Math 297 Discussion 2 Notes

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Metric Spaces

A set X, whose elements we shall call points, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p, q)$, called the *distance* from p to q, s.t.
(a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;
(b) $d(p, q) = d(q, p)$;
(c) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Examples: real line with usual distance, Euclidean metric, d_p, distance of complex numbers, Manhattan distance, supremum metric (infinity metric) $d((x_1, y_1), (x_2, y_2)) = max(|x_1 - x_2|, |y_1 - y_2|)$, metric on the space of functions. (Check these are metrics)

Open and closed sets in metric spaces These are defined in your class.
An example of a metric space in which every set is both open and closed. Let $X = \mathbb{R}$ and the discrete metric d is defined as follows: $d(x, y) = 1$ if $x \neq y$; $d(x, y) = 0$ if $x = y$.

Topological Spaces

1) Topology

A ***topology*** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a *topological space*.

2) Open and closed sets in a topological space Let $(X, \mathcal{T})$ be a topological space. A set U is said to be *open* in X if $U \in \mathcal{T}$. A set V is said to be *closed* in X if $X \setminus V \in \mathcal{T}$.

Examples
Discrete topology
Properties: The finest topology for X. Every point is an isolated point. x is not a limit point of the sequence $x, x, x, \ldots$ considered as a set, though it is an adherent point of the set. For any set $A \subset X$, $A = A^\circ = \bar{A}$. May be obtained from the discrete metric.
Indiscrete (trivial) topology
Properties: The coarsest one for X. Every point of X is a limit point for every subset of X, and every sequence converges to every point of X (we proved this later).

3) Basis and sub-basis

If X is a set, a ***basis*** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called ***basis elements***) such that

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the ***topology $\mathcal{T}$ generated by $\mathcal{B}$*** as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

A ***subbasis*** δ for a topology on X is a collection of subsets of X whose union equals X. The ***topology generated by the subbasis*** δ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of δ.

Remark. *If $\mathcal{B}$, as a basis of X, is countable, we say that X is* ***second counatble****.*

4) Hausdorff spaces
Let $(X, \mathcal{T})$ be a topological space, we say X is ***Hausdorff*** if $\forall x, y \in X$ with $x \neq y$, there are disjoint open sets $U_x, V_y \in \mathcal{T}$ such that $x \in U_x$ and $y \in V_y$.
Note: A metric space is automatically Hausdorff. (Why?)

Theorem 1. *X is Hausdorff if and only if $\Delta = \{(x, x) | x \in X\} \subset X \times X$ is closed in $X \times X$ in the product topology. (We'll prove it after introducing product topology)*