1 Optimal Power Allocation

Optimal power allocation across the transmit antennas can be formulated as a constrained maximization problem. The objective is to distribute the total available transmit power among the different antennas such that the overall achievable rate is maximized.

We consider the following general optimization problem:

$$
\begin{aligned}
\max_{\{p_n\}} \quad & \sum_{n=1}^{N} \alpha_n \log_2 (1 + a_n\, p_n) \\
\text{s. t.} \quad & \sum_{n=1}^{N} p_n = p_t \\
& \forall n \in \{1, \dots, N\} : \, p_n \geq 0
\end{aligned}
\tag{1}
$$

where p_n denotes the power allocation for antenna n, and p_t denotes the total available transmit power. The factors $\{\alpha_n\}$ represent weighting factors that allow for unequal prioritization of the individual transmission channels, while $\{a_n\}$ capture the effective signal-to-noise ratio per unit power allocated of each transmission channel.

In the specific scenario considered here, no prioritization is applied, so the weighting factors can be ignored. Furthermore, the factors a_n are defined as $\frac{\sigma_n^2}{2B\,N_0}$, where σ_n represents the channel gain associated with antenna n and N_0 denotes the noise power spectral density.

1.1 Lagragian Optimization & Waterfilling Algorithm

To solve the constrained optimization problem introduced in the previous section, the method of Lagrangian multipliers is employed. This approach allows the incorporation of the power constraints directly into the objective function by means of auxiliary variables.

$$
\mathcal{L}(\mathbf{p}, \mu, \boldsymbol{\lambda}) = \sum_{n=1}^{N} \alpha_n \log_2 (1 + a_n\, p_n) - \mu \left(\sum_{n=1}^{N} p_n - p_t \right) - \sum_{n=1}^{N} \lambda_n p_n \tag{2}
$$

where μ denotes the Lagrange multiplier associated with the total transmit power constraint, and λ_n are the Lagrange multipliers corresponding to the non-negativity constraints imposed on the individual power allocations.

Since the objective function is concave and the constraints are affine, the optimization problem is convex. Consequently, the Karush-Kuhn-Tucker (KKT) conditions constitute necessary and sufficient conditions for optimality. These conditions are given by

1. Stationarity: $\nabla_{\mathbf{p}}\, \mathcal{L}(\mathbf{p}, \mu, \boldsymbol{\lambda}) = 0$
2. Primal feasibility: $p_n \geq 0 \quad \text{AND} \quad \sum_{n=1}^{N} p_n = p_t$
3. Dual feasibility: $\lambda_n \geq 0$
4. Complementary slackness: $\begin{cases} p_n > 0 \implies \lambda_n = 0 \\ p_n = 0 \implies \lambda_n \geq 0 \end{cases}$

From the KKT conditions, we can derive the optimal power allocation for each antenna. The derivation proceeds by successively applying the stationarity, complementary slackness, and primal feasibility conditions.

The stationarity condition requires that the gradient of the Lagrangian with respect to each optimization variable vanishes at the optimum. This expresses the balance between the marginal gain in the achievable rate and the penalties introduced by the power constraints.

$$\frac{\partial \mathcal{L}}{\partial p_n} = \frac{\alpha_n a_n}{\ln(2)\,(1 + a_n p_n)} - \mu - \lambda_n = 0$$

The complementary slackness condition links the non-negativity constraint on the allocated power to its associated Lagrange multiplier. When a positive amount of power is allocated to antenna n, the corresponding multiplier λ_n is zero, and the stationarity condition reduces to

$$\begin{aligned} \frac{\alpha_n a_n}{\ln(2)(1 + a_n p_n)} - \mu = 0 \\ p_n = \frac{\alpha_n}{\mu \ln(2)} - \frac{1}{a_n} \end{aligned}$$

The primal feasibility condition enforces the non-negativity of the power allocations. As a result, the above expression for p_n must be clipped to zero whenever it becomes negative.

$$p_n = \left(\frac{\alpha_n}{\mu \ln(2)} - \frac{1}{a_n} \right)^+ \tag{3}$$

Finally, the primal feasibility condition also requires the total allocated power to equal the available transmit power p_t. By substituting the expression for p_n from equation (3) into this constraint, we obtain an implicit equation for the the Lagragian multiplier μ:

$$\sum_{n=1}^{N} \left(\frac{\alpha_n}{\mu \ln(2)} - \frac{1}{a_n} \right)^+ = p_t \tag{4}$$

Since μ cannot be obtained in closed form, it is determined numerically such that the total allocated power satisfies (4). Owing to the monotonic relationship between μ and the total allocated power, a bisection method is employed to efficiently compute its optimal value. This is illustrated in Algorithm 1.

Algorithm 1 Waterfilling Algorithm

Require: Total transmit power p_t, weighting factors $\{\alpha_n\}$, effective signal-to-noise ratios per unit power $\{a_n\}$.
Ensure: Optimal power allocation $\{p_n\}$.

Initialization

Initialize the lower μ_{low} and upper bounds μ_{high} on the Lagragian multiplier μ and after that μ itself: $\mu_{\text{low}} \leftarrow 0,\ \mu_{\text{high}} \leftarrow (\alpha_0\, a_0)/(\ln(2)(1 + a_0 p_t)),\ \mu \leftarrow (\mu_{\text{low}} + \mu_{\text{high}})/2$

Iteration

while $p_{t,\text{current}} \neq p_t$ **do**
  Update either the lower or upper bound based on the current total allocated power: **if** $p_{t,\text{current}} < p_t$ **then** $\mu_{\text{low}} \leftarrow \mu$ **else** $\mu_{\text{high}} \leftarrow \mu$ **end if**
  Update the Lagragian multiplier: $\mu \leftarrow (\mu_{\text{low}} + \mu_{\text{high}})/2$
  Compute the current total allocated power $p_{t,\text{current}}$ using equation (4).
end while

Termination

Compute the optimal power allocations $\{p_n\}$ using equation (3).
return $\{p_n\}$

The resulting solution (3) admits a waterfilling interpretation. The term $1/(\mu \ln(2))$ represents a common water level scaled by the weighting factors α_n, while $1/a_n$ defines a channel-dependent floor. Power is poored into each antenna container, so to speak, until the water level is reached. No power is allocated to antennas where the floor exceeds the water level.

This interpretation provides an intuitive understanding of the optimal power allocation strategy, where more power is allocated to antennas with better channel conditions.

1.2 Simulation Results