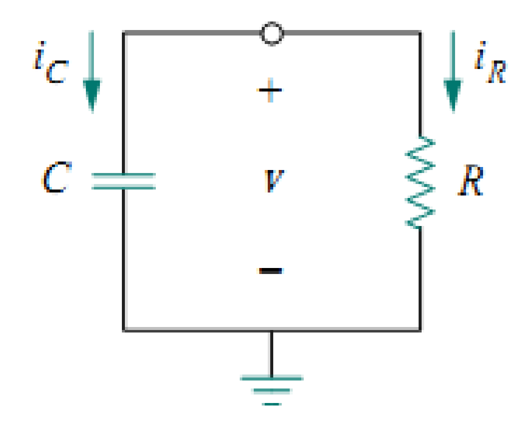
DC Circuits

First Order Circuits

The Source Free Circuits

The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

RC Circuit



$$i_C + i_R = 0$$

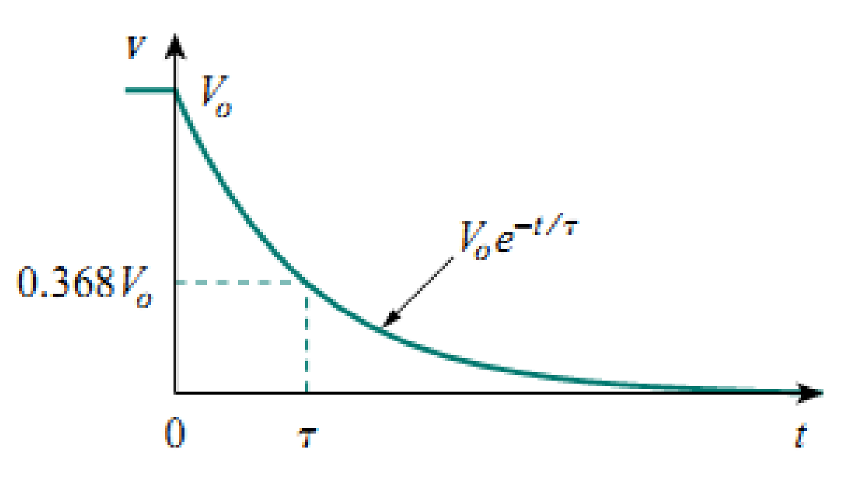
$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

$$\boxed{v(t) = V_0 e^{-t/RC}}$$

The time constant of a circuit is the time required for the response to decay by a factor of $1/e$ or 36.8 percent of its initial value.[1]

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368 V_0$$



$$\boxed{\tau = RC}$$

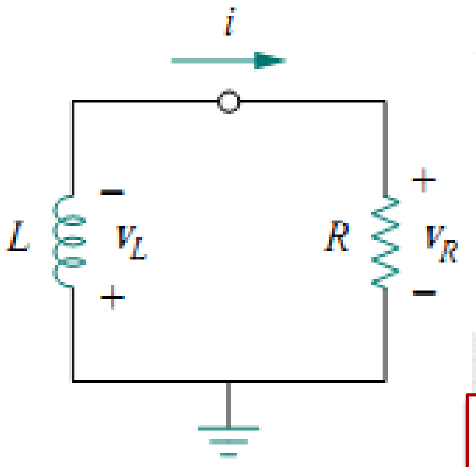
$$\boxed{v(t) = V_0 e^{-t/\tau}}$$

The Key to Working with a Source-free RC Circuit is Finding:

1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant τ.

When a circuit contains a single capacitor and several resistors and dependent sources, the Thevenin equivalent can be found at the terminals of the capacitor to form a simple RC circuit. Also, one can use Thevenin's theorem when several capacitors can be combined to form a single equivalent capacitor.

RL Circuit

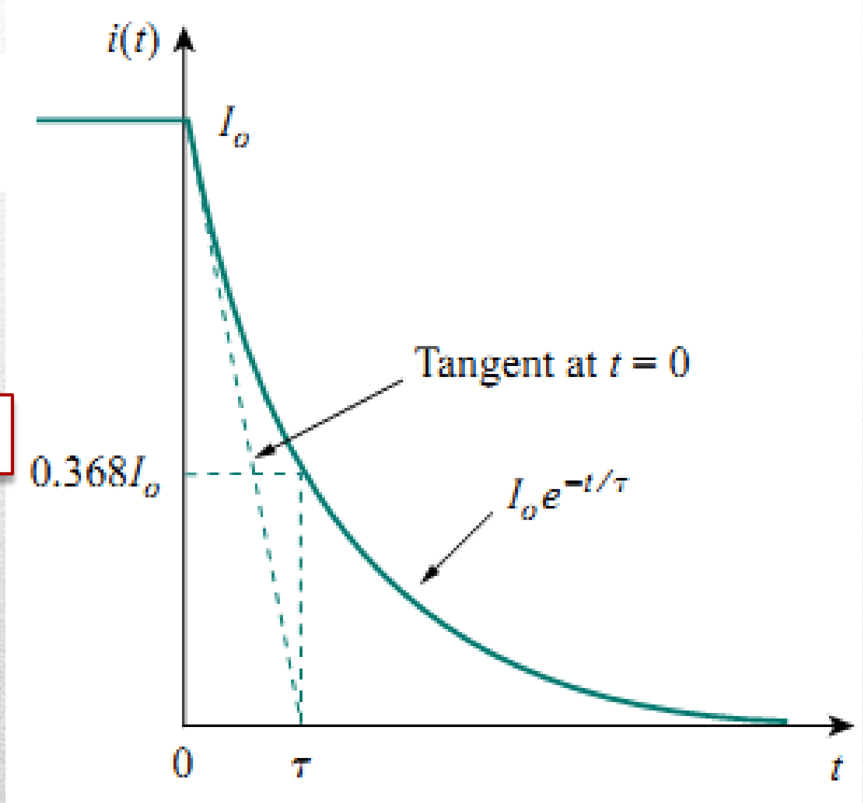


$$v_L + v_R = 0$$

$$L\frac{di}{dt} + Ri = 0$$

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

$$\boxed{i(t) = I_0 e^{-Rt/L}}$$



$$\boxed{\tau = \frac{L}{R}}$$

$$\boxed{i(t) = I_0 e^{-t/\tau}}$$

The Key to Working with a Source-free *RL* Circuit is to Find:

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant τ of the circuit.

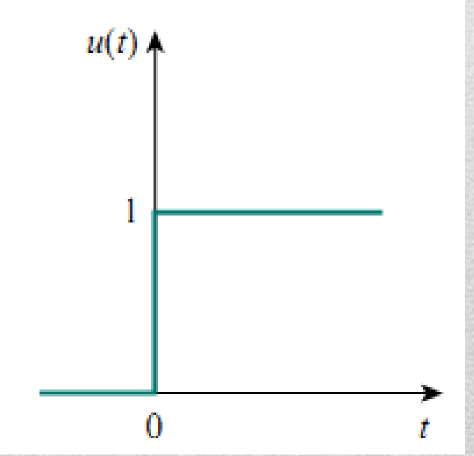
When a circuit has a single inductor and several resistors and dependent sources, the Thevenin equivalent can be found at the terminals of the inductor to form a simple RL circuit. Also, one can use Thevenin's theorem when several inductors can be combined to form a single equivalent inductor.

Singularity Functions

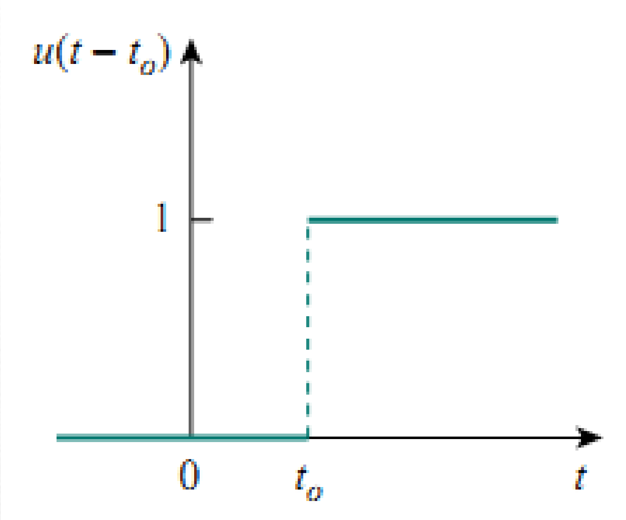
Singularity functions are functions that either are discontinuous or have discontinuous derivatives.

Unit Step Function

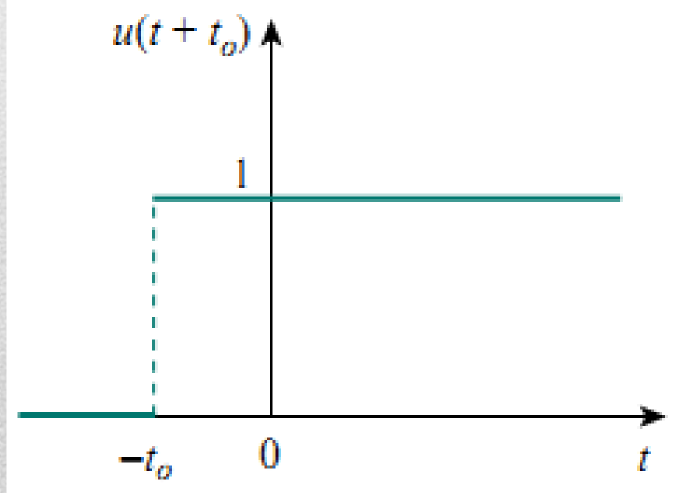
The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t.



$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



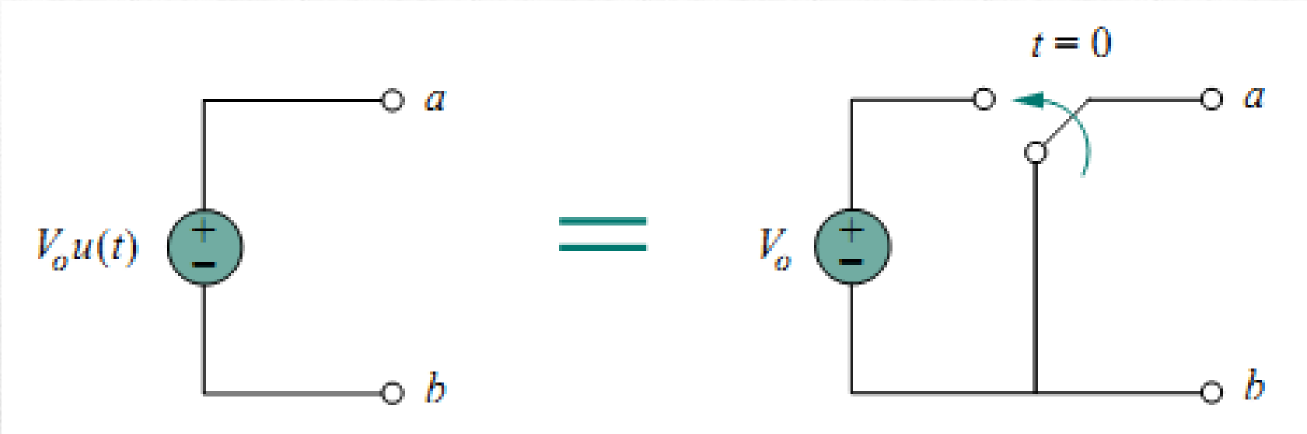
$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$

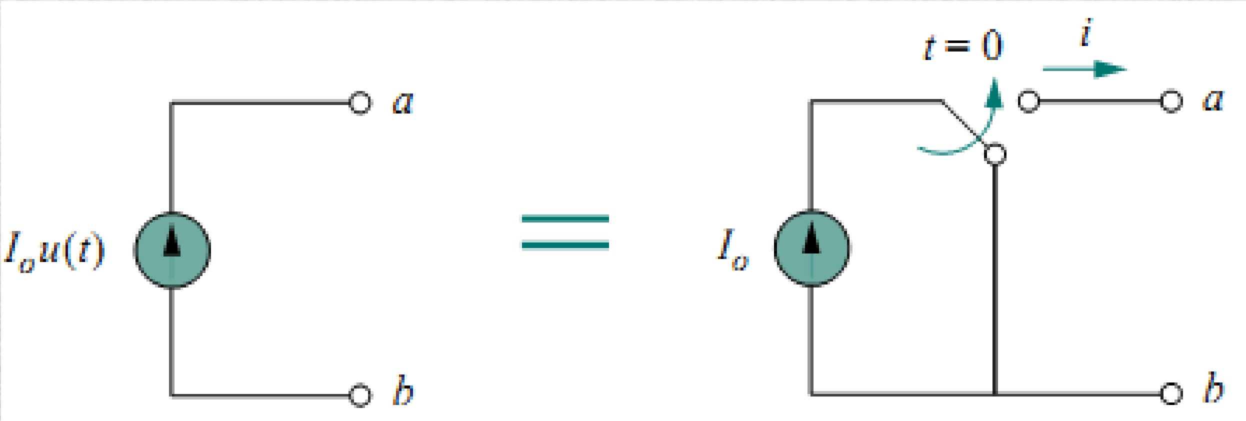


$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$

$$v(t) = V_0 u(t - t_0)$$

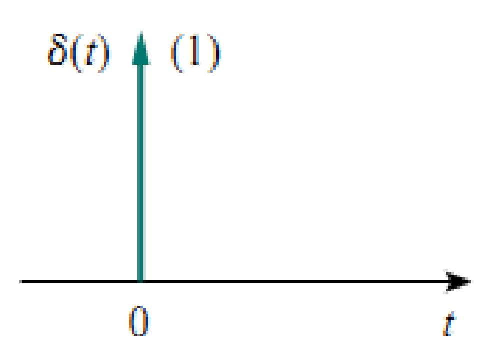
$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases}$$





Unit Impulse Function

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

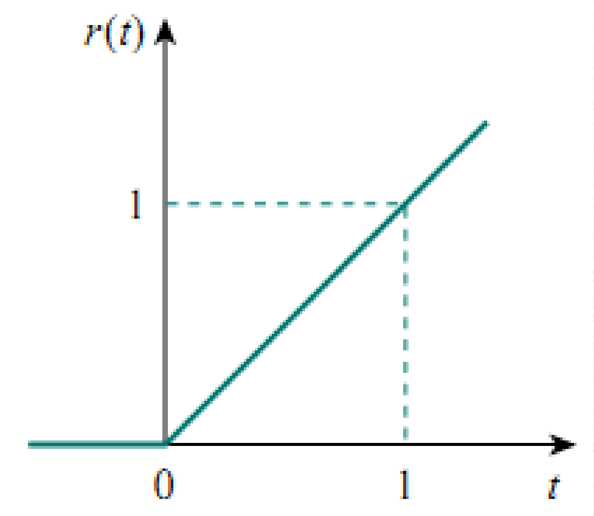


The unit impulse function $\delta(t)$ is zero everywhere except at $t = 0$, where it is undefined.

Unit Ramp Function

$$r(t) = \int_{-\infty}^{t} u(t)\,dt = tu(t)$$

$$\boxed{r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}}$$

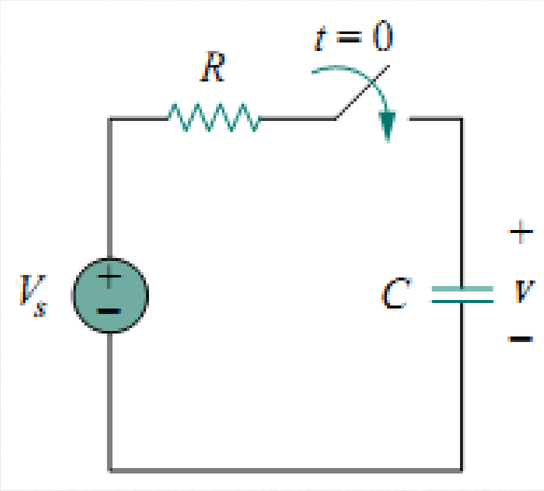


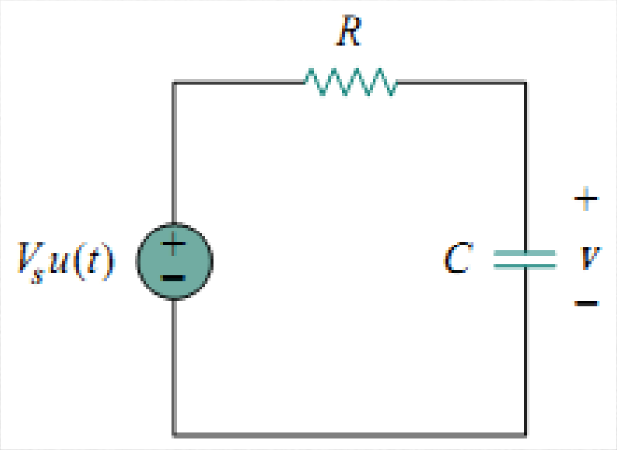
The unit ramp function is zero for negative values of t and has a unit slope for positive values of t.

Step Response

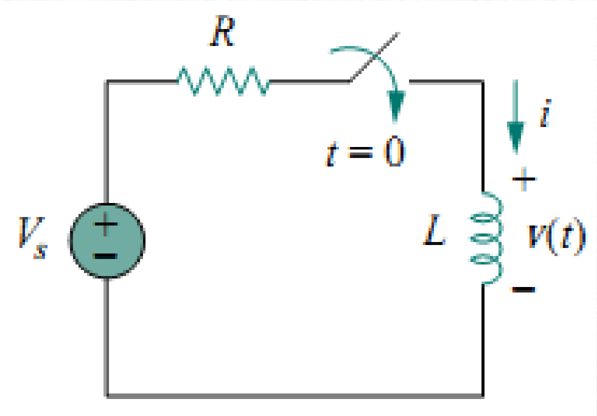
The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

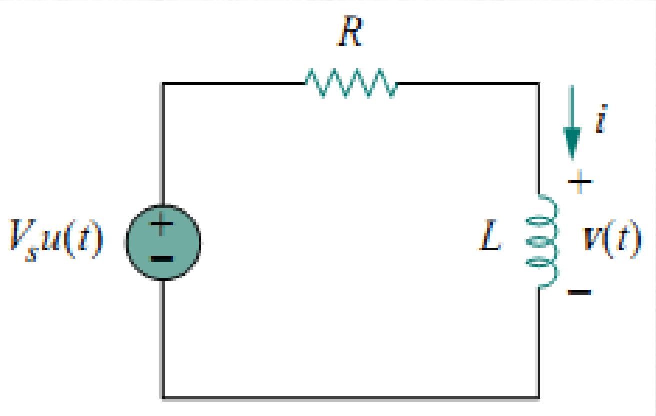
RC Circuit





RL Circuit





Complete Response

The **complete response** of the circuit is the sum of the **natural response** and the **forced response**.

$$v = v_f + v_n$$

$$i = i_n + i_f$$

The natural response or transient response is the circuit's temporary response that will die out with time.

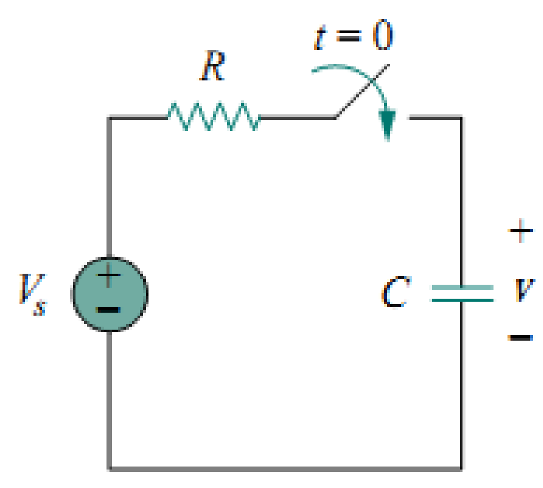
The forced response or steady-state response is the behavior of the circuit a long time after an external excitation is applied.

General Solution

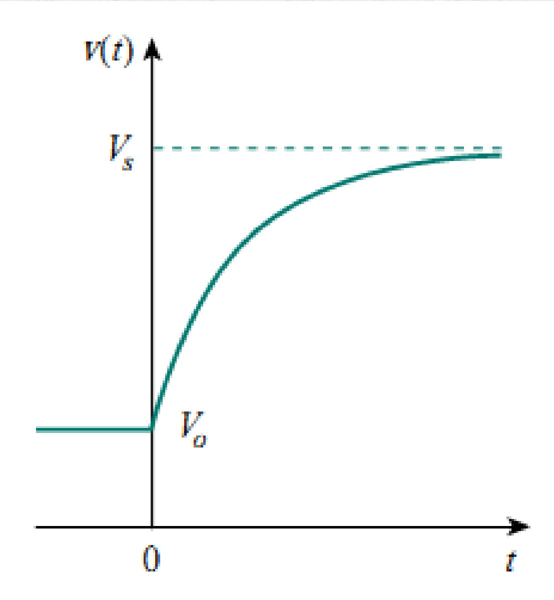
$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

Complete Response of RC Circuit

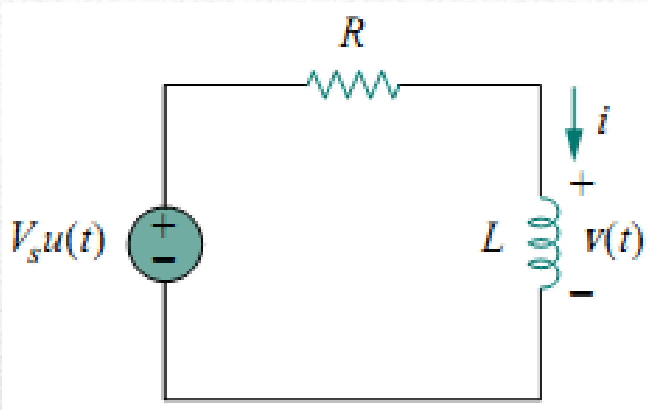


$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

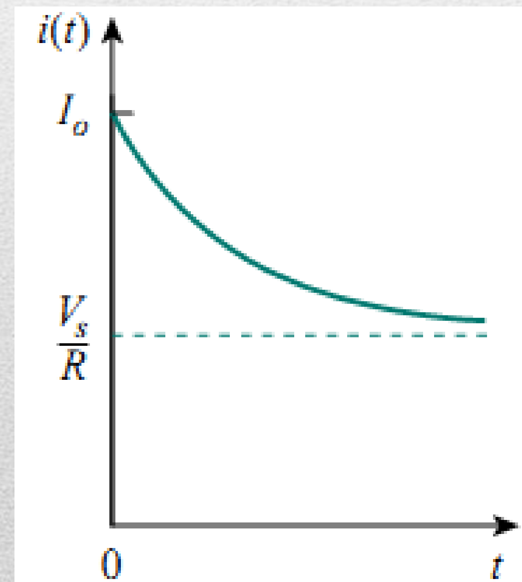


$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

Complete Response of RL Circuit



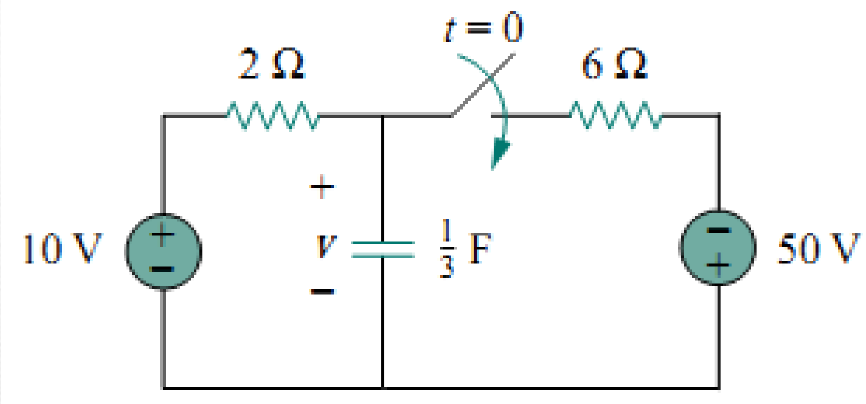
$$\boxed{i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}}$$



$$i(t) = \begin{cases} I_o & t < 0 \\ \dfrac{V_s}{R} + \left(I_o - \dfrac{V_s}{R}\right) e^{-\frac{R}{L}t} & t > 0 \end{cases}$$

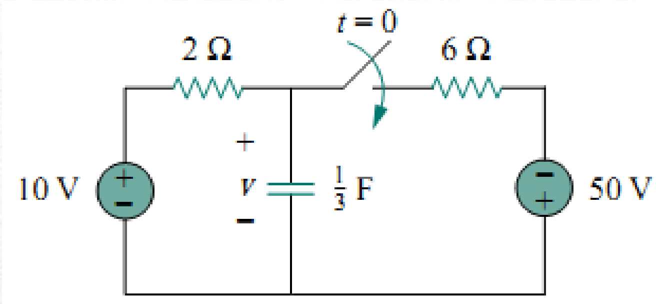
Practice Problems

1 Find $v(t)$ for $t > 0$. Assume the switch has been open for a long time and is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5$



Answer: $-5 + 15e^{-2t}$ V, 0.5182 V.

Solution



Method 1

$$v(0) = 10\text{ V}$$

$$v(\infty) = \frac{6}{6+2}10 + \frac{2}{2+6}(-50) = \frac{60-100}{8} = -5\text{ V}$$

$$\tau = (2 \parallel 6)\frac{1}{3} = \frac{12}{8}\,\frac{1}{3} = \frac{1}{2}$$

$$v(t) = v_f(t) + v_n(t)$$

$$v(t) = v\,(\infty) + Ae^{-\frac{t}{\tau}}$$

$$v(t) = -5 + Ae^{-2t}$$

For $t = 0$: $v(0) = -5 + A$

$$10 = -5 + A \quad \Longrightarrow \quad A = 15$$

$$\boxed{v(t) = -5 + 15e^{-2t}\ \text{ V}}$$

$t = 0.5$: $v(0.5) = -5 + 15e^{-2(0.5)}$

$$v(0.5) = -5 + 15 \times 0.3679$$

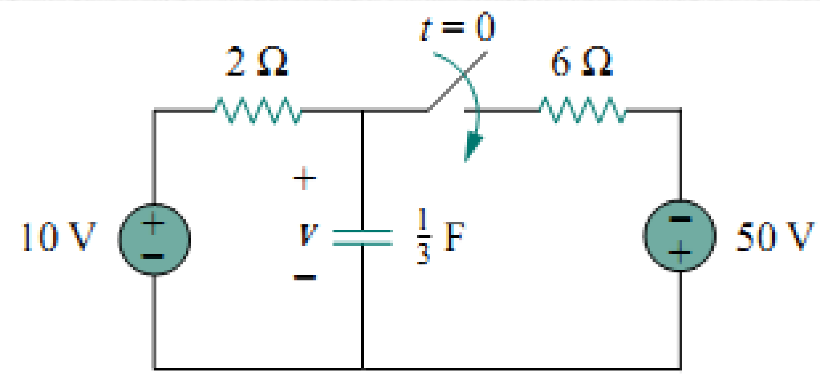
$$\boxed{v(0.5) = 0.5182\ \text{ V}}$$

Method 2

$$v(0) = 10\ \text{V}$$

$$v(\infty) = \frac{6}{6+2}10 + \frac{2}{2+6}(-50) = \frac{60-100}{8} = -5\ \text{V}$$

$$\tau = (2 \parallel 6)\frac{1}{3} = \frac{12}{8}\frac{1}{3} = \frac{1}{2}$$



$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-\frac{t}{\tau}}$$

$$v(t) = -5 + [10 - (-5)]e^{-2t}$$

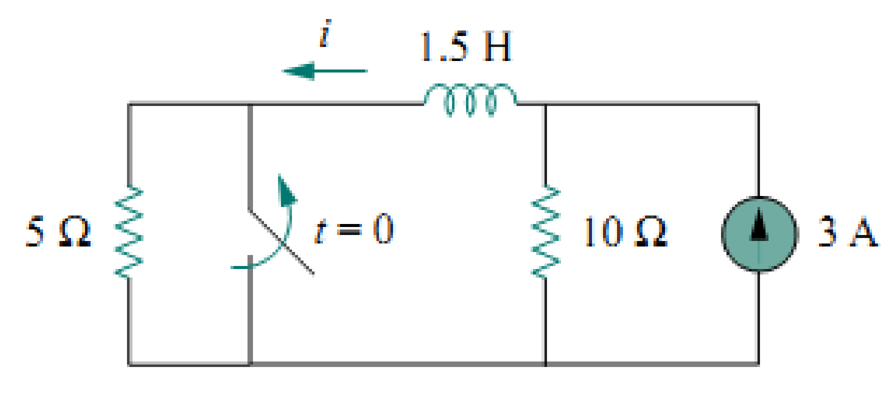
$$\boxed{v(t) = -5 + 15e^{-2t}\ \ \text{V}}$$

$t = 0.5$: $\quad v(0.5) = -5 + 15e^{-2(0.5)}$

$$v(0.5) = -5 + 15 \times 0.3679$$

$$\boxed{v(0.5) = 0.5182\ \ \text{V}}$$

2 The switch has been closed for a long time. It opens at $t = 0$. Find $i(t)$ for $t > 0$



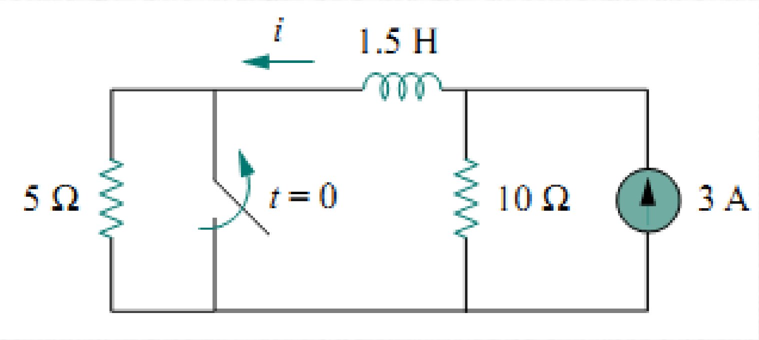
Answer: $(2 + e^{-10t})$ A, $t > 0$.

Solution

$$i(0) = 3 \text{ A}$$

$$i(\infty) = \frac{10}{5+10}(3) = 2 \text{ A}$$

$$\tau = \frac{1.5}{15} = \frac{1}{10}$$

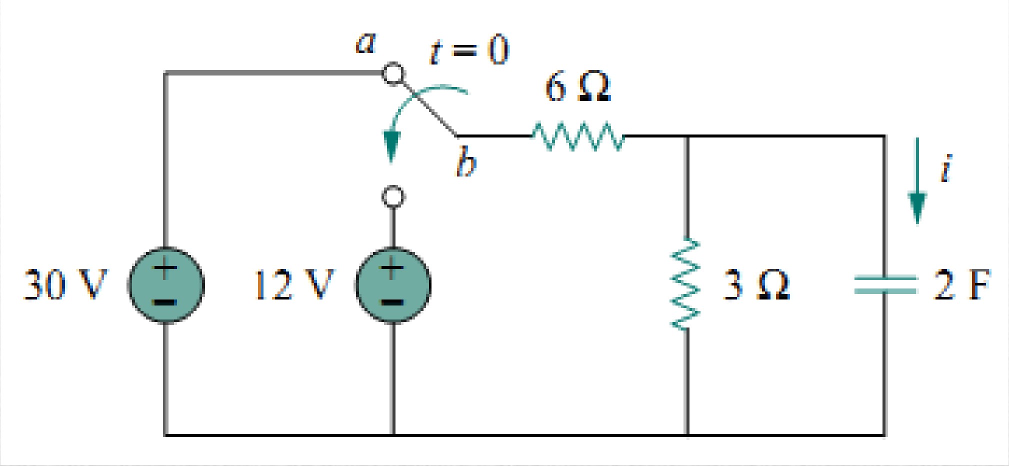


$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-\frac{t}{\tau}}$$

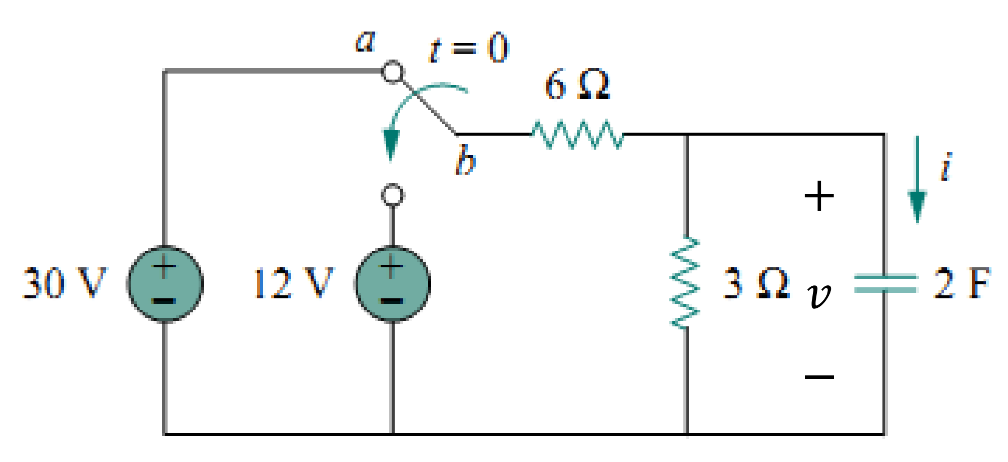
$$i(t) = 2 + [3 - 2]e^{-10t}$$

$$\boxed{i(t) = 2 + e^{-10t} \text{ A}}$$

3 The switch in has been in position a for a long time. At $t = 0$, it moves to position b. Calculate $i(t)$ for all $t > 0$.



Solution



$t < 0$ $\quad v(t) = \dfrac{3}{3+6}30 = 10\ \text{V}$

$i(t) = 0\ \text{A}$

$t = 0$ $\quad v(0) = 10\ \text{V}$

$t > 0$ $\quad v(\infty) = \dfrac{3}{3+6}12 = 4\ \text{V}$

$\tau = (6 \parallel 3)2 = 4$

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-\frac{t}{\tau}}$$

$$v(t) = 4 + [10 - 4]e^{-0.25t}$$

$$v(t) = 4 + 6e^{-0.25t}\ \text{V}$$

$$i(t) = 2\frac{dv(t)}{dt} = 2(-0.25)6e^{-0.25t}$$

$$i(t) = -3e^{-0.25t}\ \text{A}$$

$$\boxed{i(t) = \begin{cases} 0\ \text{A} & t < 0 \\ -3e^{-0.25t}\ \text{A} & t > 0 \end{cases}}$$