Institut Teknologi Sepuluh Nopember

Pengantar Robotika

REPRESENTING POSITION AND ORIENTATION

(WORK IN 2D)

Outline

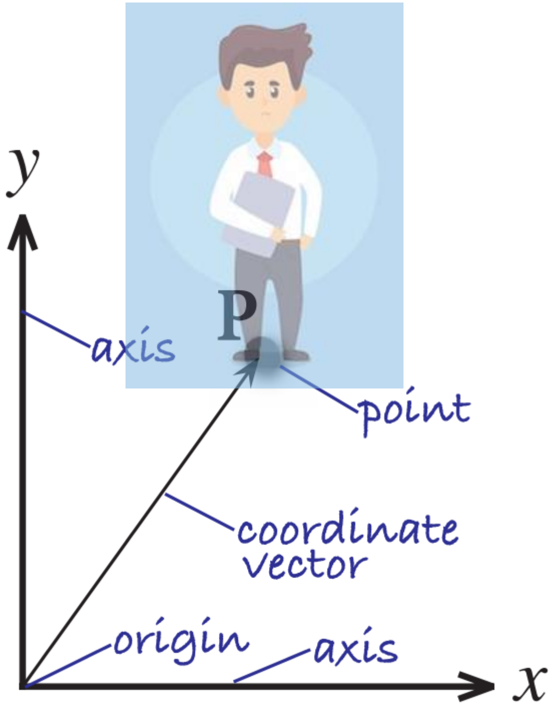
Orientation in 2D

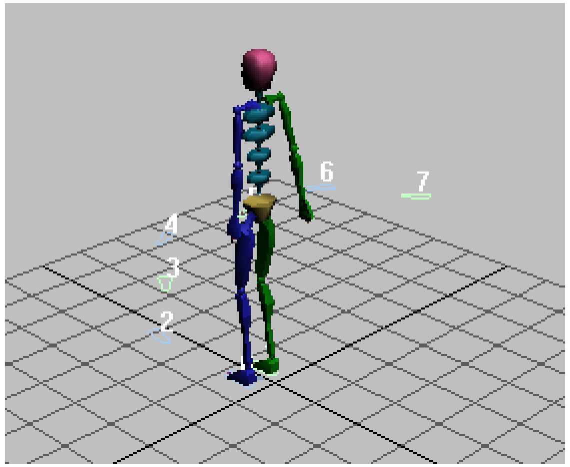
Pose in 2D

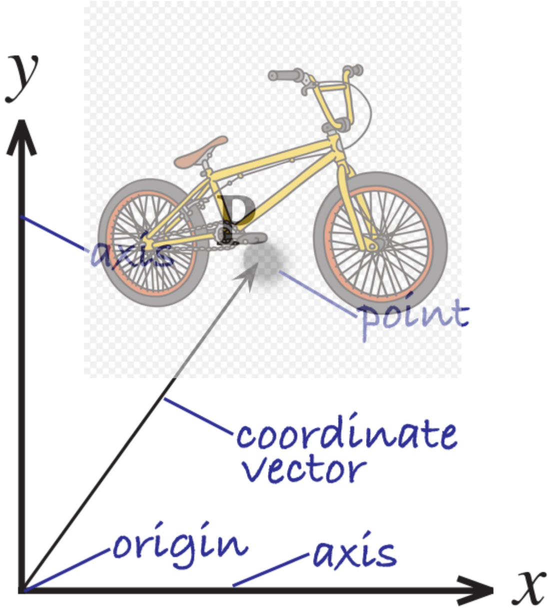
Orientation in 2D

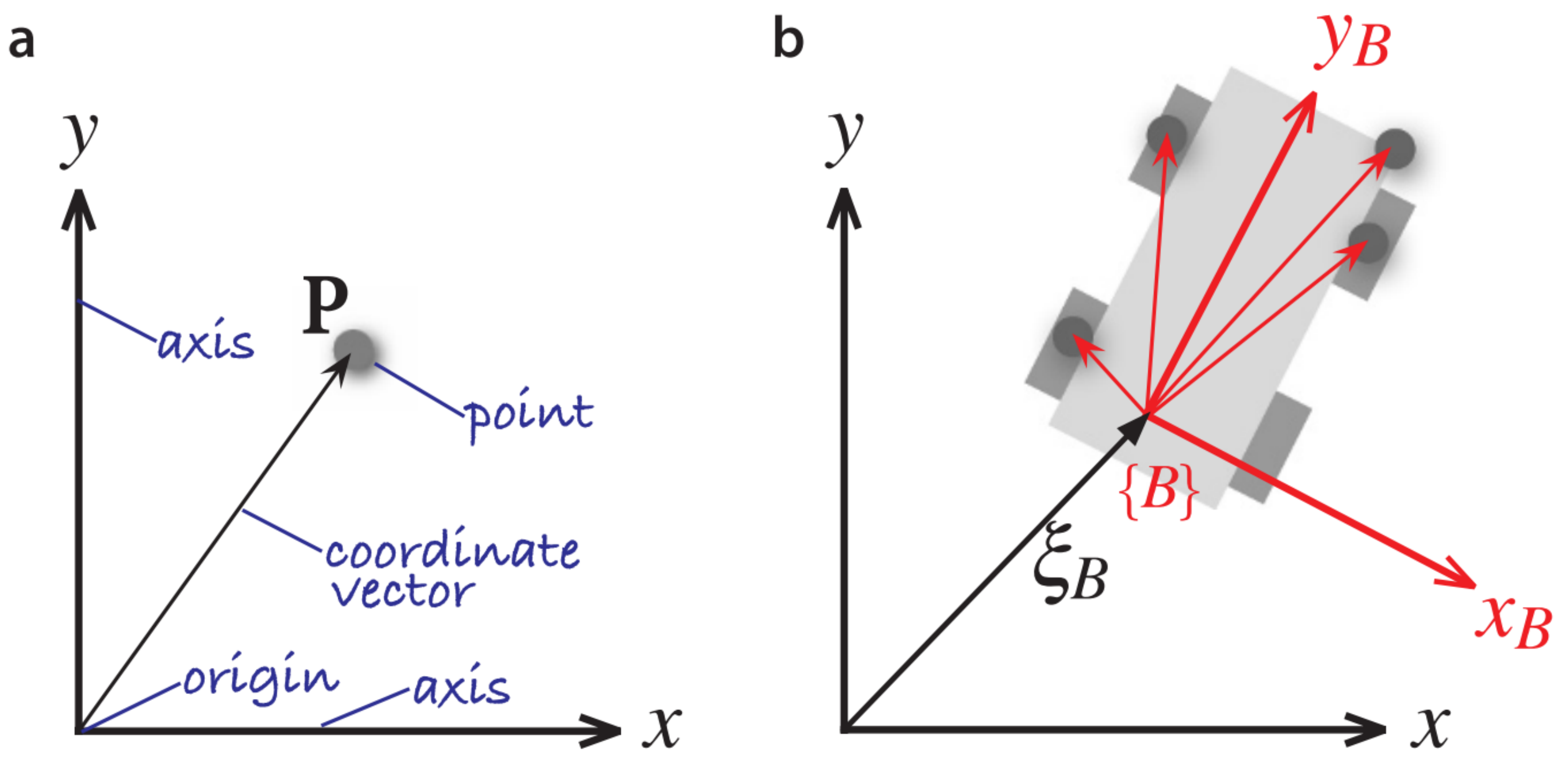
Position

- How do we know our position.
- How do we know the position of an object we want to interact with.
- How do we know the position of the object relative to our position.



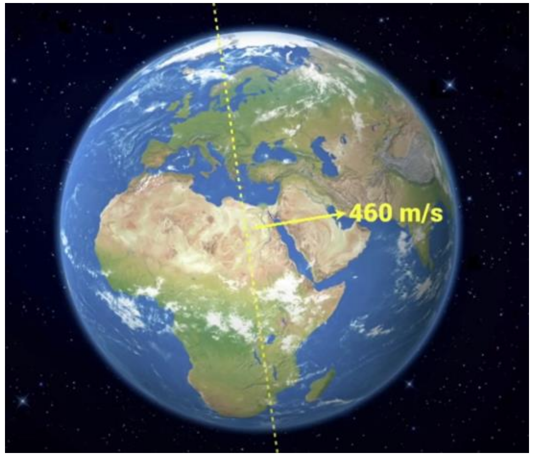


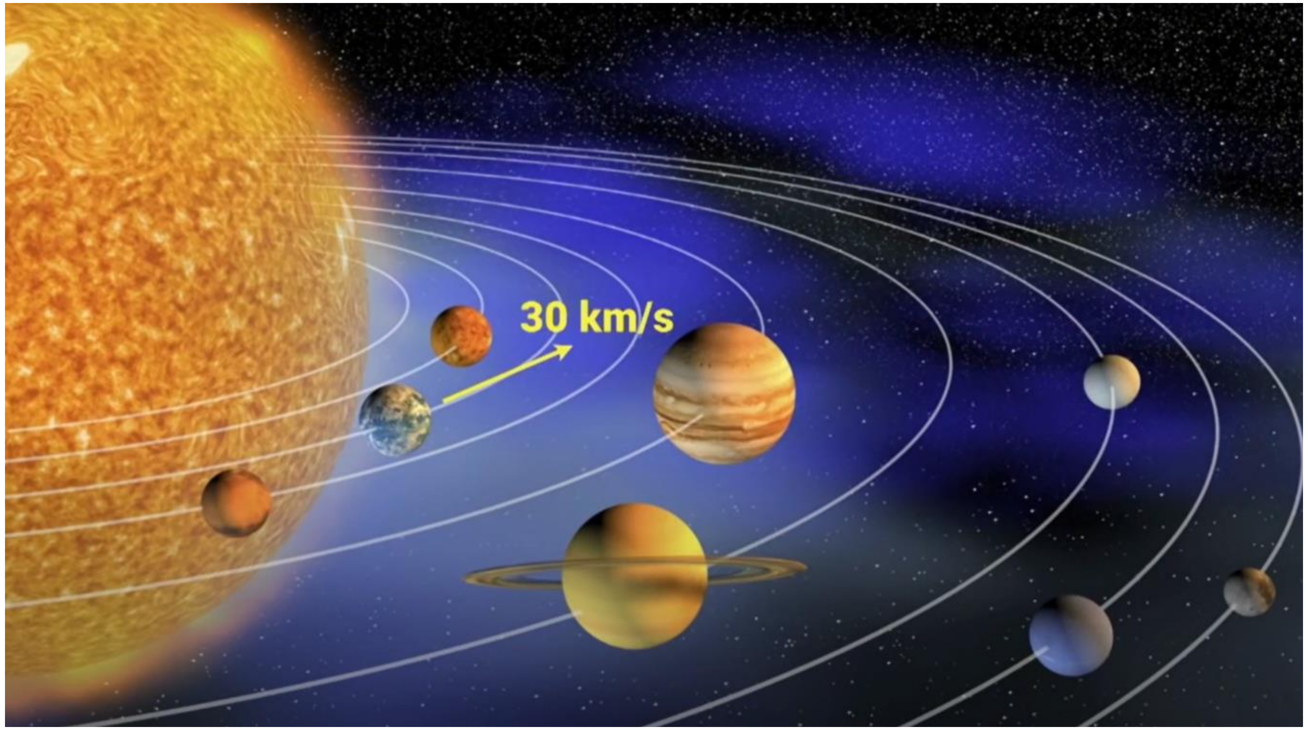


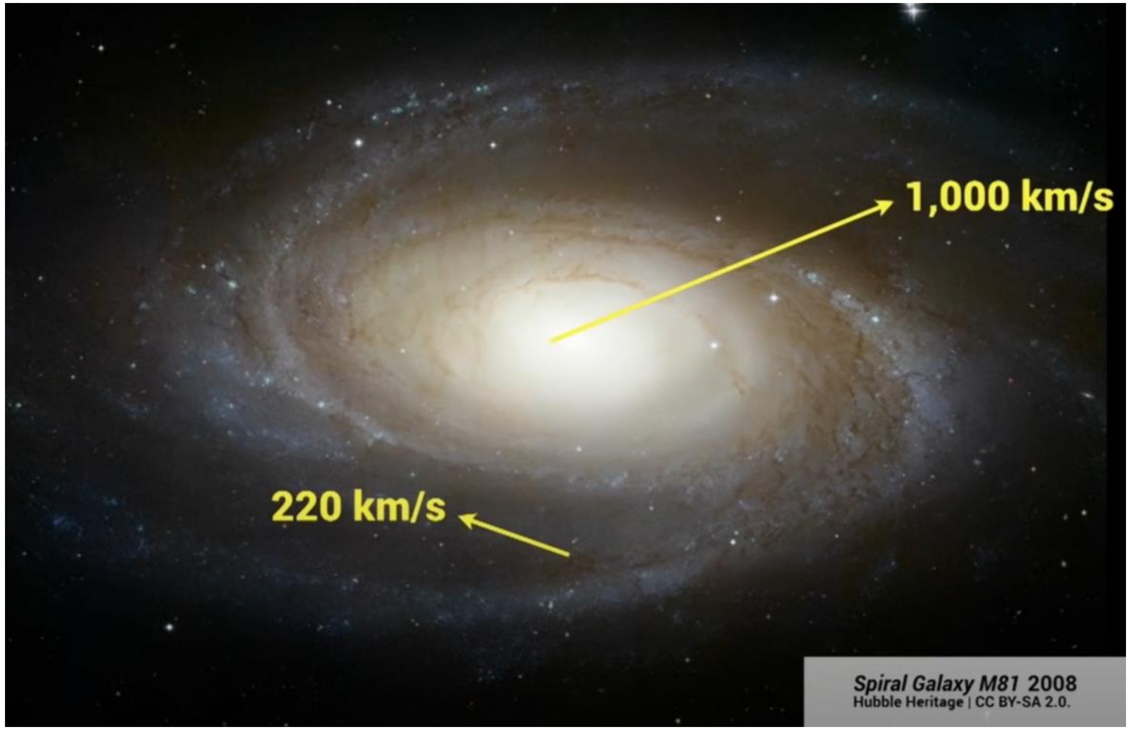

a
y
P
axis
point
coordinate
vector
origin
axis
x
b
y
y_B
$\{B\}$
ξ_B
x_B
x

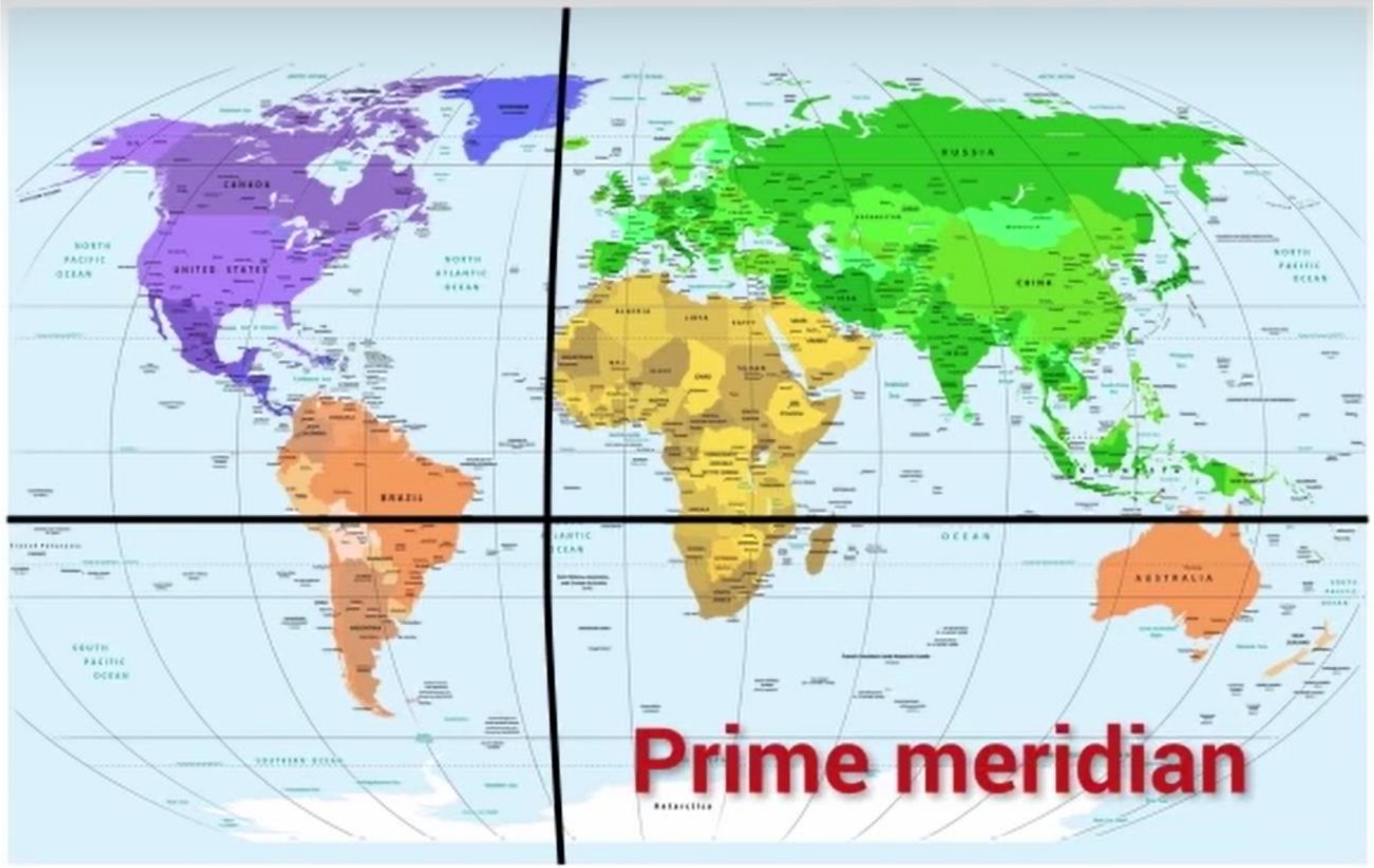
- Denominate numbers, a number plus a unit, to specify distance. It also calls this single number a scalar.
  - Ex : The object is 2 m away
- Vector, a denominate number plus a direction, to specify a locatior
  - Ex : The object is 2 m due north.
- Orientation of the object.
  - Ex : The object is 2 m due north and facing west.
- The combination of position and orientation we call Pose.

Create Your frame of Reference

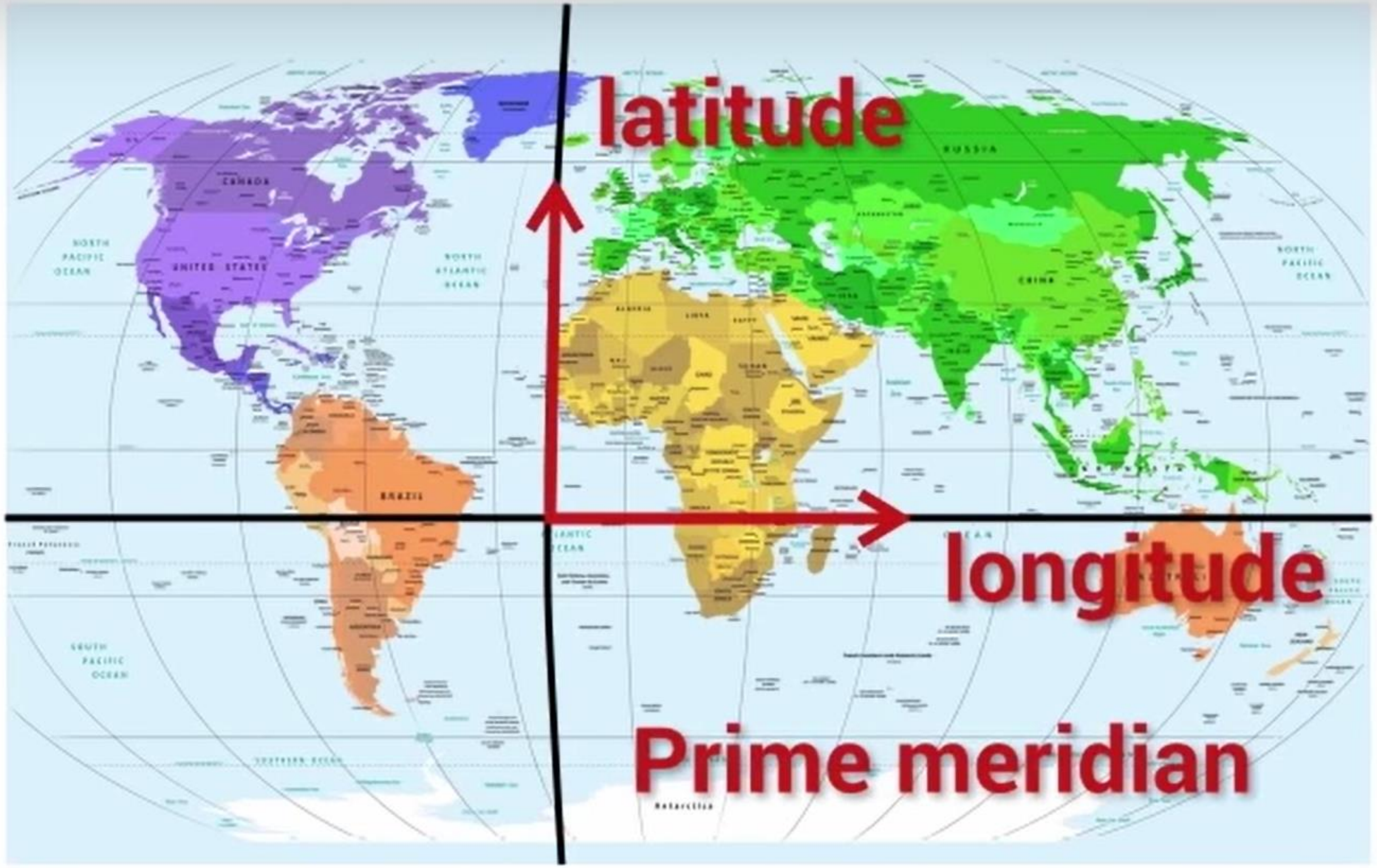

460 m/s


30 km/s


1,000 km/s
220 km/s
Spiral Galaxy M81 2008
Hubble Heritage | CC BY-SA 2.0.

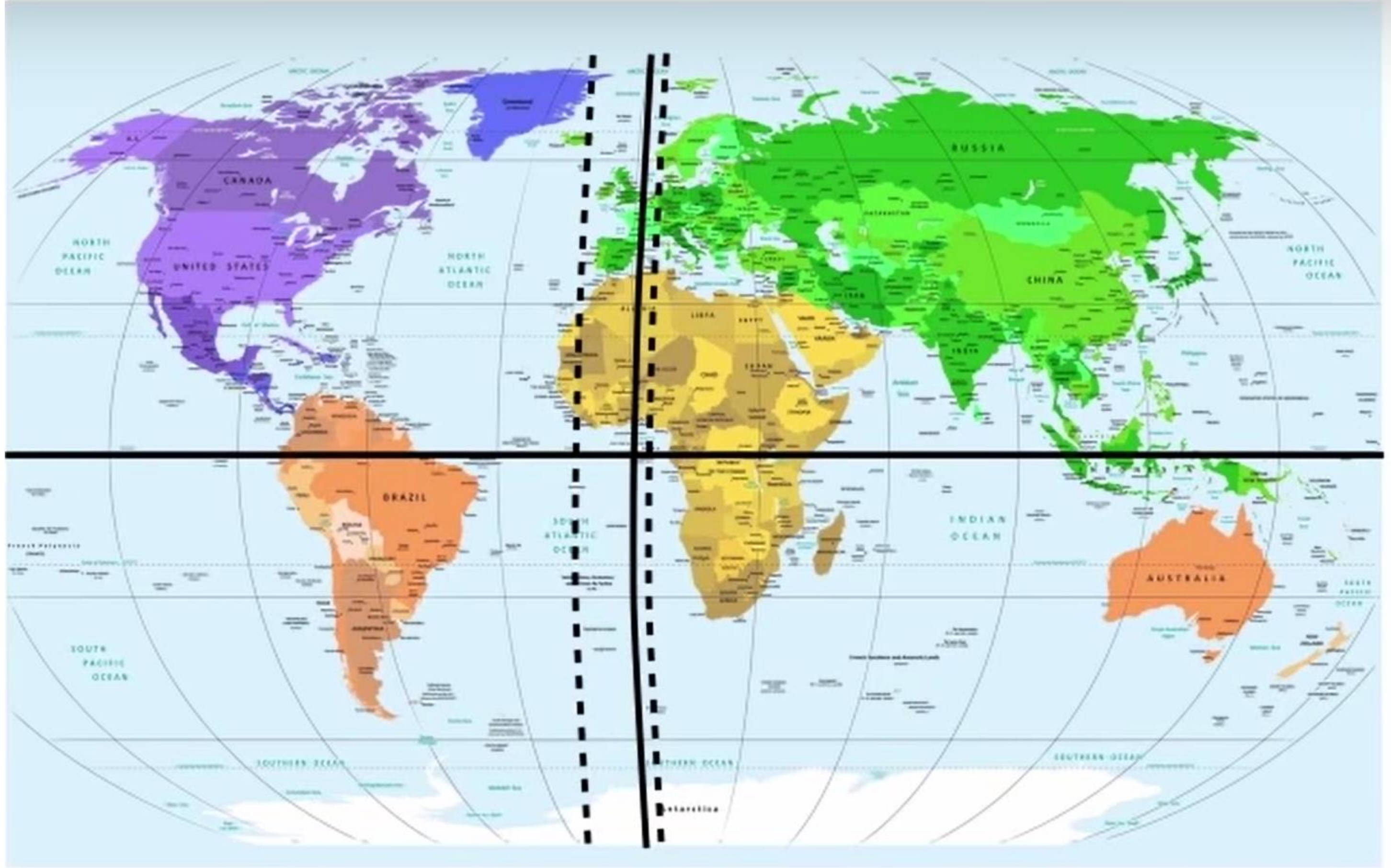

Prime meridian


ECUADOR
LATITUD: 0°- 0'- 0"
LONG. OCC. 78°- 27'-8"

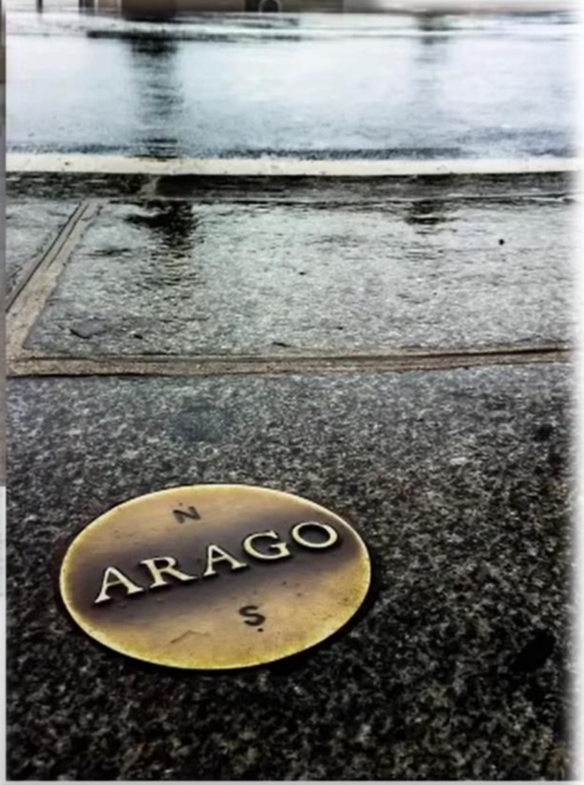


Greenwich Meridian 2008
Leon Brocard | CC BY 2.0.

Greenwich Meridian 2011
Randi Hausken | CC BY-SA 2.0.



Meridian Room at the Paris Observatory. 2013
Gilles MAIRET | CC BY-SA 3.0.



Médaillon Arago sous la pluie 2013
Jean-François Gornet | CC BY-SA 2.0.

Geometri 2D

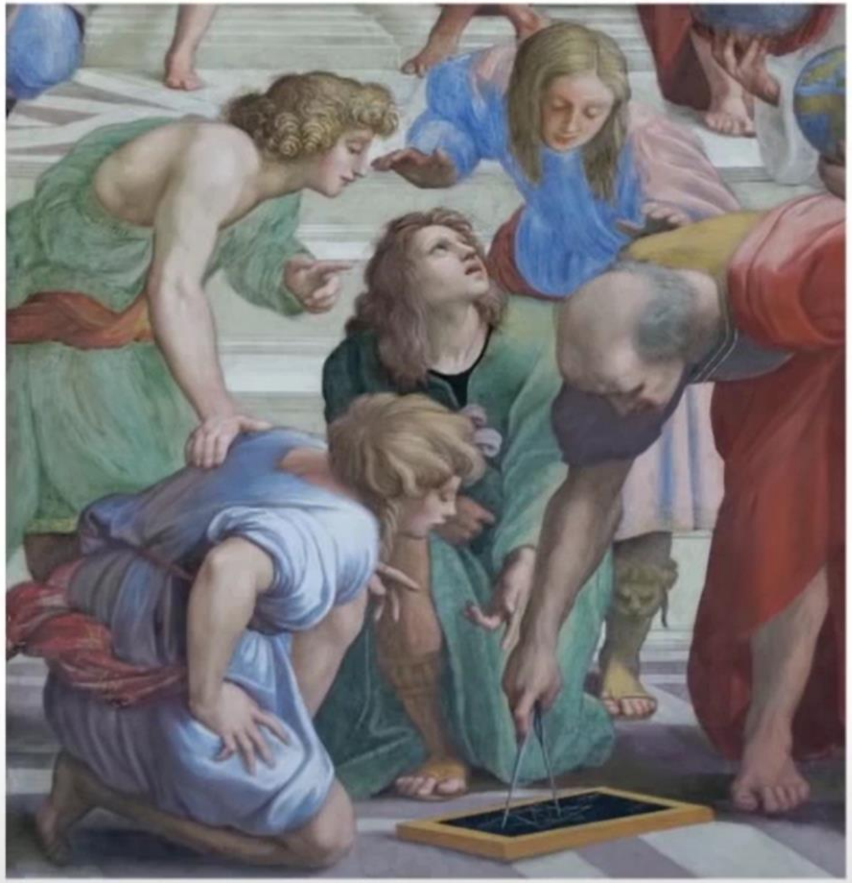
Euclidean **geometry**

Euclid of Alexandria (325-265 BCE)

- After Euclid of Alexandria

Around 300 BCE he wrote “The Elements” (13 books)

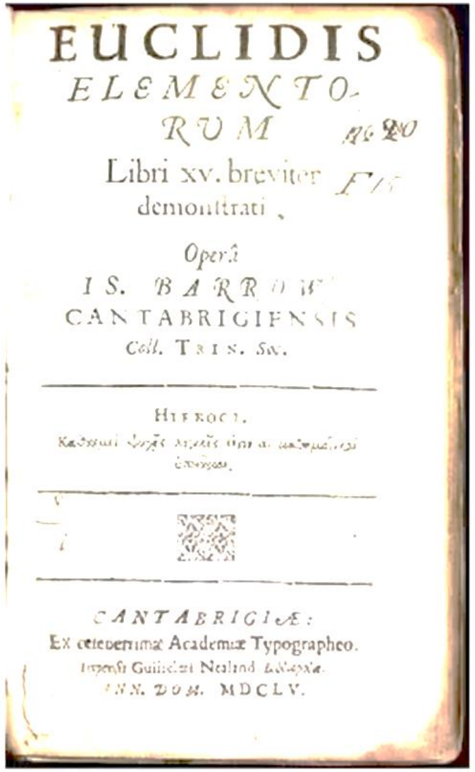
Scroll from “The Elements”
Euclid | Public domain, Wikimedia Commons



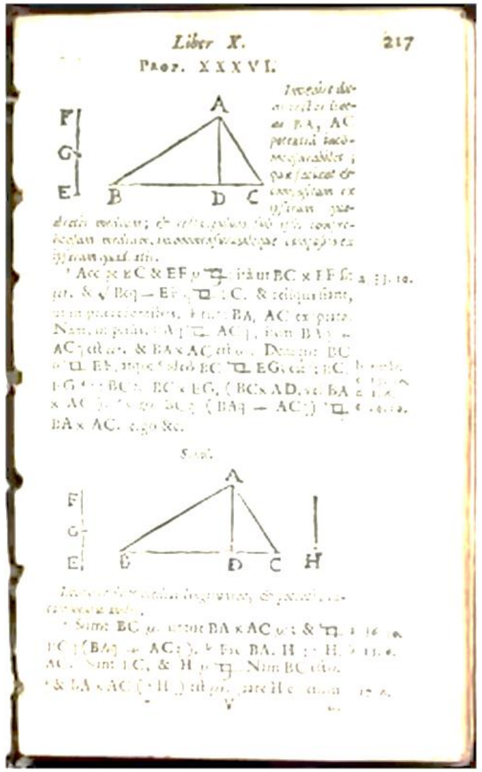
School of Athens (Raphael)
Marie-Lan Nguyen

Euclidean **geometry**

- The geometry you learnt at school
- The geometry of a **plane** and **3D space**
  - ➥ lines, triangles, circles etc.
- Based on a few axioms
- No numbers


EUCLIDIS
ELEMENTORUM
Libri xv. breviter
demonstrati,
Operâ
IS. BARROW
CANTABRIGIENSIS
Coll. Trin. Soc.

CANTABRIGIÆ:
Ex celeberrimæ Academiæ Typographeo.
Impensis Guilielmi Nealand Bibliopolæ.
ANN. DOM. MDCLV.


Liber X. 217
Prop. XXXVI.

Euclid, Elementorum Libri XV: Cantabrigiae, 1655 2005

Rene Descartes (1596-1650)

I think, therefore I am

- Philosopher
- Mathematician
- Part-time mercenary
- **The father of modern philosophy**


Eiffel Tower
Waithamai (Own work) | CC BY-SA 3.0. via Wikimedia Commons
Panoramic view of the French countryside 2013
Mary Harrsch | CC BY-SA 2.0.


Eiffel Tower
Waithamai (Own work) | CC BY-SA 3.0. via Wikimedia Commons
Panoramic view of the French countryside 2013
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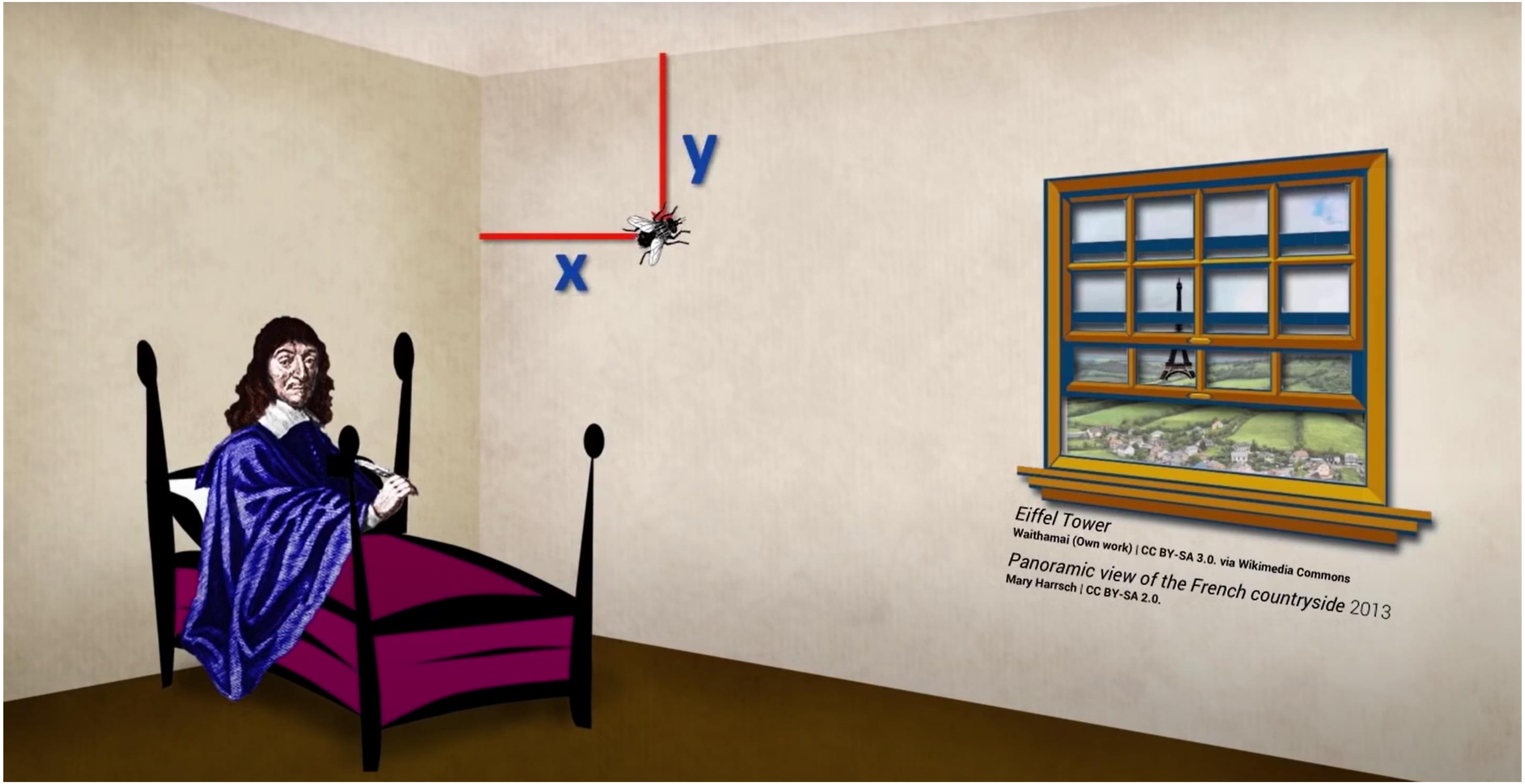
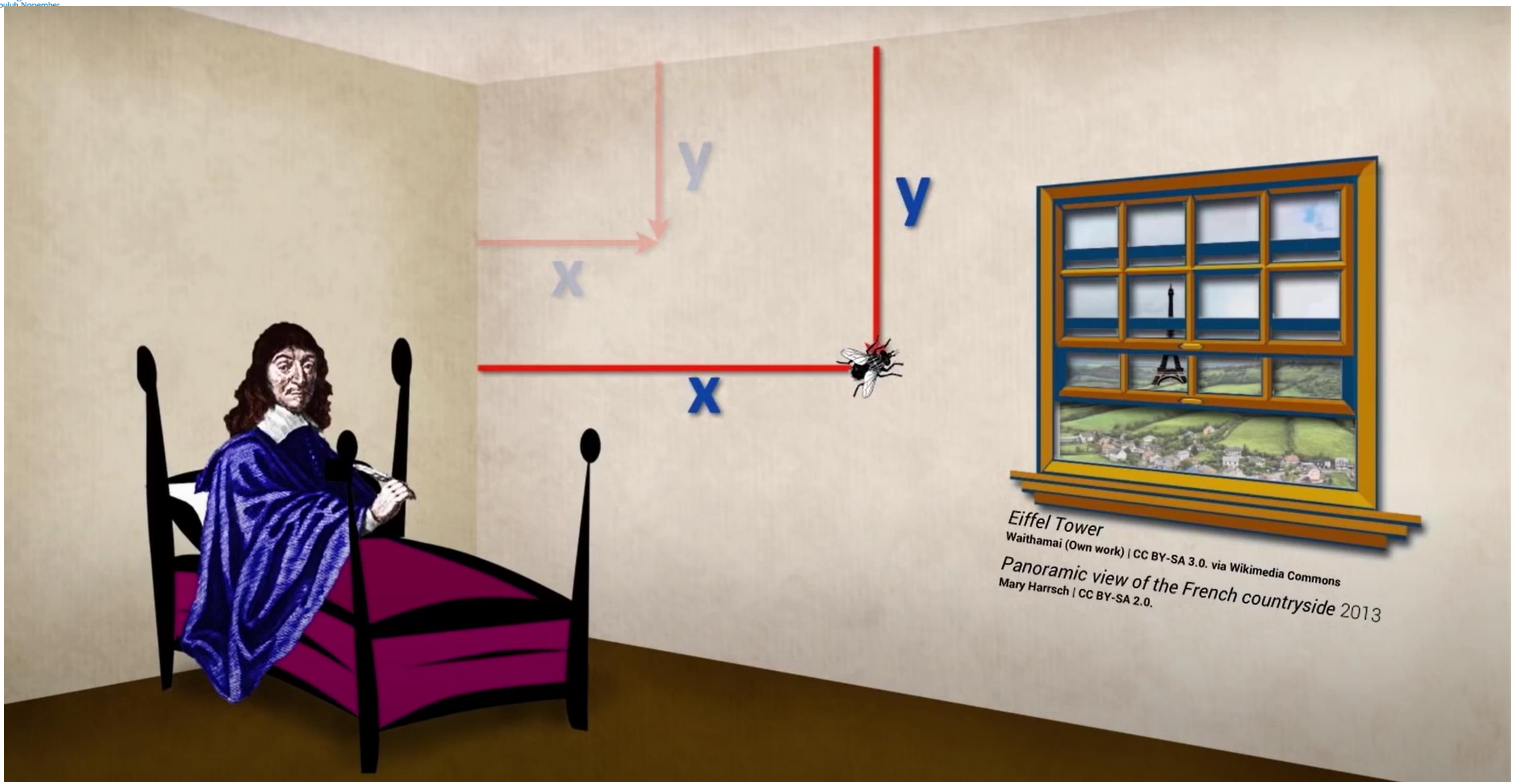

y
x
Eiffel Tower
Waithamai (Own work) | CC BY-SA 3.0. via Wikimedia Commons
Panoramic view of the French countryside 2013
Mary Harrsch | CC BY-SA 2.0.


y
x
y
x
Eiffel Tower
Waithamai (Own work) | CC BY-SA 3.0. via Wikimedia Commons
Panoramic view of the French countryside 2013
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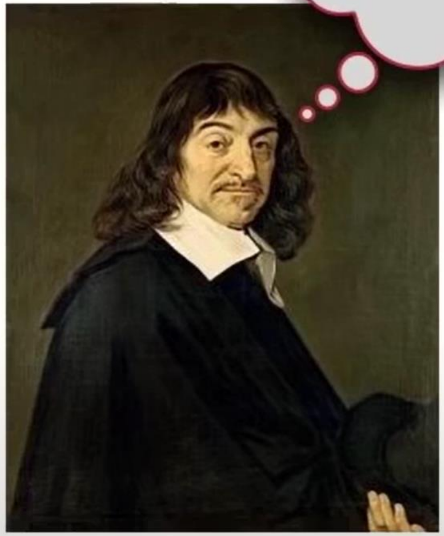
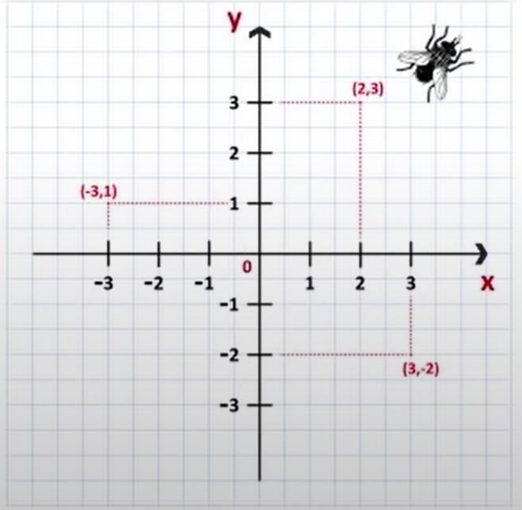
Cartesian **geometry**

$y = mx + c$



Descartes
After Frans Hals | Public domain, Wikimedia Commons

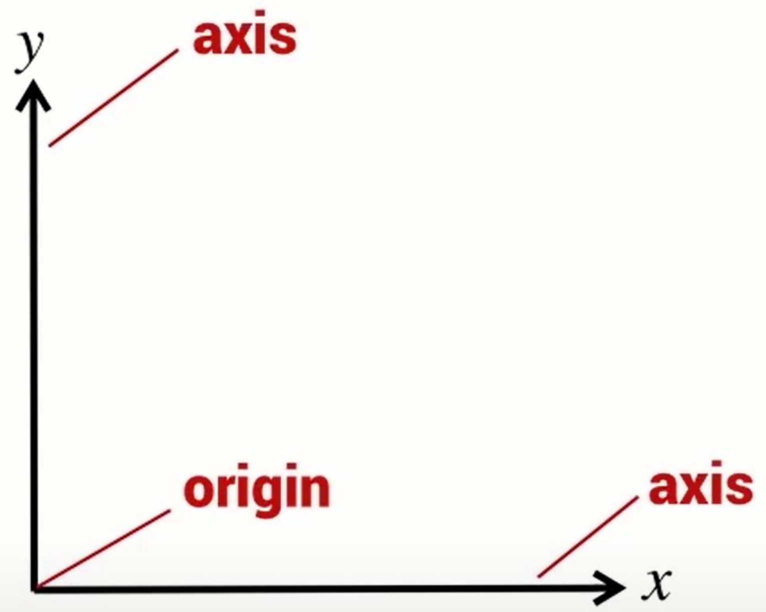


- Added algebra to Euclidean geometry
  - ➥ Shapes could be expressed by equations
  - ➥ Cartesian (or analytic) geometry

His name in Latin was Renatus Cartesius

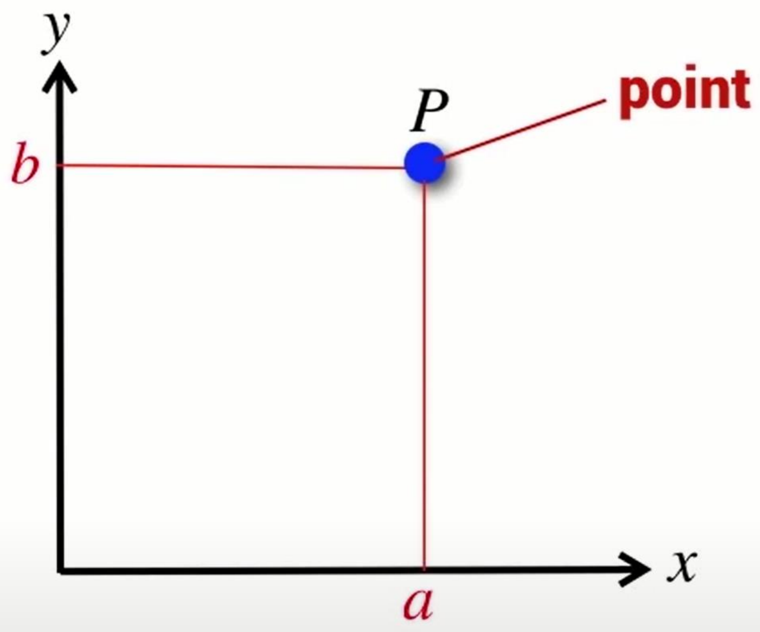
- Developed the coordinate system

2D coordinate frame



- Two axes form a 2D **coordinate frame**

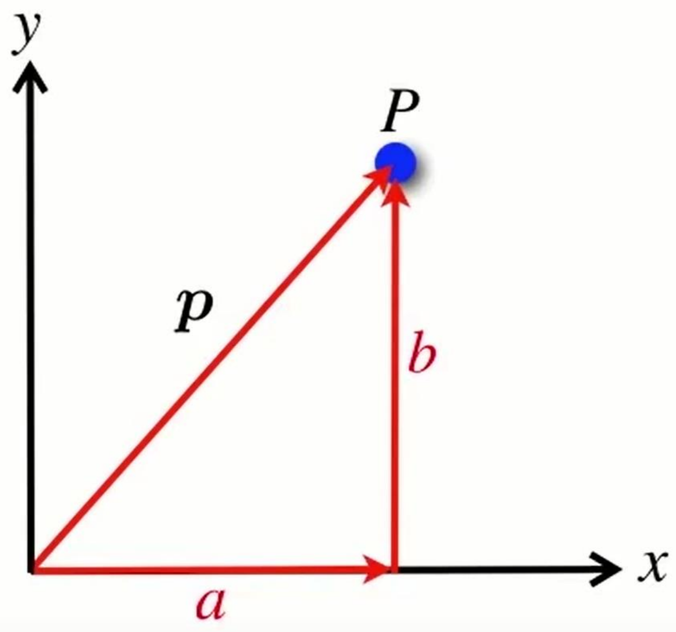
Points



$$P = (a, b)$$

- A point is a location in n-dimensional space $P \in \mathbb{R}^n$
- Represented by a Cartesian coordinate or an n-tuple $(x_1, x_2 \cdots x_n)$

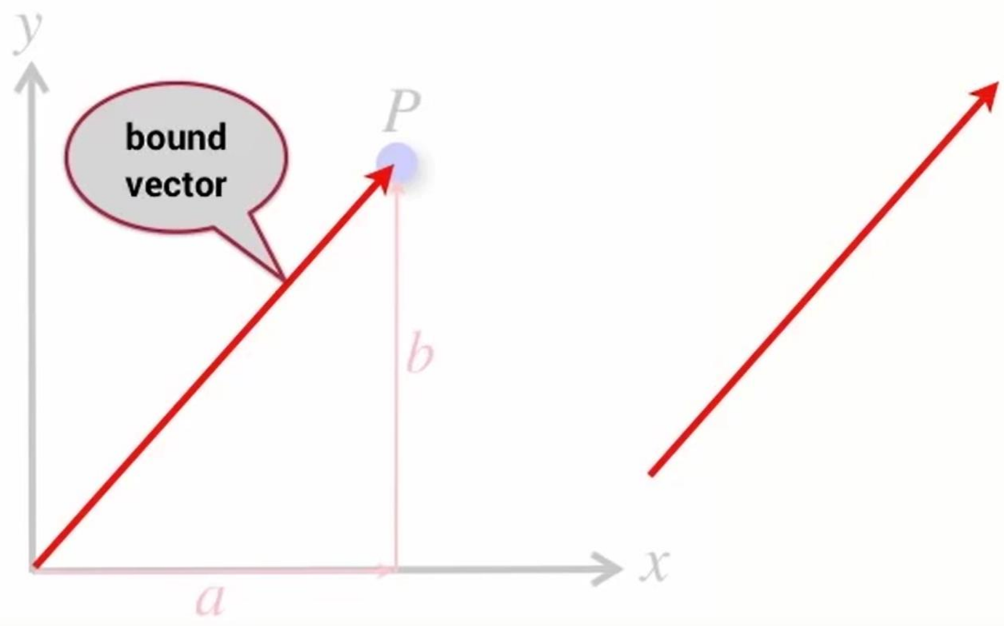
Vectors



$$\boldsymbol{p} = \begin{bmatrix} a \\ b \end{bmatrix}$$

- A vector is an element of an n-dimensional vector space $\boldsymbol{p} \in \mathbb{R}^n$
- Represented by an n-element column vector $[x_1, x_2 \cdots x_n]^T$
- Can be considered a relative displacement

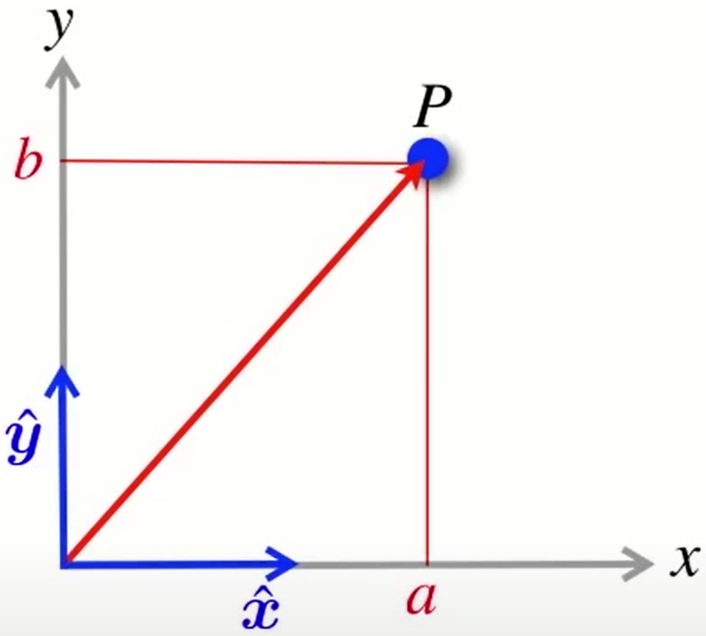
Vectors



$$\boldsymbol{p} = \begin{bmatrix} a \\ b \end{bmatrix}$$

- A vector is an element of an n-dimensional vector space $\boldsymbol{p} \in \mathbb{R}^n$
- Represented by an n-element column vector $[x_1, x_2 \cdots x_n]^T$
- Can be considered a relative displacement
- The vector has a particular starting point

Vectors

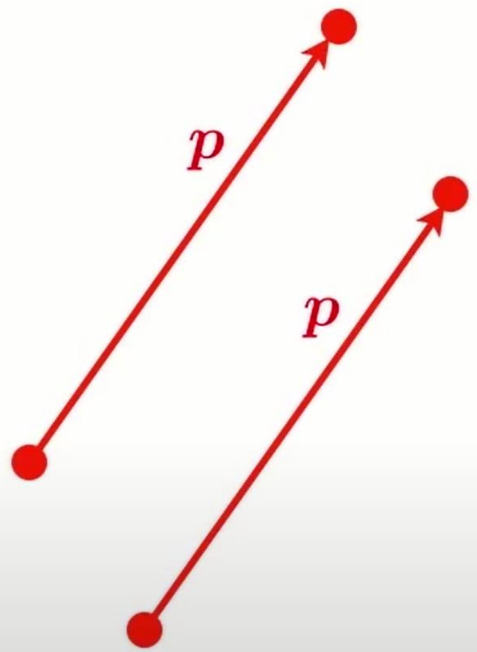


$$\mathbf{p} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\mathbf{p} = a\hat{\mathbf{x}} + b\hat{\mathbf{y}}$$

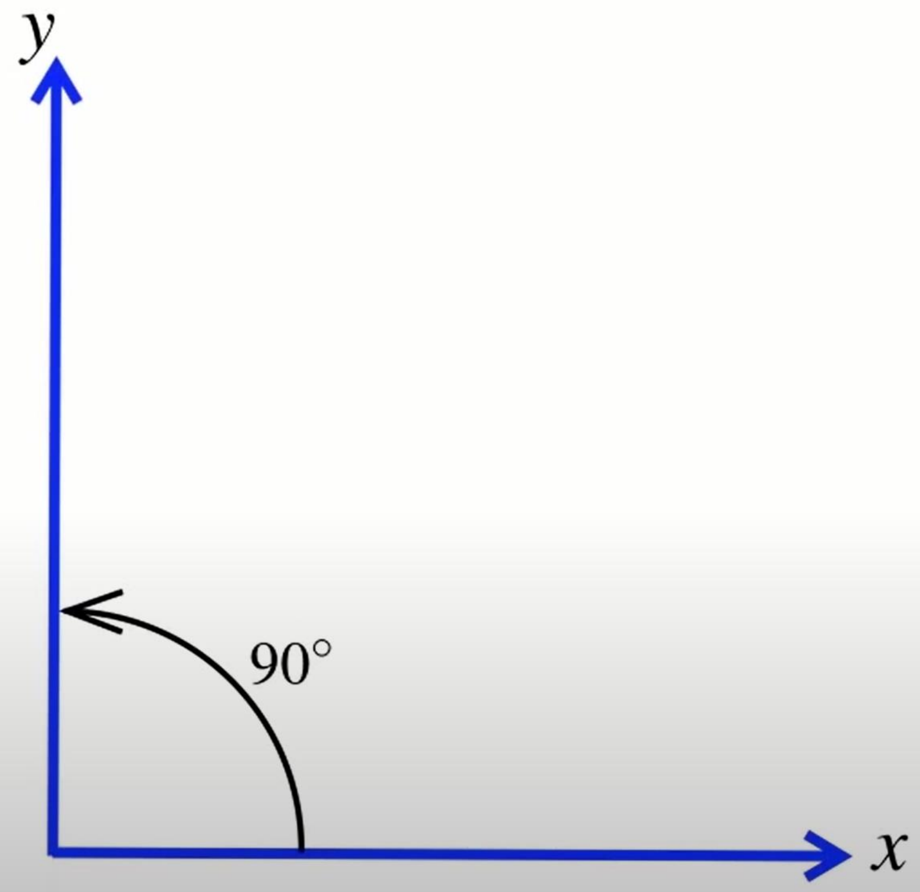
Points and vectors

- Points
  - Define a location
  - Cannot add or multiply points
- Vectors
  - *Do not* define a location
    - specify how to get from one location to another
  - Can add & subtract vectors
  - Can multiply a vector by a scalar



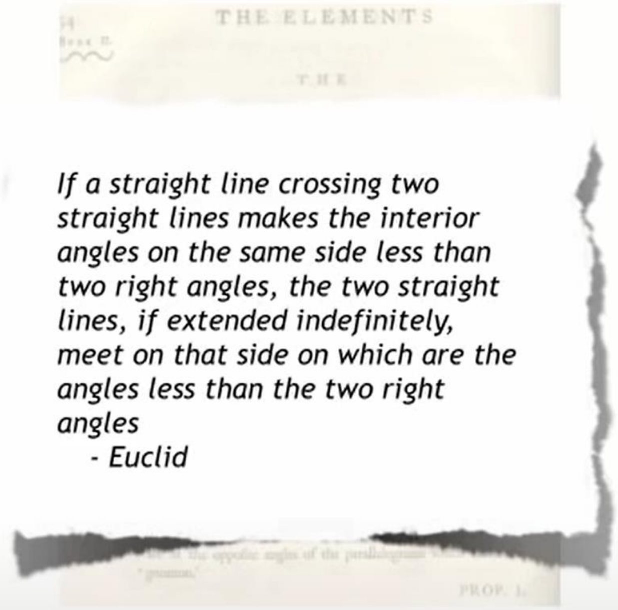
- The difference between two points is a vector
- A point is defined by a vector displacement from another point (typically the origin of a coordinate frame)
- Both can be represented by *n* real numbers

Right-handed coordinate frame



Euclidean geometry applies to planes

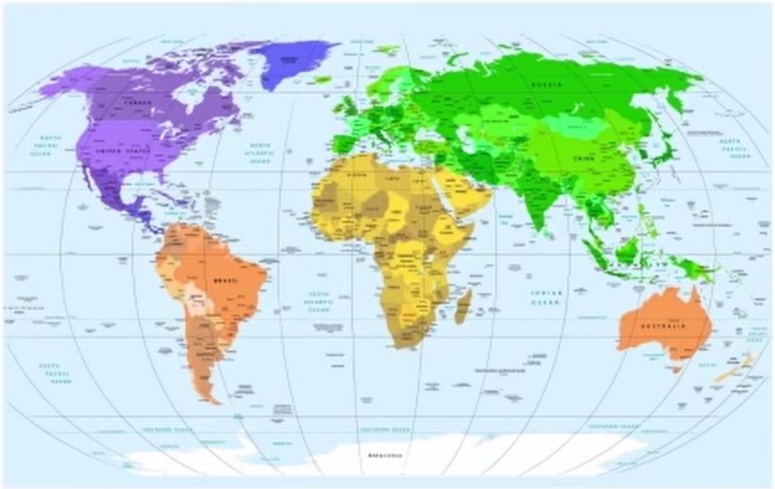
- Euclid's geometry (postulate 5)
  - ➥ parallel lines never intersect

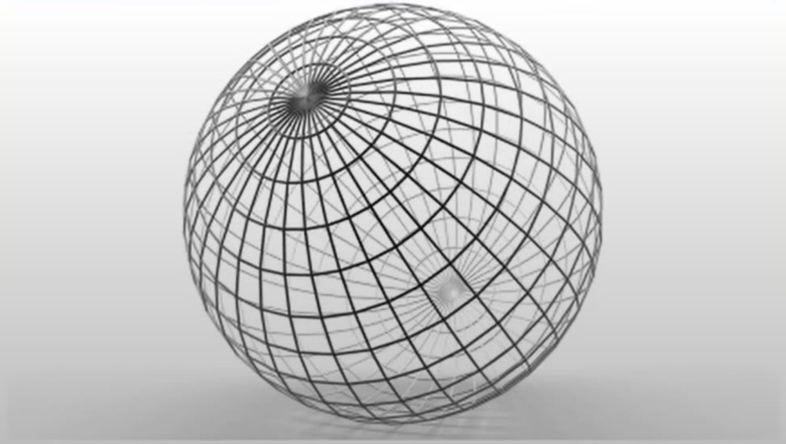


If a straight line crossing two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which are the angles less than the two right angles
- Euclid

Euclidean geometry applies to planes

- Euclid's geometry (postulate 5)
  - ➥ parallel lines never intersect
- On the surface of our planet, parallel lines at the equator will intersect at the poles
- Euclidean geometry does not apply
  - ➥ it is a non-Euclidean geometry
- Euclidean geometry is a good approximation over small areas





Position and Pose in 2D

Pose

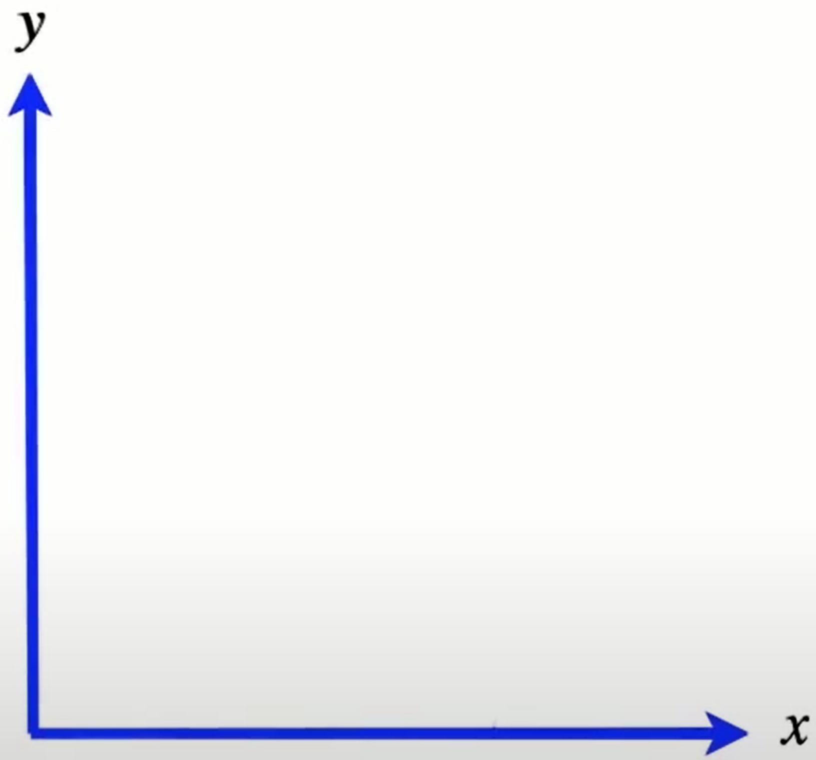
- It has 3 parameters:

$$\xi$$

pronounced **ksi**

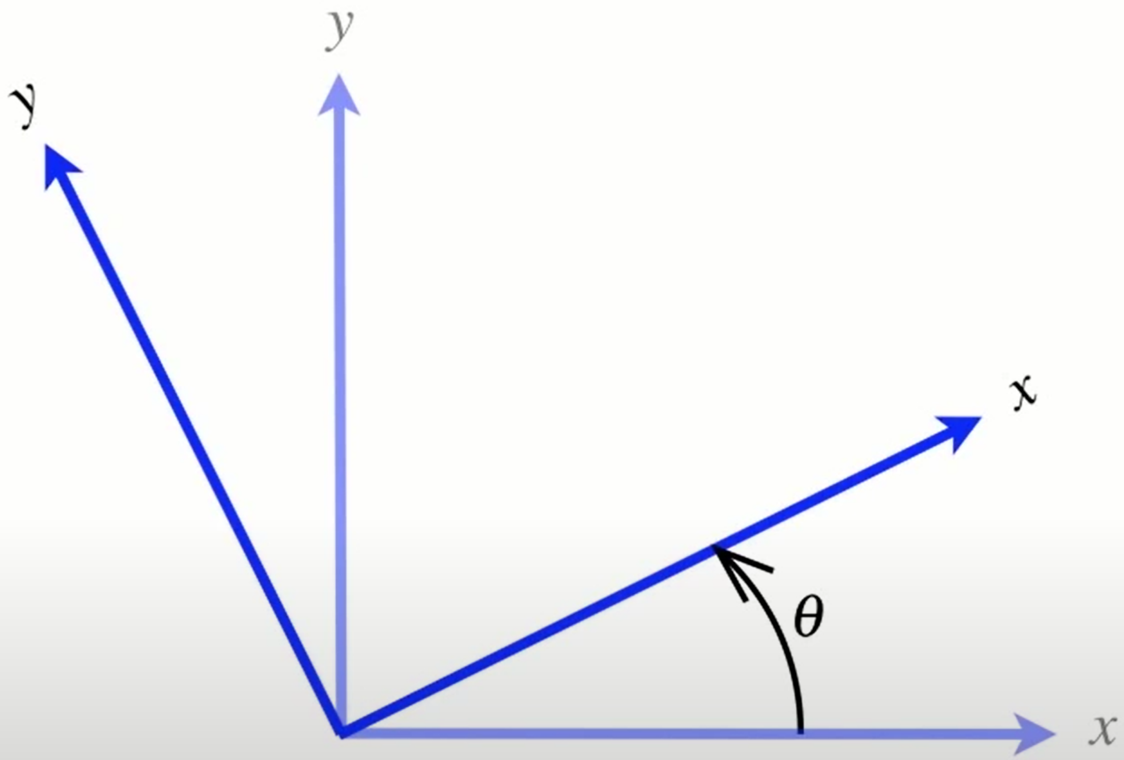
$$(x, y, \theta)$$

Coordinate frame rotation convention



- Angles increase positively in the anti-clockwise direction

Coordinate frame rotation convention



- Angles increase positively in the anti-clockwise direction

Pose

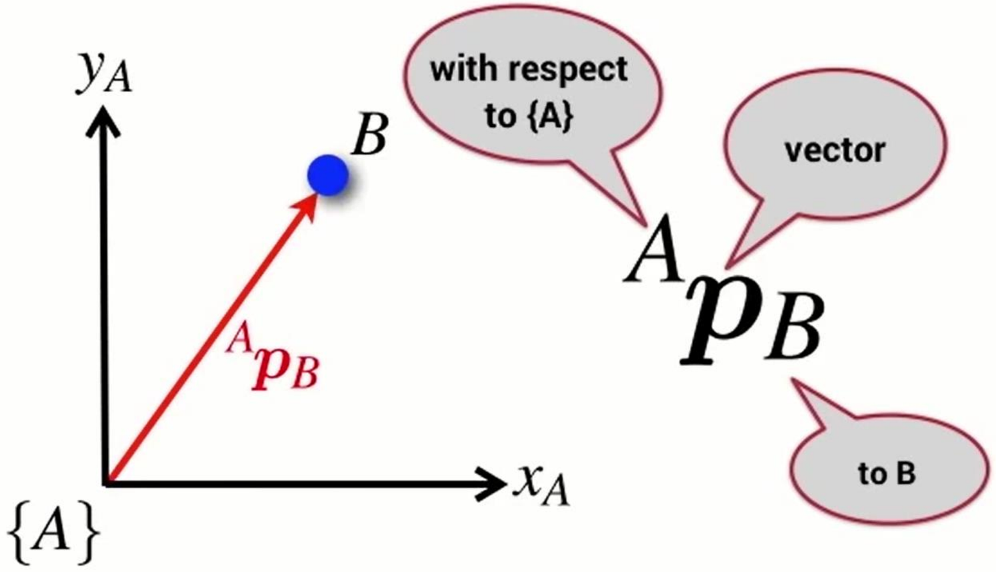
- It has 3 parameters:
- Think of it as just a move (x, y) and a twist (θ)

$$\xi$$

$$(x, y, \theta)$$

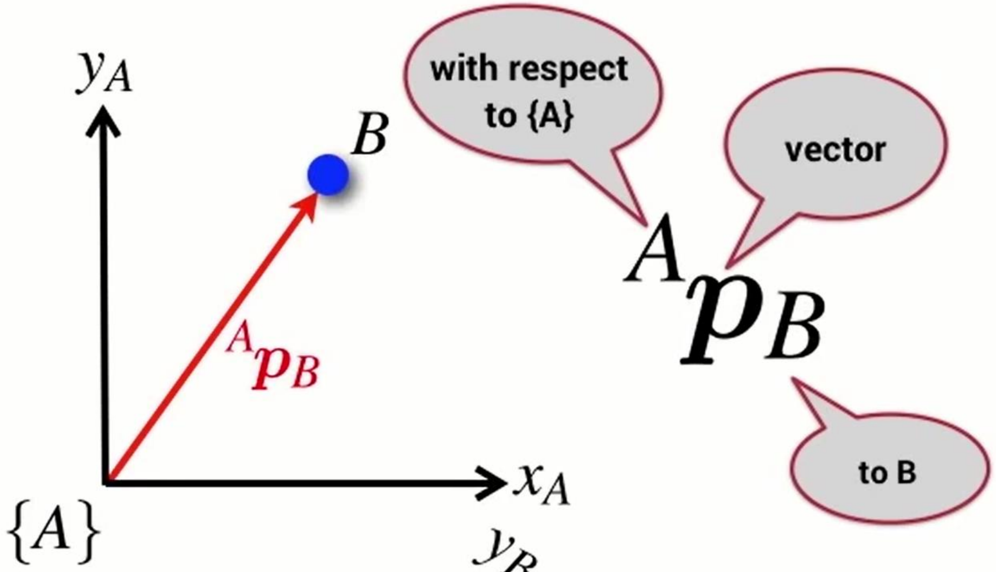
Summary

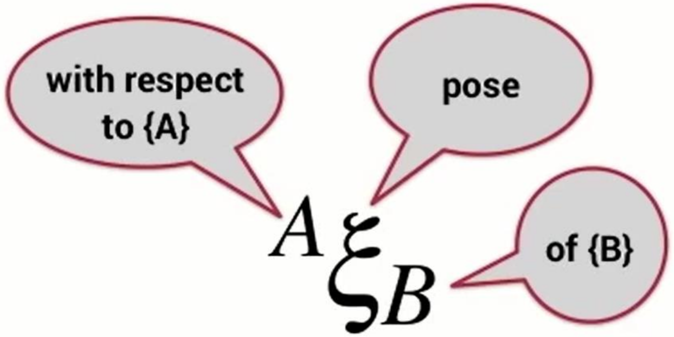
- A point is described by a vector with respect to a coordinate frame

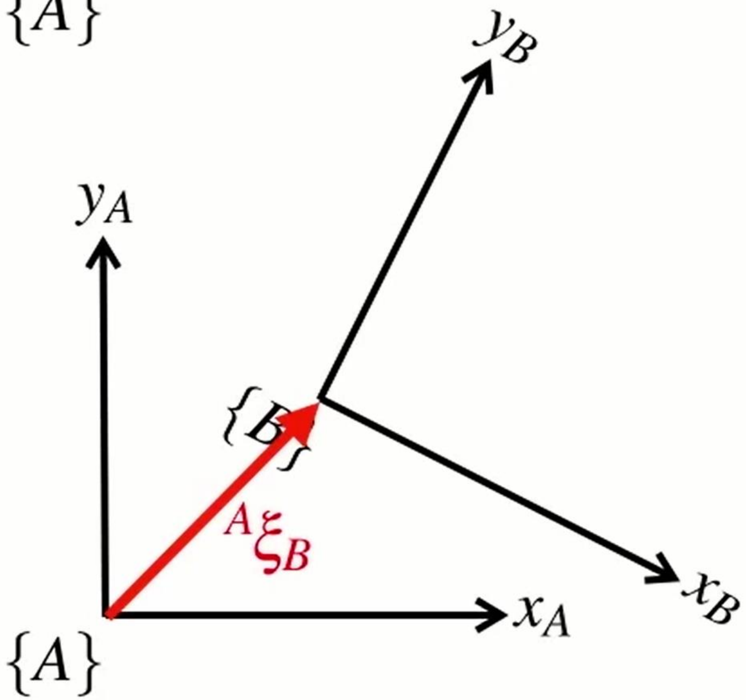


Summary

- A point is described by a vector with respect to a coordinate frame
- The pose of a coordinate frame can be described with respect to another coordinate frame

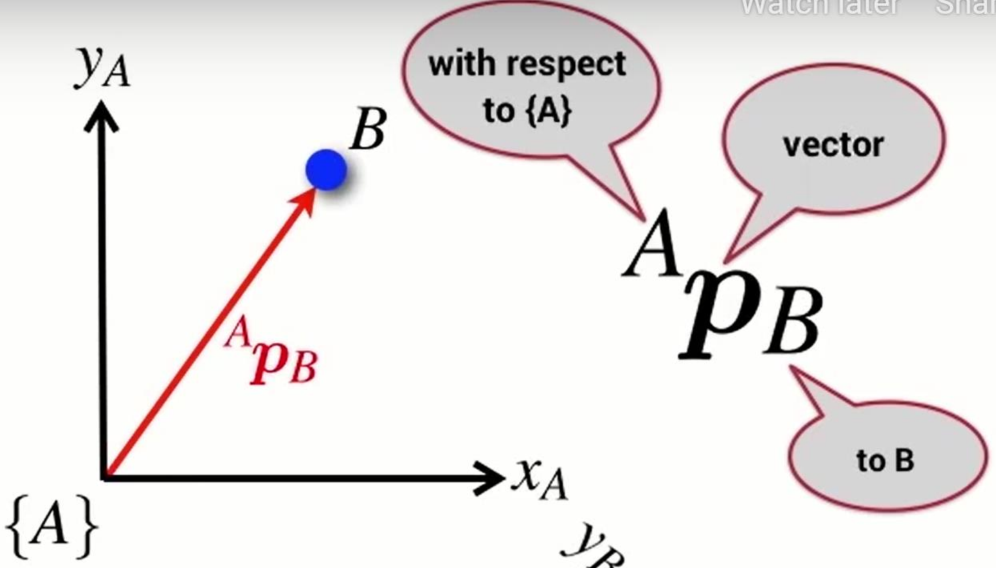


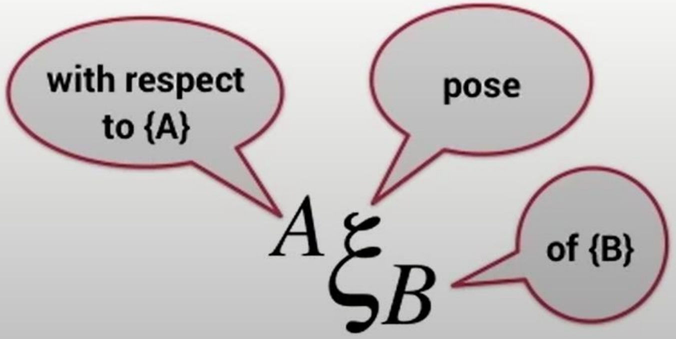


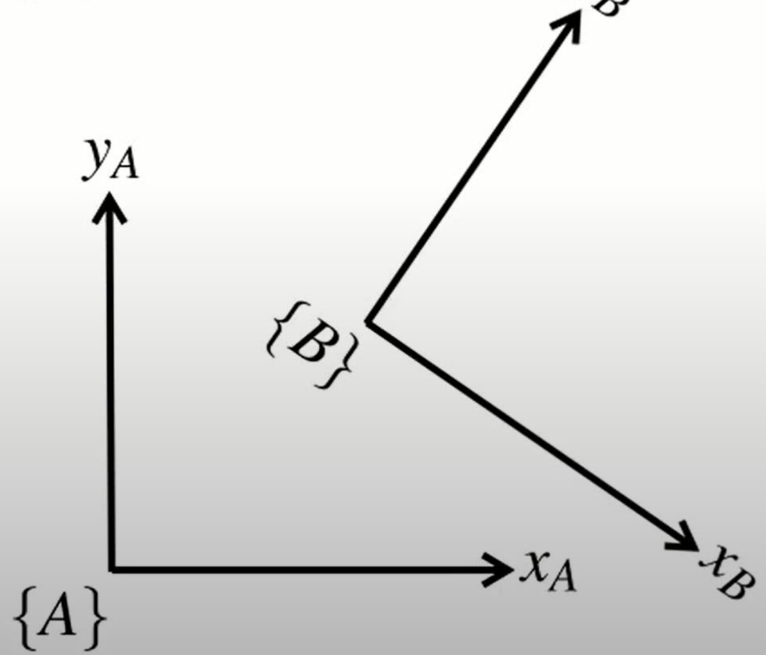


Summary

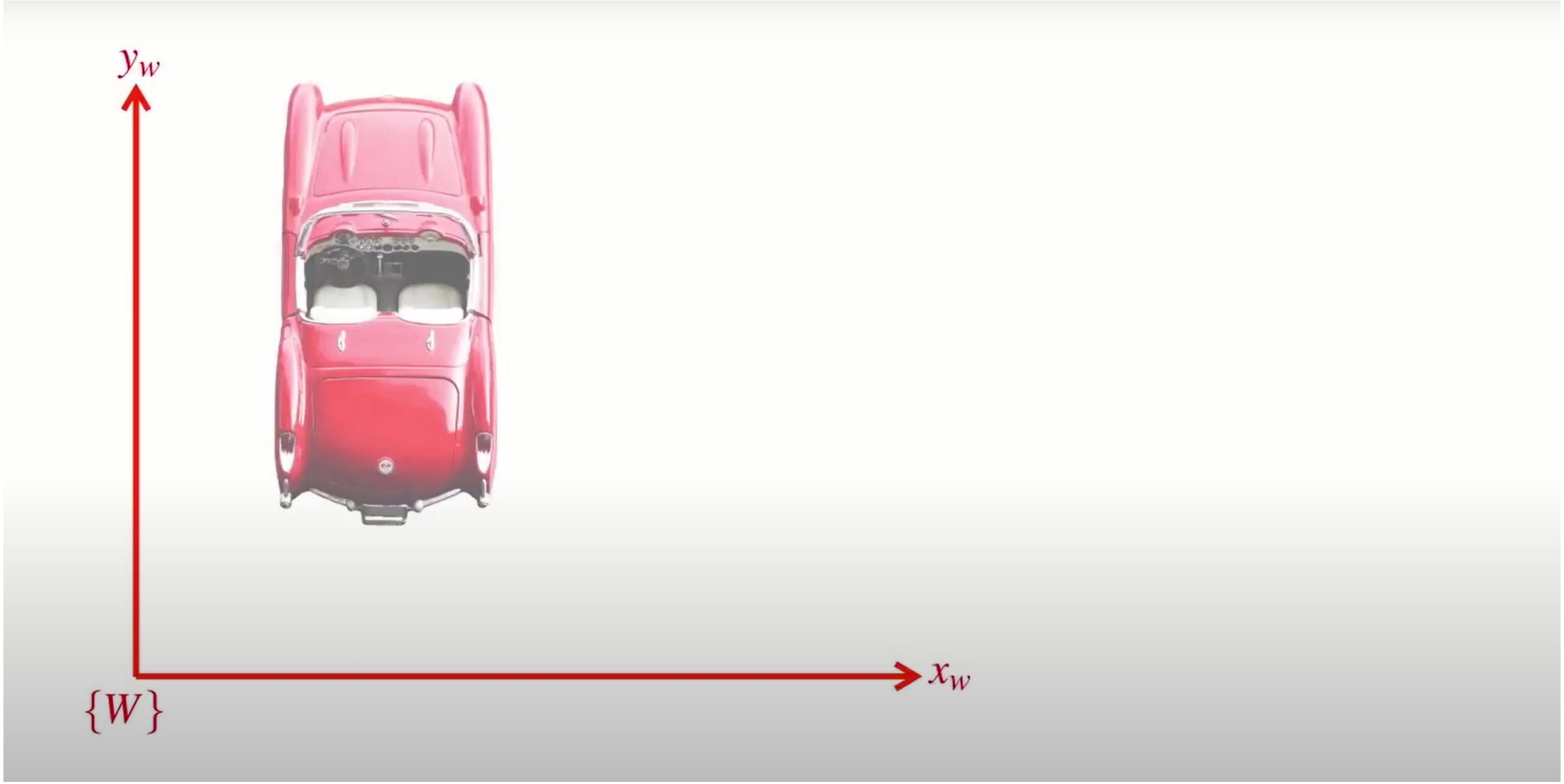
- A point is described by a vector with respect to a coordinate frame
- The pose of a coordinate frame can be described with respect to another coordinate frame
- Pose can be considered as the motion of a coordinate frame
  - ➥ a translation and a rotation

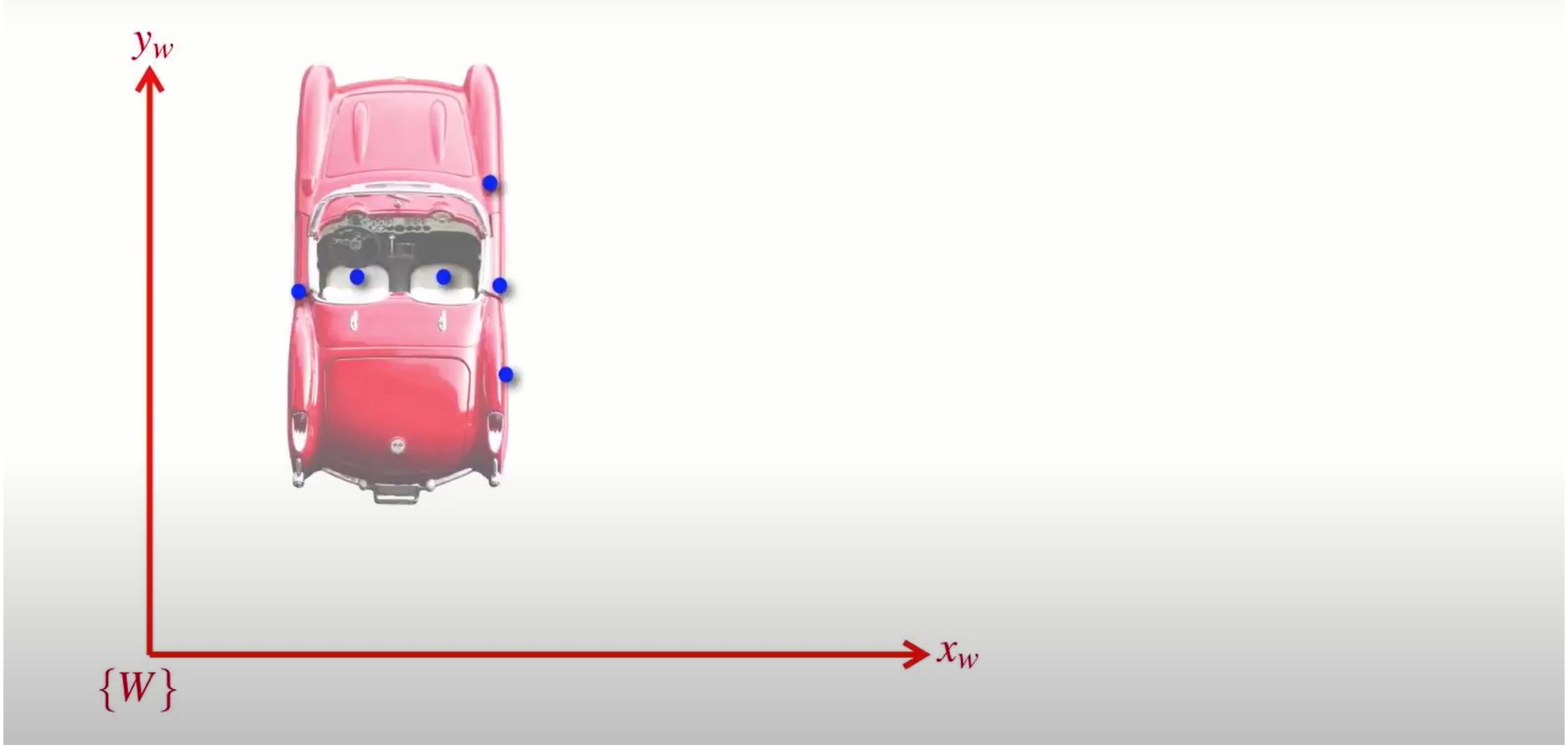




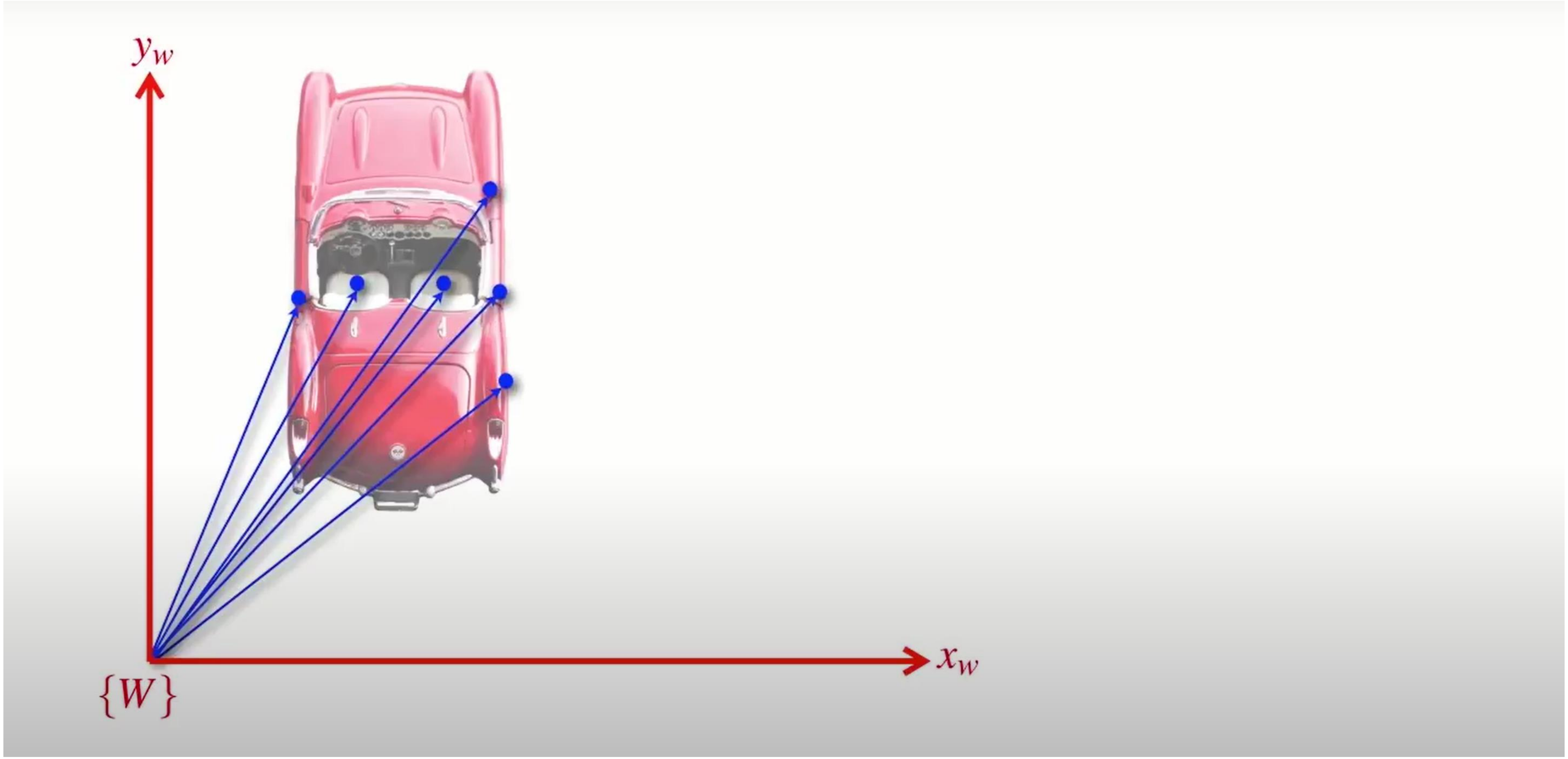


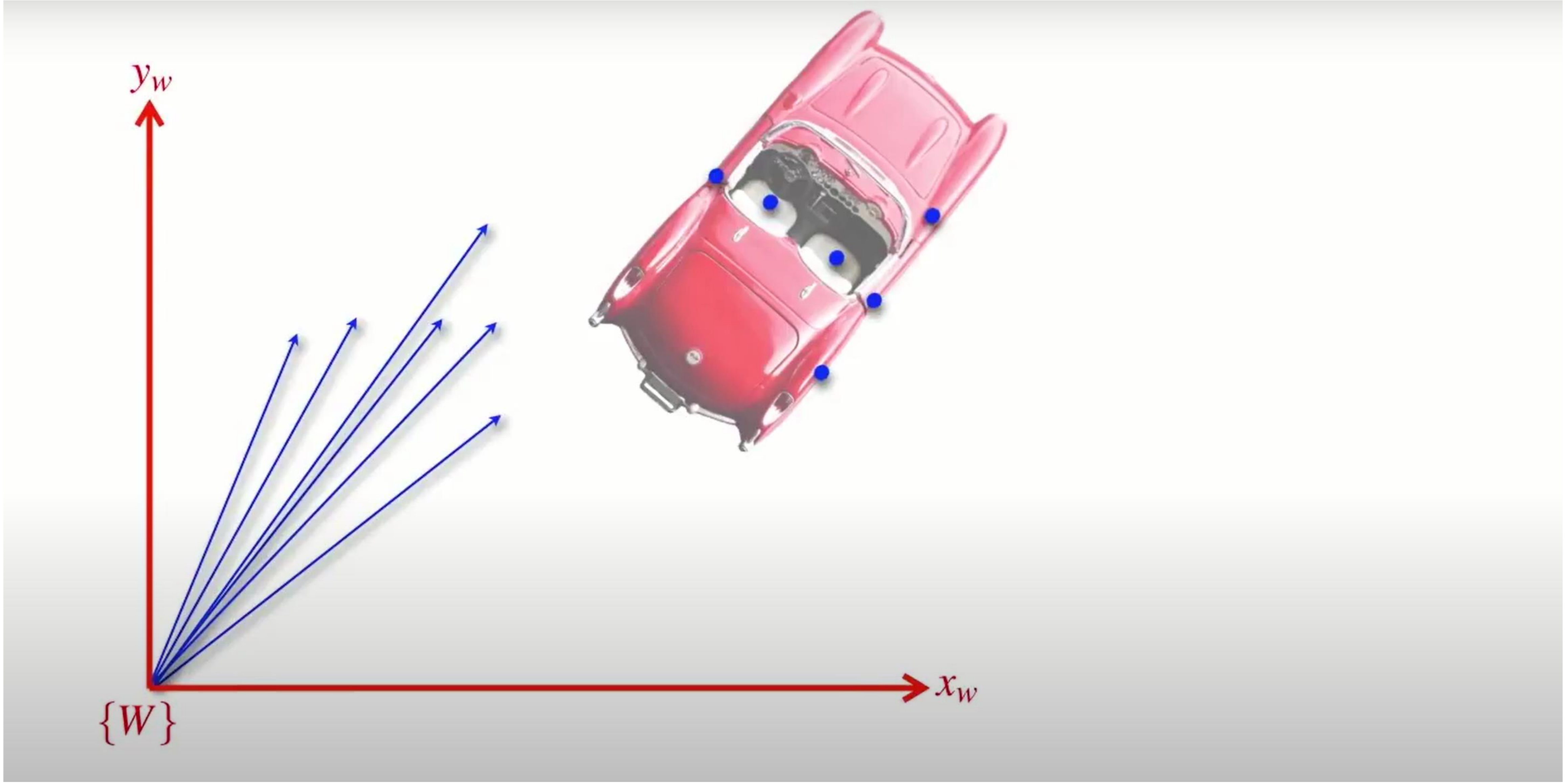
Relative Positions in 2D

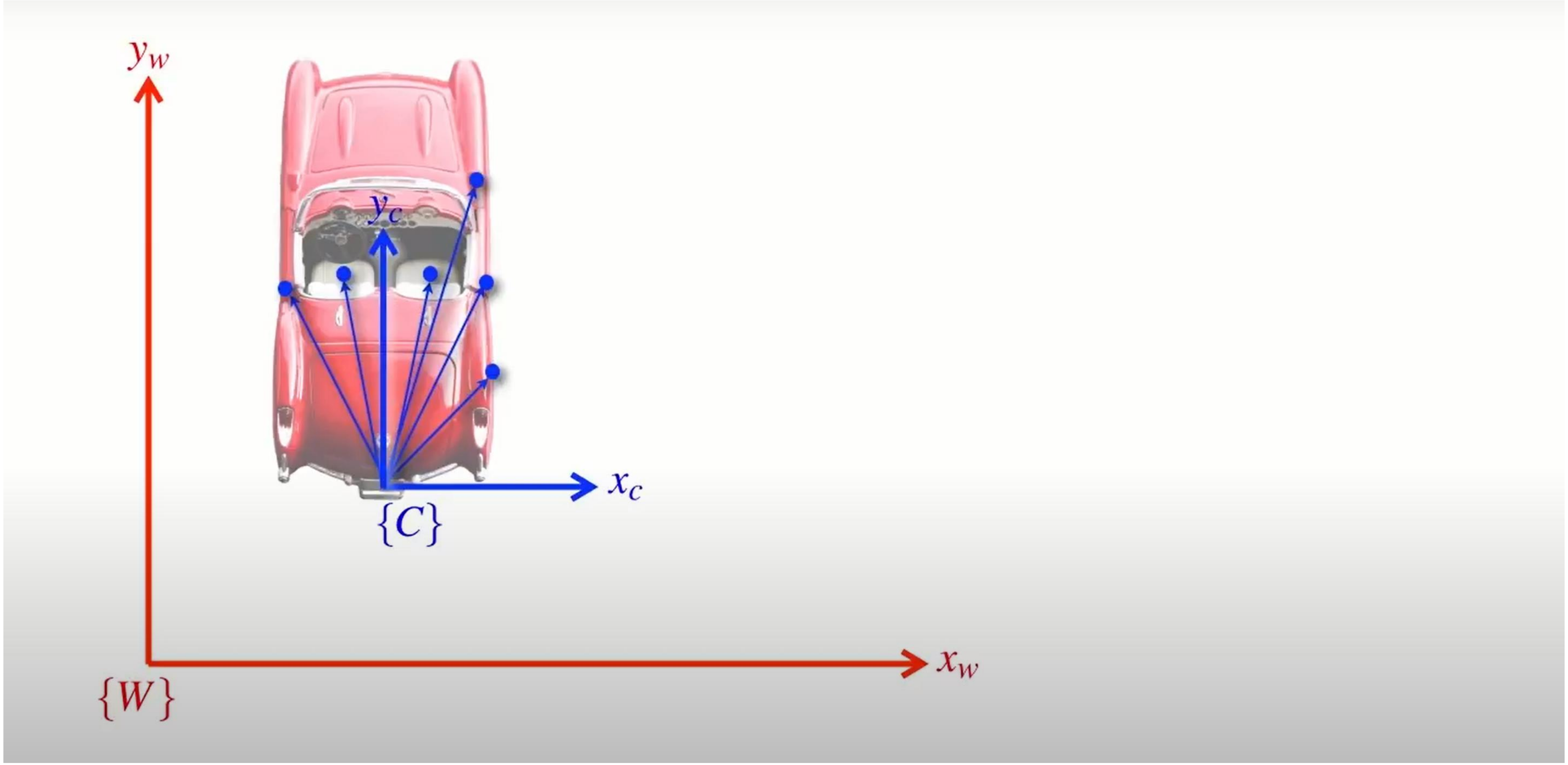

y_w
x_w
$\{W\}$



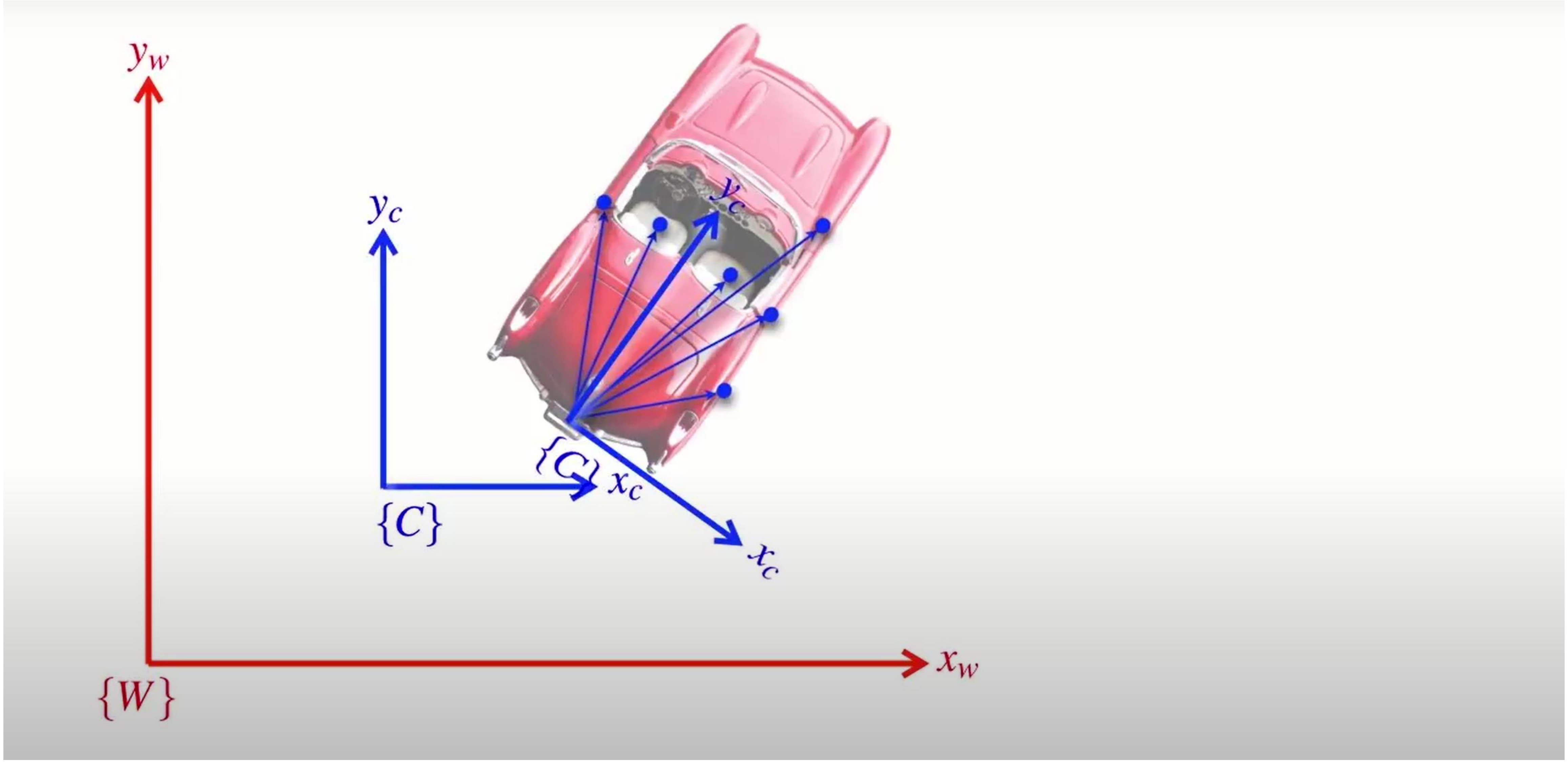
y_W
x_W
$\{W\}$


y_w
x_w
$\{W\}$

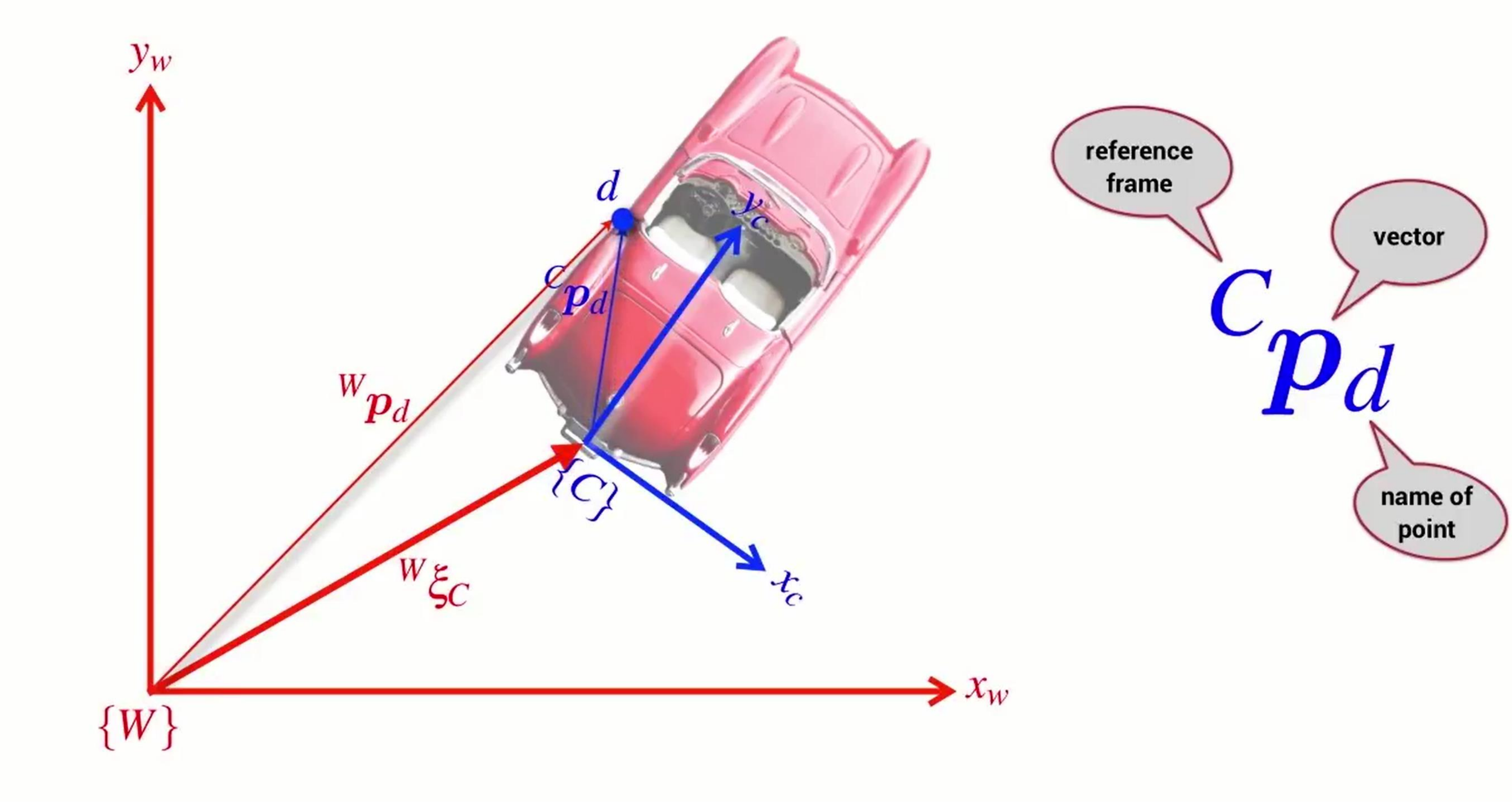

y_W
x_W
$\{W\}$

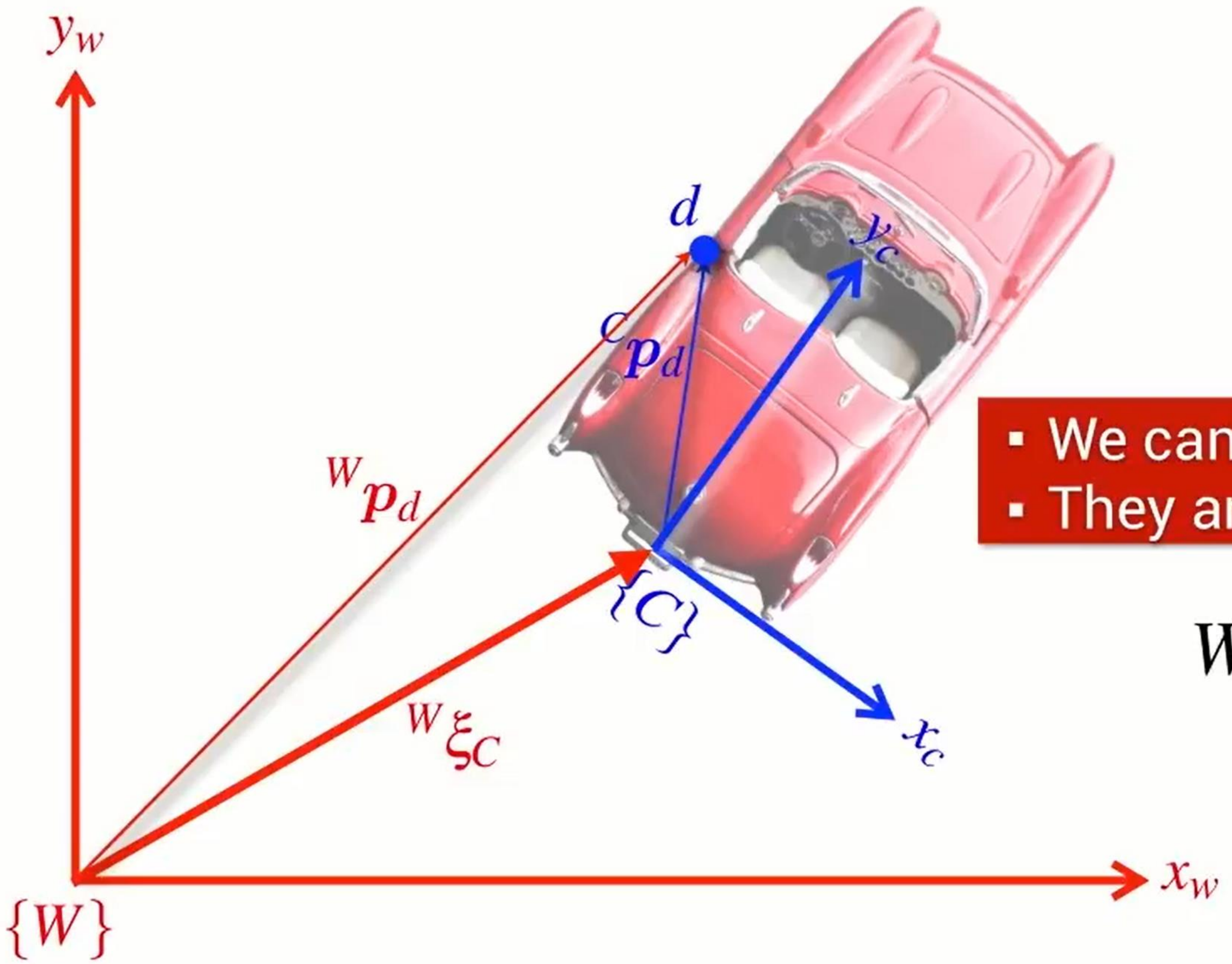


y_w
y_c
x_c
$\{C\}$
x_w
$\{W\}$



y_w
y_c
y_c
$\{C\}$
x_c
$\{C\}$
x_c
x_w
$\{W\}$


y_W
d
y_c
${}^{C}\boldsymbol{p}_d$
${}^{W}\boldsymbol{p}_d$
$\{C\}$
${}^{W}\xi_C$
x_c
x_W
$\{W\}$
reference frame
vector
${}^{C}\boldsymbol{p}_d$
name of point



reference frame

vector

$$^{C}\boldsymbol{p}_d$$

name of point

- We can't add **Poses** and **Vectors**
- They are different things

$$^{W}\boldsymbol{p}_d = {}^{W}\xi_C \cdot {}^{C}\boldsymbol{p}_d$$

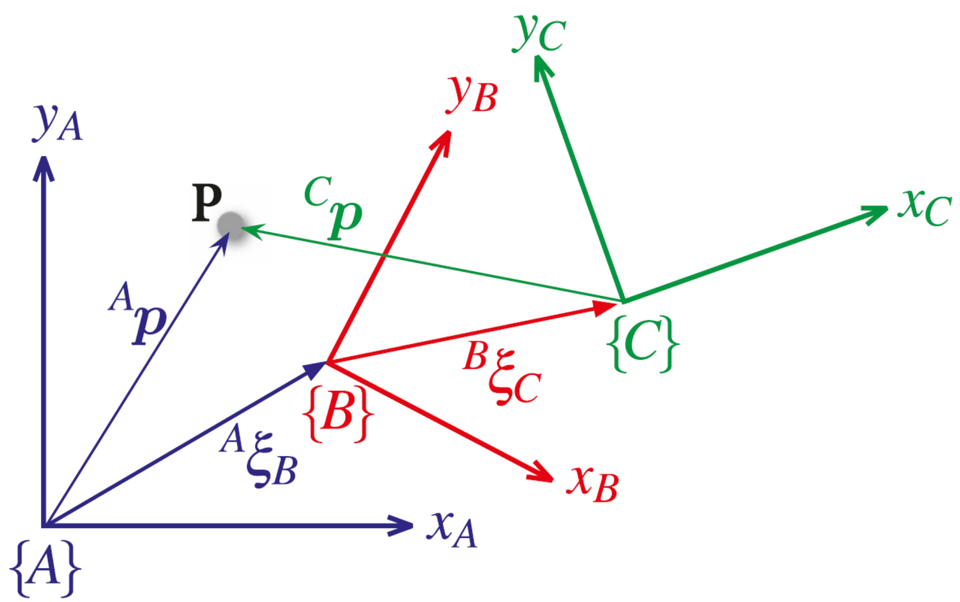
transform vector from one frame to another

Pose in 2D

- Muhtadin
- Isi : Sub Bab 2.1.2

Relative Pose in 2D

Relative points



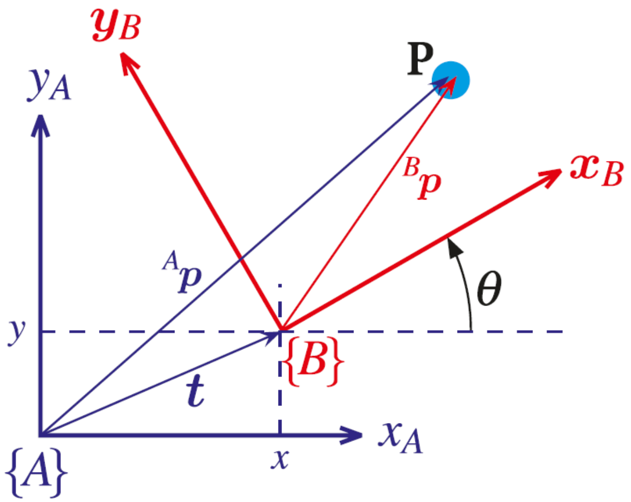
$$ {}^A p = {}^A\xi_B \cdot {}^B p $$

composition

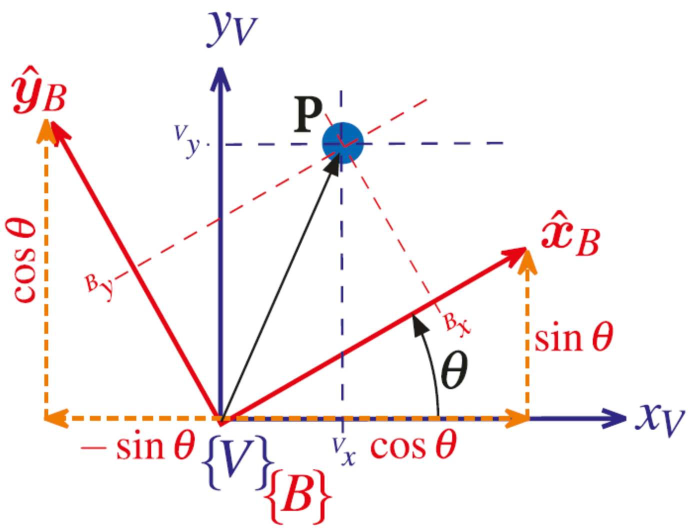
$$ {}^A\xi_C = {}^A\xi_B \oplus {}^B\xi_C $$

$$ {}^A p = \left({}^A\xi_B \oplus {}^B\xi_C \cdot {}^C p\right) $$

- Relative poses can be compounded or composed
- We can extend this approach indefinitely…

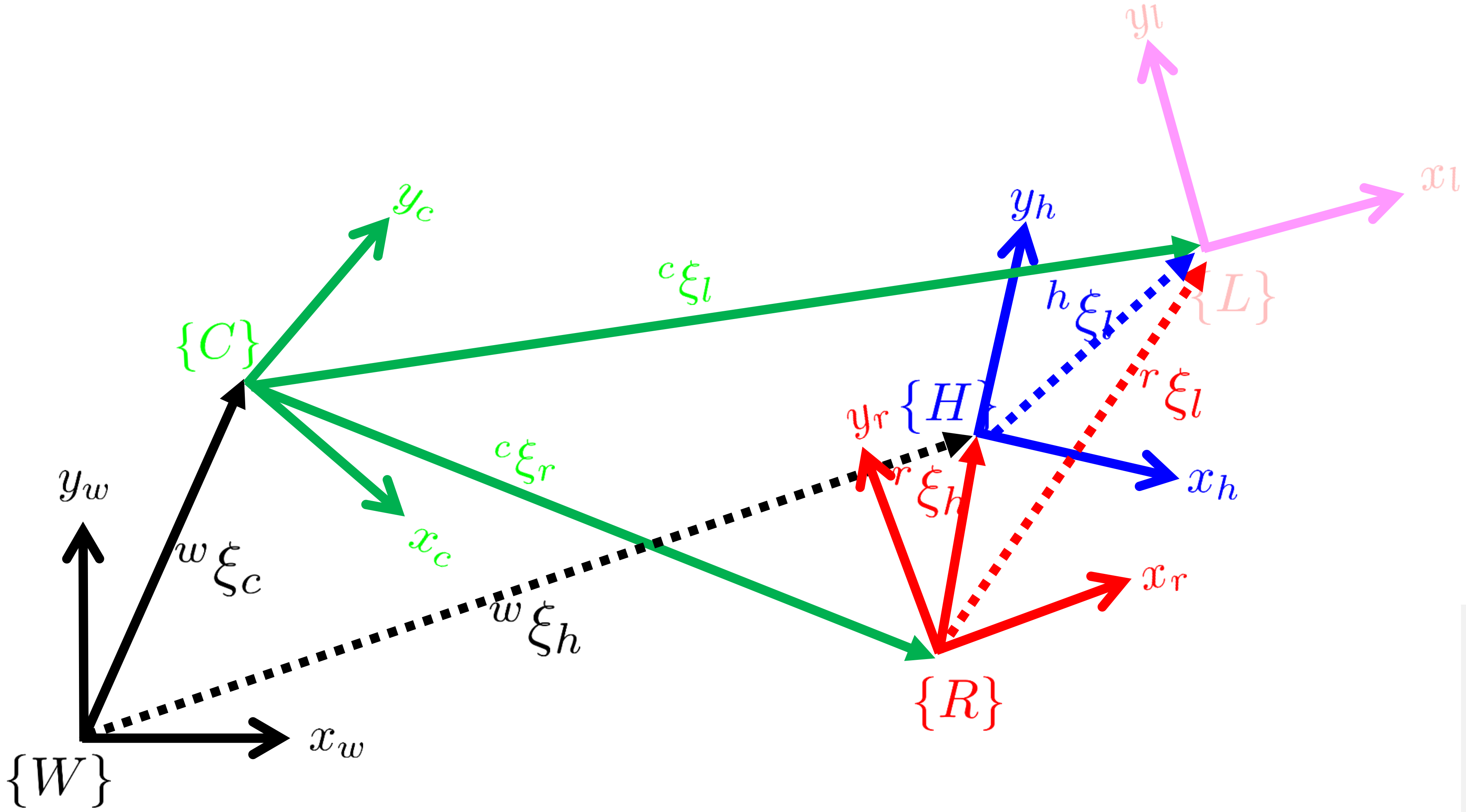


y_B
P
y_A
$^B\boldsymbol{p}$
$\boldsymbol{x}_B$
$^A\boldsymbol{p}$
θ
y
$\{B\}$
$\boldsymbol{t}$
x_A
$\{A\}$
x



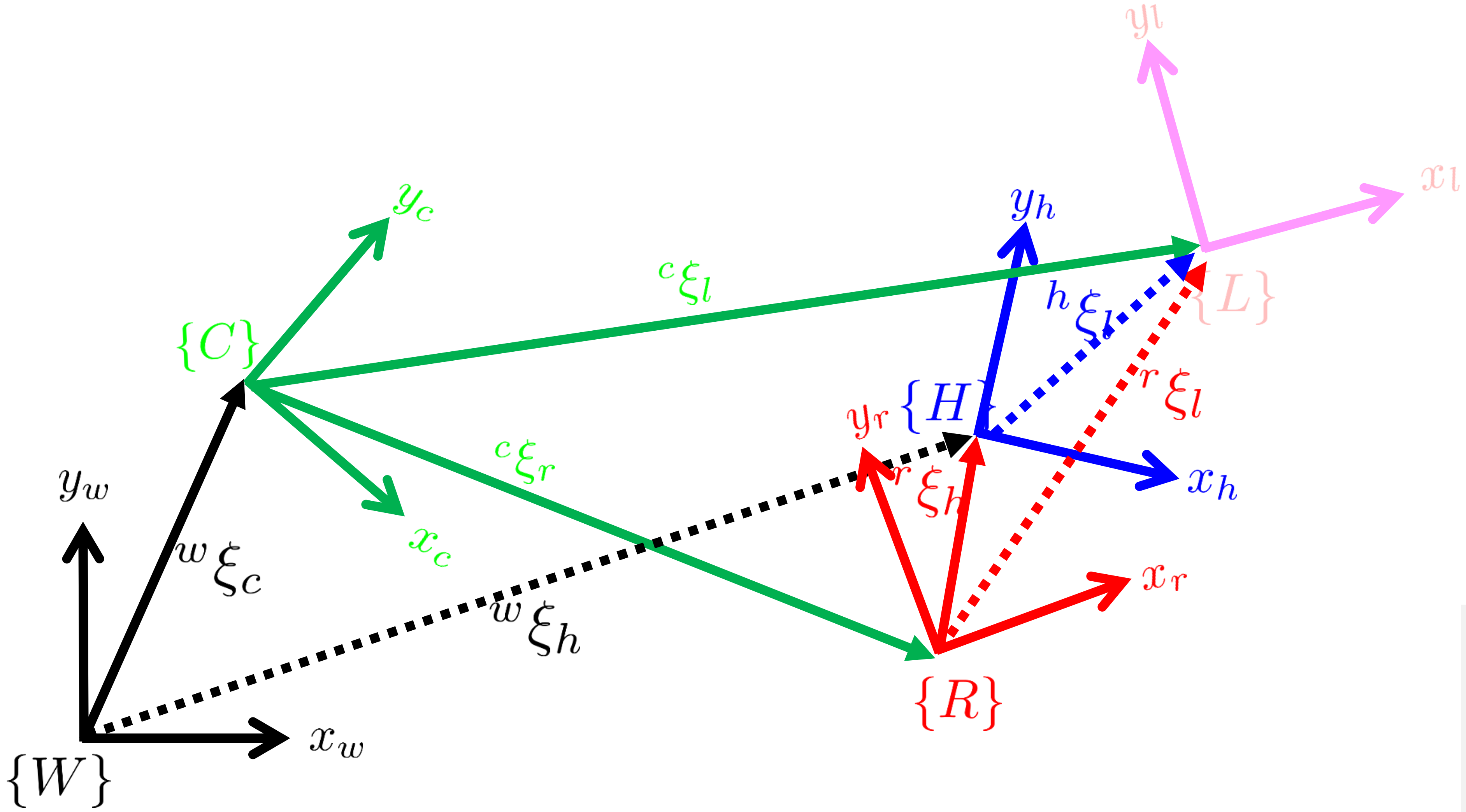
y_V
$\hat{\boldsymbol{y}}_B$
P
$^V y$
$\hat{\boldsymbol{x}}_B$
$\cos\theta$
$^B y$
$^B x$
$\sin\theta$
θ
x_V
$-\sin\theta$
$\{V\}$
$^V x$
$\cos\theta$
$\{B\}$

Complex example



W – world
C - camera
R – big robot
H – big robot's hand
L – little robot

Complex example



W – world
C - camera
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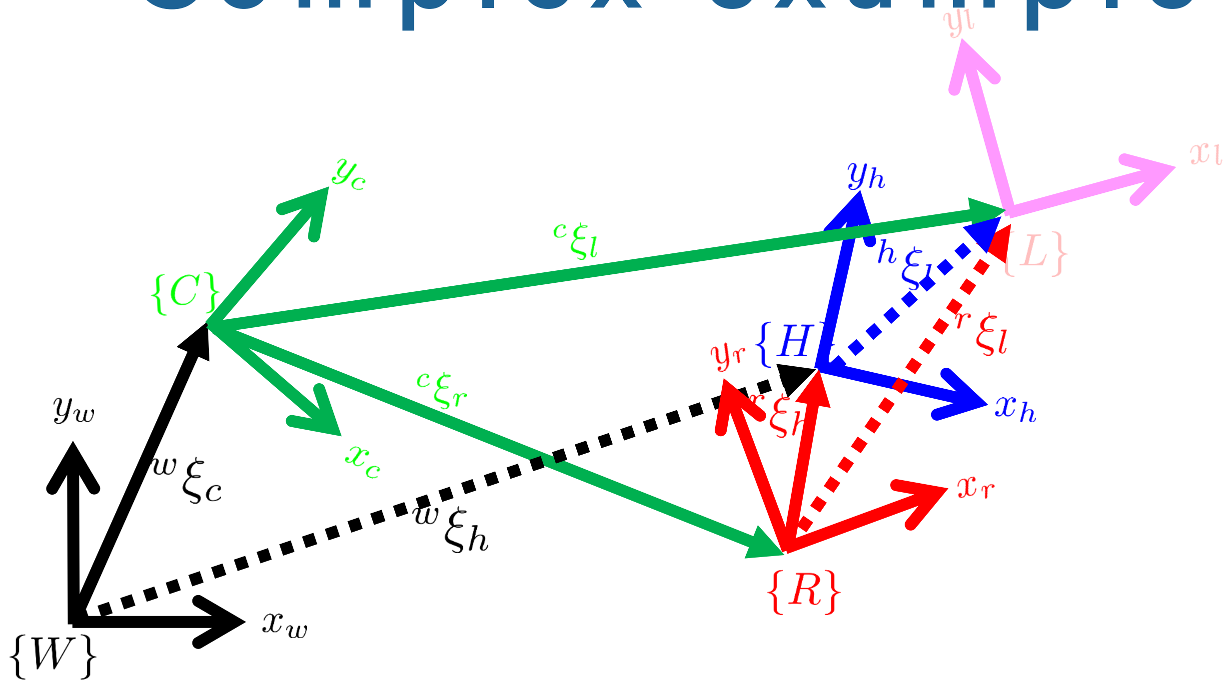
Complex example



W – world
C - camera
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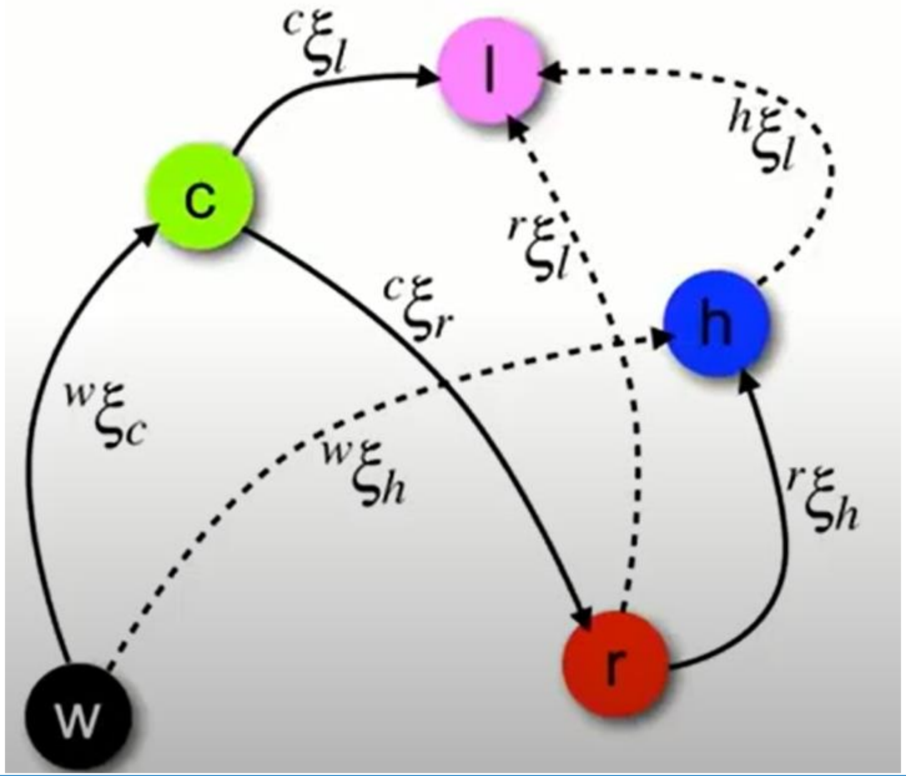
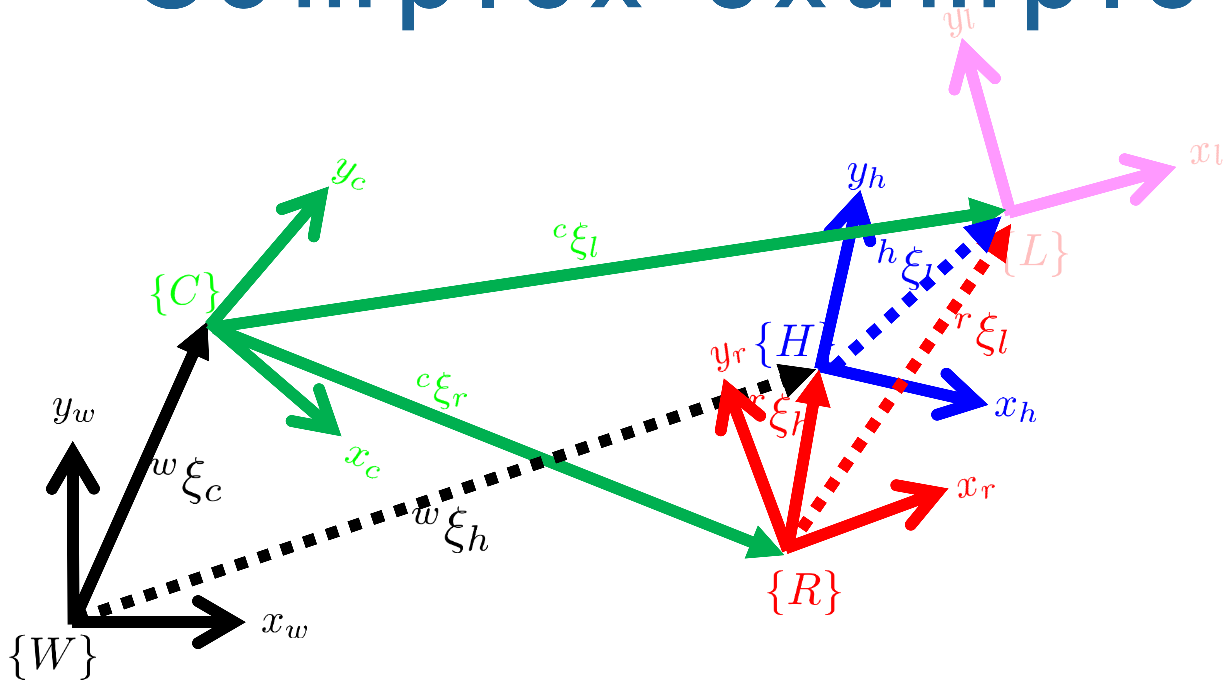
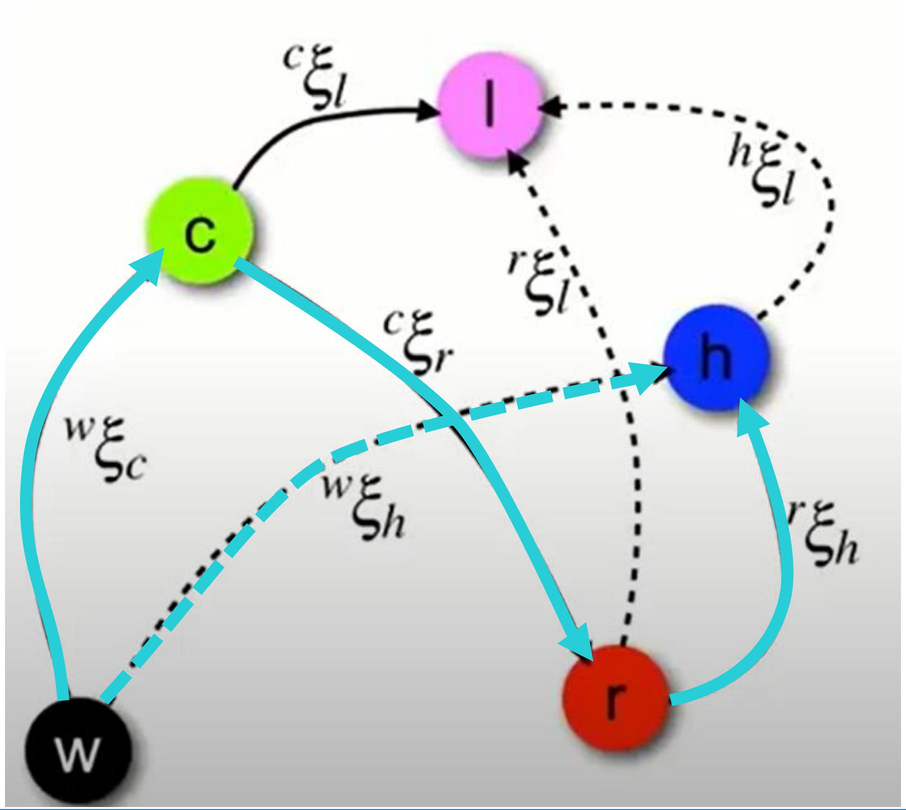
H – big robot's hand
L – little robot



Complex example

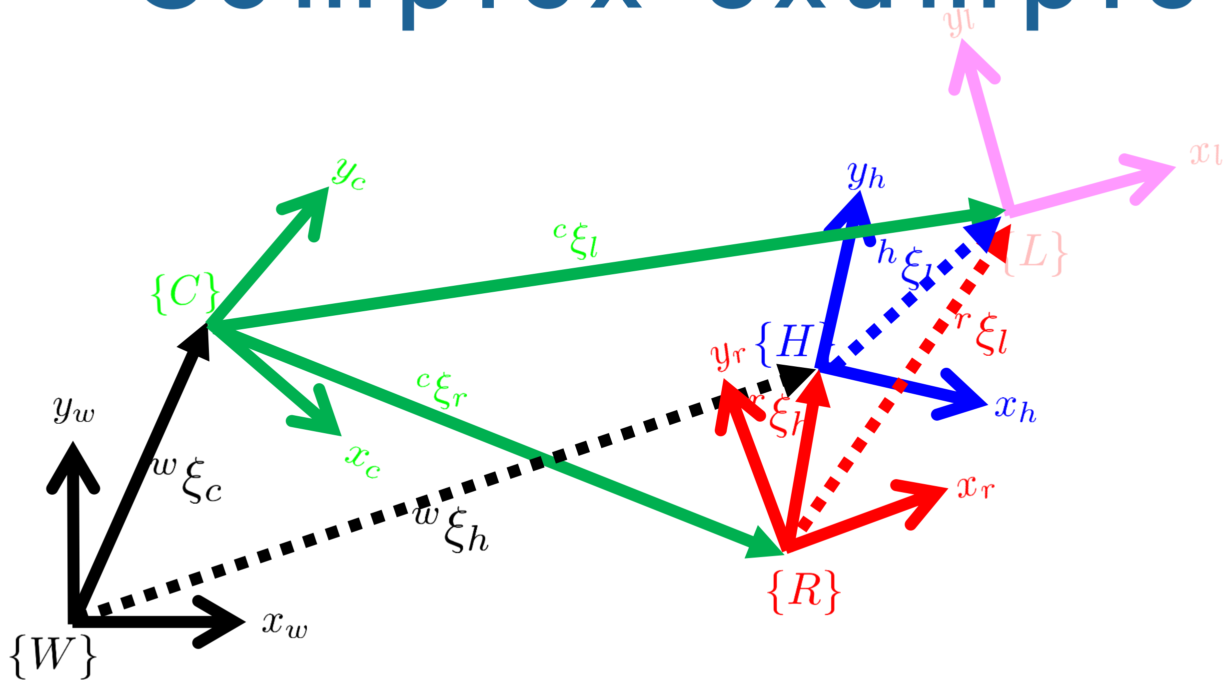


W – world
C - camera
R – big robot
H – big robot's hand
L – little robot



$$ {}^w\xi_h = {}^w\xi_c \oplus {}^c\xi_r \oplus {}^r\xi_h $$

Complex example



W – world
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R – big robot
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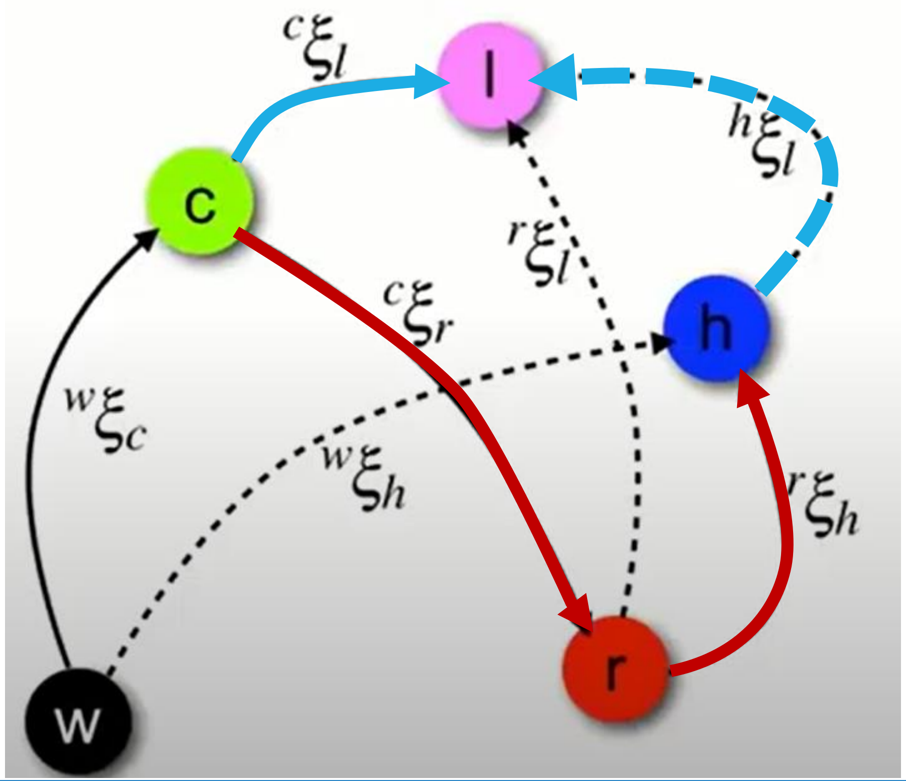


$$ {}^h\xi_l = \ominus\, {}^r\xi_h \ominus {}^c\xi_r \oplus {}^c\xi_l $$

reverse direction

Pose in algebra

- There are just a few rules

$$ {}^{X}\xi_{Y} \oplus {}^{Y}\xi_{Z} = {}^{X}\xi_{Z} \qquad {}^{X}p = {}^{X}\xi_{Y} \cdot {}^{Y}p $$

$$ \xi_1 \oplus \xi_2 \neq \xi_2 \oplus \xi_1 $$

$$ \ominus {}^{X}\xi_{Y} = {}^{Y}\xi_{X} $$

$$ \xi \ominus \xi = 0 \qquad \ominus \xi \oplus \xi = 0 $$

$$ \xi \ominus 0 = \xi \qquad \xi \oplus 0 = \xi $$

Developing a real representation of pose in 2D

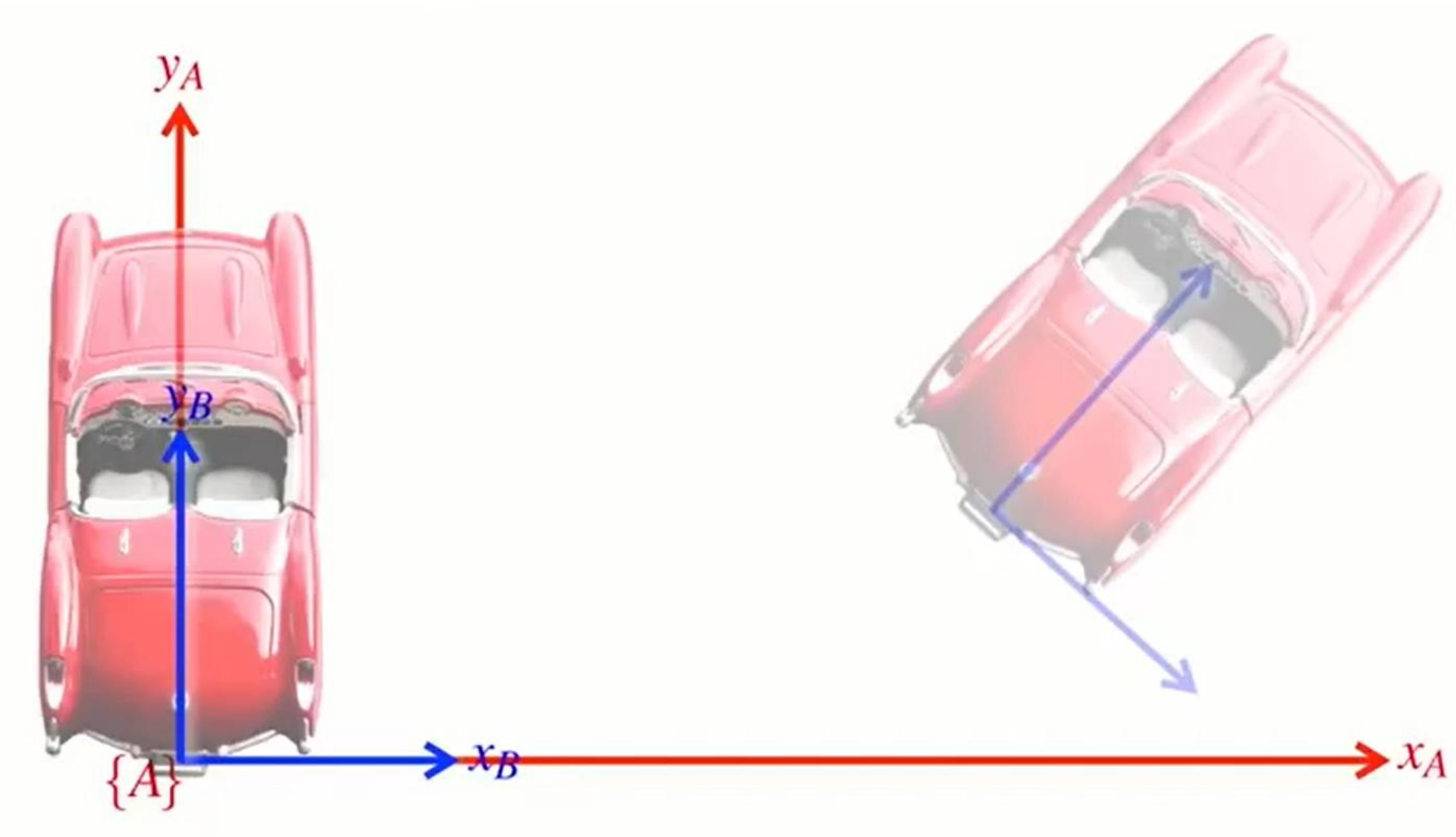
Pose

$$\xi$$

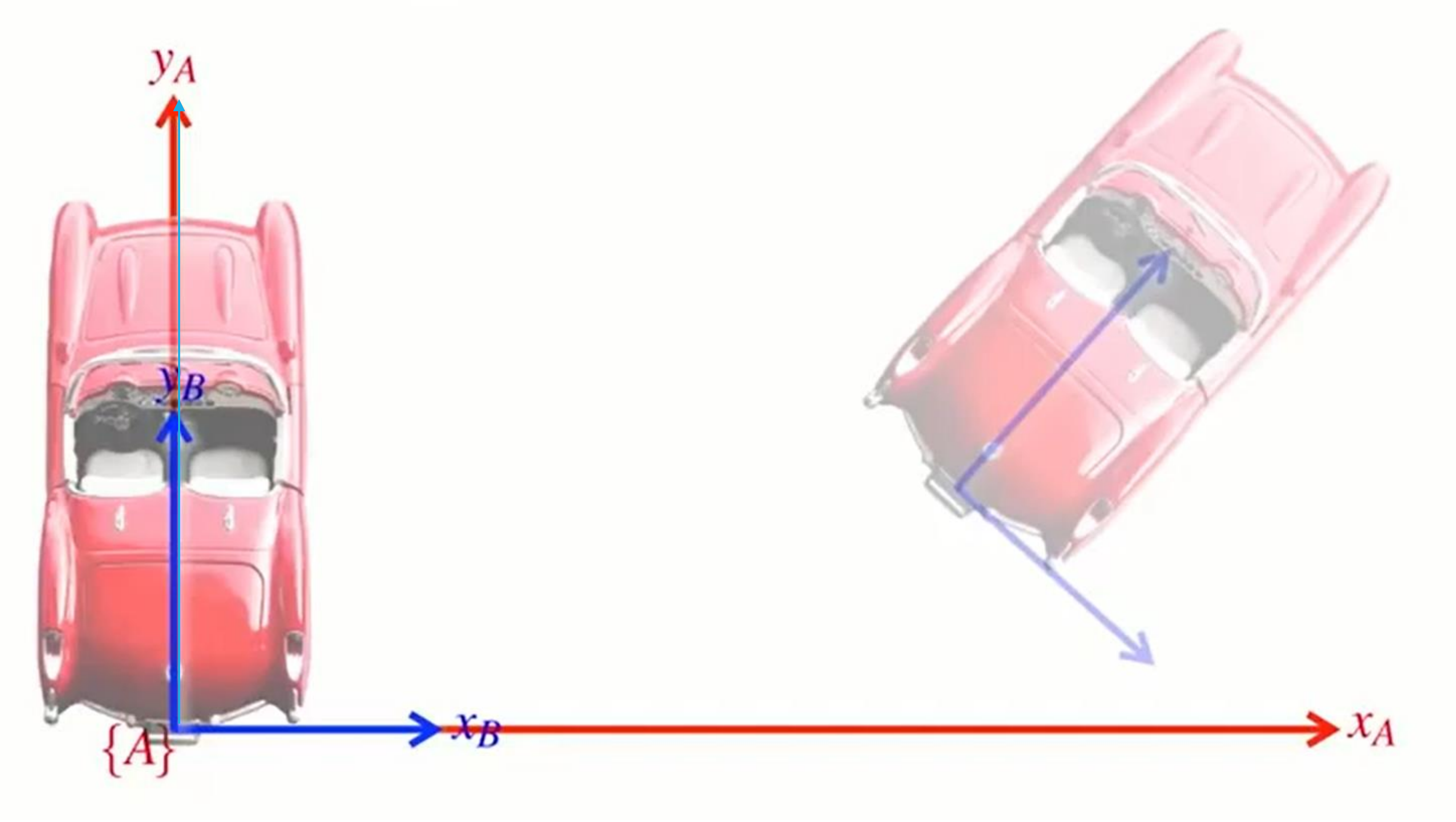
pronounced ksi

- It has 3 parameters : (x, y, θ)

Separating rotation and translation



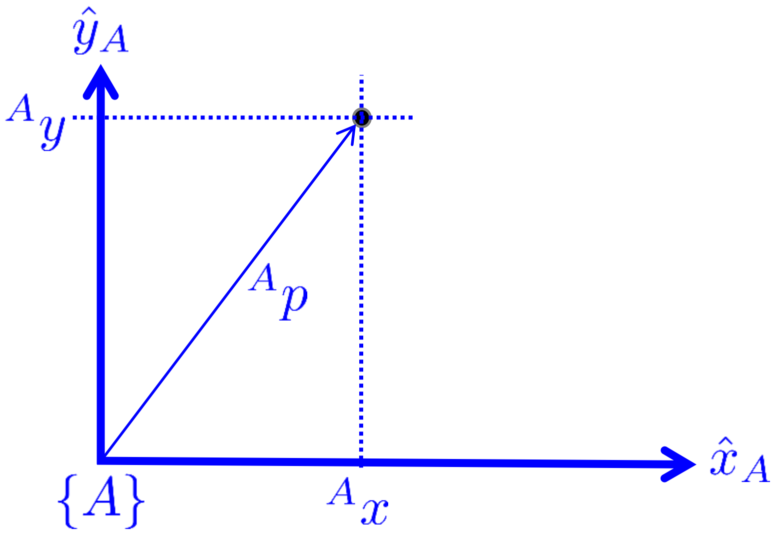
Separating rotation and translation



Describing rotation in 2D

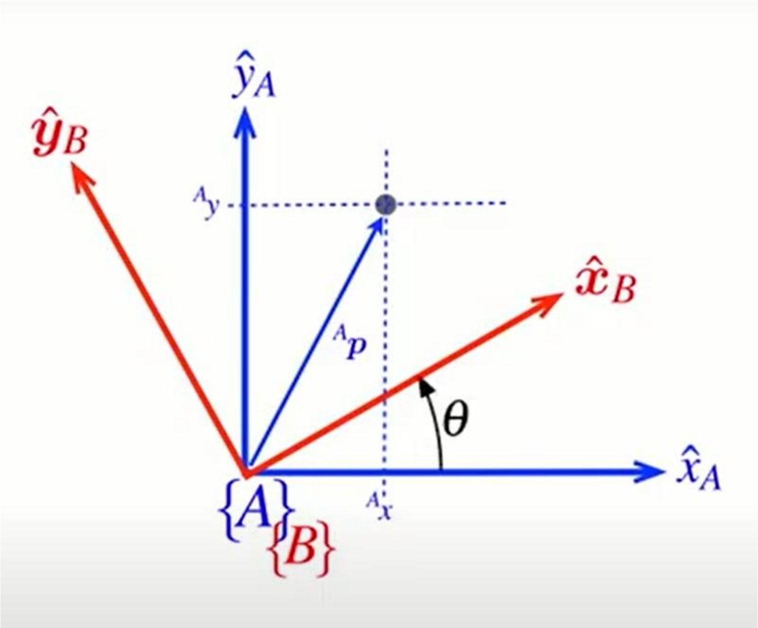
Understanding rotation

$$ {}^{A}\boldsymbol{p} = {}^{A}x\,\hat{\boldsymbol{x}}_A + {}^{A}y\,\hat{\boldsymbol{y}}_A $$



Understanding rotation

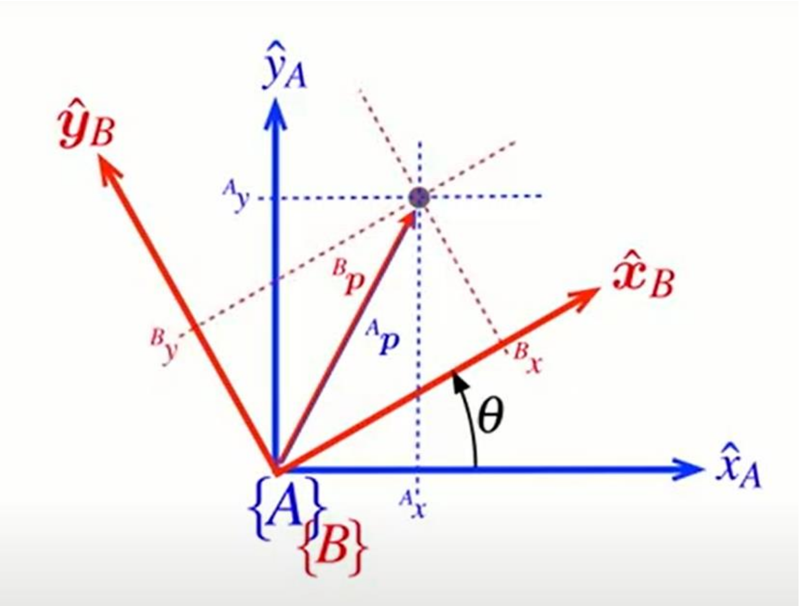
$$ {}^{A}\boldsymbol{p} = {}^{A}x\,\hat{\boldsymbol{x}}_A + {}^{A}y\,\hat{\boldsymbol{y}}_A $$



Understanding rotation

$$^{A}\boldsymbol{p} = {}^{A}x\,\hat{\boldsymbol{x}}_A + {}^{A}y\,\hat{\boldsymbol{y}}_A$$

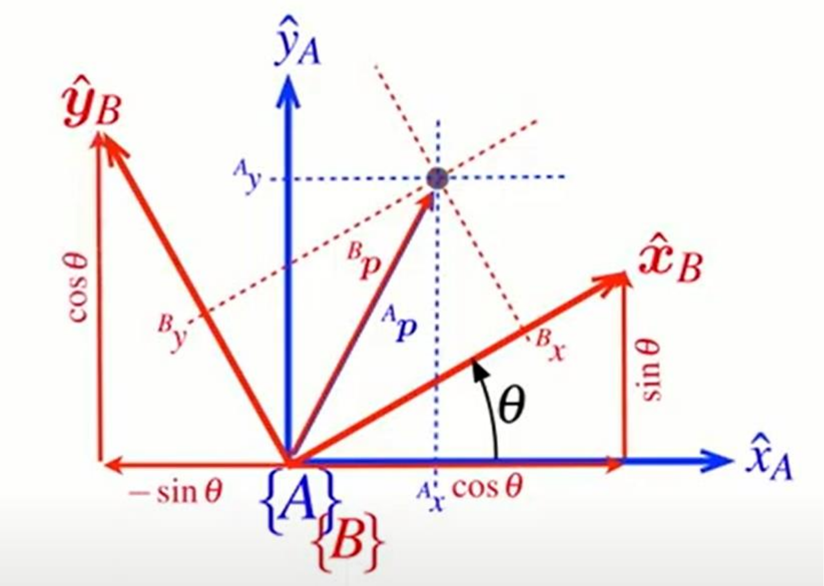
$$^{B}\boldsymbol{p} = {}^{B}x\,\hat{\boldsymbol{x}}_B + {}^{B}y\,\hat{\boldsymbol{y}}_B$$



Understanding rotation

$$^{A}\boldsymbol{p} = {}^{A}x\,\hat{\boldsymbol{x}}_A + {}^{A}y\,\hat{\boldsymbol{y}}_A$$

$$^{B}\boldsymbol{p} = {}^{B}x\,\hat{\boldsymbol{x}}_B + {}^{B}y\,\hat{\boldsymbol{y}}_B$$



$$\hat{\boldsymbol{x}}_B = \cos\theta\,\hat{\boldsymbol{x}}_A + \sin\theta\,\hat{\boldsymbol{y}}_A$$

$$\hat{\boldsymbol{y}}_B = -\sin\theta\,\hat{\boldsymbol{x}}_A + \cos\theta\,\hat{\boldsymbol{y}}_A$$

$$^{B}\boldsymbol{p} = {}^{B}x\,(\cos\theta\,\hat{\boldsymbol{x}}_A + \sin\theta\,\hat{\boldsymbol{y}}_A) + {}^{B}y\,(-\sin\theta\,\hat{\boldsymbol{x}}_A + \cos\theta\,\hat{\boldsymbol{y}}_A)$$

$$^{B}\boldsymbol{p} = \left({}^{B}x\cos\theta - {}^{B}y\sin\theta\right)\hat{\boldsymbol{x}}_A + \left({}^{B}x\sin\theta + {}^{B}y\cos\theta\right)\hat{\boldsymbol{y}}_A$$

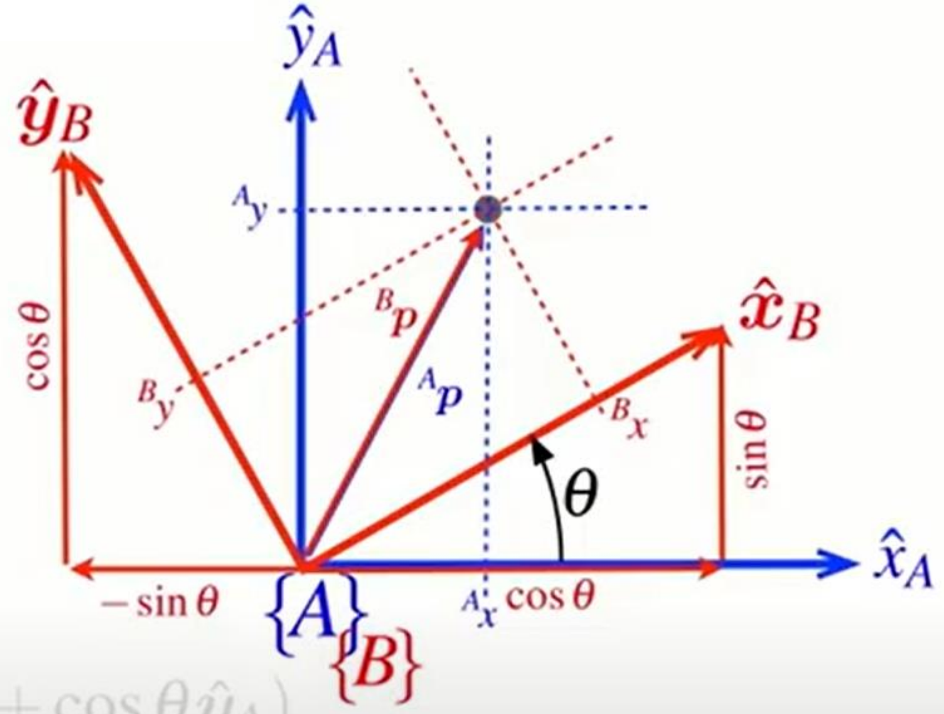
Understanding rotation

$$\boxed{{}^{A}\boldsymbol{p}} = {}^{A}x\,\hat{\boldsymbol{x}}_A + {}^{A}y\,\hat{\boldsymbol{y}}_A$$

$${}^{B}\boldsymbol{p} = {}^{B}x\,\hat{\boldsymbol{x}}_B + {}^{B}y\,\hat{\boldsymbol{y}}_B$$

$$\hat{\boldsymbol{x}}_B = \cos\theta\,\hat{\boldsymbol{x}}_A + \sin\theta\,\hat{\boldsymbol{y}}_A$$

$$\hat{\boldsymbol{y}}_B = -\sin\theta\,\hat{\boldsymbol{x}}_A + \cos\theta\,\hat{\boldsymbol{y}}_A$$



$${}^{B}\boldsymbol{p} = {}^{B}x\left(\cos\theta\,\hat{\boldsymbol{x}}_A + \sin\theta\,\hat{\boldsymbol{y}}_A\right) + {}^{B}y\left(-\sin\theta\,\hat{\boldsymbol{x}}_A + \cos\theta\,\hat{\boldsymbol{y}}_A\right)$$

$${}^{B}\boldsymbol{p} = \left({}^{B}x\cos\theta - {}^{B}y\sin\theta\right)\hat{\boldsymbol{x}}_A + \left({}^{B}x\sin\theta + {}^{B}y\cos\theta\right)\hat{\boldsymbol{y}}_A$$

$${}^{A}x = {}^{B}x\cos\theta - {}^{B}y\sin\theta \qquad {}^{A}y = {}^{B}x\sin\theta + {}^{B}y\cos\theta$$

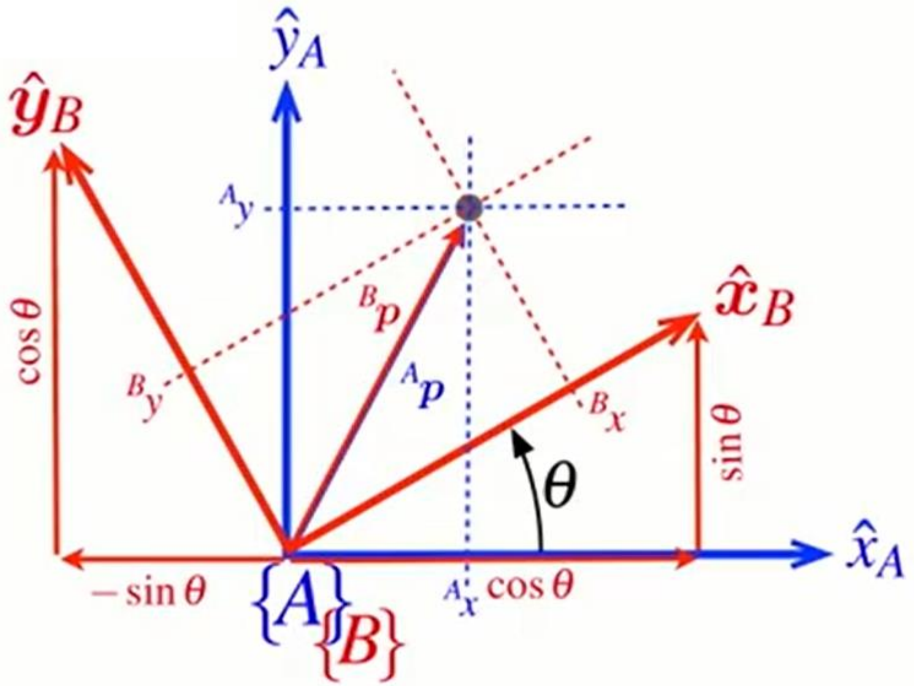
Consider Rotation First

$$^{A}x = {}^{B}x\cos\theta - {}^{B}y\sin\theta$$

$$^{A}y = {}^{B}x\sin\theta + {}^{B}y\cos\theta$$

$$\begin{pmatrix} ^{A}x \\ ^{A}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} ^{B}x \\ ^{B}y \end{pmatrix}$$

$$^{A}\boldsymbol{p} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} {}^{B}\boldsymbol{p}$$



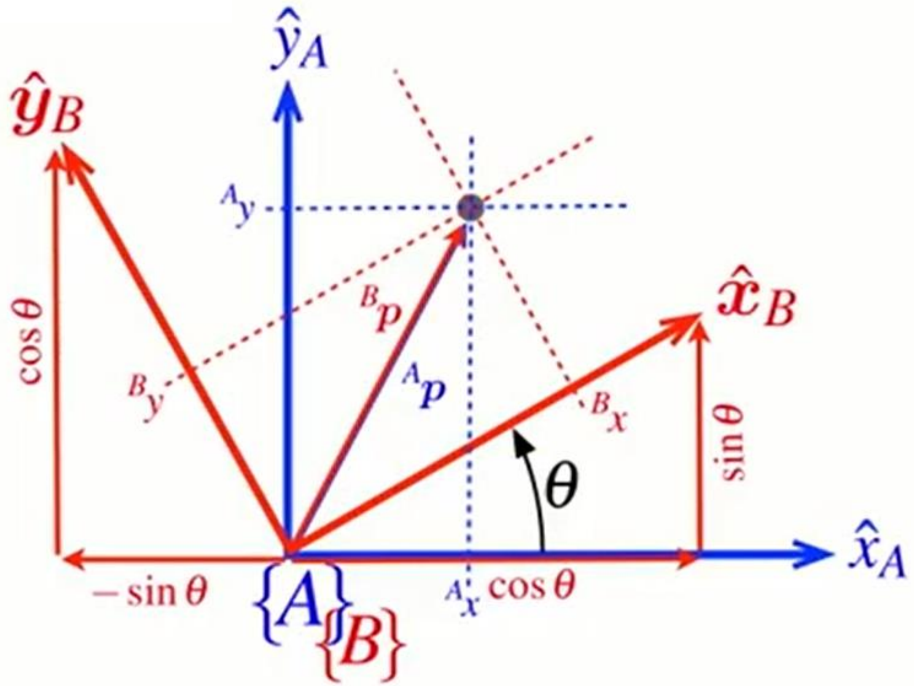
$$^{A}\boldsymbol{p} = \mathbf{R}\,{}^{B}\boldsymbol{p}$$

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Consider Rotation First

$$^{A}p = \mathbf{R}\,^{B}p \qquad \mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$^{A}p = {}^{A}\mathbf{R}_{B}\,^{B}p \qquad {}^{A}\mathbf{R}_{B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



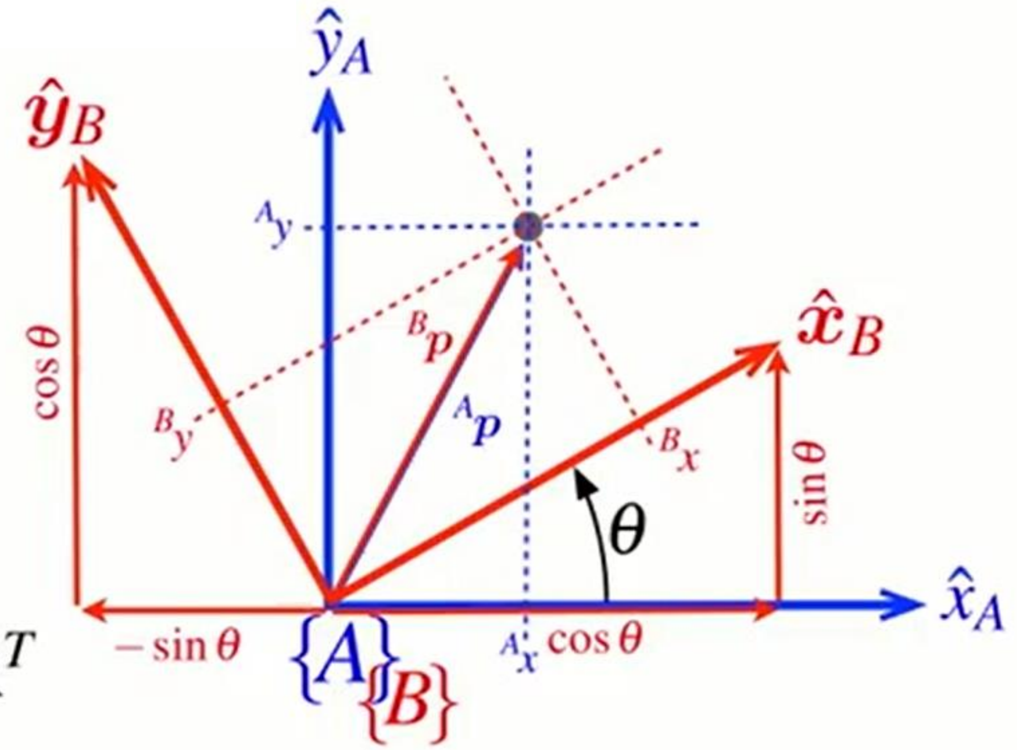
- **R** is a rotation matrix that:
  - rotates vectors from frame {B} to frame {A}
  - is a function of rotation angle from frame {A} to frame {B}

$$^{A}\mathbf{R}_{B}$$

Properties of the rotation matrix

$$
{}^{A}\mathbf{R}_{B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}
$$

- An orthogonal (orthonormal) matrix
  - ➡ Each column is a unit length vector
  - ➡ Each column is orthogonal to all other columns
- The inverse is the same as the transpose $\mathbf{R}^{-1} = \mathbf{R}^{T}$
- The determinant is +1 $det(\mathbf{R}) = 1$
  - ➡ the length of a vector is unchanged by rotation



Inverse Rotation

- To rotate a vector from frame {A} to frame {B} we use the inverse rotation matrix
  - The inverse is simply the transpose

$$ {}^{A}\boldsymbol{p} = {}^{A}\mathbf{R}_{B}\, {}^{B}\boldsymbol{p} $$

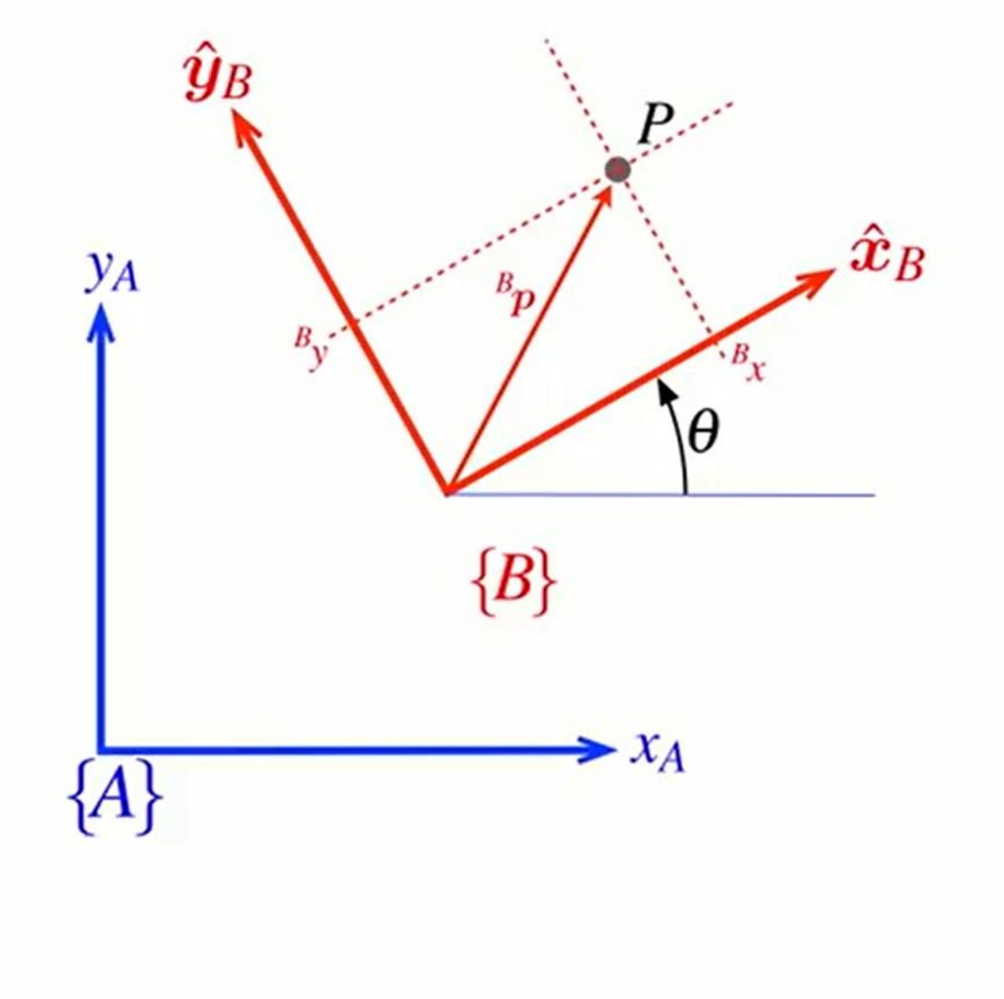
$$ {}^{B}\boldsymbol{p} = {}^{A}\mathbf{R}_{B}^{-1}\, {}^{A}\boldsymbol{p} $$

$$ {}^{B}\boldsymbol{p} = {}^{B}\mathbf{R}_{A}\, {}^{A}\boldsymbol{p} $$

$$ {}^{B}\mathbf{R}_{A} = {}^{A}\mathbf{R}_{B}^{-1} $$

Describing rotation and translation in 2D

- Coordinate frame B which is translated and rotated with respect to coordinate frame A and a point P

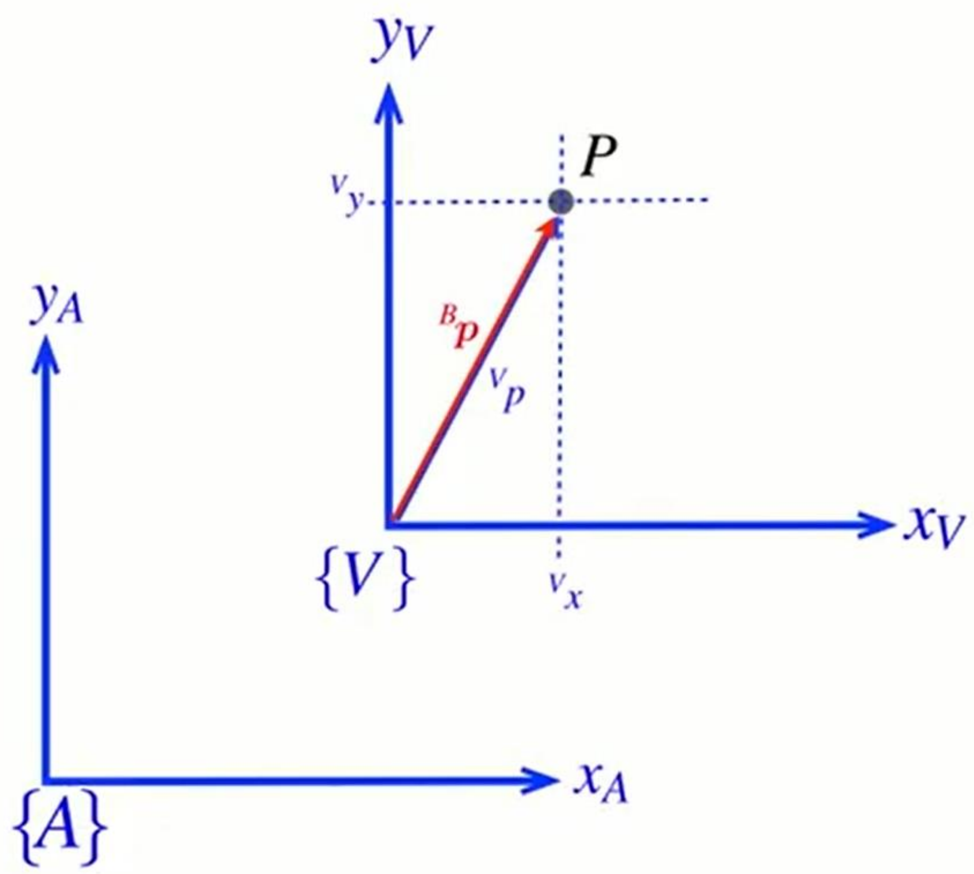


- We can rotate that vector into a new frame, coordinate frame V, and coordinate frame V has axis which are parallel to coordinate frame A

$$^{V}\boldsymbol{p} = {}^{V}\mathbf{R}_{B}\,{}^{B}\boldsymbol{p}$$

- Only add vectors in the same (or parallel) reference frames

$$^{A}\boldsymbol{p} = {}^{A}\boldsymbol{t}_{V} + {}^{V}\boldsymbol{p} \qquad \{V\} \parallel \{A\}$$

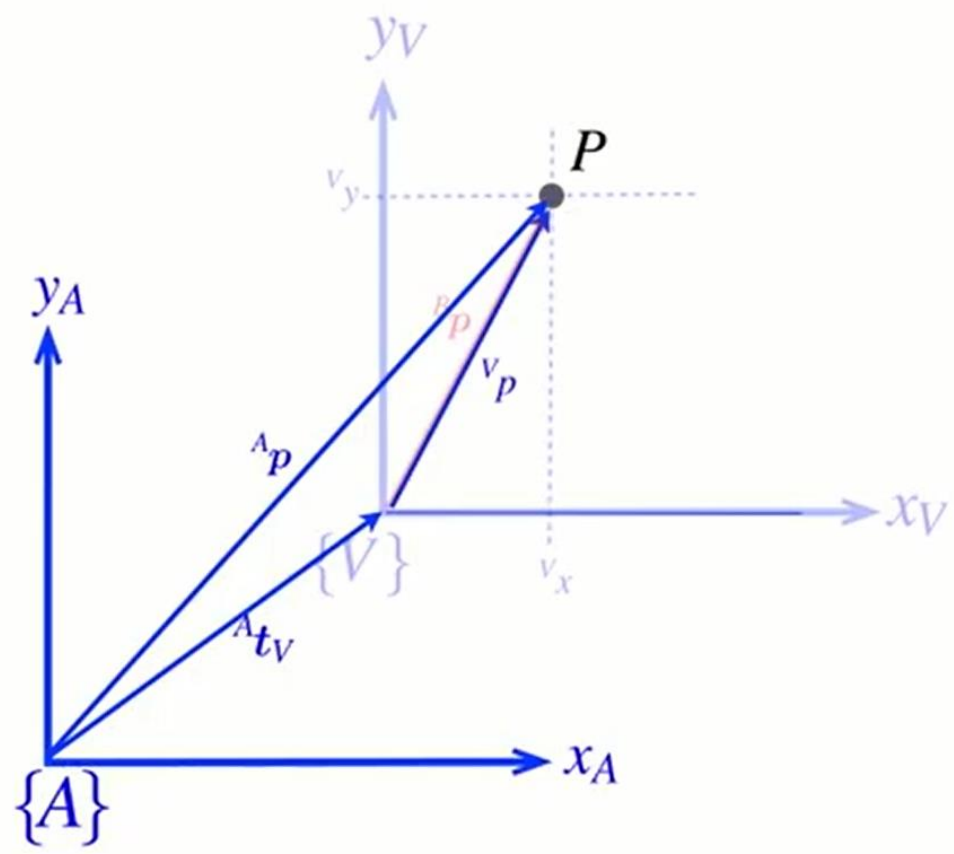


$$^{V}\boldsymbol{p} = {}^{V}\mathbf{R}_B\, {}^{B}\boldsymbol{p}$$

$$^{A}\boldsymbol{p} = {}^{A}\boldsymbol{t}_V + {}^{V}\boldsymbol{p} \qquad \{V\} \parallel \{A\}$$

$$^{A}\boldsymbol{p} = {}^{A}\boldsymbol{t}_V + {}^{V}\mathbf{R}_B\, {}^{B}\boldsymbol{p}$$

$$^{A}\boldsymbol{p} = {}^{A}\boldsymbol{t}_B + {}^{A}\mathbf{R}_B\, {}^{B}\boldsymbol{p}$$



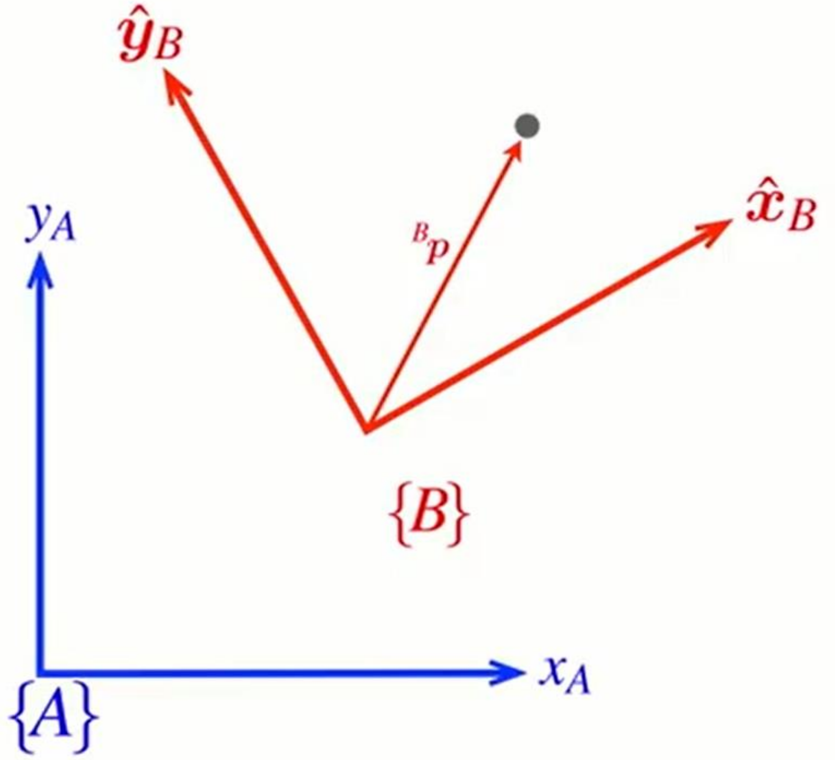
- Only add vectors in the same (or parallel) reference frames

Homogeneous form

$$^A\boldsymbol{p} = {}^A\boldsymbol{t}_V + {}^V\mathbf{R}_B\, {}^B\boldsymbol{p}$$

$$\begin{pmatrix} {}^Ax \\ {}^Ay \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} {}^Bx \\ {}^By \end{pmatrix} + \begin{pmatrix} {}^Ax_B \\ {}^Ay_B \end{pmatrix}$$

$$\begin{pmatrix} {}^Ax \\ {}^Ay \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & {}^Ax_B \\ \sin\theta & \cos\theta & {}^Ay_B \end{pmatrix} \begin{pmatrix} {}^Bx \\ {}^By \\ 1 \end{pmatrix}$$



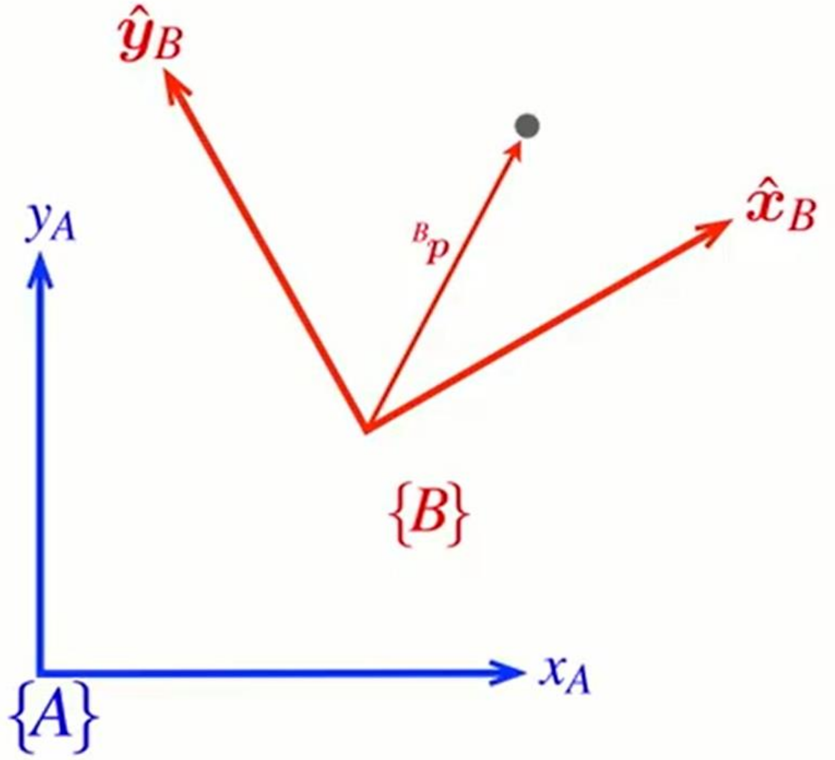
Homogeneous form

$${}^{A}\boldsymbol{p} = {}^{A}\boldsymbol{t}_{V} + {}^{V}\mathbf{R}_{B}\,{}^{B}\boldsymbol{p}$$

$$\begin{pmatrix} {}^{A}x \\ {}^{A}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix} + \begin{pmatrix} {}^{A}x_{B} \\ {}^{A}y_{B} \end{pmatrix}$$

$$\begin{pmatrix} {}^{A}x \\ {}^{A}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & {}^{A}x_{B} \\ \sin\theta & \cos\theta & {}^{A}y_{B} \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & {}^{A}x_{B} \\ \sin\theta & \cos\theta & {}^{A}y_{B} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{pmatrix}$$



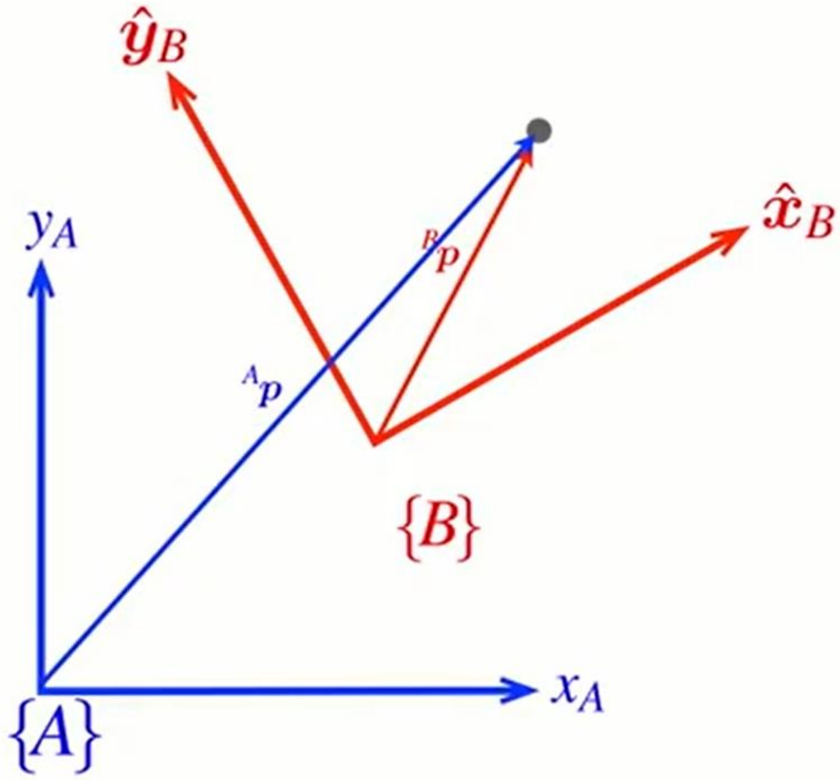
$${}^{A}\tilde{\boldsymbol{p}} = \begin{pmatrix} {}^{A}\mathbf{R}_{B} & {}^{A}\boldsymbol{t}_{B} \\ 0,0 & 1 \end{pmatrix} {}^{B}\tilde{\boldsymbol{p}}$$

$$ {}^{A}\boldsymbol{p} = {}^{A}\boldsymbol{t}_V + {}^{V}\mathbf{R}_B\, {}^{B}\boldsymbol{p} $$

$$ \begin{pmatrix} {}^{A}x \\ {}^{A}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix} + \begin{pmatrix} {}^{A}x_B \\ {}^{A}y_B \end{pmatrix} $$

$$ \begin{pmatrix} {}^{A}x \\ {}^{A}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & {}^{A}x_B \\ \sin\theta & \cos\theta & {}^{A}y_B \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{pmatrix} $$

$$ \begin{pmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & {}^{A}x_B \\ \sin\theta & \cos\theta & {}^{A}y_B \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{pmatrix} $$



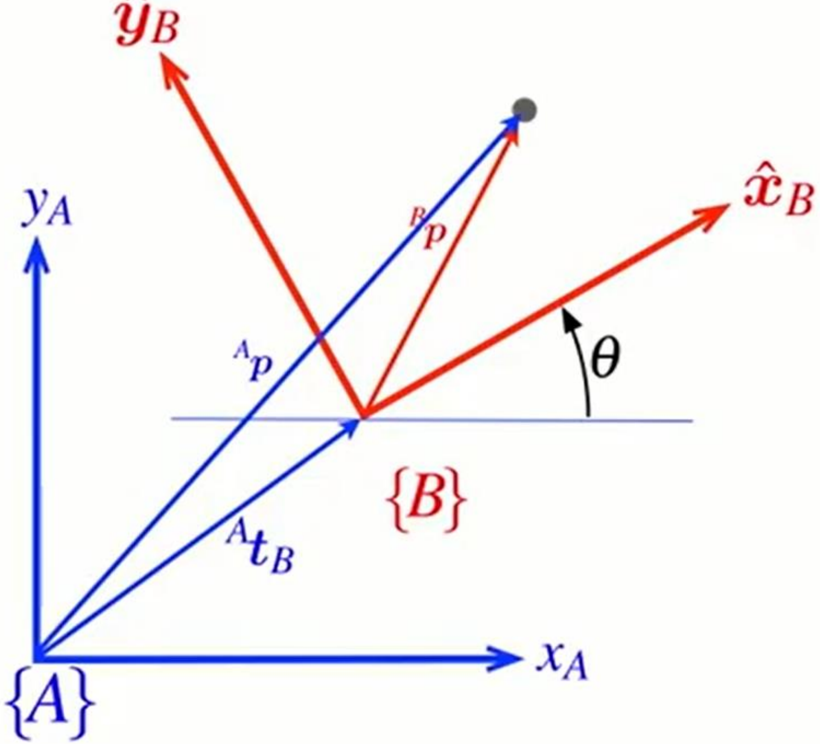
$$ {}^{A}\tilde{\boldsymbol{p}} = \begin{pmatrix} {}^{A}\mathbf{R}_B & {}^{A}\boldsymbol{t}_B \\ 0,0 & 1 \end{pmatrix} {}^{B}\tilde{\boldsymbol{p}} $$

$$ {}^{A}\boldsymbol{p} = {}^{A}\boldsymbol{t}_{V} + {}^{V}\mathbf{R}_{B}\, {}^{B}\boldsymbol{p} $$

$$ \begin{pmatrix} {}^{A}x \\ {}^{A}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix} + \begin{pmatrix} {}^{A}x_{B} \\ {}^{A}y_{B} \end{pmatrix} $$

$$ \begin{pmatrix} {}^{A}x \\ {}^{A}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & {}^{A}x_{B} \\ \sin\theta & \cos\theta & {}^{A}y_{B} \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{pmatrix} $$

$$ \begin{pmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & {}^{A}x_{B} \\ \sin\theta & \cos\theta & {}^{A}y_{B} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{pmatrix} $$



$$ {}^{A}\tilde{\boldsymbol{p}} = \begin{pmatrix} {}^{A}\mathbf{R}_{B} & {}^{A}\boldsymbol{t}_{B} \\ 0,0 & 1 \end{pmatrix} {}^{B}\tilde{\boldsymbol{p}} $$

Properties of Homogeneous Transform

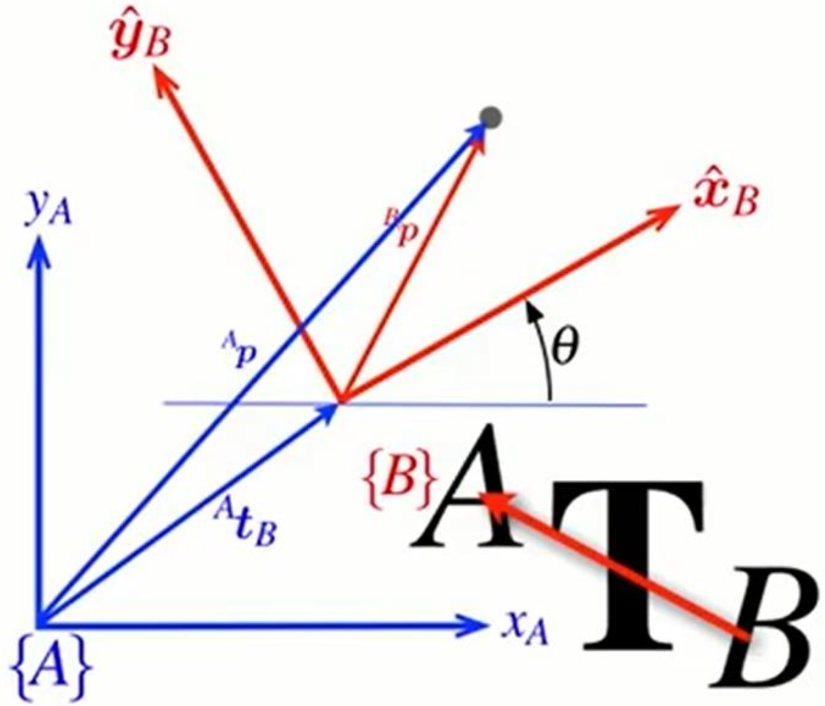
homogeneous vector

homogeneous transform

homogeneous vector

$$^A\tilde{\mathbf{p}} = \begin{pmatrix} ^A\mathbf{R}_B & ^A\boldsymbol{t}_B \\ 0,0 & 1 \end{pmatrix} {}^B\tilde{\mathbf{p}}$$

$$^A\tilde{\mathbf{p}} = {}^A\mathbf{T}_B \, {}^B\tilde{\mathbf{p}} \qquad {}^A\mathbf{T}_B = \begin{pmatrix} ^A\mathbf{R}_B & ^A\boldsymbol{t}_B \\ 0,0 & 1 \end{pmatrix}$$



- Describes a relative pose as a 3x3 matrix
- Homogeneous transforms belong to the special Euclidean group of dimension 2 $\mathbf{T} \in SE(2)$

$$\boldsymbol{p} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \tilde{\boldsymbol{p}} = \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$$

Finally we can make abstract symbols and concepts real :

- Pose is a matrix

$$^{A}\xi_{B} \sim {}^{A}\mathbf{T}_{B} = \begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

- Compounding (composition is a matrix-matrix product

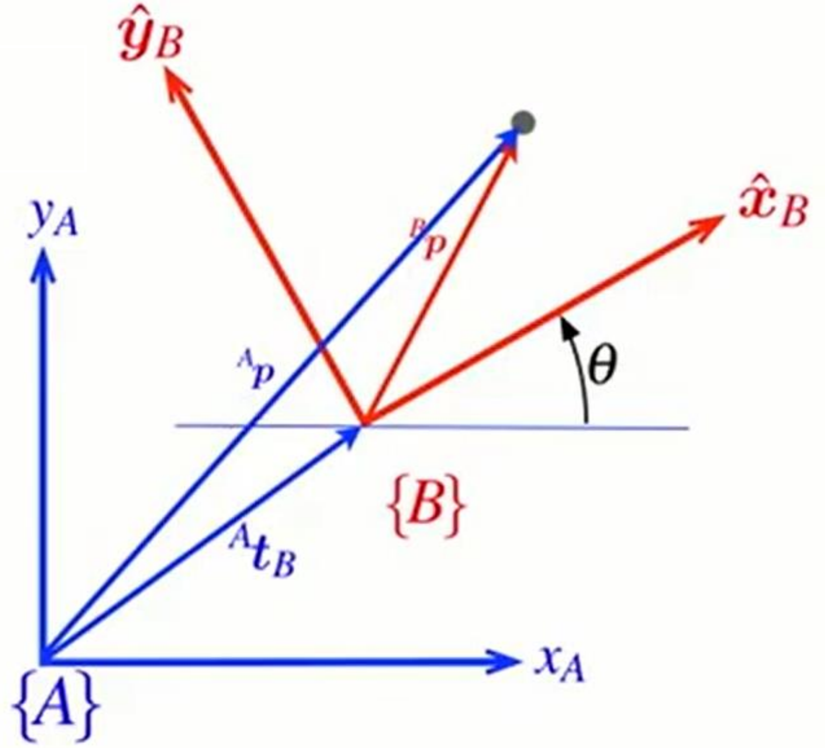
$$^{X}\xi_{Y} \oplus {}^{Y}\xi_{Z} \mapsto {}^{X}\mathbf{T}_{Y}\,{}^{Y}\mathbf{T}_{Z}$$

- Negation is a matrix inverse

$$\ominus\xi \mapsto \mathbf{T}^{-1}$$

- Vector transformation is a matrix-vector product

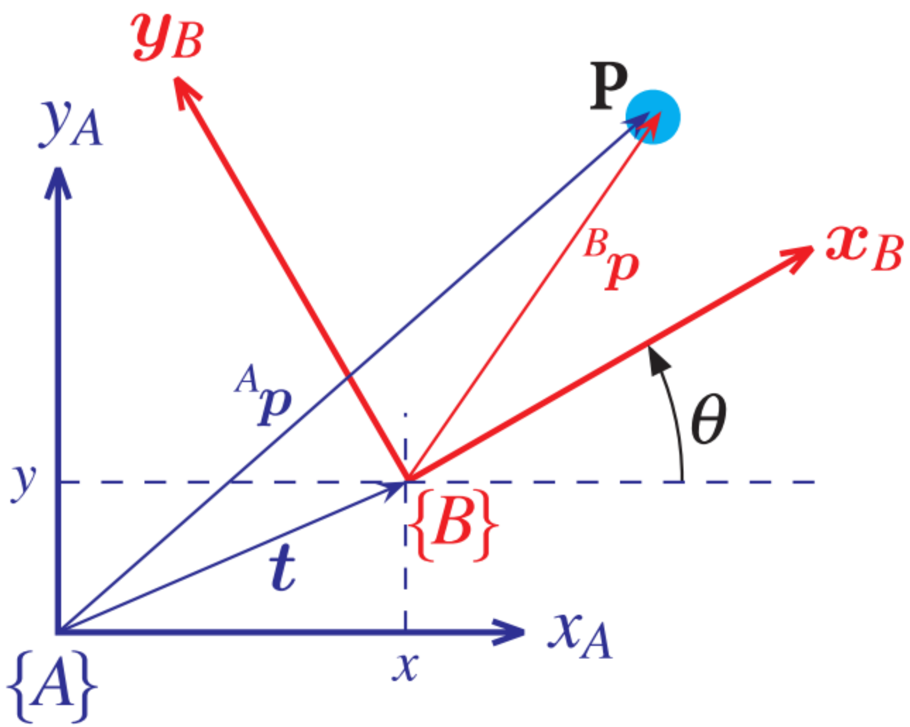
$$\xi \cdot \boldsymbol{p} \mapsto \mathbf{T}\tilde{\boldsymbol{p}}$$

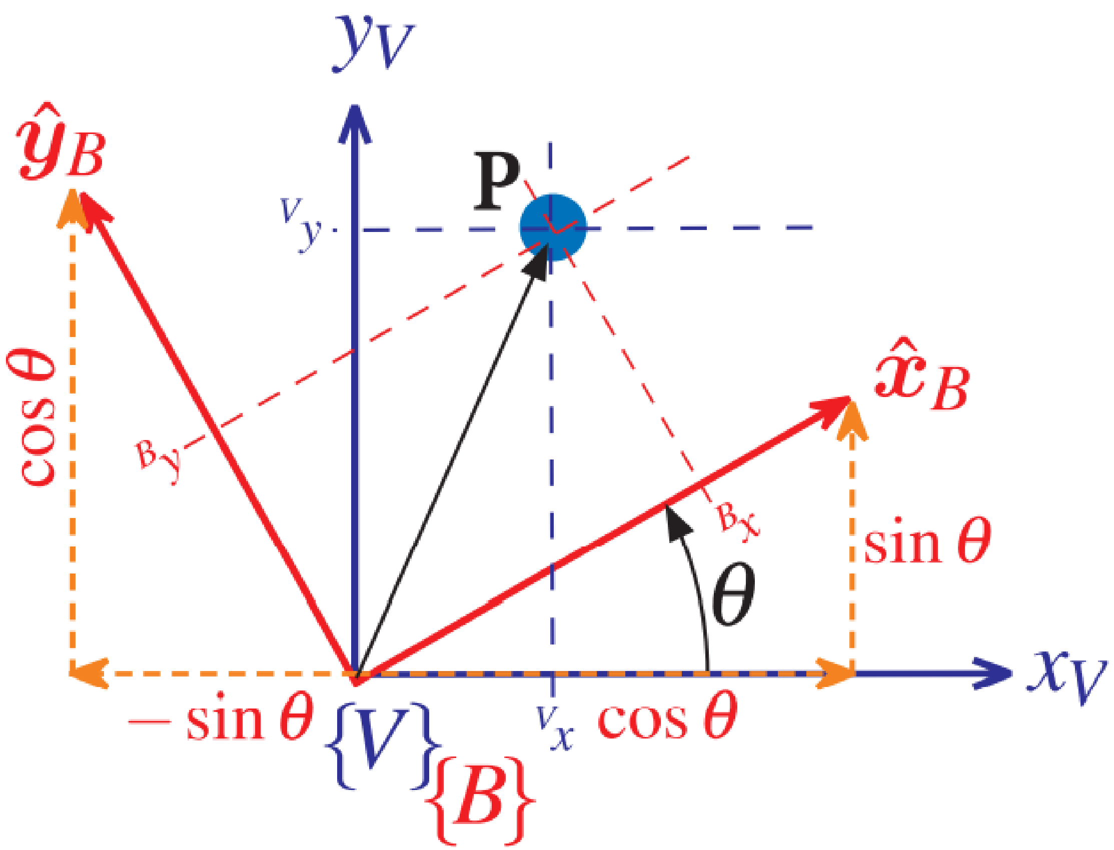


$$\boldsymbol{p} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \tilde{\boldsymbol{p}} = \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$$

Using Toolbox

Translation and Rotation





Rotation

$$^{B}p = \begin{pmatrix}\hat{x}_V & \hat{y}_V\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}^{B}x \\ ^{B}y\end{pmatrix}$$

$$\begin{pmatrix}^{V}x \\ ^{V}y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}^{B}x \\ ^{B}y\end{pmatrix}$$

$$\begin{pmatrix}^{V}x \\ ^{V}y\end{pmatrix} = {}^{V}\boldsymbol{R}_B\begin{pmatrix}^{B}x \\ ^{B}y\end{pmatrix}$$

$$^{X}\boldsymbol{R}_Y(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$

```
>> R = rot2(0.2)
R =
    0.9801   -0.1987
    0.1987    0.9801
```

where the angle is specified in radians

Determinant Matrix

$$|A| = \left|\begin{vmatrix} a & b \\ c & d \end{vmatrix}\right| = ad - bc$$

```
>> det(R)
ans =
     1
```

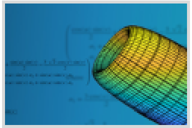
```
R =
    0.9801   -0.1987
    0.1987    0.9801
```

The product of two rotation matrices is also a rotation matrix

```
>> det(R*R)
ans =
     1
```

R is a rotation matric

Symbolic Math



Symbolic Math Toolbox version 9.1

Install add on Symbolic Math Toolbox

```
>> syms theta
>> R = rot2(theta)
R =
[ cos(theta), -sin(theta)]
[ sin(theta),  cos(theta)]
>> simplify(R*R)
ans =
[ cos(2*theta), -sin(2*theta)]
[ sin(2*theta),  cos(2*theta)]
>> simplify(det(R))
ans =
1
```

Matrix Exponential

Consider a pure rotation of 0.3 radians expressed as a rotation matrix

```
>> R = rot2(0.3)
ans =
    0.9553   -0.2955
    0.2955    0.9553
```

The logarithm of this matrix

```
>> S = logm(R)
S =
    0.0000   -0.3000
    0.3000    0.0000
```

The result is a simple matrix with two elements having a magnitude of 0.3, which intriguingly is the original rotation angle.

Skew-symmetric

a skew-symmetric (or antisymmetric or antimetric) matrix is a square matrix whose transpose equals its negative.

$$A \text{ skew-symmetric} \quad \Longleftrightarrow \quad A^{\mathsf{T}} = -A.$$

$$A \text{ skew-symmetric} \quad \Longleftrightarrow \quad a_{ji} = -a_{ij}.$$

$$[\omega]_\times = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

```
>> skew(2)
ans =
     0    -2
     2     0
```

The inverse operation is performed using the Toolbox function *vex*

```
>> vex(ans)
ans =
     2
```

```
>> R = rot2(0.3)
ans =
     0.9553   -0.2955
     0.2955    0.9553
```

```
>> S = logm(R)
S =
     0.0000   -0.3000
     0.3000    0.0000
```

The rotation angle (θ)

```
>> vex(S)
ans =
     0.3000
```

Matrix Exponential

The inverse of a logarithm

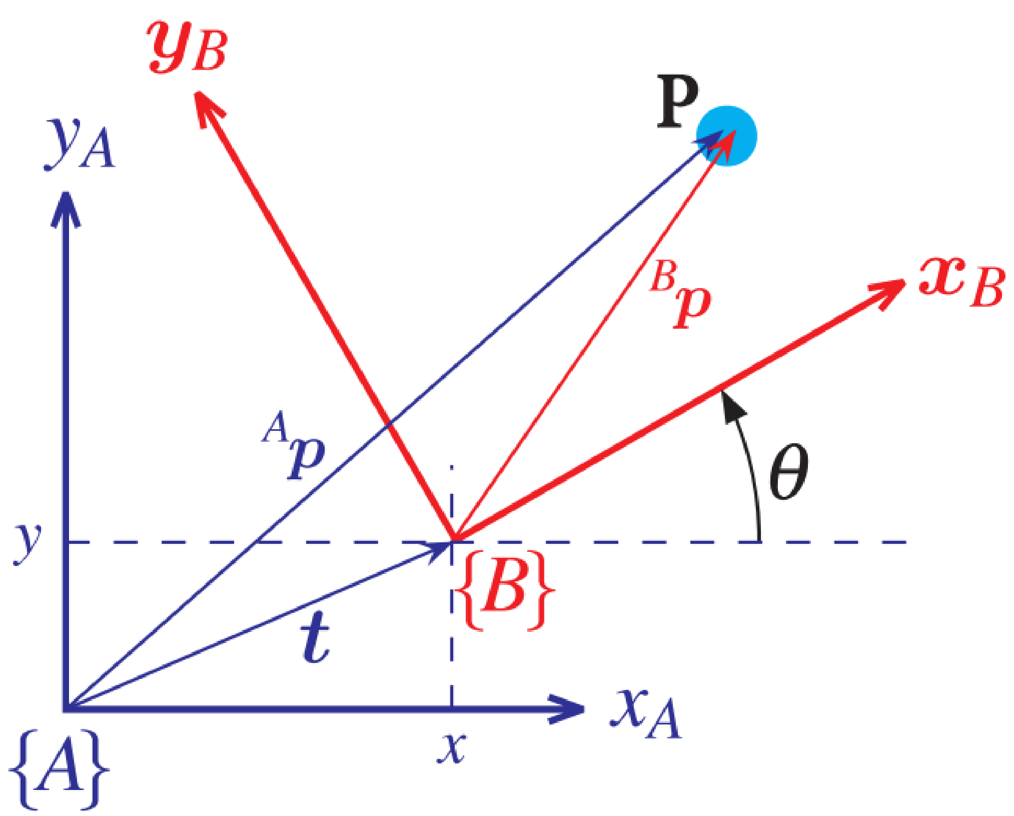
```
>> expm(S)
ans =
     0.9553   -0.2955
     0.2955    0.9553
```

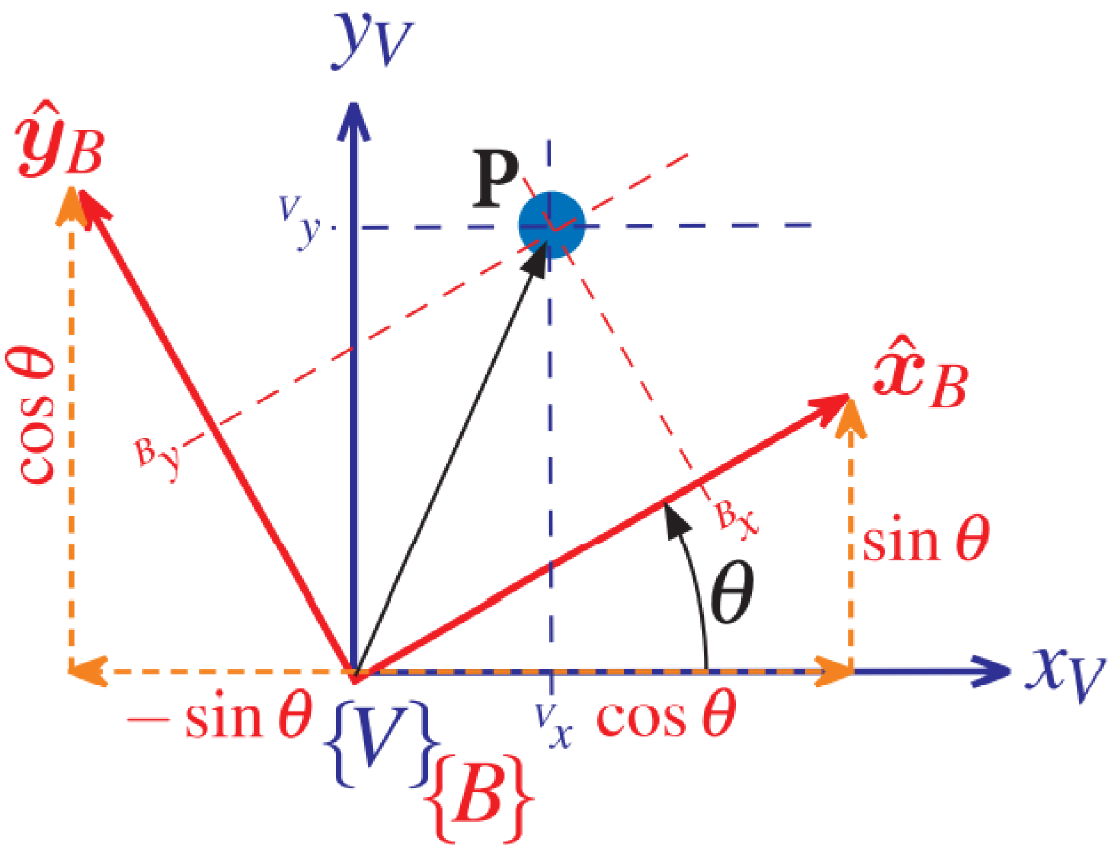
```
>> R = rot2(0.3)
ans =
     0.9553   -0.2955
     0.2955    0.9553
```

```
>> R = expm(  skew(0.3)  );
```

Formally we can write $\boldsymbol{R} = e^{[\theta]_\times} \in \mathbf{SO(2)}$

Homogeneous Transformation Matrix





Since the axes {V} and {A} are parallel, this is simply vectorial addition

$$\begin{pmatrix} {}^{A}x \\ {}^{A}y \end{pmatrix} = \begin{pmatrix} {}^{V}x \\ {}^{V}y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{pmatrix} = \begin{pmatrix} {}^{A}\boldsymbol{R}_{B} & \boldsymbol{t} \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{pmatrix}$$

t = (x, y) is the translation of the frame

the orientation is ${}^{A}\boldsymbol{R}_{B}$

$${}^{A}\boldsymbol{R}_{B} = {}^{V}\boldsymbol{R}_{B}$$

since the axes of frames {A} and {V} are parallel

$$
\begin{aligned}
{}^{A}\tilde{\boldsymbol{p}} &= \begin{pmatrix} {}^{A}\boldsymbol{R}_{B} & \boldsymbol{t} \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix} {}^{B}\tilde{\boldsymbol{p}} \\
&= {}^{A}\boldsymbol{T}_{B}\, {}^{B}\tilde{\boldsymbol{p}}
\end{aligned}
$$

A vector p= (x, y)

${}^{A}\boldsymbol{T}_{\boldsymbol{B}}$ is referred to as a homogeneous transformation

${}^{A}\boldsymbol{T}_{\boldsymbol{B}}$ represents translation and orientation or relative pose as a rigid-body motion

$$
\boldsymbol{T} = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\xi \text{ is } \xi \sim \boldsymbol{T} \in \mathbf{SE}(2) \text{ and } \boldsymbol{T}_1 \oplus \boldsymbol{T}_2 \mapsto \boldsymbol{T}_1\boldsymbol{T}_2
$$

$$
\boldsymbol{T}_1\boldsymbol{T}_2 = \begin{pmatrix} \boldsymbol{R}_1 & \boldsymbol{t}_1 \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{R}_2 & \boldsymbol{t}_2 \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_1\boldsymbol{R}_2 & \boldsymbol{t}_1 + \boldsymbol{R}_1\boldsymbol{t}_2 \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix}
$$

We know for matrices that $TT^{-1} = I$

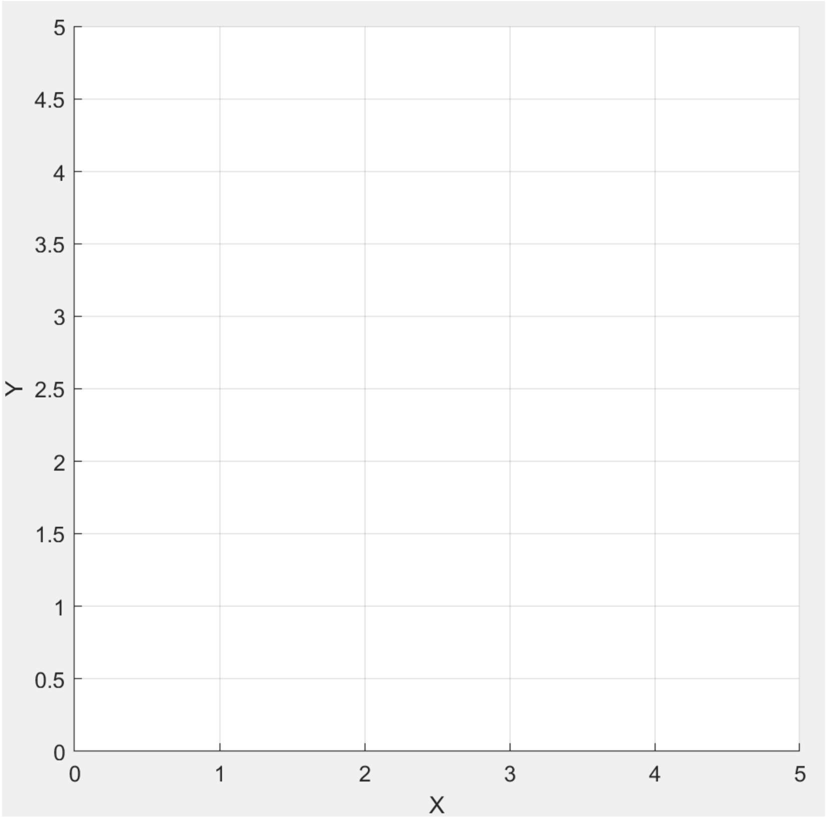
$$T^{-1} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{t} \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix}$$

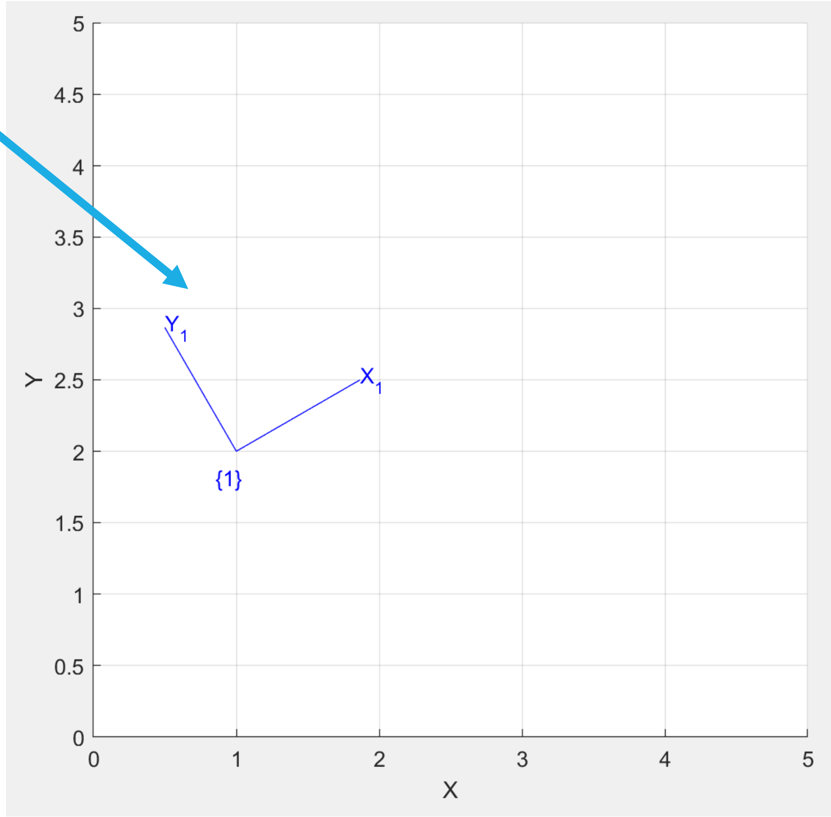
a homogeneous transformation which represents a translation of (1, 2) followed by a rotation of 30°

$$T = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}$$

```
>> T1 = transl2(1, 2) * trot2(30, 'deg')
T1 =
    0.8660   -0.5000    1.0000
    0.5000    0.8660    2.0000
         0         0    1.0000
```

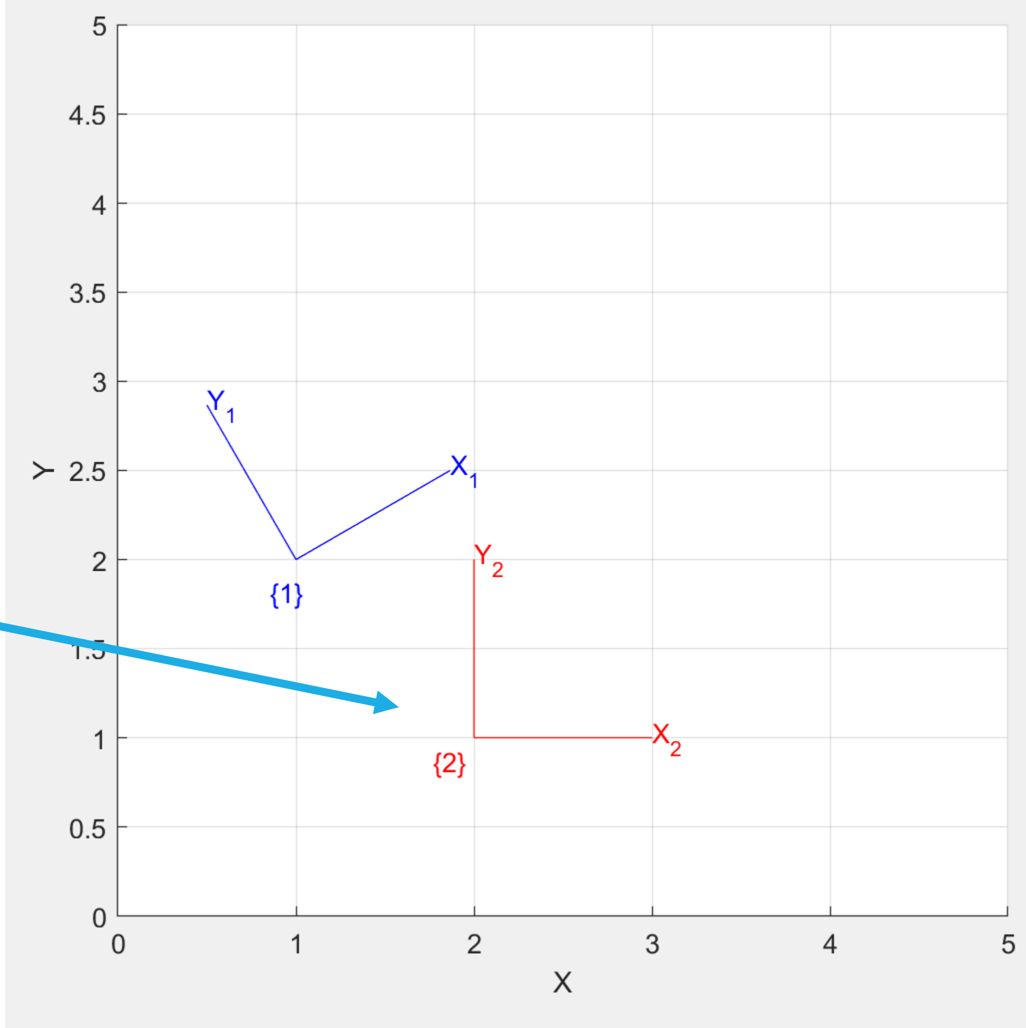
```
>> plotvol([0 5 0 5]);
>> trplot2(T1, 'frame', '1', 'color', 'b')
```





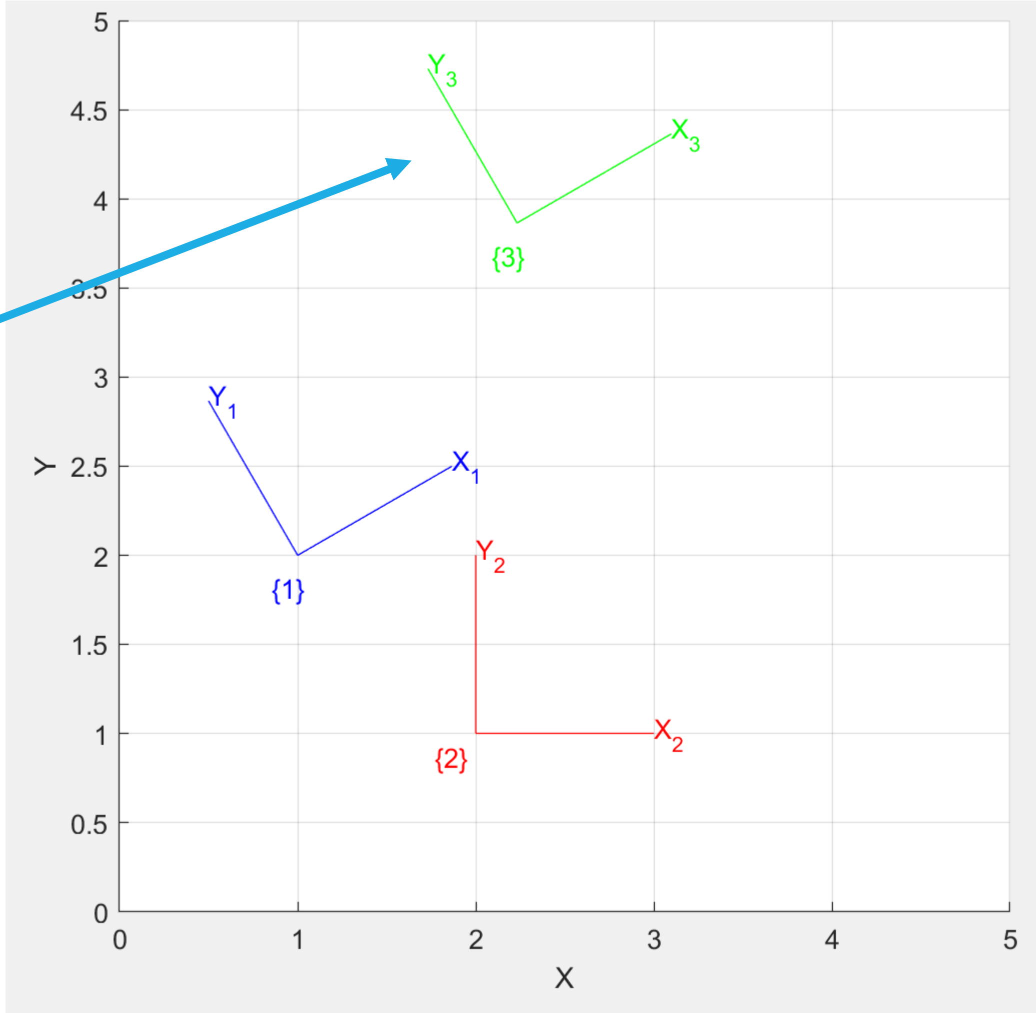
```
>> T2 = transl2(2, 1)
T2 =
     1     0     2
     0     1     1
     0     0     1
```

```
>> trplot2(T2, 'frame', '2', 'color', 'r');
```

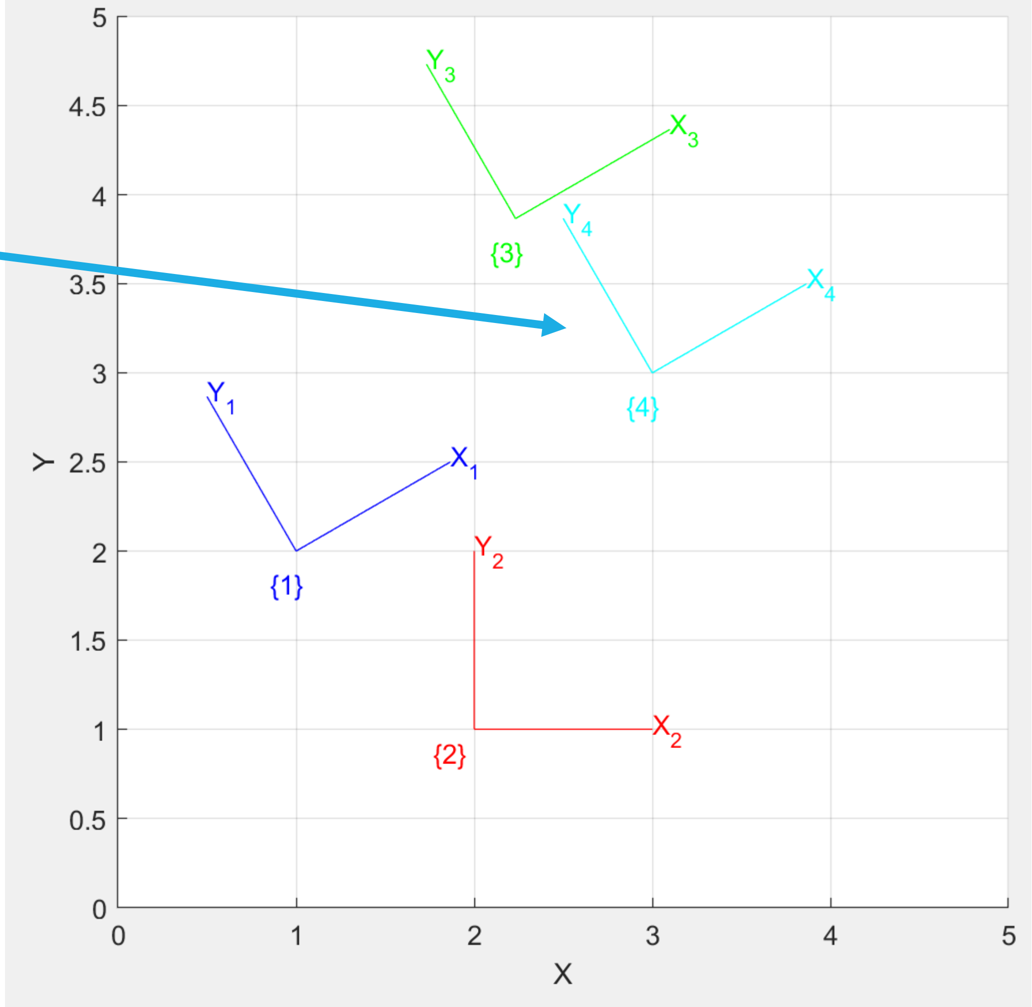


```
>> T3 = T1*T2
T3 =
      0.8660   -0.5000    2.2321
      0.5000    0.8660    3.8660
           0         0    1.0000
```

```
>> trplot2(T3, 'frame', '3', 'color', 'g');
```

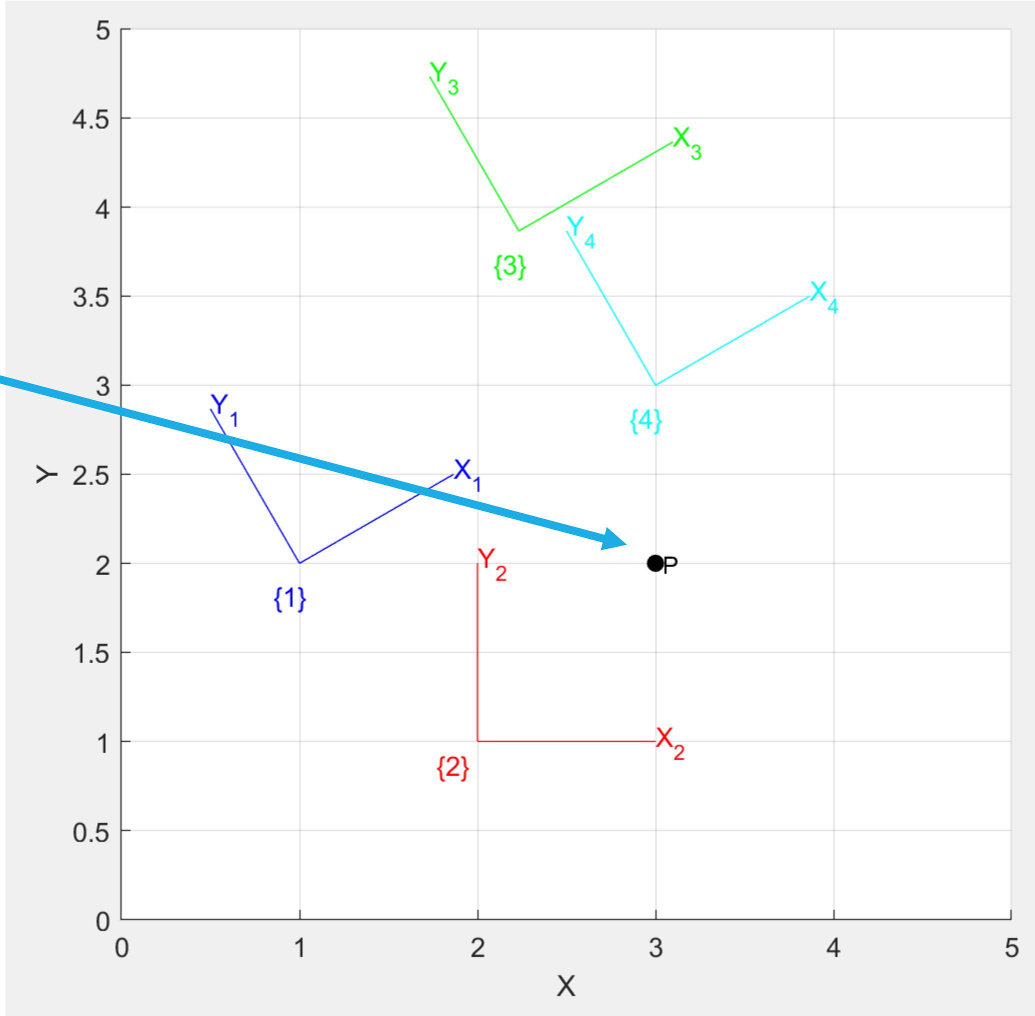


```
>> T4 = T2*T1;
>> trplot2(T4, 'frame', '4', 'color', 'c');
```



we define a point (3, 2) relative to the world frame

```
>> P = [3 ; 2 ];

>> plot_point(P, 'label', 'P', 'solid', 'ko');
```



To determine the coordinate of the point with respect to {1} $\quad {}^{0}\boldsymbol{p} = {}^{0}\xi_{1} \bullet {}^{1}\boldsymbol{p}$

and then rearrange as

$$
\begin{aligned}
{}^{1}\boldsymbol{p} &= {}^{1}\xi_{0} \bullet {}^{0}\boldsymbol{p} \\
&= \left({}^{0}\xi_{1}\right)^{-1} \bullet {}^{0}\boldsymbol{p}
\end{aligned}
$$

```
>> P1 = inv(T1) * [P; 1]
P1 =
    1.7321
   -1.0000
    1.0000
```

we first converted the Euclidean point coordinates to homogeneous form by appending a one. The result is also in homogeneous form and has a negative y-coordinate in frame {1}

```
>> h2e( inv(T1) * e2h(P) )
ans =
    1.7321
   -1.0000
```

the result is in Euclidean coordinates

function e2h converts Euclidean coordinates to homogeneous and h2e performs the inverse conversion.

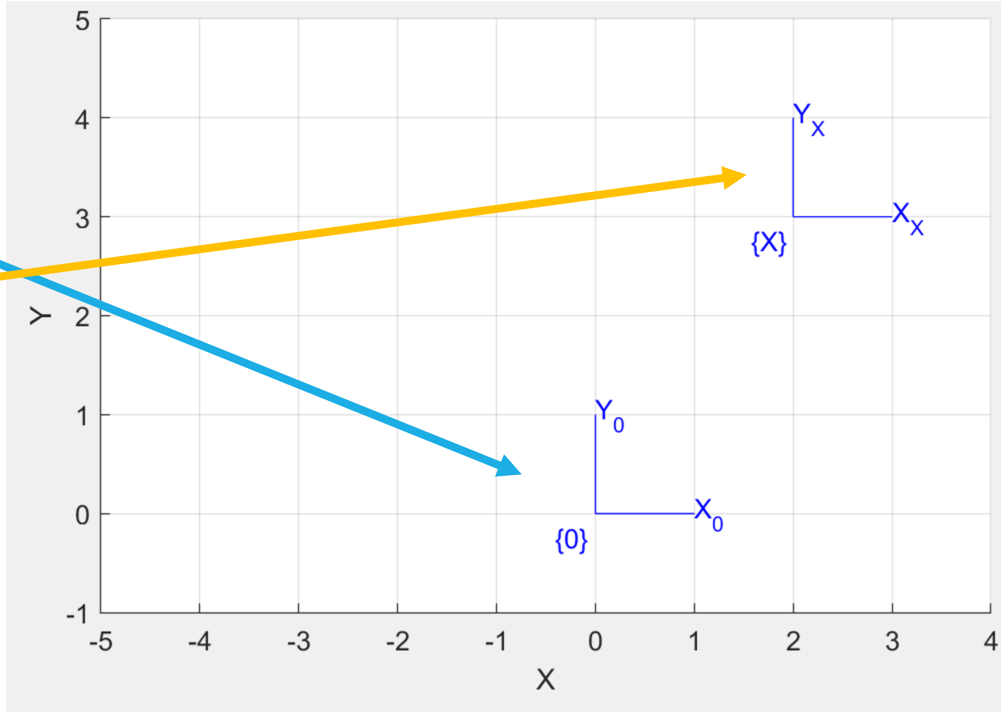
Centers of Rotation

we create and plot a reference coordinate frame {0} and a target frame {X}

```
>> plotvol([-5 4 -1 5]);
>> T0 = eye(3,3);
>> trplot2(T0, 'frame', '0');
>> X = transl2(2, 3);
>> trplot2(X, 'frame', 'X');
```

```
>> T0 = eye(3,3)

T0 =

     1     0     0
     0     1     0
     0     0     1
```

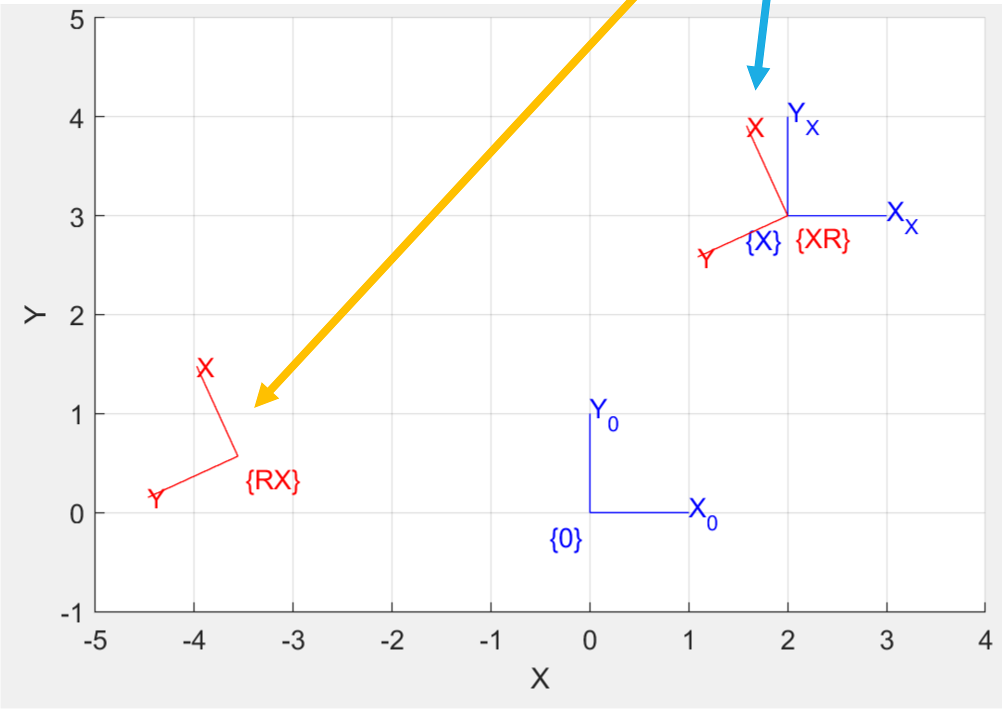
```
>> X = transl2(2, 3)

X =

     1     0     2
     0     1     3
     0     0     1
```



create a rotation of 2 radians (approximately 115°)

```
>> trplot2(R*X, 'framelabel', 'RX', 'color', 'r');
>> trplot2(X*R, 'framelabel', 'XR', 'color', 'r');
```

```
>> R = trot2(2);
>> R

R =

   -0.4161   -0.9093         0
    0.9093   -0.4161         0
         0         0    1.0000
```



if we wished to rotate a coordinate frame about an arbitrary point? First of all we will establish a new point C and display it

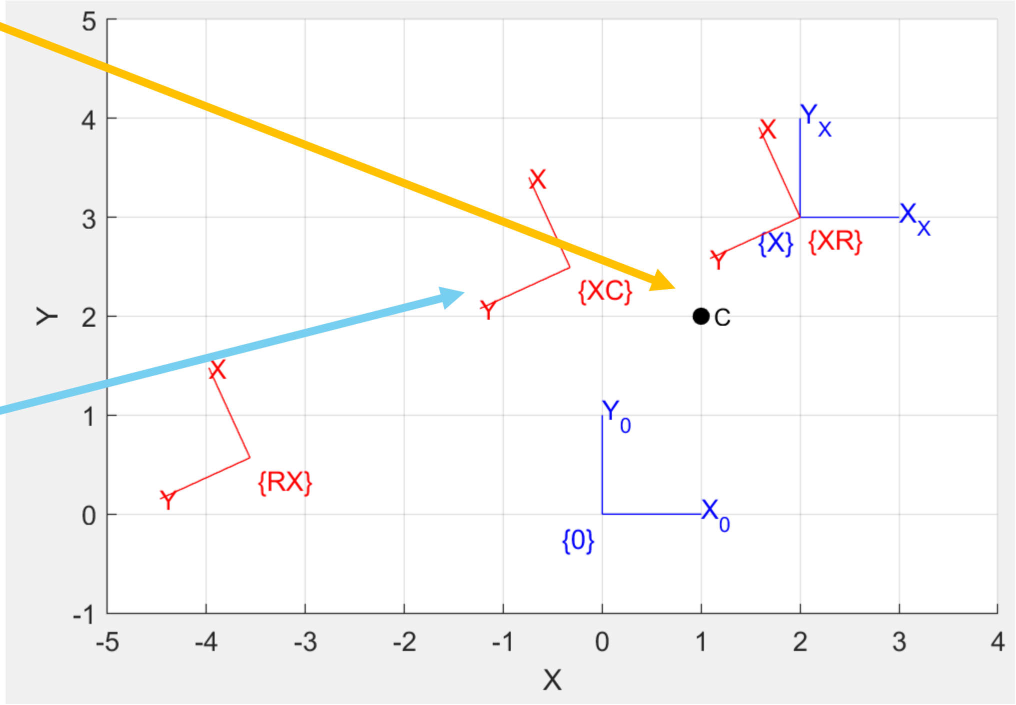
```
>> C = [1 2]';
>> plot_point(C, 'label', ' C', 'solid', 'ko')
```

```
>> C = [1 2]'

C =

     1
     2
```

compute a transform to rotate about point C

```
>> RC = transl2(C) * R * transl2(-C)
RC =
   -0.4161   -0.9093    3.2347
    0.9093   -0.4161    1.9230
         0         0    1.0000
```

```
>> trplot2(RC*X, 'framelabel', 'XC', 'color', 'r');
```



Twists in 2D

We can create a rotational twist a bout the point specified by the coordinate vector C

```
>> tw = Twist('R', C)

tw =
( 2  -1; 1 )
```

the result is a Twist object that encodes a twist vector with two components: a 2-vector moment and a 1-vector rotation

To create an SE(2) transformation for a rotation about this unit twist by 2 radians. we use the T method

```
>>  tw.T(2)

ans =

   -0.4161   -0.9093    3.2347
    0.9093   -0.4161    1.9230
         0         0    1.0000
```

```
>> tw.pole'

ans =

     1     2
```

the center of rotation

```
>> C = [1 2]';
```

```
>> RC = transl2(C) * R * transl2(-C)
RC =
   -0.4161   -0.9093    3.2347
    0.9093   -0.4161    1.9230
         0         0    1.0000
```

If we wish to perform translational motion i n the direction (1, 1)

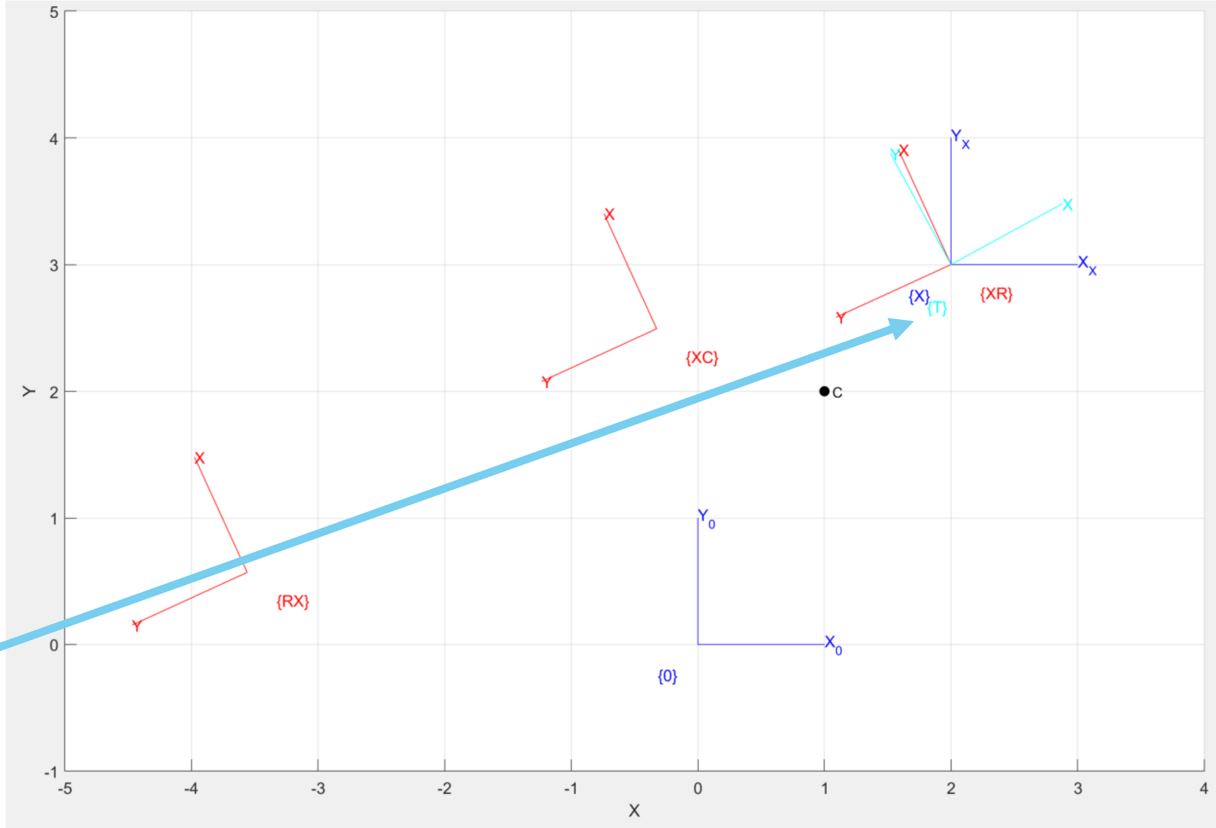
```
>> tw = Twist('T', [1 1])
tw =
( 0.70711 0.70711; 0 )
```

for a displacement of $\sqrt{2}$ in the direction defined by this twist the SE(2) transformation is

```
>> tw.T(sqrt(2))
ans =
     1     0     1
     0     1     1
     0     0     1
```

we see has a null rotation and a translation of 1 in the x- and y-directions.

```
>> T = transl2(2, 3) * trot2(0.5)
T =
    0.8776   -0.4794    2.0000
    0.4794    0.8776    3.0000
         0         0    1.0000
```

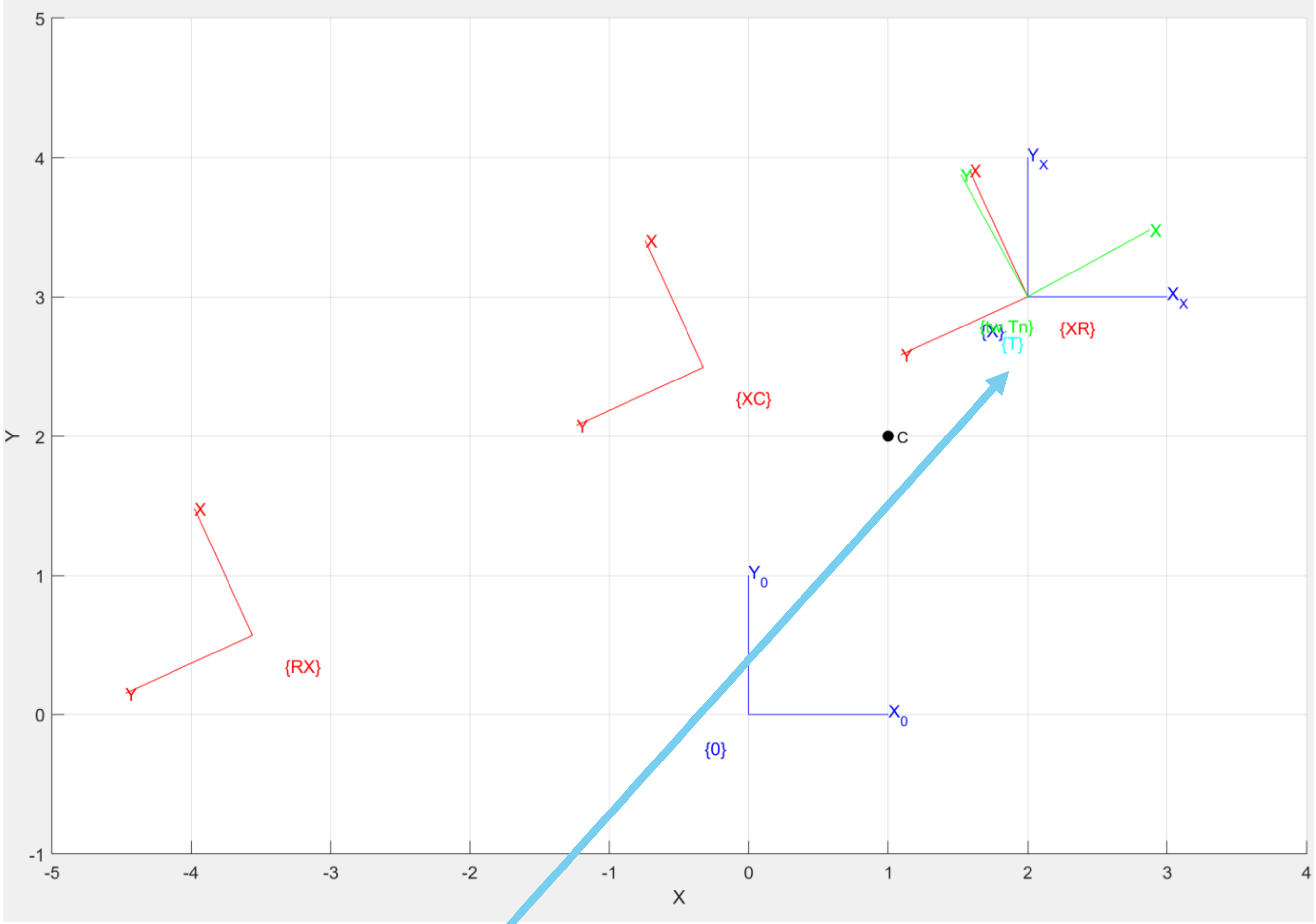


we can compute the twist vector

```
>> tw = Twist(T)
tw =
( 2.7082 2.4372; 0.5 )
```

the rotation, is not equal to one but is the required rotation angle of 0.5 radians. This is a non-unit twist. Therefore, when we convert this to an SE(2) transform w e don't need to provide a second argument since it is implicit in the twist

```
>>  tw.T
ans =
    0.8776   -0.4794    2.0000
    0.4794    0.8776    3.0000
         0         0    1.0000
```




```
>> trplot2(tw.T, 'framelabel', 'tw.Tn', 'color', 'g')
```

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