Math 297 Discussion 4*

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Recall our definition of topological spaces.

1 Topological Properties

1) Hausdorff space
Let $(X, \mathcal{T})$ be a topological space, we say X is Hausdorff if $\forall x, y \in X$ with $x \neq y$, there are disjoint open sets $U_x, V_y \in T$ such that $x \in U_x$ and $y \in V_y$.
Note: A metric space is automatically Hausdorff. (Why?)

Theorem 1. *X is Hausdorff if and only if $\Delta = \{(x, x) | x \in X\} \subset X \times X$ is closed in $X \times X$ in the product topology. (Proved later)*

2) Continuity of functions in topological spaces

Definition 1.1. *Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.*

Note: What if we change 'open' to 'closed' in the definition?
If the topology of the range space Y is given by a basis $\mathbb{B}$, then to prove continuity of f it suffices to show that the inverse image of every *basis element* is open. Why?
If the topology on Y is given by a subbasis $\mathbb{S}$, to prove continuity of f it will even suffice to show that the inverse image of each *subbasis* element is open. Why?

3) Convergence of sequences

Definition 1.2. *Let $(X, \mathcal{T})$ be a topological space and $x_n \in X$ a sequence. We say that the sequence x_n converges to $x_0 \in X$ if for every open set $U \subset X$ which contains x_0 there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the points x_n lie in U.*

4) Subspace topology

Definition 1.3. *Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection $\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$ is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X; its open sets consist of all intersections of open sets of X with Y.*

*Reference: James Munkres. *Topology.* Prentice Hall, NJ, 2000.