System Performance - Bit Error Rate Calculation

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This document presents the calculations of a theoretical upper bound and approximation for the bit error rate (BER) of the single-user multi-input multi-output (SU-MIMO) singular value decomposition (SVD) communication system.

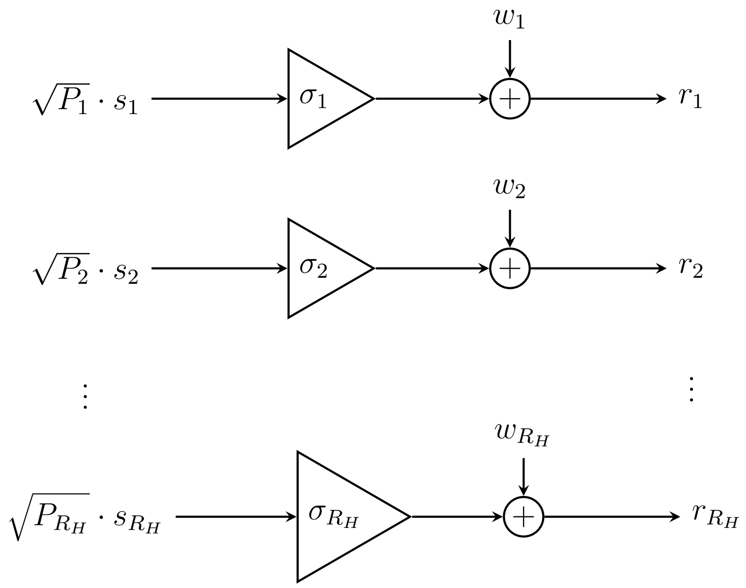

Figure 1: The R_H parallel eigenchannels of the decomposed SU-MIMO SVD system.

The BER of the complete SU-MIMO SVD system can be computed as the weighted average of the BERs of each eigenchannel of the system.

$$\mathrm{BER} = \frac{1}{C_{\text{total}}} \sum_{i=0}^{R_H} \lfloor C_i \rfloor \, \mathrm{BER}_i \tag{1}$$

The total used capacity of the system equals the sum of the used capacities of each eigenchannel: $C_{\text{total}} = \sum_{i=0}^{R_H} \lfloor C_i \rfloor$

Capacity of the eigenchannels. The capacity of each eigenchannel C_i can be computed using the Shannon capacity formula.

$$C_i = 2B \cdot \log_2(1 + \frac{P_i \sigma_i^2}{2BN_0}) \tag{2}$$

Bit Error Rate of the eigenchannels [1]. The BER of each eigenchannel BER_i equals the average amount of bit errors per received data symbol.

$$\text{BER}_i = \frac{1}{\log_2 M_i} \sum_{(\hat{\alpha},\alpha)\in\mathcal{C}_i^2} \text{N}(\hat{\alpha},\alpha) \cdot \text{P}[\hat{s}_i = \hat{\alpha} \wedge s_i = \alpha] \tag{3}$$

In the above equation:

- M_i is the size of the modulation constellation used in eigenchannel i.
- $\mathcal{C}_i^2$ is the Cartesian product of the modulation constellation used in eigenchannel i with itself, representing all possible pairs of transmitted and received symbols.
- $\text{N}(\hat{\alpha},\alpha)$ is the number of bit errors between symbol α and symbol $\hat{\alpha}$. This equals the Hamming distance between the binary representations of both symbols.
- $\text{P}[\hat{s}_i = \hat{\alpha} \wedge s_i = \alpha]$ is the joint probability that symbol α was transmitted and symbol $\hat{\alpha}$ was detected in eigenchannel i.

We can simplify this expression using Bayes' rule and by assuming that all symbols are equally likely to be transmitted. The latter is the case when the bits that arrive at the transmitter are independent and equally distributed.

$$\begin{aligned}
\text{BER}_i &= \frac{1}{\log_2 M_i} \sum_{(\hat{\alpha},\alpha)\in\mathcal{C}_i^2} \text{N}(\hat{\alpha},\alpha) \cdot \text{P}[\hat{s}_i = \hat{\alpha} \mid s_i = \alpha] \cdot \text{P}[s_i = \alpha] \\
\text{BER}_i &= \frac{1}{\log_2 M_i} \sum_{(\hat{\alpha},\alpha)\in\mathcal{C}_i^2} \text{N}(\hat{\alpha},\alpha) \cdot \text{P}[\hat{s}_i = \hat{\alpha} \mid s_i = \alpha] \cdot \frac{1}{M_i} \\
\text{BER}_i &= \frac{1}{M_i \log_2 M_i} \sum_{(\hat{\alpha},\alpha)\in\mathcal{C}_i^2} \text{N}(\hat{\alpha},\alpha) \cdot \text{P}(\hat{\alpha},\alpha) \qquad (4)
\end{aligned}$$

1 Upper Bound

At the receiver, the symbol $\hat{\alpha}$ will be detected iff the distance between the decision variable u_i and $\hat{\alpha}$ is less than the distance between u_i and any other symbol in the constellation. In other words, u_i needs to lie within the decision area $D(\hat{\alpha})$ of the symbol $\hat{\alpha}$.

$$\text{P}(\hat{\alpha},\alpha) = \text{P}(u_i \in D(\hat{\alpha}) \mid s_i = \alpha) \tag{5}$$

The decision variable u_i is given by the output of the equalizer at the receiver.

$$\begin{aligned}
u_i &= \frac{1}{\sigma_i\sqrt{P_i}} \cdot r_i \\
u_i &= \frac{1}{\sigma_i\sqrt{P_i}} \cdot (\sigma_i\sqrt{P_i} \cdot s_i + n_i) \\
u_i &= s_i + \frac{1}{\sigma_i\sqrt{P_i}} n_i \\
u_i &= s_i + n_i' \qquad (6)
\end{aligned}$$

in which $\vec{n} = \mathrm{U}^H \vec{w}$ is the noise vector after the unitary transformation at the receiver, which does not change the distribution of the noise. Therefore, the noise term n_i' is complex, circularly symmetric Gaussian distributed with $n_i' \sim \mathcal{CN}(0, \frac{N_0}{P_i \sigma_i^2})$.
So we find that the decision variable u_i is a complex, circularly symmetric Gaussian random variable with mean s_i and variance $\frac{N_0}{P_i \sigma_i^2}$.

The upper bound:

$$\mathrm{P}(\hat{\alpha}, \alpha) \leq \mathrm{P}\Big(|u_i - \hat{\alpha}| < |u_i - \alpha| \;\Big|\; s_i = \alpha \Big) \tag{7}$$

It is a necessary condition for u_i to be closer to $\hat{\alpha}$ than to α in the complex plane in order to be detected as $\hat{\alpha}$. It is however not a sufficient condition, since yet another symbol in the constellation could be even closer to u_i than $\hat{\alpha}$.
This is why the above expression is an upper bound. It can be interpreted as the probability that the noise n_i' causes the decision variable u_i to cross the decision boundary that lies halfway between α and $\hat{\alpha}$, instead of the probability that n_i' causes u_i to enter the decision area of $\hat{\alpha}$.

We simplify the right hand side of Equation 7.

$$\begin{aligned}
&= \mathrm{P}\Big(|u_i - \hat{\alpha}|^2 < |u_i - \alpha|^2 \;\Big|\; s_i = \alpha \Big) \\
&= \mathrm{P}\Big(|u_i - \alpha - (\hat{\alpha} - \alpha)|^2 < |u_i - \alpha|^2 \;\Big|\; s_i = \alpha \Big) \\
&= \mathrm{P}\Big(|u_i - \alpha - (\hat{\alpha} - \alpha)|^2 - |u_i - \alpha|^2 < 0 \;\Big|\; s_i = \alpha \Big) \\
&= \mathrm{P}\Big(|\hat{\alpha} - \alpha|^2 - 2\Re\big((u_i - \alpha)(\hat{\alpha} - \alpha)^*\big) < 0 \;\Big|\; s_i = \alpha \Big) \\
&= \mathrm{P}\Big(\Re\big((u_i - \alpha)(\hat{\alpha} - \alpha)^*\big) > \frac{|\hat{\alpha} - \alpha|^2}{2} \;\Big|\; s_i = \alpha \Big) \\
&= \mathrm{P}\Big(\Re\Big((u_i - \alpha)\frac{(\hat{\alpha} - \alpha)^*}{|\hat{\alpha} - \alpha|}\Big) > \frac{|\hat{\alpha} - \alpha|}{2} \;\Big|\; s_i = \alpha \Big) \\
&= \mathrm{P}\Big(\Re\big((u_i - \alpha)e^{j\theta}\big) > \frac{|\hat{\alpha} - \alpha|}{2} \;\Big|\; s_i = \alpha \Big) \\
&= \mathrm{Q}\Big(\frac{|\hat{\alpha} - \alpha|}{2} \cdot \frac{1}{\frac{N_0}{2 P_i \sigma_i^2}} \Big) \\
&= \mathrm{Q}\Big(\frac{P_i \sigma_i^2}{N_0} \cdot |\hat{\alpha} - \alpha| \Big)
\end{aligned} \tag{8}$$

In the penultimate step, we used the fact that $\mathrm{X} = \Re\big((u_i - \alpha)e^{j\theta}\big) \;\big|\; (s_i = \alpha)$ is a real-valued Gaussian random variable with mean 0 and variance $\sigma = \frac{N_0}{2 P_i \sigma_i^2}$.
This can be seen from the fact that $u_i - \alpha | (s_i = \alpha) = u_i - s_i = n_i'$. The noise term n_i' is circularly symmetric complex Gaussian, so rotation by $e^{j\theta}$ does not change its distribution, and the real component is normally distributed with the same mean and half the variance.

Thus, with this upper bound on $\mathrm{P}(\hat{\alpha}, \alpha)$ and using Equation 4, we can compute an upper bound on the BER of each eigenchannel BER_i as follows

$$\mathrm{BER}_i \leq \frac{1}{M_i \log_2 M_i} \sum_{(\hat{\alpha},\alpha)\in\mathcal{C}_i^2} \mathrm{N}(\hat{\alpha}, \alpha) \cdot \mathrm{Q}\Big(\frac{P_i \sigma_i^2}{N_0} \cdot |\hat{\alpha} - \alpha|\Big) \tag{9}$$

2 Approximation

We only consider symbol errors for which the decision variable u_i is converted to one of the nearest neighbors of the transmitted symbol $s_i = \alpha$. In other words, we only take into account the terms in the summation of Equation 9 (or Equation 4) for which $\hat{\alpha}$ is a nearest neighbor of α in the modulation constellation.

In this way, we can derive an accurate approximation for the BER of each eigenchannel, starting from Equation 9.

$$\mathrm{BER}_i \approx \frac{1}{M_i \log_2 M_i} \sum_{\alpha\in\mathcal{C}_i} \sum_{\hat{\alpha}\in\mathcal{S}(\alpha)} \mathrm{N}(\hat{\alpha}, \alpha) \cdot \mathrm{Q}\Big(\frac{P_i \sigma_i^2}{N_0}\ d_{\min}\Big) \tag{10}$$

In the above equation:

- $\mathcal{S}(\alpha)$ is the set of nearest neighbors of symbol α in the modulation constellation.
- d_{min} is the minimum distance between two symbols in the modulation constellation. This is a constant for a given constellation type and size.

Because we make use of gray coding during the mapping of bits to symbols, the Hamming distance $\mathrm{N}(\hat{\alpha}, \alpha)$ between two nearest neighbor symbols equals 1 for all pairs of nearest neighbors.

$$\begin{aligned}
\mathrm{BER}_i &\approx \frac{1}{M_i \log_2 M_i} \sum_{\alpha\in\mathcal{C}_i} \sum_{\hat{\alpha}\in\mathcal{S}(\alpha)} \mathrm{Q}\Big(\frac{P_i \sigma_i^2}{N_0} d_{\min}\Big) \\
\mathrm{BER}_i &\approx \frac{1}{\log_2 M_i} \mathrm{Q}\Big(\frac{P_i \sigma_i^2}{N_0} d_{\min}\Big) \cdot \frac{1}{M_i} \sum_{\alpha\in\mathcal{C}_i} \Big|\mathcal{S}(\alpha)\Big|
\end{aligned} \tag{11}$$

The last term in Equation 11 represents the average number of nearest neighbors per symbol in the constellation. This equals a constant K that only depends on the type and size of used constellation.

Therefore, we can further simplify Equation 11 to obtain a final approximation for the BER of each eigenchannel.

$$\mathrm{BER}_i \approx \frac{K}{\log_2 M_i} \mathrm{Q}\Big(\frac{P_i \sigma_i^2}{N_0} d_{\min}\Big) \tag{12}$$

The constants K and $d_{\min}$ can easily be calculated for the modulation constellations that are considered in this system. Table 1 provides the formulas.

Constellation	K	$d_{\min}$
M-PAM	$\frac{2(M-1)}{M}$	$\sqrt{\frac{12}{M^2-1}}$
M-PSK	$\begin{cases} 1, & M=2 \\ 2, & M>2 \end{cases}$	$2 \cdot \lvert \sin(\frac{\pi}{M}) \rvert$
M-QAM	$\frac{4(\sqrt{M}-1)}{\sqrt{M}}$	$\sqrt{\frac{6}{M-1}}$

Table 1: The average number of nearest neighbors K and the distance between nearest neighbors $d_{\min}$ for common constellations.

References

[1] P. M. M. Prof. Nele Noels, *Communicatietheorie*. Ghent, Belgium: Ghent University, 2022-2023.