Math 297 Discussion 3 Notes

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1 Topological Properties

1) Fine and coarse (relative)

Definition 1.1. *Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is* ***finer*** *than $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is* ***strictly finer*** *than $\mathcal{T}$. We also say that $\mathcal{T}$ is* ***coarser*** *than $\mathcal{T}'$, or* ***strictly coarser****, respectively. We say $\mathcal{T}$ is* ***comparable*** *with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}$.*

2) Continuity of a function in topological spaces

Definition 1.2. *Let X and Y be topological spaces. A function $f : X \to Y$ is said to be* ***continuous*** *if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.*

Note: If the topology of the range space Y is given by a basis $\mathcal{B}$, then to prove continuity of f it suffices to show that the inverse image of every *basis element* is open. The arbitrary open set V of Y can be written as a union of basis elements

$$V = \bigcup_{\alpha \in J} B_\alpha$$

Then

$$f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha)$$

So that $f^{-1}(V)$ is open if each set $f^{-1}(B_\alpha)$ is open.
If the topology on Y is given by a subbasis $\mathcal{S}$, to prove continuity of f it will even suffice to show that the inverse image of each *subbasis* element is open. The arbitrary basis element B for Y can be written as a finite intersection $S_1 \cap \cdots \cap S_n$ of subbasis elements; it follows from the equation

$$f^{-1}(B) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$$

that the inverse image of every basis element is open.

Exercise

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Proof. For each $x \in A$, by our hypothesis, $\exists U_x \in \mathcal{T}$ such that $x \in U \subset A$. Then $A = \bigcup_{x \in A} U_x$ is the union of elements of subcollection U_x in $\mathcal{T}$. By definition, A is open in X. □

2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is finer.

Solution. We compare pair-wisely from $1, 1$, and skip the not comparable pairs. $1, 1$ is comparable to all, and is coarser than all. $1, 2$ is finer than $3, 1$, $2, 1$ is coarser than $3, 2$ and $3, 3$. $1, 3$ is finer than $2, 1$ and $3, 1$, coarser than $2, 3$ and $3, 3$. $2, 1$ is coarser than $2, 3$, $3, 2$, and $3, 3$. $2, 2$ is coarser than $3, 3$. $2, 3$ is finer than $3, 1$, and coarser than $3, 3$. $3, 1$ is coaser than $3, 2$ and $3, 3$. $3, 2$ is coarser than $3, 3$. ■

3. Show that the collection $\mathcal{T}_c$ of all subsets U of X such that $X \setminus U$ either is countable or is all of X. Show that $\mathcal{T}_c$ is a topology on the set X. Is the collection $\mathcal{T}_\infty = \{U | X - U \text{ is infinite or empty or all of } X\}$ a topology on X?

Solution. $\emptyset \in \mathcal{T}_c$ because its complement is all of X. $X \in \mathcal{T}_c$ because its complement is countable. For any collection of elements U_α in $\mathcal{T}_c$, $X - U_\alpha$ is countable, so $X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$ is countable. If $U_1, \cdots, U_n$ are nonempty elements of $\mathcal{T}_c$, $X - \bigcap\limits_{i=1}^{n} U_i = \bigcup\limits_{i=1}^{n} (X - U_i)$ is a finite union of countable sets and, therefore, countable. ■

4. (a) If $\mathcal{T}_\alpha$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?
(b) Let $\mathcal{T}_\alpha$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.
(c) If $X = \{a, b, c\}$, let $\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$, and $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$. Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Solution. (a) Since $\emptyset, X$ are in every $\mathcal{T}_\alpha$, $\emptyset, X$ are in their intersection. For any collection of elements U_α in $\bigcap \mathcal{T}_\alpha$, U_α is in every $\mathcal{T}_\alpha$, so their union is in every $\mathcal{T}_\alpha$, and thus in $\bigcap \mathcal{T}_\alpha$. If $U_1, \cdots, U_n$ are nonempty elements of $\bigcap \mathcal{T}_\alpha$, then $U_1, \cdots, U_n$ are elements of each $\mathcal{T}_\alpha$, so their intersection is in each $\mathcal{T}_\alpha$, and thus in $\bigcap \mathcal{T}_\alpha$.
$\bigcup \mathcal{T}_\alpha$ is not a topology on X. For example, let $X = \{a, b, c\}.\mathcal{T}_1 = \{\emptyset, X, \{a\}\}$, and $\mathcal{T}_2 = \{\emptyset, X, \{b\}\}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b\}\}$ is not a topology on X.
(b) By (a), the largest topology contained in all $\mathcal{T}_\alpha$ is $\bigcap \mathcal{T}_\alpha$. Let $\mathcal{T}$ be the topology generated by the subbasis $\bigcup \mathcal{T}_\alpha$, then it is a topology containing all $\mathcal{T}_\alpha$. To show that it is the smallest, suppose $\exists \mathcal{T}'$ s.t. $\mathcal{T}' \subset \mathcal{T}$ but $\mathcal{T}' \neq \mathcal{T}$. Then $\exists U \in \mathcal{T}$ but $U \notin \mathcal{T}'$. By definition of $\mathcal{T}$, U is the union of finite intersections of elements of $\bigcup \mathcal{T}_\alpha$, thus making $\mathcal{T}'$ not a topology, contradiction!
(c) Smallest containing: $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Largest contained: $\{\emptyset, X, \{a\}\}$. ■

5. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ euqals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Solution. Let $\mathcal{T}$ be the topology generated by $\mathcal{A}$. By Lemma 13.1, $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{A}$. Since for any topology $\mathcal{T}'$ containing $\mathcal{A}$, any union of elements of $\mathcal{A}$ is in $\mathcal{T}'$, so $\mathcal{T} \subset \mathcal{T}'$. $\mathcal{T} \subset \bigcap \mathcal{T}'$. If $U \in \bigcap \mathcal{T}'$, then $U \in \mathcal{T}'$ for all $\mathcal{T}'$ containing $\mathcal{A}$. If $U \notin \mathcal{T}$, U is not a union of elements of $\mathcal{A}$.
If $\mathcal{A}$ is a subbasis, and $\mathcal{T}_2$ is generated by $\mathcal{A}$, it's easy to see that $\mathcal{T}_2 \in \bigcap \mathcal{T}'$ by definition. ■

6. Show that the topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable.

Solution. From Lemma 13.4 we know that there is no open interval (a, b) that contains x and lies in $[x, d)$. Then there can't be $(a, b) - K$ that contains x and lies in $[x, d)$. So $\mathbb{R}_l \not\subset \mathbb{R}_K$. On the other hand, given the basis element $B = (-1, 1) - K$ for $\mathcal{T}''$ and the point 0 of B, there is no $[c, d)$ that contains 0 and lies in B. So $\mathbb{R}_K \not\subset \mathbb{R}_l$. ■

7. Consider the following topologies on $\mathbb{R}$:
$\mathcal{T}_1$ = the standard topology,
$\mathcal{T}_2$ = the topology of $\mathbb{R}_K$,
$\mathcal{T}_3$ = the finite complement topology,
$\mathcal{T}_4$ = the upper limit topology, having all sets $(a, b]$ as basis,
$\mathcal{T}_5$ = the topology having all sets $(-\infty, a) = \{x | x < a\}$ as basis.
Determine, for each of these topologies, which of the others it contains.

Solution. $\mathcal{T}_1$ contains $\mathcal{T}_3$ and $\mathcal{T}_5$; $\mathcal{T}_2$ contains $\mathcal{T}_1$, $\mathcal{T}_3$, and $\mathcal{T}_5$; $\mathcal{T}_3$ contains nobody; $\mathcal{T}_4$ contains $\mathcal{T}_1$, $\mathcal{T}_5$; $\mathcal{T}_5$ contains nobody. ■