Math 297 Discussion 12 Notes

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Complex-valued functions

Note: identifying $\mathbb{C}$ with $\mathbb{R}^2$, the theory of limit, convergence, and continuity is analogous to $\mathbb{R}^2$.

Differentiation and integration

A complex-valued function $f(z)$ is **differentiable** at z_0 if the difference quotients

$$\frac{f(z) - f(z_0)}{z - z_0}$$

have a limit as $z \to z_0$. The limit is denoted by $f'(z_0)$, and we refer to it as the **complex derivative** of $f(z)$ at z_0.
Note: As you can verify, the complex derivative satisfies the usual rules for differentiating sums, products, and quotients. The chain rule is proved in a similar fashion.

Analytic functions

Definition 0.1. *A function $f(z)$ is* ***holomorphic (or, analytic)*** *on an open set U if $f(z)$ is differentiable at each point of U and the complex derivative $f'(z)$ is continuous on U.*

Examples of analytic functions (where defined):
(1) $f(z) = c$, where c is any complex constant
(2) $f(z) = z = x + iy$
(3) $f(z) = z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$
(4) $f(z) = e^z = e^x \cos y + ie^x \sin y$
These can all be checked by direct verification using the definition.

Non-examples:
(1) $f(z) = \bar{z}$
(2) $f(z) = \text{Re } z$
(3) $f(z) = \text{Im } z$
Exercise: Show (2)(3) above are not analytic.

Cauchy-Riemann equations

Suppose $f = u + iv$ is analytic on a domain D. Fix a point $x \in D$. We compute the complex derivative

$$f'(z) = \lim_{\Delta \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

in two ways: letting $\Delta z = \Delta x$ real, and $\Delta z = i\Delta y$ imaginary.

$$\begin{aligned}\frac{f(z+\Delta z)-f(z)}{\Delta z} &= \frac{u(x+\Delta x,y)+iv(x+\Delta x,y)-u(x,y)-iv(x,y)}{\Delta x}\\ &= \frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}+i\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}\end{aligned} \tag{1}$$

Taking the limit, $f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) \quad (2)$, $z = x+iy$
Note that the x-derivatives of u, v are continuous.
Doing the same thing for $\delta z = i\Delta y$, we have

$$f'(z) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y), \quad z = x+iy.$$

Now the two expressions have to equal. So we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are called the **Cauchy-Riemann equations** for u and v.

Theorem. *Let $f = u+iv$ be defined on a domain $D \subset \mathbb{C}$, where u, v are real-valued. Then $f(z)$ is analytic on D if and only if $u(x,y)$ and $v(x,y)$ have continuous first-order partial derivatives that satisfy the Cauchy-Riemann equations.*

Proof. We've shown the forward direction above. It remains to show that if the partial derivatives of u, v exist, are continuous, and satisfy the CR equations, then $f = u+iv$ is analytic. We use Taylor's theorem. Fix $z \in D$.

$$u(x+\Delta x, y+\Delta y) = u(x,y) + \frac{\partial u}{\partial x}(x,y)\Delta x + \frac{\partial u}{\partial y}(x,y)\Delta y + R(\Delta x, \Delta y),$$

where $R(\Delta x, \Delta y)/|\Delta z|$ goes to 0 as Δz approaches 0.
Similarly,

$$v(x+\Delta x, y+\Delta y) = v(x,y) + \frac{\partial v}{\partial x}(x,y)\Delta x + \frac{\partial v}{\partial y}(x,y)\Delta y + S(\Delta x, \Delta y),$$

where $S(\Delta x, \Delta y)/|\Delta z| \to 0$ as $\Delta z \to 0$. Thus

$$f(z+\Delta z) = f(z) + \frac{\partial u}{\partial x}(x,y)\Delta x + \frac{\partial u}{\partial y}(x,y)\Delta y + R(\Delta z) + i\frac{\partial v}{\partial x}(x,y)\Delta x + i\frac{\partial v}{\partial y}(x,y)\Delta y + iS(\Delta z)$$

Using the CR equations to replace the y-derivatives by x-derivatives, the identity becomes

$$f(z+\Delta z) = f(z) + (\frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y))\Delta z + R(\Delta z) + iS(\Delta z).$$

Thus

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) + \frac{R(\Delta z)+iS(\Delta z)}{\Delta z},$$

which tends to

$$\frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)$$

as Δz tends to 0. Thus $f'(z)$ exists and is given by (2), so $f'(z)$ is continuous, thus $f(z)$ is analytic. $\square$

Examples

(1) The functions $u(x, y) = x$ and $v(x, y) = y$ satisfy the CR equations
(2) The functions $u(x, y) = x$ and $v(x, y) = -y$ do not satisfy the CR equations
(3) The function e^z is analytic and satisfies $\frac{d}{dz}e^z = e^z$
Exercise: prove the above.
Note: From the chain rule, e^{az}, where a is a complex constant, is analytic and

$$\frac{d}{dz}e^{az} = ae^{az}$$

Note: Linear combinations of complex exponential functions are also analytic, and the usual formulae for the derivatives hold:

$$\frac{d}{dz}\sin z = \cos z$$
$$\frac{d}{dz}\cos z = -\sin z$$
$$\frac{d}{dz}\sinh z = \cosh z$$
$$\frac{d}{dz}\cosh z = \sinh z$$

Exercise: verify the above.
For instance,

$$\frac{d}{dz}\sin z = \frac{d}{dz}\frac{e^{iz} - e^{-iz}}{2i} = \frac{ie^{iz} + ie^{-iz}}{2i} = \cos z$$

Two important theorems that I believe you are able to prove:

Theorem. *If $f(z)$ is analytic on a domain D, and if $f'(z) = 0$ on D, then $f(z)$ is constant.*

Theorem. *If $f(z)$ is analytic and real-valued on a domain D, then $f(z)$ is constant.*

More exercises

1. Find the derivatives of the following functions.
(a) $\tan z = \frac{\sin z}{\cos z}$ (b) $\tanh z = \frac{\sinh z}{\cosh z}$ (c) $\sec z = 1/\cos z$
2. Show that if f and $\bar{f}$ are both analytic on a domain D, then f is constant.
3. Show that if f is analytic on a domain D, and if $|f|$ is constant, the f is constant. *Hint.* Write $\bar{f} = |f|^2/f$
4. Show that $u = \sin x \sinh y$ and $v = \cos x \cosh y$ satisfy the CR equations. Do you recognize the analytic function $f = u + iv$?