Math 297 Discussion 3*

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1 Important Definitions

Recall our definition of metric space.

Definition 1.1. *A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:*

1. *$\emptyset$ and X are in $\mathcal{T}$.*
2. *The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.*
3. *The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.*

A set X for which a topology $\mathcal{T}$ has been specified is called a topological space.

2 Topological Properties

1) Fine and coarse (relative)

2) Continuity of function in topological spaces

Definition 2.1. *Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.*

Note: If the topology of the range space Y is given by a basis $\mathcal{B}$, then to prove continuity of f it suffices to show that the inverse image of every *basis element* is open. Why?
If the topology on Y is given by a subbasis $\mathcal{S}$, to prove continuity of f it will even suffice to show that the inverse image of each *subbasis* element is open. Why?

Exercises

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

2. Consider the nine topologies on the board. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is finer.

3. Show that the collection $\mathcal{T}_c$ of all subsets U of X such that $X \setminus U$ either is countable or is all of X. Show that $\mathcal{T}_c$ is a topology on the set X. Is the collection $\mathcal{T}_\infty = \{U | X - U \text{ is infinite or empty or all of } X\}$ a topology on X?

4. (a) If $\mathcal{T}_\alpha$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?
(b) If $X = \{a, b, c\}$, let $\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$, and $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$. Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

*Reference: Walter Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, NY, 2003.