Math 297 Discussion 10

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Group

A **group** is a set G with a binary operation $\cdot : G \times G \to G$ s.t.
i) $\cdot$ is associative $x(yz) = (xy)z \; \forall x, y, z \in G$
ii) $\exists$ identity element $1 \in G$, $1 \cdot x = x \cdot 1 = x \; \forall x \in G$
iii) Every element x has an inverse, i.e. an element y s.t. $xy = yx = 1$
Note: Commutativity not required. Commutative groups are also called **abelian groups**.

Examples

1) Zero group 2) $G = \mathbb{Z}, \quad \cdot = +$
3) Non-example: $\mathbb{Z}$ with multiplication 4) $S_n -$ **symmetric group** on n letters
5) A vector space V along with vector addition
6) $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{Z}^+$
7) $(\mathbb{Z}/n\mathbb{Z})^\times$, $n \in \mathbb{Z}^+$ 8) $S_n -$ **symmetric group** on n letters
9) F_2 **free groups** in a, b
10) **direct product** of groups

Basic Properties

You can prove some general facts about group, using the same techniques from 297:
the identity is unique.
the inverse of each element is unique.
$(a^{-1})^{-1} = a \; \forall a \in G$
$(a \cdot b)^{-1} = (b^{-1})(a^{-1})$
generalized associative law holds.
cancellation rule: if $au = av$, then $u = v$; and if $ub = vb$, $u = v$.

Order

For G a group and $x \in G$, define the **order** of x to be the smallest positive integer n s.t. $x^n = 1$, and denote the integer by $|x|$. Also say x is of order n. If no positive power of x is the identity, the order of x is defined to be infinity.

Examples

1) $x \in G$ has order 1 if and only if x is the identity.
2) In the additive groups $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, every nonzero (nonidentity) element has infinite order.
3) In the multiplicative group $\mathbb{R} - \{0\}$ or $\mathbb{Q} - \{0\}$ the element -1 has order 2 and all other nonidentity elements have infinite order.
4) In the additive group $\mathbb{Z}/9\mathbb{Z}$, $\bar{6}$ has order 3 (Why?); the order of $\bar{5}$ is 9.
5) In the multiplicative group $(\mathbb{Z}/7\mathbb{Z})^{\times}$, $\bar{2}$ has order 3 (Why?); $\bar{3}$ has order 6.
6) S_n: using cycle notation, (1 2 3) has order 3.