Math 297 Discussion 8*

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Path-connectedness

Intermediate value theorem

Examples and Counterexamples

1. Give an example of a closed and bounded metric space which is not sequentially compact.
2. Give an example of a topological space that is not Hausdorff.
3. Can we make $\mathbb{R}$ disconnected? Consider the following topology: take a basis of the topology to be all intervals of the form $[a, b)$ with $a < b \in \mathbb{R}$. Why is $\mathbb{R}$ under this topology disconnected?
4. Is the union of two connected subsets connected? How about intersection?
5. Give an example of an infinite union of open sets that is closed.
6. Give an example of a union of compact spaces that is not compact.
7. Give an example of a topology of $\mathbb{N}$ such that there are exactly n open sets in this space.

Exercises

1. (Basis) Show that the set of rational balls $\mathbb{B} = \{B(r, q) | r \in \mathbb{Q}^n, q \in \mathbb{Q}\}$ is a basis for $\mathbb{R}^n$ (with Euclidean topology).
2. (Connectedness) Prove the ***Intermediate Value Theorem***: Let $(X, \mathcal{T})$ be a connected topological space and $f : X \to \mathbb{R}$ be a continuous map. For all $x, y \in X$, and for all t between $f(x)$ and $f(y)$, there exists $z \in X$ such that $f(z) = t$.
3. (Homeomorphism) Prove that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic. (Hint: What properties are preserved by homeomorphism? You may also find the following lemma helpful.)
4. (Lemma, take it for free) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a homeomorphism. Let $A \subset X$ and $B = f(A) \subset Y$. Then the restriction map $f|_A : A \to B$ is a homeomorphism $f|_A : (A, \mathcal{T}_A) \to (B, \mathcal{T}_B)$, where $\mathcal{T}_A$ and $\mathcal{T}_B$ are the subspace topologies of A and B repectively. (You may try to prove this lemma. It wil help you understand subspace topology better).

*Reference: James Munkres. *Topology.* Prentice Hall, NJ, 2000.

Product Topology

Product Topology

The leading question: suppose we have two topological spaces X and Y, we can easily get their Cartesian product as a set $X \times Y = \{(x, y) | x \in X, y \in Y\}$. Can we put a topology on $X \times Y$ to make it a "meaningful" topological space?

Definition 0.1. *Suppose* $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$ *are topological spaces. We define a topology* $\mathcal{T}_{X \times Y}$ *on* $X \times Y$ *in the following way:* $W \in \mathcal{T}_{X \times Y}$ *if for all* $(x, y) \in W$*, there exist* $U_x \in \mathcal{T}_X$ *and* $V_y \in \mathcal{T}_Y$ *such that* $(x, y) \in U_x \times V_y \subset W$*. We call* $\mathcal{T}_{X \times Y}$ *the* ***product topology*** *on* $X \times Y$*.*

Projection maps

Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces. We equip $X \times Y$ with the product topology defined above. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be maps given by $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$.

Exercises

1. Show that $\mathcal{T}_{X \times Y}$ defined above is actually a topology. Can you give a basis for it?
2. What properties can be preserved by taking a product? (Hausdorffness? Compactness? Connectedness? Path-connectedness?)
3. Are the projection maps continuous? open? closed? Prove or disprove.