Math 297 Discussion 5 Notes

Annie Xu

Feburary 5, 2019

Exercises

1. **Topological closure.** Suppose $(X, \mathcal{T})$ is a topological space an $A \subset X$. The topological closure of A, denoted $\bar{A}$, is defined to be the intersection of all closed subsets of X that contains A.
(a) Show that $\bar{A}$ is the smallest closed subset of X that contains A.
$A \subset \bar{A}$ is trivial. To show $\bar{A}$ is closed, use De Morgan's law to show its complement is open. To show it is the smallest such set, suppose $C \subset X$ is closed with $A \subset C$, and prove that $\bar{A} \subset C$.
(b) Show that if $A \subset B \subset X$, then $\bar{A} \subset \bar{B}$.
$A \subset \bar{B}$, $\bar{B}$ is closed $\implies$ $\bar{A} \subset \bar{B}$, by (a).
(c) Show that $x \in \bar{A}$ if and only if for all $U \in \mathcal{T}$ that contain x we have $U \cap A \neq \emptyset$.

2. **The interior of a set.** Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$. The *interior* of A, denoted $\int(A)$, is defined to be the union of all open sets (open in X) contained in A. Show that $\int(A)$ is the largest open subset of X contained in A.
(a) Consider $\mathbb{R}$ with the Euclidean topology. What is the topological closure of $A = (0, 1]$? What is its interior?
Closure: $[0, 1]$; interior: $(0, 1)$. Can you prove it?

3. **Finite complement topology.** Consider the collection $\mathcal{T} := \{U \subset \mathbb{R} | (\mathbb{R} \setminus U) \text{ is finite}\} \cup \{\emptyset\}$. Show this is a topology on $\mathbb{R}$. Now consider the sequence $n \mapsto 1/n$ in this topological space. Does this sequence converge? If so, what does it converge to?
This sequence converges to every point in $\mathbb{R}$.

4. Let X be a Hausdorff space and $x_n \in X$ a convergent sequence. Then the limit $\lim\limits_{n\to\infty} x_n$ is unique.

Proof. Suppose that there are two (or more) limits, say a and b. Since X is Hausdorff, we can find disjoint open sets U_a and U_b with $a \in U_a$ and $b \in U_b$. Let $n_a \in \mathbb{N}$ be such that $x_n \in U_a$ for all $n \geq n_a$ and $n_b \in \mathbb{N}$ have the property that $x_n \in U_b$ for all $n \geq n_b$. Then for all $n \geq \max\{n_a, n_b\}$ we have that $x_n \in U_a \bigcap U_b$ which is a contradiction since $U_a \bigcap U_b = \emptyset$. $\square$

The converse is not generally true.
Non-example Double origin real line
We take two distinct symbols $0_1, 0_2 \notin \mathbb{R}$, and let $X = (\mathbb{R} \setminus \{0\}) \cup \{0_1, 0_2\}$. As a basis of open sets, we take the intervals $(a, b) \subset \mathbb{R} \setminus \{0\}$, and the sets of the form $(-\epsilon, 0) \cup \{0_k\} \cup (0, \epsilon)$ for $\epsilon > 0$ and $k = 1, 2$. The open sets are then unions of such sets. The space is not Hausdorff, because every neighbourhood of 0_1 intersects every neighbourhood of 0_2 - the intersection contains a set of the form $(-\delta, 0) \cup (0, \delta)$ for a $\delta > 0$, and the sequence $(2^{-n})_{n\in\mathbb{N}}$ for example converges to both 0_1 and 0_2. The line with the doubled origin serves as an example of a space that is locally homeomorphic to $\mathbb{R}$ - every point has an open neighbourhood that is homeomorphic to $\mathbb{R}$ - but not Hausdorff.

5. If we identify $Mat_{n\times n}\mathbb{R}$ with $\mathbb{R}^{n^2}$ $(n \geq 2)$, which of the following sets are open? Which are closed? $GL_n\mathbb{R}$, O_n, the set of matrices with rank 1.
$GL_n\mathbb{R}$ is open but not closed; O_n is closed but not open; the set of matrices with rank 1 is neither open nor close. Need continuous functions.

6. Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces. Suppose that $f : X \to Y$ is continuous.
(a) Suppose that A is a subset of X. Show that $\mathrm{res}_A f$, the restriction of f to A, is continuous. (The restriction of f to A is the function from A to Y that sends $a \in A$ to $f(a)$.)
(b) Note that, with respect to the subspace topology on $\mathbb{Q}$, the function $g : \mathbb{Q} \to \mathbb{Q}$ that maps q to $q^2 - 2$ is continuous.
(c) Show that the function $f : X \to f(X)$ is continuous.
Note: This is a good exercise to pracitice familiarity with the definition.