Math 297 Discussion 9 Notes

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Product Topology

Product Topology

The leading question: suppose we have two topological spaces X and Y, we can easily get their Cartesian product as a set $X \times Y = \{(x, y) | x \in X, y \in Y\}$. Can we put a topology on $X \times Y$ to make it a "meaningful" topological space?

Definition 0.1. *Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces. We define a topology $\mathcal{T}_{X\times Y}$ on $X \times Y$ in the following way: $W \in \mathcal{T}_{X\times Y}$ if for all $(x, y) \in W$, there exist $U_x \in \mathcal{T}_X$ and $V_y \in \mathcal{T}_Y$ such that $(x, y) \in U_x \times V_y \subset W$. We call $\mathcal{T}_{X\times Y}$ the **product topology** on $X \times Y$.*

Projection maps

Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces. We equip $X \times Y$ with the product topology defined above. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be maps given by $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$.

Exercises

1. Show that $\mathcal{T}_{X\times Y}$ defined above is actually a topology. Can you give a basis for it?
Basis: cross product of basis
2. What properties can be preserved by taking a product? (Hausdorffness? Compactness? Connectedness? Path-connectedness?)
Hausdorffness. Compactness. Connectedness. Path-connectedness.
3. Are the projection maps continuous? open? closed? Prove or disprove.
They are continuous, open, but not generally closed (consider the closed set $\{(x, y) \in \mathbb{R}^2 | xy = 1\}$, whose projections onto both axes are $\mathbb{R} \setminus \{0\}$). However, the converse of being open is not true: if W is a subspace of the product space whose projections down to all the X_i are open, then W need not be open in X. (Consider for instance $W = \mathbb{R}^2 \setminus (0, 1)^2$.
4. Consider another interesting topology we can put on $X \times Y$, namely, the **box topology**. It's defined in this way: we say $W \in \mathcal{T}$ if W is an arbitrary union of finite intersection of sets of form $\pi_{()}^{-1}(U)$, where U is an open set in $() = X$ or Y in correspondence with the subscript. Prove that this is actually a topology and coincides with $\mathcal{T}_{X\times Y}$.

5. To generalize box topology in a finite product case: Let $(X_i, \mathcal{T}_i), = 1, 2, \ldots, n$ be topological spaces and $\pi_j : \prod_{i=1}^n X_i \to X_j$ projection maps defined above, we put box topology on $\prod_{i=1}^n X_i$ in the following way: we say W is open if W is an arbitrary union of finite intersection of sets of form $\pi_i^{-1}(U_i)$, where U_i is an open set in X_i.

Quotient Maps

Let $(X, \mathcal{T}_X)$ and $(Y.\mathcal{T}_Y)$ be topological spaces. $f : X \to Y$ is a quotient map if f is surjective and $f^{-1}(U)$ is open in X iff U is open in Y.
Eg: A continuous surjective open map is a quotient map.

Quotient Topology

Let $(X, \mathcal{T}_X)$ and $(Y.\mathcal{T}_Y)$ be topological spaces and $q : X \to Y$ be a quotient map. We define the quotient topology $\mathcal{T}_q$ on $Y : U \in \mathcal{T}_q$ if $q^{-1}(U)$ is open in X. Then we say $(Y, \mathcal{T}_q)$ is a quotient space of X via q.
Equivalently, a map f is a quotient map if it is onto and Y is equipped with the quotient topology with respect to f.

Exercises

1. Prove that $\mathcal{T}_q$ is a topology and is the unique topology defined on Y that makes q a quotient map.
2. Let $(X, \mathcal{T})$ be a topological space and $\sim$ be an equivalent relation on X. The canonical projection $\pi : X \to X/\sim$ defined by $\pi(x) = [x]$ is the quotient map and $X/\sim$ is a quotient space of X.
3. Let $(X, \mathcal{T}_X)$ and $(Y.\mathcal{T}_Y)$ be topological spaces and $q : X \to Y$ be a quotient map. Prove that $f : A \to Y$ is continuous if and only if $f \circ q : X \to Y$ is continuous.
4. What properties can be preserved by quotient maps? (Hausdorff, compactness, connectedness, and path-connectedness)