Math 297 Discussion 5*

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Exercises

1. **Topological closure.** Suppose $(X, \mathcal{T})$ is a topological space an $A \subset X$. The topological closure of A, denoted $\bar{A}$, is defined to be the intersection of all closed subsets of X that contains A.
(a) Show that $\bar{A}$ is the smallest closed subset of X that contains A.
(b) Show that if $A \subset B \subset X$, the $\bar{A} \subset \bar{B}$.
(c) Show that $x \in \bar{A}$ if and only if for all $U \in \mathcal{T}$ that contain x we have $U \cap A \neq \emptyset$.

2. **The interior of a set.** Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$. The *interior* of A, denoted Int (A), is defined to be the union of all open sets (open in X) contained in A. Show that Int (A) is the largest open subset of X contained in A.
(a) Consider $\mathbb{R}$ with the Euclidean topology. What is the topological closure of $A = (0, 1]$? What is its interior?

3. **Finite complement topology.** Consider the collection $\mathcal{T} := \{U \subset \mathbb{R} | (\mathbb{R} \setminus U) \text{ is finite}\} \cup \{\emptyset\}$. Show this is a topology on $\mathbb{R}$. Now consider the sequence $n \mapsto 1/n$ in this top. space. Does this sequence converge? If so, what does it converge to?

4. Let X be a Hausdorff space and $x_n \in X$ a convergent sequence. Then the limit $\lim\limits_{n\to\infty} x_n$ is unique.

Proof. Suppose that there are two (or more) limits, say a and b. Since X is Hausdorff, we can find disjoint open sets U_a and U_b with $a \in U_a$ and $b \in U_b$. Let $n_a \in \mathbb{N}$ be such that $x_n \in U_a$ for all $n \geq n_a$ and $n_b \in \mathbb{N}$ have the property that $x_n \in U_b$ for all $n \geq n_b$. Then for all $n \geq \max\{n_a, n_b\}$ we have that $x_n \in U_a \bigcap U_b$ which is a contradiction since $U_a \bigcap U_b = \emptyset$. □

Non-example Double origin real line

5. If we identify $Mat_{n\times n}\mathbb{R}$ with $\mathbb{R}^{n^2}$ $(n \geq 2)$, which of the following sets are open? Which are closed? $GL_n\mathbb{R}$, O_n, the set of matrices with rank 1.

6. Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces. Suppose that $f : X \to Y$ is continuous.
(a) Suppose that A is a subset of X. Show that $\text{res}_A f$, the restriction of f to A, is continuous. (The restriction of f to A is the function from A to Y that sends $a \in A$ to $f(a)$.)
(b) Note that, with respect to the subspace topology on $\mathbb{Q}$, the function $g : \mathbb{Q} \to \mathbb{Q}$ that maps q to $q^2 - 2$ is continuous.
(c) Show that the function $f : X \to f(X)$ is continuous.

*Reference: James Munkres. *Topology.* Prentice Hall, NJ, 2000.