified by its state vector, which is a unit vector in the state space.

$$|\psi\rangle = \sum_i \alpha_i \, |e_i\rangle \qquad \sum_i |\alpha_i|^2 = 1$$

Single Qubit

The two standard basis vectors represent distinct states.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ -- ground}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ -- excited}$$

A superposition of these two states is also a state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \text{ where } |\alpha|^2 + |\beta|^2 = 1$$

Composite System

Suppose system A has Hilbert space H_A.

Suppose system B has Hilbert space H_B.

The state space of the two systems together is $H_A \otimes H_B$

where $\otimes$ represents the **tensor product**, also called the Kronecker product.

Tensor Product

$$A \otimes B = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \otimes \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

$$= \begin{pmatrix} a_{00}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} & a_{01}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \\ a_{10}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} & a_{11}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_{00}b_{00} & a_{00}b_{01} & a_{01}b_{00} & a_{01}b_{01} \\ a_{00}b_{10} & a_{00}b_{11} & a_{01}b_{10} & a_{01}b_{11} \\ a_{10}b_{00} & a_{10}b_{01} & a_{11}b_{00} & a_{11}b_{01} \\ a_{10}b_{10} & a_{10}b_{11} & a_{11}b_{10} & a_{11}b_{11} \end{pmatrix}$$

Standard Basis for 2-qubit System

$$|0\rangle_A \otimes |0\rangle_B = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix}1\\0\end{pmatrix} \\ 0\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = |00\rangle_{AB}$$

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Standard Basis for 2-qubit System

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

State space for n qubits has 2^n dimensions.

Basis states = { $|j\rangle$ }, where j = $0..2^n-1$

State Vector for n-qubit System

$$|\psi\rangle_n = \alpha_0|\ldots 000\rangle + \alpha_1|\ldots 001\rangle + \alpha_2|\ldots 010\rangle + \alpha_3|\ldots 011\rangle + \ldots$$

$$|\psi\rangle_n = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle + \alpha_3|3\rangle + \ldots$$

$$|\psi\rangle_n = \sum_j \alpha_j |j\rangle$$

Separable vs. Entangled States

There are states that cannot be expressed as the tensor product of independent qubit states.

Separable	Entangled
$\frac{1}{\sqrt{2}}\|01\rangle + \frac{1}{\sqrt{2}}\|11\rangle$ $= \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle) \otimes \|1\rangle$	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$

Postulate 2

Every observable attribute of a physical system is described by an operator that acts on the kets that describe the system.

Observable is something that can be measured.
E.g., spin, position, polarization, charge, ...

Operator is described by a Hermitian matrix.

Eigenstate and Eigenvalue

In general, applying an operator to a state will change the state.

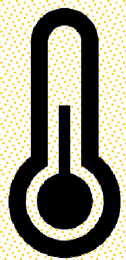
There are some states that are not changed, except for a scalar multiplier:

$$M|a_j\rangle = \lambda_j|a_j\rangle$$

These are called **eigenstates**, and the scalar value λ is the **eigenvalue** that corresponds to that eigenstate.

Postulate 3

When an observable is measured, the only possible outcome will be an eigenvalue of the operator for that observable.



Physical measurements are always real numbers.

Operator is described by a Hermitian matrix, which has real eigenvalues!

Measurement Eigenstates

This postulate is the basis for the “quantized” (discrete) nature of measured quantities.

The eigenstates a_j of a Hermitian operator form a basis:

- They are orthogonal -- $\langle a_j | a_k \rangle = \delta_{jk}$
- Any state can be expressed as a linear combination (with complex coefficients) of the eigenstates.

Measuring in the Standard Basis

Consider this measurement operator:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Two eigenstates, $|0\rangle$ and $|1\rangle$, with eigenvalues +1 and -1.

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle$$

Postulate 4

For a system in state $|\psi\rangle$, when an observable is measured, the probability of obtaining eigenvalue λ_j (corresponding to eigenstate $|a_j\rangle$) is:

$$|\langle a_j|\psi\rangle|^2$$

The complex number $\langle a_j|\psi\rangle$ is a "probability amplitude." It cannot be directly measured. Must be squared to get a real value.

Measuring in the Standard Basis

If we measure $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in the standard basis...

$\langle 0|\psi\rangle = \alpha$, and $\langle 1|\psi\rangle = \beta$, so:

Probability of measuring +1 = $|\alpha|^2$

Probablity of measuring -1 = $|\beta|^2$

Suppose $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. The probability of measuring +1 or -1 would be 0.5 for each.

Expected Value

A system prepared in state $|\psi\rangle$ and measured repeated using operator A will yield a variety of results, each with probability $|\langle a_j|\psi\rangle|^2$.

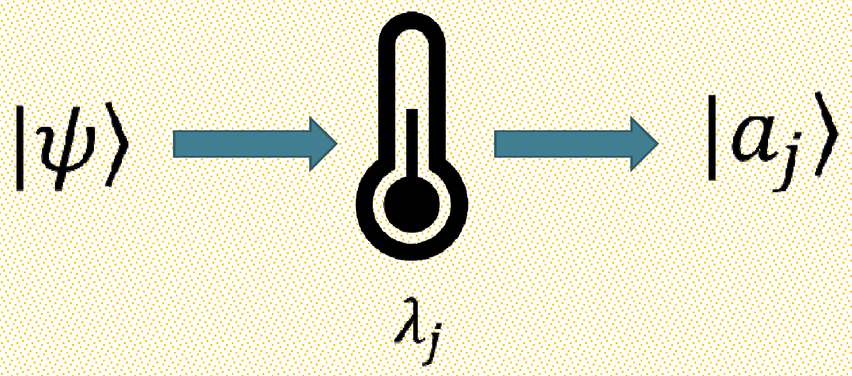
Running the same experiment (computation) twice is not expected to yield the same result!

Expected value:

$$\langle\psi\rangle \equiv \sum_j \lambda_j \cdot \Pr(\,|a_j\rangle) = \langle\psi|A|\psi\rangle$$

Postulate 5

When an observable is measured, obtaining eigenvalue λ_j, the state of the system is changed to become the corresponding eigenstate $|a_j\rangle$.



Collapse of the Wave Function

This postulate describes the “collapse” of the wave function that occurs when a measurement occurs.

All information that was embodied in the superposition is lost.

Permanently!

Implications

We never have access to the complete quantum state.

Any measurement/observation extracts one classical value and destroys the quantum state.

Can't probe or insert print statements.

Can't pause, probe, and resume to debug.

Can't checkpoint and restart.

Photo by eberhard grossgasteiger on Unsplash

Partial Measurement

What happens if we measure one qubit in a multi-qubit system?

(1) Forces that qubit to $|0\rangle$ or $|1\rangle$, based on probability amplitudes.

(2) “Removes” other cases where qubit is not the measured value.

(3) Renormalizes the remaining components.

$$|\psi\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

Measure: Qubit 0 = 1

Prob = 1/2

$$|\psi\rangle = \frac{|01\rangle + |11\rangle}{\sqrt{2}}$$

Another Example

$$|\psi\rangle = \frac{1}{4}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{3+i}{4}|10\rangle - \frac{i\sqrt{3}}{4}|11\rangle$$

Prob = 1/16

Prob = 2/16

Prob = 10/16

Prob = 3/16

Measure: Qubit 1 = 1

Prob = 13/16

$$|\psi\rangle = \frac{3+i}{\sqrt{16}}|10\rangle - \frac{i\sqrt{3}}{\sqrt{16}}|11\rangle$$

Entanglement Revisited

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Measure: Qubit 1 = 1

Prob = 1/2

$$|\psi\rangle = |11\rangle$$

When we measure qubit 1 = 1, we know (without measuring!) that qubit 0 = 1, with 100% certainty. But... there was a 50/50 chance that qubit 0 was 0 or 1 before measurement.

This is true even when the two entangled qubits are separate in space.

Postulate 6

The evolution of a closed quantum system over time is described by a unitary transformation.

The state $|\psi\rangle$ of the system at time t_1
is related to the state $|\psi'\rangle$ of the system at time t_2
by a unitary U which depends only on times t_1 and t_2.

$$|\psi'\rangle = U|\psi\rangle$$

Implications for Quantum Computing

Program = step by step evolution from initial state to end state

Each step must be unitary.

Preserves norm, so sum of probabilities = 1.

Has an inverse, so reversible.

System must be closed: no interaction with environment.

(Measurement is not reversible.)

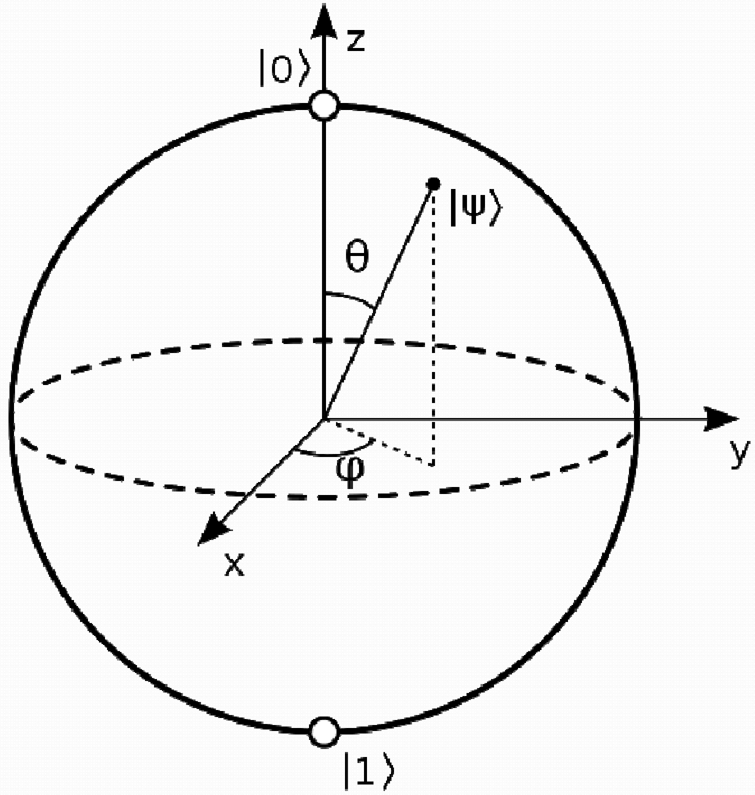
Unitary transform is a “rotation” of the vector.

Bloch Sphere Representation

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Two parameters:
θ is angle of rotation around X-axis.
φ is angle of rotation around Z-axis.

Reminder: $e^{i\varphi} = \cos\varphi + i\sin\varphi$



https://en.wikipedia.org/wiki/Bloch_sphere

Some Unitary Transformations

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

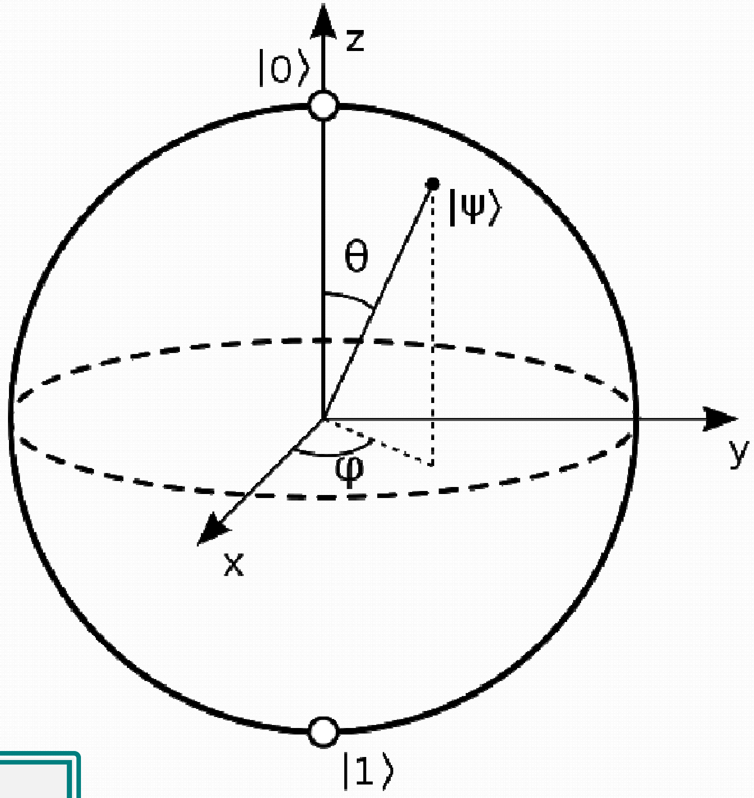
Rotation around the X axis.
(swaps $|0\rangle$ and $|1\rangle$ coefficients, $\theta = \pi$)

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Rotation around the Z axis.
(negates $|1\rangle$ coefficient, $\varphi = \pi$)

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Rotation around the Y axis.
($Y = iXZ$)



Pauli matrices

Quirk demo:
https://algassert.com/quirk

Two-Qubit Transform

Perform X on $\text{qubit}_1 = 1$, and Z on qubit_0.

$$X \otimes Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Start	*End*
$\|00\rangle$	$\|10\rangle$
$\|01\rangle$	$-\|11\rangle$
$\|10\rangle$	$\|00\rangle$
$\|11\rangle$	$-\|01\rangle$

Two-Qubit Transform

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

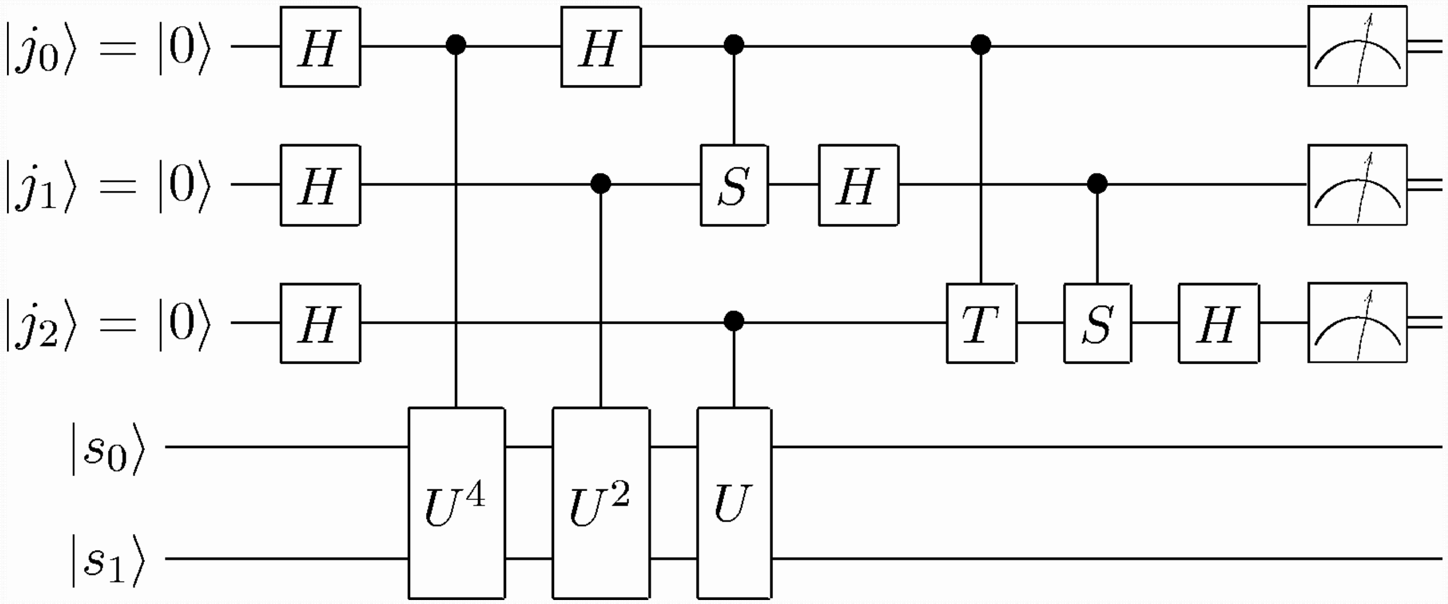
If $\text{qubit}_1 = 1$, flip qubit_0.

Start	*End*
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

Gate-Model Program = Circuit

Program = step-by-step evolution from initial state to final state

Each step (gate) is unitary.



Measurement

Summary

Qubits

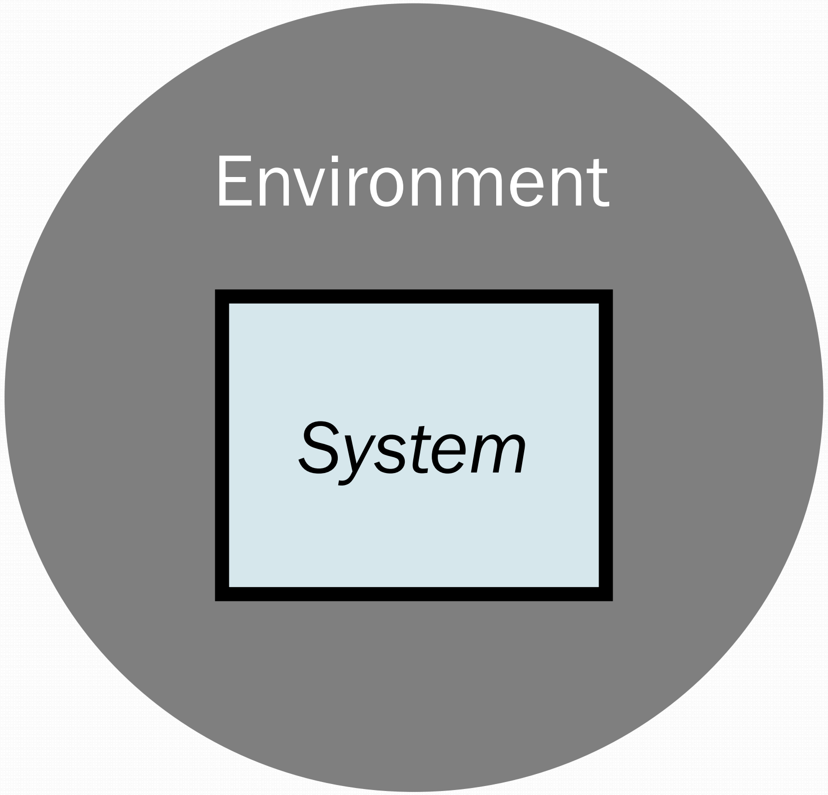
Qubit is represented by a vector in a complex Hilbert space.

Systems are composed using tensor product.

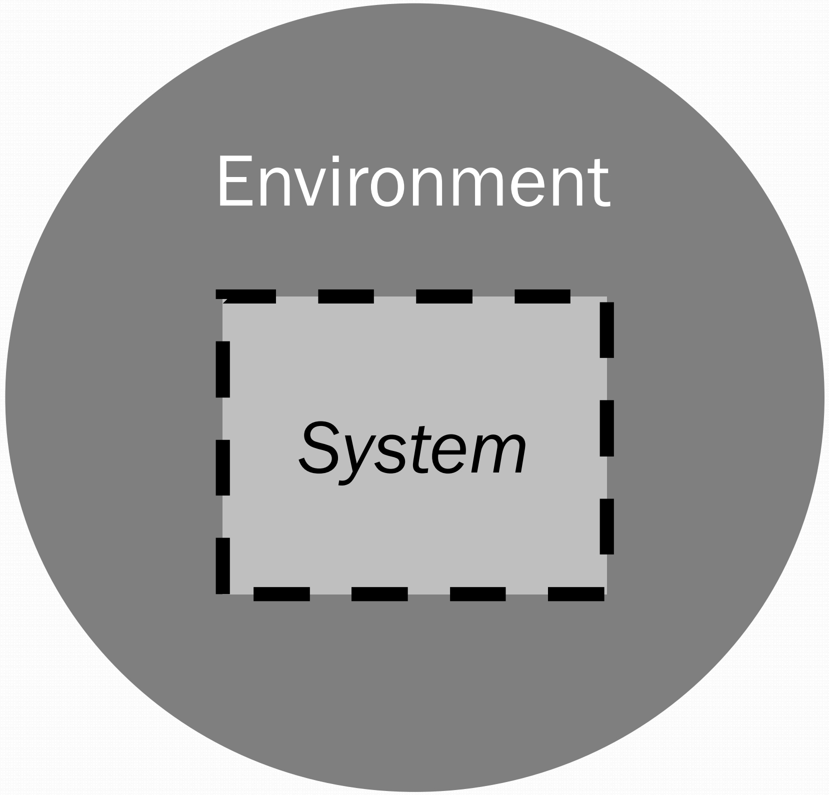
State evolves by unitary transforms (rotations).

Measurement projects state to an eigenstate of the measurement operator.

Decoherence



Theory: closed system



Reality: not closed