Math 297 Discussion 7*

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Connected Subspaces of $\mathbb{R}$

We switch to IBL format now.
1. Prove: $(\mathbb{R}, d_{\text{Euc}})$ is connected.
Hint. Suppose that $\mathbb{R}$ is disconnected. Then there exists nonempty, open sets $A, B \subset \mathbb{R}$ so that $\mathbb{R} = A \cup B$, and $A \cap B = \emptyset$. Since A and B are nonempty, choose $a \in A$ and choose $b \in B$. WOLOG, suppose that $a < b$. Define the set

$$C := \{x \in \mathbb{R} : [a, x] \subset A\}.$$

2. Prove that if $f : X \to Y$ is continuous, and if X is connected, then $f(X) \subset Y$ is connected. Does the converse hold? That is, is there a map $f : X \to Y$ which maps every connected set to a connected set necessarily continuous?

3. *Immediate application* Prove that $X = \mathbb{R}$ with the Euclidean metric and the space $\mathbb{R} \setminus \{0\}$ with the Euclidean metric are not homeomorphic.

Connected subsets of $\mathbb{R}$
What do they look like? Certainly an interval is connected $(a, b) \subset \mathbb{R}$, but are there any other connected subsets of $\mathbb{R}$?

Theorem 1. *A subset $S \subset \mathbb{R}$ is connected if and only if it has the following property: If $x, y \in S$, then for all $x \in (x, y)$, we have $z \in S$.*

This is saying the only connected subsets of $\mathbb{R}$ are intervals.

Question. *Do these all fit in here? Single points? Open intervals? Closed intervals? Half-open intervals? Intervals with infinite endpoints?*

4. First prove that if $S \subset \mathbb{R}$ does not have the 'interval property' defined above, then S is disconnected.

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*Reference: James Munkres. *Topology.* Prentice Hall, NJ, 2000.

Theorem 2. ***Intermediate Value Theorem.*** *Let $(X, \mathcal{T})$ be a connected topological space, and let $f : X \to \mathbb{R}$ be a continuous map (here we give $\mathbb{R}$ the Euclidean topology). If $a, b \in X$, then f achieves every intermediate value between $f(a)$ and $f(b)$.*

5. Prove it!

More generally...

Theorem 3. *The image of a connected space under a continuous map is connected.*