Math 297 Discussion 6 Notes

Annie Xu

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Recall examples of topologies we've seen: indiscrete topology, discrete topology, metric topology, subspace topology.

More Topological Concepts

Recall the definition of continuity of functions defined on topological spaces.

Definition 0.1. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The map $f : X \to Y$ is a **homeomorphism** if f is a continuous bijective map with continuous inverse. If there's such f between X and Y, we say X and Y are homeomorphic.*

From the above definition, if f is a continuous bijective open(closed) map, then f is a homeomorphism. (Why?)

Definition 0.2. *A topological space X is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, X is said to be connected.*

An alternate definition of connectedness is the following:
A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself. (Why?)

Examples
1. Let X denote a two-point space in the indiscrete topology. Is X connected?
2. Let Y denote the subspace $[-1, 0) \cup (0, 1]$ of the real line $\mathbb{R}$. Each of the sets $[-1, 0)$ and $(0, 1]$ is nonempty and open in Y; therefore, Y is disconnected.
3. Let X be the subspace $[-1, 1]$ of the real line. The sets $[-1, 0]$ and $(0, 1]$ are disjoint and nonempty, but they are not both open. In fact, the space X is connected.
4. The rationals $\mathbb{Q}$ are not connected. In fact, the only connected subspaces of $\mathbb{Q}$ are the one-pont sets: If Y is a subspace of $\mathbb{Q}$ containing two points p and q, one can choose an irrational number a lying between p and q, and write Y as the union of the open sets $Y \cap (-\infty, a)$ and $Y \cap (a, +\infty)$.

Question. *Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup A_n$ is connected.*

Solution: Define $B_n = \cup_{i=1}^n A_i$. *We will prove that* B_n *is connected by induction on* n. *The case* $n = 1$ *is trivial. Now assume that* $n > 1$ *and that* B_{n-1} *is connected. We have* $B_n = B_{n-1} \cup A_n$. *Moreover, both* B_{n-1} *and* A_n *are connected and* $B_{n-1} \cap A_n \subset A_{n-1} \cap A_n \neq \emptyset$. *Thus* B_n *is connected, as desired. Pick some arbitrary* $x \in A_1$. *Observe now that for all* $n, m \geq 1$, *we have* $x \in A_1 \subset B_n \cap B_m$. *Thus* $\cup A_n = \cup B_n$ *is connected.*

Question. *A space is totally disconnected if its only connected subspaces are one-point sets. Show that if* X *has the discrete topology, then* X *is totally disconnected. Does the converse hold?*
Solution: Assume that X *has the discrete topology. If* $A \subset X$ *contains more than one point and* $x \in A$, *then* $\{x\}$ *and* $A \setminus \{x\}$ *are open (since* X *is discrete) nonempty disjoint sets whose union is* A, *so* A *is not connected.*
The converse is false. A counterexample is $\mathbb{Q}$.