Math 297 Discussion 2*

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1 Finite, infinite, and uncountable sets

A set A is called *countable* if there is a bijection between A and a subset of the natural numbers $\mathbb{N} := \{1, 2, 3, \dots\}$.

Question. *Prove the following.*
(i) A is countable if and only if $A = \emptyset$ or there is a surjective function $f : \mathbb{N} \to A$.
(ii) If A is infinite and countable then there is a bijection between A and $\mathbb{N}$.

Theorem 1. *Every infinite subset of a countable set A is countable.*

Remark. *To prove this theorem, you will probably need the notion of a sequence.*

Union and intersection of sets

Question. *Prove the following:*
(i) $B \setminus A = A^c \setminus B^c$.
(ii) $(\bigcup_\alpha B_\alpha) \setminus (\bigcup_\alpha A_\alpha) \subset \bigcup_\alpha (B_\alpha \setminus A_\alpha)$.

Theorem 2. *Let $\{E_n\}, n = 1, 2, 3, \dots,$ be a sequence of countable sets, and put*

$$S = \bigcup_{n=1}^{\infty} E_n$$

Then S is countable.

Corollary 3. *Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Put*

$$T = \bigcup_{\alpha \in A} B_\alpha$$

Then T is at most countable.

Theorem 4. *If $A_1, \dots, A_k$ are countable sets then the Cartesian product $A_1 \times \cdots \times A_k$ is countable.*

Corollary 5. *The set of all rational numbers is countable.*

Note: In fact, even the set of all algebraic numbers is countable.

Theorem 6. *Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable.*

Metric Spaces

Open and closed sets in metric spaces

Question. *Is a set necessarily either open or closed? Can a set be both? Can it be neither?*

*Reference: Walter Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, NY, 1976.

Topological Spaces

1) Topology

Question. *Let* $W = \{B, H, P\}$ *and set* $\tau_W = \{\emptyset, W, \{H\}, \{B, P\}\}$*. Verify that* (W, τ_W) *is a topological space.*

Question. *Let* $\tau_R := \{(a, \infty) | a \in \mathbb{R})\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$*. Show that* $(\mathbb{R}, \tau_R)$*is a topological space.*

2) Open and closed sets in a topological space

3) Basis and sub-basis

Remark. *If* $\mathcal{B}$*, as a basis of* X*, is countable, we say that* X *is* ***second counatble****.*

Question. *Show that the set of rational balls* $\mathcal{B} = \{B(r, q) | r \in \mathbb{Q}^n, q \in \mathbb{Q}\}$ *is a basis for* $\mathbb{R}^n$ *(with Euclidean Topology).*

4) Hausdorff spaces
Let $(X, \mathbb{T})$ be a topological space, we say X is *Hausdorff* if $\forall x, y \in X$ with $x \neq y$, there are disjoint open sets $U_x, V_y \in \mathbb{T}$ such that $x \in U_x$ and $y \in V_y$.
Note: A metric space is automatically Hausdorff (Why?)

Theorem 7. *X is Hausdorff if and only if* $\Delta = \{(x, x) | x \in X\} \subset X \times X$ *is closed in* $X \times X$ *in the product topology. (We'll prove it after introducing product topology)*