Math 297 Discussion 11 Notes

Annie Xu

April 3, 2019

Group homomorphism

Definition 0.1. *G, H are groups, a* ***group homomorphism*** *$f : G \to H$ is a function satisfying $f(xy) = f(x)f(y)\ \forall x, y \in G$ and $f(1) = 1$.*

Examples

1) General Linear Group
$GL_n(\mathbb{R}) = \{n \times n$ matrices with real entries that are invertible $(\det \neq 0)\}$
$\cdot =$ matrix multiplication
2) Let x be a plane figure, i.e. $x \in \mathbb{R}^2$
A **rigid motion** of $\mathbb{R}^2$ is a bijective continuous distance-preserving map.
The set of all rigid motions Γ forms a group under composition.
Consider $\{r \in \Gamma | r(x) = x\} = G$
Claim: G is a group under composition:
i) composition is associative
ii) identity is in G
iii) inverse: $x = r^{-1}(x)$ so $r \in G \implies r^{-1} \in G$

Subgroup

Definition 0.2. *If Γ is a group, a* ***subgroup*** *of Γ is a subset G containing 1, closed under composition ($x, y \in G \implies xy \in G$), has inverses ($x \in G \implies x^{-1} \in G$).*

Example: regular hexagon
G contains rotation by $60°$ (or any multiple of $60°$) and reflections through certain lines.
Let $a \in G$ be any reflection, $b \in G$ be rotation by $60°$, every element of G has the form b^k or ab^k for a unique $0 \leq k \leq 5$.
$\#G = 12$
$a^2 = 1, b^6 = 1$
$aba = b^{-1} \implies ab = b^{-1}a^{-1} = b^{-1}a$, i.e. $ab^{-1} = ba$
$b^2ab = b(ba)b = b(ab^{-1})b = (ba)b^{-1}b = ab^{-1}b^{-1}b = ab^{-1} = ab^5$
Presentation: $D_2 =< a, b | a^2 = 1, b^6 = 1, ab = b^{-1}a >$
Generators | relations

$D_n =$ automorphisms of regular $n-$gon
$\#D_n = 2n$
$< a, b | a^2 = b^n = 1, ab = b^{-1}a >$

Isomorphism

Definition 0.3. *An* ***isomorphism*** *between groups G and H is a bijective homomorphism.*

Definition 0.4. *G is a group. Then* ***center*** *of G is $Z(G) = \{x \in G | \forall y \in G, xy = yx\}$*

ex: $Z(G)$ is a subgroup of G
ex: $Z(Q) = \{1, -1\} \quad Z(D_4) = \{1, b^2\}$ (rotation by 180°)

Examples

1) Quarternion group: $Q = \{1, -1, i, -i, j, -j, k, -k\}$
$i^2 = -1$ $(-i)^2 = -1$ $j^2 = -1$ $(-j)^2 = -1$ $k^2 = -1$ $(-k)^2 = -1$
$ij = -ji$ $-1 \cdot i = -i = i(-1)$ $ij = k$
$ik = -ki$ $-1 \cdot j = -j$ $-j = ik$
$jk = -kj$ $-1 \cdot k = -k$
2) Cyclic group of order n

Definition 0.5. *Suppose G is a group and H is a subgroup of G. A* ***left coset*** *of H in G is a set of the form $gH = \{gh | h \in H\}$ for some $g \in G$.*

Note: right coset Hg is defined similarly.
ex: $G = \mathbb{Z}, H = 2\mathbb{Z}$, even integers
The coset of H is a set of the form $n + H$ for some $n \in \mathbb{Z}$
There are exactly 2 cosets:
$0 + H =$ all even numbers
$1 + H =$ all odd numbers
Fact: any two left cosets are either equal or disjoint
$g, g' \in G$, either $gH = g'H$ or $gH \cap g'H = \emptyset \implies$ left cosets partition G.

Definition 0.6. *The* ***index*** *of H in G, denoted $[G : H]$, is the number of left cosets (= number of right).*

Fact: Any left or right coset of H has cardinality $\#H$
We find $[G : H] = \frac{\#G}{\#H}$ (if G finite) (Lagrange Theorem)
Corollary If G is a finite group, H is a subgroup, $\#H | \#G$
Corollary If $g \in G$, $\text{ord}(g) | G$