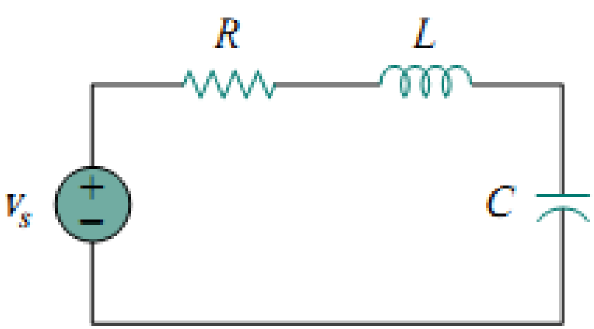
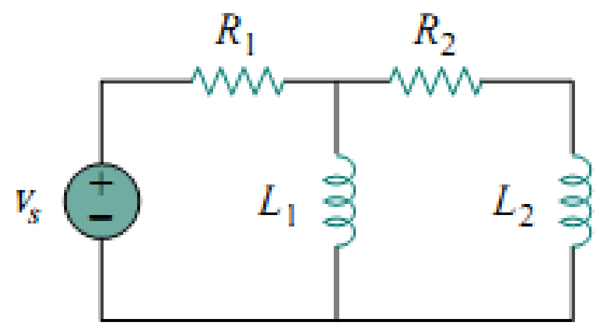
DC Circuits

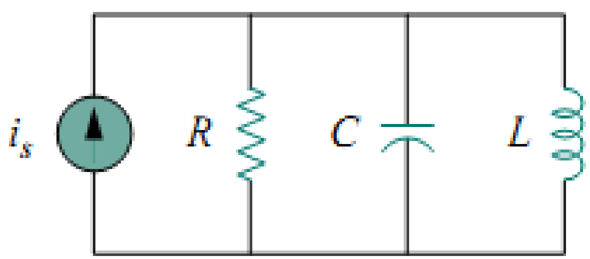
Second Order Circuits

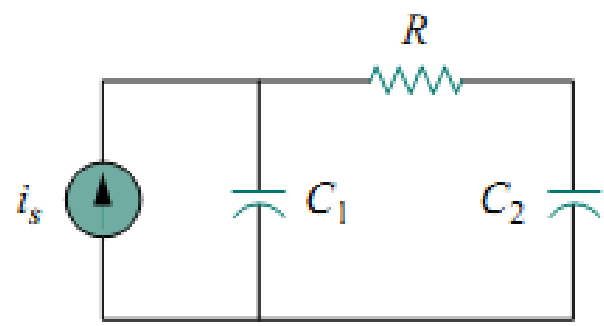
Introduction

A second-order circuit is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements.

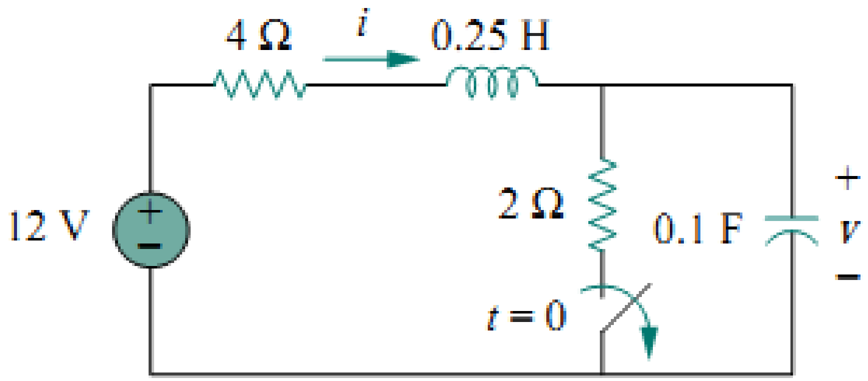








Finding Initial and Final Values



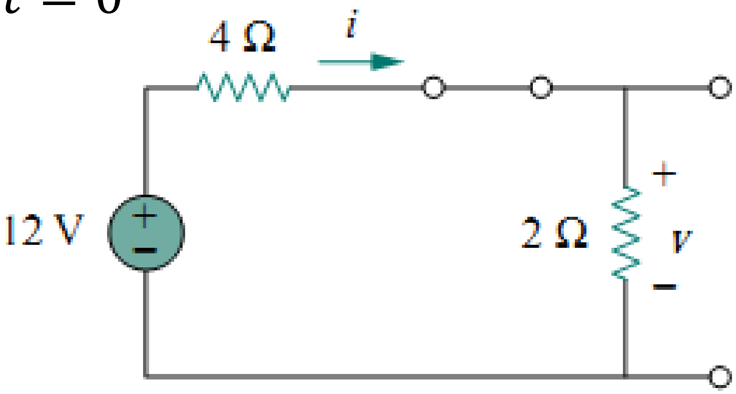
Find: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

The capacitor voltage is always continuous ➡ $v(0^+) = v(0^-)$

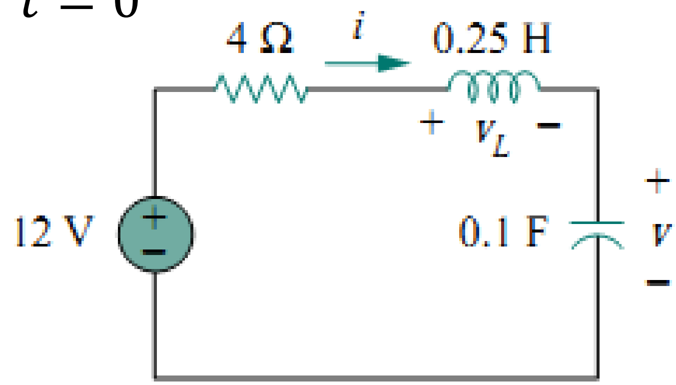
The inductor current is always continuous ➡ $i(0^+) = i(0^-)$

Equivalent circuit

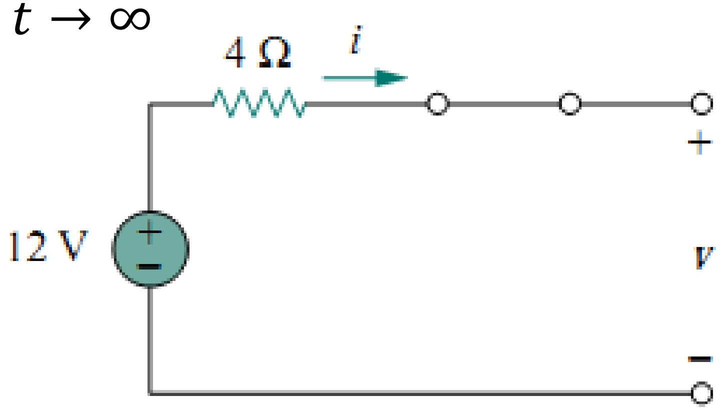
$t = 0^{-}$



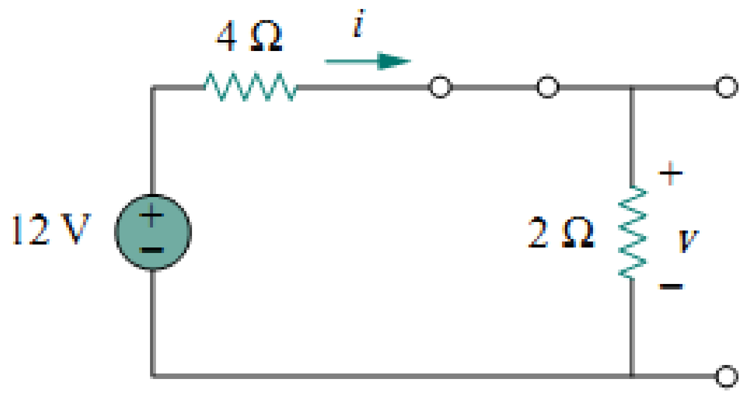
$t = 0^{+}$



$t \to \infty$



$t = 0^-$

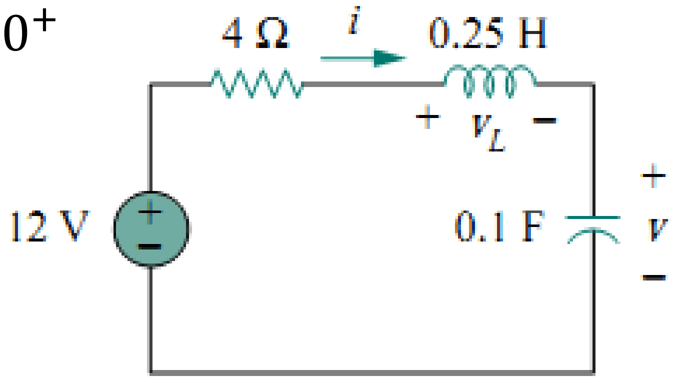


$$i(0^-) = \frac{12}{4+2} = 2\text{ A}, \qquad v(0^-) = 2i(0^-) = 4\text{ V}$$

As the inductor current and the capacitor voltage cannot change abruptly

$$i(0^+) = i(0^-) = 2\text{ A}, \qquad v(0^+) = v(0^-) = 4\text{ V}$$

$t = 0^+$



The same current flows through both the inductor and capacitor

$$i_C(0^+) = i(0^+) = 2\text{ A}$$

$C\, dv/dt = i_C,\ dv/dt = i_C/C,$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20\text{ V/s}$$

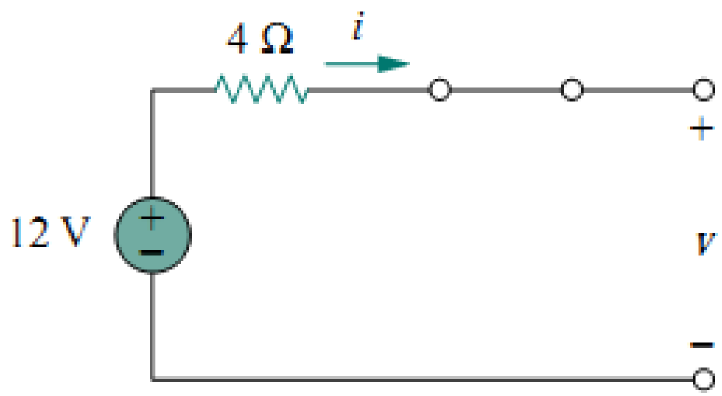
$L\, di/dt = v_L,\ di/dt = v_L/L$

$$-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0$$

$$v_L(0^+) = 12 - 8 - 4 = 0$$

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0\text{ A/s}$$

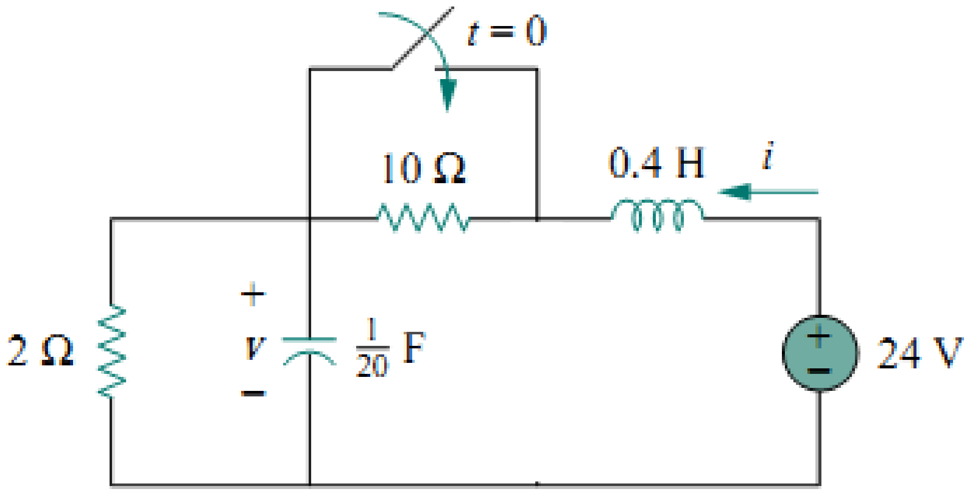
$t \to \infty$



$$i(\infty) = 0 \text{ A}, \qquad v(\infty) = 12 \text{ V}$$

Practice Problems

1

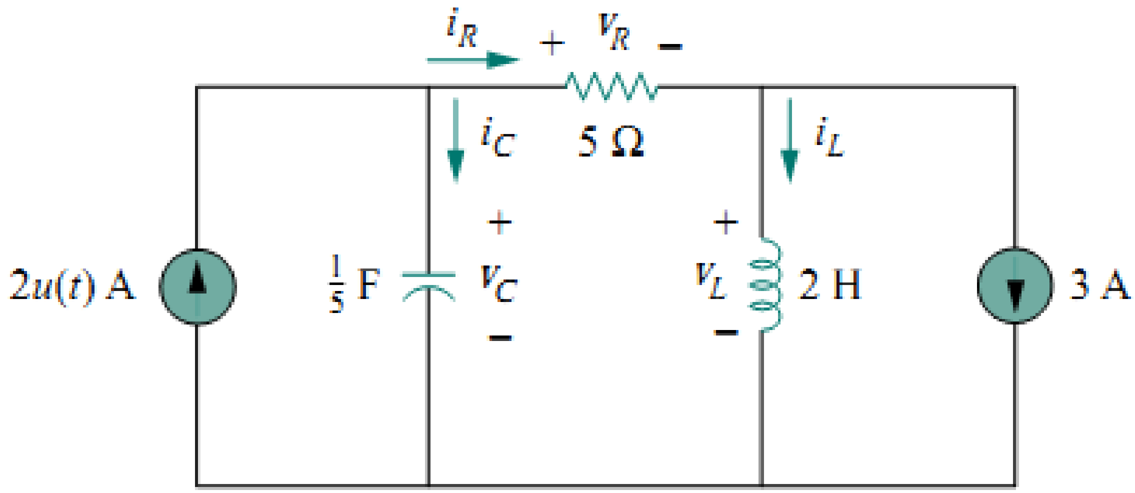


Determine:

(a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

Answer: (a) 2 A, 4 V, (b) 50 A/s, 0 V/s, (c) 12 A, 24 V.

2



Find: (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$,

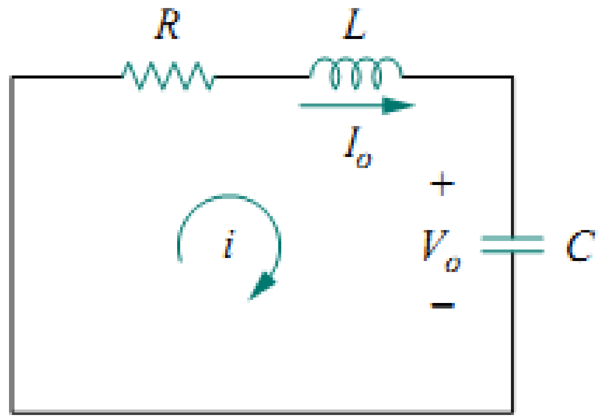
(b) $di_L(0^+)/dt, dv_C(0^+)/dt, dv_R(0^+)/dt,$

(c) $i_L(\infty), v_C(\infty), v_R(\infty)$.

Answer: (a) −3 A, 0, 0, (b) 0, 10 V/s, 0, (c) −1 A, 10 V, 10 V.

Source Free RLC Circuits

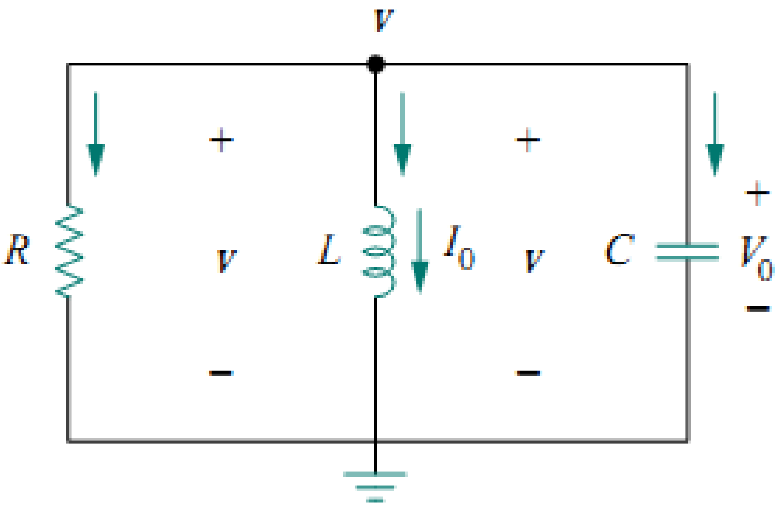
Series RLC Circuit



$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\, dt = 0$$

$$\boxed{\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0}$$

Parallel RLC Circuit



$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\, dt + C\frac{dv}{dt} = 0$$

$$\boxed{\frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0}$$

General Equation

$$\boxed{\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^{\,2}x = 0}$$

Series RLC Circuit

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Parallel RLC Circuit

$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

General Solution

Overdamped Case ($\alpha > \omega_0$)

$$\boxed{x = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

$$\frac{dx}{dt} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2}$$

Critically Damped Case ($\alpha = \omega_0$)

$$\boxed{x = e^{-\alpha t}\left(A_1 t + A_2\right)}$$

$$\frac{dx}{dt} = e^{-\alpha t}\left(A_1 - \alpha(A_1 t + A_2)\right)$$

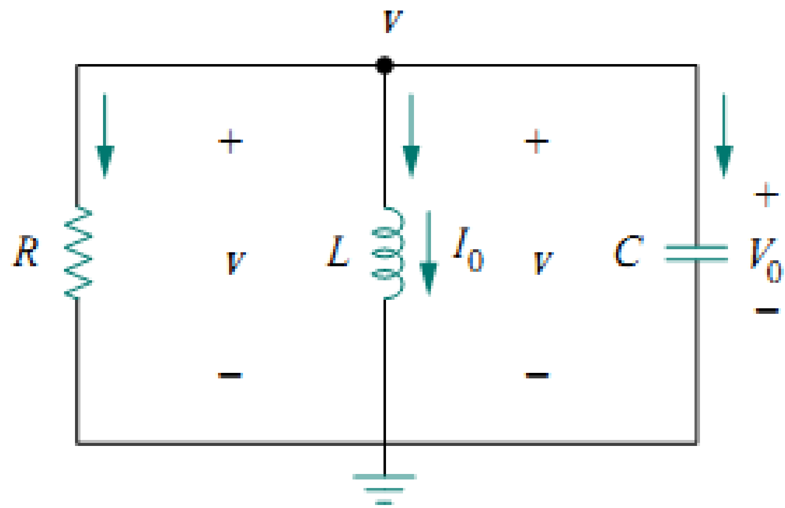
Underdamped Case ($\alpha < \omega_0$)

$$\boxed{x = e^{-\alpha t}\left(A_1 \cos \omega_d t + A_2 \sin \omega_d t\right)}$$

$$\frac{dx}{dt} = e^{-\alpha t}\left((A_2\omega_d - A_1\alpha)\cos\omega_d t - (A_1\omega_d + A_2\alpha)\sin\omega_d t\right)$$

$$\omega_d = \sqrt{\omega_0{}^2 - \alpha^2}$$

Example

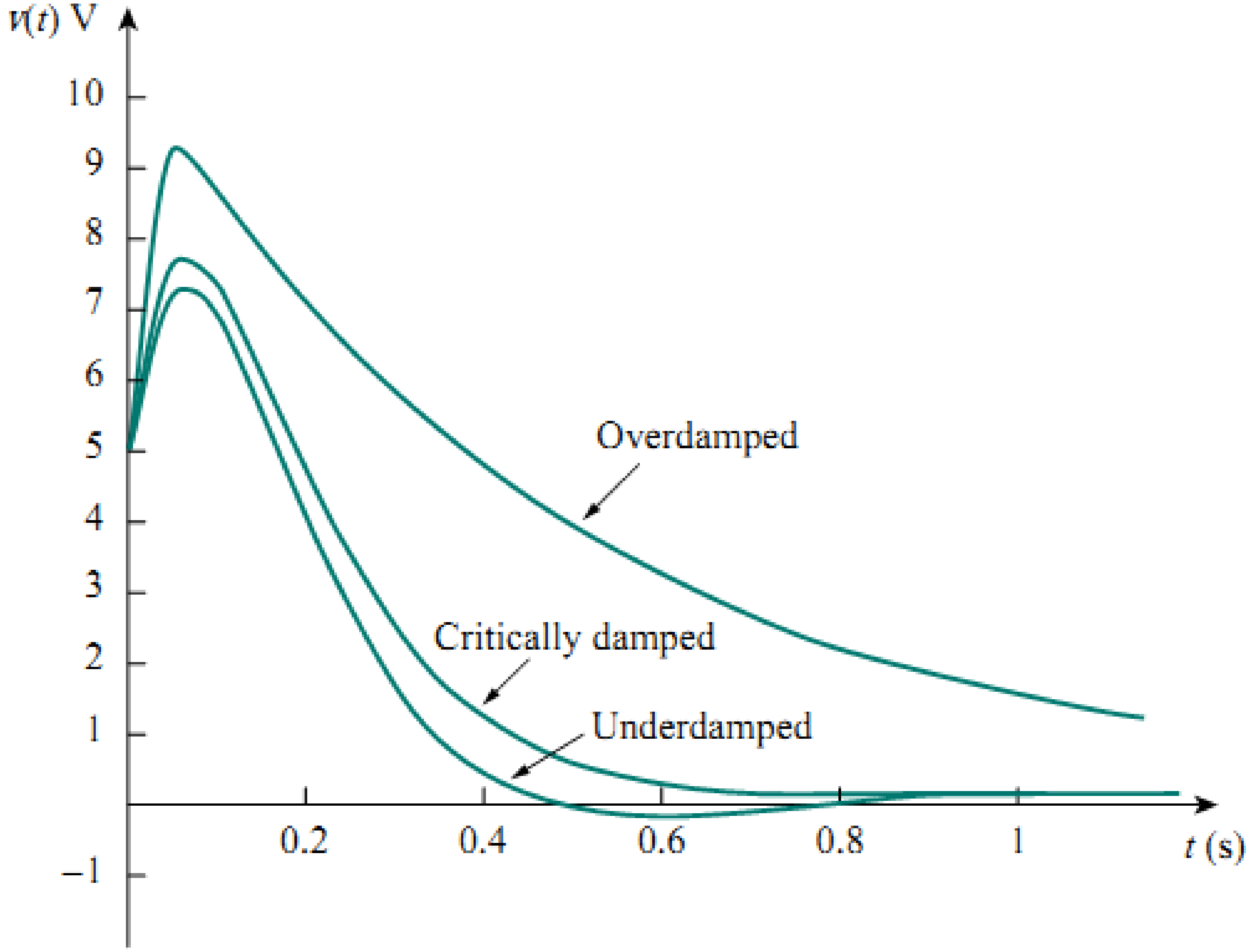


find $v(t)$ for $t > 0$, assuming $v(0) = 5$ V, $i(0) = 0$, $L = 1$ H, and $C = 10$ mF.
Consider these cases: $R = 1.923\ \Omega$, $R = 5\ \Omega$, and $R = 6.25\ \Omega$.

CASE 1 If $R = 1.923\ \Omega$, $v(t) = 10.625e^{-2t} - 5.625e^{-50t}$

CASE 2 When $R = 5\ \Omega$, $v(t) = (5 + 150t)e^{-10t}$ V

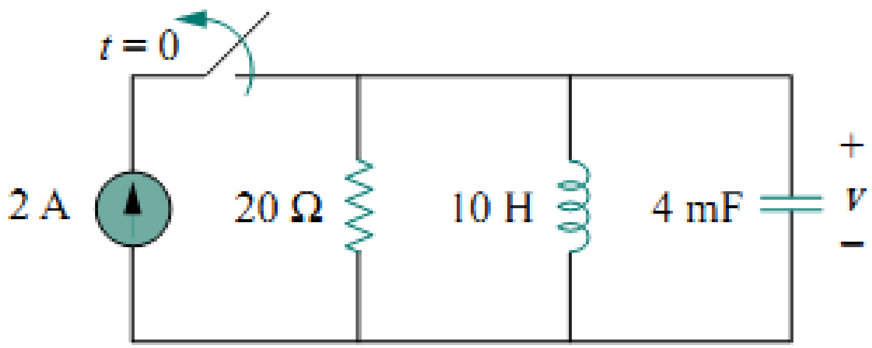
CASE 3 When $R = 6.25\ \Omega$, $v(t) = (5\cos 6t + 20\sin 6t)e^{-8t}$



v(t) V
10
9
8
7
6
5
4
3
2
1
−1
0.2
0.4
0.6
0.8
1
t (s)
Overdamped
Critically damped
Underdamped

Practice Problem

Find $v(t)$ for $t > 0$.



Answer: $66.67(e^{-10t} - e^{-2.5t})$ V.

Step Response of RLC Circuits

The complete response of the circuit is the sum of the natural response and the forced response.

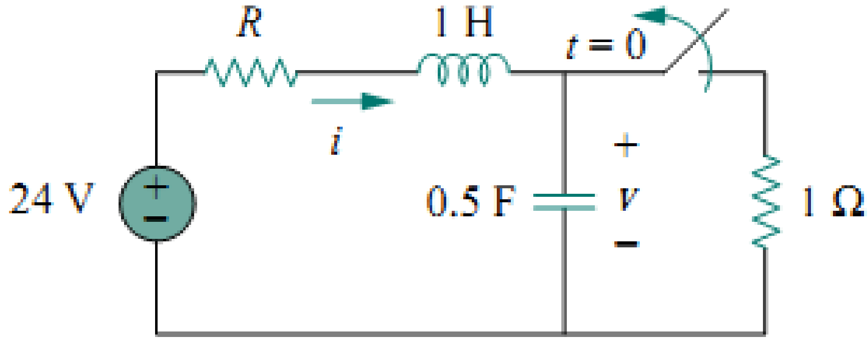
$$\boxed{x(t) = x_f(t) + x_n(t)}$$

the $x_f = x(\infty)$ is the final value and $x_n(t)$ is the natural response.

The natural response or transient response is the circuit's temporary response that will die out with time.

The forced response or steady-state response is the behavior of the circuit a long time after an external excitation is applied.

Example 1



find $v(t)$ and $i(t)$ for $t > 0$. Consider these cases: $R = 5\ \Omega$, $R = 4\ \Omega$, and $R = 1\ \Omega$.

CASE 1 When $R = 5\ \Omega$

$$v(t) = 24 + \frac{4}{3}(-16e^{-t} + e^{-4t})\ \text{V}$$

$$i(t) = \frac{4}{3}(4e^{-t} - e^{-4t})\ \text{A}$$

CASE 2 When $R = 4\ \Omega$

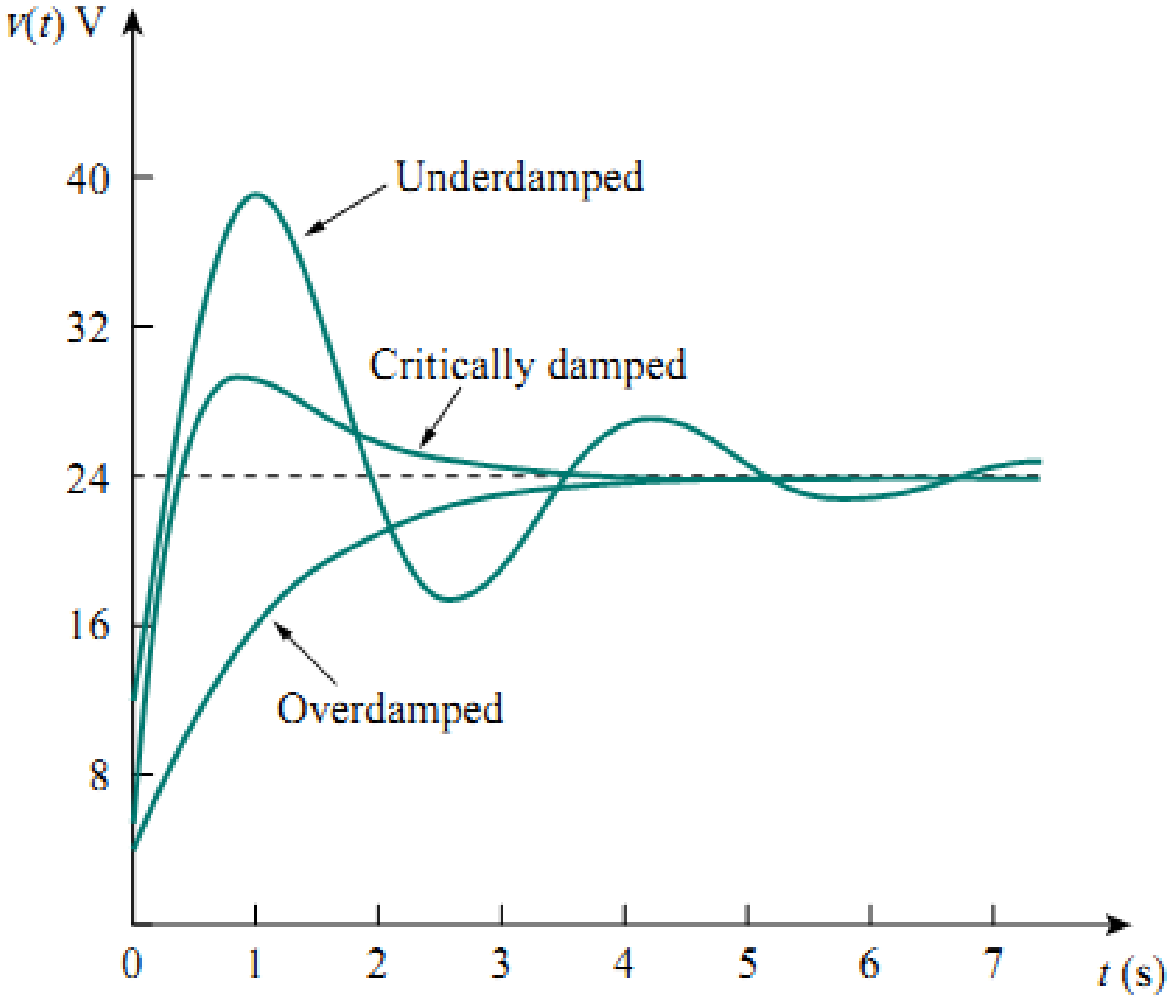
$$v(t) = 24 + (-19.5 + 57t)e^{-2t}\ \text{V}$$

$$i(t) = (4.5 - 28.5t)e^{-2t}\ \text{A}$$

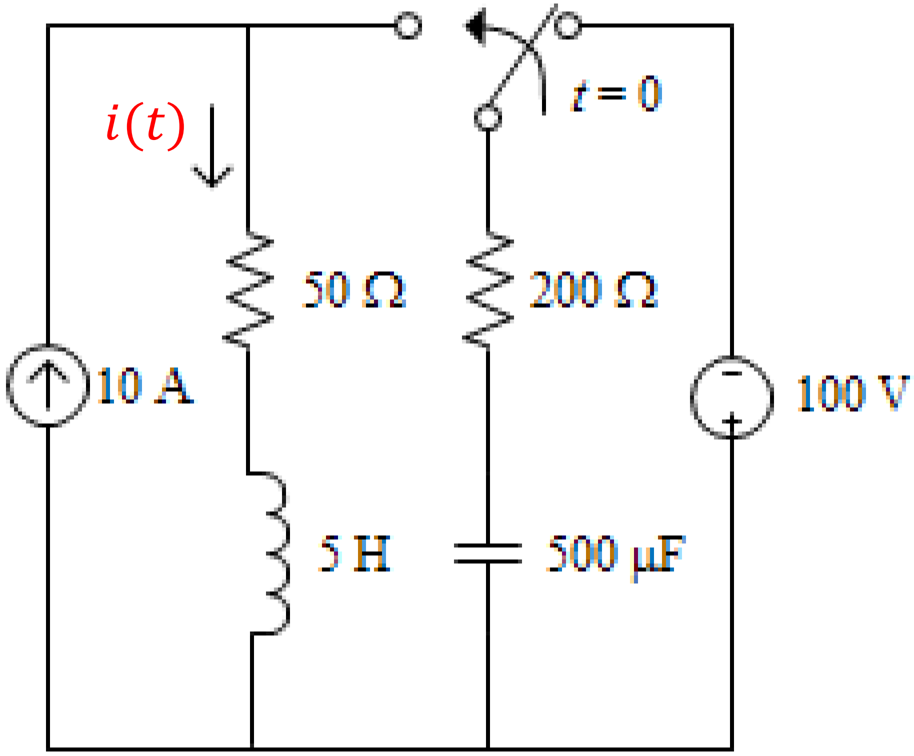
CASE 3 When $R = 1\ \Omega$

$$v(t) = 24 + (21.694 \sin 1.936t - 12 \cos 1.936t)e^{-0.5t}\ \text{V}$$

$$i(t) = (3.1 \sin 1.936t + 12 \cos 1.936t)e^{-0.5t}\ \text{A}$$

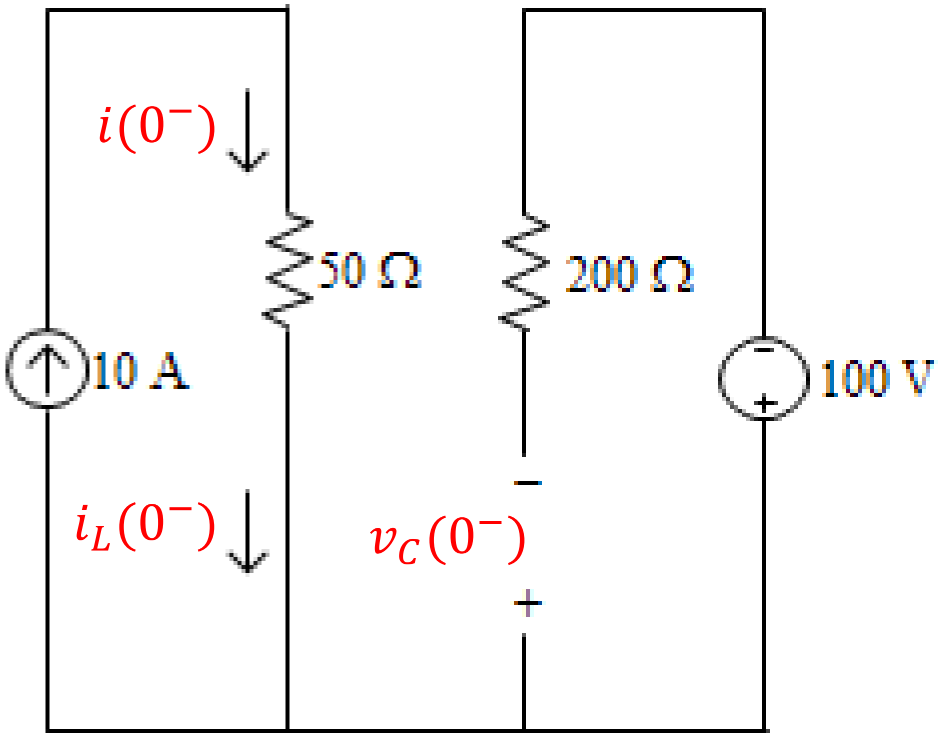

v(t) V
40
32
24
16
8
0
1
2
3
4
5
6
7
t (s)
Underdamped
Critically damped
Overdamped

Example 2



Find $i(t)$ for $-\infty < t < \infty$

$t = 0^-$

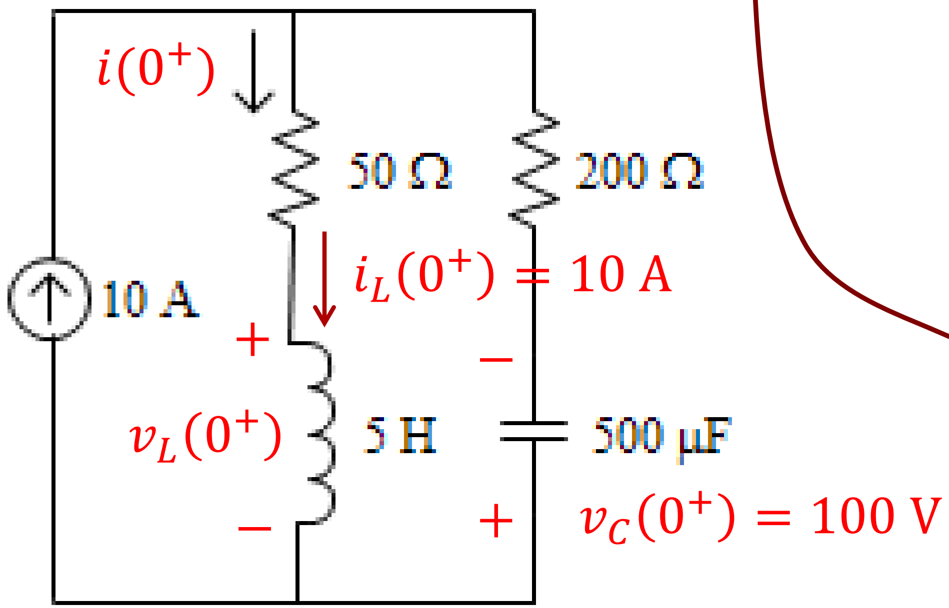


$i(0^-) = 10\ \text{A}$

$i_L(0^-) = i(0^-) = 10\ \text{A}$

$v_C(0^-) = 100\ \text{V}$

$t = 0^+$

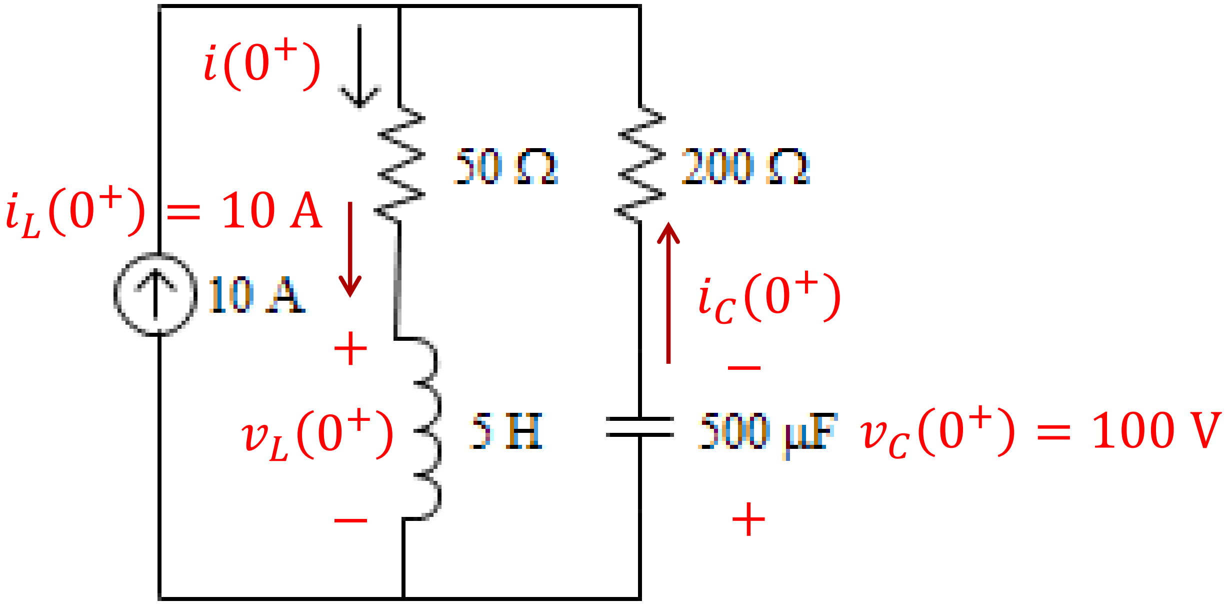


$$i_L(0^+) = i_L(0^-) = 10 \text{ A}$$

$$v_C(0^+) = v_C(0^-) = 100 \text{ V}$$

$$\boxed{i(0^+) = i_L(0^+) = 10 \text{ A}}$$

$$\frac{di(0^+)}{dt} = \frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$



$$50\, i(0^+) + v_L(0^+) + v_C(0^+) + 200\, i_C(0^+) = 0$$

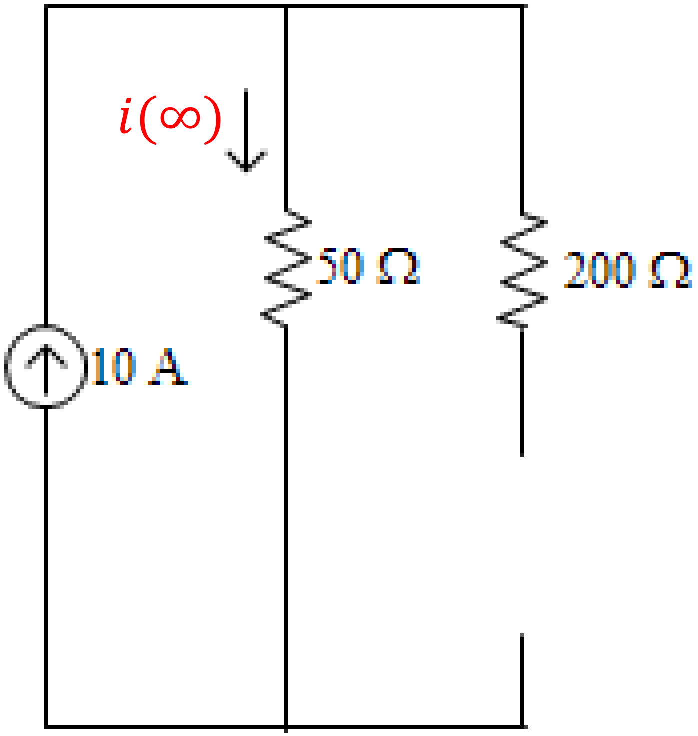
$$50\,(10) + v_L(0^+) + 100 + 200\,(0) = 0$$

$$v_L(0^+) = -600 \text{ V}$$

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{-600}{5}$$

$$\boxed{\frac{di(0^+)}{dt} = -120 \text{ A/s}}$$

$t \rightarrow \infty$



$$i(\infty) = 10\ \text{A}$$

The natural response

$$\alpha = \frac{R}{2L} = \frac{(200 + 50)}{2\,(5)} = 25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{5(500 \cdot 10^{-6})}} = \frac{100}{5} = 20$$

$$\alpha > \omega_0 \quad \Rightarrow \quad \text{Overdamped}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -25 \pm \sqrt{625 - 400} = -25 \pm 15$$

$$s_1 = -10 \qquad s_2 = -40$$

$$\boxed{i_n(t) = A_1 e^{-10t} + A_2 e^{-40t}\ \text{A}}$$

The forced response

$$\boxed{i_f(t) = i(\infty) = 10\ \text{A}}$$

The complete response

$$i(t) = i_f(t) + i_n(t)$$

$$\boxed{i(t) = 10 + A_1 e^{-10t} + A_2 e^{-40t}\ \mathrm{A}}$$

$$\frac{di(t)}{dt} = -10\,A_1 e^{-10t} - 40\,A_2 e^{-40t}\ \mathrm{A/s}$$

$t = 0^+$

$$\left.\begin{aligned} i(0^+) &= 10 + A_1 + A_2 \\ 10 &= 10 + A_1 + A_2 \end{aligned}\right\} \quad A_1 + A_2 = 0 \qquad (1)$$

$$\left.\begin{aligned} \frac{di(0^+)}{dt} &= -10\,A_1 - 40\,A_2 \\ -120 &= -10\,A_1 - 40\,A_2 \end{aligned}\right\} \quad A_1 + 4\,A_2 = 12 \qquad (2)$$

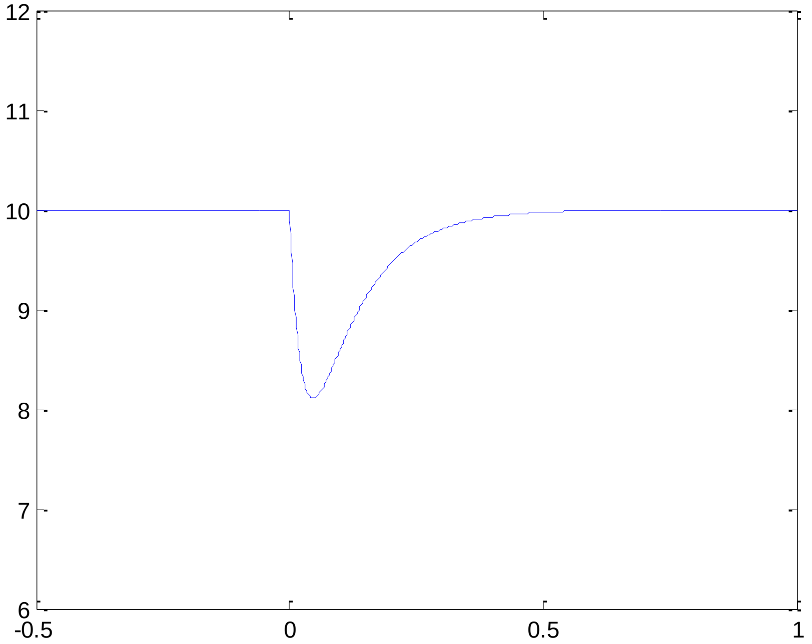
From equation (1) and (2) : $A_1 = -4 \quad A_2 = 4$

The complete response

$$i(t) = 10 - 4\, e^{-10t} + 4\, e^{-40t}\ \mathrm{A} \quad (t > 0)$$

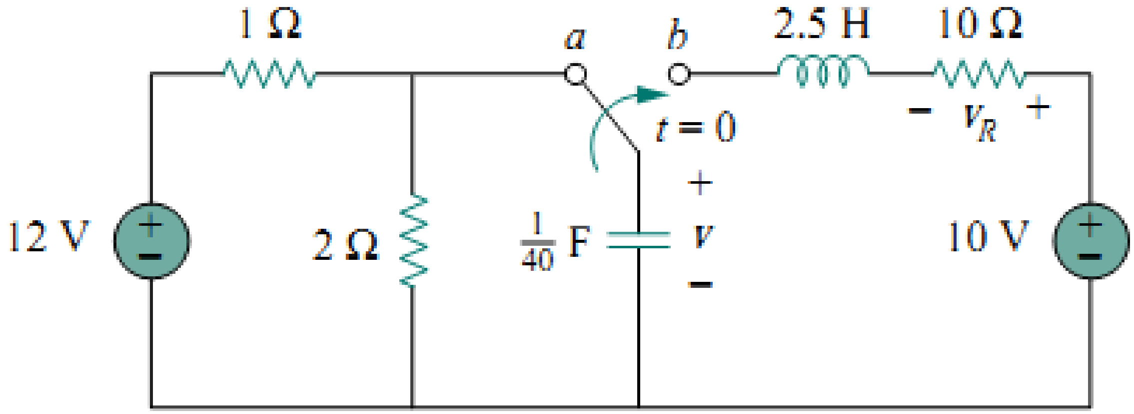
The complete response for $-\infty < t < \infty$

$$i(t) = \begin{cases} 10\ \mathrm{A} & t < 0 \\ 10 - 4\, e^{-10t} + 4\, e^{-40t}\ \mathrm{A} & t > 0 \end{cases}$$



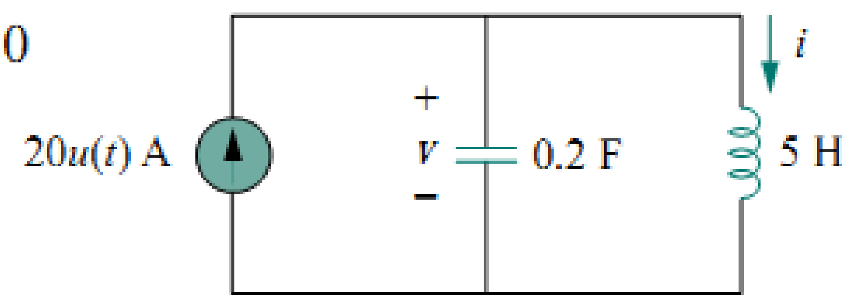
Practice Problem

1 Find $v(t)$ and $v_R(t)$ for $t > 0$.



Answer: $10 - (1.1547 \sin 3.464t + 2 \cos 3.464t)e^{-2t}$ V, $2.31e^{-2t} \sin 3.464t$ V.

2 Find $i(t)$ and $v(t)$ for $t > 0$



Answer: $20(1 - \cos t)$ A, $100 \sin t$ V.